Weak Solutions for a Class of Quadratic Operator-Differential Equations

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Abstract

"*Mathematics is the most beautiful*
and most powerful creation
of the human spirit"
(Stefan Banach)

This monograph presents results of existence and uniqueness of generalized solutions for two classes of quadratic operator-differential equations with constant coefficients:

$$Au(t) + Bu'(t) - Du''(t) = 0 \tag{1}$$

and

$$Au(t) + iBu'(t) + Du''(t) = 0, \tag{2}$$

where A, B and D are self-adjoint operators which satisfy certain conditions under which the equation (1) is called *elliptic-hyperbolic* and the equation (2) is called *hyperbolic*. The main result is the existence and uniqueness of weak solutions, on the positive real axis or on the negative real axis, for the class of operator differential equations called *elliptic-hyperbolic*. These solutions, on the positive real axis, decay exponentially to zero in the infinity. A similar result is obtained on the negative real axis.

We also give results of existence of solutions of (1) and (2) in the case in which A, B and D are self-adjoint bounded operator. These results, specifically for the elliptic-hyperbolic case, are related with a factorization of the quadratic pencil $A + \lambda B + \lambda^2 C$ (where $C = -D$) and of its associated pencil $L(\lambda) = \lambda^2 A + \lambda B + C$. The factorization of this latter pencil is related to the existence of roots for the operator equation

$$AZ^2 + BZ + C = 0. \tag{3}$$

In order to prove that (3) has a solution Z, we define the so-called linearizer $\mathbf{L}$ of the pencil. Then we use the theory of Krein spaces to establish existence of certain $\mathbf{L}$-invariant subspaces which finally lead to the existence of solutions Z of (3).

Additionally, some results on completeness and basis property of the system of eigenvectors associated to the pencil $A + \lambda B - \lambda^2 D$ are presented.

This work is based on the papers of Shkalikov [11] and Langer [4], and on the books of Bognár [1], Azizov [2], and Markus [6].

Dedication

To my mother,
Alix Ardila.

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Contents

Chapter 1

Introduction

"*No one shall expel us from Paradise that Cantor has created for us.*"
(David Hilbert)

This monograph is based on the papers of Shkalikov [11] and Langer [4], and on the books of Bognár [1], Azizov [2], and Markus [6]. The main result is the existence and uniqueness of weak solutions, on the positive real axis or on the negative real axis, for a class of operator differential equations called *elliptic-hyperbolic*. These solutions, on the positive real axis, decay exponentially to zero in the infinity. A similar result is obtained on the negative real axis.

Operator-differential equations of the form (1) and (2) can be considered as a generalization of ordinary lineal differential equations to the infinite-dimensional case [21]. They arise for example in the study of strongly damped mechanical systems [20], electrical systems and signal processing [23], in the study of partial differential equations (Wave and Heat Equations, for example) and in the study of Abstract Cauchy Problems [22].

The spectral analysis of the associated pencil $L(\lambda) = \lambda^2 A + \lambda B + C$ (where $D = -C$) generalizes the study of the *quadratic eigenvalue problem* which is very important in the previously mentioned applications to physics [23]. For the case where A, B and C are bounded self-adjoint operators acting in a separable Hilbert space $\mathcal{H}$, such that $L(\lambda)$ is strongly damped, the spectral analysis and linearization of the pencil $L(\lambda)$ can be realized with geometric methods using Krein space theory [4]. Analogous to the finite-dimensional case, the problem of linearization of the pencil $L(\lambda)$ is related to the problem of factorization of the operator equation (3) and this latter is related to the problem of existence of invariant maximal semi-definite subspaces for an associated linearizing operator $\mathbf{L}$ [4] [6].

1.1 Preliminaries

1. By a *vector space* we shall always mean a complex vector space. A *subspace* of a vector space X is a set $U \subseteq X$ such that it is itself a vector space with the operations defined in X.

A *norm* on a vector space X is a function $\|\cdot\| : X \longrightarrow \mathbb{R}$ such that, for every x, $y \in X$ and $\alpha \in \mathbb{C}$:

$$
\begin{aligned}
&i) && \|x\| = 0 \text{ implies } x = 0,\\
&ii) && \|\alpha x\| = |\alpha|\|x\|,\\
&iii) && \|x+y\| \leq \|x\| + \|y\|.
\end{aligned}
$$

The pair $(X, \|\cdot\|)$ is called a *normed space.*

A subspace U of a normed space $(X, \|\cdot\|)$ is called *closed* (*open*) if and only if it is a closed (open) subset in the topology induced on X by the norm $\|\cdot\|$. Note that the subspace $U \subseteq X$ is closed if, and only if, for every sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ and $u_0 \in X$, $u_n \to u_0$ in the norm implies $u_0 \in U$. A normed space X is called a *Banach space* if, and only if, every Cauchy sequence in X is convergent. A subspace U of a Banach space $(X, \|\cdot\|)$ is closed if and only if $(U, \|\cdot\|)$ is in itself a Banach space.

A *pre-Hilbert space* is a pair $(X, \langle\cdot,\cdot\rangle)$ where X is a vector space and $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{C}$ is a funtion such that for every x, y, $z \in X$ and α, $\beta \in \mathbb{C}$:

$$
\begin{aligned}
&i) && \langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle,\\
&ii) && \langle y, x\rangle = \overline{\langle x, y\rangle},\\
&iii) && \langle x, x\rangle > 0 \text{ if } x \neq 0.
\end{aligned}
$$

The function $\langle\cdot,\cdot\rangle$ is called a *positive definite inner product* (see section 2.2 for the general definition of an inner product).

Every pre-Hilbert space $(X, \langle\cdot,\cdot\rangle)$ is a normed space with the norm induced by $\langle\cdot,\cdot\rangle$; this norm is defined by the formula $\|x\| = \sqrt{\langle x, x\rangle}$, $x \in X$. In this case, if $(X, \|\cdot\|)$ is a Banach space then $(X, \langle\cdot,\cdot\rangle)$ is called a *Hilbert space.*

2. By a *linear operator* from a vector space X into a vector space Y we mean a function $T : \mathscr{D}(T) \subseteq X \longrightarrow Y$, where $\mathscr{D}(T) \subseteq X$ is a subspace of X (called the *domain of* T), such that $T(\alpha u + \beta v) = \alpha T u + \beta T v$ for every u, $v \in \mathscr{D}(T)$ and α, $\beta \in \mathbb{C}$. In the case $Y = X$ we say that T is a linear operator *acting* in X.

Let X be a vector space. The linear operator $P : X \longrightarrow X$ is called a *projector* if and only if $P^2 = P$.

If $T : \mathscr{D}(T) \subseteq X \longrightarrow Y$ is a linear operator then the subspaces $\ker(T) \subseteq X$ and $\mathscr{R}(T) \subseteq Y$ are defined by:

$$\begin{aligned} \ker(T) &= \{x \in \mathscr{D}(T) \mid Tx = 0\}, \\ \mathscr{R}(T) &= \{Tx \mid x \in \mathscr{D}(T)\}. \end{aligned}$$

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let T be a linear operator from X into Y. The operator T is called *bounded* if and only if there exists $M \geq 0$ such that $\|Tx\|_Y \leq M\|x\|_X$ for every $x \in \mathscr{D}(T)$. The bounded operator T is called *compact* if, and only if, for every bounded sequence $(x_n)_{n\in\mathbb{N}} \in \mathscr{D}(T)$, the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y. An operator T is called *closed* if, and only if, for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{D}(T)$ such that $(x_n)_{n\in\mathbb{N}}$ is convergent in X and $(Tx_n)_{n\in\mathbb{N}}$ is convergent in Y, then:

$$\lim_{n\to\infty} x_n \in \mathscr{D}(T) \;\text{ and }\; \lim_{n\to\infty} Tx_n = T\big(\lim_{n\to\infty} x_n\big).$$

If X and Y are normed spaces then we set:

- $\mathcal{B}(X, Y) = \{T : X \longrightarrow Y \mid T \textit{ is a bounded linear operator}\}$.
- $\mathcal{B}_\infty(X, Y) = \{T : X \longrightarrow Y \mid T \textit{ is a compact linear operator}\}$.
- $\mathcal{B}(X) = \mathcal{B}(X, X)$.
- $\mathcal{B}_\infty(X) = \mathcal{B}_\infty(X, Y)$.

3. Let T be a densely defined linear operator acting in a Hilbert space $\mathcal{H}$. Define the *adjoint* T^* of T by:

- $\mathscr{D}(T^*) = \{y \in \mathcal{H} \mid x \mapsto \langle Tx, y\rangle$ is a bounded operator on $\mathscr{D}(T)\}$.
- If $y \in \mathscr{D}(T^*)$, then T^*y is the unique u such that $\langle Tx, y\rangle = \langle x, u\rangle$ for every $x \in \mathscr{D}(T)$.

Clearly, since T is densely defined, T^* is well-defined. Moreover T^* is a densely defined closed operator.

If $T \in \mathcal{B}(\mathcal{H})$, then we have

$$\mathcal{H} = \ker(T) \oplus \overline{\mathscr{R}(T^*)} = \ker(T^*) \oplus \overline{\mathscr{R}(T)}.$$

4. The densely defined linear operator T acting in a Hilbert space $\mathcal{H}$ is called:

- *symmetric* if and only if $T \subseteq T^*$.
- *essentially self-adjoint* if and only if $\overline{T} = T^*$.
- *self-adjoint* if and only if $T = T^*$.
- *unitary* if, and only if, $T \in \mathcal{B}(\mathcal{H})$ and $TT^* = T^*T = I$.

Chapter 2

Inner Product Spaces

2.1 Vector Spaces

Definition 2.1.1. Let X be a vector space and $U_1, \dots, U_n$ subspaces of X. Then:

(*i*) $U_1 + \cdots + U_n := \{x_1 + \cdots + x_n \mid x_j \in U_j,\ j = 1, \dots, n\}$ is the *vector sum* of subspaces $U_1, \dots, U_n$.

(*ii*) $U_1, \dots, U_n$ are *linearly independent subspaces* if, and only if, $x_1 + \cdots + x_n = 0$ with $x_j \in U_j$ $(j = 1, \dots, n)$ implies $x_1 = \cdots = x_n = 0$.

Remarks:

- Clearly, the vector sum of subspaces of X is again a subspace of X.
- In the definition 2.1.1, (*ii*) is equivalent to:

$$(U_1 + \cdots + U_{j-1}) \cap U_j = \{0\} \text{ for every } j = 2, \dots, n.$$

- The vector sum of linearly independent subspaces $U_1, \dots, U_n \subseteq X$ is called a *direct vector sum* and is denoted by $U_1 \dotplus \cdots \dotplus U_n$.
- By Zorn's lemma, if $U \subseteq X$ is a subspace then there exists a subspace $V \subseteq X$ such that $U \dotplus V = X$ (and therefore $U \cap V = \{0\}$). V is called a *complementary subspace* for U

with respect to X. Moreover, if $W \subseteq X$ is a subspace with $U \cap W = \{0\}$ then there exists $V \supseteq W$ subspace of X such that $U \dot{+} V = X$.

Definition 2.1.2. Let X be a vector space, $U \subseteq X$ a subspace and X/U the quotient space of X with respect to U. Then the dimension of X/U is called the *codimension* of U with respect to X and is denoted by $\mathrm{codim}(U)$; this is,

$$\mathrm{codim}(U) := \dim(X/U).$$

Proposition 2.1.3. Let X be a vector space and $U \subseteq X$ a subspace. Then every complementary subspace for U with respect to X is isomorphic to X/U.

Proof. Let $V \subseteq X$ be a complementary subspace for U with respect to X. Let $T : V \longrightarrow X/U$ be such that, for every $v \in V$, $Tv := [v]$ (here $[v]$ denote the class of equivalency of v); clearly T is well-defined and it is a linear transformation. T is injective since $\ker(T) = \{v \in V : [v] = 0\} = \{v \in V \mid v \in U\} = U \cap V = \{0\}$. T is surjective because if $w \in X/U$ then there exists $x \in X$ and, therefore, $u \in U$ and $v \in V$ such that $w = [x] = [u + v] = [v] = Tv$. So T is an isomorphism between V and X/U. □

Corollary 2.1.4. The dimension of every complementary subspace for a subspace U is equal to $\mathrm{codim}(U)$.

2.2 Inner Products on Vector Spaces

Definition 2.2.1. An *inner product* on a vector space X is a function

$$[\cdot,\cdot] : X \times X \longrightarrow \mathbb{C}$$

such that:

(i) For every x, y in X, $[y, x] = \overline{[x, y]}$.

(ii) For every x, y, z in X and α, β in $\mathbb{C}$, $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$.

Remark. The pair $(X, [\cdot,\cdot])$ where X is a vector space and $[\cdot,\cdot]$ is an inner product on X will be called an *inner product space*. If $(X, [\cdot,\cdot])$ is an inner product space then $(X, -[\cdot,\cdot])$ is called the *anti-space* of $(X, [\cdot,\cdot])$.

Definition 2.2.2. Let $(X, [\cdot,\cdot])$ and $(X', [\cdot,\cdot]')$ be inner product spaces. A linear operator $T : X \to X'$ is called an *isometry* if, and only if, $[Tx, Ty]' = [x, y]$ for every $x, y \in X$. If further T is a linear isomorphism then X and X' are said be *isometrically isomorphic*. In this case, T is called an *isometrical isomorphism* between X and X'.

Proposition 2.2.3 (Polarization Formula)**.** Let X be an inner product space. Then, for every x, y in X:

$$4[x, y] = [x + y,\ x + y] - [x - y,\ x - y] + i[x + iy,\ x + iy] - i[x - iy,\ x - iy].$$

The proof of the polarization formula is a straightforward calculation of the right hand side.

Remark. If X is an inner product space then, for every $x \in X$, $[x, x] \in \mathbb{R}$ since $[x, x] = \overline{[x, x]}$.

Definition 2.2.4. Let X be an inner product space and $x \in X$. Then x is called:

(i) *positive* if $[x, x] > 0$.

(ii) *negative* if $[x, x] < 0$.

(iii) *neutral* if $[x, x] = 0$.

Remark. Clearly, 0 is neutral. Note that if X is an inner product space and x is a positive (negative, neutral) vector then, for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, the vector αx is positive (negative, neutral) too.

Definition 2.2.5. Let $(X, [\cdot,\cdot])$ be an inner product space. $[\cdot,\cdot]$ is called *indefinite* if and only if there exist x, y in X such that $[x, x] > 0$ and $[y, y] < 0$.

Example 1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let U, V be nontrivial subspaces of $\mathcal{H}$ such that $\mathcal{H} = U \dot{\oplus} V$ (where $\dot{\oplus}$ denotes the orthogonal direct sum of subspaces). Then, for every $x \in \mathcal{H}$, there are unique $u \in U$ and $v \in V$ such that $x = u + v$. Now, if x, y are elements in $\mathcal{H}$ with

$x = u_1 + v_1$ and $y = u_2 + v_2$, then define $[x, y] := \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle$. Clearly, $[\cdot,\cdot]$ is an inner product on $\mathcal{H}$. Note that, for every $u \in U \setminus \{0\}$, $[u, u] > 0$ and, for every $v \in V \setminus \{0\}$, $[v, v] < 0$. So, $[\cdot,\cdot]$ is an indefinite inner product on $\mathcal{H}$.

Proposition 2.2.6. Let X be an indefinite inner product space. Then there exists a non-zero neutral vector in X.

Proof. Let x, $y \in X$ such that $[x, x] > 0$ and $[y, y] < 0$. Note that, for every $\lambda \in \mathbb{R}$:

$$[x + \lambda y,\ x + \lambda y] = [y, y]\lambda^2 + 2\,\mathrm{Re}([x, y])\lambda + [x, x].$$

Since $\mathrm{Re}([x, y])^2 - [x, x][y, y] > 0$, the discriminant of the quadratic equation $[y, y]\lambda^2 + 2\,\mathrm{Re}([x, y])\lambda + [x, x] = 0$ is positive and, therefore, there exists $\lambda_0 \in \mathbb{R}$ such that $[x + \lambda_0 y,\ x + \lambda_0 y] = 0$, hence $z_0 := x + \lambda_0 y$ is a neutral vector. Suppose, towards a contradiction, that $z_0 = 0$; then $x = -\lambda_0 y$ and, therefore, $[x, x] = \lambda_0^2 [y, y]$ ↯. □

Definition 2.2.7. Let $(X, [\cdot,\cdot])$ be an inner product space. If $[\cdot,\cdot]$ is not indefinite then it is called *semi-definite* and $(X, [\cdot,\cdot])$ is called a *semi-definite inner product space*.

Remark. If $[\cdot,\cdot]$ is a semi-definite inner product on a vector space X then, for every $x \in X$, $[x, x] \geq 0$ (and in this case $[\cdot,\cdot]$ is called *positive semi-definite*) or, for every $x \in X$, $[x, x] \leq 0$ (and in this case $[\cdot,\cdot]$ is called *negative semi-definite*). If, for every $x \in X$, $[x, x] = 0$ then $[\cdot,\cdot]$ is called *neutral* (note that a neutral inner product is both a positive and a negative semi-definite inner product).

Proposition 2.2.8. Let X be a semi-definite inner product space and let $x \in X$ be a neutral vector. Then, for every $y \in X$, $[x, y] = 0$.

Proof. Suppose $x \neq 0$ (since the case $x = 0$ is clear) and suppose, without loss of generality, that the space is positive semi-definite (otherwise take $-[\cdot,\cdot]$). Let $y \in X$ be any vector.
If $[y, y] = 0$ then, for every $\lambda \in \mathbb{R}$, $0 \leq [x + \lambda y,\ x + \lambda y] = 2\lambda\,\mathrm{Re}([x, y])$ and $0 \leq [x + i\lambda y,\ x + i\lambda y] = 2\lambda\,\mathrm{Im}([x, y])$; so $\mathrm{Re}([x, y]) = 0$ and $\mathrm{Im}([x, y]) = 0$. Now, if $[y, y] > 0$ then, for every $\lambda \in \mathbb{R}$, $q(\lambda) := [x + \lambda y,\ x + \lambda y] = \lambda^2 [y, y] + 2\lambda\,\mathrm{Re}([x, y]) \geq 0$ and $p(\lambda) := [x + i\lambda y,\ x + i\lambda y] = \lambda^2 [y, y] + 2\lambda\,\mathrm{Im}([x, y]) \geq 0$; this is, $q(\lambda)$ and $p(\lambda)$ represent parabolas with a double zero at

$\lambda = 0$ and so $\text{Re}([x,y])^2$ and $\text{Im}([x,y])^2$, the discriminants, are zero; this is, $\text{Re}([x,y]) = 0$ and $\text{Im}([x,y]) = 0$. Therefore, $[x,y] = 0$. □

Corollary 2.2.9. Let X be an inner product space and let x, $y \in X$ such that $[x,x] = 0$ and $[x,y] \neq 0$. Then the subspace $\text{Span}(\{x,y\})$ is indefinite.

Proof. Let $U := \text{Span}(\{x,y\})$. If U is not indefinite then U is a semi-definite subspace and, by proposition 2.2.8, $[x,y] = 0$ ↯. □

Corollary 2.2.10. If X is a neutral inner product space then, for every x, $y \in X$, $[x,y] = 0$.

Proposition 2.2.11 (Schwarz Inequality)**.** Let X be a semi-definite inner product space. Then, for every x, $y \in X$:

$$|[x,y]|^2 \leq [x,x][y,y].$$

Proof. Suppose, without loss of generality, that the inner product is positive semi-definite. If x or y is a neutral vector then, by proposition 2.2.8, the inequality holds.
Suppose that x and y are not neutral vectors. Let $u = \frac{[y,x]}{[x,x]}x$ and $v = y - \frac{[y,x]}{[x,x]}x$, then $[u,v] = \left[\frac{[y,x]}{[x,x]}x,\ y - \frac{[y,x]}{[x,x]}x\right] = 0$, $y = u + v$ and, therefore, $[y,y] = [u,u] + [v,v] \geq [u,u] = \frac{|[x,y]|^2}{[x,x]}$. So, $|[x,y]|^2 \leq [x,x][y,y]$. □

Definition 2.2.12. An inner product on a vector space X is called *definite* if and only if, for every $x \in X$, $[x,x] = 0$ implies $x = 0$.

Remarks:

(i) If U is a definite subspace of an inner product space $(X, [\cdot,\cdot])$, then U can be endowed with a norm, $\|\cdot\|_U$, defined by the formula $\|x\|_U = \sqrt{|[x,x]|}$, $x \in U$. This norm is called the *norm induced* by the inner product on U.

(ii) By proposition 2.2.6, a definite inner product is not indefinite; this is, a definite inner product is always semi-definite. So, if the inner product is positive definite then, for every $x \neq 0$, $[x,x] > 0$ and if the inner product is negative definite then, for every $x \neq 0$, $[x,x] < 0$.

(iii) A positive definite inner product space is what is known classically as a pre-Hilbert space (see page 2).

(iv) Clearly, if $[\cdot,\cdot]$ is a positive definite inner product on a vector space X then $-[\cdot,\cdot]$ is a negative definite inner product on X.

Proposition 2.2.13. Let X be an inner product space. If X contains at least one positive vector, then every element of X is the sum of two positive vectors.

Proof. Let $x_0 \in X$ be a positive vector. Then, for every $\alpha \in \mathbb{R}\setminus\{0\}$, αx_0 is positive. If $x \in X$, then, clearly, there is $\alpha \in \mathbb{R}\setminus\{0\}$ such that $[x+\alpha x_0, x+\alpha x_0] = [x,x] + 2\operatorname{Re}([x,x_0])\alpha + [x_0,x_0]\alpha^2$ is positive. So there exists $\alpha \in \mathbb{R}\setminus\{0\}$ such that $-\alpha x_0$ and $x+\alpha x_0$ are positive, and $x = (x+\alpha x_0) + (-\alpha x_0)$. $\square$

Remark. The proposition 2.2.13 holds if positive is replaced by negative.

2.3 Orthogonality

Definition 2.3.1. Let X be an inner product space; x, $y \in X$ and A, $B \subseteq X$. Then:

(i) x and y are *orthogonal* ($x \perp y$) if and only if $[x,y] = 0$.

(ii) A and B are *orthogonal* ($A \perp B$) if and only if, for every $x \in A$ and $y \in B$, $x \perp y$.

Definition 2.3.2. Let X be an inner product space and $A \subseteq X$. The *orthogonal complement* of A, denoted $A^\perp$, is defined by

$$A^\perp := \{x \in X : x \perp A\} = \{x \in X : x \perp y \text{ for all } y \in A\}.$$

The following proposition is clear so we omit its proof.

Proposition 2.3.3. Let X an inner product space. Then:

(i) For every $A \subseteq X$, $A^\perp$ is a suspace of X.

(ii) For every $A, B \subseteq X$ with $A \subseteq B$, $B^\perp \subseteq A^\perp$.

(iii) For every U, V subspaces of X, $(U+V)^{\perp} = U^{\perp} \cap V^{\perp}$.

(iv) For every $A \subseteq X$, $A \subseteq (A^{\perp})^{\perp} =: A^{\perp\perp}$.

(v) For every $A \subseteq X$, $A^{\perp} = A^{\perp\perp\perp}$.

Example 2. Let X be a two-dimensional vector space with basis $\{e_1, e_2\}$. On X define an inner product by the relations $[e_1, e_1] = 1$, $[e_2, e_2] = -1$ and $[e_1, e_2] = 0$. Let $U := \text{Span}(e_1 + e_2)$. Then U is a subspace of X such that $U^{\perp} = U$ (because, $[\alpha e_1 + \beta e_2, e_1 + e_2] = 0 \Leftrightarrow \alpha = \beta$).

Definition 2.3.4. Let X be an inner product space and let U, $V \subseteq X$ be subspaces. U and V are *dual companions* (or U and V form a *dual pair*) if and only if $U \cap V^{\perp} = U^{\perp} \cap V = \{0\}$.

Remarks:

- If X is a semi-definite inner product space and $x \in X$ is a neutral vector then, by proposition 2.2.8, $x \in A^{\perp}$ for every $A \subseteq X$.

- If $x, y \in X$ are orthogonal then $[x+y, x+y] = [x,x] + [y,y]$.

- If U is a subspace of X then it is possible that $U \cap U^{\perp} \neq \{0\}$ (see example 2).

- By example 2, if $U \subseteq X$ is a subspace and $V \subseteq X$ is its orthogonal complement with respect to X then U and V are not necessarily linearly independent (nevertheless, if X is a definite inner product space and $U \subseteq X$ is a subspace then its orthogonal complement is a complementary subspace for U).

- The vector sum (resp. direct vector sum) of pairwise orthogonal subspaces $U_1, \ldots, U_n \subseteq X$ is called an *orthogonal sum* (resp. *orthogonal direct sum*) and it is denoted by $U_1 \oplus \cdots \oplus U_n$ (resp. $U_1 \dot{\oplus} \ldots \dot{\oplus} U_n$).

- Let $(X_j, [\cdot,\cdot]_j)$ be inner product spaces, $j = 1, \ldots, n$. Let $X := X_1 \times \cdots \times X_n$ and for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ in X define $[x, y] := \sum_{j=1}^{n} [x_j, y_j]_j$. Clearly $(X, [\cdot,\cdot])$ is an inner product space. Now, for every $k = 1, \ldots, n$, let $X_{(k)} := \{(x_1, \ldots, x_n) \in X : x_j = 0, j \neq k\}$. So, $X_{(k)}$ is a subspaces of X which is isometrically isomorphic to X_k ($k = 1, \ldots, n$) and $X = X_{(1)} \dot{\oplus} \cdots \dot{\oplus} X_{(n)}$.

2.4 Isotropic Vectors

Definition 2.4.1. Let X be an inner product space and $U \subseteq X$ a subspace. The *isotropic part* of U is the subspace $U^0 := U \cap U^\perp$.

Remarks:

- If $x \in U^0$ then x is called an *isotropic vector* of U.
- If x is an isotropic vector for some subspace then x is a neutral vector, but not every neutral vector is isotropic (in the example 2, $e_1 + e_2$ is neutral but is not isotropic for X).
- If $U^0 \neq \{0\}$ then U is called *degenerate*.
- X is degenerate if, and only if, its isotropic part, $X^0 = X^\perp$, is not equal to $\{0\}$.
- In the example 2, X is not degenerate (since $X^\perp = \{0\}$) but U is a degenerate subspace of X.
- If U is a definite subspace of an inner product space X, then U is non-degenerate.
- In X/X^0 an inner product is defined by $[\tilde{x}, \tilde{y}]_\sim := [x, y]$, where $\tilde{x},\ \tilde{y} \in X/X^0$, $x \in \tilde{x}$ and $y \in \tilde{y}$. Note that $[\cdot,\cdot]_\sim$ is well-defined since if $x, u \in \tilde{x}$ and $y, v \in \tilde{y}$ then $x - u,\ y - v \in X^0$ and, therefore, $[x, y] - [u, v] = [x - u, y] + [u, y - v] = 0$. That $[\cdot,\cdot]_\sim$ is an inner product is clear since $[\cdot,\cdot]$ is so.

Theorem 2.4.2. Let $(X, [\cdot,\cdot])$ be an inner product space and let U be a non-degenerate subspace of X such that $U = U_+ + U_-$ with U_+ a positive semi-definite subspace and U_- a negative semi-definite subspace. Then $U = U_+ \dot{+} U_-$, U_+ is maximal positive semi-definite in $(U, [\cdot,\cdot])$ and U_- is maximal negative semi-definite in $(U, [\cdot,\cdot])$.

Proof. Suppose, towards a contradiction, that there exists $w \in U_+ \cap U_-$ such that $w \neq 0$. Then w is neutral and, therefore, $w \in U_+^\perp$ and $w \in U_-^\perp$ (see proposition 2.2.8); thus $w \in U^0$ $\lightning$. Hence, $U = U_+ \dot{+} U_-$.

Suppose, towards a contradiction, that U_+ is not maximal positive semi-definite in U, and let $V \subseteq U$ be a positive semi-definite subspace such that $V \supsetneq U_+$. Since $U = U_+ \dot{+} U_-$, there exists $v \in V \cap U_-$ (therefore, v is neutral) such that $v \neq 0$. Thus, $v \perp V$, $v \perp U_-$ and, therefore, $v \perp U$ ↯. Hence, U_+ is maximal positive semi-definite and, analogously, U_- is maximal negative semi-definite. □

Proposition 2.4.3. Let X be an inner product space and U, U_1, …, U_n subspaces of X such that $U = U_1 \dot{\oplus} \cdots \dot{\oplus} U_n$. Then $U^0 = U_1^0 \dot{\oplus} \cdots \dot{\oplus} U_n^0$.

Proof. Let $i, j \in \{1, \dots, n\}$ with $i \neq j$. If $x \in U_i^0$ and $y \in U_j^0$ then $x \in U_i$ and $y \in U_j$. So $[x, y] = 0$ since U_i and U_j are orthogonal. Hence U_i^0 and U_j^0 are orthogonal.
If $x_1 + \cdots + x_n = 0$ with $x_j \in U_j^0$ then $x_1 = \cdots = x_n = 0$ since $x_j \in U_j$ and U_1, …, U_n are linearly independent. Hence U_1^0, …, U_n^0 are linearly independent.
So far, U_1^0, …, U_n^0 are mutually orthogonal and linearly independent subspaces of X.
Let $x = x_1 + \cdots + x_n \in U_1^0 \dot{\oplus} \dots \dot{\oplus} U_n^0$. Then $x \in U_1 \dot{\oplus} \dots \dot{\oplus} U_n = U$ and, for every $u_j \in U_j$ ($j = 1, \dots, n$), $[u_1 + \cdots + u_n, x] = [u_1, x] + \cdots + [u_n, x] = [u_1, x_1] + \cdots + [u_n, x_n] = 0$ since $x_j \in U_j^\perp$ ($j = 1, \dots, n$). Then $x \in U^\perp$ and, therefore, $x \in U^0$. So, $U_1^0 \dot{\oplus} \dots \dot{\oplus} U_n^0 \subseteq U^0$.
Let $x \in U^0 = U \cap U^\perp$, then there exists $x_j \in U_j$ ($j = 1, \dots, n$) such that $x = x_1 + \cdots + x_n$. Fix $j \in \{1, \dots, n\}$; then for every $u \in U_j \subseteq U$ we have $[x, u] = 0$. But, if $u \in U_j$ then $[x, u] = [x_j, u]$; so $x_j \in U_j^\perp$ and, therefore, $x_j \in U_j^0$. Then, $x \in U_1^0 \dot{\oplus} \dots \dot{\oplus} U_n^0$. So, $U^0 \subseteq U_1^0 \dot{\oplus} \dots \dot{\oplus} U_n^0$.
Therefore, $U^0 = U_1^0 \dot{\oplus} \dots \dot{\oplus} U_n^0$. □

Corollary 2.4.4. The orthogonal direct sum of non-degenerate subspaces is non-degenerate.

Proposition 2.4.5. Let X be a semi-definite inner product space. Then,

$$X^0 = \{x \in X \mid [x, x] = 0\}.$$

Proof. Clearly, $X^0 \subseteq \{x \in X \mid [x, x] = 0\}$ and, by proposition 2.2.8, $\{x \in X \mid [x, x] = 0\} \subseteq X^\perp = X^0$. So, $X^0 = \{x \in X : [x, x] = 0\}$. □

Corollary 2.4.6. Let X be a neutral inner product space (this is, $[x, x] = 0$ for every $x \in X$). Then, $X^0 = X$.

Proof. This follows from proposition 2.4.5 because every neutral space is a semi-definite space. □

Proposition 2.4.7. Let X be an inner product space and $U \subseteq X$ a subspace. If U is neutral then $U^{\perp\perp}$ is neutral.

Proof. If U is neutral then, by proposition 2.2.8, $U \subseteq U^{\perp}$. Therefore, $U^{\perp\perp} \subseteq (U^{\perp\perp})^{\perp}$, which shows that $U^{\perp\perp}$ is neutral. □

Proposition 2.4.8. Let U be a degenerate subspace of an inner product space X. Then $U^{\perp\perp}$ is degenerate.

Proof. As $U \subseteq U^{\perp\perp}$ and $U^{\perp} = U^{\perp\perp\perp}$, it follows that $\{0\} \neq U \cap U^{\perp} \subseteq U^{\perp\perp} \cap U^{\perp\perp\perp}$. So $U^{\perp\perp}$ is degenerate. □

2.5 Maximal Non-Degenerate Subspaces

Proposition 2.5.1. Let X be an inner product space; X^0 its isotropic part and V a complementary subspace for X^0. Then V is a non-degenerate subspace and $X = X^0 \dot{\oplus} V$.

Proof. Since $X^0 = X^{\perp}$ then, for every $u \in X^0$ and $v \in V$, $[u, v] = 0$. So, X^0 and V are orthogonal subspaces and, as V is a complementary subspace for X^0, we have $X = X^0 \dot{\oplus} V$.
Suppose, towards a contradiction, that V is degenerate, and let $x \neq 0$ be such that $x \in V^0$. Then, for every $u \in X^0$ and $v \in V$, $[u + v, x] = [u, x] + [v, x] = 0 + 0 = 0$. So $x \in X^0$, but this is a contradiction since X^0 and V are linearly independent. □

Proposition 2.5.2. Let X be an inner product space; X^0 its isotropic part and V a complementary subspace for X^0. Then V is isometrically isomorphic to X/X^0.

Proof. By proposition 2.1.3, V is isomorphic to X/X^0 and an isomorphism is given by $T : V \longrightarrow X/X^0$, $Tx := \tilde{x}$. Note that, for every $x, y \in V$, $[Tx, Ty]_{\sim} = [\tilde{x}, \tilde{y}]_{\sim} = [x, y]$ (see remark before definition 2.4.1). So T is an isometrical isomorphism. □

Corollary 2.5.3. If X is an inner product space, then X/X^0 is non-degenerate.

Theorem 2.5.4. Let X be an inner product space and $V \subseteq X$. V is a complementary subspace of X^0 if and only if V is a maximal non-degenerate subspace of X.

Proof. $\boxed{\Rightarrow}$ If V is a complementary subspace for X^0 then, by proposition 2.5.1, V is a non-degenerate subspace of X and $X = X^0 \dot{\oplus} V$.

Suppose, towards a contradiction, that there exists $W \subseteq X$ non-degenerate subspace such that $V \subsetneq W$. As $X = X^0 \dot{+} V$ and $V \subsetneq W$ then $W \cap X^0 \neq \{0\}$. Let $w \in W \cap X^0$ with $w \neq 0$; then, for every $x \in W$, $[x, w] = 0$. So $w \in W^0$ and, therefore, $W^0 \neq \{0\}$ ↯.

$\boxed{\Leftarrow}$ As V is a non-degenerate subspace of X it follows that $V \cap X^0 = \{0\}$, so V and X^0 are linearly independent. If $X^0 \dot{+} V \neq X$ then, by the remark before definition 2.1.1, there exists a subspace $W \subseteq X$ with $V \subsetneq W$ such that $X = X^0 \dot{+} W$. So $W \supsetneq V$ and, by proposition 2.5.1, W is non-degenerate ↯. □

Corollary 2.5.5. If X is an inner product space then X contains maximal non-degenerate subspaces and every non-degenerate subspace admits a maximal non-degenerate extension.

2.6 Maximal Semi-Definite Subspaces

Proposition 2.6.1. Let X be an inner product space and $U, V \subseteq X$ be two maximal positive semi-definite subspaces (or two maximal negative semi-definite subspaces). Then $(U + V)^0 = \{x \in U \cap V \mid [x, x] = 0\}$.

Proof. Suppose, without loss of generality, that U and V are both maximal positive semi-definite subspaces.

Let $x \in U \cap V$ be such that $[x, x] = 0$; clearly $x \in U + V$. As U and V are semi-definite subspaces then, by proposition 2.2.8, $x \in U^0 \cap V^0$ and, therefore, for every $u \in U$ and $v \in V$, $[u + v,\ x] = [u, x] + [v, x] = 0 + 0 = 0$. So $x \in (U + V)^0$.

Let $x \in (U + V)^0$, then $[x, x] = 0$ and $x \in U^\perp \cap V^\perp$. Suppose, towards a contradiction, that $x \notin U$; as, for every $u \in U$ and $\alpha \in \mathbb{C}$, $[u + \alpha x,\ u + \alpha x] = [u, u] \geq 0$ then $U + \text{Span}(\{x\}) \supsetneq U$ and $U + \text{Span}(\{x\})$ is a positive semi-definite subspace ↯ (since U is a maximal positive semi-definite subspace). So $x \in U$ and, analogously, $x \in V$. Therefore, $(U + V)^0 = \{x \in U \cap V \mid [x, x] = 0\}$. □

Proposition 2.6.2. Let X be an inner product space and let $U, V \subseteq X$ be subspaces such that $X = U \dotplus V$. If U is positive definite and V is negative semi-definite, then U is a maximal positive definite subspace and V is a maximal negative semi-definite subspace.

Analogously, if U is positive semi-definite and V is negative definite, then U is a maximal positive semi-definite subspace and V is a maximal negative definite subspace.

Proof. Suppose that U is positive definite and V is negative semi-definite, and suppose, towards a contradiction, that U is not a maximal positive definite subspace. Let $W \subseteq X$ be a positive definite subspace such that $U \subsetneq W$; as $X = U \dotplus V$ then $W \cap V \neq \{0\}$ and, therefore, there exists $w \in W$ such that $[w, w] > 0$ and $[w, w] \leq 0$ ↯. So U is maximal positive definite and, analogously, V is maximal negative semi-definite.

The case where U is positive semi-definite and V is negative definite is similar. □

Proposition 2.6.3. Let X be an inner product space and $U \subseteq X$. If U is a maximal positive definite subspace, then $U^{\perp}$ is negative semi-definite.

Analogously, if U is a maximal negative definite subspace then $U^{\perp}$ is positive semi-definite.

Proof. Let U be a maximal positive definite subspace. Suppose, towards a contradiction, that there exists $v \in U^{\perp}$ such that $[v, v] > 0$. Then $U + \mathrm{Span}(\{v\})$ is a positive definite subspace (since, for every $u \in U$ and $\alpha \in \mathbb{C}$, $[u + \alpha v,\ u + \alpha v] = [u, u] + |\alpha|^2 [v, v]$) and, as $v \notin U$ (since $U^0 = \{0\}$), $U + \mathrm{Span}(\{v\}) \supsetneq U$ ↯.

The case where U is maximal negative definite is similar. □

Corollary 2.6.4. Let X be an inner product space and $U \subseteq X$. If U is a positive definite and maximal positive semi-definite (resp., negative definite and maximal negative semi-definite) subspace, then $U^{\perp}$ is negative definite (resp., positive definite).

Proof. Suppose that U is a positive definite and maximal positive semi-definite subspace; then $U \cap U^{\perp} = \{0\}$, U is maximal positive definite (by the maximality as positive semi-definite subspace) and, by proposition 2.6.3, $U^{\perp}$ is negative semi-definite. Let $v \in U^{\perp}$ such that $[v, v] = 0$ and suppose, towards a contradiction, that $v \neq 0$; then $U + \mathrm{Span}(\{v\})$ is positive semi-definite and $U + \mathrm{Span}(\{v\}) \supsetneq U$ ↯. So $U^{\perp}$ is negative definite.

The case where U is negative definite and maximal negative semi-definite is analogous. □

2.7 Projections of Vectors on Subspaces

Definition 2.7.1. Let X be an inner product space; $U \subseteq X$ be a subspace and $x \in X$. If there exists $u \in U$ and $v \in U^{\perp}$ such that $x = u + v$ then u is called a *projection* of x on U.

Remarks:

- Note that, given a subspace U of an inner product space X and given $x \in X$ then, x has a projection on U if and only if $x \in U + U^{\perp}$. So a element $x \in X$ does not need to have a projection on U since, in general, it is possible that $X \neq U + U^{\perp}$ (see example 2).

- If $u \in U$ is a projection of the vector x on the subspace U and $v \in U^{\perp}$ is such that $x = u + v$ then, for every $w \in U^0$, $u + w$ is a projection of x on U (since $u + w \in U$ and $v - w \in U^{\perp}$).

- By the previous remark, the projection of a vector on a subspace, if it exists, is not unique in general; this is the case when the subspace is degenerate. So, if there exists $x \in X$ having exactly one projection on a subspace $U \subseteq X$ then U is non-degenerate. Moreover, if the subspace U is non-degenerate then every vector x has at most one projection on U.

- Clearly, if $u_1, u_2 \in U$ are projections of the vector x on the subspace U then $u_1 - u_2 \in U^0$ (since there exist $v_1, v_2 \in U^{\perp}$ such that $x = u_1 + v_1 = u_2 + v_2$ and so $u_1 - u_2 = v_2 - v_1 \in U^{\perp}$; therefore $u_1 - u_2 \in U \cap U^{\perp} = U^0$).

Proposition 2.7.2. Let X be an inner product space; $U \subseteq X$ a neutral subspace and $x \in X$. Then x admits a projection on U if and only if $x \perp U$. In this case, every element of U is a projection of x on U.

Proof. As U is neutral, by corollary 2.2.10, $U \subseteq U^{\perp}$. So, by the preceding remark, x has a projection on U if and only in $x \in U^{\perp}$. □

Theorem 2.7.3. Let X be an inner product space; $U \subseteq X$ a positive definite subspace and $x \in X$. Then x admits a projection on U if and only if the function $\varphi : U \longrightarrow \mathbb{R}$, $\varphi(u) := [x - u, x - u]$ attains its minimum at a unique $u_0 \in U$. This u_0 is the unique projection of x on U.

Proof. As U is a definite subspace, U is non-degenerate.

$\boxed{\Rightarrow}$ Suppose that x admits a projection on U; as U is non-degenerate, the projection of x on U is unique. Let u_0 and v_0 be the unique vectors in U and $U^\perp$, respectively, such that $x = u_0 + v_0$. Note that $\varphi(u_0) = [v_0, v_0]$ and, for every $u \in U$ with $u \neq u_0$, as U is positive definite and $u - u_0 \neq 0$, $\varphi(u) = [(u_0 + v_0) - u, (u_0 + v_0) - u] = [v_0 + (u_0 - u), v_0 + (u_0 - u)] = [v_0, v_0] + [u_0 - u, u_0 - u] > [v_0, v_0] = \varphi(u_0)$. So u_0 is the unique point of minimum for φ.

$\boxed{\Leftarrow}$ Suppose that φ has a unique point of minimum $u_0 \in U$. Suppose, towards a contradiction, that $x - u_0 \notin U^\perp$, and let $u \in U$ such that $[x - u_0, u] \neq 0$ (therefore $u \neq 0$). Then, for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $[x - (u_0 + \lambda u), x - (u_0 + \lambda u)] > [x - u_0, x - u_0]$; this is, for every $\lambda \neq 0$, $|\lambda|^2[u, u] - \lambda[u, x - u_0] - \bar{\lambda}[x - u_0, u] > 0$. So, taking $\lambda = \alpha[x - u_0, u]$, we have that for every $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $\alpha^2[u, u] - \alpha > 0$ ↯. Therefore $x - u_0 \in U^\perp$ and so u_0 is a projection of x on U. Clearly this projection is unique since U is non-degenerate. □

Corollary 2.7.4. Let X be an inner product space; $U \subseteq X$ be a negative definite subspace and $x \in X$. Then x admits a projection on U if and only if the function $\varphi : U \longrightarrow \mathbb{R}$, $\varphi(u) := [x - u,\ x - u]$ attains its maximum at a unique $u_0 \in U$. This u_0 is the unique projection of x on U.

Theorem 2.7.5. Let X be an inner product space and U, U_1, $\ldots$, $U_n \subseteq X$ be subspaces such that $U = U_1 \dot{\oplus} \cdots \dot{\oplus} U_n$. Let $x \in X$ and $u \in U$; then u is a projection of x on U if and only if there exists u_i projection of x on U_i $(i = 1, \ldots, n)$ such that $u = u_1 + \cdots + u_n$.

Proof. Note that, for every $i = 1, \ldots, n$, $U^\perp \subseteq U_i^\perp$ since $U_i \subseteq U$.

$\boxed{\Rightarrow}$ Suppose that u is a projection of x on U and let $v \in U^\perp$ such that $x = u + v$. As $U = U_1 \dot{\oplus} \cdots \dot{\oplus} U_n$, there exist unique vectors u_1, $\ldots$, u_n such that $u = u_1 + \cdots + u_n$ and, for $i \neq j$, $[u_i, u_j] = 0$. So $x = u_1 + \cdots + u_n + v$ and u_1, $\ldots$, u_n, v are orthogonal to each other. Therefore, for every $i = 1, \ldots, n$, u_i is a projection of x on U_i.

$\boxed{\Leftarrow}$ Suppose that there exists $u_i \in U_i$ projection of x on U_i $(i = 1, \ldots, n)$ such that $u = u_1 + \cdots + u_n$. Note that for every $i = 1, \ldots, n$, if $w_i \in U_i$ then $[w_i, x - u] = [w_i, x - u_i] = 0$ since $x - u_i \in U_i^\perp$; so $x - u \perp U_i$, $i = 1, \ldots, n$, and therefore $x - u \perp U$. Hence u is a projection of x on U. □

2.8 Ortho-Complemented Subspaces

Definition 2.8.1. Let X be an inner product space and $U \subseteq X$ be a subspace. If $U + U^{\perp} = X$ then U is called an *ortho-complemented* subspace of X.

Remarks:

- Note that a subspace is ortho-complemented if and only if it admits an orthogonal complementary subspace (this is a complementary subspace contained in $U^{\perp}$).
- If $U \subseteq X$ is a non-degenerate subspace then U is ortho-complemented if and only if $X = U \dot{\oplus} U^{\perp}$.
- If U is a neutral subspace then, as $U \subseteq U^{\perp}$, U is ortho-complemented if and only if $U^{\perp} = X$ if and only if $U \subseteq X^0$.
- If $U, U_1, \ldots, U_n \subseteq X$ are subspaces such that $U = U_1 \dot{\oplus} \cdots \dot{\oplus} U_n$, then U is ortho-complemented if and only if each U_i $(i = 1, \ldots, n)$ is ortho-complemented.
- Clearly, if U is an ortho-complemented subspace of an inner product space X then every $x \in X$ has at least one projection on U.
- If U is ortho-complemented then, as $U \subseteq U^{\perp\perp}$, $U^{\perp}$ is ortho-complemented too.

Proposition 2.8.2. Let X be an inner product space and $U \subseteq X$ be a subspace. If U is ortho-complemented then $U^0 \subseteq X^0$.

Proof. Suppose that U is ortho-complemented, that is $X = U + U^{\perp}$, and let $u \in U^0 = U \cap U^{\perp}$. For every $w \in U$ and $z \in U^{\perp}$ we have $[w + z, u] = [w, u] + [z, u] = 0 + 0 = 0$; so $U^0 \subseteq X^{\perp} = X^0$. □

Corollary 2.8.3. If X is a non-degenerate inner product space then every ortho-complemented subspace is non-degenerate.

Proposition 2.8.4. Let X be a non-degenerate inner product space and $U \subseteq X$ an ortho-complemented subspace. Then $U^{\perp\perp} = U$.

Proof. By corollary 2.8.3, as X is non-degenerate, U and $U^{\perp}$ are non-degenerate subspaces such that $X = U \dot{\oplus} U^{\perp}$.

Clearly $U \subseteq U^{\perp\perp}$. Suppose, towards a contradiction, that $U \subsetneq U^{\perp\perp}$; then $U^{\perp} \cap U^{\perp\perp} \neq \{0\}$ ↯. □

Proposition 2.8.5. Every definite subspace of finite dimension is ortho-complemented.

Proof. Let U be a definite subspace of an inner product space X such that $n = \dim(U) < \infty$. Suppose, without loss of generality, that U is positive definite. Since U is finite-dimensional, it has a orthonormal basis $(u_1, \ldots, u_n)$. Note that if $x \in X$ then $u = [x, u_1]u_1 + \cdots + [x, u_n]u_n$ is a projection of x of U since $[x - u, u] = 0$. Hence, U is ortho-complemented. □

2.9 Fundamental Decompositions

Definition 2.9.1. An inner product space X is *decomposable* if and only if there exists a positive definite subspace $X^+ \subseteq X$ and a negative definite subspace $X^- \subseteq X$ such that $X = X^0 \dot{\oplus} X^+ \dot{\oplus} X^-$. A decomposition of this form is called a *fundamental decomposition* of X.

Remark. If X is a non-degenerate and decomposable inner product space, then every fundamental decomposition of X is of the form $X = X^+ \dot{\oplus} X^-$ where X^+ is a positive definite subspace and X^- is a negative definite subspace.

Lemma 2.9.2. Let X be an inner product space such that $X = N \dot{\oplus} X^+ \dot{\oplus} X^-$ where N is a neutral subspace, X^+ is a positive definite subspace and X^- is a negative definite subspace. Then $N = X^0$ and therefore X is decomposable.

Proof. Since N is neutral then $N^0 = N$ (by corollary 2.2.10) and, as X^+ and X^- are definite subspaces, $(X^+)^0 = (X^-)^0 = \{0\}$. Then, by proposition 2.4.2, $X^0 = N^0 \dot{\oplus} (X^+)^0 \dot{\oplus} (X^-)^0 = N$. So $X = X^0 \dot{\oplus} X^+ \dot{\oplus} X^-$ and therefore X is decomposable. □

Theorem 2.9.3. Let X be an inner product space and $U \subseteq X$ be a positive definite (negative definite) subspace. Then X has a fundamental decomposition with $X^+ = U$ ($X^- = U$) if and only if:

i) U is maximal positive definite (maximal negative definite);

ii) U is ortho-complemented.

Proof. Suppose, without loss of generality, that U is positive definite.

$\boxed{\Rightarrow}$ Suppose that X has a fundamental decomposition $X = X^0 \dot{\oplus} X^+ \dot{\oplus} X^-$ with $X^+ = U$; clearly, U is ortho-complemented. Suppose, towards a contradicition, that U is not maximal positive definite and let $W \subseteq X$ be a positive definite subspace such that $W \supsetneq U$; then $W \cap (X^0 \dot{\oplus} X^-) \neq \{0\}$. Let $w \neq 0$ be such that $w \in W \cap (X^0 \dot{\oplus} X^-)$, then $[w, w] > 0$ (since W is positive definite) and $[w, w] \leq 0$ (since $X^0 \dot{\oplus} X^-$ is negative semi-definite) $\lightning$.

$\boxed{\Leftarrow}$ Suppose that U is a maximal positive definite and ortho-complemented subspace; then $U^\perp$ is negative (by proposition 2.6.3) and there exists $V \subseteq U^\perp$ subspace such that $X = U \dot{\oplus} V$. Let $N \subseteq V$ a maximal neutral subspace of V, then there exists $X^- \subseteq V$ subspace such that $V = N \dot{\oplus} X^-$ (by proposition 2.2.8, $N \perp V$ since V is semi-definite); clearly X^- is negative definite.

So, setting $X^+ = U$, we have $X = N \dot{\oplus} X^+ \dot{\oplus} X^-$ where X^+ is positive definite and X^- is negative definite. By lemma 2.9.2, $N = X^0$. Therefore X has a fundamental decomposition with $X^+ = U$. □

Remarks:

- By theorem 2.9.3, an inner product space X is decomposable if and only if it contains ortho-complemented maximal definite subspaces.

- If X is decomposable with fundamental decomposition $X = X^0 \dot{\oplus} X^+ \dot{\oplus} X^-$ then, by theorem 2.9.3, X^+ is maximal positive definite and X^- is maximal negative definite.

- Let X be a non-degenerate and decomposable inner product space and let $X = X^+ \dot{\oplus} X^+$ be a fundamental decomposition of X. Let $\|\cdot\|_{X^+}$ and $\|\cdot\|_{X^-}$ be the norms induced by the inner product on X^+ and X^-, respectively (see remarks before definition 2.2.12). Then X is endowed with a norm given by the formula $\|x\| = \|x^+\|_{X^+} + \|x^-\|_{X^-}$, where $x \in X$, $x = x^+ + x^-$ with $x^+ \in X^+$ and $x^- \in X^-$.

Corollary 2.9.4. Every inner product space of finite dimension is decomposable.

Proof. Let X be an inner product space of finite dimension. Let U be a maximal positive definite subspace of X (such a subspace exists since X is finite-dimensional); then U is finite-dimensional and, by proposition 2.8.5, U is orthocomplemented. Hence, by the theorem 2.9.3, X is decomposable. □

Corollary 2.9.5. Every finite dimensional non-degenerate subspace of an inner product space is ortho-complemented.

Proof. Let X be an inner product space and let U be a non-degenerate and finite-dimensional subspace of X. Then, by corollary 2.9.4, U is decomposable; that is, $U = U^+ \dot{\oplus} U^-$. Since U^+ is a definite subspace of X with finite dimension then, by proposition 2.8.5, U^+ is ortho-complemented in X. So, $X = U^+ \dot{\oplus} (U^+)^\perp$.
Note that $U^- \subseteq (U^+)^\perp$ and, again by proposition 2.8.5, U^- is ortho-complemented in $(U^+)^\perp$; that is, there exists $V \subseteq (U^+)^\perp$ such that $(U^+)^\perp = U^- \dot{\oplus} V$. Therefore, $X = U^+ \dot{\oplus} U^- \dot{\oplus} V = U \dot{\oplus} V$; that is, U is ortho-complemented. □

2.10 Orthonormal Systems

Definition 2.10.1. Let $(X, [\cdot, \cdot])$ be a inner product space and let $(e_\gamma)_{\gamma \in \Gamma}$ be a indexed family of vectors of X. The family $(e_\gamma)_{\gamma \in \Gamma}$ is called an *orthonormal system* if and only if $|[e_\gamma, e_{\gamma'}]| = \delta_{\gamma, \gamma'}$.

Remark. Clearly, every orthonormal system in an inner product space is a linearly independent subset.

Lemma 2.10.2. Let X be a non-degenerate inner product space and let $x \in X$ be a non-zero neutral vector. Then there exists an orthonormal system $(u, v) \subseteq X$ such that $x \in \text{Span}(\{u, v\})$.

Proof. Since X is non-degenerate, there exists $y \in X$ such that $[x, y] \neq 0$ and, by the corollary 2.2.9, $U = \text{Span}(\{x, y\})$ is a indefinite subspace. Clearly, $\dim(U) = 2$.
Let $u \in U$ be such that $[u, u] = 1$. Since $\text{Span}(\{u\})$ is a positive definite subspace of U, it is ortho-complemented (by the proposition 2.8.5) and, moreover, it is a maximal positive definite

subspace of U. So, by the proposition 2.6.3, $\mathrm{Span}(\{u\})^\perp$ is a negative semi-definite 1-dimensional subspace of U and, since U is indefinite, it is negative definite. Let $v \in \mathrm{Span}(\{u\})^\perp$ be such that $[v, v] = -1$. Then $(u, v) \subseteq U \subseteq X$ is an orthonormal system such that $U = \mathrm{Span}(\{u, v\})$ (since $\{u, v\}$ is a linearly independent subset) and, therefore, $x \in \mathrm{Span}(\{u, v\})$. □

Theorem 2.10.3. Let X be a non-degenerate inner product space and $(x_n)_{n\in\mathbb{N}} \subseteq X$. Then there exists a countable orthonormal system $(e_j)_j$ such that, for every $n \in \mathbb{N}$, $x_n \in \mathrm{Span}((e_j)_j)$.

Proof. Suppose, without loss of generality, that $x_n \neq 0$ for every $n \in \mathbb{N}$.

- If x_1 is not neutral, let $e_1 = |[x_1, x_1]|^{-1/2} x_1$. Otherwise, by the lemma 2.10.2, there is an orthonormal system $\{e_1, e_2)\}$ such that $x_1 \in \mathrm{Span}(\{e_1, e_2\})$.

- Suppose that for some $n \in \mathbb{N}$ there is a finite orthonormal system $\{e_1, \ldots, e_k\}$ such that its span U_k contains the vectors $x_1, \ldots, x_n$. If $x_j \in U_k$ for every $j \in \mathbb{N}$, then the assertion is clear. Otherwise, let j_n be the first index with $x_{j_n} \notin U_k$. Let $y_n = x_{j_n} - \sum_{j=1}^{k} [e_j, e_j][x_{j_n}, e_j] e_j$; then $y_n \neq 0$ and $y_n \perp U_k$.

  Since $U_k = \mathrm{Span}(\{e_1\}) \dot{\oplus} \cdots \dot{\oplus} \mathrm{Span}(\{e_k\})$ then, by corollary 2.4.3, U_k is non-degenerate. By corollary 2.9.5, U_k is ortho-complemented. Therefore, $U_k^\perp$ is non-degenerate. If y_n is not neutral then let $e_{k+1} = |[y_n, y_n]|^{-1/2} y_n$. Otherwise, by the lemma 2.10.2, there is a orthonormal system $\{e_{k+1}, e_{k+2}\} \subseteq U_k^\perp$ such that $y_n \in \mathrm{Span}(\{e_{k+1}, e_{k+2}\})$. In any case, a finite extension for the orthonormal system $\{e_1, \ldots, e_k\}$ is obtained such that its span contains x_{n+1}. □

Chapter 3

Linear Operators on Inner Product Spaces

3.1 Linear Operators

Definition 3.1.1. Let X be a vector space and let T be a linear operator acting in X. Then:

i) T is *invertible* if and only if T is injective [that is, $\ker(T) = \{0\}$].

ii) T is *completely invertible* if and only if $\mathscr{D}(T) = \mathscr{R}(T) = X$ and T is invertible.

Definition 3.1.2. Let T be a linear operator acting in a vector space X and let $\lambda \in \mathbb{C}$. Then:

i) The subspace $X_\lambda := \bigcup_{n\in\mathbb{N}} \ker(T - \lambda I)^n$ is called the *principal subspace* (or *root subspace*) of T associated to λ, and its non-zero vectors are called the *principal vectors* (or *root vectors*) of T associated to λ. The dimension of X_λ is called the *algebraic multiplicity* of λ.

ii) The complex number λ is called an *eigenvalue* of T if and only if the linear operator $T - \lambda I$ is not invertible. In this case the subspace $\ker(T - \lambda I)$ is called the *eigenspace* associated to λ, and its non-zero elements are called the *eigenvectors* of T associated to λ. The dimension of $\ker(T - \lambda I)$ is called the *geometric multiplicity* of λ.

Remarks:

- Clearly $\ker(T - \lambda I) \subseteq X_\lambda$ and, therefore, the geometric multiplicity is less than or equal to the algebraic multiplicity for every $\lambda \in \mathbb{C}$.

- Clearly if $\ker(T-\lambda I) \neq \{0\}$ then $X_\lambda \neq \{0\}$; moreover, if $X_\lambda \neq \{0\}$ then $\ker(T-\lambda I) \neq \{0\}$ [let $x \neq 0$ be such that $x \in X_\lambda$ and let $p \in N$ the minimum positive integer such that $(T-\lambda I)^p x = 0$; if $p = 1$ then $x \in \ker(T-\lambda I)$ and if $p > 1$ then $y := (T-\lambda I)^{p-1}x \neq 0$ and $y \in \ker(T-\lambda I)$].

- An eigenvalue λ of T is called *semi-simple* if, and only if, $X_\lambda = \ker(T-\lambda I)$.

- Clearly, the principal subspaces of a linear operator are linearly independent.

- The set of all eigenvalues of the operator T is denoted by $\sigma_p(T)$.

Definition 3.1.3. Let T be a linear operator acting in a vector space X and let $\lambda \in \mathbb{C}$ be an eigenvalue of T. A *Jordan chain* of T associated to λ with *length* $p \in \mathbb{N}$ is a finite sequence of non-zero vectors in X_λ, $(x_1, \ldots, x_p)$, such that $x_{k+1} = (T-\lambda I)x_k$ $(k = 1, \ldots, p-1)$ and $(T-\lambda I)x_p = 0$.

Proposition 3.1.4. Let $(x_1, \ldots, x_p)$ be a Jordan chain of T associated to the eigenvalue $\lambda \in \mathbb{C}$. Then x_1, …, x_p are linearly independent vectors.

Proof. Let $\alpha_1, \ldots, \alpha_p \in \mathbb{C}$ such that $\alpha_1 x_1 + \cdots + \alpha_p x_p = 0$; applying $(T-\lambda I)^{p-1}$ we have $\alpha_1 x_p = 0$ and, as $x_p \neq 0$, $\alpha_1 = 0$. Let $k \leq p$ and suppose that $\alpha_1 = \cdots = \alpha_{k-1} = 0$. Applying $(T-\lambda I)^{p-k}$ we obtain $\alpha_k x_p = 0$ and thus $\alpha_k = 0$. Therefore x_1, …, x_p are linearly independent vectors. □

Definition 3.1.5. Let T be a linear operator acting in a vector space X and let $U \subseteq X$ be a subspace. We say that the subspace U is *invariant under* T (or is T-*invariant*) if and only if $T(U \cap \mathscr{D}(T)) \subseteq U$.

Definition 3.1.6. Let T be a linear operator acting in X and U_1, …, $U_n \subseteq X$ subspaces such that $X = U_1 \dotplus \cdots \dotplus U_n$. The direct decomposition $X = U_1 \dotplus \cdots \dotplus U_n$ *reduces* the operator T if and only if:

1) The subspaces $U_1, \ldots, U_n$ are T-invariant;

2) $\mathscr{D}(T) = (\mathscr{D}(T) \cap U_1) \dotplus \cdots \dotplus (\mathscr{D}(T) \cap U_n)$.

In this case T is called the *direct sum* of the linear operator $T\restriction_{U_1}$, …, $T\restriction_{U_n}$.

Remarks:

- Note that if $Y, U_1, \ldots, U_n \subseteq X$ are subspaces with $X = U_1 \dotplus \cdots \dotplus U_n$ then $Y \supseteq (Y \cap U_1) \dotplus \cdots \dotplus (Y \cap U_n)$; but is not necessary that $Y = (Y \cap U_1) \dotplus \cdots \dotplus (Y \cap U_n)$. For example take $X = \mathbb{C}^2$, $U_1 = \{(u, 0) : u \in \mathbb{C}\}$, $U_2 = \{(0, v) : v \in \mathbb{C}\}$) and $Y = \{(u, u) : u \in \mathbb{C}\}$.

- Let $U_1, \ldots, U_n$ be subspaces of X such that $X = U_1 \dotplus \cdots \dotplus U_n$ and let T be a linear operator acting in X with $\mathscr{D}(T) = (\mathscr{D}(T) \cap U_1) \dotplus \cdots \dotplus (\mathscr{D}(T) \cap U_n)$. Then, given $u \in \mathscr{D}(T)$ there exist unique $u_i, v_i \in U_i$ $(i = 1, \ldots, n)$ such that $u = u_1 + \cdots + u_n$ and $T(u) = v_1 + \cdots + v_n$.

  Let $T_{ji} : \mathscr{D}(T) \cap U_i \longrightarrow U_j$ be the linear operator such that $u \in U_i \longmapsto T_{ji}(u) \in U_j$ where $T_{ji}(u)$ is the j-component in the direct decomposition of $T(u)$ in $U_1 \dotplus \cdots \dotplus U_n$. So, if $u = u_1 + \cdots + u_n \in (\mathscr{D}(T) \cap U_1) \dotplus \cdots \dotplus (\mathscr{D}(T) \cap U_1) = \mathscr{D}(T)$ then:

$$T(u) = \textstyle\sum_{i=1}^{n} T(u_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ji}(u_i) = \sum_{j=1}^{n} \sum_{i=1}^{n} T_{ji}(u_i).$$

3.2 Isometric Operators

Recall 3.2.1. A linear operator $T : \mathscr{D}(T) \subseteq X \to X$ is an *isometry* if and only if $[Tx, Ty] = [x, y]$ for every $x, y \in \mathscr{D}(T)$.

Remarks:

- Clearly, an isometric operator acting in an inner product space maps positive (negative, neutral) vectors in positive (negative, neutral) vectors.

- Note that an isometric linear operator need not be invertible. For example, in a non-zero neutral inner product space the mapping 0 is isometric but it is not invertible.

- If T is an invertible isometric linear operator, then it is clear that T^{-1} is isometric.

Proposition 3.2.2. Let T be an isometric linear operator acting in X and let u, v be isotropic and non-isotropic vectors of $\mathscr{D}(T)$, respectively. Then Tu, Tv are isotropic and non-isotropic vectors of $\mathscr{R}(T)$, respectively.

Proof. For every $x \in \mathscr{D}(T)$, as T is isometric and u is an isotropic vector of $\mathscr{D}(T)$, $[Tx, Tu] = [x, u] = 0$; so Tu is an isotropic vector of $\mathscr{R}(T)$.

As $v \in \mathscr{D}(T)$ is a non-isotropic vector of $\mathscr{D}(T)$, let $x_0 \in \mathscr{D}(T)$ such that $[x_0, v] \neq 0$; then, since T is isometric, $[Tx_0, Tv] = [x_0, v] \neq 0$. So Tv is a non-isotropic vector of $\mathscr{R}(T)$. □

Corollary 3.2.3. If T is an isometric linear operator acting in an inner product space X and $\mathscr{D}(T)$ is non-degenerate, then T is invertible.

Proof. Let $x \in \mathscr{D}(T)$ with $x \neq 0$; as x is non-isotropic then, by proposition 3.2.2, Tx is non-isotropic and therefore $Tx \neq 0$. □

Theorem 3.2.4. Let T be a isometric operator acting in a inner product space X and let $\lambda, \mu \in \mathbb{C}$ be eigenvalues of T such that $\lambda\overline{\mu} \neq 1$. Then $X_\lambda \perp X_\mu$.

Proof. Given $u \in X_\lambda$ and $v \in X_\mu$ there are $p, q \in \mathbb{N}$ such that $(T-\lambda)^p u = 0$ and $(T-\mu)^q v = 0$, then we have show that $[u, v] = 0$. By induction on $n := p + q$:

- If $n = 2$ then $p = q = 1$; so if $u \in X_\lambda$ and $v \in X_\mu$ are such that $(T-\lambda)u = 0$ and $(T-\mu)v = 0$, then $[u, v] = [Tu, Tv] = [\lambda u, \mu v] = \lambda\overline{\mu}[u, v]$. Thus, as $\lambda\overline{\mu} \neq 1$, $[u, v] = 0$.

- If $n \geq 2$, suppose that the affirmation is true for all $p + q \leq n$. Now assume that $u \in X_\lambda$, $v \in X_\mu$ are such that $(T-\lambda)^p u = 0$ and $(T-\mu)^q v = 0$ with $p + q = n + 1$. So with $u' = (T-\lambda)u$ and $v' = (T-\lambda)v$ we have $[u, v'] = [u', v] = [u', v'] = 0$ and, therefore, $[u, v] = [Tu, Tv] = [u' + \lambda u, v' + \mu v] = \lambda\overline{\mu}[u, v]$. Thus $[u, v] = 0$ since $\lambda\overline{\mu} \neq 1$, . □

Corollary 3.2.5. If T is an isometric operator acting in an inner product space X and $\lambda \in \mathbb{C}$ with $|\lambda| \neq 1$ is an eigenvalue of T, then X_λ is neutral.

Proposition 3.2.6. Let T be an isometric operator acting in an inner product space X and let $U \subseteq X$ be a subspace such that $U \subseteq T(U \cap \mathscr{D}(T))$. Then $U^\perp$ is T-invariant.

Proof. Let $v \in U^\perp \cap \mathscr{D}(T)$. If $u \in U$ then there exists $w \in U \cap \mathscr{D}(T)$ such that $u = Tw$; so $[u, Tv] = [Tw, Tv] = [w, v] = 0$. Thus, $Tv \in U^\perp$. □

Proposition 3.2.7. Let T be an isometric operator acting in an inner product space X and let $\lambda \in \mathbb{C}$ be an eigenvalue of T. If $\ker(T - \lambda I)$ is a definite subspace then $|\lambda| = 1$ and λ is semi-simple.

Proof. Suppose that $\ker(T - \lambda)$ is a definite subspace; then, by corollary 3.2.5, $|\lambda| = 1$. Now suppose, towards a contradiction, that λ is not semi-simple. Let $x \in X_\lambda \setminus \ker(T - \lambda)$ and $p \geq 2$ such that $(T - \lambda)^p x = 0$ and $(T - \lambda)^{p-1} x \neq 0$. Then,

$$\begin{aligned} 0 &= -[(T-\lambda)^p x,\ \lambda T(T-\lambda)^{p-2}x] \\ &= -[T(T-\lambda)^{p-1}x,\ \lambda T(T-\lambda)^{p-2}x] + \lambda[(T-\lambda)^{p-1}x,\ \lambda T(T-\lambda)^{p-2}x] \\ &= -[(T-\lambda)^{p-1}x,\ \lambda(T-\lambda)^{p-2}x] + [(T-\lambda)^{p-1}x,\ T(T-\lambda)^{p-2}x] \\ &= [(T-\lambda)^{p-1}x,\ (T-\lambda)^{p-1}x] \neq 0 \ \ (\lightning), \end{aligned}$$

where the last inequality follows because $(T - \lambda)^{p-1} x$ belongs to the definite space $\ker(T - \lambda)$. □

3.3 Symmetric Operators

Definition 3.3.1. Let T be a linear operator acting in an inner product space X. Then T is called *symmetric* if and only if $[Tx, y] = [x, Ty]$ for every $x, y \in \mathscr{D}(T)$.

Remarks:

- If T is a symmetric operator acting in a Hilbert space then $\sigma_p(T) \subseteq \mathbb{R}$, but in an arbitrary inner product space this can be false. For example, in a two-dimensional inner product space $(X, [\cdot, \cdot])$ with basis $\{e_1, e_2\}$ such that $[e_1, e_1] = [e_2, e_2] = 0$ and $[e_1, e_2] = 1$, we have that the linear operator T which maps e_1 to ie_1 and e_2 to $-ie_2$ is symmetric (since $[T(\alpha_1 e_1 + \alpha_2 e_2), \beta_1 e_1 + \beta_2 e_2] = [i\alpha_1 e_1 - i\alpha_2 e_2, \beta_1 e_1 + \beta_2 e_2] = i\alpha_1\overline{\beta_2} - i\alpha_2\overline{\beta_1} = [\alpha_1 e_1 + \alpha_2 e_2, i\beta_1 e_1 - i\beta_2 e_2] = [\alpha_1 e_1 + \alpha_2 e_2, T(\beta_1 e_1 + \beta_2 e_2)]$), but its eigenvalues are $\pm i \in \mathbb{C} \setminus \mathbb{R}$.

- If T is a symmetric operator acting in a Hilbert space $\mathcal{H}$ and λ, μ are eigenvalues of T with $\lambda \neq \mu$ then $\mathcal{H}_\lambda \perp \mathcal{H}_\mu$. The following proposition generalizes this for arbitrary inner product spaces.

Proposition 3.3.2. Let T be a symmetric operator acting in an inner product space X and let $\lambda, \mu \in \mathbb{C}$ be eigenvalues of T such that $\lambda \neq \overline{\mu}$. Then $X_\lambda \perp X_\mu$.

Proof. Suppose, without loss of generality, that $\lambda \neq 0$. Let $u \in X_\lambda$, $v \in X_\mu$ and $p, q \in \mathbb{N}$ such that $(T-\lambda)^p u = 0$ and $(T-\mu)^q v = 0$. We have to show that $[u, v] = 0$. By induction on $n := p + q$:

- If $n = 2$ then $p = q = 1$; so if $u \in X_\lambda$ and $v \in X_\mu$ are such that $(T-\lambda)u = 0$ and $(T-\mu)v = 0$, then $[u, v] = \frac{1}{\lambda}[\lambda u, v] = \frac{1}{\lambda}[Tu, v] = \frac{1}{\lambda}[u, Tv] = \frac{1}{\lambda}[u, \mu v] = \frac{\overline{\mu}}{\lambda}[u, v]$. Thus, as $\frac{\overline{\mu}}{\lambda} \neq 1$, $[u, v] = 0$.

- If $n \geq 2$, suppose that the affirmation is true for all $p + q \leq n$. Now let $u \in X_\lambda$, $v \in X_\mu$ and assume $(T-\lambda)^p u = 0$ and $(T-\mu)^q v = 0$ for integers p, q such that $p + q = n + 1$. So with $u' = (T-\lambda)u$ and $v' = (T-\lambda)v$ we have $[u, v'] = [u', v] = [u', v'] = 0$ and, therefore, $[u, v] = \frac{1}{\lambda}[Tu - u', v] = \frac{1}{\lambda}[Tu, v] = \frac{1}{\lambda}[u, Tv] = \frac{1}{\lambda}[u, v' + \mu v] = \frac{1}{\lambda}[u, \mu v] = \frac{\overline{\mu}}{\lambda}[u, v]$. Thus, as $\frac{\overline{\mu}}{\lambda} \neq 1$, $[u, v] = 0$. □

Corollary 3.3.3. If λ is a non-real eigenvalue of a symmetric operator T acting in an inner product space X, then X_λ is neutral.

The following proposition is similar to the proposition 3.2.6.

Proposition 3.3.4. Let T be a symmetric operator acting in an inner product space X and let $U \subseteq X$ be a T-invariant subspace such that $U \subseteq \mathscr{D}(T)$. Then $U^\perp$ is T-invariant.

Proof. Let $v \in U^\perp \cap \mathscr{D}(T)$. For every $u \in U$, as U is T-invariant and $U \subseteq \mathscr{D}(T)$, $[u, Tv] = [Tu, v] = 0$. Thus, $U^\perp$ is T-invariant. □

Proposition 3.3.5. Let T be a symmetric operator acting in an inner product space X and let $\lambda \in \mathbb{C}$ be an eigenvalue of T. If $\ker(T - \lambda I)$ is a definite subspace, then $\lambda \in \mathbb{R}$ and λ is semi-simple.

Proof. Suppose that $\ker(T-\lambda)$ is a definite subspace; then, by corollary 3.3.3, $\lambda \in \mathbb{R}$.
Now suppose, towards a contradiction, that λ is not semi-simple. Let $x \in X_\lambda \setminus \ker(T-\lambda)$ and $p \geq 2$ such that $(T-\lambda)^p x = 0$ and $(T-\lambda)^{p-1} x \neq 0$. So, $0 \neq [(T-\lambda)^{p-1}x,\ (T-\lambda)^{p-1}x] =$

$[(T-\lambda)^{p-1}x,\ (T-\lambda)(T-\lambda)^{p-2}x] = [(T-\lambda)^{p-1}x,\ T(T-\lambda)^{p-2}x] - \lambda[(T-\lambda)^{p-1}x,\ (T-\lambda)^{p-2}x] = [T(T-\lambda)^{p-1}x,\ (T-\lambda)^{p-2}x] - \lambda[(T-\lambda)^{p-1}x,\ (T-\lambda)^{p-2}x] = \lambda[(T-\lambda)^{p-1}x,\ (T-\lambda)^{p-2}x] - \lambda[(T-\lambda)^{p-1}x,\ (T-\lambda)^{p-2}x] = 0$ ↯. □

Definition 3.3.6. Let $(X, [\cdot,\cdot])$ be an inner product space and let $G : X \longrightarrow X$ be a symmetric linear operator. In X we define an inner product $[\cdot,\cdot]_G$ by $[x,y]_G := [Gx, y]$ for every $x, y \in X$. This inner product is called the *G-inner product* on X.

Remark. Every concept related to the G-inner product is prefixed by G and every symbol has the subindex G. For example, the vectors x and y are G-orthogonal (write $x \perp_G y$) if and only $[x,y]_G = 0$.

Proposition 3.3.7. Let X be an inner product space; $U, V \subseteq X$ be subspaces and $G : X \longrightarrow X$ be a symmetric operator. If $U \perp V$ and U is G-invariant, then $U \perp_G V$.

Proof. For every $u \in U$ and $v \in V$ we have $[u,v]_G = [Gu, v] = 0$ since $G(U) \subseteq U$ and $U \perp V$. □

3.4 Orthogonal and Fundamental Projectors

Definition 3.4.1. Let X be an inner product space and let $P : X \longrightarrow X$ be a linear operator. P is called an *orthogonal projector* in X if and only if P is symmetric and $P^2 = P$.

Theorem 3.4.2. Let P be an orthogonal projector in an inner product space X. Then $\mathscr{R}(P)$ is ortho-complemented and, for every $x \in X$, Px is a projection of x on $\mathscr{R}(P)$.
Conversely, if X is a non-degenerate inner product space, $U \subseteq X$ is an ortho-complemented subspace and P_U is the mapping that carries each vector of X into its projection on U, then P_U is an orthogonal projector in X with $\mathscr{R}(P_U) = U$.

Proof. Let $x \in X$, then $x = Px + (x - Px)$ and, for every $z \in X$, as P is an orthogonal projector, $[Pz, x - Px] = [z, Px - P^2x] = [z, Px - Px] = 0$. So $x = Px + (x - Px)$ with $Px \in \mathscr{R}(P)$ and $x - Px \in \mathscr{R}(P)^{\perp}$. Thus $\mathscr{R}(P)$ is ortho-complemented and, for every $x \in X$, Px is a projection of x on $\mathscr{R}(P)$.

Conversely, suppose that X is a non-degenerate inner product space and $U \subseteq X$ is an ortho-complemented subspace; then U is non-degenerate and $X = U\dot{\oplus}U^{\perp}$. So the mapping P_U that carries each vector of X into its projection on U is well-defined, clearly it is linear, $\mathscr{R}(P_U) = U$ and $P^2 = P$. Now, if $x = u+v \in U\dot{\oplus}U^{\perp}$ and $y = z+w \in U\dot{\oplus}U^{\perp}$ then $[P_U x, y] = [u, z+w] = [u, z]$ and $[x, P_U y] = [u+v, z] = [u, z]$; thus P_U is symmetric. Therefore P_U is an orthogonal projector in X. $\square$

Remark. Let X be a non-degenerate and decomposable inner product space and let $X = X^{+}\dot{\oplus}X^{-}$ be a fundamental decomposition of X. Then, by the theorem 3.4.2, there exist P^{+}, P^{-} orthogonal projectors in X such that $\mathscr{R}(P^{+}) = X^{+}$ and $\mathscr{R}(P^{-}) = X^{-}$. Clearly, $P^{+} + P^{-} = I$ and $P^{+}P^{-} = P^{-}P^{+} = 0$.

Definition 3.4.3. The orthogonal projectors, P^{+} and P^{-}, in the preceding remark are called the *fundamental projectors* associated with the fundamental decomposition $X = X^{+}\dot{\oplus}X^{-}$.

Remark. Let X be a non-degenerate and decomposable inner product space and let P^{+} and P^{-} be the fundamental projectors associated with the fundamental decomposition $X = X^{+}\dot{\oplus}X^{-}$. Let $\|\cdot\|$, $\|\cdot\|_{X^{+}}$ and $\|\cdot\|_{X^{-}}$ be the norms in the remarks after the theorem 2.9.3; then, for every $x \in X$, $\|P^{+}x\| = \|x^{+}\| = \|x^{+}\|_{X^{+}} \leq \|x\|$ and therefore P^{+} is a bounded linear operator in the norm space $(X, \|\cdot\|)$. Analogously, P^{-} is so.

Theorem 3.4.4. Let X be a non-degenerate and decomposable inner product space, let $X = X^{+}\dot{\oplus}X^{-}$ be a fundamental decomposition of X, and let P^{+}, P^{-} be the fundamental projectors associated with $X = X^{+}\dot{\oplus}X^{-}$. If U, $V \subseteq X$ are positive and negative semi-definite subspaces, respectively, then $P^{+}\restriction_U$ and $P^{-}\restriction_V$ are invertible. Further, $\dim(U) \leq \dim(X^{+})$ and $\dim(V) \leq \dim(X^{-})$.

Proof. Let U be a positive semi-definite subspace of X. Let $u \in U$, $u = u^{+} + u^{-}$ where $u^{+} \in X^{+}$ and $u^{-} \in X^{-}$. If $P^{+}u = 0$ then $0 = P^{+}(u^{+} + u^{-}) = u^{+}$, therefore $u = u^{-}$ and so $u \in U \cap X^{-}$. Hence $u = 0$; that is $P^{+}\restriction_U$ is invertible.

Analogously, if V is a negative semi-definite subspace of X then $P^{-}\restriction_V$ is invertible.

Now, clearly, $\dim(U) = \dim(P^+(U)) \leq \dim(X^+)$ since $P^+(U) \subseteq X^+$. Analogously, $\dim(V) \leq \dim(X^-)$. $\square$

Remark. Clearly, the theorem 3.4.4. holds if U and V are definite subspaces.

3.5 Fundamental Symmetries

Definition 3.5.1. Let X be a non-degenerate and decomposable inner product space, $X = X^+\dot{\oplus}X^-$ a fundamental decomposition of X and P^+, P^- the fundamental projectors associated with $X = X^+\dot{\oplus}X^-$. The operator $J := P^+ - P^-$ is called the *fundamental symmetry* associated with $X = X^+\dot{\oplus}X^-$.

Remarks:

- If J is the fundamental symmetry associated with the fundamental decomposition $X = X^+\dot{\oplus}X^-$, then for every $x = x^+ + x^-$ we have $Jx = x^+ - x^-$, where $x^+ \in X^+$ and $x^- \in X^-$.
- Clearly, if $X^+ \neq \{0\}$ then 1 is an eigenvalue of J and its associated eigenspace is X^+; similarly, if $X^- \neq \{0\}$ then -1 is an eigenvalue of J and its associated eigenspace is X^-. Further, $\sigma(J) = \sigma_p(J) \subseteq \{-1, 1\}$.

Thus, if the fundamental symmetry associated with a fundamental decomposition is known then the fundamental decomposition can be determined, since then $x^+ = \frac{1}{2}(x+Jx)$ and $x^- = \frac{1}{2}(x-Jx)$ for every $x \in X$.

Proposition 3.5.2. Every fundamental symmetry J of a non-degenerate and decomposable inner product space X is completely invertible with $J^{-1} = J$. Moreover, J is symmetric and isometric.

Proof. Let $X = X^+\dot{\oplus}X^-$ be a fundamental decomposition of X and let J be its associated fundamental symmetry. Clearly, $\mathscr{D}(J) = \mathscr{R}(J) = X$ and $J^2 = I$; therefore J is completely invertible with $J^{-1} = J$.

If $x = x^+ + x^-$ and $y = y^+ + y^-$, with x^+, $y^+ \in X^+$ and x^-, $y^- \in X^-$, then

$$[Jx, Jy] = [x^+ - x^-,\ y^+ - y^-] = [x^+,\ y^+] + [x^-,\ y^-] = [x^+ + x^-,\ y^+ + y^-] = [x,\ y]$$

and

$$[Jx,\ y] = [x^+ - x^-,\ y^+ + y^-] = [x^+,\ y^+] - [x^-,\ y^-] = [x^+ + x^-,\ y^+ - y^-] = [x, Jy].$$

Therefore J is symmetric as well as isometric. □

Remarks:

- Let $X = X^+ \dot{\oplus} X^-$ be a fundamental decomposition of a non-degenerate and decomposable inner product space X and let J be its associated fundamental symmetry. Since J is symmetric then, as is shown in the proof of the proposition 3.5.2, the J-inner product on X is given by the formula $[x, y]_J = [x^+,\ y^+] - [x^-,\ y^-]$ for every $x = x^+ + x^-$ and $y = y^+ + y^-$, where x^+, $y^+ \in X^+$ and x^-, $y^- \in X^-$.

- Note that, since $J^2 = I$, $[Jx, y]_J = [J^2x, y] = [x, y]$ for every x, $y \in X$.

Proposition 3.5.3. Let $X = X^+ \dot{\oplus} X^-$ be a fundamental decomposition of a non-degenerate and decomposable inner product space X and let J be its associated fundamental symmetry. Then X^+ and X^- are J-orthogonal.

Proof. Let $x \in X^+$ and $y \in X^-$, then $x^+ = x$, $x^- = 0$, $y^+ = 0$ and $y^- = y$. Thus, $[x, y]_J = 0$. □

The following proposition is very important because it will enable us later to build pre-Hilbert spaces from non-degenerate and decomposable inner product spaces.

Proposition 3.5.4. Let $X = X^+ \dot{\oplus} X^-$ be a fundamental decomposition of a non-degenerate and decomposable inner product space X and let J be its associated fundamental symmetry. Then the J-inner product is positive definite.

Proof. Let $x \in X$ be such that $[x, x]_J = 0$ and let $x^+ \in X^+$ and $x^- \in X^-$ be such that $x = x^+ + x^-$. Since $[x, x]_J = [x^+, x^+] - [x^-, x^-] = |[x^+, x^+]| + |[x^-, x^-]|$, it follows that $[x^+, x^+] = [x^-, x^-] = 0$. Thus, since X^+ and X^- are definite subspaces, $x^+ = x^- = 0$ and, therefore, $x = 0$. Hence, the J-inner product is definite and, clearly, is positive. □

Remarks:

- Let X be a non-degenerate and decomposable inner product space. If $X = X^+ \dot{\oplus} X^-$ is a fundamental decomposition of X and J is its associated fundamental symmetry, then, since the J-inner product is positive definite, we can define a norm on X, called the J-norm and denoted by $\|\cdot\|_J$, by the formula $\|x\|_J = \sqrt{[x,x]_J}$, $x \in X$. Note that, for every $x^+ \in X^+$ and $x^- \in X^-$, we have $\|x^+\|_J^2 = [x^+, x^+]_J = [x^+, x^+]$, $\|x^-\|_J^2 = [x^-, x^-]_J = -[x^-, x^-]$ and, therefore, $\|x^+ + x^-\|_J^2 = [x^+ + x^-, x^+ + x^-]_J = [x^+, x^+] - [x^-, x^-] = \|x^+\|_J^2 + \|x^-\|_J^2$.

- Let $X = X^+ \dot{\oplus} X^-$ be a fundamental decomposition of a non-degenerate and decomposable inner product space $(X, [\cdot,\cdot])$, and let J be its fundamental symmetry. Then, for every x, $y \in X$, we have $[x,y] = [Jx, y]_J = [x^+ - x^-, y^+ + y^-]_J = [x^+, y^+]_J - [x^-, y^-]_J$, where $x = x^+ + x^-$ and $y = y^+ + y^-$ with x^+, $y^+ \in X^+$ and x^-, $y^- \in X^-$. In particular, $[x,x] = \|x^+\|_J^2 - \|x^-\|_J^2$ and therefore x is positive (negative, neutral) if and only if $\|x^+\|_J > \|x^-\|_J$ $(\|x^+\|_J < \|x^-\|_J$, $\|x^+\|_J = \|x^-\|_J)$.

- If P^+ and P^- are the fundamental projectors associated with the fundamental decomposition $X = X^+ \dot{\oplus} X^+$, and J is its fundamental symmetry, then $\|P^+x\|_J = \|x^+\|_J \leq \|x\|_J$ and $\|P^-x\|_J = \|x^-\|_J \leq \|x\|_J$. That is, P^+ and P^- are bounded operators acting in the normed space $(X, \|\cdot\|_J)$. Therefore, J is also a bounded linear operator acting in the normed space $(X, \|\cdot\|_J)$.

- Let $U \subseteq X$ be a positive semi-definite subspace. By the theorem 3.4.4., $P^+\restriction_U$ is invertible and, by the previous remark, it is continuous. Let $u \in U$. Then u is non-negative and $\|u^+\|_J \geq \|u^-\|_J$; hence $\|P^+u\|_J^2 = \|u^+\|_J^2 = \frac{1}{2}(\|u^+\|_J^2 + \|u^+\|_J^2) \geq \frac{1}{2}(\|u^+\|_J^2 + \|u^-\|_J^2) = \frac{1}{2}(\|u\|_J^2)$. Therefore, $P^+\restriction_U$ has a bounded inverse.

- Analogous to the previous remark, if V is a negative semi-definite subspace then $P^-\restriction_V$ is invertible and bounded with bounded inverse.

By the remarks above, the next proposition holds:

Proposition 3.5.5. Let $(X, [\cdot,\cdot])$ be a non-degenerate and decomposable inner product space, $X = X^+ \dot{\oplus} X^-$ a fundamental decompositon of X; P^+, P^- its associated fundamental projectors and J its associated fundamental symmetry. If $U \subseteq X$ is a positive semi-definite subspace and $V \subseteq X$ is a negative semi-definite subspace, then $P^+\restriction_U$ is a homeomorphism between U and $P^+(U)$, and $P^-\restriction_V$ is a homeomorphism between V and $P^-(V)$.

Remark. Clearly, the proposition 3.5.5. holds if U and V are definite subspaces.

Proposition 3.5.6. If $(X, [\cdot,\cdot])$ is a non-degenerate and decomposable inner product space and J is the fundamental symmetry associated with the fundamental decomposition $X = X^+ \dot{\oplus} X^-$, then for every x, $y \in X$:

$$|[x, y]| \leq \|x\|_J \|y\|_J.$$

Proof. Since $(X, [\cdot,\cdot]_J)$ is a semi-definite space then, by Schwarz Inequality, for every x, $y \in X$:

$$|[x, y]| = |[Jx, y]_J| \leq \|Jx\|_J \|y\|_J = \|x\|_J \|y\|_J.$$ □

Theorem 3.5.7. Let $(X, [\cdot,\cdot])$ be a non-degenerate and decomposable inner product space and let $\|\cdot\|_j$ $(j = 1, 2)$ be a norm on X such that $(X, \|\cdot\|_j)$ is a Banach space and $|[x, y]| \leq \alpha_j \|x\|_j \|y\|_j$ for every x, $y \in X$, where α_j is a positive real constant. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. Let $\|\cdot\| = \|\cdot\|_1 + \|\cdot\|_2$. If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X with respect to the norm $\|\cdot\|$, then clearly it is a Cauchy sequence in X with respect to the norm $\|\cdot\|_j$, $j = 1, 2$. Let $u_j \in X$ be such that $\|x_n - u_j\|_j \to 0$ $(n \to \infty)$, $j = 1, 2$.

Note that, for every $y \in X$, $|[u_1 - u_2, y]| \leq |[u_1 - x_n, y]| + |[x_n - u_2, y]| \leq \alpha_1 \|x_n - u_1\|_1 \|y\|_1 + \alpha_2 \|x_n - u_2\|_2 \|y\|_2 \to 0$ $(n \to \infty)$. So, $|[u_1 - u_2, y]| = 0$ for every $y \in X$. Then, since X is non-degenerate, $u_1 = u_2$ and, therefore, $x_n \to u_1$ in the norm $\|\cdot\|$. That is, $(X, \|\cdot\|)$ is a Banach space.

For $j = 1, 2$, as $\|\cdot\|_j \leq \|\cdot\|$ then, by theorem A.1.3, $\|\cdot\|_j$ is equivalent to $\|\cdot\|$. Hence, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. □

Theorem 3.5.8. Let X be a non-degenerate and decomposable inner product space and let U be a subspace. Then U has a dual companion which is isometrically isomorphic to U.

Proof. Let $X = X^+ \dot{\oplus} X^-$ be a fundamental decomposition of X and let J be its associated fundamental symmetry; let $V = J(U)$. Clearly V is isometrically isomorphic to U (since J is an isometric isomorphism).

If $x \in U \cap V^\perp$ then $x \perp Jx$ and therefore $[x,x]_J = [Jx,x] = 0$. Since the J-inner product is definite, this implies $x = 0$.

If $x \in U^\perp \cap V$, there exists $y \in U$ such that $x = Jy$ and therefore $y \perp Jy$; so $[y,y]_J = [Jy,y] = 0$ and, since the J-inner product is definite, it follows that $y = 0$ and thus $x = 0$.

Therefore $U \cap V^\perp = U^\perp \cap V = \{0\}$; that is, U and V are dual companions. □

Definition 3.5.9. Let $(X, [\cdot,\cdot])$ be a non-degenerate and decomposable inner product space with a fundamental decomposition $X = X^+ \dot{\oplus} X^-$ and let J be its associated fundamental symmetry. The subspace $U \subseteq X$ is called *uniformly positive* (resp. *uniformly negative*) if and only if there exists $\gamma > 0$ such that $[x,x] \geq \gamma \|x\|_J^2$ (resp., $[x,x] \leq -\gamma \|x\|_J^2$) for every $x \in U$.

A subspace which is uniformly positive or uniformly negative is called *uniformly definite.*

Proposition 3.5.10. Let $(X, [\cdot,\cdot])$ be a non-degenerate and decomposable inner product space with a fundamental decomposition $X = X^+ \dot{\oplus} X^-$ and let J be its associated fundamental symmetry. Then X^+ and X^- are uniformly definite subspaces of X.

Proof. Note that, for every $x \in X^+, [x,x] = [Jx,x] = [x,x]_J = \|x\|_J$. Hence, X^+ is uniformly positive. Analogously, X^- is uniformly negative. □

Remarks:

- Clearly, if $U \subseteq X$ is uniformly positive (uniformly negative) then every subspace of U is so.
- By the proposition 3.5.6, the map $x \in X \mapsto [x,x] \in \mathbb{R}$ is continuous (since for every x, $y \in X$, $|[x,x] - [y,y]| = |[x-y,x] + [y,x-y]| \leq |[x-y,x]| + |[y,x-y]| \leq \|x-y\|_J \|x\|_J + \|y\|_J \|x-y\|_J$). Therefore, if $U \subseteq X$ is a uniformly positive (uniformly negative) subspace, then $\overline{U}$ is so.

The previous remark implies immediately the following proposition:

Proposition 3.5.11. Let $(X, [\cdot,\cdot])$ be a non-degenerate and decomposable inner product space with fundamental decomposition $X = X^+ \dot{\oplus} X^-$ and let J be its associated fundamental symmetry. Then a subspace U is uniformly definite if and only if $\overline{U}$ is uniformly definite.

3.6 Angular Operators

Remark. Let X be a non-degenerate and decomposable inner product space with fundamental decomposition $X = X^+ \dot{\oplus} X^-$ and let P^+ and P^- be its associated fundamental projectors. Let U, V be subspaces of X:

- If $P^+ \restriction_U$ is invertible, then for every $w \in P^+(U)$ there exists a unique $x \in U$ such that $w = P^+x$. Therefore,

  $$K^+ : P^+(U) \longrightarrow X, \quad P^+x \longmapsto P^-x$$

  is a well-defined linear operator such that $\mathscr{R}(K^+) = P^-(U)$ and, for every $x \in U$, $K^+P^+x = P^-x$. Note that:

  $$U = \left\{P^+x + P^-x \mid x \in U\right\} = \left\{P^+x + K^+P^+x \mid x \in U\right\} = \left\{w + K^+w \mid w \in P^+(U)\right\}.$$

  Conversely, if there exists a subspace $W \subseteq X^+$ and a linear operator $K^+ : W \longrightarrow X$ with $\mathscr{R}(K^+) \subseteq X^-$ such that $U = \{w + K^+w \mid w \in W\}$, then $P^+\restriction_U$ is invertible (since, for every $w \in W$, $P^+(w + K^+w) = 0$ implies $w = 0$) and, clearly, $W = P^+(U)$. Moreover, note that $K^+(P^+x) = P^-x$ for every $x \in U$; so $\mathscr{R}(K^+) = P^-(U)$ and $K^+P^+x = P^-x$ for every $x \in U$.

- Analogously, if $P^-\restriction_V$ is invertible, then for every $z \in P^-(V)$ there exists a unique $x \in V$ such that $z = P^-x$. Therefore,

  $$K^- : P^-(V) \longrightarrow X, \quad P^-x \longmapsto P^+x$$

is a well-defined linear operator such that $\mathscr{R}(K^-) = P^+(V)$ and, for every $x \in V$, $K^-P^-x = P^+x$.

Conversely, if there exists a subspace $Z \subseteq X^-$ and a linear operator $K^- : Z \longrightarrow X$ with $\mathcal{R}(K^-) \subseteq X^+$ such that $V = \{z + K^-z \mid z \in Z\}$, then $P^-\restriction_V$ is invertible (since, for every $z \in Z$, $P^-(z + K^-z) = 0$ implies $z = 0$) and, clearly, $Z = P^-(V)$. Moreover, note that $K^-(P^-x) = P^+x$ for every $x \in V$; so $\mathcal{R}(K^-) = P^+(V)$ and $K^-P^-x = P^+x$ for every $x \in V$.

Definition 3.6.1. The operator K^+ (resp. K^-) in the previous remark is called the *angular operator* of the subspace U (resp. V) with respect to X^+ (resp. X^-).

The next theorem summarizes the previous remarks:

Theorem 3.6.2. Let X be a non-degenerate and decomposable inner product space with fundamental decomposition $X = X^+\dot{\oplus}X^-$ and let P^+, P^- be its associated fundamental projectors. Let U, V be subspaces of X.
If $P^+\restriction_U$ and $P^-\restriction_V$ are invertible, then there exist $W \subseteq X^+$ and $Z \subseteq X^-$ such that:

$$U = \left\{w + K^+w \mid w \in W\right\} \tag{3.1}$$

and

$$V = \left\{z + K^-z \mid z \in Z\right\}, \tag{3.2}$$

where K^+ is the angular operator of U with respect to X^+ and K^- is the angular operator of V with respect to X^-.
Conversely, if U can be writtten in the form (3.1) for some subspace $W \subseteq X^+$ and some linear operator $K^+ : W \longrightarrow X$, with $\mathscr{R}(K^+) \subseteq X^-$, then $P^+\restriction_U$ is invertible and K^+ is the angular operator of U with respect to X^+. Analogously, if V can be written in the form (3.2) for some subspace $Z \subseteq X^-$ ans some linear operator $K^- : Z \longrightarrow X$, with $\mathcal{R}(K^-) \subseteq X^+$, then $P^-\restriction_V$ is invertible and K^- is the angular operator of V with respect to X^-.

Theorem 3.6.3. Let X be a non-degenerate and decomposable inner product space with fundamental decomposition $X = X^+ \dot{\oplus} X^-$ and let P^+ and P^- be its associated fundamental projectors and let J be its associated fundamental symmetry. If U is a subspace of X, then:

i) U is positive semi-definite if, and only if, the angular operator K^+ of U with respect to X^+ exists and $\|K^+x^+\|_J \leq \|x^+\|_J$ for every $x^+ \in P^+(U)$.

ii) U is positive definite if, and only if, the angular operator K^+ of U with respect to X^+ exists and $\|K^+x^+\|_J < \|x^+\|_J$ for every $x^+ \in P^+(U) \setminus \{0\}$.

iii) U is uniformly positive if, and only if, the angular operator K^+ of U with respect to X^+ exists and $\|K^+\| < 1$.

iv) U is neutral if, and only if, the angular operator K^+ of U with respect to X^+ exists and $\|K^+x^+\|_J = \|x^+\|_J$ for every $x^+ \in P^+(U)$.

v) V is negative semi-definite if, and only if, the angular operator K^- of V with respect to X^- exists and $\|K^-x^-\|_J \leq \|x^-\|_J$ for every $x^- \in P^-(V)$.

vi) V is negative definite if, and only if, the angular operator K^- of V with respect to X^- exists and $\|K^-x^-\|_J < \|x^-\|_J$ for every $x^- \in P^-(V) \setminus \{0\}$.

vii) V is uniformly negative if, and only if, the angular operator K^- of V with respect to X^- exists and $\|K^-\| < 1$.

Proof. Let U and V be subspaces of X.

i) Suppose that U is positive semi-definite. By theorem 3.4.4, $P^+\restriction_U$ is invertible and therefore K^+ exists. Let $x^+ \in P^+(U)$ and let $x \in U$ such that $x^+ = P^+x$, then $\|K^+x^+\|_J = \|K^+P^+x\|_J = \|P^-x\|_J \leq \|P^+x\|_J = \|x^+\|_J$ since U is positive semidefinite (see remark after proposition 3.5.4).

Now, suppose that the angular operator K^+ of U with respect to X^+ exist and $\|K^+x^+\|_J \leq \|x^+\|_J$ for every $x^+ \in P^+(U)$. Let $x \in U$, then $\|P^-x\|_J \leq \|P^+x\|_J$ and, therefore, $[x, x] \geq 0$. That is, U is positive semi-definite.

The proofs for *ii)*, *iv)*, *v)* and *vi)* are similar to the proof of *i)*.

iii) $\boxed{\Leftarrow}$ Let $0 \leq \delta < 1$ such that $\|K^+\| = \delta$; then $\|K^+x^+\|_J \leq \delta\|x^+\|_J \leq \|x^+\|_J$ for every $x^+ \in P^+(U)$. So, by i), U is positive semi-definite and, therefore, $\|P^+x\|_J \geq \|P^-x\|_J$ for every $x \in U$.

Let $x \in U$, then

$$\begin{aligned}[x,x] &= [Jx,x]_J = [P^+x - P^-x, P^+x + P^-x]_J = [P^+x, P^+x]_J - [P^-x, P^-x] \\ &= \|P^+x\|_J^2 - \|P^-x\|_J^2 = \|P^+x\|_J^2 - \|K^+P^+x\|_J^2 \geq (1-\delta^2)\|P^+x\|_J^2 \\ &= \tfrac{1-\delta^2}{2}2\|P^+x\|_J^2 \geq \tfrac{1-\delta^2}{2}(\|P^+x\|_J^2 + \|P^-x\|_J^2) = \tfrac{1-\delta^2}{2}\|x\|_J^2.\end{aligned}$$

Hence, U is uniformly positive.

$\boxed{\Rightarrow}$ Since U is uniformly positive, it is a positive definite subspace. Thus, by ii), K^+ exists and $\|K^+\| \leq 1$.

Let $\gamma > 0$ be such that, for every $x \in U$, $\gamma\|x\|_J^2 \leq [x,x]$. Suppose, towards a contradiction, that $\|K^+\| = 1$. Then there exists a bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq U$ such that $\|P^+x_n\|_J = 1$ for every $n \in \mathbb{N}$, and $\|K^+P^+x_n\|_J \to 1$ $(n \to \infty)$. Note that $x_n \nrightarrow 0$ and, therefore, there is a subsequence (x_{n_k}) and $\varepsilon > 0$ such that $\|x_{n_k}\|_J^2 \geq \varepsilon$ for every $k \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$:

$$\varepsilon\gamma \leq \gamma\|x_{n_k}\|_J^2 \leq [x_{n_k}, x_{n_k}] = \|P^+x_{n_k}\|_J^2 - \|K^+P^+x_{n_k}\|_J^2 = 1 - \|K^+P^+x_{n_k}\|_J^2 \to 0 \;\lightning.$$

Hence, $\|K^+\| < 1$.

The proof of *vii)* is similar to the proof of *iii)*. □

Chapter 4

Krein Spaces

4.1 Krein Spaces

Definition 4.1.1. Let $(\mathcal{H}, [\cdot,\cdot])$ be a non-degenerate and decomposable inner product space. Then $\mathcal{H}$ is called a *Krein space* if, and only if, it has a fundamental decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ such that $(\mathcal{H}^+, [\cdot,\cdot])$ and $(\mathcal{H}^-, -[\cdot,\cdot])$ are Hilbert spaces. This decomposition is called a *Krein decomposition* for $\mathcal{H}$.

Remarks:

- Clearly, by the proposition 3.5.10, if $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ is a Krein decomposition of the Krein space $\mathcal{H}$, then $\mathcal{H}^+$ and $\mathcal{H}^-$ are uniformly definite subspaces.

- Clearly, if $\mathcal{H}$ is a Hilbert space or $\mathcal{H}$ is the anti-space of a Hilbert space, then it is a Krein space.

Theorem 4.1.2. Let $\mathcal{H}$ be a Krein space, $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ be a Krein decomposition for $\mathcal{H}$ and J its associated fundamental symmetry. Then $(\mathcal{H}, [\cdot,\cdot]_J)$ is a Hilbert space (in this case $\mathcal{H}$ is called a J-space).

Proof. By the proposition 3.5.4, $(\mathcal{H}, [\cdot,\cdot]_J)$ is a pre-Hilbert space. Note that, for every $x \in \mathcal{H}$, if $x = x^+ + x^-$, where $x^+ \in \mathcal{H}^+$ and $x^- \in \mathcal{H}^-$, then

$$\|x\|_J^2 = \|x^+\|_{\mathcal{H}^+}^2 + \|x^-\|_{\mathcal{H}^-}^2. \tag{4.1}$$

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{H}, [\cdot,\cdot]_J)$. Then there exist sequences $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^+$ and $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^-$ such that $x_n = u_n + v_n$ for each $n \in \mathbb{N}$. By (4.1), $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively, and therefore they are convergent in $\mathcal{H}^+$ and $\mathcal{H}^-$, respectively. Thus, again by (4.1), $(x_n)_{n\in\mathbb{N}}$ is convergent in the J-norm. Therefore $(\mathcal{H}, [\cdot,\cdot]_J)$ is a Hilbert space. □

Corollary 4.1.3. Let $\mathcal{H}$ be a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and associated fundamental symmetry J. Then J is a self-adjoint and unitary linear operator acting in the Hilbert space $(\mathcal{H}, [\cdot,\cdot]_J)$.

Proposition 4.1.4. Let $\mathcal{H}$ be a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$, P^+ and P^- be its associated fundamental projectors and J be its associated fundamental symmetry. Then P^+ and P^- are self-adjoint, and therefore they are orthogonal projections in the Hilbert space $(\mathcal{H}, [\cdot,\cdot]_J)$.

Proof. Let $x,\, y \in \mathcal{H}$ with $x = x^+ + x^-$ and $y = y^+ + y^-$, where $x^+,\, y^+ \in \mathcal{H}^+$ and $x^-,\, y^- \in \mathcal{H}^-$. Then

$$[P^+x, y]_J = [x^+, y]_J = [Jx^+, y] = [x^+, y^+ + y^-] = [x^+, y^+]$$

and

$$[x, P^+y]_J = [Jx, y^+] = [x^+ - x^-, y^+] = [x^+, y^+].$$

Thus, P^+ is self-adjoint in the Hilbert space $(\mathcal{H}, [\cdot,\cdot]_J)$. The proof for P^- is analogous. □

Remark. Given a Hilbert space, we can construct a Krein space as shown below. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $J : \mathcal{H} \longrightarrow \mathcal{H}$ be a self-adjoint and unitary operator. Since J is a bounded operator, its spectrum $\sigma(J)$ is non-empty and $\sigma(J) = \sigma_p(J) \subseteq \{-1, 1\}$. In $\mathcal{H}$ define an inner product $[\cdot,\cdot]$ by the formula $[x, y] = \langle Jx, y\rangle$ for every $x,\, y \in \mathcal{H}$.

- Suppose $\sigma(J) = \{-1\}$ or $\sigma(J) = \{1\}$. Then, clearly, $(\mathcal{H}, [\cdot,\cdot]_J)$ is a Krein space with fundamental symmetry J.

- Suppose $\sigma(J) = \{-1, 1\}$. Let $\mathcal{H}^+ = \ker(I - J)$, $\mathcal{H}^- = \ker(I + J)$, $P^+ = \frac{1}{2}(I + J)$ and $P^- = \frac{1}{2}(I - J)$. Clearly $\mathcal{H}^+,\, \mathcal{H}^- \neq \{0\}$ and, by the spectral theorem applied to J,

$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (orthogonal sum in the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$). Note that $P^+P^+ = P^+$, $P^-P^- = P^-$, $P^+P^- = P^-P^+ = 0$, $P^+ + P^- = I$ and $P^+ - P^- = J$.

Clearly, P^+ and P^- are mutually orthogonal projections on $\mathcal{H}^+$ and $\mathcal{H}^-$ (since they are the projectors in the spectral decomposition of J). The spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ are also $[\cdot,\cdot]$-orthogonal because for $x \in \mathcal{H}^+$ and $y \in \mathcal{H}^-$ we obtain $[x, y] = \langle Jx, y\rangle = \langle x, y\rangle = 0$.

Note that $[x, y] = \langle x, y\rangle$ for every x, $y \in \mathcal{H}^+$ and $[x, y] = -\langle x, y\rangle$ for every x, $y \in \mathcal{H}^-$. Further, since $\mathcal{H}^+$ and $\mathcal{H}^-$ are closed subspaces of $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, it follows that $(\mathcal{H}^+, [\cdot,\cdot]) = (\mathcal{H}^+, \langle\cdot,\cdot\rangle)$ and $(\mathcal{H}^-, -[\cdot,\cdot] = (\mathcal{H}^-, \langle\cdot,\cdot\rangle))$ are Hilbert spaces. Therefore, $\mathcal{H}$ is a non-degenerate and decomposable inner product space and $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ is a fundamental decomposition of $\mathcal{H}$ (with respect to inner product $[\cdot,\cdot]$). In particular, $\mathcal{H}$ is a Krein space.

Clearly, I and J are $[\cdot,\cdot]$-symmetric, hence $P^\pm = \frac{1}{2}(I \pm J)$ are $[\cdot,\cdot]$-symmetric too and therefore they are orthogonal projectors. Further, P^+ and P^- are the fundamental projectors associated with the Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and J is its associated fundamental symmetry.

Now we obtain immediately the following theorem:

Theorem 4.1.5. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $J : \mathcal{H} \longrightarrow \mathcal{H}$ be a self-adjoint and unitary operator. If $[\cdot,\cdot]$ is the inner product on $\mathcal{H}$ defined by the formula $[x, y] := \langle Jx, y\rangle$ for every x, $y \in \mathcal{H}$, then $(\mathcal{H}, [\cdot,\cdot])$ is a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and associated fundamental symmetry J. $\mathcal{H}^+$ and $\mathcal{H}^-$ are given by $\mathcal{H}^+ = \mathscr{R}(P^+)$ and $\mathcal{H}^- = \mathscr{R}(P^-)$, where $P^+ = \frac{1}{2}(I + J)$ and $P^- = \frac{1}{2}(I - J)$.

Proposition 4.1.6. Let $\mathcal{H}$ be a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and associated fundamental symmetry J. Then $\mathcal{H}^+$ and $\mathcal{H}^-$ are closed subspaces in the topology induced by $\|\cdot\|_J$ on $\mathcal{H}$.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^+$ be a Cauchy sequence with respect to the J-norm; then, by equation (4.1), $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{H}^+, \|\cdot\|_{\mathcal{H}^+})$. Since $(\mathcal{H}^+, \|\cdot\|_{\mathcal{H}^+})$ is a Banach space there exists $u_0 \in \mathcal{H}^+$ such that $u_n \to u_0$ in the norm $\|\cdot\|_{\mathcal{H}^+}$ and, again by equation (4.1), $u_n \to u_0$

in the J-norm. Thus $\mathcal{H}^+$ is a closed subspace in the topology induced by the J-norm on $\mathcal{H}$. Analogously, $\mathcal{H}^-$ is so. □

Remark. Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and J its associated fundamental symmetry. By the proposition 4.1.5, $(\mathcal{H}^+, [\cdot,\cdot]_J)$ and $(\mathcal{H}^-, [\cdot,\cdot]_J)$ are Hilbert spaces and, moreover, it is clear that $(\mathcal{H}^+, [\cdot,\cdot]_J) = (\mathcal{H}^+, [\cdot,\cdot])$ and $(\mathcal{H}^-, [\cdot,\cdot]_J) = (\mathcal{H}^-, -[\cdot,\cdot])$. Further, by the proposition 3.5.3, $\mathcal{H}^+$ and $\mathcal{H}^-$ are mutually J-orthogonal; thus $\mathcal{H} = \mathcal{H}^+ \oplus_J \mathcal{H}^-$.

Remark. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space, with $\|\cdot\|$ the norm induced by $\langle\cdot,\cdot\rangle$, and let $G \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $0 \in \rho(G)$. Let $[\cdot,\cdot]_G$ be the inner product on $\mathcal{H}$ defined by the formula $[x, y]_G = \langle Gx, y\rangle$, $x, y \in \mathcal{H}$. Since $0 \in \rho(G)$, there exists $\alpha > 0$ such that $(-\alpha, \alpha) \subseteq \rho(G)$.

- If $G > 0$ (resp., $G < 0$) then $G \geq \alpha I$ (resp. $G \leq -\alpha I$) and, clearly, $(\mathcal{H}, [\cdot,\cdot]_G)$ is a Krein space since $[\cdot,\cdot]_G$ is positive definite (resp., negative definite) and its induced norm, $\|\cdot\|_G$, is equivalent to $\|\cdot\|$ (because $\alpha\|x\|^2 \leq \|x\|_G^2 = |\langle Gx, x\rangle| \leq \|G\|\|x\|^2$ for $x \in \mathcal{H}$). Thus $(\mathcal{H}, [\cdot,\cdot]_G)$ (resp., $(\mathcal{H}, -[\cdot,\cdot]_G)$) is a Hilbert space and, therefore, a Krein space. Clearly, in this case, $P_+ = I$, $P_- = 0$ (resp., $P_+ = 0$, $P_- = I$) and, therefore, $J = I$ (resp., $J = -I$). Thus, $\|\cdot\|_J$ and $\|\cdot\|$ are equivalent.

- Suppose that there are x_0, $y_0 \in \mathcal{H}$ such that $\langle Gx_0, x_0\rangle > 0$ and $\langle Gy_0, y_0\rangle < 0$; clearly $\sigma(G) \cap \mathbb{R}_{>0} \neq \varnothing$ and $\sigma(G) \cap \mathbb{R}_{<0} \neq \varnothing$. Let P_+ and P_- be the Riesz projections on $\sigma(G) \cap \mathbb{R}_{>0}$ and $\sigma(G) \cap \mathbb{R}_{<0}$, respectively. Then, by theorem A.1.2, P^+ and P^- are orthogonal projections which commute with G, $P^+P^- = P^-P^+ = 0$, $P^+ + P^- = I$ and $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ where $\mathcal{H}^+ = \mathscr{R}(P^+)$ and $\mathcal{H}^- = \mathscr{R}(P^-)$ are closed G-invariant subspaces. Further, since $\sigma(G\restriction_{\mathcal{H}^+}) = \sigma(G) \cap \mathbb{R}_{>0}$ and $\sigma(G\restriction_{\mathcal{H}^-}) = \sigma(G) \cap \mathbb{R}_{<0}$, we have $\langle Gx, x\rangle > 0$ for every $x \in \mathcal{H}^+$ with $\|x\| = 1$ and $\langle Gx, x\rangle < 0$ for every $x \in \mathcal{H}^-$ with $\|x\| = 1$.

  Clearly, $[\cdot,\cdot]_G$ is indefinite. Note that, since $\mathcal{H}^+$ and $\mathcal{H}^-$ are G-invariant, $\mathcal{H}^+ \perp_G \mathcal{H}^-$; hence $\mathcal{H} = \mathcal{H}^+ \dot{\oplus}_G \mathcal{H}^-$ and $(\mathcal{H}^+, [\cdot,\cdot]_G)$ and $(\mathcal{H}^-, -[\cdot,\cdot]_G)$ are Hilbert spaces. That is, $(\mathcal{H}, [\cdot,\cdot]_G)$ is a Krein space.

  Clearly P^+ and P^- are the fundamental projectors associated with the Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus}_G \mathcal{H}^-$; let J be its associated fundamental symmetry. Clearly J is a bounded operator in the topology induced by $\|\cdot\|$ and, therefore, for every $x \in \mathcal{H}$, $\|x\|_J^2 = [x,x]_J = [Jx,x]_G = \langle GJx, x\rangle = |\langle GJx, x\rangle| \leq \|GJx\|\|x\| \leq \|GJx\|\|x\| \leq \|GJ\|\|x\|^2$.

As G is J-selfadjoint (since $[Gx,y]_J = [JGx,y]_G = \langle GJGx, y\rangle = \langle JGx, Gy\rangle = \langle GJx, Gy\rangle = [Jx, Gy]_G = [x, Gy]_J$) and $\mathscr{D}(G) = \mathcal{H}$, the operators G and G^{-1} are bounded in the topology induced by $\|\cdot\|_J$. Thus, $\|x\|^2 = \langle x, x\rangle = \langle GG^{-1}x, x\rangle = [G^{-1}x, x]_G = [JJ^{-1}G^{-1}x, x]_G = [J^{-1}G^{-1}x, x]_J \leq \|J^{-1}G^{-1}\|\|x\|_J^2$.

Hence, $\|\cdot\|$ and $\|\cdot\|_J$ are equivalent. Note that the equivalence of $\|\cdot\|$ and $\|\cdot\|_J$ is also a consequence of the theorem 3.5.7, since $|[x,y]_G| \leq \|x\|_J\|y\|_J$ and $|[x,y]_G| \leq \|G\|\|x\|\|y\|$ for every x, $y \in \mathcal{H}$.

4.2 Subspaces in Krein Spaces

Remark. Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space. Note that for every subspace $U \subseteq \mathcal{H}$:

- $J(U^\perp) = [J(U)]^\perp$ since $x \in J(U^\perp) \Leftrightarrow Jx \in U^\perp \Leftrightarrow [Jx, y] = 0$ for every $y \in U \Leftrightarrow [x, Jy] = 0$ for every $y \in U \Leftrightarrow x \in [J(U)]^\perp$.

- $U^\perp = [J(U)]^{\perp_J}$ since $x \in U^\perp \Leftrightarrow [x,y] = 0$ for every $y \in U \Leftrightarrow [Jx, Jy] = 0$ for every $y \in U$ $\Leftrightarrow [x, Jy]_J = 0$ for every $y \in U \Leftrightarrow x \in [J(U)]^{\perp_J}$.

- $U^{\perp_J} = [J(U)]^\perp$ since, by the preceding item, $U^{\perp_J} = (J^2(U))^{\perp_J} = (J(U))^\perp$.

Proposition 4.2.1. If U is a closed subspace of a J-space, then $U^\perp$ is closed too.

Proof. Since U is closed, $J(U)$ is closed and, therefore, $U^\perp = [J(U)]^{\perp_J}$ is so. □

Proposition 4.2.2. Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space and $U \subseteq \mathcal{H}$ be a closed subspace (in the topology induced by $\|\cdot\|_J$). Then $U^{\perp\perp} = U$.

Proof. $U^{\perp\perp} = [J(U^\perp)]^{\perp_J} = [(J(U))^\perp]^{\perp_J} = [(J^2(U))^{\perp_J}]^{\perp_J} = [U^{\perp_J}]^{\perp_J} = \overline{U} = U$. □

Remark. From the proof of the proposition 4.2.2, we have $U^{\perp\perp} = \overline{U}$ for every subspace U of a J-space.

Proposition 4.2.3. Let $\mathcal{H}$ be a J-space and U be a closed subspace of $\mathcal{H}$. Then $\mathcal{H} = \overline{U + U^\perp}$ if, and only if, U is non-degenerate.

Proof. By the preceding remark, proposition 2.3.3 and the closedness of U, we have $\overline{U + U^\perp} = (U + U^\perp)^{\perp\perp} = (U^\perp \cap U^{\perp\perp})^\perp = (U^\perp \cap U)^\perp = (U^0)^\perp$. Thus, U is non-degenerate if and only if $\overline{U + U^\perp} = \mathcal{H}$. □

Proposition 4.2.4. Let $\mathcal{H}$ be a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and let J be its associated fundamental symmetry. If U is a maximal semi-definite subspace, then U is a closed subspace in the topology induced on $\mathcal{H}$ by the norm $\|\cdot\|_J$.

Proof. Let U be a maximal semi-definite subspace. Suppose, without loss of generality, that U is positive semi-definite.

Let $u_0 \in \mathcal{H}$ such that there is a sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ with $u_n \to u_0$ in the J-norm. Then there exist $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^+$, $(y_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^-$, $x_0 \in \mathcal{H}^+$ and $y_0 \in \mathcal{H}^-$ such that, for every $n \in \mathbb{N}_0$, $u_n = x_n + y_n$. Since $\|u_n - u_0\|_J^2 = \|x_n - x_0\|_J^2 + \|y_n - y_0\|_J^2$ for every $n \in \mathbb{N}$, then $x_n \to x_0$ and $y_n \to y_0$ in the J-norm.

Since each u_n is non-negative, $\|x_n\|_J \geq \|y_n\|_J$ for every $n \in \mathbb{N}$ and, therefore, $\|x_0\|_J \geq \|y_0\|_J$. Thus, u_0 is non-negative. Hence the closure of U is a positive semi-definite subspace of $\mathcal{H}$ and, since U is maximal positive semi-definite, U is closed. □

Corollary 4.2.5 (of the proof)**.** If U is a positive (negative) semi-definite subspace, then its closure, in the topology induced on $\mathcal{H}$ by the J-norm, is a positive (negative) semi-definite subspace also.

Theorem 4.2.6. Let $\mathcal{H}$ be a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and let P^+, P^- be the associated fundamental projectors. If $U \subseteq \mathcal{H}$ is a positive semi-definite subspace then U is maximal positive semi-definite if, and only if, $P^+(U) = \mathcal{H}^+$.

Analogously, if $V \subseteq \mathcal{H}$ is a negative semi-definite subspace then V is maximal negative semi-definite if, and only if, $P^-(V) = \mathcal{H}^-$.

Proof. Let J be the fundamental symmetry associated with $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and let $U \subseteq X$ be a positive semi-definite subspace. Clearly, $P^+(U) \subseteq \mathcal{H}^+$.

Suppose that U is maximal positive semi-definite and suppose, towards a contradiction, that $P^+(U) \neq \mathcal{H}^+$. Since U is closed in $(\mathcal{H}, [\cdot,\cdot]_J)$ (by proposition 4.2.4) and $P^+\restriction_U$ is a homeomorphism between U and $P^+(U)$ (proposition 3.5.5), then $P^+(U)$ is closed also. Thus, $P^+(U)$ is a proper closed subspace of the Hilbert space $(\mathcal{H}^+, [\cdot,\cdot]_J)$. Then there exists $x \in \mathcal{H}^+$, $x \neq 0$, such that $x \perp_J P^+(U)$ and, therefore, $x \perp_J U$ (since if $u \in U$ then $[u,x]_J = [u, P^+x]_J = [P^+u, x]_J = 0$). Hence $U+\mathrm{Span}(x)$ is a positive semidefinite subspace (since $[u+\alpha x, u+\alpha x] = [J(u+\alpha x), u+\alpha x]_J = [u^+-u^-+\alpha x, u^++u^-+\alpha x]_J = [u^++\alpha x, u^++\alpha x]_J - [u^-,u^-]_J = [u^+,u^+]_J + |\alpha|^2[x,x]_J - [u^-,u^-]_J = |\alpha|^2\|x\|_J^2 + \|u^+\|_J^2 - \|u^-\|_J^2 \geq 0$ for every $u \in U$ and $\alpha \in \mathbb{C}$ because u is non-negative) and $U + \mathrm{Span}(x) \supsetneq U$ ↯.

Now suppose that $P^+(U) = \mathcal{H}^+$. By the theorem 3.4.4, $P^+\restriction_U \colon U \to \mathcal{H}^+$ is bijective. Suppose, towards a contradiction, that U is not maximal positive semi-definite and let W be a maximal positive semi-definite subspace such that $W \supsetneq U$. Thus $P^+(W) = \mathcal{H}^+$ and therefore $P^+\restriction_W \colon W \to \mathcal{H}^+$ is bijective too. Since $U \subsetneq W$, there exists a subspace $U' \neq \{0\}$ such that $W = U \dotplus U'$.
Let $u' \in U'$ with $u' \neq 0$ and let $u \in U$ be such that $P^+(u') = P^+(u)$. Then, since $u \in W$,

$$u' = (P^+\restriction_W)^{-1}(P^+(u')) = (P^+\restriction_W)^{-1}(P^+(u)) = u \text{ ↯}.$$

In the case where V is negative semi-definite the proof is analogous. □

Corollary 4.2.7. Let $\mathcal{H}$ be a Krein space and let $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ be a Krein decomposition for $\mathcal{H}$. If $U \subseteq \mathcal{H}$ is a maximal positive semi-definite subspace, then $\dim(U) = \dim(\mathcal{H}^+)$. If $V \subseteq \mathcal{H}$ is a maximal negative semi-definite subspace, then $\dim(V) = \dim(\mathcal{H}^-)$.

Theorem 4.2.8. Let $\mathcal{H}$ be a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$; let P^+, P^- be its associated fundamental projectors and J its associated fundamental symmetry. Let $U \subseteq \mathcal{H}$ be a positive definite (negative definite) subspace. Then:

i) If U is maximal positive definite (maximal negative definite), then $\overline{P^+(U)} = \mathcal{H}^+$ ($\overline{P^-(U)} = \mathcal{H}^-$).

ii) If $P^+(U) = \mathcal{H}^+$ ($P^-(U) = \mathcal{H}^-$), then U is maximal positive definite (maximal negative definite).

Proof. Suppose, without loss of generality, that U is a positive definite subspace. Clearly, since $P^+(U) \subseteq \mathcal{H}^+$ and $\mathcal{H}^+$ is closed, $\overline{P^+(U)} \subseteq \mathcal{H}^+$.

i) Suppose that U is maximal positive definite and suppose, towards a contradiction, that $\overline{P^+(U)} \neq \mathcal{H}^+$. Then there exists $x \in \mathcal{H}^+ \setminus \{0\}$ such that $x \perp_J \overline{P^+(U)}$ and, therefore, $x \notin U$. Further, note that, $x \perp_J U$ (since for every $u \in U$, $[x,u]_J = [P^+x, u]_J = [x, P^+u]_J = 0$). Therefore $U + \mathrm{Span}(x)$ is a positive definite space such that $U + \mathrm{Span}(x) \supsetneq U$ ↯.

ii) Suppose that $P^+(U) = \mathcal{H}^+$. Then U is closed and, by the theorem 4.2.6, U is a maximal positive semi-definite subspace. Suppose, towards a contradiction, that U is not a maximal positive definite subspace; then there exists a positive definite subspace $W \subseteq \mathcal{H}$, and therefore also positive semi-definite, such that $U \subsetneq W$ ↯. □

Corollary 4.2.9. If U is a maximal positive definite closed subspace of a Krein space, then U is a maximal positive semi-definite subspace.

Proposition 4.2.10. Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space and $U \subseteq X$ be a closed and definite subspace. Then U is uniformly definite if, and only if, $(U, \|\cdot\|_U)$ is a Banach space (where $\|\cdot\|_U$ is the norm induced by $[\cdot,\cdot]$ on U).

Proof. Since U is closed, $(U, \|\cdot\|_J)$ is a Banach space.

$\boxed{\Rightarrow}$ Suppose that U is uniformly definite and let $\gamma > 0$ such that $\|x\|_U^2 = |[x,x]| \geq \gamma \|x\|_J^2$ for every $x \in U$. Then, by the proposition 3.5.6, $\gamma \|x\|_J^2 \leq \|x\|_U^2 \leq \|x\|_J^2$ for every $x \in U$. That is, $\|\cdot\|_J$ and $\|\cdot\|_U$ are equivalent on U and, therefore, $(U, \|\cdot\|_U)$ is a Banach space.

$\boxed{\Leftarrow}$ Suppose that $(U, \|\cdot\|_U)$ is a Banach space. Then, by the theorem 3.5.7, $\|\cdot\|_U$ and $\|\cdot\|_J$ are equivalent and, therefore, there exists $\gamma > 0$ such that $|[x,x]| = \|x\|_U^2 \geq \gamma \|x\|_J^2$ for every $x \in U$. Hence, U is uniformly definite. □

Corollary 4.2.11. *If U is a closed and uniformly definite subspace of a J-space, then $\|\cdot\|_U$ and $\|\cdot\|_J$ are equivalent norms on U.*

Theorem 4.2.12. Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and U be a closed subspace of $\mathcal{H}$. Then, $(U, [\cdot,\cdot])$ is a Krein space if and only if U is decomposable with a fundamental decomposition $U = U^+ \dot{\oplus} U^-$ such that U^+ is uniformly positive and U^- is uniformly negative.

Proof. $\boxed{\Rightarrow}$ It is clear.

$\boxed{\Leftarrow}$ Suppose that U is decomposable with a fundamental decomposition $U = U^+ \dot{\oplus} U^-$ such that U^+ is uniformly positive and U^- is uniformly negative.

Suppose, towards a contradiction, that $U^+ \subsetneq \overline{U^+}$; thus, there exists $w \neq 0$ such that $w \in \overline{U^+} \cap U^-$. By the proposition 3.5.11, $\overline{U^+}$ is uniformly positive and, therefore, $[w, w] > 0$ and $[w, w] < 0$ $\lightning$. Hence U^+ is closed and, analogously, U^- is closed.

By the proposition 4.2.10, $(U^+, [\cdot, \cdot])$ and $(U^-, -[\cdot, \cdot])$ are Hilbert spaces and, therefore, U is a Krein space. $\square$

Theorem 4.2.13. Let $(\mathcal{H}, [\cdot, \cdot])$ be a J-space and U be a closed subspace. If $(U, [\cdot, \cdot])$ is a Krein space, then U is ortho-complemented in $\mathcal{H}$.

Proof. Suppose that $(U, [\cdot, \cdot])$ is a Krein space; let $U = U_+ \dot{\oplus} U_-$ be a Krein decomposition of U with associated fundamental symmetry J_0. Clearly, since $(U_+, [\cdot, \cdot])$ and $(U_-, -[\cdot, \cdot])$ are Hilbert spaces, $(U_+, \|\cdot\|_{U_+})$ and $(U_-, \|\cdot\|_{U_-})$ are Banach spaces.

By the corollary 4.2.11, $\|\cdot\|_{U_+}$ and $\|\cdot\|_{J_0}$ are equivalent norms on U_+ (since U_+ is a closed and uniformly positive subspace of U). Since $\|\cdot\|_{J_0}$ and $\|\cdot\|_J$ are equivalent norms on U (by the theorem 3.5.7), the norms $\|\cdot\|_{U_+}$ and $\|\cdot\|_J$ are equivalent on U_+. Let x be any vector in $\mathcal{H}$ and define the linear functional $\varphi_x : U_+ \to \mathbb{C}$, $\varphi_x(u) := [u, x]$, $u \in U_+$. Since $|\varphi_x(u)| = |[u, x]| \leq \|u\|_J \|x\|_J$ for every $u \in U_+$, then φ_x is bounded with respect to the J-norm and, therefore, with respect to $\|\cdot\|_{U_+}$. Thus, by Riesz's Theorem, there exists $u_+ \in U_+$ such that $\varphi_x(u) = [u, x] = [u, u_+]$ for every $u \in U_+$. That is, $x - u_+ \perp U_+$.

Hence, every vector of $\mathcal{H}$ has a projection on U_+ and, analogously, every vector of $\mathcal{H}$ has a projection on U_-. Therefore, every vector of $\mathcal{H}$ has a projection on U. That is, U is ortho-complemented in $\mathcal{H}$. $\square$

Theorem 4.2.14. Let $\mathcal{H}$ be a J-space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and $U \subseteq \mathcal{H}$ be a maximal semi-definite subspace. Then $U^\perp$ is a maximal semi-definite subspace too.

Proof. Suppose, without loss of generality, that U is maximal positive semi-definite; by the proposition 2.6.3, $U^\perp$ is negative semi-definite.

Let $K^+ : \mathcal{H}^+ \to \mathcal{H}^-$ be the angular operator of U with respect to $\mathcal{H}^+$, then $U = \{x + K^+x \mid x \in \mathcal{H}^+\}$. If $w \in \mathcal{H}$, $w = w_+ + w_-$ with $w_+ \in \mathcal{H}^+$ and $w_- \in \mathcal{H}^-$. Then for every $x \in \mathcal{H}^+$:

$$\begin{aligned}[x + K^+x, w] &= [x + K^+x, w_+ + w_-] = [x - K^+x, w_+ + w_-]_J \\ &= [x, w_+]_J - [K^+x, w_-]_J = [x, w_+]_J - [x, (K^+)^*w_-]_J \\ &= [x, w_+ - (K^+)^*w_-]_J.\end{aligned}$$

Thus, $w \perp U$ if and only if $(K^+)^*w_- = w_+$. Therefore, $U^\perp = \{(K^+)^*w_- + w_- \mid w_- \in \mathcal{H}^-\}$; that is, $(K^+)^* : \mathcal{H}^- \to \mathcal{H}^+$ is the angular operator of $U^\perp$ with respect to $\mathcal{H}^-$. By the theorem 4.2.6, since $P^-(U^\perp) = \mathcal{H}^-$, we have that $U^\perp$ is maximal negative semi-definite. □

4.3 The Gram Operator of a Closed Subspace

In this section $(\mathcal{H}, [\cdot,\cdot])$ is a Krein space with Krein decomposition $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ and, associated fundamental projectors and symmetry P^+, P^- and J, respectively. Also, U is a closed subspace of $\mathcal{H}$. Clearly $(U, [\cdot,\cdot]_J)$ is a Hilbert space because U is a closed subspace of $(\mathcal{H}, [\cdot,\cdot]_J)$.
Since $(\mathcal{H}, [\cdot,\cdot]_J)$ is a Hilbert space and U is closed, the J-orthogonal projection P_U of $\mathcal{H}$ on U exists.

Definition 4.3.1. The bounded linear operator $G_U : U \to U$ defined by $G_U := P_U J\restriction_U$ is called the *Gram operator* of U.

Remarks:

- Clearly, $\|G_U\| \leq 1$.
- For every $x,\ y \in U$ we have $[G_U x, y]_J = [P_U Jx, y]_J = [Jx, P_U y]_J = [Jx, y]_J = [x, y]$.
- G_U is self-adjoint in $(U, [\cdot,\cdot]_J)$ since $[G_U x, y]_J = [x, y] = \overline{[y, x]} = \overline{[G_U y, x]_J} = [x, G_U y]$ for every $x,\ y \in U$.
- By the Lax-Milgram Theorem (see [8]), G_U is the unique operator with the properties above.
- Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be the spectral resolution of G_U. Then, $G_U = \int_{-1^-}^{1} \lambda dE_\lambda$.

- Let $P_U^- = \int_{-1^-}^{+0} \lambda dE_\lambda = E_{-0}$, $P_U^+ = \int_{-0}^{1} \lambda dE_\lambda = I - E_0$ and $P_U^0 = E_0 - E_{-0}$. Clearly, P_U^-, P_U^+ and P_U^0 are orthogonal projections and they are pairwise orthogonal. Moreover, $P_U^- + P_U^+ + P_U^0 = I_U$ (where I_U is the identity operator on U).

- From the above we obtain:

$$U = U^+ \oplus_J U^- \oplus_J U^0 \;\; [\text{with } U^+ = P_U^+(U), U^- = P_U^-(U) \text{ and } U^0 = P_U^0(U)]. \tag{4.2}$$

Theorem 4.3.2. In the decomposition (4.2), U^+ and U^- are positive definite and negative definite subspaces with $U^+ \perp U^-$. Moreover, U^0 is the isotropic part of U.

Proof. (1) Note that $U^0 = (E_0 - E_{-0})(U) = \ker(G_U)$. If $x \in U$, then $x \perp U \Leftrightarrow [x, u] = 0$ for every $u \in U \Leftrightarrow [Jx, u]_J = 0$ for every $u \in U \Leftrightarrow [Jx, P_U u]_J = 0$ for every $u \in U \Leftrightarrow [P_U Jx, u]_J = 0$ for every $u \in U \Leftrightarrow [G_U x, u]_J = 0$ for every $u \in U \Leftrightarrow G_U x = 0 \Leftrightarrow x \in \ker(G_U) = U^0$. That is U^0 is the isotropic part of U.

(2) Let $x \in U^+$ with $x \neq 0$. Then $[x, x] = [G_U x, x]_J = \int_{-1^-}^{1} \lambda d[E_\lambda x, x]_J = \int_{-1^-}^{1} \lambda d[E_\lambda x, P_U^+ x]_J = \int_{-1^-}^{1} \lambda d[E_\lambda x, (I - E_0)x]_J = \int_{-1^-}^{1} \lambda d[(I - E_0)E_\lambda x, x]_J = \int_{-1^-}^{1} \lambda d[(E_\lambda - E_0 E_\lambda)x, x]_J$.

Since $[(E_\lambda - E_0 E_\lambda)x, x]_J = 0$ if $\lambda < 0$ and $[(E_\lambda - E_0 E_\lambda)x, x]_J \geq 0$ if $\lambda \geq 0$ because $t \mapsto [E_t x, x]_J$ is an increasing function and $E_\mu E_\lambda = E_\lambda E_\mu = E_{\min\{\mu,\lambda\}}$, then $[x, x] \geq 0$. Therefore, U^+ is positive semi-definite and, analogously, U^- is negative semi-definite.

Suppose, towards a contradiction, that x is neutral. Then, clearly, $x \in U^0$ $\lightning$ (since U^+, U^- and U^0 are linearly independent). Hence, U^+ is positive definite and, analogously, U^- is negative definite.

(3) Let $x \in U^+$ and $y \in U^-$. Then $[x, y] = [G_U x, y]_J = \int_{-1^-}^{1} \lambda d[E_\lambda x, y]_J = \int_{-1^-}^{1} \lambda d[E_\lambda (I - E_0)x, E_{-0} y]_J = \int_{-1^-}^{1} \lambda d[E_{-0} E_\lambda (I - E_0)x, y]_J = 0$ (since $E_{-0} E_\lambda (I - E_0) = 0$ for every $\lambda \in \mathbb{R}$). Hence $U^+ \perp U^-$. □

Corollary 4.3.3. U is non-degenerate if and only if $\ker(G_U) = \{0\}$.

Corollary 4.3.4. U is decomposable with fundamental decomposition $U = U^+ \dot{\oplus} U^- \dot{\oplus} U^0$.

Remark. Since for every $x \in U$, $[x,x] = [G_U x, x]_J$, the subspace U is positive semi-definite (negative semi-definite) if and only if G_U is a non-negative (non-positive) self-adjoint operator. Further, U is positive definite (negative definite) if and only if, for every $x \in U$ with $x \neq 0$, $[G_U x, x]_J > 0$ ($[G_U x, x]_J < 0$).

Also, it is clear that U is uniformly positive (uniformly negative) if and only if $G_U \gg 0$ ($G_U \ll 0$).

Theorem 4.3.5. If $0 \in \rho(G_U)$, then $(U, [\cdot,\cdot])$ is a Krein space.

Proof. If $0 \in \rho(G_U)$ then $\ker(G_U) = \{0\}$ and therefore $U = U^+ \dot{\oplus} U^-$. Moreover, there exists $\alpha > 0$ such that $(-\alpha, \alpha) \subseteq \rho(G_U)$ and therefore U^+ is uniformly positive and U^- is uniformly negative. Thus, by the theorem 4.2.12, $(U, [\cdot,\cdot])$ is a Krein space. □

Now we show that the converse of the theorem 4.2.13 holds.

Theorem 4.3.6. Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space and U be a closed subspace. If U is ortho-complemented in $\mathcal{H}$, then $0 \in \rho(G_U)$ and $(U, [\cdot,\cdot])$ is a Krein space.

Proof. Suppose that $\mathcal{H} = U \dot{\oplus} U^{\perp}$ and suppose, without loss of generality, that $U \neq \{0\}$. Let G_U be the Gram operator of U and let Q, $I - Q$ be the fundamental projectors associated with the decomposition $\mathcal{H} = U \dot{\oplus} U^{\perp}$.

Let $x \in U$ with $\|x\|_J = 1$ and let $y = \frac{1}{\|x\|_J} Jx$, then $1 = \|x\|_J = \frac{\|x\|_J^2}{\|x\|_J} = \frac{1}{\|x\|_J}[x,x]_J = [x,y] = [x, Qy] = [G_U x, Qy]_J \leq \|G_U x\|_J \|Q\|$. Hence, in this case, G_U is injective and its inverse is bounded and closed; thus $\mathscr{R}(G_U) = U$.

Therefore $0 \in \rho(G_u)$ and, by the theorem 4.3.5, $(U, [\cdot,\cdot])$ is a Krein space. □

Theorem 4.3.7. Let $\mathcal{H}$ be a J-space and U be a closed and definite subspace. Then, U is ortho-complemented in $\mathcal{H}$ if and only if U is uniformly definite.

Proof. Suppose, without loss of generality, that U is positive definite.

Suppose that U is ortho-complemented in $\mathcal{H}$. By the theorem 4.3.6, $0 \in \rho(G_U)$. For every $x \in U$ with $\|x\|_J = 1$ we have $[G_U x, x]_J = [x,x] > 0$. Further, since $0 \in \rho(G_U)$, there exists $\alpha > 0$ such that $[G_U x, x]_J \geq \alpha$ for every $x \in U$ with $\|x\|_J = 1$. Thus $G_U \gg 0$ and, therefore, U is uniformly definite.

Suppose that U is uniformly definite. By the proposition 4.2.10, $(U, \|\cdot\|_U)$ is a Banach space and, therefore, $(U, [\cdot,\cdot])$ is a Krein space. Thus, by the theorem 4.2.13, U is ortho-complemented in $\mathcal{H}$. □

Theorem 4.3.8. Let $(\mathcal{H}, [\cdot,\cdot])$ be a non-degenerate and decomposable inner product space. Then $\mathcal{H}$ is a Krein space if and only if, for every associated fundamental symmetry J, the J-inner product turns $\mathcal{H}$ into a Hilbert space.

Proof. $\boxed{\Rightarrow}$ Suppose that $\mathcal{H}$ is a Krein space. Let $\mathcal{H} = \mathcal{H}^+ \dot{\oplus} \mathcal{H}^-$ be a fundamental decomposition of $\mathcal{H}$ with associated fundamental symmetry J. By the proposition 4.2.4, $\mathcal{H}^+$ and $\mathcal{H}^-$ are closed subspaces. Therefore, by the theorem 4.3.6, $(\mathcal{H}^+, [\cdot,\cdot])$ and $(\mathcal{H}^-, -[\cdot,\cdot])$ are Hilbert spaces. Hence $(\mathcal{H}, [\cdot,\cdot]_J)$ is a Hilbert space.

$\boxed{\Leftarrow}$ This is an immediate consequence of the final remark in the section 4.1. □

Remark. By the theorem 4.3.8 and the propositions 3.5.6 and 3.5.7, if $(\mathcal{H}, [\cdot,\cdot])$ is a Krein space then any two associated fundamental symmetries induce the same topology on $\mathcal{H}$.

4.4 Linear Operators in Krein Spaces

Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space. If we refer to topological properties in this space, like e.g. density or closedness, we always use the topology on $\mathcal{H}$ which is induced by J.

Definition 4.4.1. Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space and let S be a closed and densely defined linear operator acting in $\mathcal{H}$. Define the *adjoint* $S^\star$ of S in the Krein space $\mathcal{H}$ by:

- $\mathscr{D}(S^\star) = \{x \in \mathcal{H} \mid (\exists x' \in \mathcal{H})(\forall y \in \mathscr{D}(S))(\ [Sy, x] = [y, x'])\}$,
- $S^\star x = x'$ for every $x \in \mathscr{D}(S^\star)$.

Remark. Clearly $S^\star$ is well-defined since, as $\mathcal{D}(S)$ is dense, if $[y, x'] = [y, x'']$ for every $y \in \mathscr{D}(S)$ then $x' = x''$ (since the inner product is non-degenerate).

Proposition 4.4.2. Let $\mathcal{H}$ be a J-space and let S be a closed and densely defined linear operator acting in $\mathcal{H}$. If S^* is the adjoint of S in the Hilbert space $(\mathcal{H}, [\cdot,\cdot]_J)$, then $S^\star = JS^*J$.

Proof. Note that $\mathscr{D}(JS^*J) = J(\mathscr{D}(S^*))$. Let $x \in J(\mathscr{D}(S^*))$, then (since $J^2 = I$) $Jx \in \mathscr{D}(S^*)$ and, for every $y \in \mathscr{D}(S)$, $[Sy, x] = [JSy, Jx] = [Sy, Jx]_J = [y, S^*Jx]_J = [Jy, S^*Jx] = [y, JS^*Jx]$; that is $x \in \mathscr{D}(S^\star)$ and $S^\star x = JS^*Jx$.

Let $x \in \mathscr{D}(S^\star)$. Note that, for every $y \in \mathscr{D}(S)$, $[Sy, Jx]_J = [JSy, Jx] = [Sy, x] = [y, S^\star x] = [Jy, JS^\star x] = [y, JS^\star x]_J = [y, JS^\star JJx]_J$; that is, $Jx \in \mathscr{D}(S^*)$ and, therefore, $x \in J(\mathscr{D}(S^*))$. Hence, $S^\star = JS^*J$. □

Corollary 4.4.3. Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space and S be a closed and densely defined linear operator acting in $\mathcal{H}$. Then:

- $S^\star$ is closed and densely defined.
- $(\alpha S)^\star = \overline{\alpha} S^\star$ for every $\alpha \in \mathbb{C}$.
- $((S - \lambda)^{-1})^\star = (S^\star - \overline{\lambda})^{-1}$ for every $\lambda \in \rho(S)$.
- $\sigma(S^\star) = \{\overline{\lambda} \mid \lambda \in \sigma(S)\}$.

Definition 4.4.4. Let $(\mathcal{H}, [\cdot,\cdot])$ be a J-space and let S be a closed and densely defined linear operator acting in $\mathcal{H}$. Then S is called:

- *symmetric* in the Krein space if and only if $S \subseteq S^\star$.
- *self-adjoint* in the Krein space if and only if $S = S^\star$.
- *unitary* in the Krein space if and only if $\mathscr{D}(S) = \mathcal{H}$ and $SS^\star = S^\star S = I$.

Remark. If the operator S is self-adjoint in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ then, by the corollary 4.4.3, $\sigma(S)$ is a symmetric subset of $\mathbb{C}$ with respect to the real axis.

Remark. If $U : \mathcal{H} \longrightarrow \mathcal{H}$ is unitary in the Krein space $\mathcal{H}$, then:

- $[Ux, Uy] = [x, y]$ for every x, $y \in \mathcal{H}$.
- $JU = UJ$ since, for every $x \in \mathcal{H}$, $JUx = J(Ux^+ + Ux^-) = Ux^+ - Ux^- = U(x^+ - x^-) = UJx$.
- U is a bounded operator in the topology induced by the J-norm, with $\|U\| = 1$, since $\|Ux\|_J^2 = [Ux, Ux]_J = [JUx, Ux] = [UJx, Ux] = [Jx, x] = [x, x]_J = \|x\|_J^2$ for every $x \in \mathcal{H}$.

Definition 4.4.5. Let S be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ with associated fundamental symmetry J. The point $\lambda \in \sigma(S)$ is called of *positive type* (resp., *negative type*) if there is a sequence $(x_n)_{n\in\mathbb{N}}$, with $\|x_n\|_J = 1$ and $[x_n, x_n] > 0$ (resp., $[x_n, x_n] < 0$) for every $n \in \mathbb{N}$, such that $(S - \lambda I)x_n \to 0$.

The set of points of positive type (resp., negative type) is denoted by $\sigma^+(S)$ (resp., $\sigma^-(S)$).

Definition 4.4.6. Let S be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ and $\lambda \in \sigma_p(S)$. Then, $\lambda \in \sigma_p^+(S)$ (resp., $\lambda \in \sigma_p^-(S)$) if, and only if, all its associated eigenvectors are positive (resp., negative).

Chapter 5

Riesz Basis

5.1 Bessel sequences

Lemma 5.1.1. Let $\mathcal{H}$ be a separable Hilbert space and let $(v_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that, for every $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$, the series $\sum_{j=1}^{\infty} \alpha_j v_j$ is convergent. Then,

$$T : l_2(\mathbb{N}) \longrightarrow \mathcal{H}, \quad T((\alpha_j)_{j\in\mathbb{N}}) := \sum_{j=1}^{\infty} \alpha_j v_j$$

is a bounded linear operator; its adjoint operator is given by

$$T^* : \mathcal{H} \longrightarrow l_2(\mathbb{N}), \quad T^*v = (\langle v, v_j\rangle)_{j\in\mathbb{N}}$$

and, for every $v \in \mathcal{H}$, $\sum_{j=1}^{\infty} |\langle v, v_j\rangle|^2 \leq \|T\|^2\|v\|^2$.

Proof. Clearly T is well-defined and it is a linear operator. For each $n \in \mathbb{N}$ let $T_n : l_2(\mathbb{N}) \longrightarrow \mathcal{H}$, $T_n((\alpha_j)_{\in\mathbb{N}}) := \sum_{j=1}^{n} \alpha_j v_j$; then each T_n is a bounded linear operator and, for every $\alpha \in l_2(\mathbb{N})$, $T_n\alpha \to T\alpha$ $(n \to \infty)$. So $(T_n)_{n\in\mathbb{N}}$ is a family of pointwise bounded operators and, by the Banach-Steinhaus theorem, this family is bounded; therefore T is bounded.

Let $v \in \mathcal{H}$ and let $(y_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$ be such that $T^*v = (y_j)_{j\in\mathbb{N}}$; then for every $\alpha = (\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$ we have $\sum_{j=1}^{\infty} \alpha_j \overline{y_j} = \langle \alpha, T^*v\rangle = \langle T\alpha, v\rangle = \sum_{j=1}^{\infty} \alpha_j \langle v_j, v\rangle$. So $T^*v = (\langle v, v_j\rangle)_{j\in\mathbb{N}}$.

As T^* is bounded (since T is so) and $\|T^*\| = \|T\|$ then, for every $v \in \mathcal{H}$, $\sum_{j=i}^{\infty} |\langle v, v_j\rangle|^2 = \|T^*v\|^2 \leq \|T\|^2\|v\|^2$. □

Definition 5.1.2. Let $\mathcal{H}$ be a Hilbert space. A sequence $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is called a *Bessel sequence* if and only if there exists a constant $B > 0$ such that, for every $v \in \mathcal{H}$, $\sum_{j=1}^{\infty} |\langle v, v_j\rangle|^2 \leq B\|v\|^2$. In this case B is called a *Bessel bound* for $(v_j)_{j\in\mathbb{N}}$.

Remarks:

- It is clear that if $(v_j)_{j\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ then a sufficient and necessary condition for $(v_j)_{j\in\mathbb{N}}$ to be a Bessel sequence is that there exists $B > 0$ and $U \subseteq \mathcal{H}$ dense subset such that $\sum_{j=1}^{\infty} |\langle u, v_j\rangle|^2 \leq B\|u\|^2$ for every $u \in U$.

- Note that, by lemma 5.1.1, if $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is a sequence such that $\sum_{j=1}^{\infty} \alpha_j v_j$ is convergent for every $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$, then $(v_j)_{j\in\mathbb{N}}$ is a Bessel sequence.

- If $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is a Bessel sequence then, for every $v \in \mathcal{H}$, the series $\sum_{j=1}^{\infty} |\langle v, v_j\rangle|^2$ is absolutely convergent and therefore unconditionally convergent. So, for every permutation φ of $\mathbb{N}$, $(v_{\varphi(j)})_{j\in\mathbb{N}}$ is a Bessel sequence.

Proposition 5.1.3. Let $\mathcal{H}$ be a Hilbert space; $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ and $B > 0$. Then $(v_j)_{j\in\mathbb{N}}$ is a Bessel sequence with Bessel bound B if and only if $T : (\alpha_j)_{j\in\mathbb{N}} \mapsto \sum_{j=1}^{\infty} \alpha_j v_j$ defines a bounded linear operator from $l_2(\mathbb{N})$ into $\mathcal{H}$ and $\|T\| \leq \sqrt{B}$.

Proof. $\boxed{\Rightarrow}$ Suppose that $(v_j)_{j\in\mathbb{N}}$ is a Bessel sequence with Bessel bound B y let $S : \mathcal{H} \longrightarrow l_2(\mathbb{N})$, $Sv := (\langle v, v_j\rangle)_{j\in\mathbb{N}}$. Clearly, as $(v_j)_{j\in\mathbb{N}}$ is a Bessel sequence, S is well-defined and it is a bounded linear operator with $\|S\| \leq \sqrt{B}$.
Note that S is the adjoint of the operator defined in lemma 5.1.1 and therefore $S^* = T : (\alpha_j)_{j\in\mathbb{N}} \mapsto \sum_{j=1}^{\infty} \alpha_j v_j$ is a bounded linear operator from $l_2(\mathbb{N})$ into $\mathcal{H}$ with $\|T\| \leq \sqrt{B}$.
$\boxed{\Leftarrow}$ Suppose that $T : (\alpha_j)_{j\in\mathbb{N}} \mapsto \sum_{j=1}^{\infty} \alpha_j v_j$ defines a bounded linear operator from $l_2(\mathbb{N})$ into $\mathcal{H}$ with $\|T\| \leq \sqrt{B}$; then, by lemma 5.1.1, for every $v \in \mathcal{H}$, $\sum_{j=1}^{\infty} |\langle v, v_j\rangle|^2 \leq \|T\|^2\|v\|^2 \leq B\|v\|^2$. So $(v_j)_{j\in\mathbb{N}}$ is a Bessel sequence with Bessel bound B. □

Corollary 5.1.4. If $\mathcal{H}$ is a Hilbert space and $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is a Bessel sequence, then $\sum_{j=1}^{\infty} \alpha_j v_j$ converges *unconditionally* for every $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$.

5.2 Basis

Definition 5.2.1. Let A be a subset of the Hilbert space $\mathcal{H}$. Then A is called a *total subset*, or a *complete system*, of $\mathcal{H}$ if and only if $\overline{\text{Span}}(A) = \mathcal{H}$.

Definition 5.2.2. Let $\mathcal{H}$ be a Hilbert space and let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ be a sequence.

i) The sequence $(u_j)_{j\in\mathbb{N}}$ is a *(Schauder) basis* for $\mathcal{H}$ if, and only if, for each $v \in \mathcal{H}$ there exists unique scalar coefficients $(\alpha_j)_{j\in\mathbb{N}}$ such that

$$v = \sum_{j=1}^{\infty} \alpha_j u_j. \tag{5.1}$$

ii) The sequence $(u_j)_{j\in\mathbb{N}}$ is an *unconditional basis* if, and only if, it is a basis and the series (5.1) converges unconditionally for each $v \in \mathcal{H}$.

iii) The sequence $(u_j)_{j\in\mathbb{N}}$ is an *orthonormal basis* if, and only if, it is a basis and it is an orthonormal system.

Remark. If $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is an orthonormal system then $(u_j)_{j\in\mathbb{N}}$ is a Bessel sequence in $\mathcal{H}$ since, for every $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$, the series $\sum_{j=1}^{\infty} \alpha_j u_j$ is convergent (since if $m, n \in \mathbb{N}$ with $m > n$ then $\left\|\sum_{j=1}^{m} \alpha_j u_j - \sum_{j=1}^{n} \alpha_j u_j\right\|^2 = \left\|\sum_{j=n+1}^{m} \alpha_j u_j\right\|^2 = \sum_{j=n+1}^{m} |\alpha_j|^2 \to 0$ if $n \to \infty$).

Theorem 5.2.3. Let $\mathcal{H}$ be a Hilbert space and let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ be an orthonormal system. The following are equivalent:

i) $(u_j)_{j\in\mathbb{N}}$ is an orthonormal basis.

ii) For every $v \in \mathcal{H}$, $v = \sum_{j=1}^{\infty} \langle v, u_j\rangle u_j$.

iii) For every $v, w \in \mathcal{H}$, $\langle v, w\rangle = \sum_{j=1}^{\infty} \langle v, u_j\rangle \langle u_j, w\rangle$.

iv) For every $v \in \mathcal{H}$, $\|v\|^2 = \sum_{j=1}^{\infty} |\langle v, u_j\rangle|^2$.

v) $(u_j)_{j\in\mathbb{N}}$ is a total subset of $\mathcal{H}$.

vi) If $\langle v, u_j\rangle = 0$ for every $j \in \mathbb{N}$, then $v = 0$.

Proof. $\boxed{\text{i)} \Rightarrow \text{ii)}}$ Let $v \in \mathcal{H}$. Since $(u_j)_{j\in\mathbb{N}}$ is a basis for $\mathcal{H}$, there exists $(\alpha_j)_{j\in\mathbb{N}}$ such that $v = \sum_{j=1}^{\infty} \alpha_j u_j$. Fix $n \in \mathbb{N}$. Since $(u_j)_{j\in\mathbb{N}}$ is an orthonormal system, it follows that $\langle v, u_n\rangle = \langle \sum_{j=1}^{\infty} \alpha_j u_j, u_n\rangle = \sum_{j=1}^{\infty} \alpha_j \langle u_j, u_n\rangle = \alpha_n$. So, $v = \sum_{j=1}^{\infty} \langle v, u_j\rangle u_j$.

$\boxed{\text{ii)} \Rightarrow \text{iii)}}$ Let $v, w \in \mathcal{H}$. As $v = \sum_{j=1}^{\infty} \langle v, u_j\rangle u_j$, then

$$\langle v, w\rangle = \langle \sum_{j=1}^{\infty} \langle v, u_j\rangle u_j, w\rangle = \sum_{j=1}^{\infty} \langle v, u_j\rangle \langle u_j, w\rangle.$$

$\boxed{\text{iii)} \Rightarrow \text{iv)}}$ Is clear.

$\boxed{\text{iv)} \Rightarrow \text{v)}}$ Let $v \in \mathcal{H}$ be such that $v \perp \overline{\text{Span}}((u_j)_{j\in\mathbb{N}})$; then, for every $j \in \mathbb{N}$, $\langle v, u_j\rangle = 0$. So $\|v\| = 0$ and therefore $v = 0$. Then, $\mathcal{H} = \overline{\text{Span}}((u_j)_{j\in\mathbb{N}})$.

$\boxed{\text{v)} \Rightarrow \text{vi)}}$ Suppose, towards a contradiction, that there exists $v \neq 0$ such that, for every $j \in \mathbb{N}$, $\langle v, u_j\rangle = 0$; then $(\overline{\text{Span}}((u_j)_{j\in\mathbb{N}}))^{\perp} \neq \{0\}$ and therefore $\mathcal{H} \neq \overline{\text{Span}}((u_j)_{j\in\mathbb{N}})$ $\lightning$.

$\boxed{\text{vi)} \Rightarrow \text{i)}}$ Let $v \in \mathcal{H}$. Then $\sum_{j=1}^{\infty} \langle v, u_j\rangle u_j \in \mathcal{H}$ because $(u_j)_{j\in\mathbb{N}}$ is a Bessel sequence (since it is an orthonormal system). Note that, for every $n \in \mathbb{N}$, $\langle v - \sum_{j=1}^{\infty} \langle v, u_j\rangle u_j, u_n\rangle = 0$; therefore $v = \sum_{j=1}^{\infty} \langle v, u_j\rangle u_j$. Clearly, if $(\alpha_j)_{j\in\mathbb{N}}$ is such that $v = \sum_{j=1}^{\infty} \alpha_j u_j$ then $\alpha_j = \langle v, u_j\rangle$ for each $j \in \mathbb{N}$ since $(u_j)_{j\in\mathbb{N}}$ is an orthonormal system. Therefore $(u_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. $\square$

Remarks:

- If $\mathcal{H}$ is a Hilbert space and $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is an orthonormal basis (and therefore a Bessel sequence), then each $v \in \mathcal{H}$ has an unconditionally convergent expansion $v = \sum_{j=1}^{\infty} \langle v, u_j\rangle u_j$.
- Clearly, every separable Hilbert space has an orthonormal basis. Further, if $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, then $\mathcal{H}$ is isometrically isomorphic to $l_2(\mathbb{N})$.

Theorem 5.2.4. Let $\mathcal{H}$ be a Hilbert space and let $(v_j)_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Then every orthonormal basis for $\mathcal{H}$ is of the form $(Uv_j)_{j\in\mathbb{N}}$, where $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a unitary operator.

Proof. Let $U : \mathcal{H} \longrightarrow \mathcal{H}$ be a unitary operator on $\mathcal{H}$; clearly $(Uv_j)_{j\in\mathbb{N}}$ is an orthonormal system. Let $w \in \mathcal{H}$ such that $\langle w, Uv_j\rangle = 0$ for every $j \in \mathbb{N}$ and let $v \in \mathcal{H}$ such that $w = Uv$, then $0 = \langle w, Uv_j\rangle = \langle Uv, Uv_j\rangle = \langle v, v_j\rangle$ for every $j \in \mathbb{N}$; so $v = 0$ and $w = 0$. Therefore, by theorem 5.2.3, $(Uv_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$.

Let $(w_j)_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Define $U : \mathcal{H} \longrightarrow \mathcal{H}$, $Uv := \sum_{j=1}^{\infty} \langle v, v_j\rangle w_j$. As $(\langle v, v_j\rangle)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$, the operator U is well-defined; moreover, clearly U is a bijective linear operator on $\mathcal{H}$ such that, for every $j \in \mathbb{N}$, $w_j = Uv_j$. Finally, it follows that U is unitary because $(w_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$ and therefore $\|Uv\|^2 = \|\sum_{j=1}^{\infty} \langle v, v_j\rangle w_j\|^2 = \sum_{j=1}^{\infty} |\langle v, v_j\rangle|^2 = \|v\|^2$ for every $v \in \mathcal{H}$. □

Proposition 5.2.5. Let $\mathcal{H}$ be a Hilbert space and let $(u_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ be a sequence of unitary vectors such that $\|v\|^2 = \sum_{j=1}^{\infty} |\langle v, u_j\rangle|^2$ for every $v \in \mathcal{H}$. Then $(u_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$.

Proof. By therorem 5.2.3 it is sufficient to show that $(u_j)_{j\in\mathbb{N}}$ is an orthogonal sequence. Fix $n \in \mathbb{N}$; then $1 = \|u_n\|^2 = \sum_{j=1}^{\infty} |\langle u_n, u_j\rangle|^2 = |\langle u_n, u_n\rangle|^2 + \sum_{j\neq n} |\langle u_n, u_j\rangle|^2 = 1 + \sum_{j\neq n} |\langle u_n, u_j\rangle|^2$. So $\sum_{j\neq n} |\langle u_n, u_j\rangle|^2 = 0$ and therefore $\langle u_n, u_j\rangle =$ for every $j \neq n$. This is, $(u_j)_{j\in\mathbb{N}}$ is an orthogonal sequence. □

5.3 Riesz Basis

Definition 5.3.1. Let $\mathcal{H}$ be a Hilbert space. A *Riesz basis* for $\mathcal{H}$ is a sequence of the form $(Uv_j)_{j\in\mathbb{N}}$ where $(v_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bijective bounded linear operator.

Remark. The operator U in definition 5.3.1 has bounded inverse by the inverse mapping theorem.

Proposition 5.3.2. Every Riesz basis for a Hilbert space is a basis.

Proof. Let $(w_j)_{j\in\mathbb{N}}$ be a Riesz basis for $\mathcal{H}$; then there exists an orthomormal basis $(v_j)_{j\in\mathbb{N}}$ for $\mathcal{H}$ and a bijective bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ such that, for every $j \in \mathbb{N}$, $w_j = Uv_j$.

Let $w \in \mathcal{H}$ and let $v \in \mathcal{H}$ such that $w = Uv$; since $(v_j)_{j\in\mathbb{N}}$ is a basis, there exist unique coefficients $(\alpha_j)_{j\in\mathbb{N}}$ such that $v = \sum_{j=1}^{\infty} \alpha_j v_j$. So, $w = Uv = U(\sum_{j=1}^{\infty} \alpha_j v_j) = \sum_{j=1}^{\infty} \alpha_j Uv_j = \sum_{j=1}^{\infty} \alpha_j w_j$. Clearly the coefficients $(\alpha_j)_{j\in\mathbb{N}}$ are unique such that $w = \sum_{j=1}^{\infty} \alpha_j w_j$ since, if $(\beta_j)_{j\in\mathbb{N}}$ is such that $w = \sum_{j=1}^{\infty} \beta_j w_j$, then, as $v = U^{-1}w$ and U^{-1} is a bounded linear operator, $v = \sum_{j=1}^{\infty} \beta_j v_j$ and therefore $(\alpha_j)_{j\in\mathbb{N}} = (\beta_j)_{j\in\mathbb{N}}$. □

Proposition 5.3.3. Let $(v_j)_{j\in\mathbb{N}}$ be a Riesz basis for a Hilbert space $\mathcal{H}$. Then there exist constants $A, B > 0$ such that, for every $x \in \mathcal{H}$, $A\|x\|^2 \leq \sum_{j=1}^{\infty} |\langle x, v_j\rangle|^2 \leq B\|x\|^2$.

Proof. Choose an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ and a bijective bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ such that $v_j = Ue_j$ for every $j \in \mathbb{N}$. Then, for every $x \in \mathcal{H}$, $\sum_{j=1}^{\infty} |\langle x, v_j\rangle|^2 = \sum_{j=1}^{\infty} |\langle x, Ue_j\rangle|^2 = \sum_{j=1}^{\infty} |\langle U^*x, e_j\rangle|^2 = \|\sum_{j=1}^{\infty} \langle U^*x, e_j\rangle e_j\|^2 = \|U^*x\|^2$. So, with $B = \|U\|^2$, we have, $\sum_{j=1}^{\infty} |\langle x, v_j\rangle|^2 = \|U^*x\|^2 \leq B\|x\|^2$.
Now, for every $x \in \mathcal{H}$, $\|x\|^2 = \|(U^*)^{-1}U^*x\|^2 \leq \|U^{-1}\|^2\|U^*x\|^2$. So, with $A = 1/\|U^{-1}\|^2$, we have $A\|x\|^2 \leq \sum_{j=1}^{\infty} |\langle x, v_j\rangle|^2$. □

Corollary 5.3.4. Every Riesz basis is a Bessel sequence.

Theorem 5.3.5. Let $\mathcal{H}$ be a Hilbert space and let $(v_j)_{j\in\mathbb{N}}$ be a Riesz basis for $\mathcal{H}$. Then there exists a unique Riesz basis for $\mathcal{H}$, $(w_j)_{j\in\mathbb{N}}$, such that for every $x \in \mathcal{H}$:

$$x = \sum_{j=1}^{\infty} \langle x, w_j\rangle v_j. \tag{5.2}$$

The series (5.2) converges unconditionally for every $x \in \mathcal{H}$.

Proof. Let $(v_j)_{j\in\mathbb{N}}$ be a Riesz basis. Choose an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ for $\mathcal{H}$ and a bijective bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ such that $v_j = Ue_j$ for every $j \in \mathbb{N}$.
Let $x \in \mathcal{H}$; as $(v_j)_{j\in\mathbb{N}}$ is a basis for $\mathcal{H}$ (by proposition 5.3.2), there exists a unique scalar sequence $(\beta_j)_{j\in\mathbb{N}}$ such that $x = \sum_{j=1}^{\infty} \beta_j v_j$. Then, as U^{-1} is a bounded linear operator, $U^{-1}x = U^{-1}(\sum_{j=1}^{\infty} \beta_j v_j) = \sum_{j=1}^{\infty} \beta_j U^{-1}v_j = \sum_{j=1}^{\infty} \beta_j e_j$; so, $\beta_j = \langle U^{-1}x, e_j\rangle = \langle x, (U^{-1})^*e_j\rangle$ for every $j \in \mathbb{N}$. Therefore, with $w_j := (U^{-1})^*e_j$ for every $j \in \mathbb{N}$, we have that $(w_j)_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ such that (5.2) holds. Clearly, if $(z_j)_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ such that $x = \sum_{j=1}^{\infty} \langle x, z_j\rangle v_j$ for every $x \in \mathcal{H}$, then $\langle x, z_j\rangle = \langle x, w_j\rangle$ for every $x \in \mathcal{H}$; so $z_j = w_j$ for each $j \in \mathbb{N}$.
Let $x \in \mathcal{H}$. Since $(w_j)_{j\in\mathbb{N}}$ is a Bessel sequence then $(\langle x, w_j\rangle)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$; so, by corollary 5.1.4, the series (5.2) converges unconditionally. □

Remark. The unique Riesz basis $(w_j)_{j\in\mathbb{N}}$ satisfying (5.2) in the theorem 5.3.5 is called the *dual Riesz basis* of $(v_j)_{j\in\mathbb{N}}$, and the proof of the theorem shows that $w_j = (U^{-1})^*e_j$ for each $j \in \mathbb{N}$.

Now the dual Riesz basis of $(w_j)_{j\in\mathbb{N}}$ is given by $(((U^{-1})^*)^{-1})^* e_j = Ue_j = v_j$ for each $j \in \mathbb{N}$; that is, $(v_j)_{j\in\mathbb{N}}$ is the dual Riesz basis of $(w_j)_{j\in\mathbb{N}}$. In this case, $(v_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ are called a *pair of dual Riesz bases* and $\sum_{j=1}^{\infty} \langle x, w_j\rangle v_j = \sum_{j=1}^{\infty} \langle x, v_j\rangle w_j$ for every $x \in \mathcal{H}$.

Corollary 5.3.6. Let $\mathcal{H}$ be a Hilbert space and let $(v_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ be a pair of dual Riesz bases for $\mathcal{H}$. Then $(v_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ are *biorthogonal* sequences; that is, $\langle v_j, w_k\rangle = \delta_{j,k}$ for every $j, k \in \mathbb{N}$.

Proof. Fix $j \in \mathbb{N}$; then $v_j = \sum_{k=1}^{\infty} \langle v_j, w_k\rangle v_k$ and, since $(v_j)_{j\in\mathbb{N}}$ is a basis for $\mathcal{H}$, $\langle v_j, w_j\rangle = 1$ and $\langle v_j, w_k\rangle = 0$ for every $k \neq j$. □

Lemma 5.3.7. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}_1$ and $(w_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}_2$. Suppose that $(v_j)_{j\in\mathbb{N}}$ is complete in $\mathcal{H}_1$, that $(w_j)_{j\in\mathbb{N}}$ is a Bessel sequence with Bessel bound B, and that there exists a constant $A > 0$ such that, for every finite scalar sequence (α_j),

$$A \sum |\alpha_j|^2 \leq \|\sum \alpha_j v_j\|^2.$$

Then the map $U : \sum \alpha_j v_j \mapsto \sum \alpha_j w_j$ defines a bounded linear operator from $\mathrm{Span}((v_j)_{j\in\mathbb{N}})$ into $\mathrm{Span}((w_j)_{j\in\mathbb{N}})$ and U has a unique extension to a bounded linear operator from $\mathcal{H}_1$ into $\mathcal{H}_2$; the norm of U as well as its extension is at most $\sqrt{B/A}$.

Proof. If (α_j), (β_j) are finite sequences (without loss of generality suppose that both have the same length) such that $\sum \alpha_j v_j = \sum \beta_j v_j$, then $A \sum |\alpha_j - \beta_j|^2 \leq \|\sum (\alpha_j - \beta_j) v_j\|^2 = 0$; so, $(\alpha_j) = (\beta_j)$ and therefore U is well-defined. Clearly U is a linear operator from $\mathrm{Span}((v_j)_{j\in\mathbb{N}})$ into $\mathrm{Span}((w_j)_{j\in\mathbb{N}})$.

Let $v \in \mathrm{Span}((v_j)_{j\in\mathbb{N}})$ and let (α_j) the finite scalar sequence such that $v = \sum \alpha_j v_j$; then $\|Uv\|^2 = \|\sum \alpha_j w_j\|^2 \leq B \sum |\alpha_j|^2$ (by proposition 5.1.3, since every finite scalar sequence can be considered as a sequence in $l_2(\mathbb{N})$). So $\|Uv\|^2 \leq \frac{B}{A} \|\sum \alpha_j v_j\|^2 = \frac{B}{A}\|v\|^2$ and therefore U is bounded with $\|U\| \leq \sqrt{\frac{B}{A}}$.

Clearly, as $(v_j)_{j\in\mathbb{N}}$ is a complete system in $\mathcal{H}_1$, the unique bounded continuous extension of U from $\mathcal{H}_1$ into $\mathcal{H}_2$ has norm smaller or equal to $\sqrt{\frac{B}{A}}$. □

Theorem 5.3.8. Let $\mathcal{H}$ be a Hilbert space and $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$. The following are equivalent:

i) $(v_j)_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$.

ii) $(v_j)_{j\in\mathbb{N}}$ is complete in $\mathcal{H}$, and there exist constants $A, B > 0$ such that, for every finite scalar sequence (α_j),

$$A\sum |\alpha_j|^2 \leq \left\|\sum \alpha_j v_j\right\|^2 \leq B\sum |\alpha_j|^2. \tag{5.3}$$

Proof. i) $\Rightarrow$ ii) Suppose that $(v_j)_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$, then it is a Schauder basis and therefore it is a complete system in $\mathcal{H}$.

Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis and let $U : \mathcal{H} \longrightarrow \mathcal{H}$ be the linear operator defined by $Ue_j = v_j$ for all $j \in \mathbb{N}$. Then U is a bijective bounded linear operator with bounded inverse, hence $\|U^{-1}\|^{-2}\|x\|^2 \leq \|Ux\|^2 \leq \|U\|^2\|x\|^2$ for every $x \in \mathcal{H}$. Therefore for every finite scalar sequence $(\alpha_j)_j$, the inequalities (5.3) hold if we set $x = \sum \alpha_j e_j$.

ii) $\Rightarrow$ i) Clearly, the left-hand inequality in (5.3) implies that $(v_j)_{j\in\mathbb{N}}$ is a linearly independent sequence.

If $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$ then, for $m > n$, $\|\sum_{j=1}^m \alpha_j v_j - \sum_{j=1}^n \alpha_j v_j\|^2 = \|\sum_{j=n+1}^m \alpha_j v_j\|^2 \leq B\sum_{j=n+1}^m |\alpha_j|^2 \to 0$ $(n \to \infty)$. Therefore $\sum_{j=1}^\infty \alpha_j v_j$ converges and (5.3) holds for every $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$; by lemma 5.1.1 and proposition 5.1.3, $(v_j)_{j\in\mathbb{N}}$ is a Bessel sequence with bound B.

As $(v_j)_{j\in\mathbb{N}}$ is a complete system in $\mathcal{H}$, the space $\mathcal{H}$ is separable; let $(e_j)_{j\in\mathbb{N}}$ an orthonormal basis for $\mathcal{H}$. Then, by lemma 5.3.7, the mappings $U : e_j \mapsto v_j$ and $V : v_j \mapsto e_j$ $(j \in \mathbb{N})$ extend to bounded operators on $\mathcal{H}$ and $UV = VU = I$; so U is bijective and therefore $(v_j)_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$. □

Remarks:

- If $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ satisfies (5.3) for all finite scalar sequences (α_j) then it is called a *Riesz sequence*.

- By theorem 5.3.8, if $(v_j)_{j\in\mathbb{N}} \subseteq \mathcal{H}$ is a Riesz sequence then $(v_j)_{j\in\mathbb{N}}$ is a Riesz basis for $\overline{\text{Span}}((v_j)_{j\in\mathbb{N}})$.

- Clearly, every sub-sequence of a Riesz sequence is a Riesz sequence.

Proposition 5.3.9. Let $\mathcal{H}$ be a Hilbert space and let the sequence $(v_j)_{j\in\mathbb{N}}$ be a total subset of $\mathcal{H}$ such that, for every finite scalar sequence, $\|\sum \alpha_j v_j\|^2 = \sum |\alpha_j|^2$. Then $(v_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$.

Proof. By theorem 5.3.8, $(v_j)_{j\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ and $\|\sum_{j=1}^{\infty} \alpha_j v_j\|^2 = \sum_{j=1}^{\infty} |\alpha_j|^2$ for every $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$.

Choose an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ for $\mathcal{H}$ and a bijective bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ such that $v_j = Ue_j$ for every $j \in \mathbb{N}$; then $\|U(\sum_{j=1}^{\infty} \alpha_j e_j)\|^2 = \|\sum_{j=1}^{\infty} \alpha_j v_j\|^2 = \sum_{j=1}^{\infty} |\alpha_j|^2 = \|\sum_{j=1}^{\infty} \alpha_j e_j\|^2$ for every $(\alpha_j)_{j\in\mathbb{N}} \in l_2(\mathbb{N})$. Therefore U is a unitary operator and thus $(v_j)_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. □

Theorem 5.3.10. Let $Z \in \mathcal{B}(\mathcal{H})$ be an operator similar to a self-adjoint operator. Then every root vector of Z is an eigenvector, and the set of all eigenvectors of Z contains a Riesz basis for its closed linear span.

Proof. Let S be a bounded self-adjoint operator such that Z is similar to S. Then there exists a linear homeomorphism T such that $Z = T^{-1}ST$.

(1) Let x be a root vector of Z. Then there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that

$$0 = (\lambda - Z)^n x = (\lambda T^{-1}T - T^{-1}ST)^n x = [T^{-1}(\lambda - S)T]^n x = T^{-1}(\lambda - S)^n Tx$$

Since T^{-1} is injective, it follows that $(\lambda - S)^n Tx = 0$. Hence Tx is a root vector of S and, since S is self-adjoint, Tx is an eigenvector of S. Therefore, $(\lambda - Z)x = (\lambda - T^{-1}ST)x = T^{-1}(\lambda - S)Tx = 0$; that is, x is an eigenvector of Z.

(2) Let U be the closed span of eigenvectors of Z. Since $U \subseteq \mathcal{H}$ is closed, $T(U)$ is a separable Hilbert space and contains an orthonormal basis $(w_n)_{n\in\mathbb{N}}$ consisting of eigenvectors of S.

Let $u_n := T^{-1}w_n$ for every $n \in \mathbb{N}$. Then $(u_n)_{n\in\mathbb{N}}$ is a Riesz basis for U consisting of eigenvectors of Z. □

Chapter 6

Quadratic Operator Pencils

In this chapter, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ denotes a separable Hilbert space. By convention, the sets $\mathbb{C} \cup \{\infty\}$ and $\mathbb{R} \cup \{\infty\}$ (the one-point compactification of $\mathbb{C}$ and $\mathbb{R}$, resp.) are denoted by $\mathbb{C}_\infty$ and $\mathbb{R}_\infty$, respectively.

6.1 Quadratic Pencils of Bounded Self-adjont Operators

Definition 6.1.1. A *quadratic operator pencil* (of bounded self-adjoint operators) in $\mathcal{H}$ is a function $L : \mathbb{C}_\infty \to \mathcal{B}(\mathcal{H})$ for which there exist fixed self-adjoint operators A, B, $C \in \mathcal{B}(\mathcal{H})$ such that:

$$L(\infty) = A \quad \text{and} \quad L(\lambda) = \lambda^2 A + \lambda B + C \text{ if } \lambda \neq \infty.$$

Remark. Note that, if L is a quadratic operator pencil, then $L(\lambda)$ is a self-adjoint operator for every $\lambda \in \mathbb{R}_\infty$.

Definition 6.1.2. Let L be a quadratic operator pencil in a Hilbert space $\mathcal{H}$. Then:

- $\rho(L) = \{\lambda \in \mathbb{C}_\infty \mid L(\lambda) \text{ is bijective}\}$ is the *resolvent set* of L.
- $\sigma(L) = \mathbb{C}_\infty \setminus \rho(L)$ is the *spectrum* of L.
- $\sigma_p(L) = \{\lambda \in \mathbb{C}_\infty \mid L(\lambda) \text{ is not injective}\}$ is the *point spectrum* of L.
- $\sigma_c(L) = \{\lambda \in \mathbb{C}_\infty \mid \lambda \notin \sigma_p(L),\ \mathscr{R}(L(\lambda)) \neq \mathcal{H} \ \&\ \overline{\mathscr{R}(L(\lambda))} = \mathcal{H}\}$ is the *continuous spectrum* of L.

- $\sigma_r(L) = \{\lambda \in \mathbb{C}_\infty \mid \lambda \notin \sigma_p(L) \ \&\ \overline{\mathscr{R}(L(\lambda))} \neq \mathcal{H}\}$ is the *residual spectrum* of L.

Remark. Clearly, if L is an operator pencil, then $\sigma(L) = \sigma_p(L) \,\dot{\cup}\, \sigma_c(L) \,\dot{\cup}\, \sigma_r(L)$.

Definition 6.1.3. Let L be a quadratic operator pencil and let $\lambda_0 \in \sigma_p(L)$. Then λ_0 is called an *eigenvalue* of the pencil L and each non-zero vector $x \in \ker(L(\lambda_0))$ is called an *eigenvector* of the pencil L associated to λ_0. The subspace $\ker(L(\lambda_0))$ is called the *eigenspace* of L associated to λ_0 and $\dim(\ker(L(\lambda_0)))$ is called the *multiplicity* of λ_0.

Theorem 6.1.4. Let $L(\lambda) = \lambda^2 A + \lambda B + C$ be a quadratic pencil.

(i) If $0 \in \rho(A)$, then $\sigma(L)$ is bounded.

(ii) If $\lambda = 0$ is an isolated point of $\sigma(A)$ and it is an eigenvalue of A with finite geometric multiplicity, then either ∞ is an isolated point of $\sigma(L)$ or there exists a neighborhood of ∞ contained in $\sigma_p(L)$.

Proof. (i) Suppose that $0 \in \rho(A)$; since $L(\infty) = A$ and A is bijective, it follows that $\infty \notin \sigma(L)$. Note that, for every $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, $L(\lambda) = \lambda^2 A(I + \frac{1}{\lambda}A^{-1}B + \frac{1}{\lambda^2}A^{-1}C)$ and, for every $|\lambda|$ sufficiently large, $\|\frac{1}{\lambda}A^{-1}B + \frac{1}{\lambda^2}A^{-1}C\| < 1$ (since A^{-1}, B and C are bounded operators); that is, there exists $\alpha > 0$ such that $\|\frac{1}{\lambda}A^{-1}B + \frac{1}{\lambda^2}A^{-1}C\| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| > \alpha$ and, therefore, $I + \frac{1}{\lambda}A^{-1}B + \frac{1}{\lambda^2}A^{-1}C$ is bijective. Hence, $\sigma(L) \subseteq B(0;\alpha)$.

(ii) Suppose that 0 is an isolated point of $\sigma(A)$ and that 0 is an eigenvalue of A with finite geometric multiplicity; so 0 is a normal point of A (see appendix A.2). Then the orthogonal projection P on $\ker(A)$ is a finite-dimensional bounded operator and therefore it is a compact operator. Thus, by the theorem A.2.4, we have $\tilde{\rho}(A) = \tilde{\rho}(A + P)$.

Since A is self-adjoint, $\mathcal{H} = \ker(A) \oplus \overline{\mathscr{R}(A)}$. If $u \in \ker(A)$ and $v \in \overline{\mathscr{R}(A)}$ are such that $(A + P)(u + v) = 0$, then $u + Av = 0$ and, therefore, $u = Av = 0$; that is $u = 0$ and $v \in \ker(A) \cap \overline{\mathscr{R}(A)}$. So, $u = v = 0$ and, therefore, $A + P$ is injective. Thus, since $0 \in \tilde{\rho}(A + P)$ but $0 \notin \sigma_p(A + P)$, $A + P$ is bijective.

Clearly, since $L(\infty) = A$ and A is not injective, we have $\infty \in \sigma_p(L)$. Note that, since $0 \in \rho(A+P)$, there exists $\alpha > 0$ such that, for every $\lambda \in \mathbb{C}$ with $|\lambda| > \alpha$, $A + P + \frac{1}{\lambda}B + \frac{1}{\lambda^2}C$ is bijective and, therefore, for every $\lambda \in \mathbb{C}$ with $|\lambda| > \alpha$:

$$\begin{aligned} L(\lambda) &= \lambda^2 A + \lambda B + C = \lambda^2(A + \tfrac{1}{\lambda}B + \tfrac{1}{\lambda^2}C) = \lambda^2(A + P + \tfrac{1}{\lambda}B + \tfrac{1}{\lambda^2}C - P) \\ &= \lambda^2[I - P(A + P + \tfrac{1}{\lambda}B + \tfrac{1}{\lambda^2}C)^{-1}](A + P + \tfrac{1}{\lambda}B + \tfrac{1}{\lambda^2}C). \end{aligned}$$

Now, let $A(\mu) = P(A+P+\mu B+\mu^2 C)^{-1}$ and $T(\mu) = I - A(\mu)$ for every $\mu \in \mathbb{C}$ such that $|\mu| < \frac{1}{\alpha}$. Since $A(\mu)$ is a holomorphic operator-valued function whose values are compact operators then, by the theorem A.1.5, there exists $\varepsilon > 0$, with $\varepsilon < \min(\frac{1}{\alpha}, 1)$, such that, for every μ with $0 < |\mu| < \varepsilon$, the equation $T(\mu)x = 0$ has the same number of linearly independent solutions.

Since $B(0;\varepsilon) \subseteq \rho(I) \subseteq \tilde{\rho}(I)$ then, by the theorem A.2.4, $B(0;\varepsilon) \subseteq \tilde{\rho}(T(\mu))$ for every $\mu \in \mathbb{C}$ with $|\mu| < \varepsilon$. So, if there exists $\mu_0 \in B(0;\varepsilon) \setminus \{0\}$ such that $T(\mu_0)$ is bijective then, by the conclusion in the previous paragraph, $T(\mu)$ is bijective for any $\mu \in B(0;\varepsilon) \setminus \{0\}$ and, therefore, in this case ∞ is an isolated point of $\sigma(L)$. Otherwise, $T(\mu)$ is not injective for every $\mu \in B(0;\varepsilon) \setminus \{0\}$ and, therefore, there exists a neighborhood of ∞ contained in $\sigma_p(L)$. □

Remark. Let a, b, c and d be complex numbers such that $ad - bc = \pm 1$ and consider the Möbius transformation:

$$\mu = \mu(\lambda) = \frac{a\lambda + b}{c\lambda + d}. \tag{6.1}$$

Clearly, $\mu(\lambda)$ is a bijective map from $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$ with inverse mapping:

$$\lambda = \lambda(\mu) = \frac{d\mu - b}{-c\mu + a}.$$

So, for $\lambda \neq \infty$ we have $c\mu - a \neq 0$ and, therefore, $(c\mu - a)^2 L(\lambda) = \mu^2(d^2 A - dcB + c^2 C) + \mu[-2bdA + (ad + bc)B - 2acC] + (b^2 A - abB + a^2 C)$. That is,

$$(c\mu - a)^2 L(\lambda) = \mu^2 \tilde{A} + \mu \tilde{B} + \tilde{C} = \tilde{L}(\mu), \tag{6.2}$$

where $\tilde{A} = d^2 A - dcB + c^2 C$, $\tilde{B} = -2bdA + (ad + bc)B - 2acC$ and $\tilde{C} = b^2 A - abB + a^2 C$.

Therefore, for every $\lambda \in \mathbb{C}$:

$$(i)\ \lambda \in \rho(L) \Leftrightarrow \mu(\lambda) \in \rho(\tilde{L}) \quad (ii)\ \lambda \in \sigma_p(L) \Leftrightarrow \mu(\lambda) \in \sigma_p(\tilde{L})$$
$$(iii)\ \lambda \in \sigma_c(L) \Leftrightarrow \mu(\lambda) \in \sigma_c(\tilde{L}) \quad (iv)\ \lambda \in \sigma_r(L) \Leftrightarrow \mu(\lambda) \in \sigma_r(\tilde{L})$$

6.2 Factorization of Quadratic Pencils

In this section let $L(\lambda) = \lambda^2 A + \lambda B + C$ with $0 \in \rho(A)$. Thus, by the theorem 6.1.4, $\sigma(L)$ is bounded.

Proposition 6.2.1. Let L be a quadratic pencil and suppose that X, Y, $Z \in \mathcal{B}(\mathcal{H})$ are such that $L(\lambda) = (\lambda X + Y)(\lambda - Z)$. Then:

(i) $\sigma_p(\mathcal{Z}) \subseteq \sigma_p(L)$.

(ii) $\sigma(\mathcal{Z}) \subseteq \sigma(L)$.

(iii) If $\sigma(\mathcal{F}) \cap \sigma(\mathcal{Z}) = \varnothing$, then $\sigma(L) = \sigma(\mathcal{F}) \cup \sigma(\lambda - Z)$.

Where $\mathcal{Z}$ and $\mathcal{F}$ are the linear pencils defined by $\mathcal{Z}(\lambda) := \lambda - Z$ and $\mathcal{F}(\lambda) := \lambda X + Y$.

Proof. (i) and (iii) are clear.
(ii) Let $\lambda_0 \in \sigma(\mathcal{Z})$. Since the operator $\lambda_0 - Z$ is not bijective, there exists $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$, with $\|x_n\| = 1$ for every $n \in \mathbb{N}$, such that $(\lambda_0 - Z)x_n \to 0$. Thus, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$, with $\|x_n\| = 1$ for every $n \in \mathbb{N}$, such that $L(\lambda_0)x_n \to 0$ and, therefore, $L(\lambda_0)$ is not bijective. Hence, $\lambda_0 \in \sigma(L)$. □

Proposition 6.2.2. Let $L(\lambda) = \lambda^2 A + \lambda B + C$ be a quadratic pencil and $Z \in \mathcal{B}(\mathcal{H})$. Then, $L(\lambda) = (\lambda X + Y)(\lambda - Z)$ for some X and $Y \in \mathcal{B}(\mathcal{H})$ if, and only if, $AZ^2 + BZ + C = 0$.

Proof. $\boxed{\Rightarrow}$ If there exist $X, Y \in \mathcal{B}(\mathcal{H})$ such that $L(\lambda) = (\lambda X + Y)(\lambda - Z)$, then $C = -YZ$, $B = Y - XZ$ and $A = X$. Thus, $AZ = Y - B$ and, therefore, $AZ^2 + BZ + C = (Y - B)Z + BZ + C = 0$.
$\boxed{\Leftarrow}$ If $AZ^2 + BZ + C = 0$ then, $L(\lambda) = L(\lambda) - 0 = \lambda^2 A + \lambda B + C - (AZ^2 + BZ + C) = A(\lambda^2 - Z^2) + B(\lambda - Z) = [A(\lambda + Z) + B](\lambda - Z) = [\lambda A + (AZ + B)](\lambda - Z)$. □

Remark. An operator $Z \in \mathcal{B}(\mathcal{H})$ which satisfies the *operator equation* $AZ^2 + BZ + C = 0$ is called an *operator root* of the pencil L.

Theorem 6.2.3. Let $L(\lambda) = \lambda^2 A + \lambda B + C$ be a quadratic pencil and, for $i = 1, 2$, let X_i, Y_i, $Z_i \in \mathcal{B}(\mathcal{H})$ be such that $L(\lambda) = (\lambda X_i + Y_i)(\lambda - Z_i)$ and $\sigma(\mathcal{F}_i) \cap \sigma(\mathcal{Z}_i) = \varnothing$, where $\mathcal{Z}_i$ and $\mathcal{F}_i$ are the linear pencils defined by $\mathcal{Z}_i(\lambda) := \lambda - Z_i$ and $\mathcal{F}_i(\lambda) := \lambda X_i + Y_i$, $i = 1, 2$. If $\sigma(\mathcal{Z}_1) = \sigma(\mathcal{Z}_2) =: \sigma$ then $Z_1 = Z_2$.

Proof. By the proposition 6.2.1, we have $\sigma(\mathcal{F}_1) = \sigma(\mathcal{F}_2) =: \sigma'$. Note that σ is compact and σ' is closed (since for every bijective operator T in $\mathcal{B}(\mathcal{H})$ there exists a neighborhood of it which contains only bijective operators); thus, σ and σ' are separated. Hence, there exists an open subset $U \subseteq \mathbb{C}$ such that $\sigma \subseteq U \subseteq \rho(\mathcal{F}_1) = \rho(\mathcal{F}_2)$. Since $(\lambda X_1 + Y_1)(\lambda - Z_1) = (\lambda X_2 + Y_2)(\lambda - Z_2)$ for every $\lambda \in \mathbb{C}$, then for $\lambda \notin \sigma \cup \sigma'$, we have $(\lambda X_2 + Y_2)^{-1}(\lambda X_1 + Y_1) = (\lambda - Z_2)(\lambda - Z_1)^{-1}$. Let $f : \mathbb{C} \to \mathcal{B}(\mathcal{H})$ such that $f(\lambda) = (\lambda X_2 + Y_2)^{-1}(\lambda X_1 + Y_1)$ for $\lambda \in \sigma$, and $f(\lambda) = (\lambda - Z_2)(\lambda - Z_1)^{-1}$ for $\lambda \in \mathbb{C} \setminus \sigma$. Clearly, f is holomorphic (because $\mathbb{C} = U \cup (\mathbb{C} \setminus \sigma)$ and the functions $f \restriction_U$ and $f \restriction_{\mathbb{C}\setminus\sigma}$ are holomorphic) and, moreover, $f(\lambda) \to I$ at infinity (because, for $|\lambda|$ sufficiently large, $f(\lambda) = (\lambda - Z_2)(\lambda - Z_1)^{-1}$). Thus, by Liouville's Theorem, $f(\lambda) = I$ for every $\lambda \in \mathbb{C}$ and, therefore, $Z_1 = Z_2$. □

6.3 Strongly Damped Pencils

Definition 6.3.1. The operator pencil $L(\lambda) = \lambda^2 A + \lambda B + C$ is called *strongly damped* if and only if

$$\langle Bx, x\rangle^2 > 4\langle Ax, x\rangle\langle Cx, x\rangle$$

for every $x \in \mathcal{H}$ with $x \neq 0$.

Proposition 6.3.2. Let $L(\lambda)$ be a strongly damped pencil. Then the sign of $\langle Bx, x\rangle$ is constant on the set $U_0 = \{x \in \mathcal{H} \setminus \{0\} \mid \langle Ax, x\rangle = 0\}$. If further $C \geq 0$, then the sign of $\langle Bx, x\rangle$ is constant on the set $U_+ = \{x \in \mathcal{H} \setminus \{0\} \mid \langle Ax, x\rangle \geq 0\}$.

Proof. Clearly, since $L(\lambda)$ is strongly damped, we have $\langle Bx, x\rangle \neq 0$ for every $x \in U_0$. Suppose, towards a contradiction, that there exist u, $v \in U_0$ such that $\langle Bu, u\rangle < 0$ and $\langle Bv, v\rangle > 0$. Note that, since $\langle B(\alpha u), \alpha u\rangle < 0$ for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, u can be choosen such that $\mathrm{Re}(\langle Au, v\rangle) = 0$. So, for every $t \in [0, 1]$, $w(t) = (1-t)u + tv \neq 0$ and

$$\begin{aligned}\langle Aw(t), w(t)\rangle &= (1-t)^2\langle Au, u\rangle + t^2\langle Av, v\rangle + (1-t)t\langle Au, v\rangle + (1-t)t\langle Av, u\rangle \\ &= 2(1-t)t\,\mathrm{Re}(\langle Au, v\rangle) = 0;\end{aligned}$$

that is, for every $t \in [0, 1]$, $w(t) \in U_0$. Let $f(t) = \langle Bw(t), w(t)\rangle$, $t \in [0, 1]$. Then f is a continuous real-valued function defined in the interval $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$. So, by Bolzano's Intermediate Value Theorem, there exists $t_0 \in (0, 1)$ such that $f(t_0) = 0$; that is, there exists $w_0 = w(t_0) \in U_0$ such that $\langle Bw_0, w_0\rangle = 0$ ↯. Hence, the sign of $\langle Bx, x\rangle$ is constant on U_0.

Now suppose that $C \geq 0$, then for every $x \in U_+$ we have $\langle Ax, x\rangle\langle Cx, x\rangle \geq 0$ and, since $L(\lambda)$ is strongly damped, $\langle Bx, x\rangle \neq 0$. By proceeding as in the case above, we obtain that the sign of $\langle Bx, x\rangle$ is constant on U_+. □

Definition 6.3.3. Let $L(\lambda)$ be a quadratic operator pencil. For each $x \in \mathcal{H} \setminus \{0\}$ we define the function $\varphi_x : \mathbb{R}_\infty \longrightarrow \mathbb{R}$, $\varphi_x(\lambda) = \langle L(\lambda)x, x\rangle$.

Remark. If $L(\lambda)$ is a strongly damped pencil and $x \in \mathcal{H}$ is such that $x \neq 0$, then:

- $\varphi_x(\infty) = \langle Ax, x\rangle$ and, for every $\lambda \in \mathbb{R}$, $\varphi_x(\lambda) = \langle Ax, x\rangle\lambda^2 + \langle Bx, x\rangle\lambda + \langle Cx, x\rangle$.
- If $\langle Ax, x\rangle \neq 0$ then φ_x has two real zeros given by the formula:

$$\frac{-\langle Bx, x\rangle \pm d(x)}{2\langle Ax, x\rangle}, \quad \text{where } d(x) = \sqrt{\langle Bx, x\rangle^2 - 4\langle Ax, x\rangle\langle Cx, x\rangle}.$$

- If $\langle Ax, x\rangle = 0$ (and therefore $\langle Bx, x\rangle \neq 0$), then $-\frac{\langle Cx, x\rangle}{\langle Bx, x\rangle}$ is a real zero of φ_x. Also, ∞ is a zero of φ_x. Note that, in this case, ∞ is a simple zero.

Proposition 6.3.4. The pencil $L(\lambda)$ is strongly damped if, and only if, for each $x \in \mathcal{H} \setminus \{0\}$, the equation $\varphi_x(\lambda) = 0$ has exactly two different zeros in $\mathbb{R}_\infty$.

Proof. $\boxed{\Rightarrow}$ Is clear by the preceding remark.

$\boxed{\Leftarrow}$ Suppose, towards a contradiction, that $L(\lambda)$ is not strongly damped; then there exists $x \in \mathcal{H}$ with $x \neq 0$ such that $\langle Bx, x\rangle^2 \leq 4\langle Ax, x\rangle\langle Cx, x\rangle$.

If $\langle Ax, x\rangle = 0$ then $\langle Bx, x\rangle = 0$ and, therefore, $\varphi_x(\lambda) = \langle Cx, x\rangle$. So, if $\langle Cx, x\rangle = 0$ then every real number is a zero of φ_x ($\lightning$) and if $\langle Cx, x\rangle \neq 0$ then ∞ is a double zero of φ_x ($\lightning$). Thus, $\langle Ax, x\rangle \neq 0$ and, therefore, the zeros of φ_x are both real and different. Thus, $\langle Bx, x\rangle^2 > 4\langle Ax, x\rangle\langle Cx, x\rangle$ $\lightning$. $\square$

Definition 6.3.5. Let $L(\lambda)$ be a strongly damped pencil. Let $p_+, p_- : \mathcal{H} \setminus \{0\} \longrightarrow \mathbb{R}_\infty$ be the functions defined by the formulas:

$$p_+(x) = \begin{cases} \frac{1}{2\langle Ax,x\rangle}[-\langle Bx, x\rangle + d(x)] & \text{if } \langle Ax, x\rangle \neq 0 \\ -\frac{\langle Cx,x\rangle}{\langle Bx,x\rangle} & \text{if } \langle Ax, x\rangle = 0 \text{ and } \langle Bx, x\rangle > 0 \\ \infty & \text{if } \langle Ax, x\rangle = 0 \text{ and } \langle Bx, x\rangle < 0 \end{cases},$$

and

$$p_-(x) = \begin{cases} \frac{1}{2\langle Ax,x\rangle}[-\langle Bx, x\rangle - d(x)] & \text{if } \langle Ax, x\rangle \neq 0 \\ \infty & \text{if } \langle Ax, x\rangle = 0 \text{ and } \langle Bx, x\rangle > 0 \\ -\frac{\langle Cx,x\rangle}{\langle Bx,x\rangle} & \text{if } \langle Ax, x\rangle = 0 \text{ and } \langle Bx, x\rangle < 0 \end{cases}.$$

The functions p_+ and p_- are called, respectively, the *first* and *second type functionals associated to L.*

Remarks:

- Clearly, for each $x \in \mathcal{H}$ with $x \neq 0$, $p_+(x)$ and $p_-(x)$ are the roots of the function φ_x.

- Let $x \in \mathcal{H} \setminus \{0\}$ be such that $\langle Ax, x\rangle \neq 0$. Then, for every $\lambda \in \mathbb{R}$, $\frac{d}{d\lambda}\varphi_x(\lambda) = 2\langle Ax, x\rangle\lambda + \langle Bx, x\rangle$. Thus, $\frac{d}{d\lambda}\varphi_x(p_+(x)) = d(x) > 0$ and $\frac{d}{d\lambda}\varphi_x(p_-(x)) = -d(x) < 0$. That is, $p_+(x)$ is the zero of φ_x where φ_x is an increasing function and $p_-(x)$ is the zero of φ_x where φ_x is a decreasing function.

- Clearly, if $x \in \mathcal{H} \setminus \{0\}$ and $\alpha \in \mathbb{C} \setminus \{0\}$, then $p_+(\alpha x) = p_+(x)$ and $p_-(\alpha x) = p_-(x)$. Therefore, the values of p_+ and p_- are determined by their values on the unit sphere of $\mathcal{H}$.

- If $\mu = \mu(\lambda)$ as in the equation (6.1) then, by equation (6.2), for every $x \in \mathcal{H} \setminus \{0\}$ we have:

$$\widetilde{\varphi_x}(\mu) = \langle \tilde{L}(\mu)x, x\rangle = (c\mu - a)^2 \langle L(\lambda)x, x\rangle = (c\mu - a)^2 \varphi_x(\lambda). \tag{6.3}$$

So, if L is a strongly damped pencil then the equation $\widetilde{\varphi_x}(\lambda) = 0$ has two different zeros in $\mathbb{R}_\infty$ for every $x \in \mathcal{H} \setminus \{0\}$ and, by the proposition 6.2.4, also $\tilde{L}$ is a strongly damped pencil. Note that, from (6.3) we have:

$$\frac{d}{d\lambda}\varphi_x(\lambda) = \frac{1}{ad - bc}\left[\frac{d}{d\mu}\widetilde{\varphi_x}(\mu) - \frac{2c}{c\mu - a}\widetilde{\varphi_x}(\mu)\right]. \tag{6.4}$$

Let $\widetilde{p_+}$ and $\widetilde{p_-}$ be the functionals of first and second type, respectively, associated to $\tilde{L}$. Then, clearly, if $ad - bc = 1$:

$$\begin{cases} \lambda = p_+(x) \Leftrightarrow & \mu(\lambda) = \widetilde{p_+}(x) \\ \lambda = p_-(x) \Leftrightarrow & \mu(\lambda) = \widetilde{p_-}(x) \end{cases},$$

and if $ad - bc = -1$, then:

$$\begin{cases} \lambda = p_+(x) \Leftrightarrow & \mu(\lambda) = \widetilde{p_-}(x) \\ \lambda = p_-(x) \Leftrightarrow & \mu(\lambda) = \widetilde{p_+}(x) \end{cases}.$$

Definition 6.3.6. Let $L(\lambda)$ be a strongly damped pencil, $\pi_+ = \mathscr{R}(p_+)$ and $\pi_- = \mathscr{R}(p_-)$. The sets π_+ and π_- are called the *spectral zones* of $L(\lambda)$.

Proposition 6.3.7. Let $L(\lambda)$ be a strongly damped pencil. Then p_+ and p_- are continuous.

Proof. Let U_0 be as in proposition 6.2.2. Suppose, without loss of generality, that $\langle Bx, x\rangle > 0$ for every $x \in U_0$. Let $x_0 \in \mathcal{H} \setminus \{0\}$ and $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H} \setminus \{0\}$ such that $x_n \to x_0$ if $n \to \infty$.
(1) Suppose that $x_0 \in U_0$; then $p_+(x_0) = -\frac{\langle Cx_0, x_0\rangle}{\langle Bx_0, x_0\rangle}$ and $p_-(x_0) = \infty$. Note that:

- If there is an $n_0 \in \mathbb{N}$ such that $x_n \in U_0$ for every $n \geq n_0$ (and without restriction suppose that $n_0 = 1$), then $p_+(x_n) = -\frac{\langle Cx_n, x_n\rangle}{\langle Bx_n, x_n\rangle} \to -\frac{\langle Cx_0, x_0\rangle}{\langle Bx_0, x_0\rangle} = p_+(x_0)$ when $n \to \infty$, and $p_-(x_n) = \infty = p_-(x_0)$ for every $n \in \mathbb{N}$.

- If there is an $n_0 \in \mathbb{N}$ such that $x_n \notin U_0$ for every $n \geq n_0$ (and without restriction suppose that $n_0 = 1$), then $p_+(x_n) = \frac{-\langle Bx_n, x_n\rangle + d(x_n)}{2\langle Ax_n, x_n\rangle} = \frac{-2\langle Cx_n, x_n\rangle}{\langle Bx_n, x_n\rangle + d(x_n)} \to -\frac{\langle Cx_0, x_0\rangle}{\langle Bx_0, x_0\rangle} = p_+(x_0)$ when $n \to \infty$ [since $d(x_n) \to \langle Bx_0, x_0\rangle$ if $n \to \infty$], and $p_-(x_n) = \frac{-\langle Bx_n, x_n\rangle - d(x_n)}{2\langle Ax_n, x_n\rangle} \to \infty = p_-(x_0)$ when $n \to \infty$ [since $\langle Bx_0, x_0\rangle \neq 0$].

- If $(x_n)_{n\in\mathbb{N}}$ can be split in two subsequences $(y_{n_k})_{k\in\mathbb{N}} \subseteq U_0$ and $(z_{n_k})_{k\in\mathbb{N}} \subseteq \mathcal{H} \setminus U_0$, then by the two preceding items we have $p_+(x_n) \to p_+(x_0)$ and $p_-(x_n) \to p_-(x_0)$ if $n \to \infty$.

(2) Suppose that $x_0 \notin U_0$; then $p_+(x_0) = \frac{-\langle Bx_0, x_0\rangle + d(x_0)}{2\langle Ax_0, x_0\rangle}$ and $p_-(x_0) = \frac{-\langle Bx_0, x_0\rangle - d(x_0)}{2\langle Ax_0, x_0\rangle}$.
Since $\langle Ax_0, x_0\rangle \neq 0$ and the function $x \mapsto \langle Ax, x\rangle$ is real-valued and continuous, there exists a neighborhood of x_0, $B(x_0; \varepsilon) \subseteq \mathcal{H} \setminus \{0\}$, such that $\langle Ax, x\rangle \neq 0$ for every $x \in B(x_0; \varepsilon)$. Since $x_n \to x_0$ ($n \to \infty$), suppose, without loss of generality, that $(x_n)_{n\in\mathbb{N}} \subseteq B(x_0; \varepsilon)$. Therefore, $p_+(x_n) = \frac{-\langle Bx_n, x_n\rangle + d(x_n)}{2\langle Ax_n, x_n\rangle} \to \frac{-\langle Bx_0, x_0\rangle + d(x_0)}{2\langle Ax_0, x_0\rangle} = p_+(x_0)$ if $n \to \infty$ and, analogously, $p_-(x_n) \to p_-(x_0)$ if $n \to \infty$.
Hence, p_+ and p_- are continuous. □

Corollary 6.3.8. Let L be a strongly damped pencil. Then π_+ and π_- are connected subsets of $\mathbb{R}_\infty$.

Proof. Since the unit sphere in $\mathcal{H}$ is a connected set and p_+, p_- are continuous, then π_+ and π_- are connected subsets of $\mathbb{R}_\infty$. □

Proposition 6.3.9. Let L be a strongly damped pencil. Then $\pi_+ \cap \pi_- = \varnothing$.

Proof. Suppose, towards a contradiction, that $\pi_+ \cap \pi_- \neq \varnothing$. Then there exist u, $v \in \mathcal{H} \setminus \{0\}$ such that $p_+(u) = p_-(v) =: \lambda_0$. Note that $\lambda_0 \neq \infty$ since, otherwise, u, $v \in U_0$ with $\langle Bu, u\rangle < 0$ and $\langle Bv, v\rangle > 0$ ↯ (by the proposition 6.2.2). Also, note that, as in the proof of the proposition 6.2.2, u and v can be choosen such that $\mathrm{Re}(\langle L(\lambda_0)u, v\rangle) = 0$. Clearly, $\varphi_u(\lambda_0) = \varphi_v(\lambda_0) = 0$, $\frac{d}{d\lambda}\varphi_u(\lambda_0) > 0$ and $\frac{d}{d\lambda}\varphi_v(\lambda_0) < 0$.

Let $w_t = (1-t)u + tv$ for $t \in [0,1]$. Then, for every $t \in [0,1]$, $w_t \neq 0$ and

$$\begin{aligned} \varphi_{w_t}(\lambda_0) &= \langle L(\lambda_0)w_t, w_t\rangle \\ &= (1-t)^2\langle L(\lambda_0)u, u\rangle + 2t(1-t)\operatorname{Re}(\langle L(\lambda_0)u, v\rangle) + t^2\langle L(\lambda_0)v, v\rangle \\ &= (1-t)^2\varphi_u(\lambda_0) + t^2\varphi_v(\lambda_0) = 0 \end{aligned}$$

Moreover, $\frac{d}{d\lambda}\varphi_{w_t}(\lambda_0)$ is a continuous function of t, with $\frac{d}{d\lambda}\varphi_{w_0}(\lambda_0) = \frac{d}{d\lambda}\varphi_u(\lambda_0) > 0$ and $\frac{d}{d\lambda}\varphi_{w_1}(\lambda_0) = \frac{d}{d\lambda}\varphi_v(\lambda_0) < 0$. Thus, there exists $t_0 \in (0,1)$ such that $\frac{d}{d\lambda}\varphi_{w_{t_0}}(\lambda_0) = 0 = \varphi_{w_{t_0}}(\lambda_0)$.

Therefore, λ_0 is a real solution of the system of equations:

$$\begin{cases} \lambda_0^2\langle Aw_{t_0}, w_{t_0}\rangle + \lambda_0\langle Bw_{t_0}, w_{t_0}\rangle + \langle Cw_{t_0}, w_{t_0}\rangle &= 0 \\ 2\lambda_0\langle Aw_{t_0}, w_{t_0}\rangle + \langle Bw_{t_0}, w_{t_0}\rangle &= 0 \end{cases}.$$

Now, if $\lambda_0 = 0$ then $\langle Bw_{t_0}, w_{t_0}\rangle = \langle Cw_{t_0}, w_{t_0}\rangle = 0$ ↯ (since $L(\lambda)$ is strongly damped), and if $\lambda_0 \neq 0$ then $\langle Aw_{t_0}, w_{t_0}\rangle = -\frac{1}{2\lambda_0}\langle Bw_{t_0}, w_{t_0}\rangle$, $\langle Cw_{t_0}, w_{t_0}\rangle = -\frac{1}{2}\lambda_0\langle Bw_{t_0}, w_{t_0}\rangle$ and, therefore, $4\langle Aw_{t_0}, w_{t_0}\rangle\langle Cw_{t_0}, w_{t_0}\rangle = \langle Bw_{t_0}, w_{t_0}\rangle^2$ ↯. □

Lemma 6.3.10. Let $L(\lambda)$ be a strongly damped pencil and suppose that $\infty \notin \pi_+$. Let $\beta = \sup(\pi_+)$ and

$$\alpha = \begin{cases} \inf(\pi_+) & \text{if } \inf(\pi_+) > -\infty \\ \infty & \text{if } \inf(\pi_+) = -\infty \end{cases}.$$

Then we have $L(\alpha) \leq 0$ and $L(\beta) \geq 0$.

Proof. Since $\infty \notin \pi_+$ then, by the corollary 6.3.8, π_+ is a connected subset of $\mathbb{R}$ and, therefore, it is a interval of $\mathbb{R}$.

(1) If $\inf(\pi_+) > -\infty$, then $\alpha = \inf(\pi_+) \in \mathbb{R}$.

Suppose, towards a contradiction, that there exists $x_0 \in \mathcal{H}$ such that $\langle L(\alpha)x_0, x_0\rangle > 0$; then $\varphi_{x_0}(\alpha) > 0$ and $\alpha < p_+(x_0)$.

- If $\langle Ax_0, x_0\rangle \neq 0$ then, clearly, $\langle Ax_0, x_0\rangle > 0$ (since for $\langle Ax_0, x_0\rangle < 0$ the parabola φ_{x_0} is concave and, as $\varphi_{x_0}(p_+(x_0)) = 0$, $\frac{d}{d\lambda}\varphi_{x_0}(p_+(x_0)) > 0$ and $\alpha < p_+(x_0)$, then $\varphi_{x_0}(\alpha) < 0$ ↯). Thus, the parabola φ_{x_0} is a convex function and, therefore, $p_-(x_0) \in (\alpha,\ p_+(x_0)) \subseteq \pi_+$; but this is a contradiction to the proposition 6.3.9.

- If $\langle Ax_0, x_0\rangle = 0$ then, since $\infty \notin \pi_+$, $\langle Bx_0, x_0\rangle > 0$. So, for every $\lambda \in \mathbb{R}$, $\varphi_{x_0}(\lambda) = \lambda\langle Bx_0, x_0\rangle + \langle Cx_0, x_0\rangle$ and, since $\alpha < p_+(x_0)$ and $\varphi_{x_0}(p_+(x_0)) = 0$, then $\varphi_{x_0}(\alpha) < 0$ ↯.

Hence, $L(\alpha) \leq 0$.

If π_+ is not bounded above, then $(\alpha, +\infty) \subseteq \pi_+$ and, therefore, $A \geq 0$ (Otherwise, if there is a $x_0 \in \mathcal{H}$ such that $\langle Ax_0, x_0\rangle < 0$, then the parabola φ_{x_0} is concave and, therefore, its zeros are such that $p_+(x_0) < p_-(x_0)$ ↯). Thus, $L(\beta) = L(\infty) = A \geq 0$.

If π_+ is bounded above, then $\beta = \sup(\pi_+) \in \mathbb{R}$. Then, by a similar argument as in the proof of $L(\alpha) \leq 0$, we have $L(\beta) \geq 0$.

(2) Suppose that $\inf(\pi_+) = -\infty$; so, $\alpha = \infty$. Note that $A \leq 0$ (since if there is a $x_0 \in \mathcal{H}$ such that $\langle Ax_0, x_0\rangle > 0$ then the parabola φ_{x_0} is a convex function and, therefore, $p_-(x_0)$ and $p_+(x_0)$ are real numbers with $p_-(x_0) < p_+(x_0)$ ↯). Thus, $L(\alpha) \leq 0$.

The proof that $L(\beta) \geq 0$ in this case is similar to the case (1). □

Proposition 6.3.11. Let $L(\lambda)$ be a strongly damped pencil and suppose that there exist α, $\beta \in \mathbb{R}_\infty$ such that $L(\alpha) \leq 0$ and $L(\beta) \geq 0$. Then the points α and β can be choosen in $\mathbb{R}$ if any of the following conditions holds:

i) A is indefinite.

ii) $0 \in \rho(A)$.

iii) $\lambda = 0$ is an isolated point of $\sigma(A)$ and, further, it is an eigenvalue of finite multiplicity.

Proof. *i)* Suppose that α cannot be choosen in $\mathbb{R}$, then $\alpha = \infty$ and $A = L(\infty) \leq 0$; so A is not indefinite. Analogously, if β cannot be choosen in $\mathbb{R}$, then $\beta = \infty$ and $A = L(\infty) \geq 0$; so A is not indefinite. Hence, if A is indefinite then α and β can be choosen in $\mathbb{R}$.

ii) Suppose that $0 \in \rho(A)$. Since A is a bounded self-adjoint operator then its spectrum $\sigma(A)$ is a non-empty and bounded subset of $\mathbb{R}$. Further, since $0 \in \rho(A)$ and $\rho(A)$ is a open subset of $\mathbb{C}$, there exists $\delta > 0$ such that $|\lambda| > \delta$ for every $\lambda \in \sigma(A)$.

- If $\lambda > \delta$ for every $\lambda \in \sigma(A)$, then $\langle Ax, x\rangle \geq \delta > 0$ for every $x \in \mathcal{H} \setminus \{0\}$. Therefore, π_+ is bounded below and $\infty \notin \pi_+$.

Suppose, towards a contradiction, that π_+ is not bounded, and let $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ be a sequence such that $p_+(x_n) \to +\infty$ and $\|x_n\| = 1$ for every $n \in \mathbb{N}$. Since $p_+(x_n) = \frac{-\langle Bx_n,x_n\rangle + d(x_n)}{\langle Ax_n,x_n\rangle}$ and $(-\langle Bx_n, x_n\rangle + d(x_n))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$ (since A, B and C are bounded self-adjoint operators) then $\langle Ax_n, x_n\rangle \to 0$ ↯.

Let $\alpha = \inf(\pi_+)$ and $\beta = \sup(\pi_+)$. Clearly α and β are real numbers and, by the lemma 2.6.10, $L(\alpha) \leq 0$ and $L(\beta) \geq 0$.

- If $\lambda < -\delta$ for every $\lambda \in \sigma(A)$, then $\langle Ax, x\rangle \leq -\delta < 0$ for every $x \in \mathcal{H} \setminus \{0\}$. Therefore, π_+ is bounded above and $\infty \notin \pi_+$. Analogous to the preceding item, we have that there exist α, $\beta \in \mathbb{R}$ such that $L(\alpha) \leq 0$ and $L(\beta) \geq 0$.

- If there are λ_1, $\lambda_2 \in \sigma(A)$ such that $\lambda_1 < -\delta$ and $\lambda_2 > \delta$, then A is indefinite and, by i), α and β can be choosen in $\mathbb{R}$.

iii) Suppose that 0 is an eigenvalue of A with finite multiplicity and, moreover, it is an isolated point of $\sigma(A)$; that is $0 \in \sigma_d(A)$. Suppose, without loss of generality, that $A \geq 0$ [since the case $A \leq 0$ is analogous and the case where A is indefinite is treated in part i)] and that $\langle Bx, x\rangle > 0$ for every $x \in U_0$, where $U_0 = \{x \in \mathcal{H} \setminus \{0\} \mid \langle Ax, x\rangle = 0\}$. Thus, $\infty \notin \pi_+$.
Suppose, towards a contradiction, that π_+ is not bounded above. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that, for every $n \in \mathbb{N}$, $\|x_n\| = 1$ and $|p_+(x_n)| > n$. If $(x_n)_{n\in\mathbb{N}}$ contains a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \notin U_0$ for every $k \in \mathbb{N}$, then $\langle Ax_{n_k}, x_{n_k}\rangle \to 0$ when $k \to \infty$. Since $A = A^*$ and $A \geq 0$ we have $Ax_{n_k} \to 0$ when $k \to \infty$ and, as $0 \in \sigma_d(A)$, $(x_{n_k})_{k\in\mathbb{N}}$ contains a convergent subsequence (without restriction, suppose that $(x_{n_k})_{k\in\mathbb{N}}$ itself is convergent). Let $x_0 \in \mathcal{H}$ be such that $x_{n_k} \to x_0$; then $\|x_0\| = 1$, $\langle Ax_0, x_0\rangle = 0$ and, therefore, $\langle Bx_0, x_0\rangle > 0$. Thus, $p_+(x_{n_k}) \to p_+(x_0) = -\frac{\langle Cx_0,x_0\rangle}{\langle Bx_0,x_0\rangle} \in \mathbb{R}$ and, for every $k \in \mathbb{N}$, $|p_+(x_{n_k})| > n_k \geq k$ ↯. Thus there exists $n_0 \in \mathbb{N}$ such that $x_n \in U_0$ for every $n \geq n_0$ (without restriction, suppose that $n_0 = 1$). Since $\langle Ax_n, x_n\rangle = 0$ for every $n \in \mathbb{N}$ then, clearly, $\langle Ax_n, x_n\rangle \to 0$ and, therefore, $Ax_n \to 0$. Thus, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $x_0 \in \mathcal{H}$, with $\|x_0\| = 1$, such that $x_{n_k} \to x_0$, $\langle Ax_0, x_0\rangle = 0$ and $\langle Bx_0, x_0\rangle > 0$. Then, again, $p_+(x_{n_k}) \to p_+(x_0) \in \mathbb{R}$ and, for every $k \in \mathbb{N}$, $|p_+(x_{n_k})| > n_k \geq k$ ↯.

Hence, π_+ is bounded and, by the lemma 6.3.10, there exist α and β in $\mathbb{R}$ such that $L(\alpha) \leq 0$ and $L(\beta) \geq 0$. □

Definition 6.3.12. Let L be a strongly damped pencil and let $\lambda_0 \in \mathbb{C}_\infty$. The point λ_0 is called a *spectral point of first* (resp. *second*) *type* for L if, and only if, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$, with $\|x_n\| = 1$ for every $n \in \mathbb{N}$, such that $L(\lambda_0)x_n \to 0$ and $p_+(x_n) \to \lambda_0$ (resp., $p_-(x_n) \to \lambda_0$) in $\mathbb{R}_\infty$.

The set of all the spectral points of first (resp., second) type for L is denoted by $\sigma^+(L)$ (resp., $\sigma^-(L)$) and it is called the *spectrum of first* (resp. *second*) *type* of L.

Remarks:

- Clearly, by the definition of spectral point of first or second type, $\sigma^+(L) \subseteq \sigma(L)$ and $\sigma^-(L) \subseteq \sigma(L)$ [thus the name is justified].

- Clearly, if λ_0 is a spectral point of first or second type, then $\lambda_0 \in \mathbb{R}_\infty$. Thus, in the topology of $\mathbb{C}_\infty$, we have $\sigma^+(L) \subseteq \overline{\sigma^+(L)} \subseteq \mathbb{R}_\infty$ and $\sigma^-(L) \subseteq \overline{\sigma^-(L)} \subseteq \mathbb{R}_\infty$.

- By the definition 6.3.12, if $\lambda_0 \in \sigma^+(L)$ then $\lambda_0 \in \overline{\pi_+}$ (closure in the topology of $\mathbb{R}_\infty$). That is, $\sigma^+(L) \subseteq \overline{\pi_+}$.

- Analogous to the preceding item, $\sigma^-(L) \subseteq \overline{\pi_-}$.

Proposition 6.3.13. Let L be a strongly damped pencil. Then $\sigma^+(L)$ and $\sigma^-(L)$ are closed in $\mathbb{R}_\infty$.

Proof. Let $\lambda_0 \in \overline{\sigma^+(L)}$. Suppose that $\lambda_0 \in \mathbb{R}$ and let $(\lambda_k)_{k\in\mathbb{N}} \subseteq \sigma^+(L)\cap\mathbb{R}$ such that $\lambda_k \to \lambda_0$. For each $k \in \mathbb{N}$, let $(x_{k,n})_{n\in\mathbb{N}} \subseteq \mathcal{H}$, sequence with $\|x_{k,n}\| = 1$ for every $n \in \mathbb{N}$, such that $L(\lambda_k)x_{k,n} \to 0$ and $p_+(x_{k,n}) \to \lambda_k$ when $n \to \infty$. Note that $\|L(\lambda_k) - L(\lambda_0)\| = \|(\lambda_k^2 - \lambda_0^2)A + (\lambda_k - \lambda_0)B\| \to 0$ when $k \to \infty$. Then, clearly, there exist $(k_m)_{m\in\mathbb{N}} \subseteq \mathbb{N}$ and $(n_{k_m})_{m\in\mathbb{N}} \subseteq \mathbb{N}$, strictly increasing sequences, such that $|\lambda_0 - \lambda_{k_m}| < \frac{1}{2m}$, $\|L(\lambda_0) - L(\lambda_{k_m})\| < \frac{1}{2m}$, $\|L(\lambda_{k_m})x_{k_m,n_{k_m}}\| < \frac{1}{2m}$ and $|\lambda_{k_m} - p_+(x_{k_m,n_{k_m}})| < \frac{1}{2m}$ for every $m \in \mathbb{N}$. Thus, $(x_{k_m,n_{k_m}})_{m\in\mathbb{N}} \subseteq \mathcal{H}$ is a normalized sequence such that $\|L(\lambda_0)x_{k_m,n_{k_m}}\| \leq \|L(\lambda_{k_m})x_{k_m,n_{k_m}}\| + \|(L(\lambda_{k_m}) - L(\lambda_0))x_{k_m,n_{k_m}}\| < \frac{1}{m} \to 0$ $(m \to \infty)$ and $|p_+(x_{k_m,n_{k_m}}) - \lambda_0| \leq |p_+(x_{k_m,n_{k_m}}) - \lambda_{k_m}| + |\lambda_{k_m} - \lambda_0| < \frac{1}{m} \to 0$ $(m \to \infty)$. That is, $\lambda_0 \in \sigma^+(L)$.

Suppose that $\lambda_0 = \infty$ is an accumulation point of $\sigma^+(L)$ in $\mathbb{R}_\infty$. Then there exists $(\lambda_k)_{k\in\mathbb{N}} \subseteq \sigma^+(L)$ such that, for every $k \in \mathbb{N}$, $|\lambda_k| > k$. For each $k \in \mathbb{N}$, let $(x_{k,n})_{n\in\mathbb{N}} \subseteq \mathcal{H}$, sequence with $\|x_{k,n}\| = 1$ for every $n \in \mathbb{N}$, such that $L(\lambda_k)x_{k,n} \to 0$ and $p_+(x_{k,n}) \to \lambda_k$ when $n \to \infty$. Clearly, there exist $(k_m)_{m\in\mathbb{N}} \subseteq \mathbb{N}$ and $(n_{k_m})_{m\in\mathbb{N}} \subseteq \mathbb{N}$, strictly increasing sequences, such that $|p_+(x_{k_m,n_{k_m}})| > m$ and $\frac{1}{|\lambda_{k_m}|^2}\|L(\lambda_{k_m})x_{k_m,n_{k_m}}\| + \frac{1}{|\lambda_{k_m}|}\|B\| + \frac{1}{|\lambda_{k_m}|^2}\|C\| < \frac{1}{m}$. Thus, $(x_{k_m,n_{k_m}})_{m\in\mathbb{N}}$ is a normalized sequence such that $p_+(x_{k_m,n_{k_m}}) \to \infty$ (in $\mathbb{R}_\infty$) and, for every $m \in \mathbb{N}$,

$$\begin{aligned}
\|L(\lambda_0)x_{k_m,n_{k_m}}\| &= \|Ax_{k_m,n_{k_m}}\| = \tfrac{1}{|\lambda_{k_m}|^2}\|\lambda_{k_m}^2 Ax_{k_m,n_{k_m}}\| \\
&\leq \tfrac{1}{|\lambda_{k_m}|^2}\|\lambda_{k_m}^2 Ax_{k_m,n_{k_m}} + \lambda_{k_m}Bx_{k_m,n_{k_m}} + Cx_{k_m,n_{k_m}} - \lambda_{k_m}Bx_{k_m,n_{k_m}} - Cx_{k_m,n_{k_m}}\| \\
&\leq \tfrac{1}{|\lambda_{k_m}|^2}\|L(\lambda_{k_m})x_{k_m,n_{k_m}}\| + \tfrac{1}{|\lambda_{k_m}|}\|B\| + \tfrac{1}{|\lambda_{k_m}|^2}\|C\| \\
&< \tfrac{1}{m}.
\end{aligned}$$

That is, $L(\lambda_0)x_{k_m,n_{k_m}} \to 0$ when $m \to +\infty$. Thus $\lambda_0 = \infty \in \sigma^+(L)$. Hence $\sigma^+(L)$ is a closed subset of $\mathbb{R}_\infty$. Analogously, $\sigma^-(L)$ is a closed subset of $\mathbb{R}_\infty$. □

Remarks:

- Let L be a strongly damped pencil. If $x_0 \in \mathcal{H}$ is an eigenvector associated with the eigenvalue $\lambda_0 \in \sigma_p(L)$, then $\langle L(\lambda_0)x_0, x_0\rangle = 0$ and, therefore, λ_0 is a root of φ_{x_0}. That is, either $\lambda_0 = p_+(x_0)$ or $\lambda_0 = p_-(x_0)$.

- By the preceding item, if L is a strongly damped pencil, then $\sigma_p(L) \subseteq \pi_+\dot{\cup}\pi_- \subseteq \mathbb{R}_\infty$.

- Clearly, if $\lambda_0 \in \sigma_p(L)$ and if there exists $x_0 \in \ker(L(\lambda_0))$ such that $\lambda_0 = p_+(x_0)$ (resp., $\lambda_0 = p_-(x_0)$) then, by the proposition 6.3.9, for every $x \in \ker(L(\lambda_0))$ we have $\lambda_0 = p_+(x)$ (resp., $\lambda_0 = p_-(x)$).

These remarks justify the following definition:

Definition 6.3.14. Let L be a strongly damped pencil and $x_0 \in \mathcal{H}$ be an eigenvector associated with the eigenvalue $\lambda_0 \in \sigma_p(L)$. If $\lambda_0 = p_+(x_0)$, then λ_0 is called an *eigenvalue of first type* and x_0 is called an *eigenvector of first type*. Similarly, if $\lambda_0 = p_-(x_0)$ then λ_0 is called an *eigenvalue of second type* and x_0 is called an *eigenvector of second type*.

The set of all the eigenvalues of first (resp., second) type of L is denoted by $\sigma_p^+(L)$ (resp., $\sigma_p^-(L)$).

Remarks:

- It is clear that $\sigma_p^+(L) \subseteq \sigma^+(L)$ and $\sigma_p^-(L) \subseteq \sigma^-(L)$.
- By definition 6.2.14, we have $\sigma_p(L) = \sigma_p^+(L) \dot{\cup} \sigma_p^-(L)$.

Proposition 6.3.15. Let L be a strongly damped pencil, then each boundary point of π_+ (resp., π_-) in $\mathbb{R}_\infty$ belongs to $\sigma^+(L)$ (resp., $\sigma^-(L)$).

Proof. Let λ_0 be a boundary point of π_+ in $\mathbb{R}_\infty$; thus, by the lemma 6.2.10, $L(\lambda_0)$ is a self-adjoint semi-definite operator. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ an normalized sequence such that $\lambda_n := p_+(x_n) \to \lambda_0$. Suppose that $\lambda_0 \neq \infty$. Clearly, since $\langle L(p_+(x))x, x\rangle = \varphi_x(p_+(x)) = 0$ for every $x \in \mathcal{H} \setminus \{0\}$, we have $\langle L(\lambda_0)x_n, x_n\rangle = \langle L(\lambda_0)x_n, x_n\rangle - \langle L(p_+(x_n))x_n, x_n\rangle = \langle (L(\lambda_0) - L(p_+(x_n)))x_n, x_n\rangle \to 0$ when $n \to \infty$. Since $L(\lambda_0)$ is a self-adjoint and semi-definite linear operator, then $L(\lambda_0)x_n \to 0$. Therefore, $\lambda_0 \in \sigma^+(L)$.
If $\lambda_0 = \infty$, suppose, without restriction, that $\lambda_n \in \mathbb{R}\setminus\{0\}$ for every $n \in \mathbb{N}$. Then, $\langle L(\infty)x_n, x_n\rangle = \langle Ax_n, x_n\rangle = \frac{1}{\lambda_n^2}\langle \lambda_n^2 Ax_n, x_n\rangle = \frac{1}{\lambda_n^2}\langle L(\lambda_n)x_n - \lambda_n Bx_n - Cx_n, x_n\rangle = -\frac{1}{\lambda_n}\langle Bx_n, x_n\rangle - \frac{1}{\lambda_n^2}\langle Cx_n, x_n\rangle \to 0$. Thus, $L(\lambda_0)x_n \to 0$ and, therefore, $\lambda_0 \in \sigma^+(L)$. ☐

Proposition 6.3.16. Let $L(\lambda) = \lambda^2 A + \lambda B + C$ be a strongly damped pencil. If $\lambda = 0$ is an isolated point of $\sigma(A)$ and if it is an eigenvalue of A with finite multiplicity, then ∞ is an isolated point of $\sigma(L)$.

Proof. Clearly $\infty \in \sigma(L)$. By the theorem 6.1.4, and specifically by its proof, it is sufficient to show that ∞ is an accumulation point of $\rho(L)$ in $\mathbb{C}_\infty$. Since L is a strongly damped pencil, we have that $\sigma_p(L) \subseteq \mathbb{R}_\infty$ and, therefore, every neighborhood of ∞ in $\mathbb{C}_\infty$ contains points of $\rho(L)$. ☐

Definition 6.3.17. Let L be a quadratic operator pencil and $x_0 \in \mathcal{H}$ be an eigenvector of L with eigenvalue λ_0. The vectors $x_1, \ldots, x_m$ are called *associated vectors* to x_0 if, and only if, for each $j = 1, \ldots, m$, $\sum_{k=0}^{j} \frac{1}{k!} \frac{d^k}{d\lambda^k} L(\lambda_0) x_{j-k} = 0$.

Proposition 6.3.18. Let L be a stronlgy damped operator with $0 \in \rho(A)$. Then no eigenvector of L has associated vectors.

Proof. Let x_0 be an eigenvector of L with eigenvalue λ_0. Clearly $\lambda_0 \in \mathbb{R}$ since $\sigma_p(L) \subseteq \mathbb{R}_\infty$ and $\infty \in \rho(L)$.

Suppose, towards a contradiction, that there exists $x_1 \in \mathcal{H}$ such that $L(\lambda_0)x_1 + \frac{d}{d\lambda}L(\lambda_0)x_0 = 0$. Thus, $\langle \frac{d}{d\lambda}L(\lambda_0)x_0, x_0\rangle = \langle -L(\lambda_0)x_1, x_0\rangle = -\langle L(\lambda_0)x_1, x_0\rangle = -\langle x_1, L(\lambda_0)x_0\rangle = 0$; that is, $\langle 2\lambda_0 Ax_0 + Bx_0, x_0\rangle = 0$ and, therefore, $\varphi'_{x_0}(\lambda_0) = 0$ ↯. □

6.4 Operators Associated to a Pencil

Definition 6.4.1. Let $L(\lambda) = \lambda^2 A + \lambda B + C$ be a quadratic operator pencil in $\mathcal{H}$ with $0 \in \rho(A)$. Let $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$ and define $\mathbf{G}, \mathbf{L} : \mathcal{H}^2 \to \mathcal{H}^2$ by the matrices:

$$\mathbf{G} = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{pmatrix}.$$

The operator $\mathbf{L}$ is called the *linearizer* of the pencil L.

Remarks:

- Clearly $\mathbf{G}$ and $\mathbf{L}$ are bounded linear operators acting in $\mathcal{H}^2$.
- $\mathbf{GL} = \begin{pmatrix} A & 0 \\ 0 & -C \end{pmatrix}$ and $\mathbf{GL}^2 = \begin{pmatrix} -B & -C \\ -C & 0 \end{pmatrix}$.
- Since A is bijective, also $\mathbf{G}$ is bijective and it is easy see that $\mathbf{G}^{-1} = \begin{pmatrix} -A^{-1}BA^{-1} & A^{-1} \\ A^{-1} & 0 \end{pmatrix}$.

  Therefore, $\mathbf{G}^{-1}$ is bounded.

- $\mathbf{G}$ is self-adjoint since, for every (x, y), $(u, v) \in \mathcal{H}^2$,

$$\begin{aligned}\langle \mathbf{G}(x,y),(u,v)\rangle_{\mathcal{H}^2} &= \langle (Ay, Ax+By),(u,v)\rangle_{\mathcal{H}^2} = \langle Ay,u\rangle + \langle Ax+By,v\rangle \\ &= \langle Ay,u\rangle + \langle Ax,v\rangle + \langle By,v\rangle = \langle y,Au\rangle + \langle x,Av\rangle + \langle y,Bv\rangle \\ &= \langle x,Av\rangle + \langle y,Au\rangle + \langle y,Bv\rangle = \langle x,Av\rangle + \langle y,Au+Bv\rangle \\ &= \langle (x,y),(Av,Au+Bv)\rangle_{\mathcal{H}^2} \\ &= \langle (x,y),\mathbf{G}(u,v)\rangle_{\mathcal{H}^2}.\end{aligned}$$

- The name linearizer for $\mathbf{L}$ is due to the following relation between L and $\mathbf{L}$: for every $\lambda \in \mathbb{C}$ and every $x \in \mathcal{H}$,

$$(\mathbf{L}-\lambda)(\lambda x, x) = (-A^{-1}L(\lambda)x, 0).$$

- Also we have the following relation: for every $\lambda \in \mathbb{R}$ and every $x \in \mathcal{H}$,

$$\langle \mathbf{G}(\lambda x, x),(\lambda x, x)\rangle_{\mathcal{H}^2} = 2\lambda\langle Ax,x\rangle + \langle Bx,x\rangle = \varphi_x'(\lambda).$$

Proposition 6.4.2. Let $L(\lambda) = \lambda^2 A + \lambda B + C$ be a quadratic operator pencil with $0 \in \rho(A)$. Then: (i) $\rho(L) = \rho(\mathbf{L})$, (ii) $\sigma_p(L) = \sigma_p(\mathbf{L})$, (iii) $\sigma_c(L) = \sigma_c(\mathbf{L})$ and (iv) $\sigma_r(L) = \sigma_r(\mathbf{L})$.

Proof. Let $\lambda \in \mathbb{C}$. Note that for every (u, v) and (w, z) in $\mathcal{H}^2$:

$$(\mathbf{L}-\lambda)(u,v) = (w,z) \Leftrightarrow \begin{cases} L(\lambda)v &= -Bz - Aw - \lambda Az \\ u &= z + \lambda v \end{cases}. \tag{6.5}$$

Thus, (i) and (ii) holds by (6.5).

Let $\lambda \in \sigma_c(\mathbf{L})$ and $w \in \mathcal{H}$.

- If $\lambda = 0$, let $((u_n, v_n))_{n\in\mathbb{N}} \subseteq \mathcal{H}^2$ be such that $\mathbf{L}(u_n, v_n) \to (-A^{-1}w, 0)$. Then $u_n \to 0$ and therefore $-A^{-1}Cv_n \to -A^{-1}w$. Thus, $L(0)v_n \to w$ and therefore $0 \in \sigma_c(L)$.

- Suppose that $\lambda \neq 0$ and let $((u_n, v_n))_{n\in\mathbb{N}} \subseteq \mathcal{H}^2$ be such that $(\mathbf{L}-\lambda)(u_n, v_n) \to (-\frac{1}{\lambda}A^{-1}w, 0)$. Then $\lambda^2 Au_n + \lambda Bu_n + \lambda Cv_n \to w$ and $Cu_n - \lambda Cv_n \to 0$. Thus summing the two sequences we have $L(\lambda)u_n \to w$ and therefore $\lambda \in \sigma_c(L)$.

Now, let $\lambda \in \sigma_c(L)$ and $(w, z) \in \mathcal{H}^2$.

- If $\lambda = 0$, let $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ be such that $Cv_n = L(0)v_n \to -Bz - Aw$. Then $\mathbf{L}(z, v_n) = (-A^{-1}Bz - A^{-1}Cv_n, z) \to (-A^{-1}Bz - A^{-1}(-Bz - Aw), z) = (w, z)$ and therefore $0 \in \sigma_c(\mathbf{L})$.

- Suppose that $\lambda \neq 0$ and let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $L(\lambda)v_n \to -Bz - Aw - \lambda Az$ and $u_n - \lambda v_n \to z$. Then

$$\begin{aligned}
(\mathbf{L} - \lambda)(u_n, v_n) &= (-A^{-1}Bu_n - \lambda u_n - A^{-1}Cv_n, u_n - \lambda v_n) \\
&= (-A^{-1}[Bu_n + \lambda A u_n + Cv_n], u_n - \lambda v_n) \\
&= (-A^{-1}[\lambda A(u_n - \lambda v_n) + B(u_n - \lambda v_n) + \lambda^2 A v_n + \lambda B v_n + C v_n], u_n - \lambda v_n) \\
&= (-A^{-1}[\lambda A(u_n - \lambda v_n) + B(u_n - \lambda v_n) + L(\lambda)v_n], u_n - \lambda v_n) \\
&\to (-A^{-1}[\lambda Az + Bz - Bz - Aw - \lambda Az], z) = (w, z).
\end{aligned}$$

  Therefore $\lambda \in \sigma_c(\mathbf{L})$.

Thus (iii) holds and therefore (iv) too. □

Remarks:

- Note that $\sigma_p(L)$ is countable because $\sigma_p(L) = \sigma_p(\mathbf{L})$ by the proposition 6.4.2, and the point spectrum of a bounded operator in a separable Hilbert space is countable.

- Since $\mathbf{G} : \mathcal{H}^2 \to \mathcal{H}^2$ is a self-adjoint operator so, by definition 3.3.6, it induces an inner product on $\mathcal{H}^2$, $[\cdot,\cdot]$, given by the formula $[\mathbf{x}, \mathbf{y}] := \langle \mathbf{G}\mathbf{x}, \mathbf{y}\rangle_{\mathcal{H}^2}$; $\mathbf{x}$, $\mathbf{y} \in \mathcal{H}^2$. By the final remark in the section 4.1 we have that $(\mathcal{H}^2, [\cdot,\cdot])$ is a Krein space since $\mathbf{G}$ bijective.

- Note that $\mathbf{L}$ is a $\mathbf{G}$-self-adjoint operator, because for every (u, v), $(x, y) \in \mathcal{H}^2$:

$$\begin{aligned}
[\mathbf{L}(u, v), (x, y)] &= \langle \mathbf{GL}(u, v), (x, y)\rangle_{\mathcal{H}^2} = \langle (Au, -Cv), (x, y)\rangle_{\mathcal{H}^2} = \langle Au, x\rangle + \langle -Cv, y\rangle \\
&= \langle u, Ax\rangle + \langle v, -Cy\rangle = \langle (u, v), (Ax, -Cy)\rangle_{\mathcal{H}^2} = \langle (u, v), \mathbf{GL}(x, y)\rangle_{\mathcal{H}^2} \\
&= \langle \mathbf{G}(u, v), \mathbf{L}(x, y)\rangle_{\mathcal{H}^2} \\
&= [(u, v), \mathbf{L}(x, y)].
\end{aligned}$$

Proposition 6.4.3. Let L be a strongly damped pencil in $\mathcal{H}$ such that $0 \in \rho(A)$. Then $\mathbf{L}$ is a definitizable operator in $(\mathcal{H}^2, [\cdot,\cdot])$ (see appendix A.3) and $\mathbf{L}$ has no $\mathbf{G}$-neutral eigenvectors.

Proof. (1) By the theorem 6.1.4, $\sigma(L)$ is bounded and therefore, by the proposition 6.4.2, $\rho(\mathbf{L}) \neq \varnothing$. Now, by the lemma 6.3.10, there exist α, $\beta \in \mathbb{R}_\infty$ such that $L(\alpha) \leq 0$ and $L(\beta) \geq 0$ and, by the proposition 6.3.11, α and β can be chosen in $\mathbb{R}$. So, let α, β in $\mathbb{R}$ such that $L(\alpha) \leq 0$ and $L(\beta) \geq 0$.

- If $\alpha \neq \beta$ then:

$$\begin{aligned}\mathbf{G}(\mathbf{L}-\alpha)(\mathbf{L}-\beta) &= \mathbf{G}\mathbf{L}^2 - (\alpha+\beta)\mathbf{G}\mathbf{L} + \alpha\beta\mathbf{G} = \begin{pmatrix} -B-(\alpha+\beta)A & -C+\alpha\beta A \\ -C+\alpha\beta A & (\alpha+\beta)C+\alpha\beta B \end{pmatrix} \\ &= \tfrac{1}{\alpha-\beta}\left\{\begin{pmatrix} L(\beta) & -\alpha L(\beta) \\ -\alpha L(\beta) & \alpha^2 L(\beta)\end{pmatrix} - \begin{pmatrix} L(\alpha) & -\beta L(\alpha) \\ -\beta L(\alpha) & \beta^2 L(\alpha)\end{pmatrix}\right\}.\end{aligned}$$

  Let $\mathbf{T}$ be the first matrix in the braces. For every $\mathbf{x} = (x, y) \in \mathcal{H}^2$,

$$\begin{aligned}\langle \mathbf{T}\mathbf{x}, \mathbf{x}\rangle_{\mathcal{H}^2} &= \langle L(\beta)x - \alpha L(\beta)y, x\rangle - \alpha\langle L(\beta)x - \alpha L(\beta)y, y\rangle \\ &= \langle L(\beta)(x-\alpha y), x-\alpha y\rangle \geq 0.\end{aligned}$$

  Thus $\mathbf{T}$ is positive semi-definite. Analogously, the second matrix in the braces is negative semi-definite. Therefore, for every $\mathbf{x} \in \mathcal{H}^2$:

  $[(\mathbf{L}-\alpha I)(\mathbf{L}-\beta I)\mathbf{x}, \mathbf{x}] \geq 0$ if $\alpha - \beta > 0$, and $[(\mathbf{L}-\alpha I)(\mathbf{L}-\beta I)\mathbf{x}, \mathbf{x}] \leq 0$ if $\alpha - \beta < 0$.

  Thus, in this case, $\mathbf{L}$ is definitizable with definitizing polynomial $p(t) = (t-\alpha)(t-\beta)$ if $\alpha - \beta > 0$ or $p(t) = -(t-\alpha)(t-\beta)$ if $\alpha - \beta < 0$.

- If $\alpha = \beta$, then $L(\alpha) = 0$ and, therefore, $-C = \alpha^2 A + \alpha B$. Thus, since L is strongly damped, for every $x \in \mathcal{H} \setminus \{0\}$:

$$\langle Bx, x\rangle^2 > 4\langle Ax, x\rangle\langle Cx, x\rangle \Leftrightarrow \langle (2\alpha A + B)x, x\rangle^2 > 0.$$

  Thus the operator $D := 2\alpha A + B$ is self-adjoint and definite, and the operator

$$\mathbf{G}(\mathbf{L}-\alpha)^2 = \begin{pmatrix} -(2\alpha A+B) & \alpha(2\alpha A+B) \\ \alpha(2\alpha A+B) & -\alpha^2(2\alpha A+B) \end{pmatrix} = \begin{pmatrix} -D & \alpha D \\ \alpha D & -\alpha^2 D \end{pmatrix}$$

is such that, for every $\mathbf{x} = (x, y) \in \mathcal{H}^2$,

$$\begin{aligned} [(\mathbf{L}-\alpha)^2\mathbf{x}, \mathbf{x}] &= \langle -Dx + \alpha Dy, x\rangle - \alpha\langle -Dx + \alpha Dy, y\rangle = \langle -Dx + \alpha Dy, x - \alpha y\rangle \\ &= \langle -D(x-\alpha y), x - \alpha y\rangle. \end{aligned}$$

Thus, in this case, $\mathbf{L}$ is definitizable with definitizing polynomial $p(t) = -(t-\alpha)^2$ if D is positive definite and $p(t) = (t-\alpha)^2$ if D is negative definite.

Hence, $\mathbf{L}$ is definitizable.

(2) Suppose, towards a contradiction, that $\mathbf{x}_0 = (u_0, v_0) \in \mathcal{H}^2 \setminus \{0\}$ is a $\mathbf{G}$-neutral eigenvector of $\mathbf{L}$ with eigenvalue λ_0. Then $u_0 = \lambda_0 v_0$ (thus, $v_0 \neq 0$), $L(\lambda_0)v_0 = 0$ and $2\lambda_0\langle Av_0, v_0\rangle + \langle Bv_0, v_0\rangle = 0$. That is, λ_0 is a real root of φ_{v_0} such that $\frac{d}{d\lambda}\varphi_{v_0}(\lambda_0) = 0$ ↯. □

Proposition 6.4.4. Let L be a strongly damped pencil in $\mathcal{H}$ with $0 \in \rho(A)$. Then,

$$\sigma^+(L) = \sigma^+(\mathbf{L}),\ \sigma^-(L) = \sigma^-(\mathbf{L}),\ \sigma_p^+(L) = \sigma_p^+(\mathbf{L}),\ \sigma_p^-(L) = \sigma_p^-(\mathbf{L}),$$

and every eigenvalue $\lambda \in \sigma_p(\mathbf{L}) = \sigma_p(L)$ has the same multiplicity for L and $\mathbf{L}$.

Proof. (1) Let $\lambda_0 \in \mathbb{R}$.

- Suppose that $\lambda_0 \in \sigma^+(L)$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ with $\|x_n\| = 1$ for every $n \in \mathbb{N}$, such that $L(\lambda_0)x_n \to 0$ and $p_+(x_n) \to \lambda_0$.

  For each $n \in \mathbb{N}$, let $\mathbf{x}_n = \frac{1}{\sqrt{1+[p_+(x_n)]^2}}(p_+(x_n)x_n, x_n) \in \mathcal{H}^2$. Note that, for every $n \in \mathbb{N}$:

$$\begin{aligned} [\mathbf{x}_n, \mathbf{x}_n] &= \langle \mathbf{G}\mathbf{x}_n, \mathbf{x}_n\rangle_{\mathcal{H}^2} = \tfrac{1}{1+[p_+(x_n)]^2}\langle \mathbf{G}(p_+(x_n), x_n), (p_+(x_n), x_n)\rangle_{\mathcal{H}^2} \\ &= \tfrac{1}{1+[p_+(x_n)]^2}\tfrac{d}{d\lambda}\varphi_{x_n}(p_+(x_n)) > 0, \end{aligned}$$

  and $(\mathbf{L}-\lambda_0)\mathbf{x}_n = \frac{1}{\sqrt{1+[p_+(x_n)]^2}}(-A^{-1}[p_+(x_n)\lambda_0 A x_n + p_+(x_n)Bx_n + Cx_n],\ [p_+(x_n)-\lambda_0]x_n)$.

Therefore, for every $n \in \mathbb{N}$, $[\mathbf{x}_n, \mathbf{x}_n] > 0$ and $(\mathbf{L} - \lambda_0)\mathbf{x}_n \to 0$ when $n \to \infty$. Moreover, for every $n \in \mathbb{N}$, $\|\mathbf{x}_n\|_{\mathcal{H}^2} = 1$. The J-norm of the Krein space $(\mathcal{H}^2, [\cdot,\cdot])$ is equivalent to $\|\cdot\|_{\mathcal{H}^2}$, so if we set $\mathbf{y}_n = \frac{1}{\|\mathbf{x}_n\|_J}\mathbf{x}_n$ for every $n \in \mathbb{N}$, we have $\|\mathbf{y}_n\|_J = 1$, $[\mathbf{y}_n, \mathbf{y}_n] > 0$ and $(\mathbf{L} - \lambda_0)\mathbf{y}_n \to 0$ when $n \to \infty$. That is, $\lambda_0 \in \sigma^+(\mathbf{L})$ (see definition 4.4.5).

- Suppose that $\lambda_0 \in \sigma^+(\mathbf{L})$ and let $(\mathbf{x}_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}^2$ (with $\mathbf{x}_n = (u_n, v_n)$) be a sequence such that, for every $n \in \mathbb{N}$, $\|\mathbf{x}_n\|_J = 1$ and $[\mathbf{x}_n, \mathbf{x}_n] > 0$, and such that $(\mathbf{L} - \lambda_0)\mathbf{x}_n \to 0$ when $n \to \infty$. Thus $u_n - \lambda_0 v_n \to 0$, $\lambda_0 A u_n + B u_n + C v_n \to 0$ and, therefore,

$$
\begin{aligned}
L(\lambda_0)v_n &= \lambda_0^2 A v_n + \lambda_0 B v_n + C v_n = -\lambda_0 A(u_n - \lambda_0 v_n - u_n) - B(u_n - \lambda_0 v_n - u_n) + C v_n \\
&= (-\lambda_0 A - B)(u_n - \lambda_0 v_n) + [\lambda_0 A u_n + B u_n + C v_n] \to 0.
\end{aligned}
$$

If λ_0 is not a critical point of $\mathbf{L}$ then, by the proposition A.3.7, the sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ can be choosen such that $[\mathbf{x}_n, \mathbf{x}_n] \geq \delta$ for some $\delta > 0$ and, therefore, $[\mathbf{x}_n, \mathbf{x}_n] = 2\,\mathrm{Re}\langle A v_n, u_n\rangle + \langle B v_n, v_n\rangle \geq \delta$. Thus, since $2\,\mathrm{Re}\langle A v_n, \lambda_0 v_n - u_n\rangle \to 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$:

$$
\frac{d}{d\lambda}\varphi_{v_n}(\lambda_0) = 2\,\mathrm{Re}\langle A v_n, u_n\rangle + \langle B v_n, v_n\rangle + 2\,\mathrm{Re}\langle A v_n, \lambda_0 v_n - u_n\rangle \geq \tfrac{\delta}{2}.
$$

If for every $n \geq n_0$ we have $\langle A v_n, v_n\rangle = 0$, then $\langle B v_n, v_n\rangle \geq \frac{\delta}{2} > 0$ and $\lambda_0\langle B v_n, v_n\rangle + \langle C v_n, v_n\rangle \to 0$; thus, $p_+(v_n) = -\frac{\langle C v_n, v_n\rangle}{\langle B v_n, v_n\rangle} \to \lambda_0$. Now, if for every $n \geq n_0$ we have $\langle A v_n, v_n\rangle \neq 0$, then $\langle A v_n, v_n\rangle(\lambda_0 - p_-(v_n)) = \lambda_0\langle A v_n, v_n\rangle - [-\frac{1}{2}\langle B v_n, v_n\rangle - \frac{1}{2}d(v_n)] = \frac{1}{2}[2\lambda_0\langle A v_n, v_n\rangle + \langle B v_n, v_n\rangle] + \frac{1}{2}d(v_n) \geq \frac{\delta}{4}$. Thus, since $\langle L(\lambda_0)v_n, v_n\rangle = \langle A v_n, v_n\rangle(\lambda_0 - p_-(v_n))(\lambda_0 - p_+(v_n)) \to 0$, we have $p_+(v_n) \to \lambda_0$. Therefore, $\lambda_0 \in \sigma^+(L)$.

Suppose that λ_0 is a critical point of $\mathbf{L}$, then $\lambda_0 \in \{\alpha, \beta\}$ where α and β are as in the proof of the proposition 6.4.3. Thus, by the proposition 6.3.15, $\lambda_0 \in \sigma^+(L)$.

Hence $\sigma^+(L) = \sigma^+(\mathbf{L})$ and, analogously, $\sigma^-(L) = \sigma^-(\mathbf{L})$.

(2) Let $\lambda_0 \in \mathbb{R}$. By the proposition 6.4.2 (and its proof) $\sigma_p(L) = \sigma_p(\mathbf{L})$ and (u, v) is an eigenvector of $\mathbf{L}$ with eigenvalue λ_0 if, and only if, v is an eigenvector of L with eigenvalue λ_0 and $u = \lambda_0 v$.

Note that $[(\lambda_0 v, v), (\lambda_0 v, v)] = 2\lambda_0\langle Av, v\rangle + \langle Bv, v\rangle = \frac{d}{d\lambda}\varphi_v(\lambda_0)$ and, therefore, $\lambda_0 \in \sigma_p^+(\mathbf{L}) \Leftrightarrow \lambda_0 \in \sigma_p^+(L)$.

Hence, $\sigma_p^+(\mathbf{L}) = \sigma_p^+(L)$ and, analogously, $\sigma_p^-(\mathbf{L}) = \sigma_p^-(L)$. Further, the correspondence between eigenvector of L and eigenvectors of $\mathbf{L}$ shows that, all $\lambda \in \sigma_p(L) = \sigma_p(\mathbf{L})$ has the same multiplicity for L and $\mathbf{L}$. □

Remark. If $0 \in \rho(A)$, then $0 \in \rho(A^2)$ and, therefore, A^2 is a positive self-adjoint operator with $\frac{1}{\|(A^{-1})^2\|} = \operatorname{dist}(0, \sigma(A^2))$.

- Let $\gamma = \|A^{-1}\|^2(1 + \|B\|)$; then, for every $x \in \mathcal{H}$ with $\|x\| = 1$, $\langle(2\gamma A^2 + B)x, x\rangle = 2\gamma\langle A^2 x, x\rangle + \langle Bx, x\rangle \geq 2\gamma \operatorname{dist}(0, \sigma(A^2)) - \|B\| = 2\gamma\frac{1}{\|(A^{-1})^2\|} - \|B\| \geq 2\gamma\frac{1}{\|(A^{-1})\|^2} - \|B\| > 2 + \|B\| \geq 2$.

  That is, $2\gamma A^2 + B$ is a uniformly positive self-adjoint operator and therefore it is a bijective operator.

- Thus, the function $(x, y) \in \mathcal{H}^2 \mapsto \langle(2\gamma A^2 + B)x, y\rangle$ defines a positive definite inner product on $\mathcal{H}$.

- Let $|||\cdot|||$ be the norm induced by the inner product of the preceding item. Note that, for every $x \in \mathcal{H}$, $\sqrt{2}\|x\| \leq |||x||| \leq \sqrt{\|2\gamma A^2 + B\|}\|x\|$. Therefore $|||\cdot|||$ and $\|\cdot\|$ are equivalent; that is, $(\mathcal{H}, |||\cdot|||)$ is a Banach space.

In the rest the this section it is assumed that $0 \in \rho(A)$.

Definition 6.4.5. The norm $|||\cdot|||$ defined in the preceding remark is called the **G**-*norm* on $\mathcal{H}$ and a contraction with respect to this **G**-norm is called a **G**-*contraction*.

Remark. Let U_0 be the subspace of $(\mathcal{H}^2, [\cdot,\cdot])$ given by the equation

$$U_0 = \left\{ \mathbf{x} \in \mathcal{H}^2 : \mathbf{x} = \begin{bmatrix} \gamma A x \\ x \end{bmatrix}, x \in \mathcal{H} \right\}. \tag{6.6}$$

Its orthogonal complement $U_0^\perp$ is given by

$$U_0^\perp = \left\{ \mathbf{x} \in \mathcal{H}^2 : \mathbf{x} = \begin{bmatrix} -\gamma A x - A^{-1} B x \\ x \end{bmatrix}, x \in \mathcal{H} \right\} \tag{6.7}$$

since $(u,v) \perp U_0 \Leftrightarrow [(u,v),(\gamma Ax, x)] = 0$ for every $x \in \mathcal{H} \Leftrightarrow \langle G(u,v),(\gamma Ax,x)\rangle_{\mathcal{H}^2} = 0$ for every $x \in \mathcal{H} \Leftrightarrow \langle (Av, Au+Bv),(\gamma Ax,x)\rangle_{\mathcal{H}^2} = 0$ for every $x \in \mathcal{H} \Leftrightarrow \langle Av, \gamma Ax\rangle + \langle Au, x\rangle + \langle Bv, x\rangle = 0$ for every $x \in \mathcal{H} \Leftrightarrow \langle \gamma A^2 v + Bv + Au, x\rangle = 0$ for every $x \in \mathcal{H} \Leftrightarrow \gamma A^2 v + Bv + Au = 0 \Leftrightarrow u = -\gamma Av - A^{-1}Bv$.

- If $\mathbf{x} \in U_0$, then there exists $x \in \mathcal{H}$ such that $\mathbf{x} = (\gamma Ax, x)$ and, therefore, $[\mathbf{x},\mathbf{x}] = \langle \mathbf{G}\mathbf{x},\mathbf{x}\rangle_{\mathcal{H}^2} = \langle Ax, \gamma Ax\rangle + \langle \gamma A^2 x + Bx, x\rangle = \langle (2\gamma A^2 + B)x, x\rangle = |||x|||^2$. Hence $(U_0, [\cdot,\cdot])$ is a positive definite subspace of $\mathcal{H}$ and, further, as a normed space (with the norm induced by $[\cdot,\cdot]$ on U_0) it is isometrically isomorphic to $(\mathcal{H}, |||\cdot|||)$. Since $(\mathcal{H}, |||\cdot|||)$ is a Banach space, then $(U_0, [\cdot,\cdot])$ is a Hilbert space.

- If $\mathbf{x} \in U_0^\perp$, then there exists $x \in \mathcal{H}$ such that $\mathbf{x} = (-\gamma Ax - A^{-1}Bx, x)$ and, therefore, $[\mathbf{x},\mathbf{x}] = \langle \mathbf{G}\mathbf{x},\mathbf{x}\rangle_{\mathcal{H}^2} = \langle Ax, -\gamma Ax - A^{-1}Bx\rangle + \langle -\gamma A^2 x, x\rangle = \langle -(2\gamma A^2 + B)x, x\rangle = -|||x|||^2$. Hence $(U_0^\perp, [\cdot,\cdot])$ is a negative definite subspace. Thus, $(U_0^\perp, -[\cdot,\cdot])$ is a positive definite subspace which as a normed space (with the norm induced by $-[\cdot,\cdot]$ on $U_0^\perp$) is isometrically isomorphic to $(\mathcal{H}, |||\cdot|||)$. Since $(\mathcal{H}, |||\cdot|||)$ is a Banach space, then $(U_0^\perp, -[\cdot,\cdot])$ is a Hilbert space.

- By the preceding items, $U_0 \cap U_0^\perp = \{0\}$.

- Note that if u, $v \in \mathcal{H}$, then the system of equations

$$\begin{cases} u &= \gamma Ax - \gamma Ay - A^{-1}By \\ v &= x + y \end{cases}$$

has a unique solution (since $2\gamma A^2 + B$ is bijective) given by

$$\begin{cases} x &= v - (2\gamma A^2 + B)^{-1}[-Au + \gamma A^2 v] \\ y &= (2\gamma A^2 + B)^{-1}[-Au + \gamma A^2 v] \end{cases}.$$

That is, $\mathcal{H}^2 = U_0 + U_0^{\perp}$.

Hence $\mathcal{H}^2 = U_0 \dot{\oplus} U_0^{\perp}$ is a Krein decomposition of $(\mathcal{H}^2, [\cdot,\cdot])$. Let P^+ and P^- be the fundamental projectors on U_0 and $U_0^{\perp}$, respectively.

Proposition 6.4.6. For every maximal positive semi-definite subspace U of $(\mathcal{H}^2, [\cdot,\cdot])$ there exists a **G**-contraction K in $\mathcal{H}$ such that

$$U = \left\{ \mathbf{x} \in \mathcal{H}^2 : \mathbf{x} = \begin{bmatrix} \gamma A x - \gamma A K x - A^{-1} B K x \\ x + K x \end{bmatrix}, x \in \mathcal{H} \right\}. \tag{6.8}$$

Proof. Let U be a maximal positive semi-definite subspace of $(\mathcal{H}^2, [\cdot,\cdot])$ and let $\mathbf{K}^+$ the angular operator of U with respect to U_0. By the theorem 4.2.4, we have $P^+(U) = U_0$ and, therefore, $U = \{\mathbf{x}_+ + \mathbf{K}^+\mathbf{x}_+ \mid \mathbf{x}_+ \in U_0\}$. Moreover, by the theorem 3.6.3, note that $-[\mathbf{K}^+\mathbf{x}_+, \mathbf{K}^+\mathbf{x}_+] \leq [\mathbf{x}_+, \mathbf{x}_+]$ for every $\mathbf{x}_+ \in U_0$.

Note also that for each $x \in \mathcal{H}$ there exists a unique $y \in \mathcal{H}$ such that

$$\mathbf{K}^+ \begin{bmatrix} \gamma A x \\ x \end{bmatrix} = \begin{bmatrix} -\gamma A y - A^{-1} B y \\ y \end{bmatrix}. \tag{6.9}$$

So, define $K : \mathcal{H} \to \mathcal{H}$ by $Kx :=$ *the unique* $y \in \mathcal{H}$ *such that* (6.9) *holds*, $x \in \mathcal{H}$. Clearly, K is a linear operator.

Let $x \in \mathcal{H}$, $y = Kx$ and $\mathbf{x}_+ = (\gamma A x, x)$. Since

$$\begin{aligned} -[\mathbf{K}^+\mathbf{x}_+, \mathbf{K}^+\mathbf{x}_+] &= -\langle \mathbf{G}\mathbf{K}^+\mathbf{x}_+, \mathbf{K}^+\mathbf{x}_+ \rangle_{\mathcal{H}^2} = -\langle (Ay, -\gamma A^2 y), (-\gamma A y - A^{-1} B y, y) \rangle_{\mathcal{H}^2} \\ &= \langle (2\gamma A^2 + B) y, y \rangle = |||y|||^2 = |||Kx|||^2 \end{aligned}$$

and $[\mathbf{x}_+, \mathbf{x}_+] = \langle (Ax, \gamma A^2 x + Bx), (\gamma A x, x) \rangle_{\mathcal{H}^2} = \langle (2\gamma A^2 + B) x, x \rangle = |||x|||^2$, then $|||Kx||| \leq |||x|||$. Hence, K is a **G**-contraction such that (6.8) holds. □

Remark. By imitating of the proof of the proposition 6.3.6, is clear that if U is a closed and maximal positive definite subspace of $(\mathcal{H}^2, [\cdot,\cdot])$, then (6.8) holds with a **G**-contraction K such that $|||Kx||| < |||x|||$ for every $x \in \mathcal{H}$ with $x \neq 0$. Thus K has no eigenvalues of norm 1 and, by the theorem A.1.6, the subspace $(I+K)(\mathcal{H})$ is dense in $\mathcal{H}$. Therefore the operator

$$Z = (\gamma A - \gamma AK - A^{-1}BK)(I+K)^{-1} \tag{6.10}$$

is a densely defined and closed linear operator acting in $\mathcal{H}$.

Proposition 6.4.7. Let U be a closed and maximal positive definite subspace of $(\mathcal{H}^2, [\cdot,\cdot])$. Then there exists a densely defined and closed linear operator Z acting in $\mathcal{H}$ such that

$$U = \left\{ \begin{bmatrix} Zx \\ x \end{bmatrix} : x \in \mathscr{D}(Z) \right\}. \tag{6.11}$$

Moreover,

$$U^{\perp} = \left\{ \begin{bmatrix} -A^{-1}Z^*Ax - A^{-1}Bx \\ x \end{bmatrix} : x \in \mathscr{D}(Z^*A) \right\}. \tag{6.12}$$

Proof. The existence of a Z that satisfies (6.11) is clear from the preceding remark and (6.8). It is given by (6.10).

Now, $(u,v) \perp U \Leftrightarrow [(u,v),(Zx,x)] = 0$ for every $x \in \mathscr{D}(Z) \Leftrightarrow \langle (Av, Au+Bv), (Zx,x) \rangle_{\mathcal{H}^2} = 0$ for every $x \in \mathscr{D}(Z) \Leftrightarrow \langle Av, Zx \rangle + \langle Au+Bv, x \rangle = 0$ for every $x \in \mathscr{D}(Z) \Leftrightarrow v \in \mathscr{D}(Z^*A)$ and $\langle Z^*Av + Au + Bv, x \rangle = 0$ for every $x \in \mathscr{D}(Z) \Leftrightarrow v \in \mathscr{D}(Z^*A)$ and $Z^*Av + Au + Bv = 0$ (since $\mathscr{D}(Z)$ is dense in $\mathcal{H}$) $\Leftrightarrow v \in \mathscr{D}(Z^*A)$ and $u = A^{-1}Z^*Av - A^{-1}Bv$. Thus, (6.12) holds. □

Theorem 6.4.8. Let Z be a bounded linear operator in $\mathcal{H}$. Then the subspace

$$U = \left\{ \begin{bmatrix} Zx \\ x \end{bmatrix} : x \in \mathcal{H} \right\} \tag{6.13}$$

is invariant under the operator $\mathbf{L}$ if, and only if,

$$AZ^2 + BZ + C = 0.$$

In this case, we have that $\sigma_p(Z) = \sigma_p(\mathbf{L}\restriction_U)$, $\sigma_c(Z) = \sigma_c(\mathbf{L}\restriction_U)$ and $\sigma_r(Z) = \sigma_r(\mathbf{L}\restriction_U)$. Moreover, a vector $x \in \mathcal{H}$ is an eigenvector of Z with eigenvalue λ if and only if $(\lambda x, x)$ is an eigenvector of the restriction of $\mathbf{L}$ to U.

Proof. (1) U is $\mathbf{L}$-invariant $\Leftrightarrow -A^{-1}BZx - A^{-1}Cx = Z(Zx)$ for every $x \in \mathcal{H} \Leftrightarrow AZ^2x + BZx + Cx = 0$ for every $x \in \mathcal{H} \Leftrightarrow AZ^2 + BZ + C = 0$.

(2) Suppose that U is $\mathbf{L}$-invariant. Then, for every $x,\ y \in \mathcal{H}$, $(\mathbf{L} - \lambda)(Zx, x) = (Zy, y) \Leftrightarrow (Z - \lambda)x = y$. Therefore, $\sigma_p(Z) = \sigma_p(\mathbf{L} \restriction_U)$, $\sigma_c(Z) = \sigma_c(\mathbf{L} \restriction_U)$ and $\sigma_r(Z) = \sigma_r(\mathbf{L} \restriction_U)$. Clearly, $x \in \mathcal{H}$ is an eigenvector of Z with eigenvalue λ if and only if $(\lambda x, x)$ is an eigenvector of $\mathbf{L} \restriction_U$. $\square$

Remarks:

- Let U be a subspace of $\mathcal{H}^2$ and $Z : \mathcal{H} \to \mathcal{H}$ be a linear operator. We say that U and Z *correspond to each other* if, and only if, the equation (6.13) holds.

- Clearly, if $Z : \mathcal{H} \to \mathcal{H}$ is a bounded operator, then the subspace U given by the formula (6.13) is homeomorphic to $\mathcal{H}$.

Definition 6.4.9. The strongly damped pencil L satisfies the property (S) if, and only if, for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ the following is true:

$$x_n \xrightarrow{w} 0,\ \ (Ax_n, x_n) \to 0,\ \ (Bx_n, x_n) \to 0 \ \text{ and } \ (Cx_n, x_n) \to 0 \ \text{ implies } \ x_n \to 0. \tag{S}$$

Theorem 6.4.10. Let L be a strongly damped pencil with the property (S) and with $0 \in \rho(A)$. Then there exist bounded operators Z_1 and Z_2 which satisfy the equation

$$AZ^2 + BZ + C = 0, \tag{6.14}$$

and additionally

$$Z_1^*A + AZ_1 + B > 0 \ \text{ and } \ Z_2^*A + AZ_2 + B < 0. \tag{6.15}$$

Moreover $\sigma(Z_1) = \sigma^+(L)$ and $\sigma(Z_2) = \sigma^-(L)$. The eigenvalues of Z_1 (resp. Z_2) are exactly the eigenvalues of first type (resp., second type) of the pencil L and the corresponding eigenvectors coincide. In particular, we can choose Z_1 and Z_2 such that they satisfy

$$Z_2 = -A^{-1}Z_1^*A - A^{-1}B. \tag{6.16}$$

In this case, $\lambda^2 A + \lambda B + C = (\lambda - Z_2^*)A(\lambda - Z_1) = (\lambda - Z_1^*)A(\lambda - Z_2)$.

Proof. (1) By the proposition 6.4.3 and the theorem A.3.9, the operator $\mathbf{L}$ has an $\mathbf{L}$-invariant, maximal positive semi-definite and positive definite subspace $U \subseteq \mathcal{H}^2$ such that $\sigma(\mathbf{L}\restriction_U) = \sigma^+(\mathbf{L})$. By the proposition 6.7.7, there exists a densely defined and closed linear operator Z_1 acting in $\mathcal{H}$ such that

$$U = \left\{ \begin{bmatrix} Z_1 x \\ x \end{bmatrix} : x \in \mathscr{D}(Z_1) \right\}.$$

Suppose, towards a contradiction, that Z_1 is not bounded; then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathscr{D}(Z_1)$ such that $x_n \to 0$ and $\|Z_1 x_n\| = 1$ for every $n \in \mathbb{N}$. Without loss of generality, we may suppose that $(Z_1 x_n)_{n\in\mathbb{N}}$ is weakly convergent since every bounded sequence in a Hilbert space contains a weakly convergent subsequence. Clearly $Z_1 x_n \xrightarrow{w} 0$.

Let $\mathbf{x}_n = (Z_1 x_n, x_n) \in U$ $(n \in \mathbb{N})$. Then $[\mathbf{x}_n, \mathbf{x}_n] = \langle A x_n, Z_1 x_n\rangle + \langle A Z_1 x_n, x_n\rangle + \langle B x_n, x_n\rangle \to 0$ and, by the proposition 2.2.11 and since U is an $\mathbf{L}$-invariant semi-definite space, we have

$$\begin{aligned} |[\mathbf{L}\mathbf{x}_n, \mathbf{x}_n]|^2 &\leq [\mathbf{L}\mathbf{x}_n, \mathbf{L}\mathbf{x}_n][\mathbf{x}_n, \mathbf{x}_n] = [\mathbf{L}^2\mathbf{x}_n, \mathbf{x}_n][\mathbf{x}_n, \mathbf{x}_n] = \langle \mathbf{G}\mathbf{L}^2\mathbf{x}_n, \mathbf{x}_n\rangle_{\mathcal{H}^2}[\mathbf{x}_n, \mathbf{x}_n] \\ &= (\langle -BZ_1x_n - Cx_n, Z_1x_n\rangle - \langle CZ_1x_n, x_n\rangle)[\mathbf{x}_n, \mathbf{x}_n] \to 0. \end{aligned}$$

Thus, $[\mathbf{L}\mathbf{x}_n, \mathbf{x}_n] \to 0$ and, analogously, $[\mathbf{L}\mathbf{x}_n, \mathbf{L}\mathbf{x}_n] \to 0$. Therefore, $\langle AZ_1x_n, Z_1x_n\rangle \to 0$ and $\langle BZ_1x_n, Z_1x_n\rangle \to 0$.

By the proof of the proposition 6.3.11, since $0 \in \rho(A)$, there exist α and β in $\mathbb{R}$ such that $L(\alpha) \leq 0$ and $L(\beta) \geq 0$. Thus, for every $n \in \mathbb{N}$, $\alpha^2\langle AZ_1x_n, Z_1x_n\rangle + \alpha\langle BZ_1x_n, Z_1x_n\rangle \leq -\langle CZ_1x_n, Z_1x_n\rangle \leq \beta^2\langle AZ_1x_n, Z_1x_n\rangle + \beta\langle BZ_1x_n, Z_1x_n\rangle$. Therefore, $\langle CZ_1x_n, Z_1x_n\rangle \to 0$. Since L satifies the property (S) we have $Z_1x_n \to 0$ $\lightning$. Thus Z_1 is a densely defined, closed and bounded operator; therefore, $\mathscr{D}(Z_1) = \mathcal{H}$.

By the theorem 6.3.8, we have $AZ_1^2 + BZ_1 + C = 0$ and $\sigma(Z_1) = \sigma(\mathbf{L}\restriction_U)$.

Clearly, $Z_1^*A + AZ_1 + B$ is a self-adjoint operator. Note that $Z_1^*A + AZ_1 + B > 0 \Leftrightarrow \langle (Z_1^*A + AZ_1 + B)x, x\rangle > 0$ for every $x \in \mathcal{H} \setminus \{0\} \Leftrightarrow \langle Ax, Z_1x\rangle + \langle AZ_1x, x\rangle + \langle Bx, x\rangle > 0$ for every $x \in \mathcal{H} \setminus \{0\} \Leftrightarrow [(Z_1x, x), (Z_1x, x)] = 0$ for every $x \in \mathcal{H} \setminus \{0\} \Leftrightarrow [\mathbf{x}, \mathbf{x}] > 0$ for every $\mathbf{x} \in U \setminus \{0\}$. Thus, since U is positive definite, we have $Z_1^*A + AZ_1 + B > 0$.

(2) Since U is maximal positive semi-definite then, by the theorem 4.2.14, $U^\perp$ is maximal negative semi-definite. Moreover, since $U^\perp$ is $\mathbf{L}$-invariant (because U is $\mathbf{L}$-invariant and $\mathbf{L}$ is $\mathbf{G}$-selfadjoint), then $\sigma(\mathbf{L} \restriction_{U^\perp}) = \sigma^-(\mathbf{L})$.

Let $Z_2 = -A^{-1}Z_1^*A - A^{-1}B \in \mathcal{B}(\mathcal{H})$. Note that Z_2 is the unique operator in $\mathcal{B}(\mathcal{H})$ such that $U^\perp$ has the form (6.13). By the proposition 6.3.7 and the theorem 6.3.8, $AZ_2 + BZ_2 + C = 0$ and $\sigma(Z_2) = \sigma(\mathbf{L} \restriction_{U^\perp})$. Note that $Z_2^*A + AZ_2 + B = -(Z_1^*A + AZ_1 + B) < 0$. Further, $(\lambda - Z_2^*)A(\lambda - Z_1) = \lambda^2 A + \lambda B + C = (\lambda - Z_1^*)A(\lambda - Z_2)$.

(3) By the proposition A.3.9, $\sigma(\mathbf{L} \restriction_{U^\perp}) = \sigma^-(\mathbf{L})$; thus, $\sigma(Z_1) = \sigma^+(\mathbf{L})$ and $\sigma(Z_2) = \sigma^-(\mathbf{L})$. Since $\sigma^+(\mathbf{L}) = \sigma^+(L)$ and $\sigma^-(\mathbf{L}) = \sigma^-(L)$ (by the proposition 6.3.4), then $\sigma(Z_1) = \sigma^+(L)$ and $\sigma(Z_2) = \sigma^-(L)$.

Clearly, since $AZ_1^2 + BZ_1 + C = 0$, we have that $Z_1x = \lambda x$ implies $L(\lambda)x = 0$; that is, $\sigma_p(Z_1) \subseteq \sigma_p^+(L)$ and every eigenvector of Z_1 is also an eigenvector of L. Analogously, since $AZ_2^2+BZ_2+C = 0$, we have that $Z_2x = \lambda x$ implies $L(\lambda)x = 0$; that is, $\sigma_p(Z_2) \subseteq \sigma_p^-(L)$ and every eigenvector of Z_2 is also an eigenvector of L.

Let $\lambda \in \sigma_p^+(L)$ and $x \in \mathcal{H}$ with $x \neq 0$ such that $L(\lambda)x = 0$; note that $\lambda \notin \sigma_p(Z_2)$. Since $L(\lambda)x = (\lambda - Z_2^*)A(\lambda - Z_1)x$, then $\lambda \in \sigma_p(Z_1)$ and x is an eigenvector of Z_1. Thus, $\sigma_p(Z_1) = \sigma_p^+(L)$ and the corresponding eigenvectors coincide.

Analogously, $\sigma_p(Z_2) = \sigma_p^-(L)$ and the corresponding eigenvectors coincide. □

Corollary 6.4.11. Let L, Z_1 and Z_2 be as in the theorem 6.4.10. Then $\sigma_p(L) = \sigma_p(\mathcal{Z}_1) \cup \sigma_p(\mathcal{Z}_2)$, where $\mathcal{Z}_i$ is the linear pencil defined by $\mathcal{Z}_i(\lambda) = \lambda - Z_i$, $i = 1, 2$.

Proof. $\sigma_p(L) = \sigma_p^+(L) \cup \sigma_p^-(L) = \sigma_p(Z_1) \cup \sigma_p(Z_2) = \sigma_p(\mathcal{Z}_1) \cup \sigma_p(\mathcal{Z}_2)$. □

Remark. Let L be a strongly damped pencil with $0 \in \rho(A)$ and assume that it satisfies the property (S). Then:

- If $Z \in \mathcal{B}(\mathcal{H})$ satisfies the equation $AZ^2 + BZ + C = 0$ and $Z^*A + AZ + B > 0$ then the subspace given by the formula (6.13) is **L**-invariant and maximal positive semi-definite (the maximality is due to the fact that U is homeomorphic to $\mathcal{H}$). Moreover, $\sigma(Z) = \sigma^+(L)$.

- If, further, **L** does not have singular critical points then, by the theorem A.3.9, U and, therefore, Z are uniquely determined and U is uniformly positive. Thus, Z is uniquely determined by the property $Z^*A + AZ + B > 0$. Further, from of the proof of the theorem 6.4.9 and the uniform positivity of U, we have that the operator $T := Z^*A + AZ + B$ is self-adjoint, uniformly positive and, therefore, it is a homeomorphism from $\mathcal{H}$ in itself.

- Note that $TZ = Z^*AZ - C = Z^*T = (TZ)^*$. Therefore, Z is similar to a self-adjoint operator since $Z = T^{-1/2}(T^{-1/2}TZT^{-1/2})T^{1/2}$ and $T^{-1/2}TZT^{-1/2}$ is self-adjoint.

This remark shows the following theorem.

Theorem 6.4.12. Let L be a strongly damped pencil with $0 \in \rho(A)$ such that it satisfies the property (S). If the linearizer **L** does not have singular critical points, then the operators Z_1 and Z_2 in the theorem 6.4.10 are uniquely determined by the property (6.15). Further, they are similar to self-adjoint operators.

6.5 Operator-Differential Equations Associated to a Quadratic Pencil

Given a quadratic pencil $L(\lambda) = \lambda^2 A + \lambda B + C$ we can associate to it the following operator differential equation

$$Au''(t) + Bu'(t) + Cu(t) = 0, \tag{6.17}$$

where u is a $\mathcal{H}$-valued function.

Remarks:

- If $Z \in \mathcal{B}(\mathcal{H})$, then the function $T : \mathbb{R} \to \mathcal{B}(\mathcal{H})$, $t \in \mathbb{R} \mapsto e^{tZ}$, is infinitely differentiable with $\frac{d^n}{dt^n}(e^{tZ}) = Z^n e^{tZ}$ for every $n \in \mathbb{N}_0$ (see [14]).

- By the previous item, if $Z \in \mathcal{B}(\mathcal{H})$ and $x_0 \in \mathcal{H}$ is a fixed vector, then the function $u : \mathbb{R} \to \mathcal{H}$, $u(t) = e^{tZ}x_0$, is infinitely differentiable with $u^{(n)}(t) = Z^n u(t)$ for every $n \in \mathbb{N}_0$.

Proposition 6.5.1. Let $Z \in \mathcal{B}(\mathcal{H})$. The vector function $u(t) = e^{tZ}x_0$ is a solution of (6.17) for any $x_0 \in \mathcal{H}$ if, and only if, $AZ^2 + BZ + C = 0$.

Proof. $\boxed{\Leftarrow}$ Let $x_0 \in \mathcal{H}$ and suppose that $AZ^2 + BZ + C = 0$, then if $u(t) = e^{tZ}x_0$ we have $Au''(t) + Bu'(t) + Cu(t) = AZ^2u(t) + BZu(t) + Cu(t) = (AZ^2 + BZ + C)u(t) = 0$. That is, the function u is a solution of (6.17).

$\boxed{\Rightarrow}$ Suppose that for any $x_0 \in \mathcal{H}$ the function $u(t) = e^{tZ}x_0$ is solution of (6.17). Then, for every $x \in \mathcal{H}$ and for every $t \in \mathbb{R}$, $(AZ^2 + BZ + C)e^{tZ}x = 0$. Thus with $t = 0$ we have that, for every $x \in \mathcal{H}$, $(AZ^2 + BZ + C)x = 0$ and, therefore, $AZ^2 + BZ + C = 0$. □

Remark. Suppose that the pencil L satisfies the conditions of the theorem 6.4.10. Then there exist Z_1, $Z_2 \in \mathcal{B}(\mathcal{H})$ such that $AZ_j^2 + BZ_j + C = 0$ $(j = 1, 2)$ and, therefore, for any x_1, $x_2 \in \mathcal{H}$, the functions $u_j(t) = e^{tZ_j}x_j$ $(j = 1, 2)$ are solutions of (6.17). Further, for any x_1, $x_2 \in \mathcal{H}$,

$$u(t) = e^{tZ_1}x_1 + e^{tZ_2}x_2 \tag{6.18}$$

is a solution of (6.17).

Remark. Consider the Cauchy problem in $\mathcal{H}^2$ given by the formula

$$w'(t) - \mathbf{L}w(t) = 0, \quad w(0) = (u_0, v_0), \tag{6.19}$$

where $\mathbf{L}$ is the linearizer of the the pencil L (see [14]). The unique solution to this problem is given by $w(t) = e^{t\mathbf{L}}w_0$. Note that if $w(t) = (u(t), v(t))$ then $u'(t) + A^{-1}Bu(t) + A^{-1}Cv(t) = 0$ and $v'(t) - u(t) = 0$. Therefore, $v(t)$ satisfies $Av''(t) + Bv'(t) + Cv(t) = 0$ with $v(0) = v_0$ and $v'(0) = u_0$.

Conversely, if $Av''(t) + Bv'(t) + Cv(t) = 0$ with $v(0) = v_0$ and $v'(0) = u_0$, then setting $u(t) = v'(t)$ we have that $w(t) = (u(t), v(t)))$ satisfies the Cauchy problem (6.19).

Theorem 6.5.2. Suppose that the pencil L satisfies the conditions of the theorem 6.4.10. Let Z_1 and Z_2 operator roots of L. Then the following are equivalent:

1) Any solution of (6.17) has the form (6.18).

2) $\mathscr{R}(W) = \mathcal{H}^2$; where $W = \begin{pmatrix} Z_1 & Z_2 \\ I & I \end{pmatrix}$.

Proof. $\boxed{1) \Rightarrow 2)}$ Let $(u_0, v_0) \in \mathcal{H}^2$ and let $w(t) = (u(t), v(t))$ be the unique solution of the Cauchy problem (6.19). Since $v(t)$ satisfies (6.17), there exist x_1, $x_2 \in \mathcal{H}$ such that $v(t) = e^{tZ_1}x_1 + e^{tZ_2}x_2$ and, therefore, $Z_1x_1 + Z_2x_2 = v'(0) = u_0$ and $x_1 + x_2 = v_0$; that is $(u_0, v_0) \in \mathscr{R}(W)$. Hence $\mathscr{R}(W) = \mathcal{H}^2$.

$\boxed{2) \Rightarrow 1)}$ Let $v(t)$ a solution of (6.17). Since $\mathscr{R}(W) = \mathcal{H}^2$, there exist x_1, $x_2 \in \mathcal{H}$ such that $Z_1x_1 + Z_2x_2 = v'(0)$ and $x_1 + x_2 = v(0)$. Since the Cauchy problem (6.19), with $u_0 = v'(0)$ and $v_0 = v(0)$, has a unique solution and $(f'(t), f(t))$, where $f(t) = e^{tZ_1}x_1 + e^{tZ_2}x_2$, is a solution of (6.19), then $v(t) = f(t)$. That is, $v(t)$ is of the form (6.18). □

Theorem 6.5.3. Let L, Z_1, Z_2 and W be as in the theorem 6.5.2. Then the following are equivalent:

1) Every solution of (6.17) of the form (6.18) admits a unique such representation.

2) $\ker(W) = 0$.

Proof. It is clear. □

Corollary 6.5.4. Under conditions of theorem 6.5.3, the following are equivalent:

1) Any solution of (6.17) has a unique representation of the form (6.18).

2) W is bijective.

Remark. Clearly, W is bijective if and only if the operator $Z_1 - Z_2$ is bijective. Moreover, by the proof of theorem 6.4.10, $Z_1 - Z_2$ is bijective if and only if $AZ_1 + Z_1^*A + B$ is bijective. Since $AZ_1 + Z_1^*A + B > 0$ then $AZ_1 + Z_1^*A + B$ is bijective if and only if $AZ_1 + Z_1^*A + B$ is an uniformly definite operator.

6.6 A special case

In this section we assume that the operator A satisfies $A \gg 0$; that is, there exists $\delta > 0$ such that $A \geq \delta I$. Moreover, the operator C satisfies $C \leq 0$.

Remarks:

- For every $x \in \mathcal{H}$ with $x \neq 0$ we have that $p_+(x)$ and $p_-(x)$ are real and distinct.
- π_+ and π_- are bounded subsets of $\mathbb{R}$. Further, $\pi_+ \subseteq [0, +\infty)$ and $\pi_- \subseteq (-\infty, 0]$.
- $\sigma^+(L)$ and $\sigma^-(L)$ are compact subsets of $\mathbb{R}$.
- Clearly $0 \in \rho(A)$, and if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ such that $x_n \xrightarrow{w} 0$ and $\langle Ax_n, x_n\rangle \to 0$, then $x_n \to 0$. Thus, L satisfies the property (S) and, therefore, the conditions of the theorem 6.3.10.
- For every $x \neq 0$, $\langle AA^{-1}x, A^{-1}x\rangle \geq \delta\|A^{-1}x\|^2 \geq \delta\frac{\|x\|^2}{\|A\|^2} = \frac{\delta}{\|A\|^2}\|x\|^2$; that is $\langle A^{-1}x, x\rangle = \langle x, A^{-1}x\rangle \geq \frac{\delta}{\|A\|^2}\|x\|^2$. Therefore, $A^{-1} \gg 0$.
- Note that, $\mathbf{x} = (u, v) \in \ker(\mathbf{L}) \Leftrightarrow u = 0$ and $v \in \ker(C)$.

Proposition 6.6.1. Let $\lambda_0 \in \mathbb{R}$.

(i) If $\lambda_0 \notin \pi_+ \cup \pi_-$, then $L(\lambda_0) > 0$ or $L(\lambda_0) < 0$.

(ii) If $\lambda_0 \in \overline{\mathbb{R} \setminus [\pi_+ \cup \pi_-]}$, then $L(\lambda_0) \geq 0$ or $L(\lambda_0) \leq 0$.

(iii) If $\lambda_0 \notin \overline{\pi_+ \cup \pi_-}$, then $L(\lambda_0)$ is uniformly definite.

Proof. Since $\lambda_0 \in \mathbb{R}$ then, clearly, $L(\lambda_0)$ is self-adjoint.

(i) Suppose, towards a contradiction, that there exists $x \neq 0$ such that $\langle L(\lambda_0)x, x\rangle = 0$. Then λ_0 is a root of φ_x and, therefore, $\lambda_0 \in \pi_+ \cup \pi_-$ $\lightning$.

(ii) It is clear from (i).

(iii) If $\lambda_0 \notin \overline{\pi_+ \cup \pi_-}$ then, by (i), $L(\lambda_0) > 0$ or $L(\lambda_0) < 0$. Suppose, without loss of generality, that $L(\lambda_0) > 0$.

Suppose, towards a contradiction, that $L(\lambda_0)$ is not uniformly definite. Then there exists $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$, with $\|x_n\| = 1$ for every $n \in \mathbb{N}$, such that $\langle L(\lambda_0)x_n, x_n\rangle \to 0$. That is, $\lambda_0^2\langle Ax_n, x_n\rangle + \lambda_0\langle Bx_n, x_n\rangle + \langle Cx_n, x_n\rangle \to 0$.

For each $n \in \mathbb{N}$ set $q_n(\lambda) := \lambda^2\langle Ax_n, x_n\rangle + \lambda\langle Bx_n, x_n\rangle + \langle Cx_n, x_n\rangle$. Since A, B and C are bounded and self-adjoint operators, there exists a subsequence q_{n_k} and a polynomial q_0 such that $q_{n_k} \to q_0$ when $k \to \infty$. Note that, since $A \gg 0$, we have $q_0 \neq 0$ and, moreover, $q_0(\lambda_0) = 0$.

By the Hurwitz theorem, see [12], every neighborhood of λ_0 contains at least one root of every polynomial q_{n_k} with a sufficiently large index k. Thus, $\lambda_0 \in \overline{\pi_+ \cup \pi_-}$ $\lightning$. □

Chapter 7

Hyperbolic and Elliptic-hyperbolic Pencils

In this chapter $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space; A, B, $C \in \mathcal{B}(\mathcal{H})$ are self-adjoint operators and the quadratic pencil operator L has the form:

$$L(\lambda) = A + \lambda B + \lambda^2 C. \tag{7.1}$$

With $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}^+_\infty$ and $\mathbb{R}^-_\infty$ we denote, respectively, the subsets $[0, +\infty)$, $(-\infty, 0]$, $[0, +\infty) \cup \{\infty\}$ and $(-\infty, 0] \cup \{\infty\}$ of $\mathbb{R}_\infty$.

7.1 Hyperbolic Pencils

Definition 7.1.1. The pencil operator L is called *hyperbolic* if and only if $A \gg 0$, $C \leq 0$, $C \not\equiv 0$ and $\langle Bx, x \rangle \neq 0$ for every $x \in \ker(C)$ with $x \neq 0$.

Remarks:

- Clearly, every hyperbolic quadratic pencil is strongly damped.
- From the definitions 6.3.6 and 6.3.5 we have $\pi_+ \subseteq \mathbb{R}^-_\infty$ and $\pi_- \subseteq \mathbb{R}^+_\infty$.
- $0 \notin \sigma(L)$.
- Clearly, if $A \gg 0$ then $0 \in \rho(A)$ and, therefore, we can associate to the hyperbolic pencil the operators:

$$\mathbf{G} = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} -A^{-1}B & -A^{-1}C \\ I & 0 \end{pmatrix}.$$

- As in the chapter 6, $[\mathbf{x}, \mathbf{y}] := \langle \mathbf{G}\mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}^2}$ ($\mathbf{x}$, $\mathbf{y} \in \mathcal{H}^2$). Thus, $(\mathcal{H}^2, [\cdot, \cdot])$ is a Krein space.
- Let $\mu = \frac{1}{\lambda}$. Then we have $L(\lambda) = \lambda^2 L_1(\mu)$, where $L_1(\mu) = \mu^2 A + \mu B + C$.

The proof of the following proposition is basically rewrite the one of the proposition 6.3.2.

Proposition 7.1.2. If L is a hyperbolic pencil then the sign of the form $\langle Bx, x \rangle$ is constant on $\ker(C) \setminus \{0\}$.

Theorem 7.1.3 (Langer Factorization)**.** If L is a hyperbolic pencil operator, then there exist operators Z_1 and Z_2 such that:

(1) $L(\lambda) = (I - \lambda Z_2^*)A(I - \lambda Z_1) = (I - \lambda Z_1^*)A(I - \lambda Z_2)$.

(2) $\sigma(\mathcal{Z}_1) \subseteq \mathbb{R}_\infty^+$ and $\sigma(\mathcal{Z}_2) \subseteq \mathbb{R}_\infty^-$, where $\mathcal{Z}_i$ is the linear pencil defined by $\mathcal{Z}_i(\lambda) = I - \lambda Z_i$, $i = 1, 2$.

(3) The operator $S = A(Z_1 - Z_2)$ is self-adjoint and positive.

(4) $U^+ = \{(Z_1 x, x) \mid x \in \mathcal{H}\}$ and $U^- = \{(Z_2 x, x) \mid x \in \mathcal{H}\}$ are maximal positive definite and negative definite subspaces of $(\mathcal{H}^2, [\cdot, \cdot])$, respectively. Further, U^+ and U^- are $\mathbf{L}$-invariant.

Proof. Consider the pencil $L_1(\mu) = \mu^2 A + \mu B + C$ (where $\mu = \frac{1}{\lambda}$). Clearly L_1 is strongly damped and, since $A \gg 0$, L_1 satisfies the property (S) of definition 6.4.5. By the theorem 6.4.10 (and its proof) there exist Z_1, $Z_2 \in \mathcal{B}(\mathcal{H})$, with $Z_2 = -A^{-1} Z_1^* A - A^{-1} B$, such that $L_1(\mu) = (\mu - Z_2^*)A(\mu - Z_1) = (\mu - Z_1^*)A(\mu - Z_2)$, $\sigma(\mathcal{F}_1) = \sigma^+(L_1)$, $\sigma(\mathcal{F}_2) = \sigma^-(L_1)$, $Z_1^* A + A Z_1 + B > 0$ and $Z_2^* A + A Z_2 + B < 0$, where $\mathcal{F}_i$ is the linear pencil defined by $\mathcal{F}_i(\mu) = \mu - Z_i$, $i = 1, 2$. Further, the subspaces $U_+ = \{(Z_1 x, x) \mid x \in \mathcal{H}\}$ and $U_- = \{(Z_2 x, x) \mid x \in \mathcal{H}\}$ are, respectively, maximal positive definite and maximal negative definite; also they are $\mathbf{L}$-invariant. Therefore, $L(\lambda) = (I - \lambda Z_2^*)A(I - \lambda Z_1) = (I - \lambda Z_1^*)A(I - \lambda Z_2)$ and, thus, (1) and (4) holds.
Since $\sigma(\mathcal{F}_1) = \sigma^+(L_1) \subseteq \mathbb{R}_\infty^+$ (see section 6.5) and $\lambda = \frac{1}{\mu}$, then $\sigma(\mathcal{Z}_1) \subseteq \mathbb{R}_\infty^+$. Analogously, $\sigma(\mathcal{Z}_2) \subseteq \mathbb{R}_\infty^-$ and, thus, (2) holds.

From $Z_2 = -A^{-1}Z_1^*A - A^{-1}B$ we obtain $-AZ_2 = Z_1^*A + B$. Therefore, $S = A(Z_1 - Z_2) = AZ_1 - AZ_2 = AZ_1 + Z_1^*A + B > 0$. Thus, (3) holds. $\square$

Corollary 7.1.4. Let L be a hyperbolic pencil and let Z_1 and Z_2 be the operators in its Langer Factorization. Then $\ker(Z_1) \subseteq \ker(C)$ and $\ker(Z_2) \subseteq \ker(C)$. Moreover, if $\langle Bx, x\rangle > 0$ (< 0) for every $x \in \ker(C) \setminus \{0\}$, then $\ker(Z_2) = \{0\}$ ($\ker(Z_1) = \{0\}$).

Proof. By (1) of the theorem 7.1.3, $C = Z_2^*AZ_1 = Z_1^*AZ_2$ and, therefore, $\ker(Z_1)$ and $\ker(Z_2)$ are subsets of $\ker(C)$.

Suppose that $\langle Bx, x\rangle > 0$ for every $x \in \ker(C) \setminus \{0\}$. Thus for every $x \in \ker(Z_2) \setminus \{0\}$ we have $0 < \langle Bx, x\rangle = \langle(-Z_1^*A - AZ_2)x, x\rangle = \langle x, -AZ_1x\rangle = \langle x, -Sx\rangle \leq 0 \lightning$. Hence, $\ker(Z_2) = \{0\}$.

Analogously, if $\langle Bx, x\rangle < 0$ for every $x \in \ker(C) \setminus \{0\}$, then $\ker(Z_1) = \{0\}$. $\square$

Proposition 7.1.5. If Z_1 and Z_2 are the operators in the Langer Factorization of the hyperbolic pencil L, then the operator $S = A(Z_1 - Z_2)$ satisfies the inequality $SA^{-1}S \geq 4D$, where $D = -C \geq 0$.

Proof. Since S is self-adjoint and $C = Z_2^*AZ_1 = Z_1^*AZ_2$, we have $SA^{-1}S = S^*A^{-1}S = (Z_1^* - Z_2^*)AA^{-1}A(Z_1 - Z_2) = (Z_1^* - Z_2^*)A(Z_1 - Z_2) = Z_1^*AZ_1 - Z_1^*AZ_2 - Z_2^*AZ_1 + Z_2^*AZ_2 = Z_1^*AZ_1 + Z_2^*AZ_2 - 2C$. Thus, for every $x \in \mathcal{H}$, $\langle(SA^{-1}S + 4C)x, x\rangle = \langle(Z_1^*AZ_1 + Z_2^*AZ_2 + 2C)x, x\rangle = \langle Z_1^*AZ_1x, x\rangle + \langle Z_2^*AZ_2x, x\rangle + 2\langle Cx, x\rangle = \langle A^{1/2}Z_1x, A^{1/2}Z_1x\rangle + \langle A^{1/2}Z_2x, A^{1/2}Z_2x\rangle + \langle Z_2^*AZ_1x, x\rangle + \langle Z_1^*AZ_2x, x\rangle = \|A^{1/2}Z_1x\|^2 + \|A^{1/2}Z_2x\|^2 + \langle A^{1/2}Z_1x, A^{1/2}Z_2x\rangle + \langle A^{1/2}Z_2x, A^{1/2}Z_1x\rangle = \langle A^{1/2}Z_1x + A^{1/2}Z_2x, A^{1/2}Z_1x + A^{1/2}Z_2x\rangle \geq 0$. Hence, $SA^{-1}S \geq 4D$ where $D = -C$. $\square$

Definition 7.1.6. Let L be a hyperbolic pencil.

- $\sigma_d(L) := \{\lambda \in \mathbb{R}_\infty \mid \lambda$ is an isolated point of $\sigma(L)$ and $\lambda \in \sigma_p(L)$ has finite multiplicity$\}$ is called the *discrete spectrum* of L.

- $E^+ := \{x \in \mathcal{H} \mid x$ is an eigenvector with associated eigenvalue $\lambda \in \sigma_d(L)$ and $\lambda > 0\}$.

- $E^- := \{x \in \mathcal{H} \mid x$ is an eigenvector with associated eigenvalue $\lambda \in \sigma_d(L)$ and $\lambda < 0\}$.

Lemma 7.1.7. Let L be a hyperbolic pencil. Then there exists $\delta > 0$ such that $\|L(\lambda)^{-1}\| \leq \delta^{-1}|\sin(\theta)|^{-1}$ for every $\lambda = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Since $A \gg 0$ then $\delta := \inf_{\|x\|=1} \langle Ax, x\rangle > 0$. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Suppose that $x \in \ker(C)$ and suppose, without restriction, that $\langle Bx, x\rangle > 0$, then

$$\|L(\lambda)x\| \geq |\langle L(\lambda)x, x\rangle| = |\langle Ax + \lambda Bx, x\rangle| = |\langle Ax, x\rangle + \lambda\langle Bx, x\rangle| = r\langle Ax, x\rangle|\mu - p_+(x)|,$$

where $\mu = \frac{1}{\lambda}$ and p_+ is the functional of first type associated to the pencil $L_1(\mu) = \mu^2 A + \mu B + C$ (see definition 6.3.5). Thus,

$$\|L(\lambda)x\| \geq r\langle Ax, x\rangle|\mu|| \sin(-\theta)| = \langle Ax, x\rangle| \sin(\theta)| \geq \delta| \sin(\theta)| = \delta| \sin(\theta)|.$$

If $x \notin \ker(C)$ then, with $\mu = \frac{1}{\lambda}$ and p_+, p_- the functionals of first and second type associated to the pencil $L_1(\mu) = \mu^2 A + \mu B + C$, we have

$$\begin{aligned}
\|L(\lambda)x\| &\geq |\langle L(\lambda)x, x\rangle| = |\langle Ax, x\rangle + \lambda\langle Bx, x\rangle + \lambda^2\langle Cx, x\rangle| \\
&= r^2\langle Ax, x\rangle \left|\mu^2 + \mu\tfrac{\langle Bx,x\rangle}{\langle Ax,x\rangle} + \tfrac{\langle Cx,x\rangle}{\langle Ax,x\rangle}\right| \\
&= r^2\langle Ax, x\rangle|\mu - p_+(x)||\mu - p_-(x)| \\
&\geq r^2\langle Ax, x\rangle|\mu|^2| \sin(-\theta)| = \langle Ax, x\rangle| \sin(\theta)|,
\end{aligned}$$

since $p_+(x) \geq 0$ and $p_-(x) \leq 0$.

So, for every $y \in \mathcal{H}$, $\|y\| \leq \delta^{-1}| \sin(\theta)|^{-1}\|L(\lambda)y\|$ and therefore $\|L(\lambda)^{-1}y\| \leq \delta^{-1}| \sin(\theta)|^{-1}\|y\|$. Hence $\|L^{-1}(\lambda)\| \leq \delta^{-1}| \sin(\theta)|^{-1}$. □

Remark. From 7.1.7 we have that if $\lambda = re^{i\theta} \in \mathbb{C}\backslash\mathbb{R}$ then $|\langle L^{-1}(\lambda)x, x\rangle| \leq \|A^{-1/2}\|| \sin(\theta)|^{-1}\|x\|^2$ for every $x \in \mathcal{H}$ (independently of $|\lambda|$). However in [19] it is proved that $\|L^{-1}(\lambda)x\| = o\left(\frac{1}{|\lambda|}\right)$ as $\lambda \to \infty$ for each $x \in \mathcal{H}$.

Proposition 7.1.8. If L is a hyperbolic pencil and $\sigma(L) \cap \mathbb{R}^+ \subseteq \sigma_d(L)$, then the system E^+ is a total subset of the subspace $\overline{\mathscr{R}(Z_1)}$.

Analogously, if $\sigma(L) \cap \mathbb{R}^- \subseteq \sigma_d(L)$, then the system E^- is a total subset of the subspace $\overline{\mathscr{R}(Z_2)}$.

Proof. Let $x \in E^+$, then there exists $\lambda > 0$ such that $\lambda \in \sigma_p(L)$ and x is an eigenvector associated to λ. By the theorem 7.1.3, $(I - \lambda Z_1)x = 0$. Thus, $Z_1 x = \frac{1}{\lambda}x$ and, since $\frac{1}{\lambda} \neq 0$, $x \in \mathscr{R}(Z_1)$; that is, $E^+ \subseteq \mathscr{R}(Z_1)$ and, therefore, $\overline{\mathrm{Span}(E^+)} \subseteq \overline{\mathscr{R}(Z_1)}$.

Suppose, towards a contradiction, that E^+ is not a total subset of $\overline{\mathscr{R}(Z_1)}$. Let $x \neq 0$ such that $x \in \overline{\mathscr{R}(Z_1)}$ and $x \perp \overline{\mathrm{Span}(E^+)}$.

Consider the function $g(\lambda) = (I - \lambda Z_1^*)^{-1}x$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Clearly g is an analytic function. Further, for each $\lambda = |\lambda| e^{i\theta} \in \mathbb{C} \setminus \mathbb{R}$ we have $g(\lambda) = A(I - \lambda Z_2)L^{-1}(\lambda)x$ and, by lemma 7.1.7,

$$\|g(\lambda)\| = \|(I - \lambda Z_1^*)^{-1}x\| = \|A(I - \lambda Z_2)L^{-1}(\lambda)x\| \leq M(1 + |\lambda|)|\sin(\theta)|^{-1},$$

where M is a positive constant. Therefore $g(\lambda)$ is a polynomial of degree ≤ 1.

From the preceding remark, and since $g(\lambda) = A(I - \lambda Z_2)L^{-1}(\lambda)x$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we have that $g(\lambda)$ is constant. Let $w \in \mathcal{H}$ such that $g(\lambda) = w$ for every λ in the domain of g. Then $x = (I - \lambda Z_1^*)w$ and therefore, with $\lambda \to 0$, we have $x = w$. Thus $x \in \ker(Z_1^*)$. Since $x \in \overline{\mathscr{R}(Z_1)} \cap \ker(Z_1^*)$, we have $x = 0$ ↯. □

Theorem 7.1.9. If L is a hyperbolic pencil and $\sigma(L) \cap \mathbb{R}^+ \subseteq \sigma_d(L)$, then the system $\{D^{1/2}x \mid x \in E^+\}$ is complete in the subspace $\overline{\mathscr{R}(C)}$, where $D = -C$.

Analogously, if $\sigma(L) \cap \mathbb{R}^- \subseteq \sigma_d(L)$, then the system $\{D^{1/2}x \mid x \in E^-\}$ is complete in the subspace $\overline{\mathscr{R}(C)}$, where $D = -C$.

Proof. Suppose that $\sigma(L) \cap \mathbb{R}^+ \subseteq \sigma_d(L)$. Let Z_1 and Z_2 be the operators in the Langer Factorization and set $S = A(Z_1 - Z_2)$.

- Recall that $S > 0$, hence it is invertible. In particular, $\mathcal{H} = \ker(S) \oplus \overline{\mathscr{R}(S)} = \overline{\mathscr{R}(S)}$. Set $T = D^{1/2}S^{-1}$. Note that $\mathscr{D}(T) = \mathscr{R}(S)$ is a dense subspace of $\mathcal{H}$.

  For every $x = Su$, with $u \in \mathcal{H}$, we have $\|Tx\|^2 = \|D^{1/2}u\|^2 = \langle D^{1/2}u, D^{1/2}u\rangle = \langle Du, u\rangle \leq \frac{1}{4}\langle SA^{-1}Su, u\rangle = \frac{1}{4}\langle A^{-1}Su, Su\rangle = \frac{1}{4}\langle A^{-1}x, x\rangle = \frac{1}{4}\langle A^{-1/2}x, A^{-1/2}x\rangle = \frac{1}{4}\|A^{-1/2}x\|^2$; thus $\|Tx\| \leq \frac{1}{2}\|A^{-1/2}\|\|x\|$. So T is a bounded operator with dense domain in $\mathcal{H}$ and, therefore, T admits a bounded extension to $\mathcal{H}$ which will also be called T. Clearly $\overline{\mathscr{R}(T)} = \overline{\mathscr{R}(D^{1/2})} = \overline{\mathscr{R}(D)} = \overline{\mathscr{R}(C)}$.

- Let E^0 be a basis for the subspace $\ker(C)$. Suppose, towards a contradiction, that $S(E^0) \cup S(E^+)$ is not total in H; then there exists $w \in \mathcal{H}$, with $w \neq 0$, such that $\langle Sx, w\rangle = 0$ for every $x \in E^0 \cup E^+$. Thus $\langle x, Sw\rangle = 0$ for every $x \in E^0 \cup E^+$.

Since E^+ is a total subset of $\overline{\mathscr{R}(Z_1)}$, then $Sw \perp \overline{\mathscr{R}(Z_1)}$ and, therefore, $Sw \in \ker(Z_1^*)$; that is, $0 = Z_1^* Sw = (Z_1^* A Z_1 - Z_1^* A Z_2)w = (Z_1^* A Z_1 - C)w$. Now if $w \notin \ker(Z_1)$ then $\langle Z_1^* A Z_1 w, w\rangle > 0$ and, thus, $0 = \langle Z_1^* Sw, w\rangle = \langle (Z_1^* A Z_1 - C)w, w\rangle > 0$ ↯ (since $-C \geq 0$). Hence, $w \in \ker(Z_1)$ and, by the corollary 7.1.4, $w \in \ker(C)$.

Since $Sw \perp E^0$ and $w \in \ker(C)$, we have que $\langle Sw, w\rangle = 0$ and, since $S > 0$, $w = 0$ ↯. Hence $S(E^0) \cup S(E^+)$ is total in $\mathcal{H}$.

- Since $S(E^0) \cup S(E^+)$ is a total subset of $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, then $D^{1/2}(E^+) = T(S(E^0) \cup S(E^+))$ is a total subset of $\overline{\mathscr{R}(T)} = \overline{\mathscr{R}(C)}$. □

Theorem 7.1.10. Let L be a hyperbolic pencil. Again let $D = -C$ and suppose that B admits a representation $B = B_+ - B_-$, where $B_+ \geq 0$ and B_- satisfies that, for every x, $y \in \mathcal{H}$:

$$|\langle B_- x, y\rangle|^2 \leq 4\langle Ax, x\rangle\langle Dy, y\rangle. \tag{7.2}$$

Then the operator $-Z_2$ in the Langer Factorization of L is similar to an accretive operator and $\overline{\mathscr{R}(Z_2)} = \mathcal{H}$. In particular, if $\sigma(L) \cap \mathbb{R}_\infty^- \subseteq \sigma_d(L)$, then the system E^- is a total subset of $\mathcal{H}$.

Proof. Set $W_1 = A^{1/2} Z_2 A^{-1/2}$, $S_1 = A^{-1/2} S A^{-1/2}$ and $D_1 = A^{-1/2} D A^{-1/2}$. Thus, by the proposition 7.1.5, $S_1^2 = A^{-1/2} S A^{-1} S A^{-1/2} \geq 4A^{-1/2} D A^{-1/2} = 4D_1 \geq 0$ and, by the Löwner-Heinz inequality ([16] [17]), we have $S_1 \geq 2D_1^{1/2}$.
Since $S = A(Z_1 - Z_2) = -Z_2^* A - A Z_2 - B$, then $S_1 = -A^{1/2} Z_2 A^{-1/2} - A^{-1/2} Z_2^* A^{1/2} - A^{-1/2} B A^{-1/2} = -W_1 - W_1^* - A^{-1/2} B A^{-1/2} \geq 2D_1^{1/2}$. Thus, $-W_1 - W_1^* \geq 2D_1^{1/2} + A^{-1/2} B A^{-1/2} = 2D_1^{1/2} + A^{-1/2}(B_+ - B_-)A^{-1/2} = 2D_1^{1/2} + A^{-1/2} B_+ A^{-1/2} - A^{-1/2} B_- A^{-1/2} \geq 2D_1^{1/2} - A^{-1/2} B_- A^{-1/2} = 2D_1^{1/2} - B_1$, where $B_1 = A^{-1/2} B_- A^{-1/2}$.
By (7.2), for every $x, y \in \mathcal{H}$, we have $|\langle B_1 x, y\rangle|^2 = |\langle A^{-1/2} B_- A^{-1/2} x, y\rangle|^2 = |\langle B_- A^{-1/2} x, A^{-1/2} y\rangle|^2 \leq 4\langle A(A^{-1/2}x), A^{-1/2}x\rangle\langle DA^{-1/2}y, A^{-1/2}y\rangle = 4\langle A^{1/2}x, A^{-1/2}x\rangle\langle A^{-1/2}DA^{-1/2}y, y\rangle = 4\langle x, x\rangle\langle D_1 y, y\rangle$.
Thus, for $y \in \mathcal{H}$ fixed,

$$\langle B_1^2 y, y\rangle = \|B_1 y\|^2 = \sup_{\|x\|=1} |\langle x, B_1 y\rangle|^2 = \sup_{\|x\|=1} |\langle B_1 x, y\rangle|^2 \leq 4\langle D_1 y, y\rangle.$$

Then $B_1^2 \leq 4D_1$ and, by the Löwner-Heinz inequality, $B_1 \leq 2D_1^{1/2}$. Therefore, $-W_1 - W_1^* \geq 2D_1^{1/2} - B_1 \geq 0$; that is $-W_1$ is an accretive operator (see [13]). Since $-Z_2 = A^{1/2}(-W_1)A^{1/2}$, the operator $-Z_2$ is similar to an accretive operator.

Note that, by (7.2), for every $x \in \ker(C)$ we have $\langle B_- x, x\rangle = 0$ and, therefore, $\langle Bx, x\rangle = \langle B_+ x, x\rangle \geq 0$. Since L is hyperbolic, $\langle Bx, x\rangle > 0$ and, by the corollary 7.1.4, $\ker(Z_2) = \{0\}$. Thus $\ker(W_1) = \{0\}$ and, since $-W_1$ is accretive, $\ker(W_1^*) = \{0\}$. Therefore $\ker(Z_2^*) = \{0\}$ and $\overline{\mathscr{R}(Z_2)} = \mathcal{H}$.

Now, if $\sigma(L) \cap \mathbb{R}_\infty^- \subseteq \sigma_d(L)$ then, by the proposition 7.1.8, E^- is a total subset of $\mathcal{H}$. □

7.2 Operator-Differential Equation Associated to a Hyperbolic Pencil

Given the hyperbolic pencil (7.1) we can associate the following operator-differential equation

$$L\left(i\frac{d}{dt}\right)u(t) = Au(t) + iBu'(t) + Du''(t) = 0. \tag{7.3}$$

Definition 7.2.1. A continuously differentiable function $u : [0, +\infty) \to \mathcal{H}$ is called a *generalized solution* of the equation (7.3) if, and only if, $iBu(t) + Du'(t)$ is continuously differentiable and $(iBu(t) + Du'(t))' = -Au(t)$.

Remark. Since $A^{1/2}$ is a bounded self-adjoint operator with $A^{1/2} \gg 0$ we can define a positive definite inner product $\langle\cdot,\cdot\rangle_1$ on $\mathcal{H}$ by $\langle x, y\rangle_1 = \langle A^{1/2}x, A^{1/2}y\rangle$; x, $y \in \mathcal{H}$. Clearly the norm $\|\cdot\|_1$ induced by this inner product is equivalent to the norm $\|\cdot\|$. Thus, $\mathcal{H}_1 = (\mathcal{H}, \langle\cdot,\cdot\rangle_1)$ is a Hilbert space. Let $\mathbf{H} = \mathcal{H}_1 \times \mathcal{H}^0$, where $\mathcal{H}^0 = \overline{\mathscr{R}(C)}$, and $\mathbf{V} : \mathbf{H} \to \mathbf{H}$ the bounded operator defined by
$$\mathbf{V} = \begin{pmatrix} A^{-1}B & A^{-1}D^{1/2} \\ D^{1/2} & 0 \end{pmatrix}.$$

The operator $\mathbf{V}$ is self-adjoint since, for every $\mathbf{x}_1 = (x, y)$, $\mathbf{x}_2 = (z, w)$ in $\mathbf{H}$, we have

$$\begin{aligned}
\langle \mathbf{V}\mathbf{x}_1, \mathbf{x}_2\rangle_{\mathbf{H}} &= \langle (A^{-1}Bx + A^{-1}D^{1/2}y, D^{1/2}x), (z,w)\rangle_{\mathbf{H}} \\
&= \langle A^{-1}Bx + A^{-1}D^{1/2}y, z\rangle_1 + \langle D^{1/2}x, w\rangle \\
&= \langle Bx + D^{1/2}y, z\rangle + \langle D^{1/2}x, w\rangle = \langle Bx, z\rangle + \langle D^{1/2}x, w\rangle + \langle D^{1/2}y, z\rangle \\
&= \langle x, Bz\rangle + \langle x, D^{1/2}w\rangle + \langle y, D^{1/2}z\rangle = \langle x, Bz + D^{1/2}w\rangle + \langle y, D^{1/2}z\rangle \\
&= \langle x, A^{-1}Bz + A^{-1}D^{1/2}w\rangle_1 + \langle y, D^{1/2}z\rangle \\
&= \langle (x,y), (A^{-1}Bz + A^{-1}D^{1/2}w, D^{1/2}z)\rangle_{\mathbf{H}} \\
&= \langle \mathbf{x}_1, \mathbf{V}\mathbf{x}_2\rangle_{\mathbf{H}}.
\end{aligned}$$

The operator $\mathbf{V}$ is injective since if $(x,y) \in \ker(\mathbf{V})$ then $D^{1/2}x = 0$, $A^{-1}Bx + A^{-1}D^{1/2}y = 0$ and, therefore, $0 = \langle A^{-1}Bx + A^{-1}D^{1/2}y, x\rangle_1 = \langle Bx + D^{1/2}y, x\rangle = \langle Bx, x\rangle + \langle y, D^{1/2}x\rangle = \langle Bx, x\rangle$. Thus, $x \in \ker(D) = \ker(C)$ and $\langle Bx, x\rangle = 0$. Since the pencil is hyperbolic we have $x = 0$. Thus $y \in \overline{\mathscr{R}(C)}$ and $A^{-1}D^{1/2}y = 0$; that is, $y \in \overline{\mathscr{R}(C)}$ and $y \in \ker(C)$. Therefore, $y = 0$.

Since $(i\mathbf{V}^{-1})^* = -i\mathbf{V}^{-1}$ then, by a theorem of Stone, $i\mathbf{V}^{-1}$ generates a unitary $\mathcal{C}_0$-group (see [15]). Let $\mathbf{U}(t) = e^{it\mathbf{V}^{-1}}$, $t \geq 0$.

Given $x_1 \in \mathcal{H}_1$ and $x_0 \in \mathcal{H}^0$, the function $\mathbf{u}(t) = (u(t), v(t)) = \mathbf{U}(t)\mathbf{x}_0$ with $\mathbf{x}_0 = (x_1, -ix_0) \in \mathbf{H}$ solves the Cauchy problem

$$(\mathbf{V}\mathbf{u}(t))' - i\mathbf{u}(t) = 0$$

and $\lim_{t\to 0^+} \|\mathbf{u}(t) - \mathbf{x}_0\|_{\mathbf{H}} = 0$. Thus we have $\lim_{t\to 0^+} \|u(t) - x_1\|_1 = 0$, $\lim_{t\to 0^+} \|v(t) + ix_0\| = 0$, $D^{1/2}u'(t) = iv(t)$ and $(Bu(t) + D^{1/2}v(t))' = iAu(t)$. Hence,

$$(iBu(t) + Du'(t))' = -Au(t), \quad \lim_{t\to 0^+} \|u(t) - x_1\| = 0 \text{ and } \lim_{t\to 0^+} \|u'(t) - x_0\| = 0. \tag{7.4}$$

Suppose that $u(t)$ satisfies (7.4) with $x_1 = x_0 = 0$. Note that, for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, the following integral exists

$$\int_0^{+\infty} (iBu(t) + Du'(t))' e^{-st} dt = -\int_0^{+\infty} Au(t) e^{-st} dt.$$

After integration by parts we have

$$s\int_0^{+\infty} (iBu(t) + Du'(t))e^{-st}dt = -\int_0^{+\infty} Au(t)e^{-st}dt.$$

Denoting by $\hat{u}(s)$ the Laplace transform of $u(t)$, we have $A\hat{u}(s) + isB\hat{u}(s) + s^2D\hat{u}(s) = 0$. Thus $L(is)\hat{u}(s) = 0$ and, since $L(is)$ is bijective, we have $\hat{u}(s) \equiv 0$. Hence $u(t) \equiv 0$.

So we have shown the following theorem:

Theorem 7.2.2. For any vectors $x_1 \in \mathcal{H}$ and $x_0 \in \mathcal{H}^0$, there exists a unique generalized solution of (7.3) which satisfies (7.4).

7.3 Elliptic-hyperbolic Pencils and their Associated Operator-Differential Equation

Definition 7.3.1. The hyperbolic pencil L is called *elliptic-hyperbolic* if, and only if, the operator B admits a representation $B = B_+ - B_-$, where $B_+ \geq 0$ and B_- is such that for some $\varepsilon > 0$, with $\varepsilon < 2$, and every x, $y \in \mathcal{H}$,

$$|\langle B_-x, y\rangle|^2 \leq (2-\varepsilon)^2\langle Ax, x\rangle\langle Dy, y\rangle, \quad D = -C \geq 0. \tag{7.5}$$

Remark. From the proof of theorem 7.1.10, we have that if L is an elliptic-hyperbolic pencil, then $\ker(Z_2) = \ker(Z_2^*) = \{0\}$.

Given the elliptic-hyperbolic pencil $L(\lambda) = A + \lambda B + \lambda^2 C$ we can consider the following associated operator-differential equation:

$$Au(t) + Bu'(t) - Du''(t) = 0, \ D = -C. \tag{7.6}$$

Lemma 7.3.2. Suppose that $u \in \mathcal{C}_0^1[0, +\infty; \mathcal{H}]$ (see appendix A.4) is a solution of the equation

(7.6), then

$$\|A^{1/2}u(0)\|^2 + \|D^{1/2}u'(0)\|^2 = 4\int_0^{+\infty}\int_\alpha^{+\infty}(\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2)dtd\alpha.$$

Analogously, if $u \in \mathcal{C}_0^1[-\infty, 0; \mathcal{H}]$ is a solution of (7.6), then

$$\|A^{1/2}u(0)\|^2 + \|D^{1/2}u'(0)\|^2 = 4\int_{-\infty}^{0}\int_{-\infty}^{\beta}(\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2)dtd\beta.$$

Proof. (1) Suppose that $u \in \mathcal{C}_0^1[0, +\infty; \mathcal{H}]$ is solution of (7.6). By the lemma A.4.1, we have:

$$\begin{aligned}
0 &= -2\operatorname{Re}\textstyle\int_0^{+\infty}\langle Au(t) + Bu'(t) - Du''(t), u'(t)\rangle dt \\
&= -2\operatorname{Re}\textstyle\int_0^{+\infty}\langle Au(t), u'(t)\rangle dt - 2\operatorname{Re}\int_0^{+\infty}\langle Bu'(t), u'(t)\rangle dt + 2\operatorname{Re}\int_0^{+\infty}\langle Du''(t), u'(t)\rangle dt \\
&= \langle Au(0), u(0)\rangle - \langle Du'(0), u'(0)\rangle - 2\textstyle\int_0^{+\infty}\langle Bu'(t), u'(t)\rangle dt \\
&= \|A^{1/2}u(0)\|^2 - \|D^{1/2}u'(0)\|^2 - 2\textstyle\int_0^{+\infty}\langle Bu'(t), u'(t)\rangle dt.
\end{aligned}$$

Moreover for every $\alpha \geq 0$ we have:

$$\begin{aligned}
0 &= -2\operatorname{Re}\textstyle\int_\alpha^{+\infty}\langle Au(t) + Bu'(t) - Du''(t), u''(t)\rangle dt \\
&= -2\operatorname{Re}\textstyle\int_\alpha^{+\infty}\langle Au(t), u''(t)\rangle dt - 2\operatorname{Re}\int_\alpha^{+\infty}\langle Bu'(t), u''(t)\rangle dt + 2\operatorname{Re}\int_\alpha^{+\infty}\langle Du''(t), u''(t)\rangle dt \\
&= -2\operatorname{Re}\textstyle\int_\alpha^{+\infty}[\frac{d}{dt}\langle Au(t), u'(t)\rangle - \langle Au'(t), u'(t)\rangle]dt + \langle Bu'(\alpha), u'(\alpha)\rangle + 2\operatorname{Re}\int_\alpha^{+\infty}\langle Du''(t), u''(t)\rangle dt \\
&= 2\operatorname{Re}\langle Au(\alpha), u'(\alpha)\rangle + \langle Bu'(\alpha), u'(\alpha)\rangle + 2\textstyle\int_\alpha^{+\infty}[\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2]dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
4\textstyle\int_0^{+\infty}\int_\alpha^{+\infty}(\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2)dtd\alpha &= 2\textstyle\int_0^{+\infty}[-\langle Bu'(\alpha), u'(\alpha)\rangle - 2\operatorname{Re}\langle Au(\alpha), u'(\alpha)\rangle]d\alpha \\
&= 2\langle Au(0), u(0)\rangle - 2\textstyle\int_0^{+\infty}\langle Bu'(\alpha), u'(\alpha)\rangle d\alpha \\
&= 2\|A^{1/2}u(0)\|^2 - \|A^{1/2}u(0)\|^2 + \|D^{1/2}u'(0)\|^2 \\
&= \|A^{1/2}u(0)\|^2 + \|D^{1/2}u'(0)\|^2.
\end{aligned}$$

(2) Suppose that $u \in \mathcal{C}_0^1[-\infty, 0; \mathcal{H}]$ is solution of (7.6). By the lemma A.4.1, we have:

$$\begin{aligned} 0 &= 2\operatorname{Re}\int_{-\infty}^{0} \langle Au(t) + Bu'(t) - Du''(t), u'(t)\rangle dt \\ &= 2\operatorname{Re}\int_{-\infty}^{0} \langle Au(t), u'(t)\rangle dt + 2\operatorname{Re}\int_{-\infty}^{0} \langle Bu'(t), u'(t)\rangle dt - 2\operatorname{Re}\int_{-\infty}^{0} \langle Du''(t), u'(t)\rangle dt \\ &= \langle Au(0), u(0)\rangle - \langle Du'(0), u'(0)\rangle + 2\int_{-\infty}^{0} \langle Bu'(t), u'(t)\rangle dt \\ &= \|A^{1/2}u(0)\|^2 - \|D^{1/2}u'(0)\|^2 + 2\int_{-\infty}^{0} \langle Bu'(t), u'(t)\rangle dt. \end{aligned}$$

Moreover for every $\beta \leq 0$ we have:

$$\begin{aligned} 0 &= 2\operatorname{Re}\int_{-\infty}^{\beta} \langle Au(t) + Bu'(t) - Du''(t), u''(t)\rangle dt \\ &= 2\operatorname{Re}\int_{-\infty}^{\beta} \langle Au(t), u''(t)\rangle dt + 2\operatorname{Re}\int_{-\infty}^{\beta} \langle Bu'(t), u''(t)\rangle dt - 2\operatorname{Re}\int_{-\infty}^{\beta} \langle Du''(t), u''(t)\rangle dt \\ &= 2\operatorname{Re}\int_{-\infty}^{\beta} [\tfrac{d}{dt}\langle Au(t), u'(t)\rangle - \langle Au'(t), u'(t)\rangle]dt + \langle Bu'(\beta), u'(\beta)\rangle - 2\operatorname{Re}\int_{-\infty}^{\beta} \langle Du''(t), u''(t)\rangle dt \\ &= 2\operatorname{Re}\langle Au(\beta), u'(\beta)\rangle + \langle Bu'(\beta), u'(\beta)\rangle - 2\int_{-\infty}^{\beta} [\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2]dt. \end{aligned}$$

Thus,

$$\begin{aligned} 4\int_{-\infty}^{0}\int_{-\infty}^{\beta} (\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2)dtd\beta &= 2\int_{-\infty}^{0} [\langle Bu'(\beta), u'(\beta)\rangle + 2\operatorname{Re}\langle Au(\beta), u'(\beta)\rangle]d\beta \\ &= 2\langle Au(0), u(0)\rangle + 2\int_{-\infty}^{0} \langle Bu'(\beta), u'(\beta)\rangle d\beta \\ &= 2\|A^{1/2}u(0)\|^2 - \|A^{1/2}u(0)\|^2 + \|D^{1/2}u'(0)\|^2 \\ &= \|A^{1/2}u(0)\|^2 + \|D^{1/2}u'(0)\|^2. \end{aligned}$$

□

Proposition 7.3.3. Let L be an elliptic-hyperbolic pencil. Then for every solution $u \in \mathcal{C}_0^1[0, +\infty; \mathcal{H}]$ of the equation (7.6) there exists $\varepsilon_1 > 0$ such that, for every $t \geq 0$:

$$\varepsilon_1\|D^{1/2}u'(t)\| \leq \|A^{1/2}u(t)\|. \tag{7.7}$$

Analogously, for every solution $u \in \mathcal{C}_0^1[-\infty, 0; \mathcal{H}]$ of (7.6) there exists $\varepsilon_1 > 0$ such that, for every $t \leq 0$:

$$\varepsilon_1\|A^{1/2}u(t)\| \leq \|D^{1/2}u'(t)\|. \tag{7.8}$$

Proof. (1) Suppose that $u \in \mathcal{C}_0^1[0, +\infty; \mathcal{H}]$ is a solution of (7.6). Since L is elliptic-hyperbolic, let

$\varepsilon > 0$, with $\varepsilon < 2$, such that, for every $x,\ y \in \mathcal{H}$, $|\langle B_- x, y\rangle|^2 \leq (2-\varepsilon)^2 \langle Ax, x\rangle\langle Dy, y\rangle$. Then, since $B_+ \geq 0$ we obtain from the lemma A.4.1,

$$\begin{aligned}
-\langle Bu'(\alpha), u'(\alpha)\rangle &\leq \langle B_- u'(\alpha), u'(\alpha)\rangle = -2\operatorname{Re}\int_\alpha^{+\infty} \langle B_- u'(t), u''(t)\rangle dt \leq \int_\alpha^{+\infty} 2|\langle B_- u'(t), u''(t)\rangle| dt \\
&\leq |2-\varepsilon| \int_\alpha^{+\infty} 2\langle Au'(t), u'(t)\rangle^{1/2}\langle Du''(t), u''(t)\rangle^{1/2} dt \\
&\leq |2-\varepsilon| \int_\alpha^{+\infty} [\langle Au'(t), u'(t)\rangle + \langle Du''(t), u''(t)\rangle] dt \\
&= |2-\varepsilon| \int_\alpha^{+\infty} [\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2] dt.
\end{aligned}$$

Thus, $-2\int_0^{+\infty} \langle Bu'(\alpha), u'(\alpha)\rangle d\alpha \leq 2|2-\varepsilon| \int_0^{+\infty}\int_\alpha^{+\infty} [\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2] dt d\alpha$ and, by the lemma 7.3.2 (and its proof), $\|D^{1/2}u'(0)\|^2 - \|A^{1/2}u(0)\|^2 \leq \frac{|2-\varepsilon|}{2}[\|D^{1/2}u'(0)\|^2 + \|A^{1/2}u(0)\|^2]$. Therefore, with $\varepsilon_1 = \sqrt{\frac{\varepsilon}{4-\varepsilon}}$, we have that (7.7) holds for $t = 0$. Since clearly $u(t+t_0)$ is also solution of (7.6) for every $t_0 > 0$, we have that (7.7) holds.

(2) If $u \in \mathcal{C}_0^1[-\infty, 0; \mathcal{H}]$ is a solution of (7.6), then

$$\begin{aligned}
\langle Bu'(\beta), u'(\beta)\rangle &\geq -\langle B_- u'(\beta), u'(\beta)\rangle = -2\operatorname{Re}\int_{-\infty}^{\beta} \langle B_- u'(t), u''(t)\rangle dt \geq -\int_\alpha^{+\infty} 2|\langle B_- u'(t), u''(t)\rangle| dt \\
&\geq -|2-\varepsilon| \int_{-\infty}^{\beta} 2\langle Au'(t), u'(t)\rangle^{1/2}\langle Du''(t), u''(t)\rangle^{1/2} dt \\
&\geq -|2-\varepsilon| \int_{-\infty}^{\beta} [\langle Au'(t), u'(t)\rangle + \langle Du''(t), u''(t)\rangle] dt \\
&= -|2-\varepsilon| \int_{-\infty}^{\beta} [\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2] dt.
\end{aligned}$$

Thus, $2\int_{-\infty}^{0} \langle Bu'(\beta), u'(\beta)\rangle d\beta \geq -2|2-\varepsilon| \int_{-\infty}^{0}\int_{-\infty}^{\beta} [\|A^{1/2}u'(t)\|^2 + \|D^{1/2}u''(t)\|^2] dt d\beta$ and, by the lemma 7.3.2 (and its proof), $\|D^{1/2}u'(0)\|^2 - \|A^{1/2}u(0)\|^2 \geq -\frac{|2-\varepsilon|}{2}[\|D^{1/2}u'(0)\|^2 + \|A^{1/2}u(0)\|^2]$. Therefore, with $\varepsilon_1 = \sqrt{\frac{\varepsilon}{4-\varepsilon}}$, we have that (7.8) holds for $t = 0$. Since clearly $u(t+t_0)$ is also solution of (7.6) for every $t_0 < 0$, we have that (7.8) holds. □

Theorem 7.3.4. Let L be a elliptic-hyperbolic pencil, then the operators Z_1 and Z_2 in the Langer Factorization satisfy, for every $x \in \mathcal{H}$,

$$\varepsilon_1\|A^{1/2}Z_1 x\| \leq \|D^{1/2}x\| \leq \frac{1}{\varepsilon_1}\|A^{1/2}Z_2 x\|, \tag{7.9}$$

with $\varepsilon_1 = \sqrt{\frac{\varepsilon}{4-\varepsilon}}$.

Further, the operators $D^{1/2}Z_2^{-1}$ and $Z_1 D^{-1/2}$ are bounded on dense subsets of the respective spaces $\mathcal{H}$ and $\mathcal{H}^0 := \overline{\mathscr{R}(C)}$ and admit bounded extensions in these spaces.

Proof. Let $\mathbf{A} = \begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix}$, $\mathbf{T} = \begin{pmatrix} -B & D^{1/2} \\ D^{1/2} & 0 \end{pmatrix}$ and $\mathbf{F} = \mathbf{AT} = \begin{pmatrix} -A^{-1}B & A^{-1}D^{1/2} \\ D^{1/2} & 0 \end{pmatrix}$. Clearly $\mathbf{A}$ is a bijective and

$$\mathbf{A}^{-1} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

Note that $\mathbf{A}^{-1}$ is a bounded self-adjoint operator acting in $\mathcal{H}^2$ and $\mathbf{A}^{-1} \gg 0$. Thus, the positive definite inner product on $\mathcal{H}^2$, $\langle \cdot, \cdot \rangle_{\mathbf{A}^{-1}}$, defined by $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{A}^{-1}} = \langle \mathbf{A}^{-1}\mathbf{x}, \mathbf{y} \rangle_{\mathcal{H}^2}$ for every $\mathbf{x}$, $\mathbf{y} \in \mathcal{H}^2$, induces a topology which is equivalent to the one induced by $\langle \cdot, \cdot \rangle_{\mathcal{H}^2}$.

(1) Set $U = \{(Z_2 x, D^{1/2}x) \mid x \in \mathcal{H}\}$ and let $\overline{U}$ be the closure of U with respect to the topology induced by $\langle \cdot, \cdot \rangle_{\mathbf{A}^{-1}}$. Since $AZ_2^2 + BZ_2 - D = 0$ then, for every $x \in \mathcal{H}$,

$$\begin{aligned} \mathbf{F}(Z_2x, D^{1/2}x) &= (-A^{-1/2}BZ_2x + A^{-1/2}Dx, D^{1/2}Z_2x) = (A^{-1}[-BZ_2x + Dx], D^{1/2}Z_2x) \\ &= (A^{-1}[AZ_2^2x], D^{1/2}Z_2x) = (Z_2(Z_2x), D^{1/2}(Z_2x)) \in U. \end{aligned}$$

That is, U is an $\mathbf{F}$-invariant subspace and, therefore, $\overline{U}$ is also $\mathbf{F}$-invariant. Note that, since $\ker(Z_2) = \ker(C)$, $Z_2x = 0$ implies $x \in \ker(C)$ and, therefore, $D^{1/2}x = 0$. Thus, points of the form $(0, v)$, with $v \neq 0$, are not in U. Then, since $-D = Z_1^*AZ_2$ and $S = AZ_1 + Z_1^*A + B > 0$, we have that for every $\mathbf{x} = (Z_2x, D^{1/2}x) \neq 0$,

$$\begin{aligned} \langle \mathbf{Fx}, \mathbf{x} \rangle_{\mathbf{A}^{-1}} &= \langle \mathbf{A}^{-1}\mathbf{Fx}, \mathbf{x} \rangle_{\mathcal{H}^2} = \langle \mathbf{Tx}, \mathbf{x} \rangle_{\mathcal{H}^2} = \langle (-BZ_2x + Dx, D^{1/2}Z_2x), (Z_2x, D^{1/2}x) \rangle_{\mathcal{H}^2} \\ &= \langle -BZ_2x + Dx, Z_2x \rangle + \langle D^{1/2}Z_2x, D^{1/2}x \rangle \\ &= \langle -BZ_2x + Dx, Z_2x \rangle + \langle Z_2x, Dx \rangle \\ &= \langle -BZ_2x - Z_1^*AZ_2x, Z_2x \rangle + \langle Z_2x, -Z_1^*AZ_2x \rangle \\ &= -\langle (B + Z_1^*A)y, y \rangle - \langle y, Z_1^*Ay \rangle; \text{ where } y = Z_2x \\ &= -\langle (B + Z_1^*A)y, y \rangle - \langle AZ_1y, y \rangle \\ &= -\langle (AZ_1 + Z_1^*A + B)y, y \rangle \\ &= -\langle Sy, y \rangle < 0. \end{aligned}$$

That is, $\mathbf{F}\restriction_U < 0$ and, therefore, $\ker(\mathbf{F}) \cap U = \{0\}$. Let $\mathbf{F}_- = \mathbf{F}\restriction_{\overline{U}} \colon \overline{U} \to \overline{U}$ and $\mathbf{x}_0 =$

$(u,v) \in \overline{U} \setminus U$. If $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ such that $(Z_2x_n, D^{1/2}x_n) \xrightarrow{\|\cdot\|_{\mathbf{A}^{-1}}} (u,v)$, where $\|\cdot\|_{\mathbf{A}^{-1}}$ is the norm induced by $\langle\cdot,\cdot\rangle_{\mathbf{A}^{-1}}$, then it is clear that $Z_2x_n \to u$ and $D^{1/2}x_n \to v$; therefore we have that $\langle \mathbf{F}_-\mathbf{x}_0, \mathbf{x}_0\rangle_{\mathbf{A}^{-1}} = -\langle Su, u\rangle$. Thus, if $u \neq 0$ we have that $\langle \mathbf{F}_-\mathbf{x}_0, \mathbf{x}_0\rangle_{\mathbf{A}^{-1}} < 0$ and, therefore, $\mathbf{x}_0 \notin \ker(\mathbf{F}_-)$.

Since Z_2 is a closed operator, we have $U = \{(Z_2x, D^{1/2}x) \mid x \in \mathcal{H}\} = \{(y, D^{1/2}Z_2^{-1}y) \mid y \in \mathscr{R}(Z_2)\} =$ *graph of the closed operator* $D^{1/2}Z_2^{-1}$. Thus, since $\|\cdot\|_{\mathcal{H}^2}$ and $\|\cdot\|_{\mathbf{A}^{-1}}$ are equivalent, we have that $(0,v) \notin \overline{U}$ for every $v \in \mathcal{H} \setminus \{0\}$. Hence $\mathbf{F}_- < 0$ and, therefore, $\mathbf{F}_-$ is injective. Further, since the operator $\mathbf{F}$ is self-adjoint in $(\mathcal{H}^2, \langle\cdot,\cdot\rangle_{\mathbf{A}^{-1}})$, the densely defined operator $\mathbf{F}_-^{-1}$ is unbounded and self-adjoint in $(\overline{U}, \langle\cdot,\cdot\rangle_{\mathbf{A}^{-1}})$. Thus, $\mathbf{F}_-^{-1}$ generates a strongly stable $\mathcal{C}_0$-semigroup of contractions (see [14] [15]).

Let $\mathbf{x}_0 = (Z_2x, D^{1/2}x)$ be any vector in U, and let $w(t) = (e^{t\mathbf{F}_-^{-1}})\mathbf{x}_0$, $t \geq 0$. Then $w(t)$ is a solution of the operator-differential equation $\mathbf{F}w'(t) - w(t) = 0$ with $w(t) \to 0$ when $t \to +\infty$. Thus, if $w(t) = (u(t), v(t))$ then $u, v \in \mathcal{C}^1[0,+\infty;\mathcal{H}]$ and

$$\begin{cases} 0 &= -A^{-1}Bu'(t) + A^{-1}D^{1/2}v'(t) - u(t), \\ 0 &= D^{1/2}u'(t) - v(t). \end{cases}$$

That is, $v(t) = D^{1/2}u'(t)$ and, therefore, $Au(t) + Bu'(t) - Du''(t) = 0$. Then, by the proposition 7.2.3, $\varepsilon_1\|D^{1/2}u'(0)\| \leq \|A^{1/2}u(0)\|$ with $\varepsilon_1 = \sqrt{\varepsilon(4-\varepsilon)^{-1}}$. Note that $u(0) = Z_2x$, $D^{1/2}u'(0) = D^{1/2}x$ and, therefore, $\varepsilon_1\|D^{1/2}x\| \leq \|A^{1/2}Z_2x\|$ $(x \in \mathcal{H})$. Thus, for every $y \in \mathscr{R}(Z_2)$, we have $\varepsilon_1\|D^{1/2}Z_2^{-1}y\| \leq \|A^{1/2}y\| \leq \|A^{1/2}\|\|y\|$. So, the densely defined operator $D^{1/2}Z_2^{-1}$ is bounded and, therefore, it admits a bounded extension to $\overline{\mathcal{R}(Z_2)} = \mathcal{H}$.

(2) Analogous to the preceding proof, with $U = \{(Z_1x, D^{1/2}x) \mid x \in \mathcal{H}\}$, changing $\mathbf{F}$ by $-\mathbf{F}$ and defining $w(t)$ for $t \leq 0$, we have that $\varepsilon_1\|A^{1/2}Z_1x\| \leq \|D^{1/2}x\|$ for every $x \in \mathcal{H}$. Further, the function $Z_1D^{-1/2} := (D^{1/2}Z_1^{-1})^{-1}$ is a bounded operator with $\mathcal{D}(Z_1D^{-1/2}) = \mathscr{R}(C)$. Thus, $Z_1D^{-1/2}$ admits a bounded extension to $\mathcal{H}^0 = \overline{\mathscr{R}(C)}$. □

Remark. From the proof of the theorem 7.3.4, we have that for every $x_1 \in \mathcal{H}$ there exists a continuously differentiable function $u(t)$ for $t > 0$ which satisfies the equation (7.6) such that

$\lim_{t\to 0^+} \|u(t) - x_1\| = 0$ and $\lim_{t\to +\infty} \|u(t)\| = 0$.

Theorem 7.3.5. Let L be an elliptic-hyperbolic pencil, and let Z_1, Z_2 be the operators in the Langer Factorization. Then Z_2 is similar in $\mathcal{H}$ to a negative self-adjoint operator, and $D^{1/2} Z_1 D^{-1/2}$ is similar in the subspace $\mathcal{H}^0 = \overline{\mathscr{R}(C)} \subseteq \mathcal{H}$ to a positive self-adjoint operator.

Proof. Define the projections $P, Q : \mathcal{H}^2 \to \mathcal{H}^2$ by $P(u,v) = (u,0)$ and $Q(u,v) = (0,v)$.
(1) Let $\mathbf{F}$, U, $\overline{U}$ and $\mathbf{F}_-$ be as in the first part of the proof of 7.2.4; note that $\mathbf{F}_-$ is a negative self-adjoint operator. Let K be the bounded extension of $D^{1/2} Z_2^{-1}$ to $\mathcal{H}$. Then $\overline{U} = \{(x, Kx) \mid x \in \mathcal{H}\}$ and, since we can to identify $\mathscr{R}(P)$ with $\mathcal{H}$, the operator $P : \overline{U} \to \mathcal{H}$ is bijective. Thus, since $Z_2 x = P\mathbf{F}_- P^{-1} x$ for every $x \in \mathcal{H}$, we have $Z_2 = P\mathbf{F}_- P^{-1}$. Hence, Z_2 is similar to a negative self-adjoint operator.
(2) Let $\mathbf{F}$, U, and $\overline{U}$ be as in the second part of the proof of 7.2.4, and let $\mathbf{F}_+ := \mathbf{F} \restriction_{\overline{U}} : \overline{U} \to \overline{U}$. Thus, $\mathbf{F}_+$ is a positive self-adjoint operator. Since we can identify $\mathscr{R}(Q)$ with $\mathcal{H}$, the operator $Q : \overline{U} \to \mathcal{H}$ is bijective. Thus, for every $y \in \mathscr{R}(D^{1/2})$, we have $Q\mathbf{F}_+ Q^{-1} y = Q\mathbf{F}_+(Z_1 D^{-1/2} y, y) = Q(Z_1(Z_1 D^{-1/2} y), D^{1/2}(Z_1 D^{-1/2} y)) = D^{1/2} Z_1 D^{-1/2} y$. Then $D^{1/2} Z_1 D^{-1/2} = Q\mathbf{F}_+ Q^{-1}$ on $\mathscr{R}(D^{1/2})$ and, by continuity, $D^{1/2} Z_1 D^{-1/2} = Q\mathbf{F}_+ Q^{-1}$ on $\overline{\mathscr{R}(C)}$. Hence, $D^{1/2} Z_1 D^{-1/2}$ is similar to a positive self-adjoint operator in $\mathcal{H}^0 = \overline{\mathscr{R}(C)}$. □

Theorem 7.3.6. Let L be an elliptic-hyperbolic pencil with $\sigma(L) \cap \mathbb{R}^- \subseteq \sigma_d(L)$ (resp. $\sigma(L) \cap \mathbb{R}^+ \subseteq \sigma_d(L)$). Then the system E^- (resp., the system $\{D^{1/2} x \mid x \in E^+\}$) contains a Riesz basis for $\mathcal{H}$ (resp., for the subspace $\overline{\mathscr{R}(C)}$).

Proof. (1) If $\sigma(L) \cap \mathbb{R}^- \subseteq \sigma_d(L)$ and since Z_2 is similar to a self-adjoint operator then, by the proposition 7.1.8 and theorem 5.3.10, E^- contains a Riesz basis for $\mathcal{H}$.
(2) Suppose that $\sigma(L) \cap \mathbb{R}^+ \subseteq \sigma_d(L)$. Since $D^{1/2} Z_1 D^{-1/2}$ is similar to a positive self-adjoint operator and the system $\{D^{1/2} x \mid x \in E^+\}$ are the eigenvalues of this operator then, by the theorems 7.1.9 and 5.3.10, $\{D^{1/2} x \mid x \in E^+\}$ contains a Riesz basis for $\overline{\mathscr{R}(C)}$. □

Chapter 8

Pencils of Unbounded Self-adjoint Operators and their Associated Differential Equations

8.1 Hyperbolic and Elliptic-hyperbolic Pencils of Unbounded Self-adjoint Operators

Remark. Let $A \gg 0$ be a self-adjoint operator acting in a Hilbert space and let $\theta \in \mathbb{R}$.

- If $\theta \geq 0$, set $\mathcal{H}_\theta = \mathscr{D}(A^{\theta/2})$ and $\|x\|_\theta = \|A^{\theta/2}x\|$ for every $x \in \mathcal{H}_\theta$. Clearly $\|\cdot\|_\theta$ defines a norm on $\mathcal{H}_\theta$ and this norm is induced by the positive definite inner product $\langle x, y\rangle_\theta := \langle A^{\theta/2}x, A^{\theta/2}y\rangle$. Further, $(\mathcal{H}_\theta, \|\cdot\|_\theta)$ is a Banach space [since if $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\theta$ is a Cauchy sequence with respect to the norm $\|\cdot\|_\theta$, then the sequence $(A^{\theta/2}x_n)$ is Cauchy with respect to the norm $\|\cdot\|$ and, therefore, there exists $x_0 \in \mathcal{H}$ such that $A^{\theta/2}x_n \xrightarrow{\|\cdot\|} x_0$; thus $x_n \xrightarrow{\|\cdot\|_\theta} A^{-\theta/2}x_0 \in \mathcal{H}_\theta$] and, therefore, $\mathcal{H}_\theta$ is a Hilbert space.

- If $\theta < 0$ then $\|x\|_\theta = \|A^{\theta/2}x\|$ $(x \in \mathcal{H})$ defines a norm on $\mathcal{H}$. Moreover, analogous the previous case, $\|\cdot\|_\theta$ is generated by a positive definite inner product. Define $\mathcal{H}_\theta$ as the completation of $\mathcal{H}$ with respect the norm $\|\cdot\|_\theta$.

Definition 8.1.1. If $A \gg 0$ then the family $\{\mathcal{H}_\theta\}_{\theta\in\mathbb{R}}$, where $\mathcal{H}_\theta$ is defined as in the remark above, is called the *scale of Hilbert spaces generated by* $A^{1/2}$.

Lemma 8.1.2. If the operators A and S are self-adjoint with $A \gg 0$ and $\mathscr{D}(S) \supseteq \mathscr{D}(A)$, then the operator $S_1 = A^{-1/2}SA^{-1/2}$ admits a bounded extension in $\mathcal{H}$.

Proof. Clearly S_1 is a well-defined and densely defined symmetric operator. Since S is closed and $\mathscr{D}(S) \supseteq \mathscr{D}(A)$, there exists a real number $M' > 0$ such that $\|Sx\| \le M'[\|Ax\| + \|x\|]$ for every $x \in \mathscr{D}(A)$ (see [16]). Then, since $A \gg 0$, there exists $M > 0$ such that $\|Sx\| \le M\|Ax\|$ for every $x \in \mathscr{D}(A)$. Thus, by the second Heinz inequality (see [16]), $|\langle Sx, x\rangle| \le M\|A^{1/2}x\|^2$ for every $x \in \mathscr{D}(A)$. Thus, for every $y \in \mathscr{D}(S_1)$ with $A^{-1/2}y \in \mathscr{D}(A)$, $|\langle S_1y, y\rangle| = |\langle A^{-1/2}SA^{-1/2}y, y\rangle| = |\langle SA^{-1/2}y, A^{-1/2}y\rangle| \le M\|y\|^2$. Then, for every y, $z \in \mathscr{D}(S_1)$ with $A^{-1/2}y$, $A^{-1/2}y \in \mathscr{D}(A)$, $|\langle S_1y, z\rangle| \le M\|y\|\|z\|$ (see [17]) and, therefore, the form $\langle S_1y, z\rangle$ extends by continuity to the whole of $\mathcal{H}$. Thus, S_1 has a bounded extension on $\mathcal{H}$. □

Remark. In the conditions of lemma 8.1.2, we have that the form $\langle Sx, x\rangle$ can be extended by continuity to $\mathscr{D}(A^{1/2})$ and, therefore, the condition in the second part of the following definition is unambiguous.

Definition 8.1.3. (1) A *quadratic pencil* L (of unbounded self-adjoint operators) in a Hilbert space $\mathcal{H}$ is an operator-valued function with $\mathscr{D}(L) = \mathbb{C}_\infty$, for which there exist fixed self-adjoint operators A, B and D, acting in $\mathcal{H}$, such that $L(\infty) = -D$ and, for every $\lambda \in \mathbb{C}$, $L(\lambda) = A + \lambda B - \lambda^2 D$.

(2) The quadratic pencil $L(\lambda) = A + \lambda B - \lambda^2 D$ is called *hyperbolic* if, and only if, $A \gg 0$, $D \ge 0$, $D \not\equiv 0$, $\mathscr{D}(D) \supseteq \mathscr{D}(A)$, $\mathscr{D}(B) \supseteq \mathscr{D}(A)$ and $\langle Bx, x\rangle \ne 0$ for every $x \in \ker(D) \cap \mathscr{D}(A^{1/2})$ with $x \ne 0$.

(3) The hyperbolic pencil $L(\lambda) = A + \lambda B - \lambda^2 D$ is called *elliptic-hyperbolic* if $B\restriction_{\mathscr{D}(A)}$ admits a representation $B\restriction_{\mathscr{D}(A)} = B_+ - B_-$, where $B_+ \ge 0$ and there exists $\varepsilon > 0$, with $\varepsilon < 2$, such that $|\langle B_-x, y\rangle|^2 \le (2-\varepsilon)^2\langle Ax, x\rangle\langle Dy, y\rangle$ for every x, $y \in \mathscr{D}(A)$.

Remark. Given the hyperbolic pencil

$$L(\lambda) = A + \lambda B - \lambda^2 D \tag{8.1}$$

define the operators B_1, D_1 and T by

$$B_1 = A^{-1/2} B A^{-1/2}, \;\; D_1 = A^{-1/2} D A^{-1/2}, \;\; T = D^{1/2} A^{-1/2}. \tag{8.2}$$

Moreover, define $\mathcal{H}^0 := \overline{\mathscr{R}(T)}$.

- By the lemma 8.1.2, B_1 and D_1 are bounded operators defined on $\mathcal{H}$. Moreover, $D_1 \geq 0$.

- Note that T is a bounded operator defined on $\mathcal{H}$, since for every $x \in \mathcal{H}$,

$$\|Tx\|^2 = \langle D^{1/2} A^{-1/2} x, D^{1/2} A^{-1/2} x\rangle = \langle A^{-1/2} D A^{-1/2} x, x\rangle = \langle D_1 x, x\rangle \leq \|D_1\| \|x\|^2.$$

- Since $T \in \mathcal{B}(\mathcal{H})$ then, by the polar representation of T, there exists an operator U, which maps $\mathcal{H}^0$ isometrically onto $\overline{\mathscr{R}(D_1^{1/2})} = \overline{\mathscr{R}(D_1)}$, such that $UT = (A^{-1/2} D A^{-1/2})^{1/2} = D_1^{1/2}$. Thus $D_1^{1/2} = U D^{1/2} A^{-1/2}$ and, therefore, $D^{1/2} = U^{-1} D_1^{1/2} A^{1/2}$.

- Note that $\mathcal{H}^0 \subseteq \overline{\mathscr{R}(D)}$.

- In the particular case in which D is bounded, we have that $\mathcal{H}^0 = \overline{\mathscr{R}(D)}$ [since if D is bounded and self-adjoint, then $\mathscr{D}(D) = \mathcal{H}$, $D^{1/2} \in \mathcal{B}(\mathcal{H})$, $\overline{\mathscr{R}(T)} = \overline{\mathscr{R}(D^{1/2})}$ and $\overline{\mathscr{R}(D^{1/2})} = \overline{\mathscr{R}(D)}$; thus, $\mathcal{H}^0 = \overline{\mathscr{R}(T)} = \overline{\mathscr{R}(D^{1/2})} = \overline{\mathscr{R}(D)}$].

- Note that, $\mathcal{H}^0 = \overline{\mathscr{R}(D)}$ if and only if $D^{1/2} \restriction_{\mathscr{D}(A^{1/2})}$ is essentially self-adjoint. Further, if $D \restriction_{\mathscr{D}(A)}$ is essentially self-adjoint then $\mathcal{H}^0 = \overline{\mathscr{R}(D)}$.

Lemma 8.1.4. Let $\{\mathcal{H}_\theta\}_{\theta \in \mathbb{R}}$ the scale of Hilbert spaces generated by $A^{1/2}$. Then the operators $D : \mathcal{H}_1 \to \mathcal{H}_{-1}$, $B : \mathcal{H}_1 \to \mathcal{H}_{-1}$ and $D^{1/2} : \mathcal{H} \to \mathcal{H}_{-1}$, understood as operators acting from one Hilbert space into another, are bounded.

Proof. (1) For every $x \in \mathcal{H}_1$, $\|Dx\|_{-1} = \|A^{-1/2}Dx\| = \|A^{-1/2}DA^{-1/2}A^{1/2}x\| \leq \|D_1A^{1/2}x\| \leq \|D_1\|\|A^{1/2}x\| = \|D_1\|\|x\|_1$. Therefore, $D \in \mathcal{B}(\mathcal{H}_1; \mathcal{H}_{-1})$

(2) Is analogous to (1).

(3) Since $T \in \mathcal{B}(\mathcal{H})$ then $T^* \in \mathcal{B}(\mathcal{H})$. Moreover $T^* = A^{-1/2}D^{1/2}$. Thus, for every $x \in \mathscr{D}(D^{1/2})$, $\|D^{1/2}x\|_{-1} = \|A^{-1/2}D^{1/2}x\| = \|T^*x\| \leq \|T^*\|\|x\|$. Therefore, extension by continuity gives $D^{1/2} \in \mathcal{B}(\mathcal{H}; \mathcal{H}_{-1})$. □

8.2 Equation Associated with a Hyperbolic Pencil

This section (its result and proof) is analogous to the section (7.3).

Given the hyperbolic pencil (8.1) we can associate the following operator-differential equation

$$L\left(i\frac{d}{dt}\right)u(t) = Au(t) + iBu'(t) + Du''(t) = 0. \tag{8.3}$$

In this section $\{\mathcal{H}_\theta\}_{\theta\in\mathbb{R}}$ denotes the scale of Hilbert spaces generated by $A^{1/2}$.

Definition 8.2.1. A continuous function $u : [0, +\infty) \to \mathcal{H}_1$ is called a *generalized solution* of equation (8.3) if $Du(t)$ is continuously differentiable on $[0, +\infty)$ as a function with values in $\mathcal{H}_{-1}$, the function $iBu(t) + Du'(t)$ has the same property, and

$$[iBu(t) + Du'(t)]' = -Au(t), \tag{8.4}$$

understood as equality of functions with values in $\mathcal{H}_{-1}$.

Theorem 8.2.2. For any pairs of vectors $x_1 \in \mathcal{H}_1$ and $x_0 \in \mathcal{H}^0$ there exists a unique generalized solution u of (8.3), in the terms of definition 8.2.1, such that u is bounded in $\mathcal{H}_1$, $D^{1/2}u'(t)$ is continuous and bounded in the space $\mathcal{H}^0 \subseteq \mathcal{H}$, and

$$\lim_{t\to 0^+}\|u(t) - x_1\|_1 = 0, \quad \lim_{t\to 0^+}\|D^{1/2}u'(t) - x_0\| = 0. \tag{8.5}$$

Proof. (1) In $\mathbf{H} = \mathcal{H}_1 \times \mathcal{H}^0$ consider the following equation

$$(\mathbf{V}\mathbf{u}(t))' - i\mathbf{u}(t) = 0; \quad \mathbf{V} = \begin{pmatrix} A^{-1}B & A^{-1}D^{1/2} \\ D^{1/2} & 0 \end{pmatrix}. \tag{8.6}$$

Since in the product space the Euclidean norm and the norm of the sum are equivalent, there exist $\alpha > 0$ such that, for every $\mathbf{x} = (x_0, y_0) \in \mathbf{H}$, by the lemma 8.1.4,

$$\begin{aligned} \|\mathbf{V}\mathbf{x}\|_{\mathbf{H}} &= \|(A^{-1}Bx_0 + A^{-1}D^{1/2}y_0, D^{1/2}x_0)\|_{\mathbf{H}} \le \alpha[\|A^{-1}Bx_0 + A^{-1}D^{1/2}y_0\|_1 + \|D^{1/2}x_0\|] \\ &= \alpha\|A^{-1/2}Bx_0 + A^{-1/2}D^{1/2}y_0\| + \alpha\|D^{1/2}A^{-1/2}A^{1/2}x_0\| \\ &= \alpha\|Bx_0 + D^{1/2}y_0\|_{-1} + \alpha\|TA^{1/2}x_0\| \le \alpha\|Bx_0\|_{-1} + \alpha\|D^{1/2}y_0\|_{-1} + \alpha\|T\|\|A^{1/2}x_0\| \\ &\le \alpha\|B\|\|x_0\|_1 + \alpha\|D^{1/2}\|\|y_0\| + \alpha\|T\|\|x_0\|_1. \end{aligned}$$

That is, $\mathbf{V}$ is a bounded operator on $\mathbf{H}$.

For (x, y), $(w, z) \in \mathbf{H}$, we have

$$\begin{aligned} \langle \mathbf{V}(x,y), (w,z)\rangle_{\mathbf{H}} &= \langle (A^{-1}Bx + A^{-1}D^{1/2}y, D^{1/2}x), (w,z)\rangle_{\mathbf{H}} \\ &= \langle A^{-1}Bx + A^{-1}D^{1/2}y, w\rangle_1 + \langle D^{1/2}x, z\rangle \\ &= \langle A^{-1}Bx, w\rangle_1 + \langle A^{-1}D^{1/2}y, w\rangle_1 + \langle D^{1/2}x, z\rangle \\ &= \langle BA^{-1}x, w\rangle_1 + \langle A^{1/2}A^{-1}D^{1/2}y, A^{1/2}w\rangle + \langle x, D^{1/2}z\rangle \\ &= \langle x, A^{-1}Bw\rangle_1 + \langle D^{1/2}y, w\rangle + \langle A^{-1}Ax, D^{1/2}z\rangle \\ &= \langle x, A^{-1}Bw\rangle_1 + \langle y, D^{1/2}w\rangle + \langle Ax, A^{-1}D^{1/2}z\rangle \\ &= \langle x, A^{-1}Bw\rangle_1 + \langle y, D^{1/2}w\rangle + \langle x, A^{-1}D^{1/2}z\rangle_1 \\ &= \langle x, A^{-1}Bw\rangle_1 + \langle x, A^{-1}D^{1/2}z\rangle_1 + \langle y, D^{1/2}w\rangle \\ &= \langle x, A^{-1}Bw + A^{-1}D^{1/2}z\rangle_1 + \langle y, D^{1/2}w\rangle \\ &= \langle (x,y), \mathbf{V}(z,w)\rangle_{\mathbf{H}}. \end{aligned}$$

Therefore, $\mathbf{V}$ is self-adjoint.

Let $(x, y) \in \ker(\mathbf{V})$. Then $D^{1/2}x = 0$, $0 = A^{-1}Bx + A^{-1}D^{1/2}y \in \mathcal{H}_1$ and $0 = \langle A^{-1}Bx + A^{-1}D^{1/2}y, x\rangle_1 = \langle Bx + D^{1/2}y, x\rangle = \langle Bx, x\rangle + \langle D^{1/2}y, x\rangle = \langle Bx, x\rangle + \langle y, D^{1/2}x\rangle = \langle Bx, x\rangle$. Thus $x \in \mathcal{H}_1 \cap \ker(D) = \mathscr{D}(A^{1/2}) \cap \ker(D)$ and $\langle Bx, x\rangle = 0$. Since the pencil is hyperbolic, we have

$x = 0$ and, therefore, $A^{-1}D^{1/2}y = 0$. Then $A^{-1/2}D^{1/2}y = 0$; that is $T^*y = 0$. Since $y \in \ker(T^*)$ and $y \in \overline{\mathscr{R}(T)}$, then $y = 0$. Hence, $\mathbf{V}$ is injective.

Since $(i\mathbf{V}^{-1})^* = -i\mathbf{V}^{-1}$ then, by a theorem of Stone, $i\mathbf{V}^{-1}$ generates a unitary $\mathcal{C}_0$-group (see [15]). Let $\mathbf{U}(t) = e^{it\mathbf{V}^{-1}}$ $(t \in \mathbb{R})$ be the unitary $\mathcal{C}_0$-group generate by the operator $i\mathbf{V}^{-1}$.

(2) *Existence.* Suppose that $x_1 \in \mathcal{H}_1$ and $x_0 \in \mathcal{H}^0$. Let $\mathbf{x}_0 = (x_1, -ix_0)$ and $\mathbf{u}(t) = (u(t), v(t)) := \mathbf{U}(t)\mathbf{x}_0$ $(t \geq 0)$. Then $\mathbf{u}(t)$ satisfies (8.6) and $\lim_{t\to 0^+} \|\mathbf{u}(t) - \mathbf{x}_0\|_{\mathbf{H}} = 0$. Thus, $\lim_{t\to 0^+} \|u(t) - x_1\|_1 = 0$, $\lim_{t\to 0^+} \|v(t) + ix_0\| = 0$ and, by (8.6), $(A^{-1}Bu(t) + A^{-1}D^{1/2}v(t))' = iu(t) \in \mathcal{H}_1$ and $D^{1/2}u'(t) = iv(t) \in \mathcal{H}^0$. Note that:

- Since $\mathbf{U}(t)$ is a $\mathcal{C}_0$-group, $u(t)$ is continuously differentiable as a function with values in $\mathcal{H}_1$.

- For $n \in \mathbb{N}_0$ and $t_0 \in [0, +\infty)$, $\|Du^{(n)}(t) - Du^{(n)}(t_0)\|_{-1} = \|A^{-1/2}D[u^{(n)}(t) - u^{(n)}(t_0)]\| = \|A^{-1/2}DA^{-1/2}A^{1/2}[u^{(n)}(t) - u^{(n)}(t_0)]\| = \|D_1A^{1/2}[u^{(n)}(t) - u^{(n)}(t_0)]\| \leq \|D_1\|\|A^{1/2}[u^{(n)}(t) - u^{(n)}(t_0)]\| = \|D_1\|\|u^{(n)}(t) - u^{(n)}(t_0)\|_1 \xrightarrow{t\to t_0} 0$ (by the preceding item). Thus, $Du(t)$ is continuously differentiable as a function with values in $\mathcal{H}_{-1}$.

- For $n \in \mathbb{N}_0$ and $t_0 \in [0, +\infty)$, $\|iBu^{(n)}(t) + Du^{(n+1)}(t) - iBu^{(n)}(t_0) - Du^{(n+1)}(t_0)\|_{-1} \leq \|Bu^{(n)}(t) - Bu^{(n)}(t_0)\|_{-1} + \|Du^{(n+1)}(t) - Du^{(n+1)}(t_0)\|_{-1}$. By the preceding item, $\|Du^{(n+1)}(t) - Du^{(n+1)}(t_0)\|_{-1} \xrightarrow{t\to t_0} 0$. Moreover, $\|B[u^{(n)}(t) - u^{(n)}(t_0)]\|_{-1} = \|A^{-1/2}B[u^{(n)}(t) - u^{(n)}(t_0)]\| = \|A^{-1/2}BA^{-1/2}A^{1/2}[u^{(n)}(t) - u^{(n)}(t_0)]\| = \|B_1A^{1/2}[u^{(n)}(t) - u^{(n)}(t_0)]\| \leq \|B_1\|\|A^{1/2}[u^{(n)}(t) - u^{(n)}(t_0)]\| = \|B_1\|\|u^{(n)}(t) - u^{(n)}(t_0)\|_1 \xrightarrow{t\to t_0} 0$ (by the first item). Thus, $iBu(t) + Du'(t)$ is continuously differentiable as a function with values in $\mathcal{H}_{-1}$.

- Since A^{-1} is bounded and $(A^{-1}Bu(t) + A^{-1}D^{1/2}v(t))' = iu(t)$, then $(Bu(t) + D^{1/2}v(t))' = iAu(t)$. Moreover, since $D^{1/2}u'(t) = iv(t)$, we have $(Bu(t) + \frac{1}{i}Du'(t))' = iAu(t)$ and, therefore, $(iBu(t) + Du'(t))' = -Au(t)$.

- Since the group $\mathbf{U}(t)$ is unitary, we have $\|\mathbf{u}(t)\|_{\mathbf{H}} = \|\mathbf{x}_0\|_{\mathbf{H}}$ $(t \geq 0)$ and, therefore, $\|u(t)\|_1 \leq \|\mathbf{x}_0\|_{\mathbf{H}}$ for every $t \geq 0$. That is, $u(t)$ is bounded in $\mathcal{H}_1$.

- Since $D^{1/2}u'(t) = iv(t)$ then $D^{1/2}u'(t)$ is continuous as function with values in $\mathcal{H}^0$. Moreover, as in the preceding item, $D^{1/2}u'(t)$ is also bounded in $\mathcal{H}^0$.

- $\lim_{t\to 0^+}\|u(t)-x_1\|_1=0$ and, since that $\lim_{t\to 0^+}\|v(t)+ix_0\|=0$, $\lim_{t\to 0^+}\|D^{1/2}u'(t)-x_0\|=\lim_{t\to 0^+}\|iv(t)-x_0\|=\lim_{t\to 0^+}\|v(t)+ix_0\|=0$.

(3) *Uniqueness.* Suppose that $u(t)$ is a generalized solution of (8.3) which satisfies the conditions of the theorem with $x_1=x_0=0$. Then, for every $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, the following integral exists in $\mathcal{H}_{-1}$

$$\int_0^{+\infty}(iBu(t)+Du'(t))'e^{-st}dt=-\int_0^{+\infty}Au(t)e^{-st}dt. \tag{8.7}$$

Since $B:\mathcal{H}_1\to\mathcal{H}_{-1}$ and $D^{1/2}:\mathcal{H}\to\mathcal{H}_{-1}$ are bounded (lemma 8.1.4), $\lim_{t\to 0^+}\|u(t)\|_1=0$ and $\lim_{t\to 0^+}\|D^{1/2}u'(t)\|=0$, then (8.7), after integration by parts, becomes

$$s\int_0^{+\infty}(iBu(t)+Du'(t))e^{-st}dt=-\int_0^{+\infty}Au(t)e^{-st}dt.$$

Thus, denoting with $\hat{u}(s)$ the Laplace Transform of $u(t)$, we have $A\hat{u}(s)+isB\hat{u}(s)+s^2D\hat{u}(s)=0$; that is, $L(is)\hat{u}(s)=0$.

Consider the pencil $L_1(\lambda)=A^{-1/2}L(\lambda)A^{-1/2}=A^{-1/2}L(\lambda)A^{-1/2}=I+\lambda B_1-\lambda^2D_1$. Note that $L_1(\lambda)$ is a quadratic pencil of bounded and self-adjoint operators. Moreover, L_1 is a hyperbolic pencil. Thus, for every $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$, we have that $L_1(is)$ is invertible and, therefore, $L(is)$ is invertible also. Then $\hat{u}\equiv 0$ and, therefore, $u\equiv 0$. □

8.3 Equation Associated with a Elliptic-hyperbolic Pencil

Given the elliptic-hyperbolic pencil

$$L(\lambda)=A+\lambda B-\lambda^2D, \tag{8.8}$$

we can associate the following operator-differential equation

$$L\left(\frac{d}{dt}\right)u(t)=Au(t)+Bu'(t)-Du''(t)=0. \tag{8.9}$$

In this section $\{\mathcal{H}_\theta\}_{\theta\in\mathbb{R}}$ denotes the scale of Hilbert spaces generated by $A^{1/2}$.

Definition 8.3.1. A function $u(t)$ is called a *generalized solution* of (8.9) on the interval (a, b) if $u(t)$ is continuously differentiable as a function with values in $\mathcal{H}_1$ and has a Bochner-integrable second derivative in the distribution theory sense in $\mathcal{H}_1$ on any compact subset of (a, b), and (8.9) holds for almost all $t \in (a, b)$ in the space $\mathcal{H}_{-1}$.

Theorem 8.3.2. For any $x_1 \in \mathcal{H}_1$ there exists a unique generalized solution $u(t)$ of (8.9) on $(0, +\infty)$ (in the terms of the definition 8.3.1) such that $u(t)$ is a continuous function in $\mathcal{H}_1$ and

$$\lim_{t\to 0^+} \|u(t) - x_1\|_1 = 0, \qquad \lim_{t\to+\infty} \|u(t)\|_1 = 0. \tag{8.10}$$

Moreover, $u(t)$ is infinitely differentiable as a function with values in $\mathcal{H}_1$ for $t > 0$ and decreases exponentially as $t \to +\infty$. If $\mathscr{D}(B) \supseteq \mathcal{H}_1$ and $\mathscr{D}(D) \supseteq \mathcal{H}_1$, then the solution $u(t)$ is the classical solution for $t > 0$.

Analogously, for any $x_0 \in \mathcal{H}^0$ there exists a unique generalized solution $u(t)$ of (8.9) on $(-\infty, 0)$ (in the terms of the definition 8.3.1) such that $D^{1/2}u'(t)$ is a continuous function in $\mathcal{H}^0$ and

$$\lim_{t\to 0^-} \|D^{1/2}u'(t) - x_0\| = 0, \qquad \lim_{t\to-\infty} \|u(t)\|_1 = 0. \tag{8.11}$$

Moreover, $u(t)$ is infinitely differentiable as a function with values in $\mathcal{H}_1$ for $t < 0$ and decreases exponentially as $t \to -\infty$. If $\mathscr{D}(B) \supseteq \mathcal{H}_1$ and $\mathscr{D}(D) \supseteq \mathcal{H}_1$, then the solution $u(t)$ is the classical solution for $t < 0$.

Proof. 1. *Solution of* (8.9), (8.11):

Let B_+, B_- and $\varepsilon > 0$, with $\varepsilon < 2$, such that $B\restriction_{\mathscr{D}(A)} = B_+ - B_-$, $B_+ \geq 0$ and $|\langle B_- x, y\rangle|^2 \leq (2-\varepsilon)^2 \langle Ax, x\rangle \langle Dy, y\rangle$ for every x, $y \in \mathscr{D}(A)$.

Let $B_1^+ = A^{-1/2}B_+A^{-1/2}$ and $B_1^- = A^{-1/2}B_-A^{-1/2}$. Then, for every x with $A^{-1/2}x \in \mathscr{D}(A)$, $B_1 x = A^{-1/2}B_+A^{-1/2}x - A^{-1/2}B_-A^{-1/2}x = B_1^+x - B_1^-x$. Thus, by the lemma 8.1.2 and extending by continuity, $B_1 = B_1^+ - B_1^-$.

Clearly $B_1^+ \geq 0$ and, for every $x,\, y \in \mathcal{H}$ with $A^{-1/2}x,\, A^{-1/2}y \in \mathcal{H}_2$,

$$\begin{aligned}
|\langle B_1^- x, y\rangle|^2 &= |\langle A^{-1/2}B_- A^{-1/2}x, y\rangle|^2 = |\langle B_- A^{-1/2}x, A^{-1/2}y\rangle|^2 \\
&\leq (2-\varepsilon)^2 \langle AA^{-1/2}x, A^{-1/2}x\rangle \langle DA^{-1/2}y, A^{-1/2}y\rangle \\
&= (2-\varepsilon)^2 \langle x, x\rangle \langle A^{-1/2}DA^{-1/2}y, y\rangle \\
&= (2-\varepsilon)^2 \langle x, x\rangle \langle D_1 y, y\rangle.
\end{aligned}$$

Therefore, extending by continuity, we have $|\langle B_1^- x, y\rangle|^2 \leq (2-\varepsilon)^2 \langle x, x\rangle \langle D_1 y, y\rangle$ for every $x,\, y \in \mathcal{H}$. Thus, the hyperbolic pencil of bounded operators $L_1(\lambda) = I + \lambda B_1 - \lambda^2 D_1$, is elliptic-hyperbolic (B_1 and D_1 are defined as in (8.2)). Note that $L_1(\lambda) = A^{-1/2}L(\lambda)A^{-1/2}$.

Existence. Consider in $\mathcal{H}^2$ the following operator-differential equation

$$\mathbf{F}\mathbf{u}'(t) - \mathbf{u}(t) = 0, \quad \mathbf{F} = \begin{pmatrix} -B_1 & D_1^{1/2} \\ D_1^{1/2} & 0 \end{pmatrix}. \tag{8.12}$$

Similar to the proofs of theorems 7.3.4 and 7.3.5, there exists a closed subspace $\mathbf{U}_+ \subseteq \mathcal{H}^2$ such that $\mathbf{F}_+ := \mathbf{F}\restriction_{\mathbf{U}_+} \colon \mathbf{U}_+ \to \mathbf{U}_+$ is a positive operator. Moreover, $\mathbf{U}_+ = \{(Kx, x) \mid x \in \overline{\mathscr{R}(D_1)}\}$, where K is a bounded operator on the subspace $\overline{\mathscr{R}(D_1)} \subseteq \mathcal{H}$.

Let U be the operator which maps $\mathcal{H}^0$ isometrically onto $\overline{\mathscr{R}(D_1)}$ (see remark before lemma 8.1.4). If $x_0 \in \mathcal{H}^0$ then the function $\mathbf{u}(t) = (u_1(t), u_2(t)) = (e^{t\mathbf{F}_+^{-1}})\mathbf{x}_0$ $(t < 0)$, where $\mathbf{x}_0 = (KUx_0, Ux_0)$, is solution of (8.12). Further, $\mathbf{u}(t)$ is infinitely differentiable and

$$\lim_{t\to 0^-} \|\mathbf{u}(t) - \mathbf{x}_0\|_{\mathcal{H}^2} = 0, \qquad \lim_{t\to -\infty} \|\mathbf{u}(t)\|_{\mathcal{H}^2} = 0. \tag{8.13}$$

From (8.12) we have $-B_1 u_1'(t) + D_1^{1/2}u_2'(t) - u_1(t) = 0$, $D_1^{1/2}u_1'(t) - u_2(t) = 0$ and, therefore, in $\mathcal{H}$

$$L_1\left(\frac{d}{dt}\right) u_1(t) = u_1(t) + B_1 u_1'(t) - D_1 u_1''(t) = 0.$$

Let $u(t) = A^{-1/2}u_1(t)$. Then,

$$L\left(\frac{d}{dt}\right)u(t) = A^{1/2}L_1\left(\frac{d}{dt}\right)A^{1/2}A^{-1/2}u_1(t) = A^{1/2}L_1\left(\frac{d}{dt}\right)u_1(t) = 0.$$

That is, $u(t)$ is satisfies (8.9) and it is a function with values in $\mathcal{H}_1$. Clearly, $u(t)$ is infinitely differentiable for $t < 0$ as a function with values in $\mathcal{H}_1$ and, therefore, is a generalized solution of (8.9) in the sense of the definition 8.3.1 (since, clearly, $\mathcal{H}_1$ is dense in $\mathcal{H}_{-1}$). Since $\mathbf{F}_+$ is bounded and $\mathbf{F}_+^{-1} > 0$ then, $\mathbf{F}_+^{-1} \gg 0$ and, therefore, $u(t)$ is exponentially decreasing as $t \to -\infty$ (see [14],[15]).

By (8.13), since $u_2(t) = D_1^{1/2}u_1'(t) = D_1^{1/2}u'(t)$ and $D_1^{1/2} = UD^{1/2}A^{-1/2}$, $0 = \lim_{t\to 0^-}\|D_1^{1/2}u_1'(t) - Ux_0\| = \lim_{t\to 0^-}\|UD^{1/2}A^{-1/2}u_1'(t) - Ux_0\| = \lim_{t\to 0^-}\|UD^{1/2}u'(t) - Ux_0\|$. Thus, since U is invertible on $\mathcal{H}^0$, $\lim_{t\to 0^-}\|D^{1/2}u'(t) - x_0\| = 0$. Moreover, $\lim_{t\to-\infty}\|u(t)\|_1 = \lim_{t\to-\infty}\|A^{1/2}u(t)\| = \lim_{t\to-\infty}\|u_1(t)\| = 0$.

Now, if $\mathscr{D}(B) \supseteq \mathcal{H}_1$ and $\mathscr{D}(D) \supseteq \mathcal{H}_1$, then $Bu'(t)$ and $Du''(t)$ are infinitely differentiable as functions with values in $\mathcal{H}$ for $t < 0$. Therefore, $Au(t) = -Bu'(t) - Du''(t)$ has the same property. Thus, $u(t)$ takes values in $\mathscr{D}(A)$ and, therefore, is a classical solution for $t < 0$.

Uniqueness. Suppose that $u(t)$ satisfies (8.9) and (8.11) with $x_0 = 0$. Then $u_1(t) = A^{1/2}u(t)$ satisfies $u_1(t) + B_1u_1'(t) - D_1u_1''(t) = 0$ in $\mathcal{H}$ and for $t < 0$, and $u_1(t) \in \mathcal{C}_0^1[-\infty, -\delta]$ for any $\delta > 0$ (see section A.4). By the proposition 7.3.3, for every $t < 0$, $\varepsilon_1\|u_1(t)\| \leq \|D_1^{1/2}u_1'(t)\|$ and, since $D^{1/2} = U^{-1}D_1^{1/2}A^{1/2}$, we have $\varepsilon_1\|u(t)\|_1 \leq \|D^{1/2}u'(t)\|$ for every $t < 0$. Thus, since $\lim_{t\to 0^-}\|D^{1/2}u'(t)\| = 0$, we have $\lim_{t\to 0^-}\|u(t)\|_1 = 0$. Thus, by an argument analogous to the proof of uniqueness in the theorem 8.2.2, we have $u(t) \equiv 0$.

2. *Solution of* (8.9), (8.10):

Existence. Let $x_1 \in \mathcal{H}_1$. By the proof of the theorem 7.3.4, there exists an infinitely differentiable function $u_1(t)$ which satisfies $u_1(t) + B_1u_1'(t) - D_1u_1''(t) = 0$ $(t > 0)$, $\lim_{t\to+\infty}\|u_1(t)\| = 0$ and $\lim_{t\to 0^+}\|u_1(t) - A^{1/2}x_1\| = 0$.

Let $u(t) = A^{-1/2}u_1(t)$ $(t > 0)$. Then, $u(t)$ is an infinitely differentiable function with values in $\mathcal{H}_1$. Clearly, $Au(t) + Bu'(t) - Du''(t) = 0$ $(t > 0)$. Note that,

$$\lim_{t\to 0^+}\|u(t)-x_1\|_1 = \lim_{t\to 0^+}\|A^{1/2}u(t)-A^{1/2}x_1\| = \lim_{t\to 0^+}\|u_1(t)-A^{1/2}x_1\| = 0$$

and,

$$\lim_{t\to +\infty}\|u(t)\|_1 = \lim_{t\to +\infty}\|A^{1/2}u(t)\| = \lim_{t\to +\infty}\|u_1(t)\| = 0.$$

If $\mathscr{D}(B) \supseteq \mathcal{H}_1$ and $\mathscr{D}(D) \supseteq \mathcal{H}_1$ then, as in the previous case, $u(t)$ is the classical solution.

Uniqueness. It is similar to the proof of uniqueness for (8.9), (8.11). □

8.4 Completeness and Basis Property

Let E^+ (E^-) denote the set of all eigenvectors of the hyperbolic pencil $L(\lambda) = A + \lambda B - \lambda^2 D$ corresponding to its positive (negative) eigenvalues.

Theorem 8.4.1. If $\sigma(L)\cap\mathbb{R}^+ \subseteq \sigma_d(L)$ (resp. $\sigma(L)\cap\mathbb{R}^- \subseteq \sigma_d(L)$), then the system $\{D^{1/2}x\}_{x\in E^+}$ (resp. $\{D^{1/2}x\}_{x\in E^-}$) is complete in the subspace $\mathcal{H}^0 \subseteq \mathcal{H}$.

If $B \restriction_{\mathscr{D}(A)}$ admits the representation $B \restriction_{\mathscr{D}(A)} = B_+ - B_-$, where $B_+ \geq 0$ and the operator B_- is such that

$$|\langle B_- x, y\rangle| \leq 4\langle Ax, x\rangle\langle Dy, y\rangle \;\text{ for every }\; x, y \in \mathscr{D}(A),$$

then the system E^- is complete in $\mathcal{H}_1$.

Proof. Let B_1 and D_1 be the operators defined in (8.2) and let $L_1(\lambda) = I + \lambda B_1 - \lambda^2 D_1$. Since $L_1(\lambda) = A^{-1/2}L(\lambda)A^{-1/2}$ we have that the eigenvalues of $L(\lambda)$ and $L_1(\lambda)$ coincide. Moreover, $x \in \mathcal{H}$ is an eigenvector of L_1 associated to the eigenvalue λ if and only if $A^{-1/2}x$ is an eigenvector of L with eigenvalue λ. Therefore, the affirmation holds by the theorems 7.1.9 and 7.1.10. □

From theorem 7.3.6 and the formulas (8.2) the next theorem follows.

Theorem 8.4.2. If $L(\lambda)$ is an elliptic-hyperbolic pencil and its spectrum on $\mathbb{R}^-$ ($\mathbb{R}^+$) is discrete, then the system E^- (the system $\{D^{1/2}x\}_{x\in E^+}$) contains a Riesz basis in the space $\mathcal{H}_1$ (in the subspace $\mathcal{H}^0 \subseteq \mathcal{H}$).

Appendix A

Some Results on Functional Analysis and Operator Theory

A.1 Functional Analysis and Operator Theory

Theorem A.1.1. Let T a self-adjoint linear operator acting in a Hilbert space and let α, $\beta \in \mathbb{R}$. Then:

i) $\langle Tx, x\rangle \leq \beta$ for every $x \in D(T)$ with $\|x\| = 1$ if and only if $(\beta, \infty) \subseteq \rho(T)$.

ii) $\langle Tx, x\rangle \geq \alpha$ for every $x \in D(T)$ with $\|x\| = 1$ if and only if $(-\infty, \alpha) \subseteq \rho(T)$.

Theorem A.1.2 (Riesz Projections)**.** Let T be a bounded linear operator acting in a Hilbert space $\mathcal{H}$. Suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$ where σ_1 and σ_2 are disjoint and non-empty sets such that $\sigma(T) \setminus \sigma_1$ and $\sigma(T) \setminus \sigma_2$ are open. Then there exist P_1, $P_2 \in \mathcal{B}(\mathcal{H})$ orthogonal projections such that:

i) $P_1P_2 = P_2P_1 = 0$ and $P_1 + P_2 = I$.

ii) $\mathcal{H} = \mathscr{R}(P_1)\dot{\oplus}\mathscr{R}(P_2)$.

iii) T commutes with P_i, $i = 1, 2$.

iv) $\mathscr{R}(P_i)$ is T-invariant, $i = 1, 2$.

v) $\sigma(T \restriction_{\mathscr{R}(P_i)}) = \sigma_i$, $i = 1, 2$.

Theorem A.1.3 (Banach)**.** Let X be a vector space and let $\|\cdot\|_1$, $\|\cdot\|_2$ be norms on X such that $(X, \|\cdot\|_j)$ is a Banach space, $j = 1, 2$. If there is an $\alpha > 0$ such that $\|\cdot\|_1 \leq \alpha\|\cdot\|_2$, then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Theorem A.1.4. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. If $U \subseteq \mathcal{H}$ is a closed T-invariant subspace, then $U^\perp$ is T^*-invariant.

Theorem A.1.5. Let $A(\mu)$ be an operator valued function, holomorphic in a neighborhood U_{μ_0} of the point μ_0, and suppose that all the values of $A(\mu)$ are compact operators. Then there exists $\varepsilon > 0$ such that, for all values μ with $0 < |\mu - \mu_0| < \varepsilon$, the equation $(I - A(\mu))x = 0$ has the same number of linearly independent solutions.

Theorem A.1.6. Let $\mathcal{H}$ be a Hilbert space and $K : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator such that $\|Kx\| < \|x\|$ for every $x \neq 0$. Then $(I + K)(\mathcal{H})$ is dense in $\mathcal{H}$.

A.2 Normal Points of a Bounded Operator

Definition A.2.1. Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \sigma_p(A)$. λ is called a *normal eigenvalue* of A if and only if:

(1) The algebraic multiplicity of λ is finite, and

(2) $\mathcal{H} = \mathcal{H}_\lambda \dot{+} Y_\lambda$, where $\mathcal{H}_\lambda$ is the root space of $\mathcal{H}$ associated to λ and Y_λ is an A-invariant subspace such that $(A - \lambda I) \restriction_{Y_\lambda}$ is bijective.

Theorem A.2.2. Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \sigma_p(A)$. Then, λ is a normal eigenvalue of A if, and only if, it is an isolated point of $\sigma(A)$ and the corresponding Riesz projector P_λ corresponding to λ is of finite rank. In this case, $\mathscr{R}(P_\lambda) = \mathcal{H}_\lambda$.

Definition A.2.3. Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. λ is called a *normal point* of A if and only if $\lambda \in \rho(A)$ or it is a normal eigenvalue of A.

Remark. The set of all normal points of A is denoted by $\tilde{\rho}(A)$. Clearly, $\tilde{\rho}(A)$ is open.

Theorem A.2.4. Let $\mathcal{H}$ be a Hilbert space, $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator and $P \in \mathcal{B}_\infty(\mathcal{H})$. Then $\tilde{\rho}(A) = \tilde{\rho}(A + P)$.

A.3 Some Results about Definitizable Operators

Definition A.3.1. Let $(\mathcal{H}, [\cdot, \cdot])$ be a J-space and let T be a self-adjoint operator in the Krein space $\mathcal{H}$. T is called *definitizable* if, and only if, $\rho(T) \neq \varnothing$ and there exists a non-zero real polynomial $p(\lambda)$ such that $[p(T)x, x] \geq 0$ for every $x \in \mathscr{D}(p(T))$. In this case $p(\lambda)$ is called a *definitizing polynomial* for T.

Theorem A.3.2. Let T be a definitizable operator and S a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$. If $\rho(T) \cap \rho(S) \neq \varnothing$ and if $(T - \lambda)^{-1} - (S - \lambda)^{-1}$ has finite-dimensional range for some (and hence for all) $\lambda \in \rho(T) \cap \rho(S)$, then S is definitizable.

Theorem A.3.3. Let T be a definitizable self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$. If there exists a definitizing polynomial such that all its zeros are real, then $\sigma(T) \subseteq \mathbb{R}$ and $\sigma_r(T) = \varnothing$. Further, in this case, there exists a function $t \in \mathbb{R} \setminus \{\text{zeros of } p\} \mapsto E_t$, called the *spectral function* of T, such that for each $t \in \mathbb{R}$, with $p(t) \neq 0$, E_t is a self-adjoint operator in the Krein space; morevoer, if $s, t \in \mathbb{R}$ are such that $p(t), p(s) \neq 0$ then:

- For every x: $E_t x \to 0$ $(t \to -\infty)$, $E_t x \to x$ $(t \to \infty)$ and $E_t x \to E_{t_0} x$ $(t \to t_0^+)$.
- $E_t E_s = E_{\min(s,t)}$.
- If $x \in \mathcal{H}$ is fixed, then the function $[E_t x, x]$ is monotone non-decreasing in $t = t_0$ if $p(t_0) > 0$ and is monotone non-increasing in $t = t_0$ if $p(t_0) < 0$.
- $E_t T = T E_t$.
- $\sigma(T \restriction_{E_t(\mathcal{H})}) \subseteq (-\infty, t]$.

Definition A.3.4. The set $c(T)$ of all *critical points* of T is defined by

$$c(T) = [\bigcap N(p)] \cap \sigma(T) \cap \mathbb{R},$$

where $N(p)$ is the set of zeros of the polynomial p and the intersection $\bigcap N(p)$ runs over all the definitizing polynomials p of T.

Theorem A.3.5. Let T, $p(t)$ and E_t be as in the theorem A.3.6, and let $\alpha \in \mathbb{R}$. Then $\alpha \in c(T)$ if, and only if, for all $\varepsilon > 0$, the subspace $(E_{\alpha+\varepsilon} - E_{\alpha-\varepsilon})(\mathcal{H})$ contains positive and negative elements. In this case, if the strong limits $E_{\alpha+0}$ and $E_{\alpha-0}$ exist, then α is called a *regular critical point* of L; otherwise the critical point α is called *singular*.

Theorem A.3.6. Suppose that T is a non-negative operator and self-adjoint in the Krein space $\mathcal{H}$. Then T is similar to a self-adjoint operator in a Hilbert space if and only if zero is not a singular critical point of T and $\ker(T^2) = \ker(T)$.

Proposition A.3.7. Let T be a definitizable operator with definitizing polynomial p and let $t_0 \in \sigma(T)$ be such that $p(t_0) \neq 0$. Then λ_0 is of positive type (resp., negative type) if $p(t_0) > 0$ (resp., if $p(t_0) < 0$). In this case the sequence (x_n) in the definition 4.3.5 can be choosen such that $[x_n, x_n] \geq \delta$ (resp., $[x_n, x_n] \leq -\delta$) for some $\delta > 0$ given.

Theorem A.3.8. Let T, U_+ and U_- as in the theorem A.3.9., and let α be a critical point. Then α is regular if, and only if, T has no neutral eigenvectors and for at least one of of the invariant subspaces U_+ or U_-, name it U, we have that α is an isolate point of $\sigma(T \restriction_U)$ and α is an eigenvalue of $T \restriction_U$ with finite multiplicity.

Theorem A.3.9. Let T be a definitizable operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$. Then there exists U_+ (resp., U_-) T-invariant subspace of $\mathcal{H}$ such that U_+ (resp., U_-) is a maximal positive semi-definite (resp., a maximal negative semi-definite) and, therefore, closed subspace.
For each such subspace U_+ (resp., U_-) we have $\sigma(T \restriction_{U_+}) = \sigma^+(T)$ (resp., $\sigma(T \restriction_{U_-}) = \sigma^-(T)$) and $\sigma_r(T \restriction_{U_+})$ (resp., $\sigma_r(T \restriction_{U_-})$) contains only singular critical points.
Further, if the operator T has no neutral eigenvectors then U_+ is positive definite (resp., U_- is negative definite). If, moreover, all its critical points are regular, then the subspace U_+ (resp., U_-) is uniquely determined and U_+ is uniformly positive (resp., U_- is uniformly negative).

A.4 Distributions and Operators

If $-\infty \le a < b \le +\infty$, then with $\mathcal{C}_0^k[a, b; \mathcal{H}]$ we denote the set of $\mathcal{H}$-valued functions that have k continuous derivatives on $[a, b]$ and a $(k + 1)$-st derivative in the sense of distribution theory which is Bochner-integrable on any compact subset of the interior of $[a, b]$. Moreover, if $b = +\infty$ (resp., if $a = -\infty$) then all k derivatives tend strongly to zero as $t \to +\infty$ (resp., $t \to -\infty$).

Lemma A.4.1. If T is a bounded self-adjoint operator on $\mathcal{H}$, and $u \in \mathcal{C}_0^0[\alpha, +\infty; \mathcal{H}]$, then

$$2 \operatorname{Re} \int_{\alpha}^{+\infty} \langle Tu'(t), u(t)\rangle dt = 2 \operatorname{Re} \int_{\alpha}^{+\infty} \langle Tu(t), u'(t)\rangle dt = -\langle Tu(\alpha), u(\alpha)\rangle.$$

Analogously, if $u \in \mathcal{C}_0^0[-\infty, \beta; \mathcal{H}]$, then

$$2 \operatorname{Re} \int_{-\infty}^{\beta} \langle Tu'(t), u(t)\rangle dt = 2 \operatorname{Re} \int_{-\infty}^{\beta} \langle Tu(t), u'(t)\rangle dt = \langle Tu(\beta), u(\beta)\rangle.$$

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