On the regularity of the Navier-Stokes equations

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Chapter 1

Introduction

The Navier-Stokes equations describe the motion of an incompressible fluid under the action of an external force f. Derived from basic principles of Newtonian mechanics, they take the following form

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \triangle u + f, \quad \nabla \cdot u = 0,$$

where u is a velocity field in $\mathbb{R}^n$, p is the fluid's pressure and ν the kinematic viscosity.

These equations model the evolution in time of a fluid's velocity, therefore one can ask whether any physically reasonable initial state always leads, under this system, to a physically reasonable solution (state). For the two dimensional case $\mathbb{R}^2$, an affirmative answer to this question was given by O. Ladyzhenskaya in bounded domains [8]. However, the case of $\mathbb{R}^3$ remains open, and it is one of the *Millenium prize problems.* In a concrete way, for the three dimensional problem, one would like to prove that for any smooth initial velocity field in $L^2(\mathbb{R}^3)$ there is always a unique solution, or one would like to find a counterexample: a finite energy smooth initial data which becomes singular (non-unique) at a finite time T. This problem can also be formulated for periodic functions in which the domain of the velocity field u can be taken as $\mathbb{T}^3 = [0,1]^3$. We must point out that uniqueness of solutions is closely related to their regularity, if one has a solution with a high degree of differentiability it must be unique.

For the three dimensional case some partial results have been obtained. Leray, in his famous paper *'Sur le mouvement d'un liquide visqueux emplissant l'espace'* [9], proved the existence of weak solutions, which are regular solutions inside a possibly small interval $[0,\eta)\,, \eta > 0$ and might or might not become singular at some time outside this interval. By singular we mean that the solution loses its regularity, this phenomenon is called blow-up. Leray even showed that the set of blow-up times must be very small in some sense, and he conjectured the existence of bounds on the possible time of blow-up, depending on norms of the data, this means that if a solution goes beyond this time without exhibiting blow-up, then it will never become singular at any future time.

Let us call T, the *blow-up time*, the supremum of $\delta > 0$ such that u a solution to the Navier-Stokes equations is regular in the time interval $[0, \delta)$. Leray stated without a proof that, for $p > 3$, there exist a constant c_p such that

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} > \frac{c_p}{(T-t)^{\frac{p-3}{2p}}}. \tag{1.1}$$

In 1984, Kato extended Leray's estimates for the borderline case $p = 3$ in [7], showing that there exists some small constant $\epsilon_3 > 0$, such that if

$$\|u(\cdot,t)\|_{L^3(\mathbb{R})} < \epsilon_3, \quad \forall t > 0, \tag{1.2}$$

then $u(x,t)$ is smooth at all times. In 2003 Escauriaza, Seregin and Šverák, in [3], improved this result by showing that this holds for any constant: if the L^3 norm of a solution remains bounded at all times the solution is smooth. Also it is known that the energy of the velocity field (its L^2 norm) is a decreasing function of time

$$\int_\Omega |u(x,t)|^2 dx \leq \int_\Omega |u(x,0)|^2 dx, \quad \forall t > 0.$$

However this inequality has not given results towards regularity. The continuous embeddings in the homogeneous Sobolev spaces $\dot{H}^s \hookrightarrow L^{\frac{3}{6-2s}}$ imply by (1.1) the following blow-up rates

$$\|u(\cdot,t)\|_{\dot{H}^s(\Omega)} \geq \frac{c_s}{(T-t)^{\frac{2s-1}{4}}}, \tag{1.3}$$

for $\frac{1}{2} < s < \frac{3}{2}$, and the result in [3] implies $s = \frac{1}{2}$. Robinson, Sadowski and Silva in [10] extended these estimates to the Sobolev spaces $\dot{H}^s(\Omega)$, for $\frac{3}{2} < s < \frac{5}{2}$.

In [2], the following estimate is obtained for the missing exponents $s = \frac{3}{2}, \frac{5}{2}$ with a logarithmic correction:

$$\|u(\cdot,t)\|_{H^s(\Omega)} \geq \frac{c_s}{|\log(T-t)|^{\frac{2s-1}{4}}(T-t)^{\frac{2s-1}{4}}}. \tag{1.4}$$

In this work we will present a weaker formulation for $s = \frac{3}{2}$, stating that if u becomes singular at time T, then there exists a sequence of times $t_j \to T$ such that

$$\|u(\cdot,t_j)\|_{H^s(\Omega)} \geq \frac{c_s}{|\log(T-t_j)|^{\frac{2s-1}{4}}(T-t_j)^{\frac{2s-1}{4}}}, \tag{1.5}$$

and present a proof of the corresponding statement for the exponent $s = \frac{1}{2}$. However the proof of (1.5) gave the right exponent to the stronger result (1.4).

These characterizations of singular solutions and the energy inequality seem to narrow the possible set of singularities. The hope is to be able to prove that one of the conditions mentioned above fails to be true for a weak solution, thus proving regularity, however this does not seem to be an easy task. Since this problem has remained open for a long time it is very likely that a successful proof comes with very different techniques, for instance dealing with the non-linearity in a finer way which explodes all its structure.

Also an original proof of the classical result of O. Ladyzhenskaya is presented in the document: smooth initial data in $\mathbb{T}^2$ presents no blow-up.

Chapter 2

Preliminaries

In the following section we introduce some notation and prove some facts which will be used throughout this document.

- Ω denotes a domain in $\mathbb{R}^n$, $\mathbb{T}^n = [0,1]^n$ or $\mathbb{Z}^n$.
- Summation over latin indices j, l, m runs over space coordinates indices $1, 2, ..., n$.
- Summation over greek letters α, ξ runs over frequency numbers on $\mathbb{R}^n$ or $\mathbb{Z}^n$.
- $\langle x, y\rangle = x_1y_1 + x_2y_2 + ... + x_ny_n$ denotes the usual dot product in $\mathbb{R}^n$.
- D^α, $\alpha \in \mathbb{N}^n$ denotes the partial derivative $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$; with $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.
- $C^\infty(\Omega)$ is the vector space of functions with continuous derivatives of any order.
- $C_0^\infty(\Omega)$ is the vector sub-space of functions in $C^\infty(\Omega)$ with compact support.

2.1 Sobolev Spaces

Consider the following differential equation in some domain $\Omega \in \mathbb{R}^n$, with f a given function over Ω

$$\triangle u = f, \tag{2.1}$$

one says that u is a solution if it satisfies the above equality. In order to be able to solve it, some boundary conditions should also be given. One would like to know about regularity of solutions u and uniqueness. But a prior question is existence. Are there always solutions to (2.1) given f in some function space? Notice that for (2.1) to make sense u must be twice differentiable over Ω. However we could try a different definition for a solution to (2.1) which does not require u to be twice differentiable. If we multiply (2.1) by $\phi \in C_0^\infty(\Omega)$ and integrate by parts we get

$$\int_\Omega \phi \triangle u dx = -\int_\Omega (\nabla\phi \cdot \nabla u) dx = \int_\Omega \phi f dx,$$

Alternatively we could say that u is a solution of (2.1) if it satisfies the last equality for any such $\phi \in C_0^\infty(\Omega)$. Now we may have more solutions than in the initial statement, as we only require u to be once differentiable. And we could take a deeper stage of generality; if we integrate by parts once more, then

$$\int_\Omega u \triangle \phi dx = \int_\Omega \phi f dx.$$

In this case u is called a *weak solution* of (2.1).

The reason to enlarge the set of solutions, is that in these bigger spaces there are more available tools to prove existence, so it becomes easier. Once that one has existence, the hope is to be able to show that this weak solutions are in fact the classical ones: the solutions one was looking for in the first formulation of the problem. This last part can be achieved by properties that weak solutions to the particular equation satisfy, and embedding theorems over the spaces one is considering.

We say that g is a weak derivative with respect to x_j of f if it satisfies

$$\int_\Omega g\phi dx = -\int_\Omega f\frac{\partial \phi}{\partial x_j}dx, \quad \forall \phi \in C_0^\infty(\Omega).$$

The ϕ's are called test functions. Here, an interesting point of view borrowed from physics comes handy: in calculus, a (real) function is interpreted as an object which assigns a (real) value to each point on its domain. Now one can see functions as objects which assign values to each test function, i.e, test functions are used to measure them. In a physical situation f could be a physical quantity hence $\int f\phi$ would be a measurement on a particular experiment, an expected value over the physical state ϕ. This point of view is presented in Rudin's *Functional Analysis* [11].

Definition 2.1.1. *The Sobolev space $W^{k,p}(\Omega)$ is the vector space of functions u whose partial derivatives $D^\alpha u$ exist in the weak sense and belong to $L^p(\Omega)$, for all multi-inidices $|\alpha| \leq k$.*

Definition 2.1.2. *For $u \in W^{k,p}(\Omega)$, $1 \leq p < \infty$ define its norm by*

$$\|u\|_{W^{k,p}} := \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

In the case $p = \infty$

$$\|u\|_{W^{k,\infty}} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

In the case $p = 2$, the Sobolev space is denoted $H^s(\Omega) = W^{s,2}(\Omega)$. With the inner product defined by

$$\langle f, g\rangle := \sum_{|\alpha|\leq s} \int_\Omega D^\alpha f \overline{D^\alpha g} dx, \quad f, g \in H^s(\Omega),$$

$H^s(\Omega)$ is a Hilbert space [4].

2.2 Fourier series and the Fourier transform

We will consider the Navier-Stokes equations in the periodic case $\Omega = \mathbb{T}^n$, and in the whole space $\Omega = \mathbb{R}^n$; for $n = 2, 3$. In the two cases we will work with the decomposition in frequency modes. For the former we have the Fourier series and for the latter the Fourier transform. We will introduce these tools and present some basic properties which these decompositions satisfy.

2.2.1 Fourier series

A 2π-periodic function f over $\mathbb{R}$ could be consider as a function defined over the interval $I = [-\pi, \pi]$, such that $f(-\pi) = f(\pi)$. Let us consider the Hilbert space $L^2(I)$, endowed with the following inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)dx.$$

The family of functions

$$\{e^{-ixn}\}_{n\in\mathbb{Z}},$$

is an orthonormal dense set in $L^2(I)$. The Fourier series is the decomposition of a function in terms of this family.

Definition 2.2.1. *The Fourier transform $\mathcal{F} : L^2([-\pi;\pi]) \to L^2(\mathbb{Z})$ is the following linear map*

$$\mathcal{F}(f)_n = \hat{f}_n := \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad \forall n \in \mathbb{Z}.$$

Proposition 2.2.1. *The Fourier transform $\mathcal{F} : L^2([-\pi;\pi]) \to L^2(\mathbb{Z})$, is a linear isometric isomorphism.*

A proof of the preceding proposition can be found in [5], page 248.

Now let us recall some properties of the Fourier transform:
Let $f \in C^1([-\pi;\pi])$,

$$\frac{d}{dx}f = \frac{d}{dx}\mathcal{F}^{-1}(\mathcal{F}f) = \frac{d}{dx}\sum_n \hat{f}_n e^{inx} = \sum_n \frac{d}{dx}\hat{f}_n e^{inx} = \sum_n in\hat{f}_n e^{inx},$$

$$(\mathcal{F}\frac{df}{dx})_n = in\hat{f}_n = in(\mathcal{F}f)_n.$$

Differentiation translates into multiplication by the frequency number over the Fourier transform. This property will play a key role in the treatment of differential equations. Some equations become easier to handle in Fourier space.

In a more general setting we could consider f, a periodic function defined over $\mathbb{R}^n, n \geq 1$. Under a proper scaling of the period, f can be treated as a function defined over $\mathbb{T}^n = [0,1]^n$, such that f agrees at opposite ends of the boundary $\partial\mathbb{T}^n$. Here we have the Fourier transform

$$\mathcal{F} : L^2(\mathbb{T}^n) \to L^2(\mathbb{Z}^n),$$

$$(\mathcal{F}f)_\xi := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{T}^n} f(x)e^{-i\langle x,\xi\rangle}dx, \quad \xi \in \mathbb{Z}^n.$$

As in the 1-dimensional case, $\mathcal{F}$ is a linear isometric isomorphism, with inverse $\mathcal{F}^{-1}$ given by

$$\mathcal{F}^{-1} : L^2(\mathbb{Z}^n) \to L^2(\mathbb{T}^n),$$

$$(\mathcal{F}^{-1}\hat{f}_\xi)(x) = \frac{1}{(2\pi)^{n/2}}\sum_{\xi\in\mathbb{Z}^n}\hat{f}_\xi e^{i\langle x,\xi\rangle}, \quad x \in \mathbb{T}^n.$$

Similarly for differentiation we have

$$(\mathcal{F}\frac{\partial f}{\partial x_j})_\xi = i\xi_j\hat{f}_\xi = i\xi_j(\mathcal{F}f)_\xi.$$

Remark 1. *The Sobolev spaces $H^s(\mathbb{T}^n)$ can also be defined via the Fourier series [4]*

$$H^s(\mathbb{T}^n) := \{f \in L^2(\mathbb{T}^n) | \; |\xi|^s \hat{f}_\xi \in L^2(\mathbb{Z}^n)\},$$

and the homogeneous Sobolev space $\dot{H}^s(\mathbb{T}^n)$, is the vector space of zero average periodic functions, defined as

$$\dot{H}^s(\mathbb{T}^n) := \{f \in H^s(\mathbb{T}^n) | \; \hat{f}_0 = 0\}.$$

The Fourier series also encodes some properties about continuity of a function. If we were to estimate the difference $f(x) - f(y)$ for $x, y \in \mathbb{T}^n$

$$|f(x) - f(y)| = \left| \frac{1}{(2\pi)^{n/2}} \sum_\xi \hat{f}_\xi e^{i\langle x,\xi\rangle} - \frac{1}{(2\pi)^{n/2}} \sum_\xi \hat{f}_\xi e^{i\langle y,\xi\rangle} \right| = \frac{1}{(2\pi)^{n/2}} \left| \sum_\xi \hat{f}_\xi (e^{i\langle x,\xi\rangle} - e^{i\langle y,\xi\rangle}) \right|,$$

suppose that $(\hat{f}_\xi)_{\xi\in\mathbb{Z}^n} \in L^1(\mathbb{Z}^n)$, then for any given $\epsilon > 0$, there exists a $K > 0$ such that:

$$\sum_{|\xi|>K} |\hat{f}_\xi| < \frac{\epsilon}{4},$$

and we have

$$\begin{aligned}
(2\pi)^{n/2} |f(x) - f(y)| &\leq \sum_{|\xi|\leq K} \left| \hat{f}_\xi (e^{i\langle x,\xi\rangle} - e^{i\langle y,\xi\rangle}) \right| + \sum_{|\xi|>K} \left| \hat{f}_\xi e^{i\langle x,\xi\rangle} \right| + \sum_{|\xi|>K} \left| \hat{f}_\xi e^{i\langle y,\xi\rangle} \right| \\
&\leq \sup_{|\zeta|<K} |e^{i\langle x,\zeta\rangle} - e^{i\langle y,\zeta\rangle}| \sum_{|\xi|\leq K} |\hat{f}_\xi| + 2 \sum_{|\xi|>K} |\hat{f}_\xi| \\
&\leq \sup_{|\zeta|<K} |e^{i\langle x,\zeta\rangle} - e^{i\langle y,\zeta\rangle}| \|\hat{f}_\xi\|_{L^1(\mathbb{Z}^n)} + \frac{\epsilon}{2},
\end{aligned}$$

which can be made smaller than ϵ by the continuity of the exponential and the compactness of the subset $\{\xi \in \mathbb{Z}^n, |\xi| < K\}$.

Alternatively suppose that for some power $s \geq 1$ $|\xi|^s \hat{f}_\xi \in L^2(\mathbb{Z}^n)$, then by means of the Cauchy inequality we can bound the difference (notice that the term $\xi = 0$ is not considered in the sum)

$$\begin{aligned}
(2\pi)^{n/2} |f(x) - f(y)| &= \left| \sum_\xi \frac{|\xi|^s}{|\xi|^s} \hat{f}_\xi (e^{i\langle x,\xi\rangle} - e^{i\langle y,\xi\rangle}) \right| \\
&\leq \left(\sum_\xi \frac{|e^{i\langle x,\xi\rangle} - e^{i\langle y,\xi\rangle}|^2}{|\xi|^{2s}} \right)^{\frac{1}{2}} \left(\sum_\xi |\xi|^{2s} |\hat{f}_\xi|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
(2\pi)^n |f(x) - f(y)|^2 &\leq \left(\sum_{|\xi|\leq K} \frac{|e^{i\langle x,\xi\rangle} - e^{i\langle y,\xi\rangle}|^2}{|\xi|^{2s}} \right) \left(\sum_{|\xi|\leq K} |\xi|^{2s} |\hat{f}_\xi|^2 \right) \\
&+ \left(\sum_{|\xi|>K} \frac{|e^{i\langle x,\xi\rangle} - e^{i\langle y,\xi\rangle}|^2}{|\xi|^{2s}} \right) \left(\sum_{|\xi|>K} |\xi|^{2s} |\hat{f}_\xi|^2 \right) \\
&\leq \sup_{|\zeta|<K} |e^{i\langle x,\zeta\rangle} - e^{i\langle y,\zeta\rangle}|^2 \left(\sum_\xi \frac{1}{|\xi|^{2s}} \right) \left(\sum_\xi |\xi|^{2s} |\hat{f}_\xi|^2 \right) \\
&+ \left(\sum_\xi \frac{1}{|\xi|^{2s}} \right) \left(\sum_{|\xi|>K} |\xi|^{2s} |\hat{f}_\xi|^2 \right),
\end{aligned}$$

which again by a proper choice of K can be made as smaller as one desires for x, y close enough. It is important to notice that we also require $\sum_{|\xi| \in \mathbb{Z}^n} |\xi|^{-2s}$ to be finite, which is true if $s > n/2$ this can be seen by integrating in polar coordinates:

$$\sum_{\xi \neq 0} \frac{1}{|\xi|^{2s}} \leq C_n \int_1^\infty \frac{r^{n-1} dr}{r^{2s}} = \frac{C_n}{n-2s} r^{n-2s} \Big|_1^\infty,$$

by these considerations the next proposition follows.

Proposition 2.2.2. *Let $f \in L^2(\mathbb{T}^n)$ be a periodic function in $\mathbb{R}^n$, if $f \in H^s(\mathbb{T}^n)$ for $s > n/2$ or if $\mathcal{F}f \in L^1(\mathbb{Z}^n)$, then f is uniformly continuous.*

Proof. If $\mathcal{F}f \in L^1(\mathbb{Z}^n)$ or $|\xi|^s(\mathcal{F}f)_\xi \in L^2(\mathbb{Z}^n)$, then f is uniformly continuous by the estimates in the preceding lines. And if f belongs to $H^s(\mathbb{T}^n) \Rightarrow |\xi|^s(\mathcal{F}f)_\xi \in L^2(\mathbb{Z}^n)$ since $\mathcal{F}$ is an isometry. □

With these examples it is straightforward to make conclusions about f derivatives and their continuity: take $h \in \mathbb{R}, h \neq 0, x \in \Omega$, and $x_j \in \mathbb{R}^n, |x_j| = 1$ a unitary space direction,

$$\frac{|f(x + hx_j) - f(x)|}{h} \leq \frac{1}{h} \left| \sum_\xi \hat{f}_\xi (e^{i\langle x + hx_j, \xi \rangle} - e^{i\langle x, \xi \rangle}) \right| = \frac{1}{h} \left| \sum_\xi \hat{f}_\xi e^{i\langle x, \xi \rangle} (1 - e^{ih\xi_j}) \right|$$

with the estimate $|1 - \exp(ih\xi_j)| < h|\xi_j|$, valid for small h

$$\frac{|f(x + hx_j) - f(x)|}{h} \leq \frac{1}{h} \sum_\xi \left| \hat{f}_\xi e^{i\langle x, \xi \rangle} \right| |h\xi_j|,$$

by dominated convergence we could take the limit $h \to 0$.

Proposition 2.2.3. *Let $f \in L^2(\mathbb{T}^n)$ be a periodic function in $\mathbb{R}^n$, if $f \in H^{s+k}(\mathbb{T}^n)$ for $s > n/2$ or if $|\xi|^k \mathcal{F}f \in L^1(\mathbb{Z}^n)$ then the k-th derivative of f exists and is continuous.*

2.2.2 The Fourier transform

Recall the 1-dimensional case for the Fourier series: we considered functions of period $2\pi = L$, and we did a decomposition in terms of a family of functions with the same period $\{\exp(i2\pi xn/L)\}_{n \in \mathbb{Z}}$,

$$\hat{f}_n = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} f(x) \exp(i2\pi xn/L) dx,$$

if we have a function defined over $\mathbb{R}$ (not necessarily periodic) we can see it as the limit of functions of period L as $L \to \infty$. By this process one gets the Fourier transform in $L^2(\mathbb{R})$, and in a more general case in $\mathbb{R}^n$,

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

$$\mathcal{F} : f(x) \mapsto \hat{f}_\xi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x, \xi \rangle} dx, \quad \forall \xi \in \mathbb{R}^n.$$

$\mathcal{F}$ is a linear isometry with inverse

$$\mathcal{F}^{-1} \hat{f}_\xi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}_\xi e^{i\langle x, \xi \rangle} d\xi, \quad \forall x \in \mathbb{R}^n.$$

The Fourier transform satisfies the same properties which were proved in the last section for the Fourier series. This properties will be used in the chapter to follow.

Remark 2. *We will also consider Besov spaces $\Phi(\alpha)$, defined through the frequency decomposition by the pseudo-norm,*

$$\sup_{\xi\in\mathbb{R}^n} |\xi|^\alpha |\hat{f}_\xi|,$$

and for f periodic

$$\sup_{\xi\in\mathbb{Z}^n} |\xi|^\alpha |\hat{f}_\xi|,$$

in this case we require f to have zero average ($\hat{f}_0 = 0$). These spaces turn out to be Banach spaces, and by the embeddings $\Phi(\alpha)(\Omega) \hookrightarrow H^s(\Omega)$ for $\alpha > s + n/2$, $\Omega = \mathbb{R}^n,\ \mathbb{T}^n$, will be useful in proving regularity.

Chapter 3

The Equations General setting

In this chapter we introduce the Navier-Stokes equations in $\mathbb{T}^n$ and $\mathbb{R}^n$, translate them to a system of equations for the Fourier modes and present their integral form. Then we will explain the method we use to prove our regularity results.

Recall that the Navier-Stokes equations with no force, $f = 0$, are the following system of equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \triangle u, \quad \nabla \cdot u = 0, \tag{3.1}$$

where $u(x, t)$ gives the velocity at the point $x \in \Omega$ at the time t; p is a scalar field which represents the pressure exert by the fluid and ν is the fluid's kinetic viscosity, which we will take as $\nu = 1$ for practical purposes.

3.1 Navier-Stokes in Fourier modes

Some manipulations will be performed on (3.1) such as to multiply by test functions and integrate, this process will lead us to a different system of equations which we will ask our solutions to satisfy. Then we will end up with *weak solutions*, as it was explained in the previous chapter. However if a *weak solution* is smooth it is actually a classical one, meaning that it satisfies the original equation. This will be our case, by the results in [9] there is always an interval of regularity near zero, we will be considering solutions inside this interval so that the manipulations that we perform encode the same information that the original equation.

We will define a *Leray-Hopf solution* as in [1], which will be the weak solutions we will work with for the particular case of the Navier-Stokes equations.

Definition 3.1.1. *A Leray-Hopf solution of (3.1) with initial data $\psi \in L^2(\mathbb{T}^3)$ is a function $u : [0, T) \longrightarrow L^2(\mathbb{T}^3)$ such that*

(a) u is weakly continuous; that is:

$$F_\varphi : t \to \langle \varphi, u(t) \rangle_{L^2(\mathbb{T}^3)},$$

defines map continuous in t for all $\varphi \in L^2(\mathbb{T}^3)$;

(b)

$$u \in L^\infty\left(0, T; L^2\left(\mathbb{T}^3\right)\right) \cap L^2\left(0, T; H^1\left(\mathbb{T}^3\right)\right);$$

(c) u satisfies the weak formulation

$$\langle u(t), \varphi(t)\rangle_{L^2(\mathbb{T}^3)} + \int_0^t -\left\langle u(\tau), \frac{\partial}{\partial\tau}\varphi(\tau)\right\rangle_{L^2(\mathbb{T}^3)} - \langle \nabla u(\tau), \nabla\varphi(\tau)\rangle_{L^2(\mathbb{T}^3)} + \langle u\cdot\nabla u, \varphi\rangle_{L^2(\mathbb{T}^3)}\, d\tau = \langle \psi, \varphi(0)\rangle_{L^2(\mathbb{T}^3)}$$

for all $\varphi \in C^\infty(\mathbb{T}^3 \times [0,T))$ *and periodic with* $\operatorname{div}\varphi = 0$*; and*

(d) the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{T}^3)} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{T}^3)}\, d\tau \le \|\psi\|^2_{L^2(\mathbb{T}^3)}$$

holds.

With this in mind consider the Navier-Stokes equations in $\Omega = \mathbb{T}^n$, by applying the Fourier transform to (3.1), we have

$$\sum_\xi \partial_t \hat{u}^j_\xi e^{i\langle x,\xi\rangle} - \sum_{l=1}^n \partial^2_{l,l} \sum_\xi \hat{u}^j_\xi e^{i\langle x,\xi\rangle} + \sum_{l=1}^n \sum_\xi \hat{u}^l_\xi e^{i\langle x,\xi\rangle} \partial_l \sum_\alpha \hat{u}^j_\alpha e^{i\langle x,\alpha\rangle} + \sum_\xi \partial_j \hat{p}_\xi e^{i\langle x,\xi\rangle} = 0,$$

where ξ, α are to be summed over all $2\pi\mathbb{Z}^n$, and the indices l, j correspond to space coordinates; ∂_l must be understood as derivation with respect to the space coordinate x_l. From the previous expression we obtain

$$\sum_\xi \partial_t \hat{u}^j_\xi e^{i\langle x,\xi\rangle} + \sum_\xi |\xi|^2 \hat{u}^j_\xi e^{i\langle x,\xi\rangle} + i\sum_{l=1}^n \sum_\xi \hat{u}^l_\xi e^{i\langle x,\xi\rangle} \sum_\alpha \alpha_l \hat{u}^j_\alpha e^{i\langle x,\alpha\rangle} + i\sum_\xi \xi_j \hat{p}_\xi e^{i\langle x,\xi\rangle} = 0,$$

and multiplying by $e^{-i\langle\cdot,\xi\rangle}$ and integrating over $\mathbb{T}^n$ we get

$$\partial_t \hat{u}^j_\xi + |\xi|^2 \hat{u}^j_\xi + i\sum_\alpha \sum_{l=1}^n \hat{u}^l_{\xi-\alpha} \hat{u}^j_\alpha \alpha_l + i\xi_j \hat{p}_\xi = 0. \tag{3.2}$$

The divergence free condition in frequency space says that $u_\xi \perp \xi$ at all times. Then we have

$$\sum_j u^j_\xi \xi_j = 0.$$

Therefore if we multiply the last equation by $i\xi_j$ and sum over j, we get

$$i\sum_j \partial_t \hat{u}^j_\xi \xi_j + i\sum_j |\xi|^2 \hat{u}^j_\xi \xi_j - \sum_\alpha \sum_{l,j} \hat{u}^l_{\xi-\alpha} \hat{u}^j_\alpha \alpha_l \xi_j - |\xi|^2 \hat{p}_\xi = 0$$

$$\Rightarrow \hat{p}_\xi = -\frac{1}{|\xi|^2} \sum_\alpha \sum_{l,j} \hat{u}^l_{\xi-\alpha} \hat{u}^j_\alpha \alpha_l \xi_j = -\frac{1}{|\xi|^2} \sum_\alpha \sum_{l,j} \hat{u}^l_{\xi-\alpha} \hat{u}^j_\alpha (\alpha_l - \xi_l + \xi_l)\xi_j$$

$$= -\frac{1}{|\xi|^2} \sum_\alpha \sum_{l,j} \hat{u}^l_{\xi-\alpha} \hat{u}^j_\alpha \xi_l \xi_j + \frac{1}{|\xi|^2} \sum_\alpha \sum_j \hat{u}^j_\alpha \xi_j \sum_l \hat{u}^l_{\xi-\alpha} (\xi-\alpha)_l,$$

and this last sum is zero by the divergence free condition.

Therefore we obtain the following expression for $\hat{p}_\xi$:

$$\hat{p}_\xi = -\frac{1}{|\xi|^2} \sum_\alpha \sum_{l,j} \hat{u}^l_{\xi-\alpha} \hat{u}^j_\alpha \xi_l \xi_j,$$

which we can replace back into (3.2), to obtain

$$\frac{\partial \hat{u}^j_\xi}{\partial t} + |\xi|^2 \hat{u}^j_\xi + i \sum_{\alpha\in\mathbb{Z}^n} \sum_{k=1}^{n} \hat{u}^k_{\xi-\alpha} \hat{u}^j_\alpha \xi_k - \frac{i\xi_j}{|\xi|^2} \sum_{\alpha\in\mathbb{Z}^n} \sum_{k,l=1}^{n} \hat{u}^k_{\xi-\alpha} \hat{u}^l_\alpha \xi_k \xi_l = 0,$$

this equation can be written as

$$\frac{\partial \hat{u}^j_\xi}{\partial t} = -|\xi|^2 \hat{u}^j_\xi + i \sum_{\alpha} \sum_{l,k} \hat{u}^k_{\xi-\alpha} \hat{u}^l_\alpha \xi_k \left(\frac{\xi_l \xi_j}{|\xi|^2} - \delta_{l,j} \right).$$

If we solve the equation above for some given an initial velocity field $u(x, t = 0) = \phi(x)$, we get the integral equation

$$\hat{u}^j_\xi(t) = e^{-|\xi|^2 t} \hat{\phi}^j_\xi + e^{-|\xi|^2 t} \int_0^t i \sum_{\alpha\in\mathbb{Z}^n} \sum_{k,l=1}^{n} \hat{u}^k_\alpha(s) \hat{u}^l_{\xi-\alpha}(s) \xi_k \left(\frac{\xi_l \xi_j}{|\xi|^2} - \delta_{j,l} \right) e^{|\xi|^2 s} ds, \tag{I-NS}$$

where $\hat{\phi}^j_q$, are the Fourier modes of the initial data $\phi(x)$. This equation will be our almost starting point.

Remark 3. *For N-S in the whole space the equation is much the same, but the frequency modes run over real valued vectors ξ in $\mathbb{R}^n$. In this case the velocity function is decomposed as*

$$u(x, t) = \int_{\xi\in\mathbb{R}^n} \hat{u}_\xi e^{i\langle x,\xi\rangle} d\xi,$$

in which case we arrive to the 'similar' equation

$$\hat{u}^j_\xi(t) = e^{-|\xi|^2 t} \hat{\phi}^j_\xi + ie^{-|\xi|^2 t} \int_0^t \int_{\mathbb{R}^n} \sum_{k,l=1}^{n} \hat{u}^k_\alpha(s) \hat{u}^l_{\xi-\alpha}(s) \xi_k \left(\frac{\xi_l \xi_j}{|\xi|^2} - \delta_{j,l} \right) e^{|\xi|^2 s} d\alpha ds, \tag{3.3}$$

3.2 The general method

As it was shown in the previous chapter, the continuity of a function can be obtained from the of integrability of certain expressions involving its Fourier transform (**Proposition** 2.2.2 and 2.2.3). Some interpolation inequalities derived from the integral equation (I-NS) will be used in a systematic way in several proofs to come. These interpolations will show certain decay rates for the Fourier transform of a solution, thus solving the question of continuity and differentiability.

The method assumes that the frequency modes satisfy a given bound M in the time interval $[T - \rho, T]$. Then by an increasing sequence of times, we can improve the bound by successive use of the integral equation, considered over the intervals defined by this sequence of times. And by a limit process one obtains a limit bound which is reached at some time $t^* \in [T - \rho, T]$.

This is brief sketch of this process: Let $(t_n)_{n\in\mathbb{N}} \subset [T - \rho; T]$ be an increasing sequence of times, and assume that we have a weak solution u, whose Fourier modes $(\hat{u}_\xi)$ satisfy (I-NS) and the bound

$$|\hat{u}_\xi(t)| \le M(\xi), \quad \forall t \in [T - \rho; T],$$

then for $t \geq t_1$,

$$\left|\hat{u}_\xi^m(t)\right| \leq e^{-|\xi|^2(t-t_0)}\left|\hat{u}_\xi^m(t_0)\right| + \left|\int_{t_0}^{t} e^{-|\xi|^2(t-s)}\sum_{\alpha\ j,l}\hat{u}_\alpha^j(s)\hat{u}_{\xi-\alpha}^l(s)\xi_j(\frac{\xi_l\xi_m}{|\xi|^2}-\delta_{l,m})ds\right|,$$

$$\leq e^{-|\xi|^2(t_1-t_0)}\left|\hat{u}_\xi^m(t_0)\right| + \sup_{t_0<s<t}|\sum_{\alpha\ j,l}\hat{u}_\alpha^j\hat{u}_{\xi-\alpha}^l(s)\xi_j(\frac{\xi_l\xi_m}{|\xi|^2}-\delta_{l,m})|\int_{t_0}^{t} e^{-|\xi|^2(t-s)}ds,$$

$$\leq e^{-|\xi|^2(t_1-t_0)}\left|\hat{u}_\xi^m(t_0)\right| + \frac{1}{|\xi|^2}\sup_{t_0<s<t}|\sum_{\alpha\ j,l}\hat{u}_\alpha^j\hat{u}_{\xi-\alpha}^l(s)\xi_j(\frac{\xi_l\xi_m}{|\xi|^2}-\delta_{l,m})|.$$

We proceed to bound the non-linear term,

$$|\sum_{\alpha\ j,l}\hat{u}_\alpha^j\hat{u}_{\xi-\alpha}^l\xi_j(\frac{\xi_l\xi_m}{|\xi|^2}-\delta_{l,m})| \leq C\sum_{\alpha\ j,l}|\hat{u}_\alpha^j||\hat{u}_{\xi-\alpha}^l||\xi_j(\frac{\xi_l\xi_m}{|\xi|^2}-\delta_{l,m})| \leq C\sum_{\alpha}|\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\xi|.$$

Notice that we could only take care for absolute values, therefore $\sum_\alpha \hat{u}_{\xi-\alpha}\hat{u}_\alpha\xi$ will denote this sum; keeping in mind that by the divergence free condition it could be taken as $\sum_\alpha \hat{u}_{\xi-\alpha}\hat{u}_\alpha(\xi-\alpha)$, (c.f (I-NS)), sometimes it will be more convenient to work with this expression.
By applying the bound over the Fourier modes inside this sum we get a particular bound, something like

$$|\sum_{\alpha\in\mathbb{Z}^n}\hat{u}_{\xi-\alpha}\hat{u}_\alpha\xi| \leq K(\xi),$$

and overall

$$\left|\hat{u}_\xi^m(t)\right| \leq e^{-|\xi|^2(t-t_0)}M(\xi) + \frac{K(\xi)}{|\xi|^2} \leq D_1(\xi),\ \forall t \geq t_1,$$

we have the *base case* for our inductive process, which is obtained from the hypotheses assumed for the solution, then we assume some bound for t_n:

$$|\hat{u}_\xi(t)| \leq D_n(\xi), \quad \forall t \geq t_n,$$

and to do the inductive step, take $t \geq t_{n+1}$. It holds

$$\left|\hat{u}_\xi^m(t)\right| \leq e^{-|\xi|^2(t-t_n)}\left|\hat{u}_\xi^m(t_n)\right| + \left|\int_{t_n}^{t} e^{-|\xi|^2(t-s)}\sum_{\alpha\ j,l}\hat{u}_\alpha^j\hat{u}_{\xi-\alpha}^l\xi_j(\frac{\xi_l\xi_m}{|\xi|^2}-\delta_{l,m})ds\right|,$$

we use the D_n bound for the non-linear term and we must get, for times inside the interval $[t_n, t]$ a better bound

$$|\sum_{\alpha\in\mathbb{Z}^n}\hat{u}_{\xi-\alpha}\hat{u}_\alpha\xi| \leq K_n(\xi).$$

With these considerations

$$\left|\hat{u}_\xi^m(t)\right| \leq e^{-|\xi|^2(t_{n+1}-t_n)}D_n(\xi) + \frac{K_n(\xi)}{|\xi|^2} \leq D_{n+1}(\xi),\ \forall t \geq t_{n+1},$$

therefore if $t_n \to t^* \in [T-\rho, T]$, we conclude

$$|u_\xi(t)| \leq D(\xi) = \lim_{n\to\infty} D_n(\xi), \quad \forall t \in [t^*, T].$$

Given that the above limit exists and is finite.

Chapter 4

Navier Stokes in 2-D

In this section we prove regularity for solutions to 3.1 in the two dimensional torus $\Omega = \mathbb{T}^2$. The main result is that given an initial bounded vorticity (which is obtained from differentiation of the velocity field) singularities cannot occur.
The first proof for regularity in two dimensions was obtained by O.Ladyzhenskaya in [8]. Using the vorticity she showed that solutions of the Navier-Stokes equations in bounded domains of $\mathbb{R}^2$ are regular. For our proof we will use the same quantity (the vorticity), with the aid of our favourite tool, the Fourier transform.
Let us introduce the vorticity ω and show that it satisfies a similar equation to the original system. Then we shall show that this quantity is bounded, and derive some estimates for u a solution to the system (3.1) in terms of the norm of ω; this estimates together with the boundedness of ω will imply the regularity of u.
Consider (3.1) in two dimensions, explicitly $u = (u^1, u^2)$ satisfy

$$\frac{d}{dt}u^1 - \left(\partial_1^2 + \partial_2^2\right) u^1 + \left(u^1\partial_1 + u^2\partial_2\right) u^1 + \partial_1 p = 0, \tag{4.1}$$

$$\frac{d}{dt}u^2 - \left(\partial_1^2 + \partial_2^2\right) u^2 + \left(u^1\partial_1 + u^2\partial_2\right) u^2 + \partial_2 p = 0 \tag{4.2}$$

with a combination of (4.2) and (4.2) we can take the pressure out of the picture

$$\frac{d}{dt}\left(\partial_2 u^1 - \partial_1 u^2\right) - \left(\partial_1^2 + \partial_2^2\right)\left(\partial_2 u^1 - \partial_1 u^2\right) + \left(u^1\partial_1 + u^2\partial_2\right)\left(\partial_2 u^1 - \partial_1 u^2\right) = 0,$$

this is the evolution equation for the vorticity $\omega = (\partial_2 u^1 - \partial_1 u^2)$,

$$\omega_t - \Delta\omega + (u \cdot \nabla)\omega = 0. \tag{4.3}$$

This new quantity not only simplifies the equations but will give us the regularity of the velocity field since it turns out to be bounded, which is the content of the following lemma.

Lemma 4.0.1. *The L_2-norm of the vorticity remains bounded for all times given that it is finite for the initial data, moreover it is decreasing as a function of time.*

Proof. We use (4.3) to calculate the evolution of the vorticity L_2-norm.

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} \omega^2 dx = \int_{\mathbb{T}^2} \omega\omega_t dx = \int_{\mathbb{T}^2} \omega\Delta\omega dx - \int_{\mathbb{T}^2} \omega(u \cdot \nabla)\omega dx,$$

given that u is divergence free we can write

$$\omega(u \cdot \nabla)\omega = \frac{1}{2}\text{div } (\omega^2 u),$$

and use integration by parts on the evolution equation (4.3)

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{T}^2}\omega^2 dx = \int\limits_{\partial\mathbb{T}^2}\omega\partial_\nu\omega d\sigma - \int\limits_{\mathbb{T}^2}|\nabla\omega|^2\, dx - \frac{1}{2}\int\limits_{\partial\mathbb{T}^2}(\omega u\cdot\nu)d\sigma,$$

the integrals over the boundaries are all zero, at opposed ends the functions coincide by the periodicity and ν has opposite direction

$$\omega u\cdot\nu|_{[0,1]\times\{0\}} = -\omega u\cdot\nu|_{[0,1]\times\{1\}}, \quad \omega\partial_\nu\omega|_{[0,1]\times\{0\}} = -\omega\partial_\nu\omega|_{[0,1]\times\{0\}}$$

so that they cancel and we get the desired result

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{T}^2}\omega^2 dx = -\int\limits_{\mathbb{T}^2}|\nabla\omega|^2\, dx \leq 0.$$

□

Now we apply the Fourier transform and by expressing u in terms of the vorticity and by *Lemma 4.0.1* we will derive some bounds which give the regularity of the velocity field.
The infinite dimensional ODE system (3.1) in Fourier modes is given by

$$\frac{d}{dt}\hat{u}_\xi^m = -|\xi|^2\,\hat{u}_\xi^m + i\sum_{\alpha\ j,l}\hat{u}_\alpha^j\hat{u}_{\xi-\alpha}^l\xi_j(\frac{\xi_l\xi_m}{|\xi|^2} - \delta_{l,m}).$$

Where $\hat{u}_\xi = (\hat{u}_\xi^1, \hat{u}_\xi^2)$ is the Fourier transform of velocity field

$$u^m(x,t) = \sum_{\xi\in\mathbb{Z}^2}\hat{u}_\xi^m(t)e^{i(x,\xi)} \quad x\in\mathbb{T}^2, t\in\mathbb{R}^+.$$

For the vorticity we have that

$$\omega(x,t) = \sum_{\xi\in\mathbb{Z}^2}\hat{\omega}_\xi(t)e^{i(x,\xi)} \quad x\in\mathbb{T}^2, t\in\mathbb{R}^+,$$

where the Fourier modes satisfy the relation

$$\hat{\omega}_\xi = i(\hat{u}_\xi^1\xi_2 - \hat{u}_\xi^2\xi_1).$$

This also works the other way around: we can express the velocity field in terms of the vorticity, by the divergence free condition

$$-i\xi_2\hat{\omega}_\xi = \xi_2\xi_2\hat{u}_\xi^1 - \xi_2\xi_1\hat{u}_\xi^2 = \xi_2\xi_2\hat{u}_\xi^1 + \xi_1\xi_1\hat{u}_\xi^1 = |\xi|^2\,\hat{u}_\xi^1,$$

$$i\xi_1\hat{\omega}_\xi = |\xi|^2\,\hat{u}_\xi^2.$$

Recall the integral equation (I-NS),

$$\hat{u}_\xi^m(t) = e^{-|\xi|^2 t}\hat{u}_\xi^m(0) + i\int\limits_0^t e^{-|\xi|^2(t-s)}\sum_{\alpha\ j,l}\hat{u}_\alpha^j(s)\hat{u}_{\xi-\alpha}^l(s)\xi_j(\frac{\xi_l\xi_m}{|\xi|^2} - \delta_{l,m})ds. \tag{4.4}$$

Now under the assumption that we have weak solutions satisfying the integral equation we show that they must be smooth.

Theorem 4.0.1. *Let u be a solution to the integral equation (4.4) in the interval $[0, t^*)$, for some positive time $t^* > 0$. Then for any $p \in \mathbb{N}$ there exists a positive constant C_p such that*

$$\sup_{\xi \in \mathbb{Z}^2} |\xi|^p |\hat{u}_\xi(t)| < C_p, \quad \forall t \geq t^*.$$

The main idea for the proof is to come up with an infinite increasing sequence of times t_n going from below to t^*, for which at times bigger than a first time t_0 we have a bound on $|\xi|^{p_0} |\hat{u}_\xi|$; then from t_1 we improve to a bound on $|\xi|^{p_0+\delta} |\hat{u}_\xi|$ (for a fixed $\delta > 0$), then we continue and get a bound on $|\xi|^{p_0+2\delta} |\hat{u}_\xi|$ for times bigger than t_2, and so on. By finitely many steps we get the desired bound for times bigger than some $\rho < t^*$.

Proof. Let u be a weak solution in the interval $[0, t^*)$. Then its Fourier modes $\hat{u}_\xi^m$ satisfy (4.4) at positive times smaller than t^*. Let us define

$$t_n = t^* \left(1 - \frac{1}{2^{n+1}}\right), \quad n \in \mathbb{N},$$

and let $t \in [t_0, t^*]$. Then for all ξ, m we have by (4.4) the following bound

$$\left|\hat{u}_\xi^m(t)\right| \leq e^{-|\xi|^2 t} \left|\hat{u}_\xi^m(0)\right| + \left| \int_0^t e^{-|\xi|^2(t-s)} \sum_{\alpha\ j,l} \hat{u}_\alpha^j(s) \hat{u}_{\xi-\alpha}^l(s) \xi_j \left(\frac{\xi_l \xi_m}{|\xi|^2} - \delta_{l,m}\right) ds \right|.$$

Consider the sum inside the integral:

$$\left| \sum_{\alpha\ j,l} \hat{u}_\alpha^j(s) \hat{u}_{\xi-\alpha}^l(s) \xi_j \left(\frac{\xi_l \xi_m}{|\xi|^2} - \delta_{l,m}\right) \right| = \left| \sum_{\alpha\ j,l} \hat{u}_\alpha^j(s) \hat{u}_{\xi-\alpha}^l(s) (\xi - \alpha)_j \left(\frac{\xi_l \xi_m}{|\xi|^2} - \delta_{l,m}\right) \right|$$

$$\leq \sum_\alpha |\hat{u}_\alpha(s)| \, |\hat{u}_{\xi-\alpha}(s)| \, |\xi - \alpha| \leq \sum_\alpha \frac{|\hat{\omega}_\alpha(s)|}{|\alpha|} \frac{|\hat{\omega}_{\xi-\alpha}(s)|}{|\xi - \alpha|} |\xi - \alpha|$$

$$\leq \sum_\alpha |\hat{\omega}_\alpha(s)| \, |\hat{\omega}_{\xi-\alpha}(s)| \leq \sum_\alpha |\hat{\omega}_\alpha(s)|^2 \leq \|\omega(0)\|_2^2.$$

Here we used the fact that we can express the velocity field in terms of the vorticity, *Lemma 4.0.1* and Cauchy-Schwartz inequality in the last term. We get

$$\left|\hat{u}_\xi^m(t)\right| \leq e^{-|\xi|^2 t} \left|\hat{u}_\xi^m(0)\right| + \|\omega(0)\|_2^2 \left| \int_0^t e^{-|\xi|^2(t-s)} ds \right|$$

$$\leq e^{-|\xi|^2 t_0} \left|\hat{u}_\xi^m(0)\right| + \frac{\|\omega(0)\|_2^2}{|\xi|^2},$$

at this point we are almost done for t_0: the absolute value of the initial condition $\hat{u}_\xi^m(0)$ can be bounded by $\|\omega(0)\|_2 |\xi|^{-1}$. And for some $K > 0$ big enough

$$\exp(-|\xi|^2 t_0) < \frac{\|\omega(0)\|_2}{|\xi|}, \quad \forall |\xi| > K,$$

there are only finite frequencies below this K, hence we can find some constant $C_0 > 0$ which plays the same role as $\|\omega(0)\|_2$ in the above inequality for $|\xi| \leq K$. By taking the bigger of these constants we have the following bound, for some $D_0 > 0$

$$\sup_{\xi \in \mathbb{Z}^2} |\hat{u}_\xi(t)| < \frac{D_0}{|\xi|^2}, \quad \forall t \in [t_0, t^*].$$

Now we go for the inductive step:

Suppose that there exists some $D_n > 0, \delta \geq 2$ such that

$$\sup_{\xi \in \mathbb{Z}^2} |\hat{u}_\xi(t)| < \frac{D_n}{|\xi|^\delta}, \quad \forall t \in [t_n, t^*] ;$$

then for any time $t \in [t_{n+1}, t^*]$ we have

$$\left|\hat{u}_\xi^m(t)\right| = e^{-|\xi|^2(t-t_n)} \left|\hat{u}_\xi^m(t_n)\right| + \left| \int_{t_n}^{t} e^{-|\xi|^2(t-s)} \sum_{\alpha\ j,l} \hat{u}_\alpha^j(s) \hat{u}_{\xi-\alpha}^l(s) \xi_j \left(\frac{\xi_l \xi_m}{|\xi|^2} - \delta_{l,m}\right) ds \right| .$$

Again, we bound the sum inside the integral. Recall that $s > t_n$ so the above estimate holds. Split it as

$$\sum_\alpha \xi u_\alpha u_{\xi-\alpha} = I + II + III,$$

where

$$I = \sum_{1 \leq |\alpha| < \frac{1}{2}|\xi|} \xi u_\alpha u_{\xi-\alpha}, \quad II = \sum_{2|\xi| < |\alpha|} \xi u_\alpha u_{\xi-\alpha},$$

and

$$III = \sum_{\frac{1}{2}|\xi| \leq |\alpha| \leq 2|\xi|} \xi u_\alpha u_{\xi-\alpha}.$$

Estimate I: $\frac{1}{2}|\xi| > |\alpha|$ implies by the triangle inequality that $|\xi - \alpha| > |\xi| - |\alpha| > \frac{1}{2}|\xi|$, and hence

$$\left| \sum_{|\alpha| < \frac{1}{2}|\xi|} \xi u_\alpha u_{\xi-\alpha} \right| \leq |\xi| \sum_{|\alpha| < \frac{1}{2}|\xi|} \frac{D_n}{|\alpha|^\delta} \frac{D_n}{|\xi - \alpha|^\delta} \lesssim D_n^2 |\xi|^{1-\delta} \sum_{|\alpha| < \frac{1}{2}|\xi|} \frac{1}{|\alpha|^\delta}$$

$$\lesssim D_n^2 |\xi|^{1-\delta} \int_1^{\frac{1}{2}|\xi|} r^{1-\delta} dr \lesssim D_n^2 |\xi|^{1-\delta} \log(|\xi|) \lesssim D_n^2 |\xi|^{5/4-\delta} .$$

We have used the fact that for large x $\log x < x^{1/4}$.

Estimate II: $|\alpha| > 2|\xi|$ implies $|\xi - \alpha| > |\alpha| - |\xi| > |\xi|$, from which we can bound

$$\left| \sum_{|\alpha| > 2|\xi|} \xi u_\alpha u_{\xi-\alpha} \right| \leq |\xi| \sum_{|\alpha| > 2|\xi|} \frac{D_n}{|\alpha|^\delta} \frac{D_n}{|\xi - \alpha|^\delta} \lesssim D_n^2 |\xi| \int_{|\xi|}^{\infty} r^{1-2\delta} dr$$

$$\lesssim D_n^2 |\xi|^{3-2\delta} .$$

Estimate III: $|\alpha| \leq 2|\xi|$ implies $|\xi - \alpha| \leq |\alpha| + |\xi| < 3|\xi|$, and then

$$\left| \sum_{\frac{1}{2}|\xi| \leq |\alpha| \leq 2|\xi|} \xi u_\alpha u_{\xi-\alpha} \right| \leq |\xi| \sum_{\frac{1}{2}|\xi| \leq |\alpha| < 2|\xi|} \frac{D_n}{|\alpha|^\delta} \frac{D_n}{|\xi - \alpha|^\delta} \lesssim D_n^2 |\xi|^{1-\delta} \sum_{|\xi - \alpha| < 3|\xi|} \frac{1}{|\xi - \alpha|^\delta}$$

$$\lesssim D_n^2 |\xi|^{1-\delta} \int_1^{3|\xi|} r^{1-\delta} dr \lesssim D_n^2 |\xi|^{1-\delta} \log(|\xi|) \lesssim D_n^2 |\xi|^{5/4-\delta} .$$

Now we add all these bounds to get, for $t \in [t_{n+1}, t^*]$:

$$\left|\hat{u}_\xi^m(t)\right| \le e^{-|\xi|^2(t-t_n)} \left|\hat{u}_\xi^m(t_n)\right| + \left| \int_{t_n}^{t} e^{-|\xi|^2(t-s)} \sum_\alpha \xi u_\alpha u_{\xi-\alpha} ds \right| \lesssim e^{-|\xi|^2(t_{n+1}-t_n)} \frac{D_n}{|\xi|^\delta} + \frac{D_n^2}{|\xi|^{\delta+2-5/4}},$$

with $(t_{n+1} - t_n) = t^* 2^{-(n+2)}$ therefore by applying the same reasoning as before we can find a constant C_n such that

$$\exp(-|\xi|^2 t^* 2^{-(n+2)}) < \frac{C_n}{|\xi|^{3/4}}, \quad \forall \xi \in \mathbb{Z}^2,$$

thus obtaining a constant D_{n+1} such that

$$\sup_{\xi\in\mathbb{Z}^2} |\hat{u}_\xi(t)| < \frac{D_{n+1}}{|\xi|^{\delta+3/4}}, \quad \forall t \in [t_{n+1}, t^*] \,;$$

with this result we are done since to get the desired p we need only to apply the inductive step a finite number of times. □

The regularity of weak solutions follows as a corollary:

Corollary 4.0.1. *Let u be a solution to (I-NS) in the time interval $[0, \delta)$, for $\delta > 0$, with smooth initial data. Then u can be extended to a smooth solution in $[0, \infty)$.*

Proof. Assume that $[0, \delta)$ is the maximal interval of regularity. For any fixed time $\tilde{t} \in (0, \delta)$ and any $s > 0$ by *Theorem 4.0.1* and *Lemma 4.0.1* we have a constant C_{s+2} such that

$$\sup_{\xi\in\mathbb{Z}^2} |\hat{u}_\xi(t)| < \frac{C_{s+2}}{|\xi|^{s+2}}, \quad t > \tilde{t}$$

therefore we have a bound for the $H^s(\mathbb{T}^2)$ norm:

$$\sum_\xi |\xi|^{2s} |\hat{u}_\xi(t)|^2 < C_{s+2}^2.$$

For any positive s and $t > \tilde{t}$ these estimates hold for $u(x, t)$, then by taking the limit $t \to \delta$ we could obtain a weak solution $u(x, \delta)$ which satisfy the same estimates, therefore is smooth. And we could consider $u(x, \delta)$ as our new initial (smooth) data, which has an interval of regularity $[\delta, \delta + \epsilon)$ for some $\epsilon > 0$, then u is smooth in $[0, \delta + \epsilon)$. This contradicts that δ was maximal $\Rightarrow \delta = \infty$. □

Chapter 5

Navier Stokes in 3-D

5.1 Lower bounds

In the 3-dimensional setting the problem of regularity has remained open, but some partial results have been obtained. In [9], Leray showed the existence of weak solutions which could become singular at some times. He also stated that if a weak solution ceased to be regular at a finite time then certain norms should increase at a particular rate. For a weak solution u to the Navier-Stokes equations call T, the supremum of t such that $u(s)$ is regular for $s \in [0, t)$, the *'blow-up time'*. Leray showed the following bound in $\dot{H}^1(\mathbb{R}^3)$: if $T < \infty$, then there exists a constant c_1 such that

$$\|u(t)\|_{\dot{H}^1(\mathbb{R}^3)} > \frac{c_1}{(T-t)^{\frac{1}{4}}},$$

and he also stated, without a proof, a growth rate in $L^p(\mathbb{R}^3)$, for $p > 3$, at times near T: there exists some constants c_p such that

$$\|u(t)\|_{L^p(\mathbb{R}^3)} > \frac{c_p}{(T-t)^{\frac{p-3}{2p}}}.$$

A generalization of the results above has been given by Robinson, Sadowski and Silva for other Sobolev spaces $\dot{H}^s(\Omega)$, for the periodic case and solutions in the whole space ($\Omega = \mathbb{T}^3$ and $\Omega = \mathbb{R}^3$ respectively). In [10] they showed that if $u(t)$ has a finite blow-up time $T < \infty$ then there exist some constants c_s, for $\frac{1}{2} < s < \frac{5}{2}$, and $s \neq \frac{3}{2}$ such that for times close to T it holds

$$\|u(t)\|_{H^s(\Omega)} \geq \frac{c_s}{(T-t)^{\frac{2s-1}{4}}},$$

It is our purpose, in this chapter, to extend their results to the cases $s = \frac{1}{2}$ and $s = \frac{3}{2}$.

5.1.1 $s = \frac{1}{2}$

In this section we show a formulation for the inequality above in the borderline case $s = \frac{1}{2}$. To be more precise we have

Theorem 5.1.1. *Let u be a Leray-Hopf solution of (3.1) regular in the time interval $[0, T)$. There exist an absolute constant $c_{\frac{1}{2}}$, such that if for some $\rho > 0$, u satisfies the following estimate*

$$\sup_{t \in (T-\rho, T)} \|u(\cdot, t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)} \leq c_{\frac{1}{2}}.$$

Then u can be extended to a smooth solution in $[T - \rho, T]$.

For the proof we will need a few lemmas.

Lemma 5.1.1. *Let u be a Leray-Hopf solution of (3.1) for which the estimate in Theorem 5.1.1 holds for some $\rho > 0$. Then the following estimate holds*

$$|\xi|^{3/2}|\hat{u}_\xi(t)| \leq (\frac{1}{4}c_{\frac{1}{2}} + \frac{7}{2}c_{\frac{1}{2}}^2) \quad \forall\, t \in (T - 3\rho/4, T)\,, |\xi| > \frac{4^2}{\rho}. \tag{5.1}$$

Proof. Take $t \in [T - 3\rho/4, T)\,, \xi \in \mathbb{Z}^3$, by integrating equation (I-NS) we have

$$\begin{aligned} |\hat{u}_\xi(t)| &\leq |\hat{u}_\xi(T-\rho)|e^{-|\xi|^2(t-(T-\rho))} + \frac{1}{|\xi|^2}|\sum_\alpha \hat{u}_\alpha \hat{u}_{\xi-\alpha}(\xi - \alpha)| \\ &\leq |\hat{u}_\xi(T-\rho)|e^{-|\xi|^2\rho/4} + \frac{1}{|\xi|^2}|\sum_\alpha \hat{u}_\alpha \hat{u}_{\xi-\alpha}(\xi-\alpha)|. \end{aligned}$$

Here we can split the sum in two sets and bound them separately:

$$I := \sum_{|\alpha|<\frac{1}{2}|\xi|} \hat{u}_\alpha \hat{u}_{\xi-\alpha}(\xi-\alpha); \quad II := \sum_{|\alpha|\geq\frac{1}{2}|\xi|} \hat{u}_\alpha \hat{u}_{\xi-\alpha}(\xi-\alpha).$$

For I: $|\alpha| < \frac{1}{2}|\xi|$ implies $|\xi - \alpha| < \frac{3}{2}|\xi|$, then we have

$$|I| \leq \sum_{|\alpha|<\frac{1}{2}|\xi|} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\xi-\alpha| < \frac{3}{2}|\xi|^{1/2} \sum_{|\alpha|<\frac{1}{2}|\xi|} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\xi-\alpha|^{1/2}.$$

In II: replace $\xi - \alpha$ by ξ, therefore we have:

$$|II| = |\sum_{|\alpha|\geq\frac{1}{2}|\xi|} \hat{u}_\alpha \hat{u}_{\xi-\alpha}\xi| \geq \sum_{|\alpha|\geq\frac{1}{2}|\xi|} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\xi| < 2|\xi|^{1/2} \sum_{|\alpha|\geq\frac{1}{2}|\xi|} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\alpha|^{1/2}.$$

If we add these bounds and apply the Cauchy-Schwartz inequality to each set, I and II separately, we get the bound:

$$|\sum_\alpha \hat{u}_\alpha \hat{u}_{\xi-\alpha}(\xi-\alpha)| \leq |I| + |II| < \frac{7}{2}|\xi|^{1/2}\|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)}\|u\|_{L^2(\mathbb{T}^3)} \leq \frac{7}{2}|\xi|^{1/2}c_{\frac{1}{2}}^2,$$

then we can conclude

$$|\hat{u}_\xi(t)| \leq |\xi|^{-3/2}(|\xi|e^{-|\xi|^2\rho/4}c_{\frac{1}{2}} + Cc_{\frac{1}{2}}^2) \leq |\xi|^{-3/2}(\frac{1}{4}c_{\frac{1}{2}} + \frac{7}{2}c_{\frac{1}{2}}^2), \tag{5.2}$$

if we have that $|\xi| > \frac{4^2}{\rho}$, ($e^{-x} < x^{-1}$ for $x > 0$).

□

Now we derive a couple of estimates which will be used later.

Lemma 5.1.2. *Let $u(t) \in \dot{H}^{\frac{1}{2}}(\mathbb{T}^3)$, for all $K \geq 1$ the following estimates hold*

$$\sum_{|\xi|>K} |\hat{u}_\xi(t)|^2 \leq K^{-1}\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)}, \tag{5.3}$$

$$\sum_{|\xi|<K} |\hat{u}_\xi(t)| \leq 2\sqrt{\pi}K\|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)}. \tag{5.4}$$

Proof. Take $u(t) \in \dot{H}^{\frac{1}{2}}(\mathbb{T}^3)$, then we have

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)} \geq \sum_{|\xi|>K} |\hat{u}_\xi|^2|\xi| \geq K \sum_{|\xi|>K} |\hat{u}_\xi|^2,$$

the second estimate is an application of the Cauchy-Schwartz inequality

$$\sum_{|\xi|<K} |\hat{u}_\xi(t)| \leq \left(\sum_{|\xi|<K} |\hat{u}_\xi|^2|\xi| \sum_{|\xi|<K} \frac{1}{|\xi|} \right)^{1/2} \leq \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)} \left(4\pi \int_1^K r dr \right)^{1/2}.$$

□

Now we are ready to prove *Theorem 5.1.1.*

Proof of Theorem 5.1.1. Let u be smooth in the time interval $[T-\rho, T)$, first we will show that under some smallness condition on $c_{\frac{1}{2}}$, $u(t)$ will be small in $\Phi(2)(\mathbb{T}^3)$ for $t \geq T - \rho/2$, from which we can conclude that u is regular even at $t = T$. This will be done with the *interpolation inequalities*, of which *Lemma 5.1.1* is the first step.
To proceed, let us define

$$t_0 = T - \rho\ , \quad t_n = t_{n-1} + \frac{\rho}{2^{n+1}},$$

$$K_n = 2^n K_0, \quad K_0 = \frac{4^2}{\rho},\ n = 1, 2, 3 \dots$$

Consider $t \in [t_1, T)$, and take $\xi \in \mathbb{Z}^3$ such that $|\xi| > K_1$, then by *Lemma 5.1.1* we have:

$$|\hat{u}_\xi(t)| \leq |\xi|^{-3/2}(\frac{1}{4}c_{\frac{1}{2}} + \frac{7}{2}c^2_{\frac{1}{2}}) = D_1|\xi|^{-3/2}. \tag{5.5}$$

Now proceed with the inductive step, assume that

$$|\hat{u}_\xi(t)| \leq \frac{D_n}{|\xi|^{2-1/2^n}} \quad \forall\ |\xi| > K_n, t \in [t_n, T)\,. \tag{5.6}$$

Therefore if $t \in [t_{n+1}, T)$, by equation (I-NS) we can estimate

$$\begin{aligned} |\hat{u}_\xi(t)| &\leq |\hat{u}_\xi(t_n)|e^{-|\xi|^2(t-t_n)} + |\int_{t_1}^{t} e^{-|\xi|^2(s-t_1)} \sum_\alpha \hat{u}_\alpha \hat{u}_{\xi-\alpha} \alpha ds| \\ &\leq \frac{D_n|\xi|^{2^{-(n+1)}}}{|\xi|^{2-2^{-(n+1)}}} e^{-|\xi|^2\rho/2^{n+2}} \\ &\quad + \frac{1}{|\xi|^{3/2}} \left(\sum_{|\xi-\alpha|<\frac{1}{2}|\xi|^{1-2^{-n}}} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\alpha|^{1/2} + \sum_{|\xi-\alpha|\geq\frac{1}{2}|\xi|^{1-2^{-n}}} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\alpha|^{1/2} \right). \end{aligned}$$

Now we must bound the two sums shown above.
For frequencies above $1/2|\xi|^{1-2^{-n}}$, we can apply *Lemma 5.1.2* and the Cauchy-Schwartz inequality in the following way

$$\sum_{|\xi-\alpha|\geq\frac{1}{2}|\xi|^{1-2^{-n}}} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\alpha|^{1/2} \leq \|u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^3)} \left(\sum_{|\xi-\alpha|\geq\frac{1}{2}|\xi|^{1-2^{-n}}} |\hat{u}_{\xi-\alpha}|^2 \right)^{1/2} \leq \frac{2^{1/2}c^2_{\frac{1}{2}}}{|\xi|^{1/2-2^{-n+1}}}.$$

And for the lower ones, notice that if $|\xi-\alpha| < \frac{1}{2}|\xi|^{1-2^{-n}} < \frac{|\xi|}{2}$ then $|\alpha| > \frac{|\xi|}{2}$, therefore by (5.6), and *Lemma 5.1.2*

$$\sum_{|\xi-\alpha|<\frac{1}{2}|\xi|^{1-2^{-n}}} |\hat{u}_\alpha||\hat{u}_{\xi-\alpha}||\alpha|^{1/2} \le \frac{3D_n}{|\xi|^{3/2-2^{-n}}} \sum_{|\xi-\alpha|<\frac{1}{2}|\xi|^{1-2^{-n}}} |\hat{u}_{\xi-\alpha}|$$

$$\le \frac{3\sqrt{\pi}D_n c_{\frac{1}{2}}|\xi|^{1-2^{-n}}}{|\xi|^{3/2-2^{-n}}} = \frac{3\sqrt{\pi}D_n c_{\frac{1}{2}}}{|\xi|^{1/2}}.$$

Taking all these estimates together we get

$$|\hat{u}_\xi(t)| \;<\; \frac{D_n}{4|\xi|^{2-2^{-(n+1)}}} + \frac{1}{|\xi|^{2-2^{-(n+1)}}}\left(2^{1/2}c_{\frac{1}{2}}^2 + 3\sqrt{\pi}D_n c_{\frac{1}{2}}\right) \le \frac{D_{n+1}}{|\xi|^{2-2^{-(n+1)}}},$$

if we define D_{n+1} by:

$$D_{n+1} = \frac{D_n}{4} + c_{\frac{1}{2}}\left(2^{1/2}c_{\frac{1}{2}} + 3\sqrt{\pi}D_n\right).$$

Notice that if $c_{\frac{1}{2}} < \frac{1}{10}$, then $D_n < c_{\frac{1}{2}}$ for every n, and D_n is a bounded sequence.

Indeed, for $n = 1$:

$$D_1 = c_{\frac{1}{2}}(1/4 + 7/2c_{\frac{1}{2}}) < c_{\frac{1}{2}},$$

and if we assume that $D_n < c_{\frac{1}{2}}$, then

$$D_{n+1} = D_n/4 + c_{\frac{1}{2}}(\sqrt{2}c_{\frac{1}{2}} + 3\sqrt{\pi}D_n) < c_{\frac{1}{2}}(1/4 + 10^{-1}(\sqrt{2} + 3\sqrt{\pi})) < c_{\frac{1}{2}}.$$

We conclude that for all $n \in \mathbb{N}$ if $|\xi| > 2^n K_0$, $t \in [t_n, T)$ then

$$|\hat{u}_\xi(t)| \le \frac{D_n}{|\xi|^{2-2^{-n}}} < \frac{(2^n K_0)^{1/2^n} D_n}{|\xi|^2}.$$

There exists an integer m big enough such that $(2^m K_0)^{1/2^m} < 2$ then at times $t \in [T-\rho/2, T) \subset [t_m, T)$ and frequencies above $2^m K_0$ we have the following bound:

$$|\hat{u}_\xi(t)| \le \frac{2D_m}{|\xi|^2} < \frac{2c_{\frac{1}{2}}}{|\xi|^2}.$$

Therefore for a small choice of $c_{\frac{1}{2}}$, by the proof of Theorem 2 in [1] u can be extented to a smooth solution on $[T-\rho/2, T]$. □

5.1.2 $s = \frac{3}{2}$

Now we present the case $s = \frac{3}{2}$, for which we obtained a result closely related to the one obtained by Robinson, Sadowski and Silva in [10], with a logarithmic correction. The technique of proof is similar to the previous case.

Theorem 5.1.2. *Let u be a Leray-Hopf solution of (3.1) whose maximal interval of existence is $(0,T)$, $T < \infty$. Then, there is an absoulte constant $c_{\frac{3}{2}} > 0$, such that there is a sequence $t_j \to T$ along which the following estimate holds*

$$\|u(t_j)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \ge \frac{c_{\frac{3}{2}}}{\sqrt{(T-t_j)\,|\log(T-t_j)|}}.$$

The proof we will present does not require the divergence free condition for u to hold (as long a theL^2 norm of the solution remains bounded). Let us begin with an elementary lemma.

Lemma 5.1.3.

$$\sum_{1\le|\alpha|<R} |u_\alpha| \lesssim \|u\|_{\dot H^{\frac{3}{2}}(\mathbb{T}^3)} \sqrt{\log R}.$$

Proof. By means of the Cauchy-Schwarz inequality we obtain,

$$\left(\sum_{1\le|\alpha|<R} |u_\alpha|\right)^2 \le \sum_{1\le|\alpha|<R} |\alpha|^3 |u_\alpha|^2 \sum_{1\le|\alpha|<R} |\alpha|^{-3} \lesssim \|u\|^2_{\dot H^{\frac{3}{2}}(\mathbb{T}^3)} \int_1^R \frac{dr}{r} = \|u\|^2_{\dot H^{\frac{3}{2}}(\mathbb{T}^3)} \log R.$$

□

The following lemma gives the main estimate needed for the proof of Theorem 5.1.2.

Lemma 5.1.4. *Let $u(x,t)$ be a solution of (3.1) on $(0,T)$. There exists $L_0 > 0$ such that if there exists a $\rho_0 > 0$, such that*

$$\sup_{t\in(T-\rho_0,T)} (T-t)^{\frac{1}{2}} |\log(T-t)|^{\frac{1}{2}} \|u(t)\|_{\dot H^{\frac{3}{2}}(\mathbb{T}^3)} < L_0,$$

then there is a $k_1 := k_1(L_0,\rho_0)$ and a $\rho_1 > 0$ such that if $|\xi| \ge k_1$, then

$$|u_\xi(t)| \le \frac{D(L_0)}{|\xi|^2}, \quad t \ge T-\rho_1,$$

and $D \to 0$ as $L_0 \to 0$.

Proof. Fix $t^* \in (T-\rho_0,T)$. Let $k_0 > 0$ be large enough so that with $\beta = \sqrt{k_0}$ we have

$$\exp\left(-\left|2^j k_0\right|^2 \frac{\beta^2}{(2^n k_0)^2}\right) \le \frac{L_0}{2(2^j k_0)^2}, \quad \text{if} \quad j \ge 2n,$$

also $\dfrac{\|u(t)\|_{L^2(\mathbb{T}^3)}}{\log\log\left(\sqrt{k_0}\right)} << 1$ and $\rho_0 < \dfrac{1}{k_0}$, otherwise we could take a smaller ρ_0.
Let $\rho = \dfrac{\beta^2}{k_0^2}$, and write $t_m = t^* - \left(\dfrac{\beta}{2^m k_0}\right)^2$.

To begin with our computations, write the nonlinear term as

$$\sum_\alpha \xi u_\alpha u_{\xi-\alpha} = I_1 + II_1 + III_1.$$

We let $t > 0$ be such that $t_m \le t < t_{m+1}$, $m \ge 0$, and we let ξ be such that $2^j k_0 \le |\xi| < 2^{j+1} k_0$, $j \ge 0$. Let M_j be such that if $m < M_j$ then $\dfrac{2^{m-j}}{\beta} < 2$, and $\dfrac{2^{m-j}}{\beta} \ge 2$ if $m \ge M_j$.
We first estimate under the assumption that $m < M_j$ Let us begin with I_1. It is in this and in the next calculation (the estimation of II) that we make a fundamental use that weak solutions

to (3.1) dissipate energy, as we need to have control over the L^2 norm of the solution.

$$\begin{aligned}
|I_1| &\leq \left| \sum_{|\alpha|<\frac{|\xi|}{2}} \xi u_\alpha u_{\xi-\alpha} \right| \leq |\xi|^{-\frac{1}{2}} \sum_{|\alpha|<\frac{|\xi|}{2}} |\xi|^{\frac{3}{2}} |u_\alpha| |u_{\xi-\alpha}| \\
&\leq \left(\frac{2^m k_0}{\beta}\right)\left(\frac{\beta}{2^m k_0}\right)\left(\frac{1}{2^j k_0}\right)^{\frac{1}{2}} \left|\log\left(\frac{2^m k_0}{\beta}\right)\right|^{-\frac{1}{2}} \left|\log\left(\frac{2^m k_0}{\beta}\right)\right|^{\frac{1}{2}} \cdot \sum_{|\alpha|<\frac{|\xi|}{2}} |\xi|^{\frac{3}{2}} |u_\alpha| |u_{\xi-\alpha}| \\
&\lesssim \left(\frac{2^{m-j}}{\beta}\right) \left|\log\left(\frac{2^m k_0}{\beta}\right)\right|^{-\frac{1}{2}} (t^*-t)^{\frac{1}{2}} \left|\log(t^*-t)\right|^{\frac{1}{2}} \|u(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \cdot \|u(t)\|_{L^2(\mathbb{T}^3)} |\xi|^{\frac{1}{2}} \\
&\lesssim \left(\frac{2^{m-j}}{\beta}\right) \frac{1}{m^{\frac{1}{2}}} L_0 |\xi|^{\frac{1}{2}}.
\end{aligned}$$

Notice the use of the Cauchy-Schwarz inequality to obtain the third inequality in the previous calculation.

Estimating II_1:

$$\begin{aligned}
|II_1| &\leq \left| \sum_{\frac{|\xi|}{2}\leq|\alpha|<2|\xi|} \xi u_\alpha u_{\xi-\alpha} \right| \leq 2 \sum_{\frac{|\xi|}{2}\leq|\alpha|<2|\xi|} |\alpha u_\alpha u_{\xi-\alpha}| \\
&\lesssim |\xi|^{-\frac{1}{2}} \left(\frac{2^m k_0}{\beta}\right)\left(\frac{\beta}{2^m k_0}\right) \sum_{|\alpha|\geq 1} |\alpha|^{\frac{3}{2}} |u_\alpha| |u_{\xi-\alpha}| \\
&\lesssim \left(\frac{2^{m-j}}{\beta}\right) \left|\log\left(\frac{2^m k_0}{\beta}\right)\right|^{-\frac{1}{2}} (t^*-t)^{\frac{1}{2}} \left|\log(t^*-t)\right|^{\frac{1}{2}} \|u(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \|u(t)\|_{L^2(\mathbb{T}^3)} |\xi|^{\frac{1}{2}} \\
&\lesssim \left(\frac{2^{m-j}}{\beta}\right) \frac{1}{m^{\frac{1}{2}}} L_0 |\xi|^{\frac{1}{2}}.
\end{aligned}$$

And estimating III_1:

$$\begin{aligned}
|III_1| &\leq \left| \sum_{|\alpha|\geq 2|\xi|} \xi u_\alpha u_{\xi-\alpha} \right| \leq \frac{1}{|\xi|^2} \sum_{|\alpha|\geq 2|\xi|} |\alpha|^{\frac{3}{2}} |u_\alpha| |\xi-\alpha|^{\frac{3}{2}} |u_{\xi-\alpha}| \\
&\lesssim \left(\frac{2^{m-j}}{\beta}\right)^2 \left|\log\left(\frac{2^m k_0}{\beta}\right)\right|^{-1} \left((t^*-t)^{\frac{1}{2}} \left|\log(t^*-t)\right|^{\frac{1}{2}} \|u(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)}\right)^2 \\
&\lesssim \left(\frac{2^{m-j}}{\beta}\right)^2 \frac{1}{m} L_0^2,
\end{aligned}$$

where we have used the fact that $|\alpha| \geq 2|\xi|$ implies via the triangular inequality that $|\xi-\alpha| \geq |\xi|$, and the Cauchy-Schwarz inequality to obtain the third inequality.

Using (I-NS) and the previous estimates, we obtain

$$|u_\xi(t)| \leq |u_\xi(t_m)| e^{-|\xi|^2(t-t_m)} + \frac{A}{|\xi|^2}\left(1-e^{-|\xi|^2(t-t_m)}\right)\left(\left(\frac{2^{m-j}}{\beta}\right)\frac{L_0}{m^{\frac{1}{2}}} + \left(\frac{2^{m-j}}{\beta}\right)^2 \frac{L_0^2}{m}\right)|\xi|^{\frac{1}{2}}$$

where A is a universal constant. Define

$$\tilde{D}_{1,1}^j = L_0 + A\left(2L_0 + 4L_0^2\right),$$

and for $m \geq 1$, using the fact that $e^{-2} < \frac{1}{6}$

$$\tilde{D}^j_{1,m+1} = \frac{1}{6}\tilde{D}^j_{1,m} + A\left(2L_0 + 4L_0^2\right).$$

Then we have that

$$|u_\xi(t)| \leq \frac{\tilde{D}_{1,m}}{|\xi|^{\frac{3}{2}}},$$

when $2^j k_0 \leq |\xi| < 2^{j+1}k_0$, whenever $t \in [t_m, t_{m+1})$, under the assumption that $\frac{2^{m-j}}{\beta} < 2$. Let $\tilde{D}_{1,\infty} = \sup_{j,m} \tilde{D}^j_{1,m}$.

Now we estimate under the assumption that $m = M_j + k$. In this case we split the non-linear term as

$$\sum_\alpha \xi u_\alpha u_{\xi-\alpha} = I_2 + II_2 + III_2 + IV,$$

where

$$I_2 = \sum_{1\leq|\alpha|<\frac{1}{2}|\xi|} \xi u_\alpha u_{\xi-\alpha}, \quad II_2 = \sum_{|\xi-\alpha|<\frac{1}{2}|\xi|} \xi u_\alpha u_{\xi-\alpha},$$

and

$$III_2 = \sum_{\frac{1}{2}|\xi|\leq|\alpha|<2|\xi|\log k} \xi u_\alpha u_{\xi-\alpha}, \quad IV = \sum_{|\alpha|\geq 2|\xi|\log k} \xi u_\alpha u_{\xi-\alpha}.$$

Proceeding as before, we have an estimate

$$\begin{aligned} |I_2|, |II_2| &\lesssim \left(\frac{2^{m-j}}{\beta}\right)\left|\log\left(\frac{2^m k_0}{\beta}\right)\right|^{-\frac{1}{2}} (t^*-t)^{\frac{1}{2}} |\log(t^*-t)|^{\frac{1}{2}} \|u(t)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \cdot \|u(t)\|_{L^2(\mathbb{T}^3)} |\xi|^{\frac{1}{2}} \\ &\lesssim \left(\frac{2^{m-j}}{\beta}\right)\frac{1}{k^{\frac{1}{2}}} L_0 |\xi|^{\frac{1}{2}}. \end{aligned}$$

We estimate III_2 as follows

$$\begin{aligned} |III_2| &\leq \frac{1}{|\xi|^{\frac{1}{2}}} \sum_{\frac{1}{2}|\xi|\leq|\alpha|<2|\xi|\log k} |\alpha|^{\frac{3}{2}} |u_\alpha| |u_{\xi-\alpha}| \\ &\lesssim \left(\frac{2^{m-j}}{\beta}\right)\log\left(\frac{2^m k_0}{\beta}\right)^{-\frac{1}{2}} (t^*-t)^{\frac{1}{2}} |\log(t^*-t)|^{\frac{1}{2}} \|u\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \|u\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

And IV can be estimated as

$$|IV| \lesssim \left(\frac{2^{m-j}}{\beta}\right)^2 \frac{1}{k\log^2 k} L_0^2.$$

Integrating the equation and using the previous estimates, we obtain

$$\begin{aligned} |u_\xi(t)| \leq\ & |u_\xi(t_m)| e^{-|\xi|^2(t-t_m)} \\ & + \frac{A}{|\xi|^2}\left(1-e^{-|\xi|^2(t-t_m)}\right)\left(\left(\frac{2^{m-j}}{\beta}\right)\frac{L_0}{k^{\frac{1}{2}}} + \left(\frac{2^{m-j}}{\beta}\right)^2 \frac{L_0^2}{k\log^2 k}\frac{1}{\sqrt{2^{2n}k_0}}\right)|\xi|^{\frac{1}{2}}. \end{aligned}$$

Using the inequality $1-e^{-z}\leq z$ to bound the nonlinear term, it is not difficult to see that if $m=M_j+k$ (recall that $\frac{2^{m-j}}{\beta}>2$), and we define

$$D^j_{1,k+1}=D^j_{1,k}+A\left(\frac{1}{2}\right)^{(k+1)}L_0+\frac{AL_0^2}{k\log^2 k}.$$

where $D^j_{1,0}=\tilde{D}_{1,\infty}$. Then we have that

$$|u_\xi(t)|\leq\frac{D^j_{1,k}}{|\xi|^{\frac{3}{2}}},$$

when $2^jk_0\leq|\xi|<2^{j+1}k_0$ if $t\in[t_m,t_{m+1})$ if $\frac{2^{m-j}}{\beta}\geq 2$.

Notice again that $D^j_{1,k}$ is uniformly bounded on j and k. Then we have that there is a constant D_1 such that

$$|u_\xi|\leq\frac{D_1}{|\xi|^{\frac{3}{2}}},\quad |\xi|\geq k_0,\quad\text{for}\qquad t\in(t-\rho,t^*).$$

Now, assume that for $|\xi|\geq 2^{2n}k_0$ it holds that

$$|u_\xi(t)|\leq\frac{D_n}{|\xi|^{\frac{3}{2}}},\quad\text{for}\quad t\geq t^*-\left(\frac{\beta}{2^nk_0}\right)^2.$$

We will look at frequencies such that $2^jk_0\leq|\xi|<2^{j+1}k_0$, with $j\geq 2(n+1)$, and examine what happens on the time interval $[t_m,t_{m+1})$, $m\geq n+1$.

Let M_j be defined as before. if $m\leq M_j$ we split the nonlinear term as

$$\sum_\alpha \xi u_\alpha u_{\xi-\alpha}=I_1+II_1+III_1,$$

where I_1,II_1 and III_1 are as above. We estimate I_1:

$$\begin{aligned}|I_1| &\lesssim \frac{D_n}{|\xi|^{\frac{1}{2}}}\sum_{|\alpha|<\frac{1}{2}|\xi|}|u_\alpha|\lesssim\frac{D_n}{|\xi|^{\frac{1}{2}}}\|u\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)}\sqrt{\log(2^jk_0)}\\ &\lesssim D_n\left(\frac{2^{m-j}}{\beta}\right)\sqrt{\log(2^jk_0)}\log\left(\frac{2^mk_0}{\beta}\right)^{-\frac{1}{2}}|\xi|^{\frac{1}{2}}(t^*-t)^{\frac{1}{2}}|\log(t^*-t)|^{\frac{1}{2}}\|u\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)}\\ &\lesssim D_n\left(\frac{2^{m-j}}{\beta}\right)\sqrt{\log(2^jk_0)}\log\left(\frac{2^mk_0}{\beta}\right)^{-\frac{1}{2}}L_0|\xi|^{\frac{1}{2}}.\end{aligned}$$

II_1 can be estimated in a similar fashion to obtain:

$$|II|\lesssim D_n\left(\frac{2^{m-j}}{\beta}\right)\sqrt{\log(2^jk_0)}\log\left(\frac{2^mk_0}{\beta}\right)^{-\frac{1}{2}}L_0|\xi|^{\frac{1}{2}}.$$

We estimate III_1 as

$$|III_1|\ \lesssim\ \left(\frac{2^{m-j}}{\beta}\right)^2\frac{1}{m}L_0^2.$$

Then if we define

$$D^j_{n+1,0} = \left(\frac{1}{6} + cL_0\right) D_n + \frac{AL_0^2}{\sqrt{2^{2n} k_0}},$$

where $c > 0$ is a constant such that

$$\frac{(\log x)^{\frac{1}{2}}}{x} \Big/ \frac{(\log y)^{\frac{1}{2}}}{y} < c, \quad \text{whenever} \quad \frac{x}{y} \geq 2 \quad (x = 2^j \sqrt{k_0} \quad \text{and} \quad y = \frac{2^m k_0}{\beta}).$$

then

$$|u_\xi(t)| \leq \frac{D^j_{n+1,0}}{|\xi|^{\frac{3}{2}}},$$

when $2^j k_0 \leq |\xi| < 2^{j+1} k_0$, whenever $t \in [t_m, t_{m+1})$, $j \geq 2(n+1)$ and $m \geq n+1$, under the assumption that $\frac{2^{m-j}}{\beta} < 2$.

When $m \geq M_j$ we split the nonlinear term as

$$\sum_\alpha \xi u_\alpha u_{\xi-\alpha} = I_2 + II_2 + III_2 + IV,$$

with I_2, II_2, III_2 and IV as before. I_2 and II_2 can be estimated as before.

For III_2 we have

$$\begin{aligned} |III_2| &\leq \frac{D_n}{|\xi|^{\frac{1}{2}}} \sum_{\frac{1}{2}|\xi| \leq |\alpha| < 2|\xi| \log k} |u_\alpha| \lesssim \frac{D_n}{|\xi|^{\frac{1}{2}}} \|u\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} (\log|\xi| + \log\log k)^{\frac{1}{2}} \\ &\lesssim D_n \left(\frac{2^{m-j}}{\beta}\right) \log\left(\frac{2^m k_0}{\beta}\right)^{-\frac{1}{2}} \left(\log\left(2^j k_0\right)^{\frac{1}{2}} + \sqrt{\log\log k}\right) \\ &\quad (t^* - t)^{\frac{1}{2}} |\log(t^* - t)|^{\frac{1}{2}} \|u\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} \\ &\lesssim \left(\frac{2^{m-j}}{\beta}\right) \left(\left(\frac{\log(2^j k_0)}{\log\left(\frac{2^m k_0}{\beta}\right)}\right)^{\frac{1}{2}} + 1\right) L_0 D_n. \end{aligned}$$

Finally, as it was done before, we obtain

$$|IV| \lesssim \left(\frac{2^{m-j}}{\beta}\right)^2 \frac{1}{k \log^2 k} L_0^2 \frac{|\xi|^{\frac{1}{2}}}{\sqrt{2^{2(n+1)} k_0}}.$$

Again, integrating the differential inequality obtained for u_ξ (i.e., using (I-NS)), for $2^j k_0 \leq |\xi| < 2^{j+1} k_0$, $j \geq n+1$, and $t \in [t_m, t_{m+1})$, $m \geq n+1$, yields

$$\begin{aligned} |u_\xi(t)| &\leq |u_\xi(t_m)| e^{-|\xi|^2 (t - t_m)} \\ &+ \frac{A}{|\xi|^{\frac{3}{2}}} \left(1 - e^{-|\xi|^2 (t - t_m)}\right) \left(\left(\frac{2^{m-j}}{\beta}\right) \left(\frac{\log(2^j k_0)}{\log\left(\frac{2^m k_0}{\beta}\right)}\right)^{\frac{1}{2}} D_n L_0 + \left(\frac{2^{m-j}}{\beta}\right)^2 \frac{L_0^2}{k \log^2 k} \frac{1}{\sqrt{2^{2n} k_0}}\right). \end{aligned}$$

And as bounded above, if we define

$$D^j_{n+1,k+1} = D^j_{n+1,k} + c\left(\frac{1}{2}\right)^{\frac{k+1}{2}} D_n L_0 + \frac{AL_0^2}{k \log^2 k} \frac{1}{\sqrt{2^{2n} k_0}},$$

where $c > 0$ is a constant such that

$$\frac{x \log^{\frac{1}{2}} x}{y \log^{\frac{1}{2}} y} \leq c\sqrt{\frac{x}{y}}, \quad \text{with} \quad \frac{y}{x} \geq 2 \quad (x = 2^j \sqrt{k_0} \quad \text{and} \quad y = \frac{2^m \beta}{k_0}).$$

Notice that $D^j_{n+1,k}$ is an increasing sequence in k. If $L_0 > 0$ is small enough this sequence is bounded, so it has a limit as $k \to \infty$. Notice that this limit will not depend on j, so let us call this common limit D_{n+1}. Again, from this choices, it is not difficult to see that

$$|u_\xi| \leq \frac{D_{n+1}}{|\xi|^{\frac{3}{2}}}, \quad \text{if} \quad t \in (t_{n+1}, t^*)\,.$$

But for n large enough, $j \geq 2n$, if L_0 is small enough, it is not difficult to show that there is constant $B > 0$ independent of L_0 (for L_0 small enough) such that,

$$D_n \leq \frac{BL_0}{\sqrt{2^{2n} k_0}},$$

from which we can deduce that

$$|u_\xi\,(t)| \leq \frac{2BL_0}{|\xi|^2},$$

whenever $t \in (t_n, t^*)$ and $2^{2n} k_0 \leq |\xi| < 2^{2(n+1)} k_0$. Since u is regular on $(0, t^*]$, this implies that the previous estimate is valid at $t = t^*$, and for all $|\xi| \geq 2^{2N} k_0$, with N large enough. □

Now we can proof the Theorem.

Proof of Theorem 5.1.2. From *Lemma 5.1.4*, there is a k_1 such that if $|\xi| \geq k_1$ then

$$|u_\xi\,(t)| \leq \frac{D}{|\xi|^2},$$

for all $t \in \left(T - \frac{\rho_0}{2}, T\right)$, with a constant D which depends on L_0. Since $D \to 0$ as $L_0 \to 0$, then if $L_0 > 0$ is small enough, by the proof of Theorem 2 in [1], it follows that there is a constant $c_{\frac{3}{2},\epsilon} > 0$ such that if

$$\limsup_{t \to T^-} (T-t)^{\frac{1}{2}} \left|\log\,(T-t)\right|^{\frac{1}{2}} \left\|u\,(t)\right\|_{\dot{H}^{\frac{3}{2}}(\mathbb{T}^3)} < c_{\frac{3}{2},\epsilon},$$

since by a proper choice of $c_{\frac{3}{2},\epsilon} > 0$ the $\Phi(2)(\mathbb{T}^3)$ norm is small near T, where u can be extended smoothly. □

Conclusions

The main result in this work was motivated by the inequalities obtained by Robinson, Sadowski and Silva in [10]. The question was whether these inequalities could be extended to the borderline (missing) exponents. The answer is affirmative, but with a small modification, since we had to add a logarithmic correction. The main result (*Theorem 5.1.2*) was presented in a stronger formulation in [2], however the proof used in this document was the first approach which gave the right exponent in [2]. This proof arose after several modifications in a exciting exchange of ideas with Jean Carlos Cortissoz (advisor of this project). And the logarithmic correction was recently removed by Julio Montero, co-author of [2] (these results are under preparation).

One way to prove regularity of solutions to the Navier-Stokes equations, is to show that the Fourier modes of the solutions do not grow 'fast enough' to produce blow-up. These blow-up rates give us a rough estimate of what we mean by 'fast enough'. And the main problem to control the growth rate is in the non-linear term, which is a convolution over all the active frequencies, changing their direction in a complex way so that a great deal of cancellations must be going on; so that growth in a particular direction seems to be impossible. However the desired results of regularity seem to be far from the scope of the common techniques, which usually control the non-linearity with norms in different Banach spaces which do not take into account the presumed cancellations. A new approach to understand the behaviour of the non-linearity is needed in order to have progress towards the solution of the regularity problem in the Navier-Stokes equations.

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