Generalized Auslander-Reiten Theory and $t-$structures

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Abstract

Let $\mathbf{K}^{-}(\mathcal{C})$ be a full triangulated subcategory of $\mathbf{K}^{-}(\mathcal{A})$, for $\mathcal{C}$ a full additive subcategory of an abelian category $\mathcal{A}$. There is a $t-$structure $(\mathbf{D}^{\leq 0}_{\mathcal{A}}, \mathbf{D}^{\geq 0}_{\mathcal{A}})$ on $\mathbf{K}^{-}(\mathcal{A})$, where $\mathbf{D}^{\leq 0}_{\mathcal{A}}$ are complexes living in degrees less or equal than zero (see [6]). We define $\mathbf{D}^{\leq 0}_{\mathcal{C}}$ as the full subcategory $\mathbf{D}^{\leq 0}_{\mathcal{A}} \cap \mathbf{K}^{-}(\mathcal{C})$ of $\mathbf{K}^{-}(\mathcal{C})$, and $\mathbf{D}^{\geq 0}_{\mathcal{C}} := (\mathbf{D}^{\leq 0}_{\mathcal{C}})^{\perp}[1]$. Under certain approxmation hypothesis on $\mathcal{C}$ we show that $(\mathbf{D}^{\leq 0}_{\mathcal{C}}, \mathbf{D}^{\geq 0}_{\mathcal{C}})$ is a $t-$structure on $\mathbf{K}^{-}(\mathcal{C})$. We define Auslander-Reiten sequences to be simple objects of $\mathcal{H}_{\mathcal{C}}$. When $\mathcal{C}$ is a full subcategory of the category of modules over an Artin Algebra we show their existence. Finally, we relate our notion to Iyama's higher Auslander Reiten Theory.

Contents

Introduction

The concept of Auslander-Reiten (AR) sequences was introduced by Maurice Auslander (1926 - 1994) and Idun Reiten (1942 -) in [2] and [4]. With this notion and also with the concept of irreducible morphism it was possible to describe the category of finitely generated indecomposable modules over an artin algebra Λ. The study of these objects is known now as Auslander-Reiten Theory.

Auslander-Reiten sequences are certain type of short exact sequences in the category $\operatorname{mod}\Lambda$. It is possible to prove their existence if is given a non-projective or non-injective indecomposable module. In [6], an alternative approach for prove the existence of AR-sequences is given. Using a $t-$structure $\mathbf{D}^{\leq 0}_{\operatorname{mod}\Lambda}$ on $\mathbf{K}^{-}(\operatorname{mod}\Lambda)$ where $\mathbf{D}^{\leq 0}_{\operatorname{mod}\Lambda}$ consists of complexes in degrees ≤ 0, and $\mathbf{D}^{\geq 0}_{\operatorname{mod}\Lambda}$ consists of complexes in degrees ≥ -2 with no cohomology in degrees -2 and -1 vanishes. Its heart $\mathcal{H}_{\mathcal{A}}$ is the abelian category whose objects has the form $[\ X^{-2} \longrightarrow X^{-1} \longrightarrow X^{0}\ ]$ with no cohomology except in degree zero. These objects coincides with the usual AR-sequences. A theory of AR-duality was derived from Serre duality theory of triangulated categories.

If $\mathcal{C}$ is an additive subcategory of $\operatorname{mod}\Lambda$ there has been attempts to define "higher AR sequences" in $\mathcal{C}$. These should be certain long exact sequences generalizing the notion of almost split short exact sequence. O. Iyama, [9] and [10], constructs such sequences when $\mathcal{C}$ is a so-called maximal n-orthogonal subcategory of $\operatorname{mod}\Lambda$ (definition 1.2.12). He builds a higher AR-theory in this framework, containing the usual ingredients such as AR-duality and the dual of the transpose. He also consider several examples from finite dimensional algebras.

In this thesis we propose another approach to higher AR-theory that works in more general additive subcategories of $\operatorname{mod}\Lambda$ (or more generally of an abelian category $\mathcal{A}$) which generalizes that of [6]. We consider the triangulated subcategory $\mathbf{K}^{-}(\mathcal{C})$ of $\mathbf{K}^{-}(\mathcal{A})$. We define $\mathbf{D}^{\leq 0}_{\mathcal{C}}$ as the full subcategory $\mathbf{D}^{\leq 0}_{\mathcal{A}} \cap \mathbf{K}^{-}(\mathcal{C})$ of $\mathbf{K}^{-}(\mathcal{C})$, and let $\mathbf{D}^{\geq 0}_{\mathcal{C}} = (\mathbf{D}^{\leq 0}_{\mathcal{C}})^{\perp}[1]$. We prove that this defines a $t-$structure on $\mathbf{K}^{-}(\mathcal{C})$ under suitable general assumptions on $\mathcal{C}$ (definition 2.1.1). In the particular case when $\mathcal{A}$ is $\operatorname{mod}\Lambda$ and $\mathcal{C}$ is maximal n-orthogonal we show that the simple objects of the heart of the $t-$structure defined above coincide with the AR-sequences defined by Iyama in [9] and we reprove his higher AR-duality.

We now proceed to a more detailed outline of the contents of the thesis.

In Chapter 1 we give the basic facts concerning $t-$structures and AR-Theory.

The Chapter 2 is devoted to the main results. We start by defining a $t-$structure on $\mathbf{K}^{-}(\mathcal{C})$ and listing its main properties. Then, inthe second section, we identify the projective and simple objects in the heart of the $t-$structure. In the third section, we focus on the particular case of the category $\operatorname{mod}\Lambda$. Then we prove the existence of injectives and describe its form. In the last section, we relate our constructions constructions with Iyama's theory. And we reprove the existence of higher AR-sequences and AR-duality.

In the appendix A we present the basics about triangulated and derived categories, and list the main results. The appendix B contains proofs from first chapter. The reader is assumed to be familiar with basic facts about homological algebra.

Chapter 1

Preliminaries

In this chapter we give the basic definitions and tools which will be usefull in the rest of the text. We do not present proofs in this chapter, some proofs will be done in the appendix B, for the rest we refer the reader to [11], [1] and [9].

1.1 $t-$structures

We know that the derived category of an abelian category $\mathcal{A}$ contains $\mathcal{A}$ as a full abelian subcategory. However, sometimes happens that it also contains other abelian subcategories, beside the standard $\mathcal{A}$. In general, we can ask about the existence of natural abelian subcategories of triangulated categories, and one answer is given by the theory of $t-$structures due to Beilinson-Bernstein-Deligne, [7]. In this section, we present the main facts of this theory, because $t-$structures will be our main tool in the categorical study of higher Auslander-Reiten sequences.

In this section assume $\mathbf{D}$ to be a triangulated category. Here $[n]$ denotes the translation functor of the category $\mathbf{D}$. And in some cases, for short, a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ will be denote by $X \longrightarrow Y \longrightarrow Z \xrightarrow[+1]{}$.

Definition 1.1.1. *Let $\mathbf{D}^{\leq 0}$ and $\mathbf{D}^{\geq 0}$ be full subcategories of $\mathbf{D}$. We say that $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ is a $t-$structure on $\mathbf{D}$ if the following conditions are satisfied. Here, $\mathbf{D}^{\leq n} = \mathbf{D}^{\leq 0}[-n]$ and $\mathbf{D}^{\geq n} = \mathbf{D}^{\geq 0}[-n]$.*

i) $\mathbf{D}^{\leq -1} \subset \mathbf{D}^{\leq 0}$ *and* $\mathbf{D}^{\geq 1} \subset \mathbf{D}^{\geq 0}$.

ii) $\mathrm{Hom}_{\mathbf{D}}(X, Y) = 0$ *for* $X \in \mathbf{D}^{\leq 0}$ *and* $Y \in \mathbf{D}^{\geq 1}$.

iii) For any $X \in \mathbf{D}$, there exists a distinguished triangle $X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1]$ in $\mathbf{D}$ with $X_0 \in \mathbf{D}^{\leq 0}$ and $X_1 \in \mathbf{D}^{\geq 1}$

The full subcategory $\mathcal{C} = \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$ is called the **heart** *of the $t-$structure.*

Example 1.1.2. *Let $\mathcal{C}$ be an abelian category and let $\mathbf{D} = \mathbf{D}(\mathcal{C})$ denote its derived category. Denote by $\mathbf{D}^{\leq 0}(\mathcal{C})$ and $\mathbf{D}^{\geq 0}(\mathcal{C})$ the following full subcategoires:*

$$\mathbf{D}^{\leq 0}(\mathcal{C}) = \{X \in \mathbf{D} : \forall_{j>0}(H^j(X) = 0)\}$$
$$\mathbf{D}^{\geq 0}(\mathcal{C}) = \{X \in \mathbf{D} : \forall_{j<0}(H^j(X) = 0)\}$$

$(\mathbf{D}^{\leq 0}(\mathcal{C}), \mathbf{D}^{\geq 0}(\mathcal{C}))$ *is a* $t-$*structure on* $\mathbf{D}(\mathcal{C})$. *Furthermore, the heart of this* $t-$*structure is equivalent to* $\mathcal{C}$.

In the next, $\mathbf{D}$ will be denote a triangulated category and $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ a $t-$structure on it. Then we also have that $(\mathbf{D}^{\leq n}, \mathbf{D}^{\geq n})$ is a $t-$structure on $\mathbf{D}$.

Proposition 1.1.3. *The inclusion* $\mathbf{D}^{\leq n} \to \mathbf{D}$ *(resp.* $\mathbf{D}^{\geq n} \to \mathbf{D}$*) has a right adjoint functor* $\tau^{\leq n} : \mathbf{D} \to \mathbf{D}^{\leq n}$ *(resp. a left adjoint functor* $\tau^{\geq n} : \mathbf{D} \to \mathbf{D}^{\geq n}$*), i.e., there exists a morphism* $\tau^{\leq n} \to id_{\mathbf{D}}$ *(resp.* $id_{\mathbf{D}} \to \tau^{\geq n}$*) such that*

$$\mathrm{Hom}_{\mathbf{D}^{\leq n}}(X, \tau^{\leq n}Y) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(X, Y)$$

is an isomorphism for any $X \in \mathbf{D}^{\leq n}$ *and* $Y \in \mathbf{D}$ *(resp.* $\mathrm{Hom}_{\mathbf{D}^{\geq n}}(\tau^{\geq n}X, Y) \to \mathrm{Hom}_{\mathbf{D}}(X, Y)$ *is an isomorphism for any* $X \in \mathbf{D}$ *and* $Y \in \mathbf{D}^{\geq n}$*).*

Proof. See appendix B. □

Corollary 1.1.4. *If* $(\mathbf{D}^{\leq n}, \mathbf{D}^{\geq n})$ *is a* $t-$*structure on* $\mathbf{D}$, *then any object* $X \in \mathbf{D}$ *fits into a cononical distinguished triangle*

$$\tau^{\leq n}(X) \longrightarrow X \longrightarrow \tau^{\geq n+1}(X) \longrightarrow \tau^{\leq n}(X)[1]$$

Proof. See [11]. □

Definition 1.1.5. *The functors* $\tau^{\leq n}$ *and* $\tau^{\geq n}$ *are called the truncation functors with respect to the* $t-$*structure.*

We shall also write $\tau^{>n}$ and $\tau^{<n}$ instead of $\tau^{\geq n+1}$ and $\tau^{\leq n-1}$, respectively, and similarly for $\mathbf{D}^{>n}$ and $\mathbf{D}^{<n}$. Note that we have:

$$\tau^{\leq n}(X[m]) = \tau^{\leq n+m}(X)[m]$$
$$\tau^{\geq n}(X[m]) = \tau^{\geq n+m}(X)[m]$$

Proposition 1.1.6. *i) If* $X \in \mathbf{D}^{\leq n}$ *(resp.* $X \in \mathbf{D}^{\geq n}$*) then the morphism* $\tau^{\leq n}X \to X$ *(resp.* $X \to \tau^{\geq n}X$*) is an isomorphism.*

ii) Let $X \in \mathbf{D}$. *Then* $X \in \mathbf{D}^{\leq n}$ *(resp.* $X \in \mathbf{D}^{\geq n}$*) if and only if* $\tau^{>n}X = 0$ *(resp.* $\tau^{<n}X = 0$*).*

Proof. See [11]. □

Proposition 1.1.7. *Let* $X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$ *be a distinguished trinagle in* $\mathbf{D}$. *If* X' *and* X'' *belongs to* $\mathbf{D}^{\geq 0}$ *(resp.* $\mathbf{D}^{\leq 0}$*) then so does* X.

Proof. See [11]. □

Let $\mathcal{C}$ be the heart of $\mathbf{D}$, we now want to define a cohomological functor on $\mathbf{D}$.

Definition 1.1.8. *We define the functor $H^0 : \mathbf{D} \to \mathcal{C}$ by:*

$$H^0(X) = \tau^{\geq 0}\tau^{\leq 0}X = \tau^{\leq 0}\tau^{\geq 0}X$$

We also set:

$$H^n(X) = H^0(X[n]) \cong (\tau^{\geq n}\tau^{\leq n}X)[n]$$

Theorem 1.1.9. *The heart $\mathcal{C} = \mathbf{D}^{\geq 0} \cap \mathbf{D}^{\leq 0}$ is an abelian category.*

Proof. See appendix B. □

Recall that an additive functor $F : \mathbf{D} \to \mathcal{A}$, where $\mathcal{A}$ is abelian, is cohomological if for each distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow[+]{}$ in $\mathcal{C}$, the following, $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is a short exact sequence (see definition A.1.8).

Proposition 1.1.10. *The functor $H^0 : \mathbf{D} \to \mathcal{C}$ is cohomological.*

Proof. See [11]. □

We now present the notion of exact functor in triangulated categories endowed with $t-$structures.

Definition 1.1.11. *Let $\mathbf{D}_i$ $(i = 1, 2)$ be two traingulated categories endowed with $t-$structures $(\mathbf{D}_i^{\leq 0}, \mathbf{D}_i^{\geq 0})$, and let $\mathcal{C}_i$ be the heart of $\mathbf{D}_i$, $\varepsilon_i : \mathcal{C}_i \to \mathbf{D}_i$ the inclusion functor. Let $F : \mathbf{D}_1 \to \mathbf{D}_2$ be a functor of triangulated categories.*

i) One says F is left (resp. right) $t-$exact if $F(\mathbf{D}_1^{\geq 0}) \subseteq \mathbf{D}_2^{\geq 0}$ (resp. $F(\mathbf{D}_1^{\leq 0}) \subseteq \mathbf{D}_2^{\leq 0}$), and one says F is $t-$exact if it is both right and left $t-$exact.

ii) One sets:

$$^pF = H^0 \circ F \circ \varepsilon_1 : \mathcal{C}_1 \to \mathcal{C}_2$$

Proposition 1.1.12. *Let $\mathbf{D}_1$, $\mathbf{D}_2$ and F as in the previous definition, and assume F is left (resp. right) $t-$exact. Then:*

i) For $X \in \mathbf{D}_1^{\geq 0}$ (resp. $\mathbf{D}_1^{\leq 0}$), $H^0(F(X)) \cong {}^pF(H^0(X))$.

ii) $^pF : \mathcal{C}_1 \to \mathcal{C}_2$ is left (right) $t-$exact.

Proof. See appendix B. □

Corollary 1.1.13. *If $F : \mathbf{D}_1 \to \mathbf{D}_2$ is $t-$exact, then F sends $\mathcal{C}_1$ to $\mathcal{C}_2$ and $F|_{\mathcal{C}_1}$ is exact.*

Proof. See [11]. □

Proposition 1.1.14. *Let $\tilde{\mathbf{D}}_i$ be a triangulated category, $\mathbf{D}_i$ a full trinagulated subcategory, $(\mathbf{D}_i^{\leq 0}, \mathbf{D}_i^{\geq 0})$ a $t-$structure on $\mathbf{D}_i$, $i = 1, 2$. Let $f : \tilde{\mathbf{D}}_1 \to \tilde{\mathbf{D}}_2$ and $g : \tilde{\mathbf{D}}_1 \to \tilde{\mathbf{D}}_2$ be functors of triangulated categories with f a left adjoint to g. Assume $f(\mathbf{D}_1) \subseteq \mathbf{D}_2$ (resp. $g(\mathbf{D}_2) \subseteq \mathbf{D}_1$) and $f|_{\mathbf{D}_1}$ is right $t-$exact (resp. $g|_{\mathbf{D}_2}$ is left $t-$exact). Then for any $Y \in \mathbf{D}_2^{\geq 0}$ such that $g(Y) \in \mathbf{D}_1$, (resp. $X \in \mathbf{D}_1^{\leq 0}$ such that $f(X) \in \mathbf{D}_2$), one has $g(Y) \in \mathbf{D}_1^{\geq 0}$, (resp. $f(X) \in \mathbf{D}_2^{\leq 0}$).*

Proof. See [11]. □

Corollary 1.1.15. *Let $\mathbf{D}_i$ be a trinangulated category with a $t-$structure $(i = 1, 2)$, and let $f : \mathbf{D}_1 \to \mathbf{D}_2$ and $g : \mathbf{D}_2 \to \mathbf{D}_1$ be functors of triangulated categories, with f a left adjoint of g. Then f is right $t-$exact if and only if g is left $t-$exact.*

Proof. See [11]. □

1.2 Higher Dimensional Auslander-Reiten Theory

In this section we present the basic facts concerning Auslander-Reiten Theory. We start with the usal theory and then goes to the theory presented by Osamu Iyama in the paper "Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories" ([9]).

Throughout this section, Λ will be an artin algebra and k its center, i.e., k is an artin ring and Λ is a finitely generated $k-$module. If $\mathcal{C}$ is an additive category, we define an right ideal of $\mathcal{C}$, as the abelian subgroup $\mathcal{I}(X, Y)$ of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that for every $X, Y, Z \in \mathcal{C}$, for every $f \in \mathcal{I}(Z, Y)$ and for every $g \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$, fg belongs to $\mathcal{I}(X, Y)$. Dually, we define left ideals, and we say that $\mathcal{I}$ is a two-sided ideal if it is right and left ideal.

For $\mathcal{C}$ we define its (Jacobson) *radical* $\mathrm{rad}_{\mathcal{C}}$, as the two-sided ideal

$$\mathrm{rad}_{\mathcal{C}}(X, Y) = \{h \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid 1_X - gh \text{ is invertible for any } g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)\}$$

for all objects X and Y in $\mathcal{C}$.

For $X \in \mathcal{C}$, we denote by $[X]$ the following ideal

$$[X] = \{f : Y \to Z \mid f = hg \text{ for some } g : Y \to X^n, h : X^n \to Z \text{ and } n \geq 0\}$$

We denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated $\Lambda-$modules, by $\underline{\mathrm{mod}\,\Lambda} := (\mathrm{mod}\,\Lambda)/[\Lambda]$ the stable category and $\underline{\mathrm{Hom}}_\Lambda(X, Y)$ its Hom-set, similarly, $\overline{\mathrm{mod}}\Lambda := (\mathrm{mod}\,\Lambda)/[\Lambda^*]$ and $\overline{\mathrm{Hom}}_\Lambda(X, Y)$ its Hom-set. For a subcategory $\mathcal{C}$ of $\mathrm{mod}\,\Lambda$ we denote by $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ the corresponding subcategories of $\underline{\mathrm{mod}\,\Lambda}$ and $\overline{\mathrm{mod}\,\Lambda}$ respectively. We have the following dualities:

- $\mathrm{Hom}_\Lambda(\square, \Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{op}$ is called *the standard duality*
- For $X \in \mathrm{mod}\,\Lambda$, take a minimal projective resolution $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0$, define $\mathrm{Tr} : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda^{op}$ by $\mathrm{Tr}X := \mathrm{coker}\, f_1^*$, and call it the *transpose duality.*

- Let J be the injective hull of the $k-$module $k/\operatorname{rad}(k)$, then we have a duality $D := \operatorname{Hom}_k(\square, J) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{op}$, usually, this will be denote by $D = ()^*$.

We define the *syzygy functor* $\Omega : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$ as follows: If $X \in \underline{\mathrm{mod}}\Lambda$, consider a fixed projective cover $h_X : P(X) \to X$ and put $\Omega(X) := \ker h_X$; if $f : X \to Y$ is a morphism in $\underline{\mathrm{mod}}\Lambda$ consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega(X) \arrow[r] \arrow[d, "\rho(\tilde{f})"] & P(X) \arrow[r, "h_X"] \arrow[d, "\tilde{f}"] & X \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega(Y) \arrow[r] & P(Y) \arrow[r, "h_Y"] & X \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{f}$ exists because $P(X)$ is projective and $\rho(\tilde{f}) = \tilde{f}|_{\Omega(X)}$. Note that $\rho(\tilde{f})$ depends of the choice of $\tilde{f}$, but if $f' : P(X) \to P(Y)$ is other choice we get a new morphism $\rho(f') : \Omega(X) \to \Omega(Y)$, then $\rho(\tilde{f}) - \rho(f') \in [\Lambda](\Omega(X), \Omega(Y))$ so they coincide in $\underline{\mathrm{Hom}}(\Omega(X), \Omega(Y))$.

The following types of morphisms, are important for construct Auslander-Reiten sequences.

Definition 1.2.1.
- *A Λ-homomorphism $f : L \to M$ is called **left minimal** if every $h \in \operatorname{End} M$ such that $hf = f$ is an automorphism.*
- *A Λ-homomorphism $g : M \to N$ is called **rigth minimal** if every $k \in \operatorname{End} M$ such that $gk = g$ is an automorphism.*
- *A Λ-homomorphism $f : L \to M$ is called **left almost split** if f is not split monomorphism and for every Λ-homomorphism $u : L \to U$ that is not split monomorphism there exists $u' : M \to U$ such that $u'f = u$.*
- *A Λ-homomorphism $g : M \to N$ is called **right almost split** if g is not split epimorphism and for every Λ-homomorphism $v : V \to N$ that is not split epimorphism there exists $v' : V \to M$ such that $gv' = v$.*
- *A Λ-homomorphism $f : L \to M$ is called a **source map** if it is both left minimal and left almost split.*
- *A Λ-homomorphism $g : M \to N$ is called a **sink map** if it is both right minimal and right almost split.*

Proposition 1.2.2. _

- *If $f : L \to M$ and $f' : L \to M'$ are source maps, then there exists an isomorphism $h : M \to M'$ such that $f' = hf$.*
- *If $g : M \to N$ and $g' : M' \to N$ are sink maps, then there exists an isomorphism $k : M \to M'$ such that $g = g'k$.*
- *If $f : L \to M$ is source map then L is indecomposable.*

- *If $g : M \to N$ is sink map then N is indecomposable.*

Proof. See [1]. $\square$

The above morphisms allow us to define the main object of study, i.e., almost split sequences or Auslander-Reiten sequences.

Definition 1.2.3. *A short exact sequence in* $\operatorname{mod}\Lambda$

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

*is called an **Auslander-Reiten sequence** or an **almost split sequence** provided*

(1) *f is source map.*

(2) *g is sink map.*

Proposition 1.2.4. *Let* $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *be a short exact sequence. Then the following are equivalent.*

- *L is indecomposable and g is right almost split.*
- *N is indecomposable and f is left almost split.*
- *f is source map.*
- *g is sink map.*

Proof. See [1]. $\square$

Using the transpose duality Tr and the duality D, we define the Auslander-Reiten translations as $\tau := D\mathrm{Tr}$ and $\tau^{-} := \mathrm{Tr}D$. These have important properties, for our purpose we list the following:

Proposition 1.2.5. (1) *The Auslander-Reiten translations τ and τ^{-} induce mutually inverse equivalences between* $\underline{\operatorname{mod}\Lambda}$ *and* $\overline{\operatorname{mod}\Lambda}$.

(2) **AR-Duality.** *For $X, Y \in \operatorname{mod}\Lambda$ we have* $\underline{\operatorname{Hom}}_{\Lambda}(X,Y) \cong D\operatorname{Ext}^1_{\Lambda}(Y,\tau X)$ *and* $\overline{\operatorname{Hom}}_{\Lambda}(X,Y) \cong D\operatorname{Ext}^1_{\Lambda}(\tau^{-}Y,X)$ *which are functorial in both variables.*

Proof. See [1]. $\square$

The following theorem asserts the existence of AR-sequences in the category $\operatorname{mod}\Lambda$.

Theorem 1.2.6. (1) *For any non-projective $X \in \operatorname{Ind}(\operatorname{mod}\Lambda)$ there exists an AR-sequence* $0 \longrightarrow \tau X \longrightarrow C \longrightarrow X \longrightarrow 0$ *in* $\operatorname{mod}\Lambda$.

(2) *For any non-injective $Y \in \operatorname{Ind}(\operatorname{mod}\Lambda)$ there exists an AR-sequence* $0 \longrightarrow Y \longrightarrow C \longrightarrow \tau^{-}Y \longrightarrow 0$ *in* $\operatorname{mod}\Lambda$.

Proof. See [1]. □

AR-sequences allow us to construct a quiver which record the information we have on the category $\operatorname{mod}\Lambda$. The points are represented by isomorphims classes of indecomposable modules. The arrows are represented for a special type of morphisms called irreducible, this morphisms admit no nontrivial factorisation. Irreducible morphisms are the components of the maps which appear in AR-sequences. The quiver form in this way is called Aruslander-Reiten quiver.

We now present the theory given by Iyama, which is a generalization of the preceding theory. The point here is that $\operatorname{mod}\Lambda$ is replaced by an additive subcategory which allows define "long almost split sequences".

Definition 1.2.7. *i) For $X, Y \in \operatorname{mod}\Lambda$, we write $X \perp_n Y$ if $\operatorname{Ext}^i_\Lambda(X,Y) = 0$ for any $0 < i \leq n$.*

ii) For full subcategories $\mathcal{C}$ and $\mathcal{D}$ of $\operatorname{mod}\Lambda$, we write $\mathcal{C} \perp_n \mathcal{D}$ if for any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ we have $X \perp_n Y$.

iii) Let $\mathcal{C}^{\perp_n} := \{X \in \operatorname{mod}\Lambda \mid \mathcal{C} \perp_n X\}$, let ${}^{\perp_n}\mathcal{C} := \{X \in \operatorname{mod}\Lambda \mid X \perp_n \mathcal{C}\}$. Let $\mathcal{X}_n := {}^{\perp_n}({}_\Lambda\Lambda) \subseteq \operatorname{mod}\Lambda$ the subcategory of objects orthogonal to Λ as a left $\Lambda-$module, similarly for $\mathcal{X}_n^{op} := {}^{\perp_n}(\Lambda_\Lambda) \subseteq \operatorname{mod}\Lambda^{op}$,and $\mathcal{Y}_n := (\Lambda^*)^{\perp_n} \subseteq \operatorname{mod}\Lambda$.

We denote by Ω^{n-1} the composition $\Omega \circ \ldots \circ \Omega$ $(n-1)$-times. And we define

$$\tau_n := D\mathrm{Tr}\Omega^{n-1} : \underline{\mathcal{X}}_{n-1} \to \overline{\mathcal{Y}}_{n-1} \text{ and } \tau_n^- := \mathrm{Tr}\Omega^{n-1}D : \overline{\mathcal{Y}}_{n-1} \to \underline{\mathcal{X}}_{n-1}$$

which will be call the *$n-$Auslander-Reiten translations.*

Theorem 1.2.8. *Let Λ be an artin $k-$algebra and $n \geq 1$. τ_n and τ_n^- are mutually inverse equivalences.*

Proof. See [9]. □

Theorem 1.2.9. *Let Λ be an artin $k-$algebra and $n \geq 1$. For any i, $(0 < i < n)$, there exists fuctorial isomorphisms for any $X \in \mathcal{X}_{n-1}$, $Y \in \mathcal{Y}_{n-1}$ and $Z \in \operatorname{mod}\Lambda$:*

$$\operatorname{Ext}^{n-i}_\Lambda(X,Z) \cong D\operatorname{Ext}^i_\Lambda(Z,\tau_n X), \qquad \underline{\operatorname{Hom}}_\Lambda(X,Z) \cong D\operatorname{Ext}^n_\Lambda(Z,\tau_n X)$$

$$\operatorname{Ext}^{n-i}_\Lambda(Z,Y) \cong D\operatorname{Ext}^i_\Lambda(\tau_n^- Y,Z), \qquad \overline{\operatorname{Hom}}_\Lambda(Z,Y) \cong D\operatorname{Ext}^n_\Lambda(\tau_n^- Y,Z)$$

Proof. See [9]. □

Definition 1.2.10. *Let $\mathcal{C}$ be a full subcategory of $\operatorname{mod}\Lambda$.*

- *We call a morphism $f : C \to X$ a* right $\mathcal{C}-$approximation *of X if $C \in \mathcal{C}$ and $\operatorname{Hom}_\Lambda(\square, C) \xrightarrow{f_*} \operatorname{Hom}_\Lambda(\square, X) \longrightarrow 0$ is exact on $\mathcal{C}$.*
- *We call $\mathcal{C}$* contravariantly finite *if any object in $\operatorname{mod}\Lambda$ has a right $\mathcal{C}-$approximation.*

- *We call a complex*

$$\mathbf{A} := \cdots \longrightarrow C_2 \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} X$$

a right $C-$resolution *if $C_i \in \mathcal{C}$ and*

$$\cdots \longrightarrow \mathrm{Hom}_\Lambda(\square, C_2) \xrightarrow{f_{2*}} \mathrm{Hom}_\Lambda(\square, C_1) \xrightarrow{f_{1*}} \mathrm{Hom}_\Lambda(\square, C_0) \xrightarrow{f_{0*}} \mathrm{Hom}_\Lambda(\square, X)$$

is exact on $\mathcal{C}$. If each f_i is minimal, we call $\mathbf{A}$ minimal.

- *We write $\mathcal{C}$-$\dim X \leq n$ if $\mathbf{A}$ has a right $\mathcal{C}$-resolution with $\mathcal{C}_{n+1} = 0$.*

Dually we have the definitions for *left $\mathcal{C}$-approximation*, *covariantly finite* subcategory, *(minimal) left $\mathcal{C}$-resolution* and $\mathcal{C}^{op}$-dim.

Definition 1.2.11. *We call $\mathcal{C}$* functorially finite *if it is covariantly and contravariantly finite*

By Auslander-Buchweitz approximation theory (see [3]), if $\mathcal{C}$ is a functorially finite subcategory of $\mathrm{mod}\,\Lambda$, then each object X of $\mathcal{C}$ has a minimal right (resp. left) $C-$resolution, which is unique up to isomorphisms of complexes.

Definition 1.2.12. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\mathrm{mod}\,\Lambda$ *and $n \geq 0$. We call $\mathcal{C}$* maximal $n-$orthogonal *if $\mathcal{C} = \mathcal{C}^{\perp_n} = {}^{\perp_n}\mathcal{C}$.*

Proposition 1.2.13. *If $\mathcal{C}$ is maximal $n-$orthogonal, then satisfy the following properties:*

1. *Closed under direct summands.*
2. *$\Lambda \oplus (\Lambda)^*$ belongs to $\mathcal{C}$.*
3. *$\mathcal{C} \subseteq \mathcal{X}_n \cap \mathcal{Y}_n$.*
4. *If $\mathcal{C}$ is contained in a subcategory $\mathcal{D}$ of* $\mathrm{mod}\,\Lambda$ *satisfying $\mathcal{D} \perp_n \mathcal{D}$, then $\mathcal{C} = \mathcal{D}$.*
5. *$\mathcal{C}$ maximal $n-$orthogonal subcategory of* $\mathrm{mod}\,\Lambda$ *if and only if $(\mathcal{C})^*$ maximal $n-$orthogonal subcategory of* $\mathrm{mod}\,\Lambda^{op}$.

Proof. See appendix B. □

Proposition 1.2.14. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\mathrm{mod}\,\Lambda$. *The conditions* (1), $(2-i)$ *and* $(3-i)$ *are equivalent for any i $(0 \leq i \leq n)$.*

(1) *$\mathcal{C}$ is maximal $n-$orthogonal.*

(2-0) *$\mathcal{C} - \dim X \leq n$ for any $X \in \mathrm{mod}\,\Lambda$, $\mathcal{C} \perp_n \mathcal{C}$ and $\Lambda \oplus (\Lambda)^* \in \mathcal{C}$.*

(2-i) *$\mathcal{C} - \dim X \leq n - i$ for any $X \in \mathcal{C}^{\perp_i}\,\mathrm{mod}\,\Lambda$, $\mathcal{C} \perp_n \mathcal{C}$ and $\Lambda \oplus (\Lambda)^* \in \mathcal{C}$.*

(2-n) *$\mathcal{C} = \mathcal{C}^{\perp_n} \cap \mathrm{mod}\,\Lambda$ and $\Lambda \in \mathcal{C}$.*

(3-0) *$\mathcal{C}^{op} - \dim X \leq n$ for any $X \in \mathrm{mod}\,\Lambda$, $\mathcal{C} \perp_n \mathcal{C}$ and $\Lambda \oplus (\Lambda)^* \in \mathcal{C}$.*

(3-i) $\mathcal{C} - \dim X \leq n - i$ *for any* $X \in {}^{\perp_i}\mathcal{C} \operatorname{mod} \Lambda$, $\mathcal{C} \perp_n \mathcal{C}$ *and* $\Lambda \oplus (\Lambda)^* \in \mathcal{C}$.

(3-n) $\mathcal{C} = {}^{\perp_n}\mathcal{C} \cap \operatorname{mod} \Lambda$ *and* $(\Lambda)^* \in \mathcal{C}$.

Proof. See [9]. $\square$

Theorem 1.2.15. *Let* $\mathcal{C}$ *be a maximal* $(n-1)-$*orthogonal subcategory of* $\operatorname{mod} \Lambda$ $(n \geq 1)$. *Then*

(1) $\tau_n X \in \mathcal{C}$ *and* $\tau_n^- X \in \mathcal{C}$ *for any* $X \in \mathcal{C}$.

(2) *We have mutually inverse equivalences* $\tau_n : \underline{\mathcal{C}} \to \overline{\mathcal{C}}$ *and* $\tau_n^- : \overline{\mathcal{C}} \to \underline{\mathcal{C}}$.

(3) τ_n *gives a bijection from non-projective objects in* $\operatorname{Ind} \mathcal{C}$ *to non-injective objects in* $\operatorname{Ind} \mathcal{C}$, *and the inverse is given by* τ_n^-.

Proof. See [9]. $\square$

Theorem 1.2.16. *Let* $\mathcal{C}$ *be a maximal* $(n-1)-$*orthogonal subcategory of* $\operatorname{mod} \Lambda$ $(n \geq 1)$. *For any* $0 < i \leq n$, *there exists functorial isomorphisms for any* $X, Y \in \mathcal{C}$:

$$\underline{\operatorname{Hom}}_\Lambda(X,Y) \cong D \operatorname{Ext}^n_\Lambda(Y, \tau_n X) \qquad \overline{\operatorname{Hom}}_\Lambda(X,Y) \cong D \operatorname{Ext}^n_\Lambda(\tau_n^- Y, X)$$

Proof. See [9]. $\square$

Recall that an additive category $\mathcal{C}$ is said to be a Krull-Schmidt category if any object is isomorphic to a finite direct sum of objects whose endomorphism rings are local. Equivalently $\mathcal{C}$ has split idempotents, where an idempotent $e = e^2 \in \operatorname{Hom}_{\mathcal{C}}(X, X)$ is split if there are morphisms $\mu : Y \to X$ and $\rho : X \to Y$ such that $\rho\mu = 1_Y$ and $\mu\rho = e$. The theorem of Krull-Schmidt holds in a Krull-Schmidt category asserting the uniqueness (up to order) for direct decompositions. Note, that if $\mathcal{C}$ is a maximal $n-$orthogonal subcategory of $\operatorname{mod} \Lambda$, then it is a Krull-Schmidt category.

Definition 1.2.17. *If* $\mathcal{C}$ *is Krull-Schmidt category, we defined a* $\mathcal{C}-$*module as a contravariant additive funtor from* $\mathcal{C}$ *to the category* **Ab** *of abelian groups. If in addition* $\mathcal{C}$ *is small, then we define the category of* $\mathcal{C}-$*modules, denoted by* $\operatorname{Mod} \mathcal{C}$, *as the category which objects are* $\mathcal{C}-$*modules and morphisms are natural transformations between them. It is knwon that* $\operatorname{Mod} \mathcal{C}$ *is an abelian category.*

Remark 1.2.18. *B. Mitchel defines* $\mathcal{C}-$*modules in [14], just for* $\mathcal{C}$ *be a small additive category, and call it* $\mathcal{C}$ *a* ringoid

As a proposition we list the basic properties of $\mathcal{C}-$modules:

Proposition 1.2.19. *Let* $\mathcal{C}$ *be Krull-Schmidt category.*

(1) *The objecs* $\operatorname{Hom}_{\mathcal{C}}(\square, X)$, *for some* X *in* $\mathcal{C}$ *are the finitely generated projective objects in* $\operatorname{Mod} \mathcal{C}$.

(2) *S is simple in* $\operatorname{Mod}\mathcal{C}$*, i.e., $S \neq 0$ and does not admit proper subfuntors if and only if there exists indecomposable object $X \in \mathcal{C}$ such that* $S \cong \operatorname{Hom}_{\mathcal{C}}(\square, X)/\operatorname{rad}_{\mathcal{C}}(\square, X)$.

Proof. See [1]. $\square$

We define the main objects of study, namely, $n-$almost split sequences or $n-$Auslander-Reiten sequences.

Definition 1.2.20. *Let $\mathcal{C}$ be a full subcategory of* $\operatorname{mod}\Lambda$.

- *We call a complex* $\cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} X$ *with terms in $\mathcal{C}$ a sink sequence of X (in $\mathcal{C}$) if $f_i \in \operatorname{rad}_{\mathcal{C}}$ and the following sequence is exact:*

$$\cdots \xrightarrow{f_{2*}} \operatorname{Hom}_{\mathcal{C}}(\square, C_1) \xrightarrow{f_{1*}} \operatorname{Hom}_{\mathcal{C}}(\square, C_0) \xrightarrow{f_{0*}} \operatorname{rad}_{\mathcal{C}}(\square, X) \longrightarrow 0$$

  We call f_0 a sink map.

- *We call a complex* $X \xrightarrow{f_0'} C_0' \xrightarrow{f_1'} C_1' \xrightarrow{f_2'} \cdots$ *with terms in $\mathcal{C}$ a source sequence of X (in $\mathcal{C}$) if $f_i' \in \operatorname{rad}_{\mathcal{C}}$ and the following sequence is exact:*

$$\cdots \xrightarrow{f_2'^{*}} \operatorname{Hom}_{\mathcal{C}}(C_1', \square) \xrightarrow{f_1'^{*}} \operatorname{Hom}_{\mathcal{C}}(C_0', \square) \xrightarrow{f_0{}^{*}} \operatorname{rad}_{\mathcal{C}}(X, \square) \longrightarrow 0$$

  We call f_0' a source map.

Definition 1.2.21. *We call an exact sequence*

$$0 \longrightarrow Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$$

with terms in $\mathcal{C}$ an $n-$almost split sequence or $n-$Auslander-Reiten sequence, if it is a sink sequence of X and a source sequence of Y simultaneously.

Lemma 1.2.22. *Let $\mathcal{C}$ be a maximal $(n-1)-$orthogonal subcategory of* $\operatorname{mod}\Lambda$ *($n \geq 1$) and $X \in \operatorname{Ind}\mathcal{C}$.*

(1) If X is not projective, then the $\mathcal{C}^{op}-$module $\operatorname{Ext}^n_\Lambda(X, \square)$ *has a simple socle with*

$$(\operatorname{soc}_{\mathcal{C}^{op}} \operatorname{Ext}^n_\Lambda(X, \square))(\tau_n X) \neq 0$$

(2) If X is not injective, then the $\mathcal{C}-$module $\operatorname{Ext}^n_\Lambda(\square, X)$ *has a simple socle with*

$$(\operatorname{soc}_{\mathcal{C}} \operatorname{Ext}^n_\Lambda(\square, X))(\tau_n^- X) \neq 0$$

(3) $(\operatorname{soc}_{\mathcal{C}} \operatorname{Ext}^n_\Lambda(\square, \tau_n X))(X) = \operatorname{soc}_{\operatorname{End}_\Lambda(X)} \operatorname{Ext}^n_\Lambda(X, \tau_n X) = \operatorname{soc}_{\operatorname{End}_\Lambda(\tau_n X)^{op}} \operatorname{Ext}^n_\Lambda(X, \tau_n X) = (\operatorname{soc}_{\mathcal{C}^{op}} \operatorname{Ext}^n_\Lambda(X, \square))(\tau_n X)$

Proof. See appendix B $\square$

For $n-$AR-sequences we have the following equivalent definitions.

Proposition 1.2.23. *Let $\mathcal{C}$ be a maximal $(n-1)-$orthogonal subcategory of* $\operatorname{mod}\Lambda$ $(n \geq 1)$ *and* $\mathbf{A} : 0 \longrightarrow Y \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \longrightarrow 0$ *an exact sequence with terms in $\mathcal{C}$. If $f_i \in \operatorname{rad}_C$ and $X, Y \in \operatorname{Ind}\mathcal{C}$, then the conditions $(1) - (6)$ are equivalent.*

(1) *$\mathbf{A}$ is an $n-AR$-sequence.*

(2) *f_0 is a sink map.*

(3) *f_n is a source map.*

(4) *$\mathbf{A}$ is in $(\operatorname{soc}_{\mathcal{C}^{op}} \operatorname{Ext}^n_\Lambda(X, \square))(Y) \setminus \{0\}$.*

(5) *$\mathbf{A}$ is in $(\operatorname{soc}_{\mathcal{C}} \operatorname{Ext}^n_\Lambda(\square, Y))(X) \setminus \{0\}$.*

(6) *$Y = \tau_n X$ and $\mathbf{A}$ is in $\operatorname{soc}_{\operatorname{End}_\Lambda(X)} \operatorname{Ext}^n_\Lambda(X, \tau_n X) = \operatorname{soc}_{\operatorname{End}_\Lambda(\tau_n X)^{op}} \operatorname{Ext}^n_\Lambda(X, \tau_n X)$.*

Proof. See [9]. $\square$

The following result asserts the existence of $n-$AR-sequences.

Theorem 1.2.24. *Let $\mathcal{C}$ be a maximal $(n-1)-$orthogonal subcategory of* $\operatorname{mod}\Lambda$ $(n \geq 1)$.

(1) *For any non-projective $X \in \operatorname{Ind}\mathcal{C}$, there exists an $n-AR$-sequence* $0 \longrightarrow Y \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow X \longrightarrow 0$ *in $\mathcal{C}$.*

(2) *For any non-injective $Y \in \operatorname{Ind}\mathcal{C}$, there exists an $n-AR$-sequence* $0 \longrightarrow Y \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow X \longrightarrow 0$ *in $\mathcal{C}$.*

(3) *Any $n-AR$-sequence* $0 \longrightarrow Y \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow X \longrightarrow 0$ *satisfies $Y \cong \tau_n X$ and $X \cong \tau_n^- Y$.*

Proof. See appendix B $\square$

Chapter 2

Generalized Auslander-Reiten Theory

In this chapter we study a $t-$structure on the category $\mathbf{K}^-(\mathcal{C})$ where $\mathcal{C}$ is a full subcategory of an abelian category $\mathcal{A}$. The idea is to generalize the results obtained in [6], where the case $\mathcal{C} = \mathcal{A}$ was investigated.Then continue this spirit, understand the simple objects in the heart of the $t-$structure defined in $\mathbf{K}^-(\mathcal{C})$, since they will be related with higher Auslander-Reiten sequences defined in [9].

The first section concerns the $t-$structure of $\mathbf{K}^-(\mathcal{C})$ and its main properties. In the second section we describe projective and simple objects in its heart. In the third section we discuss existence of simple objects in the particular case when $\mathcal{A} = \operatorname{mod}\Lambda$ for an Artin algebra Λ. The final section relates our results with those of [9] and [10].

2.1 A $t-$structure on $\mathbf{K}^-(\mathcal{C})$

Let $\mathcal{A}$ be an abelian category, let $\operatorname{Proj}(\mathcal{A})$ (resp. $\operatorname{Inj}(\mathcal{A})$) be the full subcategory of projective (resp. injective) objects. Let $\mathcal{C}$ be a full additive subcategory of $\mathcal{A}$ closed under isomorphisms and direct summands. For $\mathcal{E}$ an additive category we denote by $\mathbf{D}_{\mathcal{E}}$ its homotopy category $\mathbf{K}^-(\mathcal{E})$. Λ will be an Artin Algebra and k its center, i.e., k is an commutative Artin ring and Λ is a finitely generated $k-$module.

Let $\mathbf{D}^{\leq 0}_{\mathcal{A}}$ be the full subcategory of $\mathbf{D}_{\mathcal{A}}$ whose objects are (homotopic to) complexes living in non-positive degrees and let $\mathbf{D}^{\geq 0}_{\mathcal{A}}$ be the full subcategory of $\mathbf{D}_{\mathcal{A}}$ which consists of complexes in degrees ≥ -2 and with no cohomology in degrees -1 and -2. In [6] is shown that $(\mathbf{D}^{\geq 0}_{\mathcal{A}}, \mathbf{D}^{\geq 0}_{\mathcal{A}})$ is a $t-$structure on $\mathbf{D}_{\mathcal{A}}$. Its heart consists in complexes $X^{-2} \longrightarrow X^{-1} \longrightarrow X^0$ which are exact, except in the zero position. We set $\mathbf{D}^{\leq 0} := \mathbf{D}^{\leq 0}_{\mathcal{C}} := \mathbf{D}^{\leq 0}_{\mathcal{A}} \cap \mathbf{D}_{\mathcal{C}}$ and $\mathbf{D}^{\geq 0} := \mathbf{D}^{\geq 0}_{\mathcal{C}} := (\mathbf{D}^{\leq 0})^{\perp}[1]$. Note that $\mathbf{D}^{\leq 0}$ consists of complexes living in degree ≤ 0, while $\mathbf{D}^{\geq 0}$ is not easy to describe in general.

Definition 2.1.1. *We say that $\mathcal{C}$ is $\mathcal{A}-$approximative if for any object $X \in \mathbf{D}_{\mathcal{A}}$ there exists an object $X_{\mathcal{C}} \in \mathbf{D}_{\mathcal{C}}$ and a quasi-isomorphism $X_{\mathcal{C}} \to X$ which induces an isomorphism*

$\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X_\mathcal{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, X)|_{\mathbf{D}_\mathcal{C}}$. *$X_\mathcal{C}$ is called a $\mathcal{C}-$approximation for X.*

We have the following consecuenses.

Proposition 2.1.2. *With the hypothesis of the definition, we have*

(1) *$X_\mathcal{C}$ is unique up to isomorphism.*

(2) *If $X \in \mathbf{D}^{\leq n}_\mathcal{A}$ then $X_\mathcal{C} \in \mathbf{D}^{\leq n}$.*

(3) *The assignment $X \mapsto X_\mathcal{C}$ can be made into a functor which is left exact.*

Proof. (1). Suppose $X'_\mathcal{C} \in \mathbf{D}_\mathcal{C}$ such that $X'_\mathcal{C} \to X$ induces isomorphism in Hom-sets $\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X'_\mathcal{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, X)|_{\mathbf{D}_\mathcal{C}}$, then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X_\mathcal{C}) \arrow[r, "\sim"] \arrow[d, dashed] & \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, X)|_{\mathbf{D}_\mathcal{C}} \arrow[d, "1"] \\
\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X'_\mathcal{C}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, X)|_{\mathbf{D}_\mathcal{C}}
\end{tikzcd}$$

then $\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X_\mathcal{C}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X'_\mathcal{C})$ is an isomorphism for all object in $\mathbf{D}_\mathcal{C}$, so $X'_\mathcal{C} \cong X_\mathcal{C}$.

(2). Its enough to consider the case $n = 0$. Suppose $X \in \mathbf{D}^{\leq 0}_\mathcal{A}$ and let $X_\mathcal{C}$ its approximation, say $X_\mathcal{C} := [\, \cdots \longrightarrow X^{N-1} \longrightarrow X^N \,]$ where $X^{N+i} = 0$ for all $i > 0$. If $N \leq 0$ there is nothing to do, in other case let $X^N[N]$ be the complex in $\mathbf{D}_\mathcal{C}$ concetrated in degree N, then by the $\mathcal{A}-$approximation, a morphism $X^N[N] \to X$ is the same as a morphism $X^N[N] \to X_\mathcal{C}$, but the first one is zero, then the second one is also zero, i.e., homotopic to the zero map, so $X^{N-1} \to X^N$ is split epi, because the following diagram is commutative

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X^N \arrow[r] \arrow[d, "0"', "1"] \arrow[dl, dotted] & 0 \\
\cdots \arrow[r] & X^{N-1} \arrow[r] & X^N \arrow[r] & 0
\end{tikzcd}$$

Thus, $X_\mathcal{C}$ is homotopic to a complex Y such that $Y^{N+i} = 0$ for all $i \geq 0$. Continuing with this proces we get a complex Y' homotopic to $X_\mathcal{C}$ such that belongs to $\mathbf{D}^{\leq 0}_\mathcal{A} \cap \mathbf{D}_\mathcal{C}$, i.e., $\mathbf{D}^{\leq 0}$.

(3). If $X \in \mathbf{D}_\mathcal{A}$ we have a unique $X_\mathcal{C} \in \mathbf{D}_\mathcal{C}$. If $f : X \to Y$ is a morphism in $\mathbf{D}_\mathcal{A}$, we get the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X_\mathcal{C}) \arrow[r, "\sim"] \arrow[d, dashed, "!f^*_\mathcal{C}"] & \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, X)|_{\mathbf{D}_\mathcal{C}} \arrow[d, "f^*"] \\
\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, Y_\mathcal{C}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, Y)|_{\mathbf{D}_\mathcal{C}}
\end{tikzcd}$$

Then we define $f_\mathcal{C} : X_\mathcal{C} \to Y_\mathcal{C}$ as the unique map such that the diagram is commutative. Obviously we have $(gf)_\mathcal{C} = g_\mathcal{C} f_\mathcal{C}$ because the diagram below is commutative

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X_\mathcal{C}) \arrow[r, "\sim"] \arrow[d, "f_\mathcal{C}^*"] \arrow[dd, bend right=60, "(g^*f^*)_\mathcal{C}"'] & \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, X)|_{\mathbf{D}_\mathcal{C}} \arrow[d, "f^*"] \arrow[dd, bend left=60, "g^*f^*"] \\
\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, Y_\mathcal{C}) \arrow[r, "\sim"] \arrow[d, "g_\mathcal{C}^*"] & \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, Y)|_{\mathbf{D}_\mathcal{C}} \arrow[d, "g^*"] \\
\mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, Z_\mathcal{C}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, Z)|_{\mathbf{D}_\mathcal{C}}
\end{tikzcd}$$

Then $(g^*f^*)_\mathcal{C} = (gf)_\mathcal{C}^*$, but $(g^*f^*)_\mathcal{C} = g_\mathcal{C}^* f_\mathcal{C}^* = (g_\mathcal{C} f_\mathcal{C})^*$, which implies $g_\mathcal{C} f_\mathcal{C} = (gf)_\mathcal{C}$. Finally, suppose $f : X \to Y$ be a monomorphism in $\mathbf{D}_\mathcal{A}$, and let $f_\mathcal{C} : X_\mathcal{C} \to Y_\mathcal{C}$ be its approximation. Let $\phi : Z \to X_\mathcal{C}$ be a morphism in $\mathbf{D}_\mathcal{C}$ such that $f_\mathcal{C}\phi = 0$, then the following is commutative

$$\begin{tikzcd}
Z \arrow[r, "\phi"] & X_\mathcal{C} \arrow[r, "f_\mathcal{C}"] \arrow[d, "t"] & Y_\mathcal{C} \arrow[d, "s"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

and so $ft\phi = 0$, because f is mono, we have $t\phi = 0$ and since $t^* : \mathrm{Hom}_{\mathbf{D}_\mathcal{C}}(\square, X_\mathcal{C}) \to \mathrm{Hom}_{\mathbf{D}_\mathcal{A}}(\square, X)|_{\mathbf{D}_\mathcal{C}}$ is an isomorphism we get $\phi = 0$, i.e., $f_\mathcal{C}$ is a monomorphism.

□

We will denote the functor defined in the previous proposition by $\pi_\mathcal{C} : \mathbf{K}(\mathcal{A}) \to \mathbf{D}_\mathcal{C}$. For $X \in \mathbf{D}_\mathcal{A}$, $\pi_\mathcal{C}(X) = X_\mathcal{C}$ and for $f : X \to Y$ we get $\pi_\mathcal{C}(f) = f_\mathcal{C} : X_\mathcal{C} \to Y_\mathcal{C}$. Note also, that we have the natural transformation $(\mathbf{D}_\mathcal{C} \hookrightarrow \mathbf{D}_\mathcal{A}) \circ \pi_\mathcal{C} \to Id_{\mathbf{D}_\mathcal{C}}$.

Example 2.1.3. *Let $\mathcal{C} = \mathcal{A}$, then $\mathcal{C}$ trivially is $\mathcal{A}-$approximative with $X_\mathcal{C} = X$.*

Example 2.1.4. *Let $\mathcal{C} = \mathrm{Proj}(\mathcal{A})$, and assume $\mathcal{A}$ has enough projectives, then taking $X_\mathcal{C} \to X$ be a projective resolution of X, we get $\mathcal{C}$ is $\mathcal{A}-$approximative.*

Definition 2.1.5. *Let $M \in \mathcal{A}$.*

- *A $\mathcal{C}-$cover of M is an $\mathcal{A}-$epimorphism $C \twoheadrightarrow M$, where $C \in \mathcal{C}$, such that induces epimorphism $\mathrm{Hom}_\mathcal{C}(\square, C) \twoheadrightarrow \mathrm{Hom}_\mathcal{A}(\square, M)|_\mathcal{C}$.*
- *A $\mathcal{C}-$resolution of M is an exact sequence $\cdots \longrightarrow C^{-n} \cdots \longrightarrow C^0 \longrightarrow M \longrightarrow 0$ such that $\cdots \longrightarrow \mathrm{Hom}_\mathcal{C}(\square, C^{-n}) \cdots \longrightarrow \mathrm{Hom}_\mathcal{C}(\square, C^0) \longrightarrow \mathrm{Hom}_\mathcal{A}(\square, M)|_\mathcal{C} \longrightarrow 0$ is also exact.*

Recall that the $\mathcal{C}-$dimension of M is the smallest number $n \in \mathbb{N} \cup \{\infty\}$ such that there is a $\mathcal{C}-$resolution of M with $C^i = 0$ for $i < -n$. Aslo note that if $C \twoheadrightarrow M$ is a $\mathcal{C}-$cover, then any map $C' \to M$ factors through $C' \to C$. Indeed, we have the surjection $\mathrm{Hom}_\mathcal{C}(C', C) \twoheadrightarrow \mathrm{Hom}_\mathcal{A}(C', M)|_\mathcal{C}$ which gives the desired factorization.

Lemma 2.1.6. *If for every object $M \in \mathcal{A}$ there is an epimorphism $C \twoheadrightarrow M$, where $C \in \mathcal{C}$, then for every complex $X \in \mathbf{D}_\mathcal{A}$ there is a quasi-isomorphism $P \to X$, where $P \in \mathbf{D}_\mathcal{C}$.*

Proof. By induction we construct a quasi-isomorphism $\pi : X \to Y$. For $q > 0$ let be $\pi^q = 0$ and $P^q = 0$. Assume the construction is done uo to the n spot, where $n \leq 0$. Let π^q be an epimorphism for $q \geq n$ and a quasi-isomorphism for $q \geq n+1$. We have the commutativity of the following diagram:

$$\begin{tikzcd}
X^n \arrow[rr, "d^n"] & & X^{n+1} \\
& \ker d'^{n+1} \times_{X^{n+1}} X^n \arrow[ul, "m_{n+1}"'] \arrow[dr, "k_{n+1}"] & P^{n+1} \arrow[u, "\pi^{n+1}"'] \\
P^n \arrow[uu, "\pi^n"] \arrow[ur, "c^n"'] \arrow[rr, "k_{n+1}c^n"] & & \ker d'^{n+1} \arrow[u, hook, "i^{n+1}"']
\end{tikzcd}$$

where $\ker d'^{n+1} \times_{X^{n+1}} X^n$ is the pullback of d^n and $\pi^{n+1} i^{n+1}$, and m_{n+1} is an epimorphism. Let be $d'^n = i^{n+1} k_{n+1} c^n$.

Consider the following diagram

$$\begin{tikzcd}
X^{n-1} \arrow[rr, "d^{n-1}"] & & X^n \arrow[rr, "d^n"] & & X^{n+1} \\
& & & \ker d'^{n+1} \sqcap_{X^{n+1}} X^n \arrow[ul, "m_{n+1}"'] \arrow[dr, "k_{n+1}"] & P^{n+1} \arrow[u, "\pi^{n+1}"'] \\
& \ker d'^n \sqcap_{X^n} X^{n-1} \arrow[uul, "m_n"'] \arrow[dr, "k_n"] & P^n \arrow[uu, "\pi^n"] \arrow[ur, "c^n"'] \arrow[rr, "k_{n+1}c^n"] & & \ker d'^{n+1} \arrow[u, hook, "i^{n+1}"'] \\
P^{n-1} \arrow[uuu, dotted, "\pi^{n-1}"] \arrow[ur, dotted, "c^{n-1}"'] \arrow[rr, dotted, "k_n c^{n-1}"] & & \ker d'^n \arrow[u, hook, "i^n"'] & &
\end{tikzcd}$$

Then $d^{n-1} m_n = \pi^n i^n k_n$. P^{n-1} and c^{n-1} are the object and the epimorphism which exists by the hypothesis. Let be $d'^{n-1} = i^n k_n c^{n-1}$, then we have the commutativity of the diagram and $d'^n d'^{n-1} = 0$. Let see that m_n is epimorphism and π^n is a quasi-isomorphism.

If $x \in X^{n-1}$, $d^{n-1}(x) \in X^n$. Then $(0, d^{n-1}(x)) \in \ker d'^{n+1} \times_{X^{n+1}} X^n$ since $\pi^{n+1} i^{n+1}(0) = 0 = d^n(d^{n-1}(x))$, and is a preimage of $d^{n-1}(x)$ under m_{n+1}. Because c^n is an epimorphism, there is $z \in P^n$ such that $c^n(z) = (0, d^{n-1}(x))$, thus $k_{n+1} c^n(z) = 0$ and $d'^n(z) = 0$. For this, $z \in \ker d'^n$. Moreover, $\pi^n i^n(z) = \pi^n(z) = m_{n+1} c^n(z) = d^{n-1}(x)$, i.e., $(z, x) \in \ker d'^n \times_{X^n} X^{n-1}$ and $m_n(z, x) = x$. Then m_n is epi.

Let be $\alpha \in {}^{\ker d'^n}/{}_{\operatorname{im} d'^{n-1}}$ such that $(\pi^n)^*(\alpha) = 0$. If $x \in \ker d'^n$, is a representative for α, then $\pi^n(x) \in \operatorname{im} d^{n-1}$, then there is $y \in X^{n-1}$ such that $\pi^n(x) = d^{n-1}(y)$. Since $x \in \ker d'^n$, we get $i^n(x) = x$ and so $\pi^n i^n(x) = d^{n-1}(y)$, i.e., $(x, y) \in \ker d'^n \times_{X^n} X^{n-1}$.

Then there exists $z \in P^{n-1}$ such that $c^{n-1}(z) = (x,y)$ and $d'^{n-1}(z) = i^n k_n c^{n-1}(z) = i^n k_n(x,y) = i^n(x) = x$. Thus $x \in \operatorname{im} d'^{n-1}$ and for this $\alpha = 0$.

On the other hand, let $\beta \in {}^{\ker d^n}/{}_{\operatorname{im} d^{n-1}}$. Let $y \in \ker d^n$ a representative for β. There exists $x \in \ker d'^n$ such that $c^n(x) = (0,y)$ then $\pi^n(x) = y$. Thus $\pi^n(x) + \operatorname{im} d^{n-1} = y + \operatorname{im} d^{n-1}$, and so $(\pi^n)^*(\bar{x}) = \beta$. □

Proposition 2.1.7. *The following statemants are equivalent:*

(1) *Each $M \in \mathcal{A}$ has a $\mathcal{C}-$cover.*

(2) *Each $M \in \mathcal{A}$ has a $\mathcal{C}-$resolution.*

(3) *Each $X \in \mathbf{D}_{\mathcal{A}}$ has a $\mathcal{C}-$approximation.*

Proof. (1) $\Rightarrow$ (2). Suppose $M \in \mathcal{A}$, by (1) M admits a $\mathcal{C}-$cover, say $C^0 \twoheadrightarrow M$, let $K^0 = \ker(C^0 \twoheadrightarrow M)$ and consider C^{-1} a $\mathcal{C}-$cover for K^0, then we have the diagrams

$$\begin{tikzcd} C^{-1} \arrow[rr] \arrow[dr, two heads] & & C^0 \arrow[r] & M \arrow[r] & 0 \\ & K^0 \arrow[ur, hook] & & & \end{tikzcd}$$

and

$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(\square, C^{-1}) \arrow[rr] \arrow[dr, two heads] & & \operatorname{Hom}_{\mathcal{C}}(\square, C^0) \arrow[r] & \operatorname{Hom} \mathcal{A}(\square, M)|_{\mathcal{C}} \arrow[r] & 0 \\ & \operatorname{Hom}_{\mathcal{C}}(\square, K^0) \arrow[ur, hook] & & & \end{tikzcd}$$

where the top rows in the previous diagrams are exact sequences. Now, if C^{-n} was construct, we define C^{-n-1} as the $\mathcal{C}-$cover of $K^{-n} = \ker(C^{-n} \to C^{-n+1})$ and we proceed in the same fashion. Then, we get exact sequences $\cdots \longrightarrow C^{-n} \longrightarrow \cdots \longrightarrow C^0 \longrightarrow M \longrightarrow 0$ and $\cdots \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, C^{-n}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, C^0) \longrightarrow \operatorname{Hom} \mathcal{A}(\square, M)|_{\mathcal{C}} \longrightarrow 0$, i.e., we obtain a $\mathcal{C}-$resolution for M.

(2) $\Rightarrow$ (1). If $M \in \mathcal{A}$ admits a $\mathcal{C}-$resolution, say $\cdots \longrightarrow \longrightarrow C^0 \longrightarrow M \longrightarrow 0$, then $C^0 \to M$ is a $\mathcal{C}-$cover for M.

(3) $\Rightarrow$ (2). For $M \in \mathcal{A}$ consider $P_M := M[0]$ the complex concetrated in degree zero. Let $(P_M)_{\mathcal{C}}$ be a $\mathcal{C}-$approximation for P_M. Let us say that $(P_M)_{\mathcal{C}}$ has the form

$$[\ \cdots \longrightarrow C^{-n} \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-1}} C^0\ ]$$

for some $C^i \in \mathcal{C}$, then since $H^0((P_M)_{\mathcal{C}}) = H^0(P_M) = M$ and $H^i((P_M)_{\mathcal{C}}) = H^i(P_M) = 0$ for $i \neq 0$ we have exact sequence

$$\cdots \longrightarrow C^{-n} \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} M \longrightarrow 0$$

Let see now that

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^{-n}) \xrightarrow{d_*^{-n}} \cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^0) \xrightarrow{d_*^0} \mathrm{Hom}_{\mathcal{A}}(\square, M)|_{\mathcal{C}} \longrightarrow 0$$

is exact. Suppose $Y \in \mathcal{C}$ and consider the sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, C^{-n}) \xrightarrow{d_*^{-n}} \cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, C^0) \xrightarrow{d_*^0} \mathrm{Hom}_{\mathcal{A}}(Y, M)|_{\mathcal{C}} \longrightarrow 0$$

If $f \in \mathrm{Hom}_{\mathcal{A}}(Y, M)|_{\mathcal{C}}$, then it defines $f[0] : Y[0] \to M[0]$ in $\mathbf{D}_{\mathcal{A}}$ and so by the $\mathcal{C}-$approximation hypothesis there exists $(f[0])_{\mathcal{C}} : Y[0] \to (M[0])_{\mathcal{C}}$ in $\mathbf{D}_{\mathcal{C}}$ such that $((P_M)_{\mathcal{C}} \to P_M)\circ(f[0]_{\mathcal{C}}) = f[0]$. Then, $d_*^0((f[0]_{\mathcal{C}})^0) = f$. For the other position we have $d_*^{-n} \circ d_*^{-n-1} = 0$, then $\mathrm{im}(d_*^{-n-1}) \subseteq \ker(d_*^{-n})$. For the converse, assume $f \in \ker(d_*^{-n})$, then $f : Y \to C^{-n}$ and $d_*^{-n}(f) = 0$, i.e., $d^{-n} \circ f = 0$, so f defines a morphism between $Y[-n]$ and $(P_M)_{\mathcal{C}}$ and since $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(Y[-n], (P_M)_{\mathcal{C}}) \cong \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(Y[-n], P_M)|_{\mathbf{D}_{\mathcal{C}}}$ but the last one is zero, we have that there is some $g : Y \to C^{-n-1}$ such that $d^{-n-1}g = f$, then $f \in \mathrm{im}(d_*^{-n-1})$. Thus, our desired sequence is exact and then $\cdots \longrightarrow C^0 \xrightarrow{d^0} M \longrightarrow 0$ is a $\mathcal{C}-$resolution for M.

(2) $\Rightarrow$ (3). Let $X := [\ \cdots \longrightarrow X^{-n} \longrightarrow \cdots \longrightarrow X^0\ ]$ be in $\mathbf{D}_{\mathcal{A}}$. By lemma 2.1.6 there exists $X_{\mathcal{C}} \to X$ quasi-isomorphism. Denote by $\mathcal{F}$ the complex

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\square, X^{-1})|_{\mathcal{C}} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\square, X^0)|_{\mathcal{C}} \longrightarrow 0 \longrightarrow \cdots$$

and by $\mathcal{F}_{\mathcal{C}}$ the complex

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, X_{\mathcal{C}}^{-1}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, X_{\mathcal{C}}^0) \longrightarrow 0 \longrightarrow \cdots$$

Then, proceding as in the proof of the previous lemma we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(\square, X^{-n-1}) \arrow[rr, "d_*^{-n-1}"] & & \mathrm{Hom}_{\mathcal{C}}(\square, X^{-n}) \\
 & \mathrm{Hom}_{\mathcal{A}}(\square, \ker d'^{-n} \times_{X^{-n}} X^{-n-1})|_{\mathcal{C}} \arrow[ul, "m_{-n*}"'] \arrow[dr, "k_{-n*}"] & \mathrm{Hom}_{\mathcal{C}}(\square, X_{\mathcal{C}}^{-n}) \arrow[u, "\pi_*^{-n}"'] \\
\mathrm{Hom}_{\mathcal{C}}(\square, X_{\mathcal{C}}^{-n-1}) \arrow[uu, dotted, "\pi_*^{-n-1}"'] \arrow[ur, dotted, two heads, "c_*^{-n-1}"'] \arrow[rr, dotted, "{k_{-n}c^{-n-1}}_*"] & & \mathrm{Hom}_{\mathcal{A}}(\square, \ker d'^{-n})|_{\mathcal{C}} \arrow[u, hook, "{i^{-n}}_*"']
\end{tikzcd}$$

where $\mathrm{Hom}_{\mathcal{C}}(\square, X_{\mathcal{C}}^{n-1}) \to \mathrm{Hom}_{\mathcal{A}}(\square, \ker d'^n \sqcap_{X^n} X^{n-1})|_{\mathcal{C}}$ is an epimorphism since $X_{\mathcal{C}}^{n-1}$ is a $\mathcal{C}-$cover of $\ker d'^n \sqcap_{X^n} X^{n-1}$. Then the proof of the lemma 2.1.6 gives that $\mathcal{F}_{\mathcal{C}} \to \mathcal{F}$ is a quasi-isomorphism, so $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\square, X_{\mathcal{C}}) \cong \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(\square, X)|_{\mathbf{D}_{\mathcal{C}}}$. This means, $X_{\mathcal{C}}$ is a $\mathcal{C}-$approximation of X. $\square$

Corollary 2.1.8. *Assume that every object in $\mathcal{A}$ has $\mathcal{C}-$dimension less or equal than n. Let $X := [\,0 \longrightarrow X^{-m} \xrightarrow{d^{-m}} \cdots \xrightarrow{d^{-1}} X^0\,]$ be an object in $\mathbf{D}_{\mathcal{A}}$ and let d^{-m} be a monomorphism. Then $X_{\mathcal{C}}$ has the form $[\,0 \longrightarrow X_{\mathcal{C}}^{-m-n} \longrightarrow \cdots \longrightarrow X_{\mathcal{C}}^0\,]$*

Proof. We can see in the constructions of $X_{\mathcal{C}}$ given in the lemma 2.1.6, that

$$\cdots \longrightarrow X_{\mathcal{C}}^{-m-n} \longrightarrow \cdots \longrightarrow X_{\mathcal{C}}^{-m}$$

is a $\mathcal{C}-$resolution for X^{-m}. Then $X^{-m-n-1} = 0$, and $X_{\mathcal{C}}$ has the desired form. $\square$

Theorem 2.1.9. *Assume that $\mathcal{C}$ is $\mathcal{A}-$approximative. Then $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ is a $t-$structure on $\mathbf{D}_{\mathcal{C}}$.*

Proof. By definition we have $\mathbf{D}^{\leq 0} := \mathbf{D}_{\mathcal{A}}^{\leq 0} \cap \mathbf{D}_{\mathcal{C}}$ and $\mathbf{D}^{\geq 0} := (\mathbf{D}^{\leq 0})^{\perp}[1]$, then because $(\mathbf{D}_{\mathcal{A}}^{\leq 0}, \mathbf{D}_{\mathcal{A}}^{\geq 0})$ is a $t-$structure on $\mathbf{D}_{\mathcal{A}}$ we have $\mathbf{D}_{\mathcal{A}}^{\leq -1} \subseteq \mathbf{D}_{\mathcal{A}}^{\leq 0}$, and so $\mathbf{D}^{\leq -1} \subseteq \mathbf{D}^{\leq 0}$. This implies that $\mathbf{D}^{\geq 1} = (\mathbf{D}^{\leq 0})^{\perp} \subseteq (\mathbf{D}^{\leq -1})^{\perp}$. Now consider $X \in (\mathbf{D}^{\leq -1})^{\perp}$ and let see that X belongs to $(\mathbf{D}^{\leq 0})^{\perp}[1]$, which is equivalent to $X[-1] \in (\mathbf{D}^{\leq 0})^{\perp}$. If Y belongs to $\mathbf{D}^{\leq 0}$, we have $Y[1] \in \mathbf{D}^{\leq 0}[1]$, then $\mathrm{Hom}(Y[1], X) = 0$ because $X \in (\mathbf{D}^{\leq -1})^{\perp}$, but this implies $\mathrm{Hom}(Y, X[-1]) = 0$, i.e. $X[-1]$ belongs to $(\mathbf{D}^{\leq 0})^{\perp}$, so $X \in (\mathbf{D}^{\leq 0})^{\perp}[1]$. Thus, $\mathbf{D}^{\geq 1} \subseteq \mathbf{D}^{\geq 0}$.

By definition we inmediatly have $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 1}) = 0$. It remains to prove that any $X \in \mathbf{D}_{\mathcal{C}}$ fits into a triangle $X' \longrightarrow X \longrightarrow X'' \xrightarrow[+1]{}$, where $X' \in \mathbf{D}^{\leq 0}$ and $X'' \in \mathbf{D}^{\geq 1}$. We have $\tau_{\mathcal{A}}^{\leq 0} X \in \mathbf{D}_{\mathcal{A}}^{\leq 0}$, where $\tau_{\mathcal{A}}^{\leq 0}$ is the trucation, adjoint to the inclusion which appear in the $t-$structure of $\mathbf{D}_{\mathcal{A}}$. By the $\mathcal{A}-$approximation property there exists $X' = (\tau_{\mathcal{A}}^{\leq 0} X)_{\mathcal{C}}$ in $\mathbf{D}^{\leq 0}$ and $\mathbf{D}_{\mathcal{A}}-$morphism $X' \to \tau_{\mathcal{A}}^{\leq 0} X$ such that $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\square, X') \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(\square, \tau_{\mathcal{A}}^{\leq 0} X)|_{\mathbf{D}_{\mathcal{C}}}$. By proposition 2.1.2, (3), we have a morphism $X' \to X_{\mathcal{C}} = X$, define $X'' = \mathrm{cone}(X' \to X)$, then $X' \longrightarrow X \longrightarrow X'' \xrightarrow[+1]{}$ is a distinguished triangle in $\mathbf{D}_{\mathcal{C}}$ in which X' belongs to $\mathbf{D}^{\leq 0}$. Let see that X'' belongs to $\mathbf{D}^{\geq 1}$.

We have the following commutative diagram

$$\begin{tikzcd}
\tau_{\mathcal{A}}^{\leq 0} X \arrow[r] & X \arrow[r] & \mathrm{cone}(\tau_{\mathcal{A}}^{\leq 0} X \to X) \arrow[r, "+1"'] & {} \\
X' \arrow[u] \arrow[r] & X \arrow[u] \arrow[r] & X'' \arrow[u, dotted] \arrow[r, "+1"'] & {}
\end{tikzcd}$$

where the top row is a distinguished triangle in $\mathbf{D}_{\mathcal{A}}$ with $\mathrm{cone}(\tau_{\mathcal{A}}^{\leq 0} X \to X) \in \mathbf{D}_{\mathcal{A}}^{\geq 1}$ and the dotted arrow exists by the axioms of triangulated category. Now if we apply the cohomological functor Hom we get

$$\begin{tikzcd}
\mathbf{D}_{\mathcal{A}}(\square, \tau_{\mathcal{A}}^{\leq 0}X)|_{\mathbf{D}_{\mathcal{C}}} \arrow[r] & \mathbf{D}_{\mathcal{A}}(\square, X)|_{\mathbf{D}_{\mathcal{C}}} \arrow[r] & \mathbf{D}_{\mathcal{A}}(\square, \mathrm{cone}(\tau_{\mathcal{A}}^{\leq 0}X \to X))|_{\mathbf{D}_{\mathcal{C}}} \arrow[r, "+1"'] & {} \\
\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\square, X') \arrow[r] \arrow[u, "\cong"'] & \mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\square, X) \arrow[r] \arrow[u, "\cong"'] & \mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\square, X'') \arrow[r, "+1"'] \arrow[u, dashed, "\cong"'] & {}
\end{tikzcd}$$

then we get an isomorphism $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\square, X'') \to \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(\square, \mathrm{cone}(X' \to X))|_{\mathbf{D}_{\mathcal{C}}}$ by the five lemma (because the Hom's are abelian groups). Thus, if Z belongs to $\mathbf{D}^{\leq 0}$ it is also in $\mathbf{D}_{\mathcal{A}}^{\leq 0}$, and then

$$\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(Z, X'') \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(Z, \mathrm{cone}(X' \to X))|_{\mathbf{D}_{\mathcal{C}}} = 0$$

i.e. X'' belongs to $\mathbf{D}^{\geq 1}$. $\square$

Corollary 2.1.10. *If* $X' \longrightarrow X \longrightarrow X'' \xrightarrow[+1]{}$ *is a distinguished triangle in* $\mathbf{D}_{\mathcal{A}}$ *then*

$$(\mathrm{cone}(X' \to X))_{\mathcal{C}} \cong \mathrm{cone}(X'_{\mathcal{C}} \to X_{\mathcal{C}})$$

$\square$

We denote by $\mathcal{H}_{\mathcal{C}}$ the heart of the $t-$structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$, i.e. $\mathcal{H}_{\mathcal{C}} = \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$, which is an abelian category. Note that if $X \in \mathcal{H}_{\mathcal{A}}$ then $X_{\mathcal{C}} \in \mathcal{H}_{\mathcal{C}}$, this because $X_{\mathcal{C}}$ belongs to $\mathbf{D}^{\leq 0}$ as we noted in proposition 2.1.2 (2), and also we have for $X \in \mathbf{D}_{\mathcal{A}}^{\geq 0}$ that $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\mathbf{D}^{\leq 1}, X_{\mathcal{C}}) \subseteq \mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\mathbf{D}^{\leq 1}, X) \subseteq \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(\mathbf{D}_{\mathcal{A}}^{\leq 1}, X) = 0$, showing that $X_{\mathcal{C}} \in \mathbf{D}^{\geq 0}$.

We also have the following general definition,

Definition 2.1.11. *An AR-sequence or almost split sequence in* $\mathcal{C}$ *is a simple object in* $\mathcal{H}_{\mathcal{C}}$.

Example 2.1.12. *In [6] is shown that* $\mathcal{H}_{\mathcal{A}}$ *consists of complexes* $X^{-1} \longrightarrow X^{-1} \longrightarrow X^0$ *with no cohomology except in degree* 0. *Usual AR-sequences provide simple objects.*

Example 2.1.13. *If* $\mathcal{C} = \mathrm{Proj}(\mathcal{A})$ *and* $\mathcal{A}$ *has enough projectives then* $\mathbf{D}_{\mathcal{C}} \equiv \mathbf{D}^{-}(\mathcal{A})$, *so the* $t-$*structure on* $\mathbf{D}_{\mathcal{C}}$ *corresponds with the standard on* $\mathbf{D}^{-}(\mathcal{A})$ *and so* $\mathcal{H}_{\mathcal{A}} \cong \mathcal{A}$. *The (simple) objects of* $\mathcal{H}_{\mathcal{C}}$ *are equal to projective resolutions of the (simple) objects of* $\mathcal{A}$.

Let $\tau^{\leq 0} : \mathbf{D}_{\mathcal{C}} \to \mathbf{D}^{\leq 0}$ and $\tau^{\geq 0} : \mathbf{D}_{\mathcal{C}} \to \mathbf{D}^{\geq 0}$ be the truncation functors, right and left adjoints to the inclusions $\mathbf{D}^{\leq 0} \to \mathbf{D}_{\mathcal{C}}$ and $\mathbf{D}^{\geq 0} \to \mathbf{D}_{\mathcal{C}}$ respectively. By the proof of the theorem 2.1.9 we get for X in $\mathbf{D}_{\mathcal{C}}$ a distinguised triangle

$$(\tau_{\mathcal{A}}^{\leq 0}X)_{\mathcal{C}} \longrightarrow X \longrightarrow \mathrm{cone}((\tau_{\mathcal{A}}^{\leq 0}X)_{\mathcal{C}} \to X) \xrightarrow[+1]{}$$

Then by proposition 1.1.3 we obtain $\tau^{\leq 0}X = (\tau_{\mathcal{A}}^{\leq 0}X)_{\mathcal{C}}$ and $\tau^{\geq 1}X = \mathrm{cone}((\tau_{\mathcal{A}}^{\leq 0}X)_{\mathcal{C}} \to X)$, and also $\tau^{\geq 0}X = \mathrm{cone}(\tau^{\leq -1}X \to X)$, but this is the same as $\tau^{\geq 0}X = (\tau_{\mathcal{A}}^{\geq 0}X)_{\mathcal{C}}$. Hence we have that $\mathbf{D}^{\leq 0} = \pi_{\mathcal{C}}(\mathbf{D}_{\mathcal{A}}^{\leq 0})$ and $\mathbf{D}^{\geq 0} = \pi_{\mathcal{C}}(\mathbf{D}_{\mathcal{A}}^{\geq 0})$, and for this $\mathcal{H}_{\mathcal{C}} = \pi_{\mathcal{C}}(\mathcal{H}_{\mathcal{A}})$. With this in mind we have the following.

Proposition 2.1.14. *Under the above conditions about $\mathcal{C}$ and $\mathcal{A}$ we have that*

- $\mathcal{H}_{\mathcal{C}} = \pi_{\mathcal{C}}(\mathcal{H}_{\mathcal{A}})$.
- *The objects of $\mathcal{H}_{\mathcal{C}}$ are complexes X living in non-positive degrees such that $H^i(X) = 0$ for $i < 0$.*

Proof. It remains to prove the last assertion. Suppose $V \in \mathcal{H}_{\mathcal{C}}$, then there exists $W \in \mathcal{H}_{\mathcal{A}}$ such that $V = W_{\mathcal{C}}$. By definition of the approximation, there is quasi-isomorphism $V \to W$, and because W belongs to $\mathcal{H}_{\mathcal{A}}$ is a complex with no cohomology except in degree zero, then $H^i(V) = H^i(W) = 0$ for $i < 0$. □

Recall that if $f : X \to Y$ is a morphism in $\mathcal{H}_{\mathcal{C}}$, then $\ker f = \tau^{\leq 0}(\mathrm{cone}(f)[-1])$ and $\mathrm{coker}\, f = \tau^{\geq 0}(\mathrm{cone}(f))$.

Lemma 2.1.15. *Let be $X, Y \in \mathcal{H}_{\mathcal{C}}$ and let $f : X \to Y$ be a morphism in $\mathcal{H}_{\mathcal{C}}$. Then*

(1) *The following are equivalent:*

- $X \cong 0$ *in* $\mathbf{D}_{\mathcal{C}}$.
- $X^{-1} \to X^0$ *is a split epimorphism.*
- X *belongs to* $\mathbf{D}^{\leq -1}$.

(2) *The following are equivalent:*

- f *is an epimorphism*
- $X^0 \oplus Y^{-1} \to Y^0$ *is a split epimorphism.*

Proof. Let see (1). If $X \cong 0$ in $\mathbf{D}_{\mathcal{C}}$, then X is homotopic to the zero complex, this means that there exists masps $g : X \to 0$ and $h : 0 \to X$ such that $gh \sim 1_0$ and $hg \sim 1_X$. So, we have a homotopy map s between X and X, and commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-2} \arrow[r, "d_X^{-2}"] \arrow[d, "(hg)^{-2}"'] & X^{-1} \arrow[r, "d_X^{-1}"] \arrow[d, "(hg)^{-1}"'] \arrow[dl, "s^{-1}"'] & X^0 \arrow[r, "d_X^0"] \arrow[d, "(hg)^0"'] \arrow[dl, "s^0"'] & 0 \arrow[d] \arrow[dl, "s^1"'] \\
\cdots \arrow[r] & X^{-2} \arrow[r, "d_X^{-2}"'] & X^{-1} \arrow[r, "d_X^{-1}"'] & X^0 \arrow[r, "d_X^0"'] & 0
\end{tikzcd}$$

Then, $1_{X^0} - h^0 g^0 = d_X^{-1} s^0 + s^1 d_X^0$, i.e., $1_{X^0} = d_X^{-1} s^0$, i.e., $X^{-1} \to X^0$ is a split epimorphism.

If $X^{-1} \to X^0$ is a split epimorphism, then as in the proof of proposition 2.1.2 (2), we can get a complex in $\mathbf{D}^{\leq -1}$ homotopic to X. Conversely, if $X \in \mathbf{D}^{\leq -1}$, trivially $X^{-1} \to X^0$ is split epimorphism, because $d_X^{-1} \circ 0 = 0 = 1_{X^0} = 1_0$.

Finally, if $X \in \mathbf{D}^{\leq -1}$ since $X \in \mathbf{D}^{\geq 0}$ we have $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\mathbf{D}^{\leq -1}, X) = 0$. In particular $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X^{-1}[-1], X) = 0$, and so the following diagram is commutative

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X^{-1} \arrow[r] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \arrow[dl, dotted, "s^1"'] & 0 \\ \cdots \arrow[r] & X^{-2} \arrow[r] & X^{-1} \arrow[r] & 0 \end{tikzcd}$$

thus, $1_{X^{-1}} = d_X^{-2}s^1$, i.e., d_X^{-2} is split epi. Then X is homotopic to a complex $Y_{(1)}$ in $\mathbf{D}_{\mathcal{C}}$ such that $Y_{(1)}^i = 0$ for all $i \geq -1$, i.e., $Y_{(1)} \in \mathbf{D}^{\leq -2}$. Assume by induction we can find a complex $Y_{(k)}$ in $\mathbf{D}^{\leq -(k+1)}$ for $k \geq 1$ which is homotopic to X, then $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X^{-(k+1)}[-(k+1)], Y_{(k)}) = 0$, and by the same argument as in the base step, we find homotopy between $Y_{(k)}$ and a complex $Y_{(k+1)}$ such that $Y_{(k+1)}^i = 0$ for all $i \geq -(k+1)$. Then by induction, we get that X is homotopic to the zero complex as desired.

(2). Because $\mathcal{H}_{\mathcal{C}}$ is abelian, coker f is in $\mathcal{H}_{\mathcal{C}}$ but $\mathrm{coker}(f) = \tau^{\geq 0}(\mathrm{cone}(f))$ then $\mathrm{coker}(f) = \mathrm{cone}(f)$. Thus, by (1), $X^0 \oplus Y^{-1} \to Y^0$ is split epi if and only if $\mathrm{cone}(f) \in \mathbf{D}^{\leq -1}$ if and only if $\mathrm{coker}(f) \in \mathbf{D}^{\leq -1}$ if and only if $\mathrm{coker}(f) = 0$ if and only if f is an epimorphism. □

Proposition 2.1.16. (1) *The inclusion $\mathbf{D}_{\mathcal{C}} \hookrightarrow \mathbf{D}_{\mathcal{A}}$ is right $t-$exact. Hence, $\tau_{\mathcal{A}}^{\geq 0}(\mathcal{H}_{\mathcal{C}}) \subseteq \mathcal{H}_{\mathcal{A}}$ and $\tau_{\mathcal{A}}^{\geq 0} : \mathcal{H}_{\mathcal{C}} \to \mathcal{H}_{\mathcal{A}}$ is right exact.*

(2) *$\tau_{\mathcal{A}}^{\geq 0} : \mathcal{H}_{\mathcal{C}} \to \mathcal{H}_{\mathcal{A}}$ is left adjoint to $\pi_{\mathcal{C}} : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{C}}$.*

(3) *$\pi_{\mathcal{C}} : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{C}}$ is exact.*

Proof. (1). The first assertion follows since $\mathbf{D}^{\leq 0} = \mathbf{D}_{\mathcal{A}}^{\leq 0} \cap \mathbf{D}^-(\mathcal{A}) \subseteq \mathbf{D}_{\mathcal{A}}^{\leq 0}$. The second one, because if $X \in \mathcal{H}_{\mathcal{C}}$, then $X \in \mathbf{D}^{\leq 0}$ and so is in $\mathbf{D}_{\mathcal{A}}^{\leq 0}$ and then $\tau_{\mathcal{A}}^{\geq 0}(X) \in \mathcal{H}_{\mathcal{A}}$. Right exactness follows since the inclusion is right $t-$exact.

(2). Suppose $X \in \mathcal{H}_{\mathcal{C}}$ and $Y \in \mathcal{H}_{\mathcal{A}}$, then we have the isomorphisms

$$\mathrm{Hom}_{\mathcal{H}_{\mathcal{C}}}(X, \pi_{\mathcal{C}}(Y)) \cong \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(X, Y) \cong \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}^{\geq 0}}(\tau_{\mathcal{A}}^{\geq 0} X, Y) \cong$$
$$\mathrm{Hom}_{\mathbf{D}_{\mathcal{A}}}(\tau_{\mathcal{A}}^{\geq 0} X, Y) \cong \mathrm{Hom}_{\mathcal{H}_{\mathcal{A}}}(\tau_{\mathcal{A}}^{\geq 0}, Y)$$

and so we get the adjunction property.

(3). Left exactness was shown in proposition 2.1.2(3), let us stablish the right exactness. Let $f : X \to Y$ be an epimorphism in $\mathcal{H}_{\mathcal{A}}$, then as is shown in [6] the map $\alpha : X^0 \oplus Y^{-1} \to Y^0$ is split spi, which means that there is a map $\beta : Y^0 \to Y^0$ such that $\beta\alpha = 1_{Y^0}$. Consider then the induce maps $\alpha[0] : P_{X^0 \oplus Y^{-1}} \to P_{Y^0}$ and $\beta[0] : P_{Y^0} \to P_{Y^0}$ (where P_Z is the complex concetrated in degree zero), which satisfies $\beta[0]\alpha[0] = 1_{P_{Y^0}}$. Then applying $\pi_{\mathcal{C}}$ we get $\pi_{\mathcal{C}}(\beta[0])\pi_{\mathcal{C}}(\alpha[0]) = 1_{(P_{Y^0})_{\mathcal{C}}}$, but in the zero spot we exactly obtain $X^{0,0} \oplus (Y^{-1,0} \oplus Y^{0,1}) \to Y^{0,0} \to X^{0,0} \oplus (Y^{-1,0} \oplus Y^{0,1})$, where $X^{0,0}, Y^{-1,0}, Y^{0,1}$ and $Y^{0,0}$ are the terms which appears in the $\mathcal{C}-$resolutions of X and Y (see proof of proposition 2.1.7), in other words, we have that the composite $X_{\mathcal{C}}^0 \oplus Y_{\mathcal{C}}^{-1} \to Y_{\mathcal{C}}^0 \to X_{\mathcal{C}}^0 \oplus Y_{\mathcal{C}}^{-1}$ is the identity, i.e., $X_{\mathcal{C}}^0 \oplus Y_{\mathcal{C}}^{-1} \to Y_{\mathcal{C}}^0$ is split epi, and so $\pi_{\mathcal{C}}(f)$ is an epimorphism. Thus, $\pi_{\mathcal{C}}$ is right exact.

□

2.2 Projectives, Injectives and Simples in $\mathcal{H}_\mathcal{C}$

Let us describe the projective objects in $\mathcal{H}_\mathcal{C}$. For $M \in \mathcal{C}$ let $P_M \in \mathcal{H}_\mathcal{C}$ be the complex $P_M := M[0]$ concentrated in degree zero. This gives a fully faithful functor

$$\begin{array}{rccc} P: & \mathcal{C} & \longrightarrow & \mathcal{H}_\mathcal{C} \\ & M & \mapsto & P_M \end{array}$$

We have the following

Proposition 2.2.1. *P defines an equivalence of categories $P : \mathcal{C} \to \mathrm{Proj}(\mathcal{H}_\mathcal{C})$.*

Proof. First of all, let us see that P_M is projective for M an object of $\mathcal{C}$. Let $f : X \to Y$ be an epimorphism in $\mathcal{H}_\mathcal{C}$ and $g : P_M \to Y$ a morphism, thus, g is given by $g^0 : M \to Y^0$.

Because f is epi, by lemma 2.1.15 (2) we have that $l := (f^0, d_Y^{-1}) : X^0 \oplus Y^{-1} \to Y^0$ is split epi. So, there exists $h : Y^0 \to X^0 \oplus Y^{-1}$ such that $lh = 1_{Y^0}$. Let $\pi : X^0 \oplus Y^{-1} \to X^0$ be the projection and consider de map $g' : P_M \to X$ given by $g'^0 = \pi h g^0$. We claim that fg' is homotopic to g.

Indeed, if $h = (a_0, b_0)^t$ define a homotopy between fg' and g as

$$s^n = \begin{cases} -b_0 g^0 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

then, $f^0 g'^0 - g^0 = f^0 \pi h g^0 - g^0 = (f^0 \pi h - 1_{Y^0}) g^0 = (f^0 (1_{X^0}, 0)(a_0, b_0)^t - 1_{Y^0}) g^0 = (1_{Y^0} - d_Y^{-1} b_0 - 1_{Y^0}) g^0 = -d_Y^{-1} b_0 g^0 = d_Y^{-1} s^0 = d_Y^{-1} s^0 + s^1 d_{P_M}^0$. Thus $fg' \sim g$, and in $\mathbf{K}^{-1}(\mathcal{C})$ we have $fg' = g$, i.e., P_M is projective.

Trivially, $P : \mathcal{C} \to \mathrm{Proj}(\mathcal{H}_\mathcal{C})$ is fully and faithful, now if $X = [\, \cdots \longrightarrow X^n \longrightarrow \cdots \longrightarrow X^0 \,]$ belongs to $\mathrm{Proj}(\mathcal{H}_\mathcal{C})$, we have $X = P_{X^0} \oplus \tilde{X}$, where $\tilde{X} = [\, \cdots \longrightarrow X^n \longrightarrow \cdots \longrightarrow X^{-1} \,]$, and by lemma 2.1.15 $\tilde{X} = 0$, $P(X^0) = P_{X^0} \cong X$.

So, P is an equivalence of categories. $\square$

Corollary 2.2.2. *$\mathcal{H}_\mathcal{C}$ has enough projectives. Hence the natural morphism $\mathbf{D}_\mathcal{C} \to \mathbf{D}^-(\mathcal{H}_\mathcal{C})$ is an equivalence of categories.*

Proof. If $X = [\, \cdots \longrightarrow X^n \longrightarrow \cdots \longrightarrow X^0 \,]$ belongs to $\mathcal{H}_\mathcal{C}$, consider the natural morphism $P_{X^0} \to X$ given by $1_{X^0} : X^0 \to X^0$, then $(1_{X^0}, d_X^{-1}) : X^0 \oplus X^{-1} \to X^0$ is spli epi, and by lemma 2.1.15, $P_{X^0} \to X$ is surjective. Hence, $\mathcal{H}_\mathcal{C}$ has enough projectives and so the equivalence follows, $\mathbf{D}_\mathcal{C} \equiv \mathbf{K}^-(\mathrm{Proj}(\mathcal{H}_\mathcal{C})) \equiv \mathbf{D}^-(\mathcal{H}_\mathcal{C})$. $\square$

Let now study injectives in the particular case when $\mathcal{C}$ is a full additive subcategory of the category $\mathcal{A} = \mathrm{mod}\,\Lambda$ of finitely generated $\Lambda-$modules, assume that $\mathcal{C}$ contains Λ and is $\mathcal{A}-$approximative.

Consider a morphism $f : X \to Y$ in $\mathcal{H}_\mathcal{C}$, Recall that $\ker f = \tau^{\leq 0}(\mathrm{cone}(f)[-1]) = (\tau_\mathcal{A}^{\leq 0}(\mathrm{cone}(f)[-1]))_\mathcal{C}$, and note that

$$\tau_\mathcal{A}^{\leq 0}(\mathrm{cone}(f)[-1]) := \cdots \longrightarrow X^{-2} \oplus Y^{-3} \xrightarrow{-\begin{pmatrix} -d_X^{-2} & 0 \\ f^{-2} & d_Y^{-3} \end{pmatrix}} X^{-1} \oplus Y^{-2} \xrightarrow{-\begin{pmatrix} -d_X^{-1} & 0 \\ f^{-1} & d_Y^{-2} \end{pmatrix}} X^0 \times_{Y^0} Y^{-1} \longrightarrow 0$$

where $X^0 \times_{Y^0} Y^{-1} = \ker(-(f^0, d_Y^{-1}) : X^0 \oplus Y^{-1} \to Y^0)$, i.e., $X^0 \times_{Y^0} Y^{-1} = \{(a, b) : f^0 a = -d_{Y^{-1}} b\}$.

We have $\ker(f) = (\tau_\mathcal{A}^{\leq 0}(\mathrm{cone}(f)[-1]))_\mathcal{C} = [\, \cdots \longrightarrow Z^n \longrightarrow \cdots \longrightarrow Z^0 \,]$. By lemma 2.1.15 f is monomorphism if and only if $\ker(f) = 0$, and by lemma 2.1.15 $\ker(f) = 0$ if and only if $Z^{-1} \to Z^0$ is a split epimorphism. In this case, if $L \in \mathbf{D}_\mathcal{C}$, a morphism $L \to \ker(f)$ is zero, and is the same as a morphism $L \to \tau_\mathcal{A}^{\leq 0}(\mathrm{cone}(f)[-1])$, which is also zero, then taking $L = P_\Lambda$ we get

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[dl, dotted, "s"'] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & 0 \\
\cdots \arrow[r] & X^{-1} \oplus Y^{-2} \arrow[r] & X^0 \times_{Y^0} Y^{-1} \arrow[r] & 0
\end{tikzcd}$$

which is commutative, and then $X^{-1} \oplus Y^{-2} \to X^0 \times_{Y^0} Y^{-1}$ is an epimorphism. Also, if $X^{-1} \oplus Y^{-2} \to X^0 \times_{Y^0} Y^{-1}$ is split epi, then $Z^{-1} \to Z^0$ is split epi, this because if the first is, then the following diagram is commutative

$$\begin{tikzcd}
 & X^0 \times_{Y^0} Y^{-1} \arrow[dl] \arrow[d, "1"] \\
X^{-1} \oplus Y^{-2} \arrow[r] & X^0 \times_{Y^0} Y^{-1}
\end{tikzcd}$$

and because the approximation is functorial, we have the commutativity of

$$\begin{tikzcd}
 & Z^0 \arrow[dl] \arrow[d, "1"] \\
Z^{-1} \arrow[r] & Z^0
\end{tikzcd}$$

i.e., $Z^{-1} \to Z^0$ split epi. We summarize this in the following.

Lemma 2.2.3. *Let $\mathcal{C}$ be a full additive subcategory of* $\mathrm{mod}\,\Lambda$ *which is* $\mathrm{mod}\,\Lambda-$*approximative and such that $\Lambda \in \mathcal{C}$. Let $f : X \to Y$ be a morphism in $\mathcal{H}_\mathcal{C}$.*

- *If f is a monomorphism then $X^{-1} \oplus Y^{-2} \to X^0 \times_{Y^0} Y^{-1}$ is an epimorphism.*
- *If $X^{-1} \oplus Y^{-2} \to X^0 \times_{Y^0} Y^{-1}$ is a split epimorphism then f is a monomorphism.*
- *$X^{-1} \oplus Y^{-2} \to X^0 \times_{Y^0} Y^{-1}$ is an epimorphism in* $\mathrm{mod}\,\Lambda$ *if and only if* $\mathrm{coker}(d_X^{-1}) \to \mathrm{coker}(d_Y^{-1})$ *is a monomorphism in* $\mathrm{mod}\,\Lambda$.

Proof. It remains to prove the final item. If $\bar{c} \in \operatorname{coker}(d_X^{-1})$ such that $\overline{f}^0(c) = 0$, there exists $b \in Y^{-1}$ such that $f^0(c) = d_Y^{-1}(b)$, thus $(c, -b) \in X^0 \times_{Y^0} Y^{-1}$. By hipotesis $-\begin{pmatrix} -d_X^{-1} & 0 \\ f^{-1} & d_Y^{-2} \end{pmatrix}$ is epi, then there is some $(a, b) \in X^{-1} \oplus Y^{-2}$ with $d_X^{-1}(a) = c$ and $-f^{-1}(a) - d_Y^{-2}(b) = -b$, then $\bar{c} = 0$. Conversly, if $(c, b) \in X^0 \times_{Y^0} Y^{-1}$ then $-f^0(a) = d_Y^{-1}(b)$, i.e., $f^0(c) \in \operatorname{im}(d_Y^{-1})$ which by hypothesis implies that $c \in \operatorname{im}(d_X^{-1})$, so there exist $a \in X^{-1}$ such that $d_X^{-1}(a) = c$. Now, note that $d_Y^{-1}(b) = -f^0(c) = -f^0 d_X^{-1}(a) = -d_Y^{-1} f^{-1}(a)$, i.e., $-b - f^{-1}(a) \in \ker(d_Y^{-1}) = \operatorname{im}(d_Y^{-2})$ (the last equality because Y is in $\mathcal{H}_\mathcal{C}$, then is exact except in degree zero) then there is some $\nu \in Y^{-2}$ such that $d_Y^{-2}(\nu) = -b - f^{-1}(a)$. Thus $(a, \nu) \in X^{-1} \oplus Y^{-2}$ satisfies

$$-\begin{pmatrix} -d_X^{-1} & 0 \\ f^{-1} & d_Y^{-2} \end{pmatrix} \begin{pmatrix} a \\ \nu \end{pmatrix} = -\begin{pmatrix} -c \\ f^{-1}(a) - b - f^{-1}(a) \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix}$$

as desired. □

Proposition 2.2.4. *Let $\mathcal{C}$ be a full additive subcategory of $\operatorname{mod}\Lambda$ which is $\operatorname{mod}\Lambda-$approximative and assume that $\Lambda \in \mathcal{C}$. If I is injective in $\mathcal{C}$, then P_I is injective in $\mathcal{H}_\mathcal{C}$.*

Proof. Suppose $f : X \to Y$ be a monomorphism in $\mathcal{H}_\mathcal{C}$ and let $g : X \to P_I$. Because f monomorphism, by the previous lemma, $X^{-1} \oplus Y^{-2} \to X^0 \times_{Y^0} Y^{-1}$ is an epimorphism, and then $\operatorname{coker}(d_X^{-1}) \to \operatorname{coker}(d_Y^{-1})$ is a monomorphism, so $(\operatorname{coker}(d_X^{-1}) \to \operatorname{coker}(d_Y^{-1}))_\mathcal{C}$ also is a monomorphism. We have the following diagram

$$\begin{tikzcd}
I & & \\
& (\operatorname{coker}(d_X^{-1}))_\mathcal{C} \arrow[ul] \arrow[r] & (\operatorname{coker}(d_Y^{-1}))_\mathcal{C} \arrow[ull, dotted] \\
X^0 \arrow[uu] \arrow[ur] \arrow[r] & Y^0 \arrow[ur] &
\end{tikzcd}$$

where the dotted line exists because I injective. So there is a morphism $Y \to P_I$, such that

$$\begin{tikzcd}
& P_I & \\
0 \arrow[r] & X \arrow[u] \arrow[r] & Y \arrow[ul]
\end{tikzcd}$$

is commutative, thus P_I is injective. □

Definition 2.2.5. *We say that an abelian category has enough simples if each indecomposable projective has a simple quotient.*

Proposition 2.2.6. (1) *Let $L = [\, A \longrightarrow B \longrightarrow C \,]$ be a simple object of $\mathcal{H}_\mathcal{A}$. Then, $L_\mathcal{C}$ is zero or simple in $\mathcal{H}_\mathcal{C}$. Furthermore, if $C \in \mathcal{C}$ then $L_\mathcal{C}$ is non-zero.*

(2) *Assume that* $\mathcal{A}$ *is a Krull-Schmidt category and let* $M \in \mathcal{C}$ *be indecomposable. Then all simple quotients of* P_M *are isomorphic.*

(3) *Let* $M \in \mathcal{C}$*. If* L *is a simple quotient of* P_M *in* $\mathcal{H}_\mathcal{A}$*, then* $L_\mathcal{C}$ *is a simple quotient of* P_M *in* $\mathcal{H}_\mathcal{C}$

Proof. (1). We must to show that any non-zero map $M \to L_\mathcal{C}$ in $\mathcal{H}_\mathcal{C}$ is an epimorphism. By $\mathcal{A}-$approximativity, $M \to L_\mathcal{C}$ corresponds to a non-zero map $M \to L$ and because $L \in \mathcal{H}_\mathcal{C}$, by proposition 1.1.3 corresponds to a non-zero map $\tau_\mathcal{A}^{\geq 0} M \to L$ which is epi since L is simple. Thus, $M \cong (\tau_\mathcal{A}^{\geq 0} M)_\mathcal{C} \to L_\mathcal{C}$ is also epi since $\pi_\mathcal{C}$ is exact. Finally, if $C \in \mathcal{C}$ we have that $L_\mathcal{C}$ has the form $[\cdots \longrightarrow C^{-1} \longrightarrow C\,]$, and since L is simple, $B \to C$ is not split epi and so $C^{-1} \to C$ is not split epi too, but this means that $L_\mathcal{C}$ is non-zero, i.e., $L_\mathcal{C}$ is simple.

(2). Assume L_M and L'_M simple quotients of P_M to prove that they are isomorphic, is the same as to prove that every proper maximal subobjects of P_M are the same. Let us consider K and K' two proper maximal subobjects of P_M, then we have monomorphism $K \hookrightarrow P_M$ and $K' \hookrightarrow P_M$, which gives the mono $K \oplus K' \hookrightarrow P_M$. Now, because K and K' are proper, the maps $K \hookrightarrow P_M$ and $K' \hookrightarrow P_M$ are not epimorphisms and by lemma 2.1.15 we get that $K^0 \to M$ and $K'^0 \to M$ are not split epis, then by Krull-Schmidt theomrem, $K^0 \oplus K'^0 \to M$ is not a split epi. Then $K \oplus K'$ is maximal proper subobject of P_M. Therefore, $K = K'$.

(3). First of all note that $L_\mathcal{C}$ is a quotient of P_M in $\mathcal{H}_\mathcal{C}$ since for any $F \in \mathcal{H}_\mathcal{C}$ we have the isomorphism $\mathrm{Hom}_{\mathcal{H}_\mathcal{C}}(F, L_M) \cong \mathrm{Hom}_{\mathcal{H}_\mathcal{A}}(F, L_M^\mathcal{A})$. Now, since the end term of L is in $\mathcal{C}$ by (1), $L_\mathcal{C}$ is not zero, so is simple. □

For the last results of this section assume $\mathcal{A}$ to be a Krull-Schmidt category.

Recall that $\mathrm{Mod}\,\mathcal{C}$ denotes the category of $\mathcal{C}-$modules, i.e., contravariant additive functors from $\mathcal{C}$ to **Ab**. and this is an abelian category. We denoted by $\mathrm{mod}\,\mathcal{C}$ the full subcategory which objects are finitely presented $\mathcal{C}-$modules, i.e., functors F which admit a presentation

$$\mathrm{Hom}_\mathcal{C}(\Box, Y) \longrightarrow \mathrm{Hom}_\mathcal{C}(\Box, X) \longrightarrow F \longrightarrow 0$$

for some $X, Y \in \mathcal{C}$. Also recall that the projectives in $\mathrm{mod}\,\mathcal{C}$ are of the form $\mathrm{Hom}_\mathcal{C}(\Box, X)$ for some $X \in \mathcal{C}$.

Lemma 2.2.7. $\mathrm{mod}\,\mathcal{C} = \mathrm{mod}\,\mathcal{A} \cap \mathrm{Mod}\,\mathcal{C}$.

Proof. Obviously, $\mathrm{mod}\,\mathcal{C} \subseteq \mathrm{mod}\,\mathcal{A} \cap \mathrm{Mod}\,\mathcal{C}$. Assume $F \in \mathrm{mod}\,\mathcal{A} \cap \mathrm{Mod}\,\mathcal{C}$, then $F \in \mathrm{Mod}\,\mathcal{C}$ and have presentation

$$\mathrm{Hom}_\mathcal{A}(\Box, Y) \longrightarrow \mathrm{Hom}_\mathcal{A}(\Box, X) \longrightarrow F \longrightarrow 0$$

for some $X, Y \in \mathcal{A}$. Then by proposition 2.1.7, we have approximations for these objects, and we get the presentation

$$\mathrm{Hom}_\mathcal{C}(\Box, Y_\mathcal{C}) \longrightarrow \mathrm{Hom}_\mathcal{C}(\Box, X_\mathcal{C}) \longrightarrow F \longrightarrow 0$$

then $F \in \operatorname{mod}\mathcal{C}$. $\square$

Lemma 2.2.8. (1) $\operatorname{mod}\mathcal{C}$ *is an abelian subcategory of* $\operatorname{Mod}\mathcal{C}$.

(2) *The categories* $\mathcal{C}$ *and* $\operatorname{Proj}(\operatorname{mod}\mathcal{C})$ *are equivalent.*

(3) *There exists a fully faithful functor of triangulated categories,*

$$\mathbf{D}_{\mathcal{C}} \cong \mathbf{K}^-(\operatorname{Proj}(\operatorname{mod}\mathcal{C})) \cong \mathbf{D}^-(\operatorname{mod}\mathcal{C}) \longrightarrow \mathbf{D}^-(\operatorname{Mod}\mathcal{C})$$

which is $t-$exact respect to $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$. Moreover, this induce an exact fully faithful embedding $\rho_{\mathcal{C}} : \mathcal{H}_{\mathcal{C}} \to \operatorname{Mod}\mathcal{C}$ and $\mathcal{H}_{\mathcal{C}} \cong \operatorname{mod}\mathcal{C}$.

Proof. (1). Its enough to show that $\operatorname{mod}\mathcal{C}$ is closed under kernels and cokernels.

Let $\varphi : F \to G$ be a morphism in $\operatorname{mod}\mathcal{C}$ then is in $\operatorname{mod}\mathcal{A}$ and thus φ is also in $\operatorname{Mod}\mathcal{A} = \mathcal{H}_{\operatorname{Mod}\mathcal{A}}$. Since, F and G admits presentations we have the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}(\square, Y) \arrow[r, "d^{-1}"] \arrow[d, dotted, "\varphi_2"] & \operatorname{Hom}_{\mathcal{A}}(\square, X) \arrow[r, "d^0"] \arrow[d, dotted, "\varphi_1"] & F \arrow[r] \arrow[d, "\varphi"] & 0 \\
\operatorname{Hom}_{\mathcal{A}}(\square, Y') \arrow[r, "\partial^{-1}"] & \operatorname{Hom}_{\mathcal{A}}(\square, X') \arrow[r, "\partial^0"] & G \arrow[r] & 0
\end{tikzcd}$$

where φ_1 and φ_2 exists because $\operatorname{Hom}_{\mathcal{A}}(\square, X)$ and $\operatorname{Hom}_{\mathcal{A}}(\square, Y)$ are proyectives. Then in $\mathbf{D}^-(\operatorname{Mod}\mathcal{A})$ the morphism φ is the same as the following morphism of complexes

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}(\square, K) \arrow[r, "d^{-2}"] \arrow[d, "\varphi_3"] & \operatorname{Hom}_{\mathcal{A}}(\square, Y) \arrow[r, "d^{-1}"] \arrow[d, "\varphi_2"] & \operatorname{Hom}_{\mathcal{A}}(\square, X) \arrow[d, "\varphi_1"] \\
\operatorname{Hom}_{\mathcal{A}}(\square, K') \arrow[r, "d^{-2}"] & \operatorname{Hom}_{\mathcal{A}}(\square, Y') \arrow[r, "\partial^{-1}"] & \operatorname{Hom}_{\mathcal{C}}(\square, X')
\end{tikzcd}$$

where K and K' are the kernels of $Y \to X$ and $Y' \to X'$ respectively, and so the top row is a projective resolution for F and the bottom row for G. With this in mind, we can compute the kernel and the cokernel of this chain map using the standard $t-$structure on $\mathbf{D}^-(\operatorname{Mod}\mathcal{A})$, i.e., if $\psi = (\varphi_3, \varphi_2, \varphi_1)$ we have $\ker(\psi) = \tau^{\leq 0}_{\operatorname{Mod}\mathcal{A}}(\operatorname{cone}\psi[-1])$ and $\operatorname{coker}(\psi) = \tau^{\geq 0}_{\operatorname{Mod}\mathcal{A}}(\operatorname{cone}\psi)$, thus we get

$$\ker(\psi) := 0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(\square, K) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(\square, Y \oplus K') \longrightarrow \operatorname{Hom}_{\mathcal{A}}(\square, K'')$$

$$\operatorname{coker}(\psi) := 0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(\square, X')/\operatorname{im}(\operatorname{Hom}_{\mathcal{A}}(\square, X \oplus Y') \to \operatorname{Hom}_{\mathcal{A}}(\square, X')) \longrightarrow 0$$

where $K'' = \ker(X \oplus Y' \to X')$. Then we obtain

$$\operatorname{Hom}_{\mathcal{A}}(\square, Y \oplus K') \longrightarrow \operatorname{Hom}_{\mathcal{A}}(\square, K'') \longrightarrow \ker(\varphi) \longrightarrow 0$$

$$\mathrm{Hom}_{\mathcal{A}}(\square, X \oplus Y') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\square, X') \longrightarrow \mathrm{coker}(\varphi) \longrightarrow 0$$

which are presentations for $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$, which mean that they belongs to $\mathrm{mod}\,\mathcal{A}$ and then by the previous lemma, $\ker(\varphi)$ and $\mathrm{coker}(\varphi)$ belongs to $\mathrm{mod}\,\mathcal{C}$.

(2). Follows by Yoneda lemma and the assignment $X \to \mathrm{Hom}_{\mathcal{C}}(\square, X)$.

(3). We give the proof in steps.

Step 1. By (2) we have the equivalence $\mathbf{D}_{\mathcal{C}} \equiv \mathbf{K}^{-}(\mathrm{Proj}(\mathrm{mod}\,\mathcal{C}))$. And since the $\mathbf{D}^{\leq 0}$ are complexes in non-positive degree is clearly mapped to $\mathbf{D}^{\leq 0}_{\mathrm{Proj}(\mathrm{mod}\,\mathcal{C})}$ which are also complexes in non-positive degrees. Then because $\mathbf{D}^{\geq 0} = (\mathbf{D}^{\leq 0})^{\perp}[1]$, the $t-$structure is sent to the $t-$ structure in $\mathbf{K}^{-}(\mathrm{Proj}(\mathrm{mod}\,\mathcal{C}))$.

Step 2. We have $\mathbf{K}^{-}(\mathrm{Proj}(\mathrm{mod}\,\mathcal{C}))$ equivalent to $\mathbf{D}^{-}(\mathrm{mod}\,\mathcal{C})$, then note that since $\mathbf{D}^{\leq 0}_{\mathrm{Proj}(\mathrm{mod}\,\mathcal{C})}$ are complexes in non-positive degrees does not have cohomology in positives degrees, i.e, belongs to $\mathbf{D}^{\leq 0}_{\mathrm{mod}\,\mathcal{C}}$. And because this categories are equivalent and $\mathbf{D}^{\geq 0}_{\mathrm{Proj}(\mathrm{mod}\,\mathcal{C})} = (\mathbf{D}^{\leq 0}_{\mathrm{Proj}(\mathrm{mod}\,\mathcal{C})})^{\perp}[1]$, we have that $\mathbf{D}^{\geq 0}_{\mathrm{Proj}(\mathrm{mod}\,\mathcal{C})}$ is mapped to $\mathbf{D}^{\geq 0}_{\mathrm{mod}\,\mathcal{C}}$. Then the equivalence is $t-$exact.

Step 3. We have that $\mathrm{Proj}(\mathrm{mod}\,\mathcal{C})$ is a full subcategory of $\mathrm{Proj}(\mathrm{Mod}\,\mathcal{C})$, then the inclusion $\mathrm{Proj}(\mathrm{mod}\,\mathcal{C}) \hookrightarrow \mathrm{Proj}(\mathrm{Mod}\,\mathcal{C})$ induces a fully faithful functor $\mathbf{K}^{-}(\mathrm{Proj}(\mathrm{mod}\,\mathcal{C})) \to \mathbf{K}^{-}(\mathrm{Proj}(\mathrm{Mod}\,\mathcal{C}))$, i.e., $\mathbf{D}^{-}(\mathrm{mod}\,\mathcal{C}) \to \mathbf{D}-(\mathrm{Mod}\,\mathcal{C})$. Similarly as in the previous case, the equivalence $\mathbf{D}^{-}(\mathrm{Mod}\,\mathcal{C}) \equiv \mathbf{K}^{-}(\mathrm{Proj}(\mathrm{Mod}\,\mathcal{C}))$ is $t-$exact.

Step 4. We can compose the previous functors and get $\mathbf{D}_{\mathcal{C}} \to \mathbf{D}^{-}(\mathrm{Mod}\,\mathcal{C})$ which is fully and faithful. Let us see that is $t-$exact. By the previous equivalences, we show that $\mathbf{D}^{-}(\mathrm{mod}\,\mathcal{C}) \to \mathbf{D}^{-}(\mathrm{Mod}\,\mathcal{C})$ is $t-$exact. If $V \in \mathbf{D}^{\leq 0}_{\mathrm{mod}\,\mathcal{C}}$, then is a complex in non-positives degrees, and so its cohomology vanishes in positives degrees, which means that $V \in \mathbf{D}^{\leq 0}_{\mathrm{Mod}\,\mathcal{C}}$. On the other hand, we want to show that $\mathbf{D}^{\geq 0}_{\mathrm{mod}\,\mathcal{C}} \subseteq \mathbf{D}^{\geq 0}_{\mathrm{Mod}\,\mathcal{C}}$, which is the same as to show that $\mathrm{Hom}_{\mathbf{D}^{-}(\mathrm{Mod}\,\mathcal{C})}(X, Y) = 0$ for $Y \in \mathbf{D}^{\geq 0}_{\mathrm{mod}\,\mathcal{C}}$ and all $X \in \mathbf{D}^{<0}_{\mathrm{Mod}\,\mathcal{C}}$. First of all, note that if P is projective in $\mathrm{Mod}\,\mathcal{C}$, by Yoneda's lemma we have the isomorphism $\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(\square, X), P) \cong P(X)$, then for each $\varphi \in P(X)$ we have a morphism $\mathrm{Hom}_{\mathcal{C}}(\square, X) \to P$, so we can define $\bigoplus_{\varphi \in P(X)} \mathrm{Hom}_{\mathcal{C}}(\square, X) \to P$, which is clearly epi and because P is projective we get that the morphism is split epimorphism, so P is isomorphic to $\bigoplus_I \mathrm{Hom}_{\mathcal{C}}(\square, X)$, for some I a subindex set. Now, if $P^{\bullet} \twoheadrightarrow X$ is a projective resolution for some $X \in \mathbf{D}^{\geq 0}_{\mathrm{Mod}\,\mathcal{C}}$, then $\mathrm{Hom}_{\mathbf{D}^{-}(\mathrm{Mod}\,\mathcal{C})}(X, Y) = 0$ if $\mathrm{Hom}_{\mathbf{D}^{-}(\mathrm{Mod}\,\mathcal{C})}(P^{\bullet}, Y) = 0$, but for show the last equality is enough to show $\mathrm{Hom}_{\mathbf{D}^{-}(\mathrm{Mod}\,\mathcal{C})}(\bigoplus_I \mathrm{Hom}_{\mathcal{C}}(\square, Z)[i], Y) = 0$ for each $i < 0$ since this is a generating family. And this is true since

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^-(\mathrm{Mod}\,\mathcal{C})}(\bigoplus_I \mathrm{Hom}_{\mathcal{C}}(\square, Z)[i], Y) &\cong \mathrm{Hom}_{\mathbf{D}^-(\mathrm{Mod}\,\mathcal{C})}(\varinjlim \bigoplus_{finite} \mathrm{Hom}_{\mathcal{C}}(\square, Z)[i], Y) \\
&\cong \varprojlim \mathrm{Hom}_{\mathbf{D}^-(\mathrm{Mod}\,\mathcal{C})}(\bigoplus_{finite} \mathrm{Hom}_{\mathcal{C}}(\square, Z)[i], Y) \\
&= 0
\end{aligned}$$

the last zero is because $\bigoplus_{finite} \mathrm{Hom}_{\mathcal{C}}(\square, Z)[i] \in \mathbf{D}^{<0}_{\mathrm{mod}\,\mathcal{C}}$ and $Y \in \mathbf{D}^{\geq 0}_{\mathrm{mod}\,\mathcal{C}}$. Thus, $\mathbf{D}^-(\mathrm{mod}\,\mathcal{C}) \to \mathbf{D}^-(\mathrm{Mod}\,\mathcal{C})$ is $t-$exact.

Finally, the last claim follows by the $t-$exactness of the previous functor, and then we have an exact fully faithful embedding $\rho_{\mathcal{C}} : \mathcal{H}_{\mathcal{C}} \to \mathrm{Mod}\,\mathcal{C}$ on hearts and $\mathcal{H}_{\mathcal{C}} \cong \mathrm{mod}\,\mathcal{C}$.

$\square$

Proposition 2.2.9. (1) *Let $L \in \mathcal{H}_{\mathcal{C}}$ then, L is simple if and only if $\rho_{\mathcal{C}}(L)$ is simple.*

(2) *If $X \in \mathrm{Ind}(\mathcal{C})$ and P_X admits the simple quotient L_X, then $\rho_{\mathcal{C}}(L_X) = \mathrm{Hom}_{\mathcal{C}}(\square, X)/\,\mathrm{rad}_{\mathcal{C}}(\square, X)$. Consequently, if $L_X := [\, \cdots \longrightarrow C^{-1} \longrightarrow X\,]$ then $\mathrm{im}(\mathrm{Hom}_{\mathcal{C}}(\square, C^{-1}) \to \mathrm{Hom}_{\mathcal{C}}(\square, X)) = \mathrm{rad}_{\mathcal{C}}(\square, X)$ and*

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^{-2}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^{-1}) \longrightarrow \mathrm{rad}_{\mathcal{C}}(\square, X) \longrightarrow 0$$

is exact.

(3) *If $X \in \mathrm{Ind}(\mathcal{C})$, $[\, \cdots \longrightarrow C^{-1} \longrightarrow X\,]$ is in $\mathcal{H}_{\mathcal{C}}$ and*

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^{-2}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^{-1}) \longrightarrow \mathrm{rad}_{\mathcal{C}}(\square, X) \longrightarrow 0$$

is exact, then this object is simple and thus isomorphic to L_X

Proof. (1). Assume $L \in \mathrm{mod}\,\mathcal{C}$. Let $\varphi : M \to L$ be a non-surjective map with $M \in \mathrm{Mod}\,\mathcal{C}$ We must show $\varphi = 0$. As we saw in the proof of the previous proposition, we have epimorphism $\bigoplus_{\varphi \in M(X)} \mathrm{Hom}_{\mathcal{C}}(\square, X) \twoheadrightarrow M$. We may assume M of this form. Then since $M \cong \varinjlim M_i$ where $M_i \cong \bigoplus_{finite} \mathrm{Hom}_{\mathcal{C}}(\square, X)$, where each finite sum belongs to $\mathrm{mod}\,\mathcal{C}$, we have $\varphi|_{M_i} : M_i \to L$ non-surjective, and thus is zero because L is simple in $\mathrm{mod}\,\mathcal{C}$. So, $\mathrm{im}(\varphi) \cong \mathrm{im}(\varinjlim \varphi|_{M_i}) = 0$ and we conclude that φ is simple in $\mathrm{Mod}\,\mathcal{C}$. The converse holds since $\rho_{\mathcal{C}}$ is faithful.

(2). By (1) we have that $\rho(L_X)$ is a simple quotient of $\rho(P_X) = \mathrm{Hom}_{\mathcal{C}}(\square, X)$, and since $X \in \mathrm{Ind}(\mathcal{C})$ the unique posibility is $\rho(L_X) = \mathrm{Hom}_{\mathcal{C}}(\square, X)/\,\mathrm{rad}_{\mathcal{C}}(\square, X)$. Then $\mathrm{Hom}_{\mathcal{C}}(\square, C^{-1}) \twoheadrightarrow \mathrm{rad}_{\mathcal{C}}(\square, X)$ is epi, and so

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^{-2}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\square, C^{-1}) \longrightarrow \mathrm{rad}_{\mathcal{C}}(\square, X) \longrightarrow 0$$

is exact in all other degrees by the $t-$exactness of the functor defined in Lemma 2.2.8, (3).

(3). Since X is indecomposable, $\rho(L_X) = \mathrm{Hom}_{\mathcal{C}}(\Box, X)/\operatorname{rad}_{\mathcal{C}}(\Box, X)$ is simple, then by the exactness of

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Box, C^{-2}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Box, C^{-1}) \longrightarrow \operatorname{rad}_{\mathcal{C}}(\Box, X) \longrightarrow 0$$

we obtain that the projective resolution

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Box, C^{-1}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Box, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\Box, X)/\operatorname{rad}_{\mathcal{C}}(\Box, X) \longrightarrow 0$$

for $\mathrm{Hom}_{\mathcal{C}}(\Box, X)/\operatorname{rad}_{\mathcal{C}}(\Box, X)$ is simple, and by (1), we get $[\ \cdots \longrightarrow C^{-1} \longrightarrow X\ ]$ simple, and so isomorphic to L_X.

$\Box$

2.3 The structure of $\mathcal{H}_{\mathcal{C}}$ when $\mathcal{C}$ is a subcategory of $\operatorname{mod}\Lambda$

AR-sequences are simple objects of the heart $\mathcal{H}_{\mathcal{C}}$. And we want to give conditions for their existence. Recall that an additive category $\mathcal{D}$ is said to be an $R-$linear category for some commutative ring R, if each Hom-set is an $R-$module and the composition law is $R-$bilinear. Let J be the injective hull of the $k-$module $k/\operatorname{rad}(k)$, then we have a duality $()^* := \mathrm{Hom}_k(\Box, J)$ on $\operatorname{mod} k$.

If $\mathcal{A} = \operatorname{mod}\Lambda$ then $\mathcal{A}^* = \operatorname{mod}\Lambda^{op}$ and $\mathcal{C}^* = \{M^* | M \in \mathcal{C}\}$. We can dualize the definition 2.1.1 as follows.

Definition 2.3.1. *We say that $\mathcal{C}^*$ is $\mathcal{A}^*-$approximative if for any object $X \in \mathbf{D}_{\mathcal{A}^*}$ there exists an object $X_{\mathcal{C}} \in \mathbf{D}_{\mathcal{C}}$ and a quasi-isomorphism $X_{\mathcal{C}} \to X$ which induces an isomorphism $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(\Box, X_{\mathcal{C}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}_{\mathcal{A}^*}}(\Box, X)|_{\mathbf{D}_{\mathcal{C}}}$. $X_{\mathcal{C}}$ is called a $\mathcal{C}^*-$approximation for X.*

Remark 2.3.2. *We can also have a dual version of proposition 2.1.7, then $\mathcal{C}$ is $\mathcal{A}-$approximative and $\mathcal{A}^*-$approximative is the same as be fuctorially finite*

Proposition 2.3.3. *Let k be an Artin ring, let $\mathcal{A}$ be a $k-$linear category whose Hom-sets are finitely generated $k-$modules. Assume that $0 \neq P \in \mathrm{Proj}(\mathcal{A})$, $\mathrm{End}_{\mathcal{A}}(P)$ is local and the functor $\mathrm{Hom}_{\mathcal{A}}(P, \Box)^* : \mathcal{A}^{op} \to \operatorname{mod} k$ is representable for some $S_P \in \mathcal{A}$. Then P has a simple quotient L.*

Proof. Note that $\mathrm{End}_{\mathcal{A}}(P)$ is an Artin algebra, and so an Artin ring. Also, $\mathrm{Hom}_{\mathcal{A}}(P, S_P)$ $= \mathrm{End}_{\mathcal{A}}(P)^* \neq 0$ because $P \neq 0$. By hypothesis, $\mathrm{Hom}_{\mathcal{A}}(P, S_P)$ is a finitely generated $k-$module and also a right module over the ring $\mathrm{End}_{\mathcal{A}}(P)$, but $\mathrm{End}_{\mathcal{A}}(P)$ is finitely generated $k-$module too, so $\mathrm{End}(P) \supseteq k$ and then $\mathrm{Hom}_{\mathcal{A}}(P, S_P)$ is a finitely generated $\mathrm{End}_{\mathcal{A}}(P)$ right module.

Because $\mathrm{End}_{\mathcal{A}}(P)^* \neq 0$, then $\mathrm{End}_{\mathcal{A}}(P) \neq 0$, and since this is an Artin ring there exists n such that $(\operatorname{rad}_{\mathcal{A}} \mathrm{End}(P))^n = 0$ and $(\operatorname{rad}_{\mathcal{A}} \mathrm{End}(P))^{n-1} \neq 0$. Pic some $\tilde{\tau} \in (\operatorname{rad}_{\mathcal{A}} \mathrm{End}(P))^{n-1}$, then for all $f \in \operatorname{rad}_{\mathcal{A}} \mathrm{End}(P)$ we have $\tilde{\tau} \cdot f = 0$. Then, because $()^*$ is a duality, there exists $\tau \in \mathrm{End}_{\mathcal{A}}(P)$ such that $\tau \cdot f = 0$ for all f belonging to $\operatorname{rad}_{\mathcal{A}} \mathrm{End}(P)$, i.e., for all non-units in $\operatorname{End}_{\mathcal{A}}(P)$.

Now let $L := \operatorname{im}\tau$ we show that L is simple. Let $i : L' \to L$ be an $\mathcal{A}-$monomorphism which is not an epimorphism, we must show that $L' = 0$. Because L is a subobject of S_P also is L', so $L' = 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(L', S_P) = 0$ if and only if $\operatorname{Hom}_{\mathcal{A}}(P, L') = 0$. Consider $f \in \operatorname{Hom}_{\mathcal{A}}(P, L')$, and consider the map $i \circ f : P \to L$, since P is projective there exists $h \in \operatorname{End}_{\mathcal{A}}(P)$ such that $i \circ f = \tau \circ h$. If h is a unit, we get that $i \circ f$ is epi, this implies i epi, which is impossible. Then h is not a unit and then $\tau \circ h = 0$, so $f = 0$. Thus, $L' = 0$ and L is simple □

Theorem 2.3.4. *Let $\mathcal{A} := \operatorname{mod}\Lambda$ for Λ an Artin $k-$algebra, let $\mathcal{C}$ be a full additive subcategory of $\operatorname{mod}\Lambda$ closed under isomorphism and direct summands which is $\mathcal{A}-$approximative. Then the following is true:*

(1) *For any $X \in \mathcal{C}$ the functor $\operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(P_X, \Box)^* : \mathcal{H}_{\mathcal{C}}^{op} \to \operatorname{mod} k$ is representable by some $SP_X \in \operatorname{Inj}(\mathcal{H}_{\mathcal{C}})$. This defines a functor $S : \operatorname{Proj}(\mathcal{H}_{\mathcal{C}}) \to \operatorname{Inj}(\mathcal{H}_{\mathcal{C}})$ which is an equivalence.*

(2) *$\mathcal{C}$ has AR-sequences, i.e., $\mathcal{H}_{\mathcal{C}}$ has simples. Indeed, each P_M for $M \in \mathcal{C}$ indecomposable, has a unique simple quotient $L_M \in \mathcal{H}_{\mathcal{C}}$. Moreover, $L_M \cong L_{\mathcal{C}}$ where L is the simple quotient of P_M in $\mathcal{H}_{\mathcal{A}}$*

Proof. (1). It was shown in [6] that the functor $\operatorname{Hom}_{\mathcal{H}_{\mathcal{A}}}(P_M, \Box)^* : \mathcal{H}_{\mathcal{A}}^{op} \to \operatorname{mod} k$ is representable by $S_{\mathcal{A}}P_M \in \operatorname{Inj}(\mathcal{H}_{\mathcal{A}})$. Put $SP_M := (S_{\mathcal{A}}P_M)_{\mathcal{C}}$, then we have the isomorphism $\operatorname{Hom}_{\mathbf{D}_{\mathcal{C}}}(\Box, SP_M) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}_{\mathcal{A}}}(\Box, S_{\mathcal{A}}P_M)|_{\mathbf{D}_{\mathcal{C}}}$ and so we have that $\operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(\Box, SP_M) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{H}_{\mathcal{A}}}(\Box, S_{\mathcal{A}}P_M)|_{\mathcal{H}_{\mathcal{C}}}$ is an isomorphism too. But also the following isomorphisms holds by the duality $()^*$ and the fully faithfulness $\operatorname{Hom}_{\mathcal{H}_{\mathcal{A}}}(\Box, S_{\mathcal{A}}P_M)|_{\mathcal{H}_{\mathcal{C}}} \cong \operatorname{Hom}_{\mathcal{H}_{\mathcal{A}}}(P_M, \Box)^*|_{\mathcal{H}_{\mathcal{C}}} \cong \operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(P_M, \Box)^*$, i.e., $\operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(P_M, \Box)^* \cong \operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(\Box, SP_M)$.

Finally, because P_M is projective it follows that $\operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(\Box, SP_M)$ is an exact functor, so that SP_M is injective.

(2) Let $M \in \mathcal{C}$ be indecomposable, then $\operatorname{End}_{\mathcal{C}}(M) \cong \operatorname{End}_{\mathcal{H}_{\mathcal{C}}}(P_M)$ is local because $\operatorname{End}_{\mathcal{C}}(M) = \operatorname{End}_{\operatorname{mod}\Lambda}(M)$ is finitely generated. By (1) $\operatorname{Hom}_{\mathcal{H}_{\mathcal{A}}}(P_M, \Box)^* : \mathcal{H}_{\mathcal{A}}^{op} \to \operatorname{mod} k$ is representable by some $S_{\mathcal{A}}P_M$ in $\mathcal{H}_{\mathcal{A}}$, then by the previous theorem P_M has a simple quotient $L_M^{\mathcal{A}}$ in $\mathcal{H}_{\mathcal{A}}$. Consider $L_M := (L_M^{\mathcal{A}})_{\mathcal{C}}$, then for any $F \in \mathcal{H}_{\mathcal{C}}$ and any $N \in \mathcal{C}$ we have the following isomorphisms $\operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(F, L_M) \cong \operatorname{Hom}_{\mathcal{H}_{\mathcal{A}}}(F, L_M^{\mathcal{A}})$ and $\operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(P_N, L_M) \cong \operatorname{Hom}_{\mathcal{H}_{\mathcal{A}}}(P_N, L_M^{\mathcal{A}})$, which implies that L_M is a simple quotient of P_M.

□

Remark 2.3.5. *The isomorphism $\operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(P_M, \Box)^* \cong \operatorname{Hom}_{\mathcal{H}_{\mathcal{C}}}(\Box, SP_M)$ will be called AR-duality*

Corollary 2.3.6. (1) *L_M is a simple submodule of SP_M*

(2) *The functor S defines a tringulated functor $\mathbb{S} : \mathbf{K}^b(\operatorname{Proj}(\mathcal{H}_{\mathcal{C}})) \to \mathbf{K}^b(\operatorname{Inj}(\mathcal{H}_{\mathcal{C}}))$ satisfying $\operatorname{Hom}_{\mathbf{K}^b(\mathcal{C})}(X, Y)^* \cong \operatorname{Hom}_{\mathbf{K}^b(\mathcal{C})}(Y, \mathbb{S}X)$.*

□

Proposition 2.3.7. *Let $\mathcal{A} = \operatorname{mod}\Lambda$ and assume that $\mathcal{C}$ is functorially finite. Then $\mathcal{H}_\mathcal{C}$ has enough injectives. They are of the form $(SP_C)_\mathcal{C}$ for $C \in \mathcal{C}$.*

Proof. Let $M \in \mathcal{H}_\mathcal{C}$ and $S_\mathcal{A}P_M$ be an injective hull of $\tau_\mathcal{A}^{\geq 0}M$ in $\mathcal{H}_\mathcal{A}$. Let $C^* \twoheadrightarrow X^*$ be a $\mathcal{C}^*-$cover of X^* in $\mathcal{A}^*$ and let $X \hookrightarrow C$ be the dual map. We know that SP_C is an injective of $\mathcal{H}_\mathcal{C}$, and then we have the following composite $M = (\tau_\mathcal{A}^{\geq 0})_\mathcal{C} \hookrightarrow SP_X \to SP_C$, where the first arrow is mono, let's see that the second one is mono too.

For this purpose it is enough to show that $\operatorname{Hom}_{\mathbf{D}_\mathcal{A}}(V, SP_X) \to \operatorname{Hom}_{\mathbf{D}_\mathcal{A}}(V, SP_C)$ is injective for every $V \in \mathcal{H}_\mathcal{C}$. Note that for $V \in \mathcal{H}_\mathcal{C}$ we have that $V^* \in \mathbf{K}^+(\mathcal{C}^*)$, so the natural map $\operatorname{Hom}_{\mathbf{K}^+(\mathcal{A}^*)}(V^*, P_{C^*}) \to \operatorname{Hom}_{\mathbf{K}^+(\mathcal{A}^*)}(V^*, P_{X^*})$ is an epimorphisms since $C^* \to X^*$ is a $\mathcal{C}-$cover. Then we have the following composite of maps

$$\operatorname{Hom}_{\mathbf{D}_\mathcal{A}}(V, SP_X) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{D}_\mathcal{A}}(P_X, V)^* \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{K}^+(\mathcal{A}^*)}(V^*, P_{X^*})^* \hookrightarrow$$

$$\operatorname{Hom}_{\mathbf{K}^+(\mathcal{A}^*)}(V^*, P_{C^*})^* \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{D}_\mathcal{A}}(P_C, V)^* \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{D}_\mathcal{A}}(V, SP_C)$$

Then, $M \to SP_C$ is a monomorphism, and so $\mathcal{H}_\mathcal{C}$ has enough injectives. □

2.4 Relationship with Iyama's Higher AR-sequences

In this section Λ will be an Artin $k-$algebra, $\mathcal{C}$ will be a $(n-1)-$maximal orthogonal subcategory of $\operatorname{mod}\Lambda$ which is $\operatorname{mod}\Lambda-$approximative and $(\operatorname{mod}\Lambda)^*-$approximative .

Lemma 2.4.1. *Let $0 \longrightarrow X^{-n-1} \xrightarrow{d^{-n-1}} \cdots \xrightarrow{d^{-1}} X^0 \longrightarrow 0$ be an exact sequence with terms in $\mathcal{C}$. Then the following statements are equivalent*

i) *d^{-1} is split epi.*

ii) *d^{-n-1} is split mono.*

iii) *The sequence is homotopic to zero.*

Proof. Is the same as the proof of lemma 2.1.15. □

Proposition 2.4.2. *V belongs to $\mathcal{H}_\mathcal{C}$ if and only if $V = [\ C^{-n-1} \longrightarrow \cdots \longrightarrow C^0\ ] \in \mathbf{D}_\mathcal{C}$ with C^i's belongs to $\mathcal{C}$ and $H^i(V) = 0$ for $i < 0$.*

Proof. We know that $V = [\ X \longrightarrow Y \longrightarrow Z\ ]_\mathcal{C}$ for some short exact sequence in $\mathcal{A}$, $0 \longrightarrow X \longrightarrow Y \longrightarrow Z$. Because $\mathcal{C}$ is $(n-1)-$maximal orthogonal subcategory of $\mathcal{A}$, corollary 2.1.8 implies that V has the desired lenght. Clearly, $H^i(V) = 0$ for $i < 0$ since $[\ X \longrightarrow Y \longrightarrow Z\ ]$ is quasi-isomorphic to V.

Conversly, assume that $V = [\, C^{-n-1} \longrightarrow \cdots \longrightarrow C^0 \,] \in \mathbf{D}_{\mathcal{C}}$ with $C^i \in \mathcal{C}$ and $H^i(V) = 0$ for $i < 0$. Then evidently $V \in \mathbf{D}^{\leq 0}$, let us see that $V \in \mathbf{D}^{\geq 0}$. Then we must to show that $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X, V) = 0$ for every object $X \in \mathbf{D}^{<0}$, but is enough to show that $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X[i], V) = 0$ for $X \in \mathcal{C}$ and $i < 0$.

First of all, note that morphisms between $X[i]$ and V in the homotopy category $\mathbf{D}_{\mathcal{C}}$ are diagrams of the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[dl, dotted] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{-i-1} \arrow[r, "d^{-i-1}"] & C^{-i} \arrow[r, "d^{-i}"] & C^{-i+1} \arrow[r, "-i+1"] & \cdots
\end{tikzcd}$$

this means, are morphism $X \to \mathrm{im}(d^{-i-1})$ which factorizes throughout C^{-i-1}, i.e.,

$$\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X[i], V) \cong \mathrm{Hom}_{\Lambda}(X, \mathrm{im}(d^{-i-1}))/\,\mathrm{im}(\mathrm{Hom}_{\Lambda}(X, C^{-i-1}) \to \mathrm{Hom}_{\Lambda}(X, \mathrm{im}(d^{-i-1})))$$

But, for the short exact sequence $0 \longrightarrow \ker(d^{-i-1}) = \mathrm{im}(d^{-i-2}) \longrightarrow C^{-i-1} \longrightarrow \mathrm{im}(d^{-i-1}) \longrightarrow 0$ we get the long exact Ext-sequence

$$0 \longrightarrow \mathrm{Hom}_{\Lambda}(X, \mathrm{im}(d^{-i-2})) \longrightarrow \mathrm{Hom}_{\Lambda}(X, C^{-i-1}) \longrightarrow \mathrm{Hom}_{\Lambda}(X, \mathrm{im}(d^{-i-1})) \longrightarrow$$

$$\mathrm{Ext}^1_{\Lambda}(X, \mathrm{im}(d^{-i-2})) \longrightarrow \mathrm{Ext}^1_{\Lambda}(X, C^{-i-1}) \longrightarrow \mathrm{Ext}^1_{\Lambda}(X, \mathrm{im}(d^{-i-1})) \longrightarrow \cdots$$

where $\mathrm{Ext}^1_{\Lambda}(X, C^{-i-1}) = 0$, because X and C^{-i-1} belongs to $\mathcal{C}$, then we have

$$\mathrm{Hom}_{\Lambda}(X, \mathrm{im}(d^{-i-1}))/\,\mathrm{im}(\mathrm{Hom}_{\Lambda}(X, C^{-i-1}) \to \mathrm{Hom}_{\Lambda}(X, \mathrm{im}(d^{-i-1}))) \cong \mathrm{Ext}^1_{\Lambda}(X, \mathrm{im}(d^{-i-2}))$$

i.e.,

$$\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X[i], V) \cong \mathrm{Ext}^1_{\Lambda}(X, \mathrm{im}(d^{-i-2}))$$

and since $\mathcal{C}$ is $(n-1)-$maximal orthognal we get

$$\mathrm{Ext}^1_{\Lambda}(X, \mathrm{im}(d^{-i-2})) \cong \mathrm{Ext}^2_{\Lambda}(X, \mathrm{im}(d^{-i-3})) \cong \cdots \cong \mathrm{Ext}^{n-i}_{\Lambda}(X, C^{-n-1})) = 0$$

then $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X[i], V) = 0$ for $i < 0$, and so $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X, V) = 0$ for $X \in \mathbf{D}^{<0}$. Then V belongs to $\mathbf{D}^{\geq 0}$, i.e., $V \in \mathcal{H}_{\mathcal{C}}$. □

Theorem 2.4.3. *Let $X \in \mathrm{Ind}(\mathcal{C})$ be non-projective, and L_X be the simple quotient of P_X in $\mathcal{H}_{\mathcal{C}}$. The following holds*

(1) *L_X define an $n-$AR-sequence in the sense of Iyama with end term X.*

(2) *If* $0 \longrightarrow Y \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow X \longrightarrow 0$ *is a* $n-$*AR-sequence in the sense of Iyama, then* $[\, Y \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow X \,]$ *is a simple object in* $\mathcal{H}_{\mathcal{C}}$ *isomorphic to* L_X*.*

Proof. (1). If L_X is the simple quotient of P_X in $\mathcal{H}_{\mathcal{C}}$, we have that $L_X = (L_X^{\mathcal{A}})_{\mathcal{C}}$, but $L_X^{\mathcal{A}} = [\, Z^{-2} \longrightarrow Z^{-1} \longrightarrow X \,]$ where $0 \longrightarrow Z^{-2} \longrightarrow Z^{-1} \longrightarrow X \longrightarrow 0$ is an exact sequence. Then, since $X \in \mathcal{C}$ we have that

$$L_X := [L_X^{-n-1} \longrightarrow \cdots \longrightarrow L_X^{-1} \longrightarrow X]$$

and in view of L_X is quasi-isomorphic to $L_X^{\mathcal{A}}$, L_X is acyclic. Also, by proposition 2.2.9 we have that

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, L_X^{-n-1}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, L_X^{-1}) \longrightarrow \operatorname{rad}_{\mathcal{C}}(\square, X) \longrightarrow 0$$

is exact, and with out lost of generality, we can consider this as a minimal projective resolution, which gives each morphism $L_X^j \to L_X^{j+1}$ belongs to $\operatorname{rad}_{\mathcal{C}}(L_X^j, L_X^{j+1})$.

Then because $\mathcal{C}$ is $(n-1)-$maximal orthogonal we get that

$$0 \longrightarrow L_X^{-n-1} \longrightarrow \cdots \longrightarrow L_X^{-1} \longrightarrow X \longrightarrow 0$$

is a $n-$AR-sequence.

(2). If $0 \longrightarrow Y \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow X \longrightarrow 0$ is a $n-$AR-sequence in the sense of Iyama, we get the following projective resolution for $\operatorname{Hom}_{\mathcal{C}}(\square, X)/\operatorname{rad}_{\mathcal{C}}(\square, X)$,

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, Y) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, X)/\operatorname{rad}_{\mathcal{C}}(\square, X)$$

but $\operatorname{Hom}_{\mathcal{C}}(\square, X)/\operatorname{rad}_{\mathcal{C}}(\square, X)$ is simple in $\mathbf{D}^-(\operatorname{Mod}\mathcal{C})$ as X is indecomposable, and so $0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, Y) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\square, X)$ is simple in $\operatorname{Mod}\mathcal{C}$, then by the fully faithfulness of $\rho_{\mathcal{C}}$ (see lemma 2.2.8), we have that

$$0 \longrightarrow Y \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow X \longrightarrow 0$$

is simple in $\mathcal{H}_{\mathcal{C}}$ and so isomorphic to L_X.

$\square$

The next goal is to prove the $n-$AR-dality using our AR-duality given in the theorem 2.3.4. So we wanto to prove the following:

$$\text{For any } X, Y \in \mathcal{C} \text{ we have } \underline{\operatorname{Hom}}_{\Lambda}(X, Y) \cong D\operatorname{Ext}^n_{\Lambda}(Y, \tau_n X)$$

for this we need the following previous results.

Lemma 2.4.4. *Let* $V = [\, C^{-n-1} \longrightarrow \cdots \longrightarrow C^0 \,]$ *be an oject of* $\mathcal{H}_{\mathcal{C}}$*. Then,*

- *V admits a projective resolution* $0 \longrightarrow P_{C^{-n-1}} \longrightarrow \cdots \longrightarrow P_{C^0} \longrightarrow V \longrightarrow 0$.
- $\mathrm{gl.dim}(\mathcal{H}_{\mathcal{C}}) \leq n+1$.

Proof. Define $\epsilon : P_{C^0} \to V$ as $\epsilon^0 = 1_{C^0}$ and $\epsilon^i = 0$ for $i \neq 0$, then since $\begin{pmatrix} 0 & 0 \\ 1_{C^0} & d_V^{-1} \end{pmatrix}$: $C^0 \oplus C^{-1} \to C^0$ satisfies $\begin{pmatrix} 0 & 0 \\ 1_{C^0} & d_V^{-1} \end{pmatrix} \begin{pmatrix} 1_{C^0} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1_{C^0} \end{pmatrix}$ is split epi, then by lemma 2.1.15, ϵ is epi. Also, note that $\ker(\epsilon) = [\ C^{-n-1} \longrightarrow \cdots \longrightarrow C^{-1}\ ]$, and so $P_{C^{-1}} \to \ker(\epsilon)$ is epi too. Then we get the commutative diagram

$$\begin{tikzcd} P_{C^{-1}} \arrow[rr] \arrow[dr, two heads] & & P_{C^0} \arrow[r, "\epsilon"] & V \arrow[r] & 0 \\ & \ker(\epsilon) \arrow[ur, hook] & & & \end{tikzcd}$$

and because $\mathcal{C}$ is full subcategory of $\mathrm{mod}\,\Lambda$, $P_{C^{-1}} \to P_{C^0}$ is a morphism in $\mathcal{C}$. Repeating this proces we get the long exact sequence

$$0 \longrightarrow P_{C^{-n-1}} \longrightarrow \cdots \longrightarrow P_{C^{-1}} \longrightarrow P_{C^0} \xrightarrow{\epsilon} V \longrightarrow 0$$

which is a projective resolution for V. Of course this implies that $\mathrm{gl.dim}(\mathcal{H}_{\mathcal{C}}) \leq n+1$. □

Corollary 2.4.5. *If $X \in \mathcal{C}$ then $J := [\ X \longrightarrow I^{-n} \longrightarrow \cdots \longrightarrow I^0\ ]$ where I^j's are injectives in $\mathcal{C}$, is an injective object of $\mathcal{H}_{\mathcal{C}}$.*

Proof. We have the resolution

$$0 \longrightarrow P_X \longrightarrow P_{I^{-n}} \qquad \cdots \longrightarrow P_{I^0} \longrightarrow J \longrightarrow 0$$

and because the P_{I^j}'s are injectives in $\mathcal{H}_{\mathcal{C}}$ by proposition 2.2.4 and $\mathrm{gl.dim}(\mathcal{H}_{\mathcal{C}}) \leq n+1$ we have J injective in $\mathcal{H}_{\mathcal{C}}$. □

Proposition 2.4.6. *The injectives of $\mathcal{H}_{\mathcal{C}}$ are precisely of the form $[\ X \longrightarrow I^{-n} \longrightarrow \cdots \longrightarrow I^0\ ]$ for $X \in \mathcal{C}$ and the I^j's are injectives in $\mathcal{C}$.*

Proof. Let $V = [\ C^{-n-1} \longrightarrow \cdots \longrightarrow C^0\ ]$ be an oject of $\mathcal{H}_{\mathcal{C}}$. Consider an injective resolution of C^{-n-1}, say $0 \longrightarrow C^{-n-1} \longrightarrow I^{-n} \longrightarrow \cdots \longrightarrow I^0$, where the I^j's are injectives in $\mathrm{mod}\,\Lambda$ and so in $\mathcal{C}$. Let $J = [\ C^{-n-1} \longrightarrow I^{-n} \longrightarrow \cdots \longrightarrow I^0\ ]$ be an injective in $\mathcal{H}_{\mathcal{C}}$, and note that the identity map from C^{-n-1} to C^{-n-1} lifts into a map $\varphi : V \to J$ unique up to homotopy, indeed consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & C^{-n-1} \arrow[r] \arrow[d, "1"] & C^{-n} \arrow[d, dotted] \\ 0 \arrow[r] & C^{-n-1} \arrow[r] & I^{-n} \end{tikzcd}$$

the dotted line exists because I^{-n} is injective. By exactness $\operatorname{im}(C^{-n-1} \to C^{-n}) = \ker(C^{-n} \to C^{-n+1})$, so we can repeat the process, i.e., we have

$$\begin{tikzcd} 0 \arrow[r] & \ker(C^{-n} \to C^{-n+1}) \arrow[r] \arrow[d] & C^{-n+1} \arrow[d, dotted] \\ & I^{-n} \arrow[r] & I^{-n+1} \end{tikzcd}$$

continue the process we get the desired morphism. Them we have that $\operatorname{Hom}_{\mathbf{D}_{\mathcal{C}}}(V, J) = \operatorname{Hom}_{\mathbf{D}_{\operatorname{mod}\Lambda}}(V, C^{-n-1}[-n-1])$.

Now, we can assume that $\operatorname{coker}(d_V^{-1}) \to \operatorname{coker}(d_I^{-1})$ is monomorphism, because if there is no the case we can add a direct sumand to I^0 and get the injection in the following way, consider the commutative diagram

$$\begin{tikzcd} C^{-1} \arrow[r, "d_V^{-1}"] \arrow[d, "\varphi^{-1}"] & C^0 \arrow[r, "\overline{d_V^{-1}}"] \arrow[d, "\varphi^0"] & \operatorname{coker}(d_V^{-1}) \arrow[r, hook, "i"] & I'^0 i_2 \arrow[d] \arrow[ddr, "j_2"] & \\ I^{-1} \arrow[r, "d_I^{-1}"] & I^0 \arrow[rr, "i_1"] \arrow[drrr, "j_1"'] & & S \arrow[dr, hook, "j"] & \\ & & & & I'' \end{tikzcd}$$

where I'^0 is an injective hull for $\operatorname{coker}(d_V^{-1})$, $S = I^0 \oplus I'^0 / \langle c \in C^0 : i\overline{d_V^{-1}}(c) = f^0(c) \rangle$ is the pushout, and I'' is an injective hull of S. Note that if $\overline{c} \in \operatorname{coker}(d_V^{-1})$ such that $\overline{ji_1\varphi^0}(c) = 0$, there is some $g \in I^{-1}$ such that $ji_1\varphi^0(c) = ji_2i\overline{d_V^{-1}}(c) = ji_1 d_I^{-1}(g)$, i.e., $i_1 d_I^{-1}(g) = i_2 i\overline{d_V^{-1}}(c)$, this means $(d_I^{-1}(g), 0) = (0, i\overline{d_V^{-1}}(c))$ and so $i\overline{d_V^{-1}}(c) = \overline{c} = 0$. Then our induce map is mono.

Then if we replace I^0 for I'', d_I^{-1} for $ji_1 d_I{-}1$ and φ^0 for $ji_1\varphi^0$, J is also an injective resolution, φ is a morphism and $\operatorname{coker}(d_V^{-1}) \to \operatorname{coker}(d_I^{-1})$ is monomorphism.

But by lemma 2.2.3 we have that $C^{-1} \oplus I^{-2} \to C^0 \times_{I^0} I^{-1}$ is an epimorphism, and so $\ker(\varphi) = [\, 0 \longrightarrow C^{-n} \oplus C^{-n-1} \longrightarrow \cdots \longrightarrow C^{-1} \oplus I^{-2} \longrightarrow C^0 \times_{I^0} I^{-1} \,]_{\mathcal{C}}$, where the first zero is because $\varphi^{-n-1} = 1_{C^{-n-1}}$ and is in the $-n-1$ position. Now, if

$$[\, D^{-n+1} \longrightarrow \cdots \longrightarrow D^{-1} \longrightarrow D^0 \,]$$

is an approximation for $C^0 \times_{I^0} I^{-1}$, we get that

$$\ker(\varphi) = [\, 0 \longrightarrow C^{-n} \oplus C^{-n-1} \longrightarrow \cdots \longrightarrow D^{-1} \oplus C^{-1} \oplus I^{-2} \longrightarrow D^0 \,]$$

and satisfies that

$$0 \longrightarrow 0 \longrightarrow C^{-n} \oplus C^{-n-1} \longrightarrow \cdots \longrightarrow D^{-1} \oplus C^{-1} \oplus I^{-2} \longrightarrow D^0 \longrightarrow 0$$

(where the second zero is in the $-n-1$ position), is exact because in the $0-$spot there is no cohomology as $C^{-1} \oplus I^{-2} \rightarrow C^0 \times_{I^0} I^{-1}$ is epi and $\ker(\varphi)$ is quasi-isomorphic to the complex $[\, 0 \longrightarrow C^{-n} \oplus C^{-n-1} \longrightarrow \cdots \longrightarrow C^{-1} \oplus I^{-2} \longrightarrow C^0 \times_{I^0} I^{-1} \,]$. Then because the first map of the kernel is split we have by lemma 2.4.1 that $\ker(\varphi) = 0$.

Then φ is monomorphism for every $V \in \mathcal{H}_{\mathcal{C}}$, in particular when V is injective, and in this case φ will be a split mono, which implies that V is a direct summand of J, and for this, V has the same form as J which is what we want. □

Theorem 2.4.7. *If $X, Y \in \mathcal{C}$ then $\underline{\mathrm{Hom}}_\Lambda(X, Y) \cong D\,\mathrm{Ext}^n_\Lambda(Y, X')$ for some X'.*

Proof. Let be X, Y in $\mathcal{C}$, then we have $Y \in \mathrm{mod}\,\Lambda$ and there exists projective resolution $P^\bullet \twoheadrightarrow Y$, such that as a complex $[P^\bullet \twoheadrightarrow Y]$ belongs to $\mathbf{D}_{\mathcal{C}}$, this because $\Lambda \in \mathcal{C}$ and $\mathcal{C}$ is closed under direct summands, then we can take a free resolution for Y. Consider the following truncation of $[P^\bullet \twoheadrightarrow Y]$,

$$W := 0 \longrightarrow \Omega^{-n} \longrightarrow P^{-n+1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow Y \longrightarrow 0$$

where $\Omega^{-n} = \ker(P^{-n} \to P^{-n+1})$ and which is a long exact sequence. Note that $Y_{\mathcal{C}} = Y$ and $P^i_{\mathcal{C}} = P^i$, but for Ω is not necessary true. Consider $C^\bullet := \Omega^{-n}[0]_{\mathcal{C}}$ the approximation for $\Omega^{-n}[0]$, then $V := W_{\mathcal{C}}$ has the form

$$V := \cdots \longrightarrow C^{-1} \longrightarrow C^0 \longrightarrow P^{-n+1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow Y \longrightarrow 0$$

Since P_X in concetrated in degree zero we have that

$$\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(P_X, V) \cong \mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(X[0], V) \cong \mathrm{Hom}_{\mathbf{D}_{\mathrm{mod}\,\Lambda}}(X[0], W)$$

Let be $\Omega^{-1} = \ker(P^{-1} \to P^0)$ and consider the followins short exact sequence

$$0 \longrightarrow \Omega^{-1} \longrightarrow P^0 \longrightarrow Y \longrightarrow 0$$

for this we can get the following long exact Ext $-$sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_\Lambda(X, \Omega^{-1}) \longrightarrow \mathrm{Hom}_\Lambda(X, P^0) \longrightarrow \mathrm{Hom}_\Lambda(X, Y) \longrightarrow \\ \mathrm{Ext}^1_\Lambda(X, \Omega^{-1}) \longrightarrow \mathrm{Ext}^1_\Lambda(X, P^0) \longrightarrow \mathrm{Ext}^1_\Lambda(X, Y) \longrightarrow \cdots \end{array}$$

but $\mathrm{Ext}^1_\Lambda(X, P^0) = 0$, so $\mathrm{Ext}^1_\Lambda(X, \Omega^{-1}) \cong \mathrm{Hom}_\Lambda(X, Y)/\,\mathrm{im}(\mathrm{Hom}_\Lambda(X, P^0) \to \mathrm{Hom}_\Lambda(X, Y))$ where $\mathrm{im}(\mathrm{Hom}_\Lambda(X, P^0) \to \mathrm{Hom}_\Lambda(X, Y)) = \{f : X \to Y : f = gh$ for some $h : X \to P^0, g : P^0 \to Y\}$. Note that, if Q is any projective such that every map $f : X \to Y$ factorizes throughout Q, then we have the following commutative diagram

$$
\begin{tikzcd}
Q \arrow[d, dotted] \arrow[dr] & X \arrow[l] \arrow[d] & \\
P^0 \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}
$$

where de dotted line exists because Q is projective, this means that f factorizes throughout P^0 too. Then we have that $\mathrm{Hom}_\Lambda(X,Y)/\,\mathrm{im}(\mathrm{Hom}_\Lambda(X,P^0) \to \mathrm{Hom}_\Lambda(X,Y)) \cong \underline{\mathrm{Hom}}_\Lambda(X,Y)$, i.e., $\mathrm{Ext}^1_\Lambda(X,\Omega^{-1}) \cong \underline{\mathrm{Hom}}_\Lambda(X,Y)$. Then since each map in $\underline{\mathrm{Hom}}_\Lambda(X,Y)$ determines a morphism in $\mathrm{Hom}_{\mathbf{D}_{\mathrm{mod}\,\Lambda}}(X[0],W)$, and the previous show the other direction, we get that $\mathrm{Hom}_{\mathbf{D}_{\mathrm{mod}\,\Lambda}}(X[0],W) \cong \underline{\mathrm{Hom}}_\Lambda(X,Y)$.

Let us consider now SP_X and a morphism in $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(V,SP_X)$, because SP_X is injective it has the form $[\ X^{-n-1} \longrightarrow I^{-n} \longrightarrow \cdots \longrightarrow I^0\ ]$, so by the following commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & C^{-1} \arrow[r] \arrow[d] & C^0 \arrow[r] \arrow[d] \arrow[ddl] & P^{-n+1} \arrow[r] \arrow[d] \arrow[ddl, "1"'] & \cdots \arrow[r] & P^0 \arrow[r] \arrow[d] \arrow[ddl, "1"'] & Y \arrow[r] \arrow[d] \arrow[ddl, "1"'] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & X^{-n-1} \arrow[r] & I^{-n} \arrow[r] & \cdots \arrow[r] & I^{-1} \arrow[r] & I^0 & \\
0 \arrow[r] \arrow[ur] & \Omega^{-n} \arrow[r] \arrow[ur] & P^{-n+1} \arrow[rr] \arrow[ur] & & P^0 \arrow[r] \arrow[ur] & Y \arrow[r] \arrow[ur] & 0 &
\end{tikzcd}
$$

we have that a morphism between V and SP_X is the same as a morphism between W and SP_X, i.e., $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(V,SP_X) = \mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(W,SP_X)$. Since the I^j's are injectives and W is acyclic, a morphism $\Omega^{-n} \to X^{-n-1}$ lifts to a map $W \to SP_X$ unique up to homotopy, indeed consider the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^{-n} \arrow[r] \arrow[d] & P^{-n-1} \arrow[d, dotted] \\
& X^{-n-1} \arrow[r] & I^{-n}
\end{tikzcd}
$$

the dotted line exists because I^{-n} is injective. By exactness $\mathrm{im}(\Omega^{-n} \to P^{-n+1}) = \ker(P^{-n+1} \to P^{-n+2})$, so we can repeat the process, i.e., we have

$$
\begin{tikzcd}
0 \arrow[r] & \ker(P^{-n+1} \to P^{-n+2}) \arrow[r] \arrow[d] & P^{-n+2} \arrow[d, dotted] \\
& I^{-n} \arrow[r] & I^{-n+1}
\end{tikzcd}
$$

continue the process we get the desired morphism. Them we have that $\mathrm{Hom}_{\mathbf{D}_{\mathcal{C}}}(W,SP_X) = \mathrm{Hom}_{\mathbf{D}_{\mathrm{mod}\,\Lambda}}(W,X^{-n-1}[-n-1])$. Then they are morphisms of the form

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{-n} \arrow[r] \arrow[d] & P^{-n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P^0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{-n-1} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So, $\operatorname{Ext}^n_\Lambda(Y, X^{-n-1}) \cong \operatorname{Ext}^1_\Lambda(\Omega^{-n+1}, X^{-n-1})$, and for the short exact sequence $0 \longrightarrow \Omega^{-n} \longrightarrow P^{-n} \longrightarrow \Omega^{-n+1} \longrightarrow 0$ we have

$$0 \longrightarrow \operatorname{Hom}_\Lambda(\Omega^{-n+1}, X^{-n-1}) \longrightarrow \operatorname{Hom}_\Lambda(P^{-n}, X^{-n-1}) \longrightarrow \operatorname{Hom}_\Lambda(\Omega^{-n}, X^{-n-1}) \longrightarrow$$

$$\operatorname{Ext}^1_\Lambda(\Omega^{-n+1}, X^{-n-1}) \longrightarrow \operatorname{Ext}^1_\Lambda(P^{-n}, X^{-n-1}) \longrightarrow \operatorname{Ext}^1_\Lambda(\Omega^{-n}, X^{-n-1}) \longrightarrow \cdots$$

But, $\operatorname{Ext}^1_\Lambda(P^{-n}, X^{-n-1}) = 0$ and then $\operatorname{Ext}^1_\Lambda(\Omega^{-n+1}, X^{-n-1})$ is isomorphic to $\operatorname{Hom}_\Lambda(\Omega^{-n}, X^{-n-1}) / \operatorname{im}(\operatorname{Hom}_\Lambda(P^{-n}, X^{-n-1}) \to \operatorname{Hom}_\Lambda(\Omega^{-n}, X^{-n-1}))$, i.e., are the maps from Ω^{-n} to X^{-n-1} which factorizes throughout P^{-n+1}, this means that they define a morphism up to homotopy between W and $X^{-n-1}[-n-1]$. Therefore, $\operatorname{Ext}^1_\Lambda(P^{-n}, X^{-n-1}) \cong \operatorname{Hom}_{\mathbf{D}_{\operatorname{mod}\Lambda}}(W, X^{-n-1}[-n-1]) \cong \operatorname{Ext}^n_\Lambda(Y, X^{-n-1})$.

Finally, by the AR-duality we have $\operatorname{Hom}_{\mathbf{D}_{\operatorname{mod}\Lambda}}(P_X, W)^* \cong \operatorname{Hom}_{\mathbf{D}_{\operatorname{mod}\Lambda}}(W, SP_X)$, i.e., $\operatorname{Hom}_{\mathbf{D}_{\operatorname{mod}\Lambda}}(X[0], W)^* \cong \operatorname{Hom}_{\mathbf{D}_{\operatorname{mod}\Lambda}}(W, X^{-n-1}[-n-1])$, and then we have $\underline{\operatorname{Hom}}_\Lambda(X, Y)^* = D\underline{\operatorname{Hom}}_\Lambda(X, Y) \cong \operatorname{Ext}^n_\Lambda(Y, X^{-n-1})$ as desired.

□

So because this formula was proved in [9] with $\tau_n X$ instead of X^{-n-1}, we have $X^{-n-1} \cong \tau_n X$. We have finally derived the higher AR-duality of Iyama,

Corollary 2.4.8. *For any $X, Y \in \mathcal{C}$ we have $\underline{\operatorname{Hom}}_\Lambda(X, Y) \cong D \operatorname{Ext}^n_\Lambda(Y, \tau_n X)$.*

□

Appendix A

Triangulated and Derived Categories

In This Appendix we give the basic facts concerning to Triangulated categories and Derived categories. Not all results will be proved but we will refer the reader to [7], [8], [12] and [11] where complete proofs can be found.

A.1 Triangulated Categories and Localization of Categories

In this section we will define and give some basic properties about triangulated categories.

Definition A.1.1. *Let $\mathcal{C}$ be an additive category. We say that a functor $T : \mathcal{C} \to \mathcal{C}$ is a translation functor of $\mathcal{C}$, if it is an automorphism of categories.*

Definition A.1.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two additive categories, let T and T' be its translation functors. A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be compatible with the translation functors T y T' if $FT = T'F$.*

Definition A.1.3. *Let $\mathcal{C}$ be an additive category and $T : \mathcal{C} \to \mathcal{C}$ a translation functor. A **Triangle** of $\mathcal{C}$ is an ordered sextuple (X, Y, Z, f, g, h) with $X, Y, Z \in \mathcal{C}$; $f : X \to Y$, $g : Y \to Z$ and $f : Z \to T(X)$ morphisms of $\mathcal{C}$. We represent the triangles by an exact sequence in $\mathcal{C}$ as follows*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

or as a diagram

$$\begin{tikzcd} & Z \arrow[dl, "h"'] & \\ X \arrow[rr, "f"'] & & Y \arrow[ul, "g"'] \end{tikzcd}$$

Definition A.1.4. *A* ***morphism of triangles*** $(X,Y,Z,,u,v,w)$, (X',Y',Z',u',v',w') *is a triple* (f,g,h) *such that the following diagram is commutative*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & T(X) \arrow[d, "T(f)"'] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T(X') \end{tikzcd}$$

If (f,g,h) *are isomorphisms, we say that the triangle morphism is an isomorphism.*

Definition A.1.5. *A* **Triangulated Category** *is an additive category* $\mathcal{C}$ *with tanslation functor* T *endowed with a family of triangles, called* distinguished triangles *(denoted by* τ*), which satisfay the axioms below:*

TR1 *Each trianguile isomorphic to a distinguished triangle is a distinguished triangle.*

TR2 *Each* $X \in \mathcal{C}$*, defines a distinguished traingle* $(X,X,0,1_X,0,0)$.

TR3 *Each* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ *defines a distinguished traingle* (X,Y,Z,f,g,h).

TR4 *A triangle* (X,Y,Z,f,g,h) *is distinguished if and only if* $(Y,Z,T(X),g,h,-T(f))$ *is a distinguished traingle.*

TR5 *For* $(X,Y,Z,f,g,h),(X',Y',Z',f',g',h') \in \tau$, $u \in \mathrm{Hom}_{\mathcal{C}}(X,X')$, $v \in \mathrm{Hom}_{\mathcal{C}}(Y,Y')$ *such that* $vf = f'u$*, there exists* $w \in \mathrm{Hom}_{\mathcal{C}}(Z,Z')$ *such that* $(u,v,w) : (X,Y,Z,f,g,h) \to (X',Y',Z',f',g',h')$ *is a morphism of triangles, i.e., the following is a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, dotted, "w"'] & T(X) \arrow[d, "T(u)"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X') \end{tikzcd}$$

TR6, Octahedron Axiom *Suppose* $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y), f' \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ *and* $f'' = f'f$. *If* $(X,Y,Z',f,g,h),(Y,Z,X',f',g',h'),(X,Z,Y',f'',g'',h'') \in \tau$. *Then, there exist* $u \in \mathrm{Hom}_{\mathcal{C}}(Z',Y')$, $v \in \mathrm{Hom}_{\mathcal{C}}(Y',X')$ *such that* $(Z',Y',X',u,v,T(g)h') \in \tau$ *and* $(1_X,f',u),(f,1_Z,v)$ *are morphism of triangles,i.e., the following is a commutative diagram:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_X"'] & Y \arrow[r, "g"] \arrow[d, "f'"'] & Z' \arrow[r, "h"] \arrow[d, dotted, "u"'] & T(X) \arrow[d, "T(1_X)"'] \\
X \arrow[r, "f''"] \arrow[d, "f"'] & Z \arrow[r, "g''"] \arrow[d, "1_Z"'] & Y' \arrow[r, "h''"] \arrow[d, dotted, "v"'] & T(X) \arrow[d, "T(f)"'] \\
Y \arrow[r, "f'"] & Z \arrow[r, "g'"] & X' \arrow[r, "h'"] \arrow[d, dotted, "T(g)h'"'] & T(Y) \arrow[dl, "T(g)"] \\
 & & T(Z') &
\end{tikzcd}$$

Remark A.1.6. *From **TR4** y **TR5**, if* $(X, Y, Z, f, g, h), (X', Y', Z', f', g', h') \in \tau$*, and* $v \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$ *y* $w \in \mathrm{Hom}_{\mathcal{C}}(Z, Z')$ *satisfies* $wg = g'v$*, there is some* $u \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ *such that* $(u, v, w) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ *is a morphism of triangles.*

Proposition A.1.7. *The composition of two consecutive morphism in a triangle is zero.*

□

Definition A.1.8. *Let* $\mathcal{C}$ *be a triagulated category and let* $\mathcal{A}$ *be an abelian category. We say that an additive functor* $F : \mathcal{C} \to \mathcal{A}$ *is a (covariant)* ***cohomological functor****, if for each distinguished triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ *in* $\mathcal{C}$*, the following,* $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ *is a short exact sequence in* $\mathcal{A}$*.*

Proposition A.1.9. *Let* $\mathcal{C}$ *be a triangulated category, then if* $M \in \mathcal{C}$ *the functors* $\mathrm{Hom}(M, -) : \mathcal{C} \to \mathbf{Ab}$*. and* $\mathrm{Hom}(-, M) : \mathcal{C} \to \mathbf{Ab}$*. are cohomological functors.*

□

Proposition A.1.10. *Let* $\mathcal{C}$ *be a triangulated category and* (X, Y, Z, u, v, w) *a distinguished traingle. the following are equivalent:*

i) $w = 0$.

ii) u *is split mono.*

iii) v *is spli epi.*

□

Definition A.1.11. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two triangulated categories. A functor* $F : \mathcal{C} \to \mathcal{C}'$ *is a* $\partial-$*functor if* F *is compatible with the tranlation functors and sends distinguished triangles of* $\mathcal{C}$ *to distinguished triangles of* $\mathcal{C}'$*.*

Definition A.1.12. *Let* $\mathcal{B}$ *be a full subcategory of the triangulated category* $\mathcal{C}$*. We say that* $\mathcal{B}$ *is a triangulated subcategory of* $\mathcal{C}$ *if satisfies the following conditions:*

a) *$\mathcal{B}$ is stable inder the translation functor $\mathcal{C}$, this means, if $B \in \mathcal{B}$ then $T(B) \in \mathcal{B}$.*

b) *For all triangle of $\mathcal{C}$ where the first objects belongs to $\mathcal{B}$, the third one also belongs to $\mathcal{B}$.*

We show now the triangulated structure of $\mathbf{K}(\mathcal{A})$, where $\mathcal{A}$ is an additive category.

Definition A.1.13. *Let $\mathcal{A}$ be an additive category, we define the functor $T : \mathbf{C}^*(\mathcal{A}) \to \mathbf{C}^*(\mathcal{A})$ $(* = \emptyset, +, -, b)$ as*

- *If $X^\bullet \in \mathbf{C}^*(\mathcal{A})$, then $T(X^\bullet)^i = X^{i+1}$ and $d^i_{T(X)} = -d^{i+1}_X$ for all $i \in \mathbb{Z}$.*
- *If $f : X^\bullet \to Y^\bullet$ then $T(f) : T(X^\bullet) \to T(Y^\bullet)$ is such that $T(f)^i = f^{i+1}$.*

Definition A.1.14. *Consider $u \in \mathrm{Hom}_{\mathbf{C}^*(\mathcal{A})}(X^\bullet, Y^\bullet)$, we define the* **cone** *of u, denoted by $\mathrm{cone}(u)^\bullet$, as the complex:*

- $\mathrm{cone}(u)^i = Y^i \oplus T(X)^i = Y^i \oplus X^{i+1}$
- $d^i_{\mathrm{cone}(u)} = \begin{bmatrix} d^i_Y & u^{i+1} \\ 0 & -d^{i+1}_X \end{bmatrix}$

with this in mind we can define a triangle in $\mathbf{C}^*(\mathcal{A})$ in the following way: for $u \in \mathrm{Hom}_{C^*(\mathcal{A})}(X^\bullet, Y^\bullet)$, we have

$$X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{q_Y} \mathrm{cone}(u)^\bullet \xrightarrow{p_{T(X)}} T(X^\bullet)$$

donde $q_Y = [1_Y, 0]^t$ and $p_{T(X)} = [0, 1_{T(X)}]$. We can embed this triangle in $\mathbf{K}^*(\mathcal{A})$ using the map which sends each object in $\mathbf{C}^*(\mathcal{A})$ to the same in $\mathbf{K}^*(\mathcal{A})$, and each map to its homotopy class. The triangle in $\mathbf{K}^*(\mathcal{A})$ will be called **standard triangle over** u.

Proposition A.1.15.
- *If $X^\bullet \in \mathbf{C}(\mathcal{A})$ then $\mathrm{cone}(1_X)$ is homotopic to zero.*
- *If $u \in \mathrm{Hom}_{\mathbf{C}^*(\mathcal{A})}(X^\bullet, Y^\bullet)$ is a quiasi-isomorphism, then its cone, $\mathrm{cone}(u)^\bullet$ is an acyclic complex.*

□

Remark A.1.16. *If $u = v$ in $K(\mathcal{A})$, then $C^\bullet_u \cong C^\bullet_v$. So the standard triangles over u and v are isomorphic*

Definition A.1.17. *The family of distinguished triangles in $\mathbf{K}(\mathcal{A})$, denoted by τ, is defined as the family of standard triangles in $\mathbf{K}(\mathcal{A})$ closed under isomorphisms.*

Remark A.1.18. *Evidently the translation functor $T : \mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})$ induces a translation functor $T_K : \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$.*

Theorem A.1.19. *The category $\mathbf{K}(\mathcal{A})$, the family τ and the translation functor T_K define a triangulated category.*

□

Proposition A.1.20. *If $\mathcal{A}$ is an abelian category, then $H^* : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor.*

□

Corollary A.1.21.

- *If in the axiom **TR5** the morphisms u and v are quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$, then also w.*
- *The categories $\mathbf{K}^*(\mathcal{A})$ ($* = \emptyset, +, -, b$) are triangulated subcategories of $\mathbf{K}(\mathcal{A})$.*

□

We now want to generalized the idea of localization of a ring, the purpose of this is to define the derived category of an abelian category.

Definition A.1.22. *Let $\mathcal{C}$ be a category. A multiplicative system $\Sigma_{\mathcal{C}}$ is a family of morphisms in $\mathcal{C}$ such that:*

SM1 *a) If $s, t \in \Sigma_{\mathcal{C}}$ and st is defined, then $st \in \Sigma_{\mathcal{C}}$.*

b) If $X \in \mathcal{C}$, then $1_X \in \Sigma_{\mathcal{C}}$.

SM2 *Let $X, Y, Z \in \mathcal{C}$. IF $u \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ (resp. $u \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$) and $s \in \mathrm{Hom}_{\mathcal{C}}(Z, Y) \cap \Sigma_{\mathcal{C}}$ (resp. $s \in \mathrm{Hom}_{\mathcal{C}}(Y, Z) \cap \Sigma_{\mathcal{C}}$), then there exists $W \in \mathcal{C}$ and morphisms $v \in \mathrm{Hom}_{\mathcal{C}}(W, Z)$ (resp. $v \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$) y $t \in \mathrm{Hom}_{\mathcal{C}}(W, X) \cap \Sigma_{\mathcal{C}}$ (resp. $t \in \mathrm{Hom}_{\mathcal{C}}(X, W) \cap \Sigma_{\mathcal{C}}$) such that the following diagram is commutative (resp. the right hand diagram):*

$$\begin{tikzcd} W \arrow[r, dotted, "v"] \arrow[d, dotted, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "u"'] & Y \end{tikzcd} \qquad \begin{tikzcd} W & Z \arrow[l, dotted, "v"] \\ X \arrow[u, dotted, "t"] & Y \arrow[l, "u"] \arrow[u, "s"'] \end{tikzcd}$$

SM3 *Let $X, Y \in \mathcal{C}$ and $u, v \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the following conditions are equivalent:*

i) There exists $Z \in \mathcal{C}$ and $s \in \mathrm{Hom}_{\mathcal{C}}(Y, Z) \cap \Sigma_{\mathcal{C}}$, such that $su = sv$

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \arrow[r, dotted, "s"] & Z \end{tikzcd}$$

ii) There exists $W \in \mathcal{C}$ and $t \in \mathrm{Hom}_{\mathcal{C}}(W, X) \cap \Sigma_{\mathcal{C}}$, such that $ut = uv$

$$\begin{tikzcd} W \arrow[r, dotted, "t"] & X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Y \end{tikzcd}$$

Remark A.1.23. *A morphism $s : X \to Y$ which belongs to Σ, pictorially is drawn as follows* $X \xrightarrow[\sim]{s} Y$ *.*

Since the following theorem is very important in the theory because with this we can define the derived category, we give a sketch of the proof. For the complete details the reader could refer to [8] or [12]

Theorem A.1.24 (Localization Theorem)**.** *Let Σ be a multiplicative system of a category $\mathcal{C}$, then there exists a category $\mathcal{C}_\Sigma$ and a functor $P_\Sigma : \mathcal{C} \to \mathcal{C}_\Sigma$ which satisfies the following conditions:*

i) If $s \in \Sigma$, then $P_\Sigma(s)$ is an isomorphism in $\mathcal{C}_\Sigma$.

ii) If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism in $\mathcal{D}$ for all $s \in \Sigma$, then there is a unique functor $G : \mathcal{C}_\Sigma \to \mathcal{D}$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "P_\Sigma"] \arrow[dr, "F"'] & & \mathcal{C}_\Sigma \arrow[dl, dotted, "G"] \\ & \mathcal{D} & \end{tikzcd}$$

Proof. Consider $X, Y \in \mathcal{C}$. Define M_{XY} as the family of diagrams of the form

$$X \xleftarrow[\sim]{s} Z \xrightarrow{u} Y$$

where $Z \in \mathcal{C}$, and s, u are morphisms in $\mathcal{C}$. We will represent this diagram by (Z, s, u).

For $(W, s, a), (Z, t, b) \in M_{XY}$, we define a morphism $p : (W, s, a) \to (Z, t, b)$ as a morphism $p \in \mathrm{Hom}_\mathcal{C}(W, Z)$ such that the following is commutative:

$$\begin{tikzcd} & W \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] \arrow[dd, "p"] & \\ X & & Y \\ & Z \arrow[ul, "t", "\sim"'] \arrow[ur, "b"'] & \end{tikzcd}$$

We say that two objects $(Z, s, a), (Z', s', a') \in M_{XY}$ are equivalent if there is some $(T, r, b) \in M_{XY}$ and $p : (T, r, b) \to (Z, s, a)$, $p' : (T, r, b) \to (Z', s', a')$ such that the following diagram is commutative

$$
\begin{tikzcd}
 & Z \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & \\
X & T \arrow[l, "r"', "\sim"] \arrow[u, "p"'] \arrow[r, "b"] \arrow[d, "p'"] & Y \\
 & Z' \arrow[ul, "s'", "\sim"'] \arrow[ur, "a'"'] &
\end{tikzcd}
$$

We denote this relation by $\equiv$ and we can also show that is an equivalence relation over M_{XY}. Then we can define a composition law in $M_{XY}/\equiv$ as: For $\alpha, \beta \in M_{XY}/\equiv$ which are represented by (U, s, a) and (V, t, b) respectively, form the commutative diagram

$$
\begin{tikzcd}
 & & W \arrow[dl, "r"', "\sim"] \arrow[dr, "c"] & & \\
 & U \arrow[dl, "s"', "\sim"] \arrow[dr, "a"] & & V \arrow[dl, "t"', "\sim"] \arrow[dr, "b"] & \\
X & & Y & & Z
\end{tikzcd}
$$

then the class of (W, sr, bc) in $M_{XY}/\equiv$ will be our desired composition, and will be denoted by $\beta\alpha$. This law is associative and the identity is $(X, 1_X, 1_X) \in M_{XY}$.

With this in mind, we define the localized category $\mathcal{C}_\Sigma$ in the following way:

- $\operatorname{Obj} \mathcal{C}_\Sigma = \operatorname{Obj} \mathcal{C}$
- If $X, Y \in \mathcal{C}_\Sigma$, then $\operatorname{Hom}_{\mathcal{C}_\Sigma}(X, Y) = M_{XY}/\equiv$

The localization functor $P_\Sigma : \mathcal{C} \to \mathcal{C}_\Sigma$ is defined as

- For $X \in \mathcal{C}$, put $P_\Sigma(X) = X$
- If $u \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, then $P_\Sigma(u)$ is the equivalence class of $(X, 1_X, u) \in M_{XY}$

Finally let us see that the functor P_Σ satisfies the universal property, but this follows if we define $G : \mathcal{C}_\Sigma \to \mathcal{D}$ as

- For $X \in \mathcal{C}_\Sigma$, put $G(X) = F(X)$.
- If $\alpha \in \operatorname{Hom}_{\mathcal{C}_\Sigma}(X, Y)$, such that is represented by (U, s, a), then put $G(\alpha) = F(a)F(s)^{-1}$

$\square$

Note also, that if we define for $X \in \mathcal{C}$, the category $\mathcal{I}_X$ as:

- $w \in \mathcal{I}_X \Leftrightarrow \exists_{Z\in\mathcal{C}}\exists_{s\in \mathrm{Hom}_{\mathcal{C}}(Z,X)\cap\Sigma}(w = (Z,s))$
- If $(Z,s),(Z',s') \in \mathcal{I}_X$ then

$$\mathrm{Hom}_{\mathcal{I}_X}((Z,s),(Z',s')) = \{f \in \mathrm{Hom}_{\mathcal{C}}(Z,Z') \ / \ s'f = s\}$$

then we have that

$$\mathrm{Hom}_{\mathcal{C}_\Sigma}(X,Y) = \varinjlim_{(Z,s)\in\mathcal{I}_X} \mathrm{Hom}_{\mathcal{C}}(Z,Y)$$

We finish this sections givin the basic properties of localized categories.

Proposition A.1.25. *Let $\mathcal{C}$ be an additive category.*

- *$\mathcal{C}_\Sigma$ is an additive category and the localization functor P_Σ is additive.*
- *If $u \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, the following are equivalent:*
  - *i) $P_\Sigma(u) = 0$.*
  - *ii) There is $s \in \mathrm{Hom}_{\mathcal{C}}(W,X)\cap\Sigma$ such that $us = 0$.*
  - *iii) There is $t \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)\cap\Sigma$ such that $tu = 0$.*

□

Remark A.1.26. *Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. Let $\Sigma_{\mathcal{C}}$ be a multiplicative system of $\mathcal{C}$ and denoted by $\Sigma_{\mathcal{B}}$ the set $\mathrm{Hom}(\mathcal{B})\cap\Sigma_{\mathcal{C}}$.*

If $\Sigma_{\mathcal{B}}$ is a multiplicative system of $\mathcal{B}$, then by theorem A.1.24, we can construct the localized category $\mathcal{B}_{\Sigma_{\mathcal{B}}}$ and the localization functor $P_{\Sigma_{\mathcal{B}}} : \mathcal{B} \to \mathcal{B}_{\Sigma_{\mathcal{B}}}$. Now, if we consider the inclusion functor $I : \mathcal{B} \to \mathcal{C}$ we have that $P_{\Sigma_{\mathcal{C}}}I(s)$ is an isomorphism in $\mathcal{C}_\Sigma$ for all $s \in \Sigma_{\mathcal{C}}$, the by the universal property there is unique F_Σ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "P_{\Sigma_{\mathcal{B}}}"] \arrow[dr, "P_{\Sigma_{\mathcal{C}}}I"'] & & \mathcal{B}_{\Sigma_{\mathcal{B}}} \arrow[dl, dotted, "F_\Sigma"] \\
& \mathcal{C}_{\Sigma_{\mathcal{C}}} &
\end{tikzcd}$$

Proposition A.1.27. *Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$. Let $\Sigma_{\mathcal{C}}$ be a multiplicative system of $\mathcal{C}$ and consider $\Sigma_{\mathcal{B}} = \mathrm{Hom}(\mathcal{B})\cap\Sigma_{\mathcal{C}}$, a multiplicative system of $\mathcal{B}$. Suppose that one of the following conditions hold:*

i) For all $s \in \mathrm{Hom}_{\mathcal{C}}(X,Y)\cap\Sigma_{\mathcal{C}}$, there is $f \in \mathrm{Hom}_{\mathcal{C}}(Z,X)$ with $Z \in \mathcal{B}$, such that $sf \in \Sigma_{\mathcal{B}}$.

ii) For all $s \in \operatorname{Hom}_{\mathcal{C}}(X,Y) \cap \Sigma_{\mathcal{C}}$*, there is* $f \in \operatorname{Hom}_{\mathcal{C}}(Y,Z)$ *with* $Z \in \mathcal{B}$*, such that* $fs \in \Sigma_{\mathcal{B}}$*.*

Then the functor F_{Σ} *is full and faithful.*

$\square$

Definition A.1.28. *Let* $\mathcal{C}$ *be a triangulated category, with translation functor* T *and family of distinguished traingles* τ*. Let* Σ *be a multiplicative system of* $\mathcal{C}$*. We say that the multiplicative system* Σ *is compatible with the triangulated structure of* $\mathcal{C}$*, if in addition of fulfilling **SM1, SM2** y **SM3**, also satisfies:*

SM4 $s \in \Sigma$ *if and only if* $T(s) \in \Sigma$*.*

SM5 *If in **TR5**, the morphisms* $u, v \in \Sigma$ *then* $w \in \Sigma$*.*

Theorem A.1.29. *Let* $\mathcal{C}$ *be a triangulated category,* Σ *a multiplicative system of* $\mathcal{C}$ *compatible with the triangulated structure. Then the localized category* $\mathcal{C}_{\Sigma}$ *is endow with a triangulated structure such that the localization functor* $P_{\Sigma} : \mathcal{C} \to \mathcal{C}_{\Sigma}$ *is a* $\partial-$*functor.*

$\square$

A.2 The Derived Category

In this section we will defined the derived category as the localization of the Homotopy category when we take as a multiplicative system the family of quasi-isomorphisms. We give some basic properties and proved under some conditions when the derived category and the Homotopy category coincide.

Definition A.2.1. *Let* $\mathcal{A}$ *be an abelian category. We denoted by* $\mathrm{His}_{\mathcal{A}}$ *(or just* His *if there is no place to confusion) the family of quasi-isomorphisms of* $\mathbf{K}(\mathcal{A})$*.*

Proposition A.2.2. *The family* His *is a multiplicative system of* $\mathbf{K}(\mathcal{A})$ *which is compatible with the triangulated structure.*

$\square$

Definition A.2.3. *Let* $\mathcal{A}$ *be an abelian category. The* **Derived Category** *of* $\mathcal{A}$ *denoted by* $\mathbf{D}(\mathcal{A})$*, is defined as the localization of the category* $\mathbf{K}(\mathcal{A})$ *with respect to the multiplicative system* His*, i.e.,* $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})_{His}$*.*

Remark A.2.4. *1. Explicitely we have:*

- $\operatorname{Obj} \mathbf{D}(\mathcal{A}) = \operatorname{Obj} \mathbf{K}(\mathcal{A})$
- *For* $X^{\bullet}, Y^{\bullet} \in \mathbf{D}(\mathcal{A})$*, we have*

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{A})}(X^{\bullet}, Y^{\bullet}) = \varinjlim_{(Z^{\bullet},s)\in \mathcal{I}_{X^{\bullet}}} \operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(Z^{\bullet}, Y^{\bullet})$$

2. *$P_{\mathcal{A}}$ denotes the localization functor $P_{\mathcal{A}} : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$.*

Theorem A.2.5. *Let $\mathcal{A}$ be an abelian category. Then, $\mathbf{D}(\mathcal{A})$ is an additive and triangulated category and the localization functor $P_{\mathcal{A}} : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ is additive and is also a ∂-functor, which sends quasi-isomorphisms of $\mathbf{K}(\mathcal{A})$ to isomorphisms of $\mathbf{D}(\mathcal{A})$.*

□

Remark A.2.6. *By the universal property of the localization we can extend $H^n : \mathbf{K}(\mathcal{A}) \to \mathcal{A}$ to a functor $F : \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ such that $FP_{\mathcal{A}} = H^n$, and if there is no place to confusion we call it also H^n*

Proposition A.2.7.

- *$\alpha \in \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X^\bullet, Y^\bullet)$ is an isomorphism in $\mathbf{D}(\mathcal{A})$ if and only if admits representative $(Z^\bullet, s, a)$ where s and a are quasi-isomorphism in $\mathbf{K}(\mathcal{A})$.*
- *Let $u \in \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X^\bullet, Y^\bullet)$. If $P_{\mathcal{A}}(u) = 0$, then $H^n(u) = 0$ for all $n \in \mathbb{Z}$*

□

Definition A.2.8. *Let $C_0 : \mathcal{A} \to \mathbf{K}(\mathcal{A})$ and $P_{\mathcal{A}} : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ be functors, we define the functor $D : \mathcal{A} \to \mathbf{D}(\mathcal{A})$ as $D = P_{\mathcal{A}}C_0$. This is a fully faithful functor.*

Since $\mathbf{D}(\mathcal{A})$ is triangulated, it is endowed with a translation functor T and a family of distinguished triangles. The functor T is the unique with do the following diagram commutative

$$\begin{tikzcd} K(\mathcal{A}) \arrow[rr, "P_{\mathcal{A}}"] \arrow[dr, "P_{\mathcal{A}}T_K"'] & & D(\mathcal{A}) \arrow[dl, "T"] \\ & D(\mathcal{A}) & \end{tikzcd}$$

And the distinguished triangle are the standard ones, i.e., triangles of the form

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} \mathrm{cone}(f)^\bullet \xrightarrow{h} T(X^\bullet)$$

In a similar way, we can define His^* $(* = +, -, b)$ as $\mathrm{His}^* = \mathrm{Hom}(\mathbf{K}^*(\mathcal{A})) \cap \mathrm{His}$. And also prove that this is a multiplicative system for $\mathbf{K}^*(\mathcal{A})$. Then, we have a bounded derived category $\mathbf{D}^*(\mathcal{A})$ obtain as the localization of $\mathbf{K}^*(\mathcal{A})$ with respect to His^*. The localization functor will be denoted as $P^*_{\mathcal{A}} : \mathbf{K}^*(\mathcal{A}) \to \mathbf{D}^*(\mathcal{A})$, and $F^* : \mathbf{D}^*(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ denotes the induce functors obtain by the inclusions $I^* : \mathbf{K}^*(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$. We can show that the functors F^* $(* = +, -, b)$, are fully and faithful.

We finish this section showing the principal equivalences between derived categories and homotopy categories

Theorem A.2.9. *Let $\mathcal{A}$ be an abelian category. There exists a fully faithful functor*

$$P_{\mathcal{A}} : \mathbf{K}^{-}(\mathrm{Proj}(\mathcal{A})) \to \mathbf{D}^{-}(\mathcal{A})$$

Moreover, if $\mathcal{A}$ admits enough projectives, then $P_{\mathcal{A}}$ is an equivalence of categories.

□

Theorem A.2.10. *Let $\mathcal{A}$ be an abelian category. There exists a fully faithful functor*

$$I_{\mathcal{A}} : \mathbf{K}^{+}(\mathrm{Inj}(\mathcal{A})) \to \mathbf{D}^{+}(\mathcal{A})$$

Moreover, if $\mathcal{A}$ admits enough projectives, then $I_{\mathcal{A}}$ is an equivalence of categories.

□

Appendix B

Selected Proofs of Chapter 1

In this appendix we give some proofs about $t-$structures and AR-Theory. These are important for the constructions presented or just by theoretical reasons.

Proof of proposition 1.1.3. We may assume $n = 0$, because using the translation functor $\mathbf{D}^{\leq n}[-n] \to \mathbf{D}[-n]$ is $\mathbf{D}^{\leq 0} \to \mathbf{D}$. By axiom *iii*) of $t-$structure, for $X \in \mathbf{D}$, there exists a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1]$$

where $X_0 \in \mathbf{D}^{\leq 0}$ and $X_1 \in \mathbf{D}^{\geq 1}$.

Suppose $Y \in \mathbf{D}^{\leq 0}$ (resp. $Z \in \mathbf{D}^{\geq 1}$) and consider the following sequence:

$$\mathrm{Hom}(Y, X_1[-1]) \longrightarrow \mathrm{Hom}(Y, X_0) \longrightarrow \mathrm{Hom}(Y, X) \longrightarrow \mathrm{Hom}(Y, X_1)$$

(resp. $\mathrm{Hom}(Y, X_1[-1]) \longrightarrow \mathrm{Hom}(Y, X_0) \longrightarrow \mathrm{Hom}(Y, X) \longrightarrow \mathrm{Hom}(Y, X_1)$)

So, as $X_1 \in \mathbf{D}^{\geq 1}$ and $Y \in \mathbf{D}^{\leq 0}$, we have that $\mathrm{Hom}(Y, X_1) = 0$, and since $X_1[-1] \in \mathbf{D}^{\geq 2} \subseteq \mathbf{D}^{\geq 1}$ we obtain $\mathrm{Hom}(Y, X_1[-1]) = 0$.

This gives the sequence

$$0 \longrightarrow \mathrm{Hom}(Y, X_0) \longrightarrow \mathrm{Hom}(Y, X) \longrightarrow 0$$

these means, $\mathrm{Hom}(Y, X_0) \longrightarrow \mathrm{Hom}(Y, X)$ is an isomorphism. Then taking $\tau^{\leq 0}X := X_0$ we have the desired result. (resp. $Z \in \mathbf{D}^{\geq 1}$, $X_0 \in \mathbf{D}^{\leq 1}$ and $X_0[1] \in \mathbf{D}^{\leq 0}$ gives $\mathrm{Hom}(X_0[1], Z) = \mathrm{Hom}(X_0, Z) = 0$ and $\mathrm{Hom}(X_1, Z) \longrightarrow \mathrm{Hom}(X, Z)$ is an isomorphism. Take $\tau^{\geq 1}X := X_1$).

This complete the proof. $\square$

Proof of proposition 1.1.9. By considering the following distinguised triangles, for X and Y in $\mathcal{C}$

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow X[1]$$

$$0 \longrightarrow Y \xrightarrow{1} Y \longrightarrow 0$$

We form the distinguished triangle $X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow[+1]{}$, where $X \oplus Y$ exists because $\mathbf{D}$ is additive, then $X \oplus Y$ belongs to $\mathcal{C}$, so this is an additive category (the other conditions follows for $\mathbf{D}$ be additive).

Let $f : X \to Y$ be a morphism in $\mathcal{C}$, and let us embed f into a distinguished triangle $X \xrightarrow{f} Y \longrightarrow Z \xrightarrow[+1]{}$. Then, $Y \longrightarrow Z \longrightarrow X[1] \xrightarrow[+1]{}$ is a distinguished triangle, and by proposition 1.1.7, Z belongs to $\mathbf{D}^{\leq 0} \cup \mathbf{D}^{\geq -1}$ because $Y \in \mathbf{D}^{\leq 0} \cup \mathbf{D}^{\geq 0} \subseteq \mathbf{D}^{\leq 0} \cup \mathbf{D}^{\geq -1}$ and $X[1] \in \mathbf{D}^{\leq 1} \cup \mathbf{D}^{\geq -1} \subseteq \mathbf{D}^{\leq 0} \cup \mathbf{D}^{\geq -1}$.

We shall prove:

$$\begin{cases} H^0(Z) \cong \tau^{\geq 0}\tau^{\leq 0} Z \cong \tau^{\geq 0} Z \cong \operatorname{coker} f \\ H^0(Z[-1]) \cong (\tau^{\geq -1}\tau^{\leq -1} Z)[-1] \cong (\tau^{\leq -1} Z)[-1] \cong \tau^{\leq 0}(Z[-1]) \cong \ker f \end{cases} \qquad (\star)$$

For that purpose take $W \in \mathcal{C}$ and consider the long exact sequences

$$\operatorname{Hom}(X[1], W) \longrightarrow \operatorname{Hom}(Z, W) \longrightarrow \operatorname{Hom}(Y, W) \longrightarrow \operatorname{Hom}(X, W)$$

and

$$\operatorname{Hom}(W, Y[-1]) \longrightarrow \operatorname{Hom}(W, Z[-1]) \longrightarrow \operatorname{Hom}(W, X) \longrightarrow \operatorname{Hom}(W, Y)$$

Then, we note that $X[1] \in \mathbf{D}^{\leq -1} \cup \mathbf{D}^{\geq -1}$, $W \in \mathbf{D}^{\geq 0} \cup \mathbf{D}^{\leq 0}$ and $Y[-1] \in \mathbf{D}^{\leq 1} \cup \mathbf{D}^{\geq 1}$, so $\operatorname{Hom}(X[1], W) = \operatorname{Hom}(W, Y[-1]) = 0$. And by proposition 1.1.3, $\operatorname{Hom}(Z, W) \cong \operatorname{Hom}_{\mathbf{D}^{\geq 0}}(\tau^{\geq 0} Z, W)$, $\operatorname{Hom}(W, Z[-1]) \cong \operatorname{Hom}_{\mathbf{D}^{\leq 0}}(W, \tau^{\leq 0}(Z[-1]))$ because $W \in \mathcal{C}$.

We get the exact sequences

$$0 \longrightarrow \operatorname{Hom}(\tau^{\geq 0} Z, W) \xrightarrow{h^*} \operatorname{Hom}(Y, W) \xrightarrow{f^*} \operatorname{Hom}(X, W)$$

$$0 \longrightarrow \operatorname{Hom}(W, \tau^{\leq 0}(Z[-1])) \xrightarrow{h_*} \operatorname{Hom}(W, X) \xrightarrow{f_*} \operatorname{Hom}(W, Y)$$

which implies $(\star)$, i.e. $\tau^{\geq 0} Z \cong \operatorname{coker} f$ and $\tau^{\leq 0}(Z[-1]) \cong \ker f$. This is, because if we have the kernel diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[drr, bend right, "0"'] & Y \arrow[r, "h"] \arrow[dr, "g"'] & \tau^{\geq 0} Z \arrow[d, dashed, "\varphi"] \\ & & W \end{tikzcd}$$

$gf = f^*(g) = 0$, then $g \in \operatorname{im} h^*$, so as h^* mono, exists unique φ such that $\varphi h = g$. The same if we have the cokernel diagram,

$$\begin{tikzcd} \tau \leq 0(Z[-1]) \arrow[r, "h"] & X \arrow[r, "f"] & Y \\ W \arrow[u, dashed, "\psi"] \arrow[ur, "g"] \arrow[urr, bend right, "0"'] & & \end{tikzcd}$$

$fg = f_*(g) = 0$,so then $g \in \operatorname{im} h_*$, so as h_* mono, exists unique ψ such that $h\psi = g$.

Now, let us prove that the canonical morphism $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. We embed $Y \to \tau^{\geq 0} Z$ into a distinguished triangle $I \longrightarrow Y \longrightarrow \tau^{\geq 0} Z \xrightarrow[+1]{}$. Then we have that $\tau^{\geq 0} Z[-1] \longrightarrow I \longrightarrow Y \xrightarrow[+1]{}$ is distinguished too, and $\tau^{\geq 0} Z[-1] \in \mathbf{D}^{\geq -1} \subseteq \mathbf{D}^{\geq 0}$, $Y \in \mathbf{D}^{\geq 0}$ gives $I \in \mathbf{D}^{\geq 0}$ by proposition 1.1.7.

We apply the octahedral axiom to the morphism $Y \to Z \to \tau^{\geq 0} Z$ and the triangles

$$Y \longrightarrow Z \longrightarrow X[1] \xrightarrow[+1]{}$$

$$Z \longrightarrow \tau^{\geq 0} Z \longrightarrow \tau^{\leq -1} Z[1] \xrightarrow[+1]{}$$

$$Y \longrightarrow \tau^{\geq 0} Z \longrightarrow I[1] \xrightarrow[+1]{}$$

the second is a translation of the triangle

$$\tau^{<0} Z \longrightarrow Z \longrightarrow \tau^{\geq 0} Z \xrightarrow[+1]{}$$

.

Then, there exists distinguished tringle

$$X[1] \longrightarrow I[1] \longrightarrow \tau^{\leq -1} Z[1] \xrightarrow[+1]{}$$

and we get

$$(\tau^{\leq -1} Z)[-1] \longrightarrow X \longrightarrow I \xrightarrow[+1]{}$$

but $(\tau^{\leq -1} Z)[-1] \cong \tau^{\leq 0}(Z[-1])$.

Thus we obtain the distinguished triangle

$$\tau^{\leq 0}(Z[-1]) \longrightarrow X \longrightarrow I \xrightarrow[+1]{}$$

Hence I belongs to $\mathbf{D}^{\leq 0}$, because X and $\tau^{\leq 0}(Z[-1])[1] \cong (\tau^{\leq 0} Z)[-1][1] \cong \tau^{\leq 0} Z$ belongs to $\mathbf{D}^{\leq 0}$ and $X \longrightarrow I \longrightarrow \tau^{\leq 0}(Z[-1]) \xrightarrow[+1]{}$ is a distinguished triangle. So I belongs to $\mathcal{C}$.

Since $\tau^{\leq 0}(Z[-1]) \cong \ker f$, we have the triangle $\ker f \longrightarrow X \longrightarrow I \xrightarrow[+1]{}$ and by $(\star)$, $\tau^{\geq 0}I \cong \operatorname{coker}(\ker f \to X) = \operatorname{coim} f$, but $\tau^{\geq 0}I \cong I$, so $I \cong \operatorname{coim} f$. Similarly, because $\tau^{\geq 0}Z \cong \operatorname{coker} f$, we get $I \longrightarrow Y \longrightarrow \operatorname{coker} f \xrightarrow[+1]{}$ and so $Y \longrightarrow \operatorname{coker} f \longrightarrow I[1] \xrightarrow[+1]{}$ is distinguished, and $\tau^{\leq 0}(I[1][-1]) \cong \tau^{\leq 0}I \cong \ker(Y \to \operatorname{coker} f) = \operatorname{im} f$, but $\tau^{\leq 0}I \cong I$ then $I \cong \operatorname{im} f$.

Then, $\operatorname{coim} f \cong \operatorname{im} f$ and the theorem is prove. $\square$

Proof of proposition 1.1.12. It is enough to prove the results for left $t-$exact functors.

i). Let $X \in \mathbf{D}^{\geq 0}$ and consider the distinguished triangle

$$\tau^{\leq 0}X \longrightarrow X \longrightarrow \tau^{>0}X \xrightarrow[+1]{}$$

as $\tau^{\leq 0}X \cong \tau^{\leq 0}\tau^{\geq 0}X \cong H^0(X)$ we have

$$H^0(X) \longrightarrow X \longrightarrow \tau^{>0}X \xrightarrow[+1]{}$$

applying F, we obtain the distinguished triangle

$$F(H^0(X)) \longrightarrow F(X) \longrightarrow F(\tau^{>0}X) \xrightarrow[+1]{}$$

Then, apply the cohomological functor H^0 (roughly speaking is $H^0_2 : \mathbf{D}_2 \to \mathcal{C}_2$) we obtain

$$H^0(F(H^0(X))) \longrightarrow H^0(F(X)) \longrightarrow H^0(F(\tau^{>0}X)) \xrightarrow[+1]{}$$

but $H^0(F(\tau^{>0}X)) = 0$ because $F(\tau^{>0}X)$ belongs to $\mathbf{D}_2^{>0}$, (indeed, $F(\mathbf{D}_1^{>0}) = F(\mathbf{D}_1^{\geq 1}) = F(\mathbf{D}_1^{\geq 0}[-1]_{\mathbf{D}_1}) = [-1]_{\mathbf{D}_2}F(\mathbf{D}_1^{\geq 0}) \subseteq \mathbf{D}_2^{\geq 0}[-1]_{\mathbf{D}_2} = \mathbf{D}_2^{\geq 1} = \mathbf{D}_2^{>0}$) and furtheremore $H^0(F(H^0(X))) = {}^pF(H^0(X))$ ($H^0(X) \in \mathcal{C}_1$ then $\varepsilon_1(H^0(X)) = H^0(X)$). So we have that, $H^0(F(X)) \cong {}^pF(H^0(X))$ (roughly speaking $H^0_2(F(X)) \cong {}^pF(H^0_1(X))$).

ii). Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a exact sequence in $\mathcal{C}_1$. It gives rise to a distinguished triangle $X \longrightarrow Y \longrightarrow Z \xrightarrow[+1]{}$ in $\mathbf{D}_1$. Applying F to this triangle to obtain

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \xrightarrow[+1]{}$$

and then applying the cohomological functor H^0 and the result in *i*), we get the exact sequence

$$0 \longrightarrow {}^pF(H^0(X)) \longrightarrow {}^pF(H^0(Y)) \longrightarrow {}^pF(H^0(Z))$$

but, as $X, Y, Z \in \mathcal{C}_1$, we get $H^0(X) \cong X$, $H^0(Y) \cong Y$ and $H^0(Z) \cong Z$, so

$$0 \longrightarrow {}^pF(H^0(X)) \longrightarrow {}^pF(H^0(Y)) \longrightarrow {}^pF(H^0(Z))$$

is an exact sequence. $\square$

Proof of proposition 1.2.13. The first item follows because if $\{X_\alpha\}_\alpha$ is a family of objects in $\mathcal{C}$, we have for $Y \in \mathcal{C}$ that $\mathrm{Ext}^i_\Lambda(X_\alpha, Y) = \mathrm{Ext}^i_\Lambda(Y, X_\alpha) = 0$ for all $0 < i \leq n$, then

$$\mathrm{Ext}^i_\Lambda(\bigoplus_\alpha X_\alpha, Y) \cong \prod_\alpha \mathrm{Ext}^i_\Lambda(X_\alpha, Y) = 0$$

that means $\bigoplus_\alpha X_\alpha \in \mathcal{C}$. The second follows because Λ as $\Lambda-$module is projective, and $()^*$ is a duality, so by 1. the sum $\Lambda \oplus (\Lambda)^*$ belongs to $\mathcal{C}$. The third follows by the second item. Now if we consider X in $\mathcal{C}$, then X is in $\mathcal{D}$, then for all Y in $\mathcal{D}$ we have $X \perp_n Y$ and $Y \perp_n X$, i.e., all objects in $\mathcal{D}$ are orthogonal to $\mathcal{C}$, then $\mathcal{D} = \mathcal{C}$. The last follows because $()^*$ is a duality. $\square$

Proof of lemma 1.2.22. (1) and (2). Since $Y := \tau_n^- X$ is indecomposable non-projective, $\underline{\mathrm{Hom}}_\Lambda(Y, \square)$ has a simple top $F := \mathrm{Hom}_{\mathcal{C}}(Y, \square)/\operatorname{rad}_{\mathcal{C}}(Y, \square)$. Thus $D\underline{\mathrm{Hom}}_\Lambda(Y, \square)$ $= \mathrm{Ext}_\Lambda(\square, X)$ has a simple socle DF with $DF(Y) \neq 0$. Let see (3). For any $X \in \mathcal{C}$, we have an exact functor $\mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathrm{End}_\Lambda(X)$ given by $F \mapsto F(X)$. For any sub-$\mathrm{End}_\Lambda(X)-$module M of F(X), there exists a sub-$\mathcal{C}-$module F' of F such that $M = F'(X)$. Then $(\operatorname{soc}_{\mathcal{C}} \mathrm{Ext}^n_\Lambda(\square, \tau_n X))(X)$ is a simple socle of the $\mathrm{End}_\Lambda(X)-$module $\mathrm{Ext}^n_\Lambda(X, \tau_n X)$. $\square$

Proof of lemma 1.2.24. Let see (1). If $n = 1$, $\mathcal{C} = \mathrm{mod}\,\Lambda$; then by lemma 1.2.22 we can take a non-split short exact sequence $\mathbf{A}$ which is in $(\operatorname{soc}_{\mathrm{mod}\,\Lambda} \mathrm{Ext}^1_\Lambda(X, \square))(\tau_n X)$. Then $\mathbf{A}$ is an 1-almost split sequence by proposition 1.2.23. Now assume $n > 1$. We can take a sink map $a : Y \to X$ in $\mathrm{mod}\,\Lambda$ by the case $n = 1$, and a right $\mathcal{C}-$approximation $b : Z \to Y$ of Y. Let $f_0 : C_0 \to X$ be a right minimal direct summand of ab as a complex. Then f is a sink map in $\mathcal{C}$. Since X is non-projective, f is surjective. Thus $X_1 := \ker f$ is in $\mathrm{mod}\,\Lambda$, and so there exists a minimal right $\mathcal{C}-$resolution $0 \longrightarrow C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_1 \longrightarrow X_1 \longrightarrow 0$ of X_1. By proposition 1.2.23, the sequence $0 \longrightarrow C_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} C_0 \longrightarrow X \longrightarrow 0$ is an $n-$almost split sequence.

(2) is similar. For (3), we have $(\operatorname{soc}_{\mathcal{C}^{op}} \mathrm{Ext}^n_\Lambda(X, \square))(Y) \neq 0$ by proposition 1.2.23 and by lemma 1.2.22, $Y \cong \tau_n X$ and $X \cong \tau_n^- Y$ hold. $\square$

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