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La ampliación mínimamente consistente de la regla de aceptación inmediata y de la regla de intercambios en la cima en problemas de asignación escolar

Paula Jaramillo*

Abstract

Consideramos un problema de asignación escolar. Estamos interesados en soluciones que sean *consistentes*. Considere un problema y la recomendación hecha por la solución para este problema. Suponga que algunos estudiantes son removidos con sus posiciones en las escuelas. Considere el problema “reducido” de los estudiantes restantes y las posiciones restantes. Constancia dice que en este problema reducido, la solución debe asignar a cada estudiante restante la misma escuela que se le asignó originalmente. Ni la regla de aceptación inmediata (también conocida como el mecanismo de Boston), ni la regla de intercambios en la cima es consistente. Nosotros mostramos que la solución eficiente es la solución consistente más pequeña que contiene la regla de aceptación inmediata. También es la solución consistente más pequeña que contiene la regla de intercambios en la cima.

JEL classification: C78, D47, D78

Palabras clave: Asignación escolar, consistencia, mecanismo de Boston, intercambios en la cima, ampliación mínimamente consistente

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Minimal consistent enlargements of the immediate acceptance rule and the top trading cycles rule in school choice*

Paula Jaramillo[†]

Abstract

We consider school choice problems. We are interested in solutions that satisfy *consistency*. Consider a problem and a recommendation made by the solution for the problem. Suppose some students are removed with their positions in schools. Consider the “reduced” problem consisting of the remaining students and the remaining positions. Consistency states that in the reduced problem, the solution should assign each remaining student to the same school as initially. Neither the immediate acceptance rule (also known as the Boston mechanism) nor the top trading cycles rule is consistent. We show that the efficient solution is the smallest consistent solution containing the immediate acceptance rule. It is also the smallest consistent solution containing the top trading cycles rule.

JEL classification: C78, D47, D78

Keywords: school choice, consistency, Boston mechanism, top trading cycles, minimal consistent enlargement

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1 Introduction

An important application of mechanism design is school choice. Many cities assign students to schools using rules designed to recommend allocations with desirable properties (Boston, New York City, among others).¹

Formally, a school choice problem belongs to the family of problems known as “object allocation problems”. School choice has important characteristics. First, the objects are partitioned into types (schools) of identical objects (positions in a school). Second, each object-type is endowed with a priority relation over the possible receivers (students). These priority relations are assumed to be strict. Moreover, a student can get at most one object and each object-type can have more than one student. Each student has preferences over the schools and not being assigned to a school (attending a private school or being home-schooled). Each school has a strict priority over the students. Finally, each school has a limited number of seats.

The literature on school choice has focused on three rules; the deferred acceptance rule, the immediate acceptance rule, and the top trading cycles rule. The deferred acceptance rule was initially defined for many-to-one matching problems (Gale and Shapley, 1962). The top trading cycles rule was initially defined for the reallocation of objects when each agent is endowed with one object (Shapley and Scarf, 1974). Abdulkadiroğlu and Sönmez (2003) introduce these rules to the school choice literature and define the immediate acceptance rule (known as Boston mechanism). They show that the top trading cycles rule and the immediate acceptance rule are *efficient*, but that the deferred acceptance rule is not.²

We are interested in rules satisfying the property known as *consistency*. Consider a school choice problem and the assignment recommended by a rule for that problem. Remove some students with their assignment, that is, positions in schools. Now we have a smaller group of students, the ones who remained and fewer resources to distribute among them, the positions that were not occupied by the students who were removed. This problem is a “reduction” of the original one. What should

¹See Pathak (2011) for a survey on the use of mechanism design in school choice. Abdulkadiroğlu et. al. (2005a) consider the case for New York and Abdulkadiroğlu et. al. (2005b and 2006) consider the case for the Boston.

²Other rules have been defined for the school choice problem. A family of rules containing the deferred acceptance rule and the immediate acceptance rule has been defined (Chen and Kesten, 2013). While in the deferred acceptance algorithm proposals are only accepted in the last round, in the immediate acceptance algorithm some proposals might be accepted in the first round. Each rule in the family is defined by an algorithm in which the proposals might be accepted in the first round, or in the second round, and so on. A different version of the immediate acceptance rule has been introduced, the “immediate acceptance rule with skips” (Alcalde, 1996 and Harless, 2014). For a detailed discussion, see Remark 3.1.

each student receive in this reduced problem? *Consistency* states that each student should be assigned to the same school as initially.

Consistency was first defined for coalitional games (Davis and Maschler, 1965). This property has since been studied in many different settings such as bargaining games (Harsanyi, 1959 and 1963), claims and taxation problems (Young, 1987 and 1988), and classical problems of fair division (Thomson, 1988) among others.³

When a solution is not *consistent*, we would like to know how far it is to satisfy the property. One way of answering this question is to enlarge so as to recover *consistency*.⁴ However, we would like to enlarge as little as possible and recover the property. The *minimal consistent enlargement* of the rule is the smallest *consistent* solution containing the rule (Thomson, 1994).⁵

For two-sided matching, several authors have explored the implications of *consistency*.⁶ For one-to-one problems, the deferred acceptance rule is not *consistent* (Sasaki and Toda, 1992 and Toda, 2006), but the stable solution is *consistent* (Sasaki and Toda, 1992).⁷ Moreover, when preferences on both sides are strict and being unmatched is possible, no proper subsolution of the stable solution is *consistent* (Toda, 2006). As a corollary, the *minimal consistent enlargement* of the deferred acceptance rule is the stable solution.⁸ These results carry over for the deferred acceptance when applied to the school choice model.

We first show that the efficient solution is *consistent* (Proposition 3.3), but the immediate acceptance rule and the top trading cycles rule are not (Proposition 3.1 and 3.2). We find that the *minimal consistent enlargement* of the immediate acceptance rule is the efficient solution (Theorem 3.1). So is the *minimal consistent enlargement* of the top trading cycles rule (Theorem 3.2). Since for some problems the efficient solution is large and some agents may be very well treated while others may be very badly treated, our results can be seen as negative.

³For detailed surveys on *consistency* in different models of fair allocation literature, see Thomson (1990 and 2014).

⁴When we have a solution that is not *consistent*, but has a *consistent* subsolution, we might want to know how much we need to reduce this solution to obtain *consistency*. The *maximal consistent subsolution* is the largest *consistent* subsolution (Thomson, 1994).

⁵The *minimal consistent enlargement* has been studied in several models: Roommates matching problems (Nizamogullari and Özkal-Sanver, 2014b), problems in which several objects and money are allocated among a group of people (Bevia, 1996), social choice problems (Yeh, 2006), and the classical fair allocation problem (Thomson, 1994).

⁶In the model in which some objects are allocated to a group of people and some of them own some of the objects, the “you request my house-I get your turn” rule is characterised on the basis of *consistency* (Sönmez and Ünver, 2010). In roommates problems, the core is characterised on the basis of *consistency* (Can and Klaus, 2013), but there is no *consistent* “pseudo-refinement” of the core (Özkal-Sanver, 2010).

⁷The stable solution is characterised in the basis of *consistency* (Nizamogullari and Özkal-Sanver, 2014a).

⁸These results are the same in many-to-one problems, when preferences are strict and “responsive” (Toda, 2006).

The remainder of the paper is organised as follows. In Section 2, we present the school choice model and the definitions. In Section 3, we present our results.

2 The model

Let $\mathbb{S}$ denote an infinite set of potential students and $\mathbb{C}$ an infinite set of potential schools. Let $\mathcal{S}$ be the class of all finite subsets of $\mathbb{S}$ and $\mathcal{C}$ the class of all finite subsets of $\mathbb{C}$. At any given time, a finite **set of students** $S \in \mathcal{S}$ and a finite **set of schools** $C \in \mathcal{C}$ are present. Each school $c \in C$ has a **capacity**, $q_c \geq 1$ which represents the number of seats it offers. We suppose that $\sum_c q_c \geq |S|$. Let $q \equiv (q_c)_{c \in C}$ be the capacity profile.⁹

Each student $s \in S$ has a complete, transitive, and strict **preference relation** P_s over the schools and “being unassigned” (attending a private schooled or being home-schooled), which is denoted by $\emptyset$. For each pair $c, c' \in C$, we write $c P_s c'$ if s prefers c to c' ($c \neq c'$), and $c R_s c'$ if s finds c as desirable as c' , i.e., $c P_s c'$ or $c = c'$. Let $P \equiv (P_s)_{s \in S}$ be the preference profile.

Each school $c \in C$ has a complete, transitive, and strict **priority relation** $\prec_c$ over the students S . For each pair $s, s' \in S$ and each $c \in C$, we write $s \prec_c s'$ if s has priority over s' in c ($s \neq s'$), and $s \preceq_c s'$ if $s \prec_c s'$ or $s = s'$. Let $\prec \equiv (\prec_c)_{c \in C}$ be the priority profile.

A **problem** is a list $\theta \equiv (S, C, P, \prec, q)$. Let Θ be the class of all problems. An **assignment** μ for $\theta \in \Theta$ is a function $\mu : S \cup C \rightarrow C \cup \emptyset$ such that (1) each student is assigned to one school or is unassigned, i.e., for each $s \in S$, $\mu(s) \in C \cup \emptyset$; (2) each school is assigned a set of students that does not exceeds its capacity, i.e., for each $c \in C$, $\mu(c) \subset S \cup \{\emptyset\}$,¹⁰ and if $\mu(c) \subset S$, then $|\mu(c)| \leq q_c$; and (3) a student is assigned to a school if and only if the school has the student in its assignment, i.e., for each $s \in S$ and $c \in C$, $\mu(s) = c$ if and only if $s \in \mu(c)$.¹¹ Let μ be an assignment and let $i, j \in S \cup C$. If $j \in \mu(i)$, we say that i and j are assigned to each other. We refer to $\mu(i)$ as agent i 's assignment. Let $\mathcal{M}(\theta)$ be the **set of assignments** for $\theta \in \Theta$.

A **solution** φ is a non-empty correspondence that selects for each problem a set of assignments, i.e., for each $\theta \in \Theta$, $\varphi(\theta) \subset \mathcal{M}(\theta)$ and $\varphi(\theta) \neq \emptyset$. A **rule** is a single-valued solution.

We now introduce some desirable properties of assignments, rules, and solutions.

⁹Even though we assume that the number of positions offered is greater than the number of students, the results hold for more general cases.

¹⁰When a school $c \in C$ does not have a student assigned to it, we write $\mu(c) = \emptyset$.

¹¹Note that an assignment profile is equivalent to a matching in the many-to-one matching literature.

An assignment μ is **efficient for θ** if there is no other assignment at which a student is better off and some student is worse off. Formally, μ is *efficient* for θ if there is no $\mu' \in \mathcal{M}(\theta)$ such that for each $s \in S$, $\mu'(s) R_s \mu(s)$ and there is $s' \in S$ such that $\mu'(s') P_{s'} \mu(s')$. Let $\mathbf{E}(\theta)$ be the set of all efficient assignments for θ . Let φ be a solution, if for each $\theta \in \Theta$, $\varphi(\theta) \subseteq \mathbf{E}(\theta)$, then φ is *efficient*.

We now introduce our main axiom. Consider a rule φ , a problem $\theta \in \Theta$, and the assignment φ recommends for θ , $\mu \equiv \varphi(\theta)$. Suppose some students are removed with their assignments and let us reevaluate the situation at this point. We face a “reduced” problem, θ' , with the remaining students (S') as well as schools with remaining positions (for each $c \in C$ such that $q'_c = q_c - |\mu^{-1}(c) \setminus S'| > 0$). *Consistency* requires that in this new problem, each student should be assigned the same position as initially, $\varphi(\theta') = \mu|_{S'}$.

Formally, let $\theta = (S, C, P, f, q) \in \Theta$ and $\mu \in \mathcal{M}(\theta)$. Let $S' \subset S$ be a group of students such that $S' \neq \emptyset$. The **reduced problem of θ for S' at μ , $r_{S'}^\mu(\theta)$** , is the list (S', C', P', f', q') defined as follows:

- (1) each student has his school available in the reduced problem, i.e., for each $s \in S'$, $\mu(s) \in C'$;
- (2) each student has the same preferences with respect to the schools, i.e. for each $s \in S'$, P'_s is the restriction of P_s over C' ; and
- (3) each school has the same priority with respect to the students, i.e., for each $c \in C'$, $\prec'_c$ is the restriction of $\prec_c$ over S' .
- (4) each school decreases its capacity by the number of its students who were removed, i.e., for each $c \in C'$ with $|\mu(c)| = q_c$, we have $\mu(c) \cap S' = \emptyset$, $q'_c = q_c - |\mu^{-1}(c) \setminus S'|$;
- (5) each school has a positive capacity, i.e., $q'_c > 0$;

The solution φ is **consistent** if for each $\theta \in \Theta$, each $\mu \in \varphi(\theta)$, and each $S' \subset S$ with $S' \neq \emptyset$,

$$\mu_{S'} \in \varphi(r_{S'}^\mu(\theta)).$$

As explained in the introduction, when a rule is not *consistent*, we would like to know how much it has to be enlarged to obtain *consistency* (Thomson, 1994). Since we would like to be as close as the original rule as possible, we want to find the smallest *consistent* solution containing the rule. To do this, we consider a rule φ and a new solution defined as the intersection of all the *consistent* solutions containing

φ . Since *consistency* is preserved under intersections, this operation is well defined (Thomson, 1994).

The **minimal consistent enlargement** of φ , $mce(\varphi)$ is the smallest *consistent* solution containing φ (Thomson, 1994), i.e.,

$$mce(\varphi) \equiv \bigcap_{\phi \in \Phi} \phi \quad \text{where } \Phi \equiv \{\phi \mid \varphi \subseteq \phi, \text{ and } \phi \text{ is consistent}\}.$$

This is a well-defined solution because Φ is not empty. In particular, the solution that recommends the entire set of assignments is *consistent*, i.e., the solution φ such that for each θ , $\varphi(\theta) = \mathcal{M}(\theta)$ is *consistent*. Therefore, there is a *consistent* solution containing the immediate acceptance rule. Also this solution contains the top trading cycles rule.

We now define the algorithms underlying the rules.

First is the **immediate acceptance rule, IA** .¹² Let θ be a problem. Then, $IA(\theta)$ is computed as follows:

STEP 1: Each student s applies to his favorite school or if he prefers to be unassigned, he does not apply to any school. Each school c fills its positions with its top applicants according to $\prec_c$. Each school accepts at most q_c students. It rejects all other applicants. These assignments are final. If a student does not apply to a school, he remains unassigned.

STEP k , $k \geq 2$: For each school c , let q_c^k be the number of remaining positions. Each student s applies to his favorite school among those to which he hasn't applied yet or if he prefers to be unassigned, he does not apply to any school. Each school c fills its remaining positions with its top applicants according to $\prec_c$. Each school accepts at most q_c^k students. It rejects all other applicants. These assignments are final. If a student does not apply to a school, he remains unassigned.

The algorithm terminates when no student is rejected. Then, $IA(\theta)$ is the resulting assignment.

Next, the **top trading cycle rule, TTC** which is based on the top trading cycles algorithm. Let θ be a problem. Then, $TTC(\theta)$ is computed as follows:

STEP 1: Each student s points at his favorite school or if he prefers to be unassigned, he points at himself. Each school c points at the student with the highest priority according to $\prec_c$. Remove each student who is part of a cycle and assign him to the

¹²This rule is known in the literature as the Boston mechanism (Abdulkadiroğlu and Sönmez, 2003). The name of the immediate acceptance rule comes from the fact that acceptances are final at each step of the algorithm.

school at which he is pointing.¹³ These assignments are final. If a student points at himself, he remains unassigned.

STEP k , $k \geq 2$: For each school c , let q_c^k be the number of remaining positions. Among the remaining schools, each student s points at his favorite school or if he prefers to be unassigned, he points at himself. Among the remaining students, each school c points at the student with the highest priority according to $\prec_c$. Remove each student who is part of a cycle and assign him to the school at which he is pointing. These assignments are final. If a student points at himself, he remains unassigned.

The algorithm terminates when each student is part of a cycle. Then, $TTC(\theta)$ is the resulting assignment.

3 Results

We divide this section in three parts. In Part 1, we show that the Pareto solution is *consistent*, whereas neither the immediate acceptance rule nor the top trading cycles rules are. Since IA and TTC are subsolutions of the Pareto solution, their minimal consistent enlargement is contained in the Pareto solution. In Part 2, we show that the *minimal consistent enlargement* of the immediate acceptance rule is the Pareto solution. In Part 3, we show that the Pareto solution is also the *minimal consistent enlargement* of the top trading cycles rule. Therefore, to recover *consistency* and include the immediate acceptance rule or the top trading cycle rule, it is necessary to add all other *efficient* assignments. Since for some problems the efficient solution is large, some agents may be very well treated while others may be very badly treated. Then, our results can be seen as negative.

3.1 Consistency

First, we show that the Pareto solution is *consistent*. We then show that neither IA nor TTC rules are *consistent*.

Proposition 3.1. *The Pareto solution is consistent.*

Proof. Let $\theta \equiv (S, C, P, \prec, q) \in \Theta$ and $\mu \in E(\theta)$. Let $S' \subset S$, and $r_{S'}^\mu(\theta)$ such that $\mu_{S'} \notin E(r_{S'}^\mu(\theta))$. That is, there is $\mu' \in \mathcal{M}(r_{S'}^\mu(\theta))$ such that for each $s \in S'$, $\mu'(s) R'_s \mu(s)$ and there is $s \in S'$ such that $\mu'(s) P'_s \mu(s)$.

¹³Since the number of students and schools is finite, there is always a cycle and no two cycles intersect.

Let $\hat{\mu} \in \mathcal{M}(\theta)$ be defined as follows: for each $s \in \{s_1, \dots, s_K\}$, $\hat{\mu}(s) \equiv \mu'(s)$ and for each $s \in S \setminus \{s_1, \dots, s_K\}$, $\hat{\mu}(s) \equiv \mu(s)$. Note that for each $s \in S$, we have $\hat{\mu}(s) R_s \mu(s)$ and for each $s \in \{s_1, \dots, s_K\}$, $\hat{\mu}(s) P_s \mu(s)$. Therefore, $\mu \notin E(\theta)$, which is a contradiction. $\square$

Our next results are negative.

Proposition 3.2. *The immediate acceptance rule is not consistent.*

Proof. Let $\theta \equiv (S, C, P, \prec, q) \in \Theta$ be such that $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2, c_3\}$ and $q = (1, 1, 1)$. Preferences and priorities are given in Table 1.

Original Problem						Reduced Problem			
Students			Schools			Students		Schools	
P_{s_1}	P_{s_2}	P_{s_3}	$\prec_{c_1}$	$\prec_{c_2}$	$\prec_{c_3}$	P_{s_1}	P_{s_2}	$\prec_{c_1}$	$\prec_{c_2}$
c_3	c_2	c_3	s_1	s_1	s_3	c_2	c_2	s_1	s_1
c_2	c_1	c_1	s_2	s_2	s_2	c_1	c_1	s_2	s_2
c_1	c_3	c_2	s_3	s_3	s_1				

Table 1: Preferences P and priorities $\prec$ in Proposition 3.1

It is easily verified that $IA(\theta) = \begin{pmatrix} s_1 & s_2 & s_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, which is the boxed assignment in Table 1. Let $\mu \equiv IA(\theta)$. Consider the reduced problem in which the remaining students are $S' = \{s_1, s_2\}$. For this reduced problem, $r_{S'}^\mu(\theta)$, IA recommends the boldfaced assignment in Table 1. Note that $IA(r_{S'}^\mu(\theta)) = \begin{pmatrix} s_1 & s_2 \\ c_2 & c_1 \end{pmatrix} \neq IA(\theta)|_{S'} = \begin{pmatrix} s_1 & s_2 \\ c_1 & c_2 \end{pmatrix}$. Therefore, IA is not consistent. $\square$

Proposition 3.3. *The top trading cycle rule is not consistent.*

Proof. Let $\theta \equiv (S, C, P, \prec, q) \in \Theta$ be such that $S = \{s_1, s_2, s_3, s_4\}$, $C = \{c_1, c_2, c_3, c_4\}$ and $q = (1, 1, 1, 1)$. Preferences and priorities are given in Table 2.

It is easily verified that $TTC(\theta) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ c_2 & c_3 & c_1 & c_4 \end{pmatrix}$, which is the boxed assignment in Table 2. Let $\mu \equiv TTC(\theta)$. Consider the reduced problem in which the remaining students are $S' = \{s_1, s_2, s_4\}$. For this reduced problem, $r_{S'}^\mu(\theta)$, TTC recommends the boldfaced assignment in Table 2. Note that $TTC(r_{S'}^\mu(\theta)) = \begin{pmatrix} s_1 & s_2 & s_4 \\ c_4 & c_2 & c_3 \end{pmatrix} \neq TTC(\theta)|_{S'} = \begin{pmatrix} s_1 & s_2 & s_4 \\ c_2 & c_3 & c_4 \end{pmatrix}$. Therefore, TTC is not consistent. $\square$

Original Problem							
Students				Schools			
P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	$\prec_{c_1}$	$\prec_{c_2}$	$\prec_{c_3}$	$\prec_{c_4}$
$\boxed{c_2}$	$\boxed{c_3}$	$\boxed{c_1}$	c_3	s_1	s_2	s_3	s_3
c_3	c_1	c_2	$\boxed{c_4}$	s_2	s_3	s_4	s_2
c_1	c_2	c_3	c_1	$\boxed{s_3}$	$\boxed{s_1}$	s_1	s_1
c_4	c_4	c_4	c_2	s_4	s_4	$\boxed{s_2}$	$\boxed{s_4}$

Reduced Problem							
Students				Schools			
P_{s_1}	P_{s_2}	P_{s_4}		$\prec_{c_2}$	$\prec_{c_3}$	$\prec_{c_4}$	
$\boxed{c_2}$	$\boxed{c_3}$	c_3		s_2	s_4	s_2	
c_3	c_2	$\boxed{c_4}$		$\boxed{s_1}$	s_1	s_1	
c_4	c_4	c_2		s_4	$\boxed{s_2}$	$\boxed{s_4}$	

Table 2: Preferences P and priorities $\prec$ in Proposition 3.2

3.2 Minimal consistent enlargement of the immediate acceptance rule

We show that the Pareto solution is the smallest *consistent* solution containing IA.

The proof exploits a “bracing construction” (Thomson, 2014). Consider a problem and an *efficient* assignment μ for this problem. We construct a larger problem by introducing students and schools. We extend the preferences of the original students so as to take into account the new schools, and we extend the priorities of the original schools so as to take into account the new students. We do this in such a way that the IA outcome of the augmented problem that results, when restricted to the original population, is μ . Since the Pareto solution is *consistent* and IA is *efficient*, μ should be in the minimal consistent enlargement of IA.

We provide an illustration of the proof for the immediate acceptance rule.

Example 3.1. Illustrating the bracing construction for the immediate acceptance rule.

Consider the following problem. Let $S = \{s_1, \dots, s_{10}\}$, $C = \{c_1, \dots, c_9\}$, and $q = (2, 1, 1, \dots, 1)$. Preferences are given in Table 3. The priorities of the schools are such that for each $c \in C$, $\prec_c: s_1 \ s_2 \ \dots \ s_{10}$.

Let $\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \\ c_8 & c_7 & c_6 & c_5 & c_4 & c_2 & c_1 & c_3 & c_1 & c_9 \end{pmatrix}$, the boxed assignment in Table 3. Note that μ is *efficient* and that it is not the assignment profile recommended by IA, the assignment profile in boldface in Table 3.

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}	P_{s_7}	P_{s_8}	P_{s_9}	$P_{s_{10}}$
$\mathbf{c_7}$	$\mathbf{c_3}$	$\boxed{\mathbf{c_6}}$	$\mathbf{c_4}$	$\mathbf{c_1}$	$\mathbf{c_1}$	c_3	$\boxed{c_3}$	$\boxed{c_1}$	$\boxed{\mathbf{c_9}}$
$\boxed{c_8}$	c_6	$\emptyset$	$\boxed{c_5}$	c_3	$\boxed{c_2}$	$\boxed{c_1}$	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$	c_9		$\emptyset$	$\boxed{c_4}$	$\emptyset$	$\emptyset$			
	$\boxed{c_7}$			$\emptyset$					
	$\emptyset$								

Table 3: Preferences P in Example 3.1

We partition the students according to their assignment at μ . A student s is in M_0 if he is assigned to his most preferred school. A student s is in M_1 if each school that he prefers to $\mu(s)$ is filled with students in M_0 . A student s is in M_2 if each school he prefers to $\mu(s)$ is filled with students in M_0 or M_1 . We continue in this way until no student is left. In the example, $M_0 = \{s_3, s_8, s_9, s_{10}\}$, $M_1 = \{s_2, s_7\}$, $M_2 = \{s_1, s_5, s_6\}$, and $M_3 = \{s_4\}$.

We first see how to enlarge the problem in a way that guarantees that in the new problem under the IA algorithm, each student in M_1 applies to an original school after all the students in M_0 are assigned. The same thing happens to students in M_2 and M_3 . This guarantees that no two students in different components of the partition of M apply to the same school at the same time. Therefore, priorities are not used to break ties. If $s_2 \in M_1$ applied to c_3 at the same time as $s_8 \in M_0$, for example, the priority of c_3 would be needed to break ties. In this case, since $s_2 \prec_{c_3} s_8$, s_2 would be assigned to c_3 .

Pursuing the analysis of the example, we introduce a set $\Sigma \subset (\mathcal{S} \setminus \{S\})$ of new students with $|\Sigma| = 8$ and a set $\Gamma \subset (\mathcal{C} \setminus \{C\})$ of new schools with $|\Gamma| = 8$. Let $S' \equiv S \cup \Sigma$ and $C' \equiv C \cup \Gamma$. For each $\gamma \in \Gamma$, $q'_\gamma = 1$ and for each $c \in C$, $q'_c = q_c$. Moreover, for each $\gamma \in \Gamma$, $\prec'_\gamma: \sigma_1, \dots, \sigma_8, s_1, \dots, s_{10}$ and for each $c \in C$, $\prec'_c: s_1, \dots, s_{10}, \sigma_1, \dots, \sigma_8$. Finally, the preferences of the students are in Table 4. Let $\theta' \equiv (S', C', P', \prec', q')$.

The recommendation made by IA for the enlarged problem is the boxed assignment in Table 4. Note that $IA(\theta')|_S = \mu$. Thus, $\mu \in mce(IA)$ is in the *minimal consistent enlargement* of IA of the original problem.

Now we prove our first main result.

Theorem 3.1. The Pareto solution is the *minimal consistent enlargement* of the immediate acceptance rule.

Original students										New students			
P'_{s_1}	P'_{s_2}	P'_{s_3}	P'_{s_4}	P'_{s_5}	P'_{s_6}	P'_{s_7}	P'_{s_8}	P'_{s_9}	$P'_{s_{10}}$	P'_{σ_1}	P'_{σ_2}	$\dots$	P'_{σ_8}
γ_1	γ_1	c_6	γ_1	γ_1	γ_1	γ_1	c_3	c_1	c_9	γ_1	γ_2	$\dots$	γ_8
γ_2	c_3	$\emptyset$	γ_2	γ_2	γ_2	c_3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$		$\emptyset$
γ_3	c_6		γ_3	γ_3	γ_3	c_1							
γ_4	c_9		γ_4	γ_4	γ_4	$\emptyset$							
γ_5	c_7		γ_5	γ_5	γ_5								
c_7	$\emptyset$		γ_6	c_1	c_1								
c_8			γ_7	c_3	c_2								
$\emptyset$			γ_8	c_4	$\emptyset$								
			c_4	$\emptyset$									
			c_5										
			$\emptyset$										

Table 4: Preferences P' of the enlarged problem for IA in Example 3.1

Proof. Since IA is *efficient*, for each $\theta \in \Theta$, $IA(\theta) \in E(\theta)$. Then, $mce(IA) \subseteq E$. We need to show that $mce(IA) \supseteq E$.

Let $\theta \in \Theta$ and $\mu \in E(\theta)$. We divide the proof in 3 parts. First, we enlarge θ to problem θ' . Second, we calculate $IA(\theta')$. Third, we show that the restriction of $IA(\theta')$ to the original student population is μ and that the reduced problem of θ' with respect to the original students is θ .

Part 1 is based on partitioning the students in a particular way. Let $\{M_k\}_{k=0}^K$ be the partition of the students S , defined as follows.¹⁴

- The first component consists of each student who is assigned to his most preferred school under μ . That is, $M_0 \equiv \{s \in S \mid \forall c \in C, \mu(s) R_s c\}$.
- The second component includes each student s for whom the schools he prefers to $\mu(s)$ are assigned to students in M_0 . That is, $M_1 \equiv \{s \in S \mid \forall c \in C \text{ with } c P_s \mu(s), \mu(c) \subseteq M_0\}$.
- We continue in this manner to define the other components. In general, for each $k \in \{1, \dots, K\}$, $M_k \equiv \{s \in S \mid \forall c \in C \text{ with } c P_s \mu(s), \mu(c) \subseteq \bigcup_{l=0}^{k-1} M_l\}$.

Note that $K \leq |S|$ and it is indeed a partition of S : $\bigcup_{k=0}^K M_k = S$ and for each pair $k, k' \in \{0, \dots, K\}$ such that $k \neq k'$, we have $M_k \cap M_{k'} = \emptyset$. Moreover, since μ is *efficient*, $M_0 \neq \emptyset$. Finally, for each $k \in \{1, \dots, K\}$, if $M_k \neq \emptyset$, then $M_{k-1} \neq \emptyset$.

We define for each student s the number of schools that he prefers to $\mu(s)$. That is, for each $s \in S$, let $r_s(P_s) \equiv |\{c \in C \mid c P_s \mu(s)\}|$. Finally, we define

¹⁴Formally, this partition depends on μ . To avoid unnecessary notation we denote the partition as M .

for each $k \in \{0, \dots, K\}$, the number of schools that should be introduced in the preferences of each $s \in M_k$, $t_k \equiv \max\{k, \max_{s \in M_k} r_s(P_s)\}$. Note that $t_0 = 0$. Let $T_K = \sum_{k=0}^{K-1} t_k + K$.

Part 1: Defining the augmented problem $\theta' = (S', C', P', \prec', q')$.

- Students and schools. We introduce a set $\Sigma \in \mathcal{S} \setminus \{S\}$ of new students and a set $\Gamma \in \mathcal{C} \setminus \{C\}$ of new schools with $|\Sigma| = |\Gamma| = T_K$. Let $\Sigma \equiv \{\sigma_1, \dots, \sigma_{T_K}\}$ and $\Gamma \equiv \{\gamma_1, \dots, \gamma_{T_K}\}$. Let $S' \equiv S \cup \Sigma$ and $C' \equiv C \cup \Gamma$.
- Preferences.
 - Extending the preferences of each original student to the new schools. For each $k \in \{0, \dots, K\}$ and each $s \in M_k$,
 - (1) $\gamma_1 P'_s \dots P'_s \gamma_{T_K}$;
 - (2) for each $c, c' \in C \cup \emptyset$, if $c P_s c'$ then $c P'_s c'$;
 - (3) for each $c \in C$ and each $\gamma \in \{\gamma_1, \gamma_2, \dots, \gamma_{\sum_{l=0}^{k-1} t_l + k}\}$, $\gamma P'_s c$; and
 - (4) for each $\gamma \in \Gamma \setminus \{\gamma_1, \gamma_2, \dots, \gamma_{\sum_{l=0}^{k-1} t_l + k}\}$, $\emptyset P'_s \gamma$.¹⁵
 - Defining the preferences of the new students. Each new student $\sigma_i \in \Sigma$ has a unique acceptable school; namely the new school indexed by the same number $\gamma_i \in \Gamma$. Formally, for each $i \in \{1, \dots, T_K\}$, each $\sigma_i \in \Sigma$ and each $c \in C' \setminus \{\gamma_i\}$, we have $\gamma_i P'_{\sigma_i} \emptyset P'_{\sigma_i} c$.
- Priorities.
 - Extending the priorities of the original schools to the new students. For each $c \in C$,
 - (1) for each $s, s' \in S$, if $s \prec_c s'$, then $s \prec'_c s'$;
 - (2) $\sigma_1 \prec'_c \dots \prec'_c \sigma_{T_K}$; and
 - (3) for each $s \in S$ and each $\sigma \in \Sigma$, $s \prec'_c \sigma$.
 - Defining the priorities of the new schools. For each $j \in \{1, \dots, T_K\}$ and each $\gamma_j \in \Gamma$,
 - (1) for each $s' \in S' \setminus \{\sigma_j\}$, $\sigma_j \prec'_{\gamma_j} s'$;
 - (2) $\sigma_1 \prec'_{\gamma_j} \dots \prec'_{\gamma_j} \sigma_{T_K}$;
 - (3) for each $\sigma \in \Sigma$ and each $s \in S$, $\sigma \prec'_{\gamma_j} s$; and

¹⁵For each s , we are not specifying the complete preference relation below “being unassigned”, $\emptyset$. This is not a problem because the immediate acceptance rule does not take into account those schools.

(4) for each $s, s' \in S$, if $s \prec_{\gamma_j} s'$, then $s \prec'_{\gamma_j} s'$.

- Capacities. For each $c \in C$, $q'_c = q_c$ and for each $\gamma \in \Gamma$, $q'_\gamma = 1$. Let $\theta' \equiv (S', C', P', f', q')$.

Part 2. Applying IA to θ' .

STEP 1: Each student s applies to his most preferred school according to P'_s .

Note that for each $i \in \{1, \dots, T_K\}$, each $\sigma_i \in \Sigma$ applies to $\gamma_i \in \Gamma$. Since for each $i \in \{1, \dots, T_K\}$, each $\gamma_i \in \Gamma$, and each $s \in S' \setminus \{\sigma_i\}$, $\sigma_i \prec_{\gamma_i} s$, then for each $i \in \{1, \dots, T_K\}$, $IA(\theta')(\sigma_i) = \gamma_i$. In addition, since for each $i \in \{1, \dots, T_K\}$, $\gamma_i \in \Gamma$, $q'_{\gamma_i} = 1$, $IA(\theta')(\gamma_i) = \{\sigma_i\}$. That is, in the IA outcome of the augmented problem, each new student is assigned to a new school. Moreover, no original student is assigned to a new school.

Now, we consider the original students. Each $s \in M_0$ applies to his most preferred school, $\mu(s) \in C$. Since μ is a well defined assignment, then for each $s \in M_0$, $q_{\mu(s)} \geq |\mu(\mu(s))|$. Then, for each $s \in M_0$, $IA(\theta')(s) = \mu(s)$.

Note that at Step 1 ($= t_0 + 1$), each $s \in \bigcup_{k=1}^K M_k$ applies to one of the new schools. Since each of these schools receives its highest priority student and has capacity of 1, each new school rejects all the original students applying to it. Then, each $s \in \bigcup_{k=1}^K M_k$ is rejected.

STEP 2 ($= t_0 + 1 + 1$): Since $t_0 = 0$, by definition of P' , each $s \in M_1$ applies to his most preferred school, which is one of the original schools. Since $s \in M_1$, his most preferred original school is filled with students in M_0 . Then, this school rejects all its applicants.

Each student $s \in \bigcup_{k=2}^K M_k$ applies to a new school. Since each of these schools has filled up its positions, each new school rejects all its applicants. No student is assigned at this step.

STEP $t \in \{3, \dots, t_1 + 2\}$: At each of these steps, each $s \in M_1$ applies to his most preferred school that has not rejected him before. Note that each of these schools is in the original problem.

Claim 1: Each student in M_1 is assigned at one of these steps.

Note that $t_1 = \max\{1, \max_{s \in M_1} r_s(P_s)\}$ and for each $s \in M_1$, $r_s(P_s) \geq 1$. Consider $s \in M_1$ with $r_s(P_s) = 1$, by construction of P'_s , at Step 1 he applies to γ_1 . At Step 2, he applies to his most preferred original school. Since $s \in M_1$, he is rejected because that school filled its positions at Step 1 with students in M_0 . At Step 3, he applies to $\mu(s)$. Since $\mu \in E(\theta)$, then for each $s \in M_1$, $q_{\mu(s)} \geq |\mu(\mu(s))|$. Then, s is accepted. Consider $s \in M_1$ with $r_s(P_s) = 2$. Note that Steps 1 and 2 do not

change. Moreover, by the same reasoning as at Step 2, s is rejected at Step 3. At Step 4, s is accepted by $\mu(s)$. This process continues until each $s \in M_1$ is assigned. The student with the highest $r_s(P_s)$ is the last to be assigned. He will be assigned at Step t_1+2 . Then, for each $s \in M_1$, $IA(\theta')(s) = \mu(s)$.

Claim 2: No student in $\bigcup_{k=2}^K M_k$ is assigned at these steps.

By definition of P' , no student $s \in \bigcup_{k=2}^K M_k$ applies to an original school. Then, s is rejected at each Step $t \in \{3, \dots, t_1 + 2\}$.

STEP $t = t_1 + 2 + 1$: By definition of P' , each $s \in M_2$ applies to his most preferred school, which is one of the original schools. Since $s \in M_2$, the preferred school of s is assigned to students in M_0 and M_1 . Since c has already its final assignment, c rejects all its applicants.

Each student $s \in \bigcup_{k=3}^K M_k$, applies to new schools. Since each of these schools has filled up its positions, each new school rejects all its applicants. No student is assigned at this step.

STEP $t \in \{t_1 + 4, \dots, t_1 + t_2 + 3\}$: At each of these steps, each $s \in M_2$ applies to his most preferred school that has not rejected him before. Note that each of these schools is in the original problem.

Claim 1: Each student in M_2 is assigned at one of these steps.

To show that each student in M_2 is assigned at one of these steps, note that $t_2 = \max\{2, \max_{s \in M_2} r_s(P_s)\}$ and for each $s \in M_2$, $r_s(P_s) \geq 1$. Consider $s \in M_2$ with $r_s(P_s) = 1$, by construction of P'_s , until Step $t_1 + 2$ he applies to new schools. At Step $t_1 + 3$, he applies to his most preferred original school. Since $s \in M_2$, he is rejected because that school filled its positions at previous steps with students in M_0 and M_1 . At Step $t_1 + 4$, he applies to $\mu(s)$. Since $\mu \in E(\theta)$, then for each $s \in M_2$, $q_{\mu(s)} \geq |\mu(\mu(s))|$. Then, s is accepted. Consider $s \in M_2$ with $r_s(P_s) = 2$. Note that previous steps do not change. At Step $t_1 + 5$, s is accepted by $\mu(s)$. This process continues until each $s \in M_2$ is assigned. The student with the highest $r_s(P_s)$ is the last to be assigned. Note that he will be assigned at Step $t_1 + t_2 + 3$. Then, for each $s \in M_2$, $IA(\theta')(s) = \mu(s)$.

Claim 2: No student in $\bigcup_{k=3}^K M_k$ is assigned at these steps.

By definition of P' , no student $s \in \bigcup_{k=3}^K M_k$ applies to an original school. Then, each of these students is rejected at each Step $t \in \{t_1 + 4, \dots, t_1 + t_2 + 3\}$.

$\vdots$

STEP $t = \sum_{k=0}^{T-1} t_k + T + 1$: By definition of P' , each $s \in M_T$ applies to his most preferred school, which is one of the original schools. Since $s \in M_T$, s preferred school in the original problem c is assigned to students in $\bigcup_{k=0}^{T-1} M_{T-1}$. Since c is filled up, c rejects all its applicants.

Each student $s \in \bigcup_{k=T+1}^K M_k$, applies to new schools. Since each of these schools already has his final assignment, each new school rejects all its applicants. No student is assigned at this step.

STEP $t \in \{\sum_{k=0}^{T-1} t_k + T + 2, \dots, \sum_{k=0}^T t_k + (T + 1)\}$: At each of these steps, each $s \in M_T$ applies to his most preferred school that has not rejected him before. Note that each of these schools is in the original problem.

Claim 1: Each student in M_T is assigned at one of these steps.

To show that each student in M_T is assigned at one of these steps, note that $t_T = \max\{T, \max_{s \in M_T} r_s(P_s)\}$ and for each $s \in M_T$, $r_s(P_s) \geq 1$. Consider $s \in M_T$ with $r_s(P_s) = 1$, by construction of P'_s , until Step $\sum_{k=0}^{T-1} t_k + T$ he applies to new schools. At Step $\sum_{k=0}^{T-1} t_k + T + 1$, he applies to his most preferred original school. Since $s \in M_T$, he is rejected because that school filled its positions at previous steps with students in $\bigcup_{k=0}^{T-1} M_k$. At Step $\sum_{k=0}^{T-1} t_k + T + 2$, he applies to $\mu(s)$. Since $\mu \in E(\theta)$, then for each $s \in M_T$, $q_{\mu(s)} \geq |\mu(\mu(s))|$. Then, s is accepted. Consider $s \in M_T$ with $r_s(P_s) = 2$. Note that previous steps do not change. At Step $\sum_{k=0}^{T-1} t_k + T + 3$, s is accepted by $\mu(s)$. This process continues until each $s \in M_T$ is assigned. The student with the highest $r_s(P_s)$ is the last to be assigned. Note that he will be assigned at Step $\sum_{k=0}^T t_k + T + 1$. Then, for each $s \in M_T$, $IA(\theta')(s) = \mu(s)$.

Claim 2: No student in $\bigcup_{k=T+1}^K M_k$ is assigned at these steps.

By definition of P' , no student $s \in \bigcup_{k=T+1}^K M_k$ applies to an original school. Then, he is rejected at each Step $t \in \{\sum_{k=0}^{T-1} t_k + T + 2, \dots, \sum_{k=0}^T t_k + (T + 1)\}$.

Since K is finite, the algorithm finishes at a finite number of steps.

Part 3: The restriction of $IA(\theta')$ to the original students is the assignment μ .

Note that the immediate rule gives to each student in the original problem his assignment in μ . That is, $IA(\theta')|_S = \mu$. Therefore, $\mu \in mce(IA(\theta))$ and $mce(IA) \supseteq E$. Then, $mce(IA) = E$. $\square$

Remark 3.1. We consider a different rule, the **immediate acceptance rule with skips**.¹⁶ The algorithm to find the assignment recommended by this rule is very similar to the immediate acceptance rule. Step 1 is the same. At Step k , $k \geq 2$, the only difference is that each student s applies to his most preferred school according to P_s *that is not full*. The immediate acceptance rule with skips is *efficient* (Harless, 2014). Using the same example as in Proposition 1, it is easy to see that it is not

¹⁶I thank William Thomson for raising this question. Harless (2014) gives the name to the rule and compares the properties of this rule, the deferred acceptance rule, and the immediate acceptance rule.

consistent. Moreover, the Pareto solution is its *minimal consistent enlargement*. The proof is similar, but easier than the one provided in Theorem 3.1.¹⁷

3.3 Minimal consistent enlargement of the top trading cycles rule

We follow the same logic of the bracing construction for the TTC.

Now we illustrate the proof for the top trading cycles rule.

Example 3.2. Illustrating the bracing construction for the top trading cycles rule.

Consider the problem in Example 3.1. Let μ be the boxed assignment in Table 3. Note that μ is *efficient* and that it is not the assignment profile recommended by TTC, the assignment profile in boldface in Table 3.

We now see how to enlarge the problem in a way that guarantees that under the top trading cycle algorithm, each student $s \in M_0$ is part of a cycle before each student in M_1 who prefer $\mu(s)$ to his assignment at μ . To do this, for each original student we introduce a new school that points at him and a new student who points at this new school. Each original student points according to his original preferences. The key of this construction is to change the priorities of the original schools. In this example, since for each original school s_1 has the highest priority, s_1 is assigned to the school he most prefers, c_7 . To guarantee that s_1 is assigned to $c_8 = \mu(s_1)$, c_1 cannot point at s_1 and should be assigned at an earlier step than s_1 .

Pursuing the analysis of the example, we introduce a set $\Sigma \subset (\mathcal{S} \setminus \{S\})$ of new students with $|\Sigma| = 10$ and a set $\Gamma \subset (\mathcal{C} \setminus \{C\})$ of new schools with $|\Gamma| = 10$. Let $S' \equiv S \cup \Sigma$ and $C' \equiv C \cup \Gamma$. For each $\gamma \in \Gamma$, $q'_\gamma = 1$ and for each $c \in C$, $q'_c = q_c$. Moreover, each new student only wants to be assigned to the new school indexed by the same number, i.e., for each $i \in \{1, \dots, 10\}$, $\sigma_i \in \Sigma$, $\gamma_i P'_\sigma \emptyset$. Each original student has the same preferences over the original schools and finds all the new schools worse than being unassigned, i.e., for each $s \in S$, $P'_s|_C = P_s|_C$ and for each $\gamma \in \Gamma$, $\emptyset P'_s \sigma$.

Finally, the priorities of the schools are in Table 5. Let $\theta' \equiv (S', C', P', \prec', q')$.

The recommendation made by TTC for the enlarged problem restricted to S is $TTC(\theta')|_S = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} \\ c_8 & c_7 & c_6 & c_5 & c_4 & c_2 & c_1 & c_3 & c_1 & c_9 \end{pmatrix}$. Note that $TTC(\theta')|_S = \mu$. Thus, $\mu \in mce(TTC)$ is in the *minimal consistent enlargement* of TTC of the original problem.

¹⁷The complete proof is available upon request.

Original schools									New schools			
$\prec'_{c_1}$	$\prec'_{c_2}$	$\prec'_{c_3}$	$\prec'_{c_4}$	$\prec'_{c_5}$	$\prec'_{c_6}$	$\prec'_{c_7}$	$\prec'_{c_8}$	$\prec'_{c_9}$	$\prec'_{\gamma_1}$	$\prec'_{\gamma_2}$	$\dots$	$\prec'_{\gamma_{10}}$
σ_9	σ_6	σ_8	σ_5	σ_4	σ_3	σ_2	σ_1	σ_{10}	s_1	s_2	$\dots$	s_{10}
σ_7	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	$\vdots$	$\vdots$		$\vdots$
s_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$\vdots$	s_{10}	s_{10}	s_{10}	s_{10}	s_{10}	s_{10}	s_{10}	s_{10}				
s_{10}	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$\vdots$												

Table 5: Priorities $\prec'$ of the enlarged problem for TTC in Example 3.1

Now we prove the final main result of the paper.

Theorem 3.2: The Pareto solution is the minimal consistent enlargement of the top trading cycles rule.

Proof. Since TTC is *efficient*, for each $\theta \in \Theta$, $TTC(\theta) \in E(\theta)$. Then, $mce(TTC) \subseteq E$. We need to show that $mce(TTC) \supseteq E$.

Let $\theta \in \Theta$ and $\mu \in E(\theta)$. We divide the proof in 3 parts. First, we enlarge θ to problem θ' . Second, we calculate $TTC(\theta')$. Third, we show that the restriction of $TTC(\theta')$ to the original student population is μ and that the reduced problem of θ' with respect to the original students is θ .

We define the same partition of the students S , $M = \{M_k\}_{k=0}^K$, as the proof of the Theorem 3.1.

Part 1: Defining the augmented problem $\theta' = (S', C', P', \prec', q')$.

- Students and schools. We introduce a set $\Sigma \in \mathcal{S} \setminus \{S\}$ of new students and a set $\Gamma \in \mathcal{C} \setminus \{C\}$ of new schools with $|\Sigma| = |\Gamma| = |S|$. Let $S \equiv \{s_1, \dots, s_{|S|}\}$, $C \equiv \{c_1, \dots, c_{|C|}\}$, $\Sigma \equiv \{\sigma_1, \dots, \sigma_{|S|}\}$, and $\Gamma \equiv \{\gamma_1, \dots, \gamma_{|S|}\}$. Let $S' \equiv S \cup \Sigma$ and $C' \equiv C \cup \Gamma$.
- Preferences.
 - Extending the preferences of each original student to the new schools. Each original student rates the original schools as initially and prefers being unassigned to each of the new schools. For each $s \in S$,
 - (1) for each pair $c, c' \in C$, if $c P_s c'$ then $c P'_s c'$; and
 - (2) for each $\gamma \in \Gamma$, we have $\emptyset P'_s \gamma$.

- Defining the preferences of each new students. Each new student $\sigma_i \in \Sigma$ has a unique acceptable school; namely the new school indexed by the same number $\gamma_i \in \Gamma$. Formally, for each $i \in \{1, \dots, |S|\}$, each $\sigma_i \in \Sigma$ and each $c \in C' \setminus \{\gamma_i\}$, we have $\gamma_i P'_{\sigma_i} \emptyset P'_{\sigma_i} c$.¹⁸
- Priorities. Before defining the priorities of the original schools we need more notation. For each $j \in \{1, \dots, |S|\}$ and each $c_j \in C$, let $\hat{\mu}(c_j) \equiv \{\hat{s}_i \in \Sigma \mid \exists i \in \{1, \dots, |S|\} \text{ with } s_i \in \mu(c)\}$. Let $\hat{M} = \{\hat{M}_k\}_{k=0}^K$ be a partition over Σ . For each $k \in \{0, \dots, K\}$, $\hat{M}_k \equiv \{\sigma_i \in \Sigma \mid \exists i \in \{1, \dots, |S|\} \text{ with } s_i \in M_k\}$. Note that for each $i \in \{1, \dots, |S|\}$ and each $k \in \{0, \dots, K\}$, if $s_i \in M_k$, then $\sigma_i \in \hat{M}_k$; and if $s_i \in \mu(c)$, then $\sigma_i \in \hat{\mu}(c)$.
 - Extending the priorities of each original school: Each original school has on the top of its priority the new students indexed by the same number as the ones assigned to it in μ . The order among them is established according to which component of $\hat{M}$ the new students belong. For each $c \in C$,
 - (1) for each $i \in \{1, \dots, |S|\}$ such that $\sigma_i \in \hat{\mu}(c)$ and each $s' \in S' \setminus \hat{\mu}(c)$, $\sigma_i \prec'_c s'$;
 - (2) for each $\sigma, \sigma' \in \hat{\mu}(c)$ such that $\sigma \in \hat{M}_k$, $\sigma' \in \hat{M}_l$, and $k < l$, we have $\sigma \prec'_c \sigma'$;
 - (3) for each $\sigma_i, \sigma_j \in \hat{\mu}(c)$ if there is $k \in \{0, \dots, |S|\}$ such that $\sigma_i, \sigma_j \in \hat{M}_k$ and $i < j$, we have $\sigma_i \prec'_c \sigma_j$;
 - (4) for each $s, s' \in S$, if $s \prec_s s'$, then $s \prec' s'$;
 - (5) for each $s, s' \in S$, if $s \prec_c s'$, then $s \text{ prec}' s'$; and
 - (6) for each $s \in S$ and each $\sigma \notin \hat{\mu}(c)$, $s \prec \sigma$.¹⁹
 - Defining the priorities for each new school. For each $j \in \{1, \dots, |S|\}$, each new school $\gamma_j \in \Gamma$ has on the top of its priority the original student indexed by the same number $s_j \in S$. For each $j \in \{1, \dots, |S|\}$, each $\gamma_j \in \Gamma$, and each $s' \in S' \setminus \{s_j\}$, $s_j \prec'_{\gamma_j} s'$.²⁰
- Capacities. The new schools have one position available and the original

¹⁸For each $\sigma \in \Sigma$, we are not specifying the complete preference relation below “being unassigned”, $\emptyset$. This is not a problem because the top trading cycles rule does not take into account those schools.

¹⁹For each $c \in C$, we are not specifying the complete priority order. For this proof, it is not a problem.

²⁰For each $\gamma_j \in \Gamma$, we are not specifying the complete priority order. For this proof, it is not a problem.

schools keep the same number of positions. For each $\gamma \in \Gamma$, $q'_\gamma = 1$ and for each $c \in C$, $q'_c = q_c$. Let $\theta' \equiv (S', C', P', f', q')$.

Part 2: Applying *TTC* to θ' .

STEP 1: Each student $s \in S'$ points at his most preferred school and each school $c \in C'$ points at the student with the highest priority.

Note that for each $i \in \{1, \dots, |S|\}$, $\sigma_i \in \Sigma$ points at $\gamma_i \in \Gamma$ and γ_i points at s_i . Note that these pointings do not change until s_i is assigned. Moreover, for each $i \in \{1, \dots, |S|\}$, $\mu(s_i)$ points at the student with the highest priority, who is a new student $\sigma \in \Sigma$, and s_i points at his favorite school, which is an original school $c \in C$. When a cycle is realized, each new student $\sigma \in \Sigma$ is assigned to a new school $\gamma \in \Gamma$ and each original student $s \in S$ is assigned to an original school $c \in C$. Moreover, when for each $i \in \{1, \dots, |S|\}$, s_i is assigned, also σ_i is assigned to γ_i .

Note that each original student $s \in M_0$ is pointing at his assignment in μ , and no original student $s \in \bigcup_{k=1}^K M_k$ points at his assignment in μ .

Claim 1: At least one $s \in M_0$ is assigned.

Note that by definition of the priorities of the original schools, each $c \in C$ with $\mu(c) \cap M_0 \neq \emptyset$ is pointing at a student in M_0 . With out loss of generality, we call this student s_1 . Since s_1 is pointing at $\mu(s_1) = c$, σ_1 points at γ_1 , and $\gamma_1 \in \Gamma$ points at s_1 , we have a cycle. In the final assignment, s_1 is assigned to c and σ_1 is assigned to γ_1 .

If two or more students belong to M_0 and are assigned to the same school in the original problem, only one of them is assigned at this step.

For each student $s \in S$ assigned to a school, we have that $TTC(\theta')(s) = \mu(s)$.

Claim 2: No student $s \in \bigcup_{k=1}^K M_k$ is assigned.

Each $s \in M_1$ points at a school $\mu(s)$ that points at a student in $\hat{M}_0$. With out loss of generality, let $\sigma_1 \in \hat{\mu}(\mu(s)) \cap \hat{M}_0$ be that student. By definition, $s_1 \in M_0$. Then, $s \neq s_1$ and s cannot be part of a cycle. Note that the same argument applies for each $s \in \bigcup_{k=1}^K M_k$.

STEP $t \leq |S|$: Each remaining student $s \in S'$ points at his most preferred remaining school and each remaining school $c \in C'$ points at the remaining student with the highest priority.

Note that for each $i \in \{1, \dots, |S|\}$ if s_i is present at Step t then σ_i and γ_i are also present at Step t . Moreover, σ_i points at γ_i and γ_i points at s_i .

Claim 1: At least one $s \in \bigcup_{k=0}^{t-1} M_k$ is assigned.

Case 1: There is $s \in M_0$ not assigned at Step t . By the definition of the priorities of the original schools, each $c \in C$ with $\mu(c) \cap M_0 \neq \emptyset$ is pointing at a student in M_0 .

With out loss of generality, we call this student s_1 . Since s_1 is pointing at $\mu(s_1) = c$, σ_1 points at γ_1 , and $\gamma_1 \in \Gamma$ points at s_1 , we have a cycle. In the final assignment, s_1 is assigned to c and σ_1 is assigned to γ_1 .

Case 2: Each $s \in M_0$ is assigned and there is $s \in M_1$ not assigned at Step t . By definition of the priorities of the original schools, each $c \in C$ with $\mu(c) \cap M_1 \neq \emptyset$ is pointing at a student in M_1 . With out loss of generality, we call this student s_1 . Since s_1 is pointing at $\mu(s_1) = c$, σ_1 points at γ_1 , and $\gamma_1 \in \Gamma$ points at s_1 , we have a cycle. In the final assignment, s_1 is assigned to c and σ_1 is assigned to γ_1 .

$\vdots$

Case $t - 1$: Each $s \in \bigcup_{k=0}^{t-2} M_k$ is assigned and there is $s \in M_{t-1}$ not assigned at Step t . By definition of the priorities of the original schools, each $c \in C$ with $\mu(c) \cap M_{t-1} \neq \emptyset$ is pointing at a student in M_{t-1} . With out loss of generality, we call this student s_1 . Since s_1 is pointing at $\mu(s_1) = c$, σ_1 points at γ_1 , and $\gamma_1 \in \Gamma$ points at s_1 , we have a cycle. In the final assignment, s_1 is assigned to c and σ_1 is assigned to γ_1 .

For each student $s \in S$ assigned to a school, we have that $TTC(\theta')(s) = \mu(s)$.

Claim 2: No student $s \in \bigcup_{k=t}^K M_k$ is assigned.

Note that for each $s \in M_t$ there is $s' \in \mu(s)$ such that $s' \in M_{t-1}$. Therefore, s' is part of a cycle before s . Since s' cannot be part of a cycle before the Step t , s cannot be part of a cycle at Step t or before. In general, each $s \in \bigcup_{k=t}^K M_k$ points at a school $\mu(s)$ that points at a student in $\bigcup_{k=0}^{t-1} \hat{M}_k$. With out loss of generality, let $\sigma_1 \in \hat{\mu}(\mu(s)) \cap \bigcup_{k=0}^{t-1} \hat{M}_k$ be that student. By definition, $s_1 \in \bigcup_{k=0}^{t-1} M_k$. Then, $s \neq s_1$ and s cannot be part of a cycle.

Since at each step at least one student in S is assigned, the algorithm finishes in at most $|S|$ steps.

Part 3: The restriction of $TTC(\theta')$ to the original students is the assignment μ .

Note that the TTC gives to each student in the original problem his assignment in μ . That is, $TTC(\theta')|_S = \mu$. Therefore, $\mu \in mce(TTC(\theta))$ and $mce(TTC) \supseteq E$. Then, $mce(TTC) = E$. $\square$

Remark 3.2. In the school choice model, a weaker axiom than *consistency* is “*pre-assignment invariance*” (Kojima and Ünver, 2014 and Harless, 2014).²¹ Consider a problem and the assignment recommended by a rule for that problem. Suppose that some students are “pre-assigned” to the same schools as in the original assignment, but they are not removed. Moreover, if a school is filled with “pre-assigned” students, then the school is not removed of the problem even when is filled with pre-assigned

²¹I thank William Thomson for raising this question.

students (this is the main difference with *consistency*). Let the new problem be the students (some of which are already “pre-assigned”) and all positions (some of which are already “pre-assigned”). What should each student receive in this problem? *Pre-assignment invariance* states that each student should get the same position as initially. The immediate acceptance rule satisfies this property (Kojima and Ünver, 2014 and Harless, 2014).²² However, the top trading cycles rule does not.²³ We can define a *minimal pre-assignment invariant enlargement* of the top trading cycles rule. It is easy to show that this solution is well defined. Moreover, we can show that the Pareto solution is the *minimal pre-assignment invariant enlargement* of the TTC. The proof is similar to the one in Theorem 3.2.²⁴

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²²The IA has been characterized on the basis of *pre-assignment invariance* (Kojima and Ünver, 2014). Harless (2014) describes the difference between the two properties and shows that the IA with skips is *pre-assignment invariant*.

²³The example in Proposition 3.2 shows this. In Remark 3.2, we discuss the smallest solution containing the top trading cycles rule that satisfies this property.

²⁴The complete proof is available upon request.

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