Different approaches to Radiation Back Reaction.

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1 Introduction

Electrodynamics is the classical field of physics which deals with the study of the dynamics of charged particles and the electromagnetic fields produced by them. It is one of the greatest success of theoretical physics because it is an almost self contained and consistent theory which have proven to have a very broad spectrum of applications in many different branches of modern physics and other disciplines. It is also a model theory for other fundamental interactions. Indeed gauge theories (invariance under the action of local lie groups) have played a capital role in the development of the theory of strong and weak interactions. Perhaps the only remaining theoretical unsolved problem of classical electrodynamics is the self interaction of the charged particles with its own fields, problem which arises from two different considerations: First, it is a well known fact that accelerated charged particles emit energy in form of radiation at a rate predicted by Larmor's formula, therefore the motion of the particle can not be analyzed only under the assumptions of a Lorentz force but there should be a dissipative force which accounts for this energy losses which is called the radiation back reaction force. Second when one thinks on the nature of the interaction of the particle with its own surrounding fields one is naturally lead to couple the Maxwell equations describing the fields of the particle with the Lorentz force law describing its motion. Then one finds that the corrections for the dynamics of the particle due to its self field (in the point particle model)is given by a third order equation (the Abraham-Lorentz equation) . This problem of including the self interaction in the dynamics of a charged particle has been a frequent subject of study since the beginning of the last century. Perhaps one of the earliest solution given to this problem was the one proposed in non-covariant form by Abraham and Lorentz and later generalized to covariant form by Dirac. Based mainly on energy balance arguments they arrived at the following equation which carries their name (we will call it LAD equation):

$$m\dot{\vec{v}} = \vec{F}_{ext} + \frac{2e^2}{3c^3}\ddot{\vec{v}}. \tag{1}$$

However since this is a third order equation it possesses deep conceptual difficulties such as the preacceleration phenomena and runaway solutions. Preacceleration is a phenomenon which consist in that the acceleration experienced by a charged particle at a given time depends upon the force exerted by the electromagnetic fields at future times. Clearly this is a direct consequence of the LAD equation being third order in time and evidently it represents a violation of the principle of causality. The runway solutions are solutions to the LAD equatio which under certain circumstances imply that the particle starts to self accelerate to infinite even though there is no external force present. One may guest that this difficulties arises from the fact that Maxwell's equations are divergent in the particles position, because when the fields are evaluated at this point one is wrongly lead to the conclusion that they carry an infinite electromagnetic energy which is absurd. Thus there must be something incorrect in the model used to study charged particles and their fields. The first step to the answer to this riddle is to understand that trying to build a consistent classical theory which describes the structure of the charged particle and its dynamics is almost nonsense at the classical level and that the best that can be done is to describe qualitatively how the charge is distributed in the particle by introducing a form factor which does this job. In other words as is widely known the structure of charged particles is not in the domain of classical electrodynamics.

Therefore the purpose of the present work is to study the problem of radiation back reaction, its implications in the dynamics of charged particles from an semi classical perspective and the range of validity of the results obtained. It is semi classical because relativistic covariant cases will be studied too, whereas quantum effects in principle won't be taken into account although we may rely on some of the results of this theory to perform our studies. In order to do this we will have to study first the cause for radiation trough the Larmor formula, so we will derive a covariant expression for it in order to leave clear that emission of radiation is a covariant process and not just an effect produced by the reference frame. Then we will make some derivations of the LAD equation in covariant and noncovariant form and explained the already mentioned deficiencies that it possesses. Then based on the works of Landau-Lifshitz, Ford and O'Connell, Rohrlich and Spohn we will demonstrate that the treatment which solves more satisfactorily the problem of the self force is the one done many years ago by Landau and Lifshitz which gives rise to an equation of motion for the charged particles to which some authors give their names, that is the Landau Lifshitz equation. Therefore we will derive the Landau Lifshitz equations from different points of view and this will corroborate the fact that this equation is consistent with different principles from different branches of physics. As an addition we will study the effect that radiation reaction has over the motion on the spin of the electron. Finally we will conclude that the Landau Lifshitz equation is the correct equation for the motion of the electron at a classical level and will analyze its domain of applicability.

2 Derivation of the Larmor formula in covariant form.

The radiation emitted by an accelerated non-relativistic charge is well described by the Larmor formula

$$P = \frac{2e^2}{3c^3} \mid \dot{v} \mid^2 \tag{2}$$

which is not adequate for the relativistic case because its derivation uses certain approximations which require that the velocity of the charged particle is small in the reference frame of observation and this makes the formula non covariant. So in order to include the self radiation effects in the motion of a charged particle it is useful to obtain the relation between the energy and momentum carried away by the radiation in terms of the dynamical variables of the particles. This will be done by computing the amount of energy carried away by the fields produced by the moving particle. Our beginning point is the four vector potential A^{α}, which comes from solving the covariant Maxwell equations for a charged particle

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{3}$$

and the relation between the field-strength tensor and the 4-vector potentials

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha, \tag{4}$$

whose solution gives the following retarded potentials(see Jackson chap 14 and Appendix 1):

$$A^\alpha_{ret} = \frac{4\pi}{c} \int d^4x' D_r(x - x') J^\alpha(x') \tag{5}$$

where $D_r(x - x')$ is the retarded green function. We denote by x^α the 4-coordinates of the observation point, x'^α the 4-coordinates of the volume occupied by the charge and R will be the spatial difference $R \equiv \mid \vec{x} - \vec{x'} \mid$. Also $z(\tau)$ denotes the world line of the particle and R^μ will be a light like 4-vector such that $R^\mu \equiv x^\mu - z^\mu(\tau)$ and such that $R^\mu R_\mu = 0$. J^α is the 4-vector current which for a charge with 4-velocity v^α (in the point particle model) is given by :

$$J^\alpha(x') = ec \int d\tau v^\alpha(\tau) \delta^4(x' - z(\tau)) \tag{6}$$

which together with equation (5) lead us to the next expression for the retarded potential:

$$A^\alpha(x) = \left| \frac{eV^\alpha(\tau)}{V \cdot [x' - z(\tau)]} \right|_{\tau=\tau_0} \tag{7}$$

with τ_0 defined by the light cone condition $R^\mu R_\mu = [x - z(\tau_0)]^2 = 0$. This condition expresses that the past-light cone with vertices's at x^α intersects the world line of the particle only at one point, that is at $z(\tau_0)$, in other words the contribution of the fields coming out of the particle to the observed fields at $x^{\alpha\prime}$ comes only from one point of the worldline of the particle.

Now we define a space like unit vector u^μ , orthogonal to v^μ, such that $\hat{u}$ points from the retarded position of the particle to the observation field point x^α. Since $u_\mu v^\mu = 0$ we can define a quantity ρ such that :

$$\begin{aligned} R^\mu &= \rho(u^\mu + \frac{v^\mu}{c}) \\ \rho &= u_\mu R^\mu = -v_\mu R^\mu / c. \end{aligned} \tag{8}$$

Now with this notation we can write the final expression for the potential of the fields produced by the moving particle:

$$A^\mu = \frac{e}{c} \frac{v^\mu}{\rho} \tag{9}$$

Now that we know the potentials we can calculate the field strength tensor by using equation (4). So let's first differentiate :

$$c\partial^\mu A^\nu(x) = -\frac{ev^\nu}{\rho^2}\partial^\mu\rho + \frac{e}{\rho}\partial^\mu(v^\nu) = -\frac{ev^\nu}{\rho^2}\partial^\mu\rho + \frac{e}{\rho}a^\nu\partial^\mu\tau \tag{10}$$

where a^ν is the 4-acceleration $dv^\nu/d\tau$.

Now, since R^μ is light-like we have that:

$$R^\mu R_\mu = R_0^2 - \mid \vec{R} \mid^2 = 0,$$

and differentiating with respect to τ we get that

$$0 = 2R_0\frac{dR_0}{d\tau} - 2 \mid \vec{R} \mid \frac{d \mid \vec{R} \mid}{d\tau} = 2(R_\mu\frac{dR^\mu}{d\tau}) \tag{11}$$

and from the definition of R^μ we get that

$$R_\mu(\frac{dx^\mu}{d\tau} + v^\mu) = 0 \tag{12}$$

then replacing R_μ for the expression in (8) we get

$$\rho(u^\mu + \frac{v^\mu}{c})(\frac{dx_\mu}{d\tau} - v_\mu) = 0 \tag{13}$$

then

$$(u^\mu + \frac{v^\mu}{c})\frac{dx_\mu}{d\tau} = \frac{v^\mu}{c}v_\mu = -c \tag{14}$$

Now from this equation we can obtain the reciprocal of $\frac{dx_\mu}{d\tau}$ that is:

$$\frac{\partial\tau}{c\partial x_\mu} = -(u^\mu + \frac{v^\mu}{c}) \tag{15}$$

with this we can differentiate ρ using $\partial^\mu = \frac{\partial\tau}{\partial x_\mu}\frac{d}{d\tau}$, $v_\nu v^\nu = -c^2$ and $v_\nu a^\nu = 0$ then

$$\begin{aligned}
\partial^\mu\rho &= \partial^\mu(-v_\nu R^\nu/c) = (\partial^\mu v_\nu)R^\nu/c + v_\nu\partial^\mu R^\nu/c \\
&= a_\nu(u^\nu + \frac{v^\nu}{c})\frac{R^\mu}{c^2} + \frac{v_\nu}{c}(\frac{\partial\tau}{\partial x_\nu}\frac{dx^\mu}{d\tau} - \frac{\partial\tau}{\partial x_\nu}v^\mu) \\
&= \frac{a_\nu u^\nu R^\mu}{c^2} + a_\nu v^\nu\frac{R^\mu}{c^3} + \frac{v_\mu}{c} - \frac{u^\mu v_\nu v^\nu}{c^2} + v_\mu\frac{v_\nu v^\nu}{c^3}
\end{aligned} \tag{16}$$

Now let's define

$$a_u \equiv a_\nu u^\nu \tag{17}$$

that is the projection of a^μ in the u^μ direction, with this the preceding equation simplifies to:

$$\partial^\mu \rho = u^\mu + a_u \frac{R^\mu}{c^2} = u^\mu + \frac{a_u}{c^2}\rho(u^\mu + \frac{v^\mu}{c}) \tag{18}$$

So inserting this expression in (10) and this in equation (4), we obtain an expression for the retarded field strength tensor in terms of the dynamical variables, that is the velocity and acceleration :

$$\begin{aligned} F^{\mu\nu}_{ret} &= \frac{e}{\rho^2 c}(v^\mu u^\nu - v^\nu u^\mu) + \frac{e}{\rho c^2}[(a^\mu v^\nu - a^\nu v^\mu)/c - u^\mu(\frac{v^\nu}{c}a_u + a^\nu) \\ &+ u^\nu(\frac{v^\mu}{c}a_u + a^\mu)]. \end{aligned} \tag{19}$$

In a similar fashion, with some obvious changes, we can obtain an expression for the advanced field strength tensor:

$$\begin{aligned} F^{\mu\nu}_{adv} &= \frac{e}{\rho^2 c}(v^\mu u^\nu - v^\nu u^\mu) + \frac{e}{\rho c^2}[(a^\mu v^\nu - a^\nu v^\mu)/c - u^\mu(\frac{v^\nu}{c}a_u - a^\nu) \\ &+ u^\nu(\frac{v^\mu}{c}a_u - a^\mu)]. \end{aligned} \tag{20}$$

The symmetric energy-momentum field tensor is defined by

$$\Theta^{\mu\nu}_{elm} = \frac{1}{4\pi}(F^{\mu\alpha}F^\nu_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \tag{21}$$

where $\eta^{\mu\nu}$ is the Minkowski metric $(1, -1, -1, -1)$.

Now in order to calculate the amount of energy and momentum carried away by the radiation fields it seems reasonable to integrate the energy-momentum tensor $\Theta^{\mu\nu}_{elm}$ in a volume space, but for conserving relativistic invariance the proper way to proceed is to define a 4-vector which does this job. This we will call the energy momentum 4-vector and will be defined by:

$$P^\mu \equiv \frac{1}{c}\int \Theta^{\mu\nu}_{elm} d\sigma_\nu \tag{22}$$

where $d\sigma_\nu$ is a surface element of a space-like plane. This space-like plane represents a three dimensional space for a given fixed time, so the plane differential looks like $d\sigma^\mu = n^\mu d^3\sigma$ where $d\sigma$ is a volume differential and n^μ is a time-like unit vector. But we can't integrate over the whole space because the integral would diverge as we approach the particle's position, (that is where the plane σ intersects the world line of the particle $z(\tau)$ and the integral would cover radiation emitted by the particle at different places of the world line and different instants of time. So if we want to obtain the radiation emitted by the charge in a particular instant of time τ what we do is intersect the plane σ with the future light cone of the particle at $z(\tau)$. Well at one exact instant of time not much radiation is emitted so we rather want to know the radiation emitted in an small time interval $\Delta\tau$. So the space that we are going to integrate over is the intersection of the plane σ and the intersection of the future light cones of the particle at $z(\tau)$ and at $z(\tau+\Delta\tau)$. So by doing this we obtain the quantity:

$$\Delta P^{\mu} = \frac{1}{c}\int_{\Delta\sigma} \Theta^{\mu\nu}_{elm} d\sigma_{\nu} \tag{23}$$

The integral above is significant if its value is independent of the observer, that is if it is a covariant quantity . To prove that ΔP^{μ} is covariant we need to demonstrate that it is independent of the choice of $\Delta\sigma$. We will show that the surface independence occurs for surfaces such that $\Delta\sigma \to \infty$

2.1 Sketch of the proof that ΔP^{μ} is covariant

Let σ_1 and σ_2 be two different surfaces. These surfaces intersect the world line of the particle at two different moments $z(\tau_1)$ and $z(\tau_2)$. So when we intersect these surfaces with their respective future light cones emerging from the world-line they form two different three dimensional surfaces $\Delta\sigma_1$ and $\Delta\sigma_2$. So what we want to prove is that the integrals over this two surfaces are equal, that is: $\Delta P^{\mu}(\Delta\sigma_1) = \Delta P^{\mu}(\Delta\sigma_2)$

Consider the 4-dimensional volume whose boundaries are the two three dimensional surface $\Delta\sigma_1$ and $\Delta\sigma_2$ and the future light cones from $z(\tau_1)$ and $z(\tau_2)$. We denote this 4-dimensional volume by V_4 and its 3-dimensional surface boundary by Ξ. Then by a four dimensional Gauss's theorem we have that:

$$-\int_{v_4} \partial_{\nu}\Theta^{\mu\nu}_{elm} d^4x = \int_{\Xi} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} \tag{24}$$

The momentum energy conservation law requires that $\partial_{\nu}\Theta^{\mu\nu}_{elm} d^4x = 0$ and using the definition of Ξ we have that:

$$\int_{\Xi} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} = \int_{\Delta\sigma_2} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} - \int_{\Delta\sigma_1} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} + \int_{\Lambda z(\tau_2)} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} \tag{25}$$

$$- \int_{\Lambda z(\tau_1)} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} \tag{26}$$

where $\Lambda z(\tau)$ denotes the three dimensional surface formed by the light cone emerging from $z(\tau)$. Therefore we conclude that:

$$\Delta P^{\mu}(\Delta\sigma_1) = \Delta P^{\mu}(\Delta\sigma_2) + \frac{1}{c}\int_{\Lambda z(\tau_2)} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} - \frac{1}{c}\int_{\Lambda z(\tau_1)} \Theta^{\mu\nu}_{elm} d^3\sigma \tag{27}$$

So our equality will be true if :

$$\int_{\Lambda z(\tau_2)} \Theta^{\mu\nu}_{elm} d^3\sigma_{\nu} = \int_{\Lambda z(\tau_1)} \Theta^{\mu\nu}_{elm} d^3\sigma \tag{28}$$

Now by inserting the expression that we found for $F^{\mu\nu}$ equation (20) and (19) in the definition of $\Theta^{\mu\nu}_{elm}$ equation (21) we find that:

$$\begin{aligned}\Theta^{\mu\nu}_{elm} &= \frac{e^2}{4\pi\rho^4}(u^\mu u^\nu - \frac{v^\mu v^\nu}{c^2} - \frac{1}{2}\eta_{\mu\nu}) + \frac{e^2}{2\pi\rho^3 c^2}\left[a_u\frac{R^\mu R^\nu}{\rho^2} - \frac{a_u(v^\mu R^\nu + R^\mu v^\nu)}{\rho c}\right. \\ &+ \left.\frac{a^\mu R^\nu + R^\mu a^\nu)}{c}\right] + \frac{e^2}{4\pi\rho^2 c^4}(a_u^2 - a_\lambda a^\lambda)\frac{R^\mu R^\nu}{\rho^2}\end{aligned} \tag{29}$$

Now since we want true radiation, that is the one that travels along all the space, because this is the one that really does take energy and momentum away, we want to evaluate the behavior of the integral at infinity, that is we want the observation point x to be infinitely far away from the source point $z(\tau_2 + \Delta\tau)$. This implies that $\rho \to \infty$ therefore $\Delta\sigma \to \infty$. Now in the integration over the light cones $\Lambda z(\tau_2)$and$\Lambda z(\tau_1)$ the volume element in the light cones is:

$$d^3\sigma^\mu = R^\mu d^2\omega \tag{30}$$

where $d^2\omega$ is a two dimensional invariant and R^μ is the null vector already defined . Now since the terms in $\Theta^{\mu\nu}_{elm}$ fall off as ρ^{-4},ρ^{-3} and ρ^{-2} then the terms in $\int_{\Lambda z(\tau_2)} \Theta^{\mu\nu}_{elm} d^3\sigma_\nu$ and $\int_{\Lambda z(\tau_1)} \Theta^{\mu\nu}_{elm} d^3\sigma$ fall of as ρ^{-3},ρ^{-2} and ρ^{-1} so in this limit they both vanish, therefore the equality (equation 27) hold and the quantity ΔP^μ behaves as a 4-vector.

So what was proven here is that in a observation point infinitely away from the source point, there is no flux that crosses the bounding light cones, therefore the flux through the two surfaces $\Delta\sigma_1$ and $\Delta\sigma_2$ in the limit $\rho \to \infty$ is the same.

With this in mind we are allowed to define the quantity:

$$dP^\mu \equiv -\lim_{\rho\to\infty}\frac{1}{c}\int_{d\sigma} \Theta^{\mu\nu}_{elm} d^3\sigma_\nu \tag{31}$$

So now it is left to evaluate the above integral. For this purpose it is worth mentioning that in the derivation of dP^μ it was not required that the surface σ was space-like, then the demonstration would have been done as well with time like surfaces. Therefore in making the integration we can do it over a time-like surface, which is kind of a cylindrical surface surrounding the world-line of the charge.

Then:

$$d^3\sigma^\mu = u^\mu \rho^2 d\Omega c d\tau \tag{32}$$

where $d\omega$ is a solid angle. Then the rate at which radiation is emitted is :

$$\frac{dP^\mu}{d\tau} = -\underset{\rho\to\infty}{lim}\frac{e^2}{4c^4}\int (a_\lambda a^\lambda - a_u^2)\frac{R^\mu R^\nu u_\nu}{\rho^2} d\Omega \tag{33}$$

Now keeping in mind that $R^\mu = \rho(u^\mu + \frac{v^\mu}{c})$ and $u^\mu u_\mu = 1$, $u^\mu v_\mu = 0$ we have that:

$$\frac{R^\mu R^\nu u_\nu}{\rho^2} = \frac{\rho(u^\mu + \frac{v^\mu}{c})\rho(u^\mu + \frac{v^\mu}{c})u_\nu}{\rho^2} = u^\mu + \frac{v^\mu}{c} \tag{34}$$

then the integral is independent from ρ and it is:

$$\int (a_\lambda a^\lambda - a_u^2)(u^\mu + \frac{v^\mu}{c})d\Omega \tag{35}$$

Now since we spend two pages proving that the quantity dP^μ is a four vector we are going to use this fact to perform the integration, so we will do it in the rest reference frame system of the particle and then use the fact that dP^μ must transform like the velocity to find the expression for a general system in which the particle is not are rest.

In the rest frame system $u^\mu = (0, \hat{u})$ and $v^\mu = (c, 0, 0, 0)$.Let α be the angle between $\hat{u}$ and $\vec{a}$ then $\hat{u} \cdot \vec{a} = a\cos\alpha$: First let's calculate the time component

$$\int (a_\lambda a^\lambda - a_u^2)d\Omega = \int [a^2 - (a \cdot \hat{u})^2]d\Omega = \frac{2}{3}a^2 4\pi \tag{36}$$

where the $\frac{2}{3}$ comes from the integration of the factor $\sin\alpha^3$. And the spatial component is:

$$\int [a^2 - (a \cdot \hat{u})^2]\hat{u}d\Omega = \int [a^2 \sin\alpha^2]\hat{u}d\Omega = 0 \tag{37}$$

This last part is in complete accordance with the fact that our momentum four vector rate must reduce to the predicted by Larmor formula in the non-relativistic case. Now since in the rest system $a^2 = a_\lambda a^\lambda$ and $v^\mu = (1, 0, 0, 0)$ it is easily seen that when we make the Lorentz transformation to a general reference frame our momentum 4-vector rate is:

$$\frac{dP^\mu}{d\tau} = \frac{2}{3}\frac{e^2}{c^5}a_\lambda a^\lambda v^\mu. \tag{38}$$

Now the energy rate of radiation is easily calculated from the momentum rate:

$$\Re \equiv -v_\mu \frac{dP^\mu}{d\tau} = \frac{2}{3}\frac{e^2}{c^3}a_\lambda a^\lambda \tag{39}$$

And this quantity give us the rate at which momentum detaches from the particle in form of radiation. These are the famous radiation fields which represent true radiation in contradiction with the velocity fields which travel along with the particle as it moves, so from this formula we know that independent of the observer a particle which experiences an acceleration will lost energy in form of radiation.

3 The Abraham Model

Any formal treatment of the problem of radiation reaction involves the Maxwell's equations and the Lorentz force law, this is in order to analyze the behavior of the fields produced by the charged particle and its dynamics. Although these fundamental equations don't require to have and exact model for the particle, they require to have an charge density distribution . In most problems in electrodynamics this two equations are separated , therefore the point particle model does quite well, but since radiation reaction combines this two laws in a way that it requires the evaluation of the fields in the particles position, just where the point particle diverges it is clear that some model must be introduced otherwise inconsistencies will arise as in the case of the LAD equation.

One the first model for the electron was proposed by Abraham, and it consisted of a tiny spherically symmetric body, in which the charge is uniformly distributed. Since this model is based on a macroscopic object (the sphere) it is easily seen that this model ain't a microscopic model at all, it is just a macroscopic model taken to a very small scale thus some problems may arise. The first obvious requirement that the model must fulfill is that its total charge is equal to the known electron charge e, then:

$$e = \int_V \rho(r) d^3x \tag{40}$$

where the volume of integration is the whole sphere $V = \frac{4\pi}{3}R^3$, with R its radius . But here the macroscopic features of the model become troublesome. Since every charge produces a coulomb potential which exerts forces on charges, the different parts of the charge must produce repulsive forces on the other parts of the sphere. The net result is then that the sphere should start expanding unless there is some other force that contrarest this repulsive potential. Mathematically the net result of this self force is that the electromagnetic self energy tensor $\Theta^{\mu\nu}_{elm}$ has no longer vanishing divergence, instead $\partial_\mu \Theta^{\mu\nu}_{elm}$ gives us the equation of a force density.

$$\vec{f} = \nabla \cdot T - \frac{\partial \vec{S}}{c^2 \partial t} = \rho E \frac{1}{c} \vec{J} \times \vec{B} \tag{41}$$

where T is the Maxwell stress tensor given by

$$T_{i,j} = (E_i E_j + B_i B_j - \frac{\delta_{ij}}{2}(E^2 + B^2)) \tag{42}$$

which is also the spatial component of the electromagnetic energy field tensor $\Theta^{\mu\nu}_{elm}$ (see Jackson chapter 12). If we work in the particle's rest frame $\vec{v} = 0$ hence $\vec{B} = 0$, then

$$\nabla \cdot T = \rho \vec{E} = (\frac{1}{4\pi} \nabla \cdot \vec{E})\vec{E} = \frac{d}{dr}(\frac{E^2 \hat{r}}{8\pi}) \tag{43}$$

By gauss law we know that the electric field of a sphere of uniformly distributed charge is

$$\vec{E} = \frac{2}{r^2}\hat{r} \tag{44}$$

Then the total self-force experienced by the particle is given by

$$2dF = d\Omega \int \rho E r^2 dr \;=\; d\Omega \int (\nabla \cdot T) r^2 dr = d\Omega \int_R^\infty \frac{d}{dr}(\frac{E^2 \hat{r}}{8\pi}) r^2 dr \tag{45}$$

$$= \frac{e^2}{4\pi R^2} \hat{r} d\Omega \tag{46}$$

Then we get that the internal self pressure is:

$$\wp = \frac{dF}{R^2 d\Omega} = \frac{e^2}{8\pi R^4} \hat{r} \tag{47}$$

This pressure might be written in terms of the electromagnetic self energy

$$W_{self} = \frac{1}{8\pi} \int E^2 d^3x = \frac{e^2}{2R} \tag{48}$$

then

$$\wp = \frac{1}{3} \frac{W_{self}}{V} \hat{r}$$

So as we said before the net effect of this pressure which comes from inside the particle is to expand it, unless we add to the electromagnetic tensor some factor which inserts in the equation (43) some cohesion force that opposes the electromagnetic self force. But then the addition of this factor would ruin the beauty of the idea of a purely electromagnetic model for the charge particle. Thus what we found is that a classical model for the charged particle is almost an oximoron, because the structure of particles is clearly out of the domain of classical physics and is rather competence of other fields of physics as quantum mechanics and the quantum field theory.

This incapability of describing classically the internal structure of the charge particles and its interaction with its own field its called the particle-field dichotomy. Anyway existence of this dichotomy does not imply that the description of other features of the classical charged particles are meaningless. Thus the dynamics of the charged particles under the influence of certain classical forces stills being a very interesting and meaningful problem as long as we don't make any reference or use of the internal structure because this would lead us to contradictions or inconsistencies. Thus the classical theory that we are working upon refers to the interaction of structure-less extended charged particles with the electromagnetic fields and does not intend to reconcile the dichotomy between particles and fields, that is it does not try to build up a classical theory of purely electromagnetic particles which carry along an coulomb field because this is clearly out of the domain of classical electrodynamics and rather in the domain of Quantum Field Theory. But then if the particle and its surrounding field are not clearly distiguishable objects exactly about what are we talking about when we speak of the particles motion?. The answer is of the motion of the observable mass, which as we know from quantum electrodynamics is compound of the bare mass of the particle and the energy of its surrounding fields.

4 Non covariant LAD equation and its flaws.

In this section we intend to show that the LAD equation arises very naturally when thinking in radiation reaction from two different points of view which are:

-How to account for the energy losses produced by radiation in the equation of motion of the particle in such a way that the energy of the whole system preserves.

-How the fields produced by a particle affects its own motion. This is done by incorporating the Maxwell's equations in the the Lorentz force law.

It is quite natural that the two derivations arrive at the same equation because they are based in principles which are universally accepted and very consistent between them.

4.1 Energy balance

As we have already seen when a particle is under a force it radiates energy at a rate given by Larmor's formula which in noncovariant form is:

$$P = \frac{2e^2}{3c^3} \mid \dot{v} \mid^2 \tag{49}$$

So the loss of mechanical energy $(E = T + V)$ from the particle is :

$$\frac{dE}{dt} = -P \tag{50}$$

Now V is the external potential which produces the external force therefore:

$$\frac{dV}{dt} = \vec{v} \cdot \nabla V = \vec{v} \cdot \vec{F}_{ext} \tag{51}$$

the change in kinetic energy is

$$\frac{dT}{dt} = \frac{d}{dt}(m\dot{\vec{v}} \cdot \dot{\vec{v}}) = m\vec{v} \cdot \dot{\vec{v}} \tag{52}$$

then equation (50) takes the form:

$$m\vec{v} \cdot \dot{\vec{v}} + \frac{2e^2}{3c^3} \mid \dot{v} \mid^2 = \vec{v} \cdot \vec{F}_{ext} \tag{53}$$

Now from Newtons equation we know that $m\dot{\vec{v}} = \vec{F}$, where $\vec{F}$ accounts for the total force acting on the particle. But since we know that this force comes from two sources, that is the external fields and the self fields, we may decompose it to yield:

$$m\dot{\vec{v}} = \vec{F}_{ext} + \vec{F}_{\text{self}} \tag{54}$$

therefore making dot product with $\vec{v}$ we get that:

$$m\dot{\vec{v}} \cdot \vec{v} = \vec{F}_{ext} \cdot \vec{v} + \vec{F}_{\text{self}} \cdot \vec{v} \tag{55}$$

So inserting (51) in this equation and canceling the common terms we get that

$$\vec{F}_{\text{self}} \cdot \vec{v} = -\frac{2e^2}{3c^3} \mid \dot{v} \mid^2 \tag{56}$$

So if we integrate this last equation in a time interval $t_1 < t < t_2$ what we obtain is that the work done by the self force on the particle is equal to the energy radiated away in this interval of time, therefore

$$\int_{t_1}^{t_2} \vec{F}_{\text{self}} \cdot \vec{v} dt = -\int_{t_1}^{t_2} \frac{2e^2}{3c^3} \dot{\vec{v}} \cdot \dot{\vec{v}} dt \tag{57}$$

which by integration by parts yields

$$\int_{t_1}^{t_2} \vec{F}_{\text{self}} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \frac{2e^2}{3c^3} \ddot{\vec{v}} \cdot \vec{v}\, dt - \frac{2e^2}{3c^3} \dot{\vec{v}} \cdot \vec{v} \bigg|_{t_1}^{t_2} \tag{58}$$

Now for most practical cases the last term vanishes (for instance periodic motion and forces acting perpendicular to the velocity) , therefore

$$\int_{t_1}^{t_2} (\vec{F}_{\text{self}} - \frac{2e^2}{3c^3} \ddot{\vec{v}}) \cdot \vec{v}\, dt = 0 \tag{59}$$

Then we conclude that :

$$\vec{F}_{\text{self}} = \frac{2e^2}{3c^3} \ddot{\vec{v}} \tag{60}$$

yielding the following equation of motion for the charged particle:

$$m\dot{\vec{v}} = \vec{F}_{\text{ext}} + \frac{2e^2}{3c^3} \ddot{\vec{v}} \tag{61}$$

and this is the famous Abraham-Lorentz equation. By the way if the total energy of the particle is purely electromagnetic $mc^2 = W_{self} = e^2/2R$ where m is the rest mass, then from here we could find an aproximate expression for the particle's radius:

$$R = \frac{1}{2} \frac{e^2}{mc^2} \tag{62}$$

In order to make it independent of the model used the classical electron radius has been defined as:

$$r_e = \frac{e^2}{mc^2} \tag{63}$$

from which we define the proper time τ_e

$$\tau_e = \frac{e^2}{mc^3} \tag{64}$$

which is the time it takes to light ray to cross the particles radius. As one could guess it is a very small time but it will play a prominent role in the development of our studies.

4.2 Fundamental derivation of the Abraham Lorentz equation

Now we will give a more fundamental derivation of the Abraham-Lorentz equation. The charge will be characterized by a charge density with spherical symmetry as given by the Abraham model. As we have already mentioned our beginning point is the Lorentz force law given by

$$m\dot{\vec{v}} = \vec{F}_{ext} + \vec{F}_{\text{self}} \tag{65}$$

where m is the mechanical mass of the particle and with $\vec{F}_{\text{self}}$ obtained by taking the magnetic and electric fields used in the Lorentz force law to be the own fields produced by the particle as follows:

$$\vec{F}_{\text{self}} = \int \left(\rho \vec{E}_{\text{self}} + \frac{1}{c} \vec{J} \times \vec{B}_{\text{self}} \right) d^3x \tag{66}$$

with $\vec{J}$ being the current , which for an extended charge model is given by $\vec{J}(\vec{x}', t) = \rho(\vec{x}', t)\vec{v}(t)$.

Now if we work in the instantaneous particle's rest frame $\vec{v} = 0$ thus the term associated with the magnetic potential vanish. Therefore expressing $\vec{E}$ in terms of the potentials $\vec{A}$ and ϕ (see Appendix 1) we get the following expression for the self force:

$$\vec{F}_{\text{self}} = \int \rho(\vec{x}, t) \left[\nabla\phi(\vec{x}, t) + \frac{1}{c} \frac{\partial \vec{A}(\vec{x}, t)}{\partial t} \right] \tag{67}$$

Now the retarded solutions for the Maxwell's equations in the Lorentz gauge are given by:

$$\phi = \int \frac{\rho(x', t')_{ret}}{R} d^3x' , \qquad \vec{A} = \frac{1}{c} \int \frac{\vec{J}(x', t')_{ret}}{R} d^3x' \tag{68}$$

where R means the same as in section (2) that is $\vec{R} = \vec{x} - \vec{x}\,'$ and R its norm.

So with this expressions for the potentials the force takes the form:

$$\vec{F}_{\text{self}} = \int \rho(\vec{x}, t) \left[\nabla \int \frac{\rho(x', t')_{ret}}{R} d^3x' + \frac{1}{c^2} \frac{\partial}{\partial t} \int \frac{\vec{J}(x', t')_{ret}}{R} d^3x' \right] d^3x \tag{69}$$

where the potentials should be evaluated at a retarded time $t' = t - R/c$. But since the source and the observation point of the self fields is the particle it self the retardation time should not be greater than the time it takes the light to cross the classical electron radius, which means that $R/c \leq \tau_e = e^2/mc^3$. So since the retardation time is so small it is very plausible to expand the retarded fields in Taylor series obtaining that:

$$\vec{F}_{\text{self}} = \int \rho(\vec{x},t)\left[\int \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{R}{c}\right)^n \frac{\partial^n}{\partial t^n}\left[\nabla \frac{\rho(x',t')'}{R} + \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\vec{J}(x',t')}{R}\right]_{t'=t} d^3x'\right] d^3x \tag{70}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int \rho(\vec{x},t)\left[\int \frac{\partial^n}{\partial t^n}\left[\rho(x',t')\nabla R^{n+1} + \frac{R^{n+1}}{c^2}\frac{\partial \vec{J}}{\partial t}\right]_{t'=t} d^3x'\right] d^3x$$

Now if we look to the first two terms we realize that they should vanish. The first one because it is of the form

$$\int d^3x \int d^3x' \rho(\vec{x},t)\rho(\vec{x}',t)\nabla\left(\frac{1}{R}\right) \tag{71}$$

which corresponds to the electrostatic self energy, which as we saw in the last section must vanish in order for the particle to be stable. The second one includes a gradient of R^0 which is a constant so it vanish. Then the summation indexes may be shifted yielding :

$$\vec{F}_{\text{self}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int d^3x \int d^3x' \rho(\vec{x},t) R^{n+1}\frac{\partial^{n+1}}{\partial t^{n+1}}\left[\vec{J}(\vec{x}',t) - \frac{\partial\rho}{\partial t}(\vec{x},t)\frac{\nabla R^{n+1}}{(n+2)(n+1)R^{n-1}}\right] \tag{72}$$

Now to simplify the following operations we define:

$$\vec{\xi} = \vec{J}(\vec{x}',t) - \frac{\partial\rho}{\partial t}(\vec{x}',t)\frac{\nabla R^{n+1}}{(n+2)(n+1)R^{n-1}} \tag{73}$$

So the equation of motion takes the form

$$\vec{F}_{\text{self}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\int d^3x \int d^3x' \rho(\vec{x},t) R^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}\vec{\xi} \tag{74}$$

but by the continuity equation (380)

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \tag{75}$$

therefore

$$\vec{\xi} = \vec{J}(\vec{x}',t) + (\nabla\cdot\vec{J}(\vec{x}',t)\frac{(n+1)R^n\vec{n}}{(n+2)(n+1)R^{n-1}} = \vec{J}(\vec{x}',t) + \nabla'\cdot\vec{J}(\vec{x}',t)\frac{\vec{R}}{n+2} \tag{76}$$

with this we may evaluate the integral involving the second term of $\vec{\xi}$

$$-\int d^3x' R^{n-1}\frac{\vec{R}}{n+2}\nabla'\cdot\vec{J} = \frac{1}{n+2}\int d^3x'(\vec{J}\cdot\nabla)R^{n-1}\vec{R}$$

$$= \frac{-1}{n+2}\int d^3x' R^{n-1}\left(\vec{J} + (n-1)\frac{\vec{J}\cdot\vec{R}}{R^2}\vec{R}\right) \tag{77}$$

So the whole integral gives:

$$\int d^3x' R^{n-1}\vec{\xi} = \int d^3x' R^{n-1} \left[\left(\frac{n+1}{n+2}\right) \vec{J}(\vec{x}', t) - \left(\frac{n-1}{n+2}\right) \frac{\vec{J}\cdot\vec{R}}{R^2}\vec{R} \right] \tag{78}$$

Introducing the expression for the current we get that:

$$\begin{aligned} \int d^3x' R^{n-1}\vec{\xi} &= \int d^3x' R^{n-1}\rho(\vec{x}', t)\vec{v}(t) \left[\left(\frac{n+1}{n+2}\right) - \left(\frac{n-1}{n+2}\right) \left(\frac{\vec{R}\cdot\vec{v}}{Rv}\right)^2 \right] \\ &= \int d^3x' R^{n-1}\rho(\vec{x}', t)\vec{v}(t) \left[\left(\frac{n+1}{n+2}\right) - \left(\frac{n-1}{n+2}\right) \left(\frac{1}{3}\right) \right] \\ &= \int d^3x' R^{n-1}\frac{2}{3}\rho(\vec{x}', t)\vec{v}(t) \end{aligned} \tag{79}$$

where we have replaced $\left(\frac{\vec{R}\cdot\vec{v}}{Rv}\right)^2$ by its average value 1/3. So replacing these integrals in the expression for the self force (74) we obtain that:

$$\vec{F}_{\text{self}} = \sum_{n=0}^{\infty} \frac{2(-1)^n}{3n!c^n} \frac{d^{n+1}\vec{v}}{dt^{n+1}} \int d^3x \int d^3x' \rho(\vec{x}) R^{n-1} \rho(\vec{x}') \tag{80}$$

so going again to the force law (65) we get the following expression for the external fields:

$$\vec{F}_{ext}(t) = m_0 \frac{d\vec{v}(t)}{dt} + \sum_{n=0}^{\infty} \frac{2(-1)^n}{3n!c^{n+2}} \frac{d^{n+1}\vec{v}(t)}{dt^{n+1}} \int d^3x \int d^3x' \rho(\vec{x}) R^{n-1} \rho(\vec{x}') \tag{81}$$

Now let's see how this equation behaves in the time Fourier space with:

$$\vec{v}(t) = \frac{1}{\sqrt{2\pi}} \int \tilde{\vec{v}}(\omega) e^{-i\omega t} d\omega \tag{82}$$

$$\begin{aligned} \tilde{\vec{F}}_{ext}(\omega) &= -i\omega m_0 \tilde{\vec{v}}(\omega) + \sum_{n=0}^{\infty} \frac{2(i\omega)^n}{3n!c^{n+2}} \tilde{\vec{v}}(\omega) \int d^3x \int d^3x' \rho(\vec{x}) R^{n-1} \rho(\vec{x}') \\ &= -i\omega \tilde{\vec{v}}(\omega) \left(m_0 + \sum_{n=0}^{\infty} \frac{2(i\omega)^n}{3n!c^{n+2}} \int d^3x \int d^3x' \rho(\vec{x}) R^{n-1} \rho(\vec{x}') \right) \end{aligned} \tag{83}$$

Now note that there is a Taylor expansion hidden in the last expression which is :

$$\sum_{n=0}^{\infty} \frac{(i\omega)^n}{n!c^n} R^{n-1} = e^{i\omega R/c}/R \tag{84}$$

so the expression simplifies to:

$$\tilde{\vec{F}}_{ext}(\omega) = -i\omega \tilde{\vec{v}}(\omega) \left(m_0 + \frac{2}{3c^2} \int d^3x \int d^3x' \rho(\vec{x}) \frac{e^{i\omega R/c}}{R} \rho(\vec{x}') \right) \tag{85}$$

Now if we identify the terms inside the parenthesis as a effective mass the equation takes the form

$$\tilde{\vec{F}}_{ext}(\omega) \quad = \quad -i\omega\tilde{\vec{v}}(\omega)M(\omega) \tag{86}$$

where the observable mass is given by

$$M(\omega) = m_0 + \frac{2}{3c^2}\int d^3x \int d^3x' \rho(\vec{x})\frac{e^{i\omega R/c}}{R}\rho(\vec{x}')\,. \tag{87}$$

At this point it is convenient to define the form factor f which is the Fourier transform of the charge density and characterizes the particle's structure, for a point particle it is 1.

$$\rho(x) = \frac{e}{(2\pi)^3}\int d^3k f(\vec{k})e^{i\vec{k}\cdot\vec{x}} \tag{88}$$

so introducing the form factor we get that the renormalized mass takes the form:

$$M(\omega) = m_0 + \frac{e^2}{3\pi^2c^2}\int d^3k\frac{|f(\vec{k})|^2}{k^2-(\omega/c)^2} \tag{89}$$

Here in this point it is clear that the observed mass is not solely due to the bare mass of the charged particle but that the energy of the fields take active part on it. If we perform the integral for a point particle form factor we get that the observed mass is:

$$M(\omega) = m(1 + i\omega\tau) \tag{90}$$

which depends on the frequency of the motion of the particle, therefore on the energy of the fields. Inserting this result in (86) we obtain the following equation of motion for the point particle in time Fourier space:

$$\tilde{\vec{F}}_{ext}(\omega) \quad = \quad -im\omega\tilde{\vec{v}}(\omega) + m\tau_e\omega^2\tilde{\vec{v}}(\omega) \tag{91}$$

So performing the inverse Fourier transform we get that:

$$\vec{F}_{ext}(t) \quad = \quad \dot{\vec{v}}(t) + m\tau_e\ddot{\vec{v}}(t) \tag{92}$$

which is the Abraham Lorentz equation already derived. So from this analysis it clear that the LAD equation corresponds to a point particle model thus maybe assuming a extended charge model the flaws in the equation will be alleviated.

4.3 Application of the LAD: Preacceleration and runaway solutions

Now in order to give some physical meaning to the equation derived by Abraham and Lorentz we will use it to solve the motion of a particle under various different forces. As a result we will see that the Abraham-Lorentz equation presents some unsatisfactory features which lead to unphysical situations. As a first example suppose that we have a particle under the influence of an external field so that but that at time $t = 0$ the field is suddenly turned off. Then according to the LAD equation (139) the equation of motion for times $t > 0$ is:

$$m\vec{a} = m\tau\dot{\vec{a}} \tag{93}$$

so taking the norm on both sides

$$|a| = \tau|\dot{a}| \tag{94}$$

The solution to this equation is simply

$$|a| = c_1 e^{t/\tau} \tag{95}$$

where c_1 is some positive constant. c_1 can't be zero because for time $t = 0$ it was under a force thus it had an nonzero acceleration. So we have arrived at the so called runaway solutions. It is clearly an unsatisfactory solution because it implies that once the particle emerges from the external field it begins to self accelerate to infinity which clearly implies a violation to the conservation of energy law. It is well known that if we impose that $a = 0$ for $t = -\infty$ (we will talk later on this condition)runaway solutions may be avoided, but then there arises another undesirable phenomenon that is the preacceleration , which is due to the fact that the LAD equation (139) is a third order one. Let's see this in the following example. Let's suppose that a particle is affected by an external force $\vec{f}_{ex}(t)$ for a given time interval say $0 \leq t \leq t_p$ for t_p a given time.

$$m\dot{\vec{v}} = \begin{cases} m\tau\ddot{\vec{v}}, & (t \leq 0) \\ \vec{f}_{ex}(t) + m\tau\ddot{\vec{v}}, & (0 \leq t \leq t_p) \\ m\tau\ddot{\vec{v}}, & (t \geq t_p) \end{cases} \tag{96}$$

Now imposing the asymptotic condition $a = 0$ for $t = -\infty$ we get the following solution:

$$\dot{\vec{v}} = \begin{cases} \frac{1}{m\tau}e^{t/\tau}\int_0^{t_p} e^{-t'/\tau}\vec{f}_{ex}(t')dt', & (t \leq 0) \\ \\ \frac{1}{m\tau}e^{t/\tau}\int_t^{t_p} e^{-t'/\tau}\vec{f}_{ex}(t')dt', & (0 \leq t \leq t_p) \\ \\ 0, & (t \geq t_p) \end{cases} \tag{97}$$

to get to this expression we have imposed continuity at $t = 0$ and $t = t_p$. So looking at this solution we see that for $t \leq 0$ the particle is accelerating even though that the particle is not being acted by any force, therefore there is a violation of causality. One may think that this problems are due to the non covariant nature of the Abraham-Lorentz equations but as we will see in the next two sections the relativistic generalization does not alleviate at all the problem, therefore the problem should be docked by different means.

5 Derivation of the Abraham-Lorentz-Dirac equation in covariant form.

The dynamics of the charged particle in which we are working on must be subject to certain conditions which ensures that the domain of action of our theory remains in the domain of applicability of classical electrodynamics and that situation we are working on actually corresponds to a physical one, otherwise we would end up using assumptions which would lead us to contradictions and inconsistencies. An example of such inconsistencies is the one that we already point out, that is trying to work with solid particles of radius R and then take the limit $R \to 0$, because according to the equation for the electric field produced by a charged sphere ($\vec{E} = \frac{2}{r^2}\hat{r}$), at the particles position we would have an infinite field, therefore an infinite energy. Therefore due to the particle-field dichotomy it is hopeless to try to describe classically in an exact manner the mechanism of interaction of the particles and the fields. Nevertheless we may characterize the motion of the particle independently of the model we use, as long as we do it under conditions that don't go so deeply in these interactions. An accelerated charge losses energy in form of radiation and since there must be a lower bound to its energy given by the rest energy of the particle (we will see in section 9 that the case of negative bare mass is unphysical), it is easily seen that the particle can't accelerate forever, therefore it is necessary to impose to the dynamics of the particle the so called asymptotic condition, which consist in assuming that the interaction between the different external fields and the particle decreases asymptotically, so that the particle begins and end's up at infinite in a free movement. This is expressed as:

$$\begin{aligned} \lim_{\tau\to\infty} a^\mu(\tau) &= 0 \\ \lim_{|\tau|\to-\infty} a^\mu(\tau) &= 0 \end{aligned} \tag{98}$$

This resembles to what we do in a quantum mechanics when we analyze a scattering process, there the exact interaction between the particle and the potential is unknown,but we are only interested in calculating the scattered wave in terms of the incident wave and we do it by placing the particle detector very far away from the potential, therefore the incident and scattered waves are considered a stationary asymptotic state.

In the motion of charged particles the asymptotic condition allow us to measure certain quantities as the rest energy and mass of the particle. Thus one can obtain the asymptotic momentum:

$$p^\mu_{in} = mv^\mu_{in} = \lim_{|\tau|\to-\infty} mv^\mu(\tau) \tag{99}$$

$$p^\mu_{out} = mv^\mu_{out} = \lim_{|\tau|\to+\infty} mv^\mu(\tau) \tag{100}$$

The asymptotic condition also implies that at infinity the only fields that surround the particle are its own Coulomb fields:

$$\lim_{|\tau|\to -\infty} F^{\mu\nu}_{particle} = F^{\mu\nu}_{coul,in} \tag{101}$$

$$\lim_{|\tau|\to +\infty} F^{\mu\nu}_{particle} = F^{\mu\nu}_{coul,out} \tag{102}$$

Next our task is to derive the equations of motion of a charged particle in covariant form, including the effects that its own fields might have on it . This will be done based upon the Maxwell's equations, the momentum conservation law and the asymptotic condition. In the general case the particle may interact with an incident radiation field which we denote by $F^{\mu\nu}_{in}$ which might be completely or partially electromagnetic, and with external forces of different kinds F^{μ}_{ext}.As this incident radiation fields interact with the particle they may interchange momentum and energy. Asymptotically at $\tau \to -\infty$ the only field present besides the particle velocity field is $F^{\mu\nu}_{in}$ (for instance a heat radiation field), but due to the radiation emitted by the particle during its period of acceleration the outcoming free field will be different from the incoming free field and asymptotically this will be the only free field :

$$F^{\mu\nu}_{out} = F^{\mu\nu}_{in} + F^{\mu\nu}_{rad} \tag{103}$$

Now we know that the fields produced by the charge satisfy the Maxwell's equation

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c} J^\nu \tag{104}$$

Then when we solve this equation with the condition that at $|\tau| \to -\infty$ the only free field present is $F^{\mu\nu}_{in}$ we find that the only possible solution is

$$F^{\mu\nu} = F^{\mu\nu}_{ret} + F^{\mu\nu}_{in} \tag{105}$$

with $F^{\mu\nu}_{ret}$ being the retarded potential given by (19), and with the condition that at $|\tau| \to +\infty$ the only free field present is $F^{\mu\nu}_{out}$ the only possible solution is

$$F^{\mu\nu} = F^{\mu\nu}_{adv} + F^{\mu\nu}_{out} \tag{106}$$

$F^{\mu\nu}_{adv}$ being the advanced potential given by equation (20) , then equating this two equations we obtain:

$$F^{\mu\nu}_{out} - F^{\mu\nu}_{in} = F^{\mu\nu}_{ret} - F^{\mu\nu}_{adv} \tag{107}$$

Now for future reference we define the next quantities:

$$F^{\mu\nu}_{+} \equiv \frac{1}{2}(F^{\mu\nu}_{ret} + F^{\mu\nu}_{adv}) \tag{108}$$

$$F^{\mu\nu}_{-} \equiv \frac{1}{2}(F^{\mu\nu}_{ret} - F^{\mu\nu}_{adv}) \tag{109}$$

$$\overline{F}^{\mu\nu} \equiv \frac{1}{2}(F^{\mu\nu}_{adv} + F^{\mu\nu}_{out}) \tag{110}$$

Now in a similar manner to what we did in deriving the relativistic Larmor equation for the radiation emission, we are going to calculate the amount of energy and momentum interchanged by the particle with the incoming electromagnetic field during the time interval $\tau_2 - \tau_1$.

In order to make this calculation we elaborate a surrounding tube of radius ϵ around the world line of the particle and compute the energy momentum four vector $P^{\mu\nu} = \int_1^2 \Theta^{\mu\nu}_{elm} d\sigma_\nu$ where $d\sigma_\nu$ is the surface of the tube, this surface is time-like because the world line of the particle moves in a time-like surface, it has a unit tangent space-like vector u^μ.

Now we present the basic line of reasoning that leads to the Lorenz-Abraham-Dirac equation for which we used the next relation which we will prove when we have already derived the equation of motion:

$$-\frac{1}{c}\int_1^2 \Theta^{\mu\nu}_{elm} d\sigma^3_\nu = \int_1^2 \left(\frac{dP^\mu_{Coul}}{d\tau} - \frac{e}{c}\overline{F}^{\mu\nu} v_\nu \right) d\tau \tag{111}$$

where P_{Coul} is the four-momentum due to the velocity field of the moving charge. If we do this integral for the whole world line of the particle we obtain that:

$$\int_{-\infty}^{-\infty} \left(\frac{dP^\mu_{Coul}}{d\tau} - \frac{e}{c}\overline{F}^{\mu\nu} v_\nu \right) d\tau = (P^\mu_{out} + P^\mu_{coul,out})_\infty - (P^\mu_{in} + P^\mu_{coul,in})_{-\infty} \tag{112}$$

This result is clearly a direct consequence of the asymptotic condition. From this we get that the change in the asymptotically 4-momentum vector of the free fields is given by

$$\frac{e}{c}\int_{-\infty}^{-\infty} \overline{F}^{\mu\nu} v_\nu d\tau = P^\mu_{in} - P^\mu_{out} \tag{113}$$

or equivalently

$$dP^\mu = -\frac{e}{c}\overline{F}^{\mu\nu} v_\nu d\tau \tag{114}$$

We see that this expression for the free fields is totally independent of the coulomb field, therefore is also independent of the radius ϵ of the tube surrounding the world line, and therefore the structureless features of the particle are not present here. This important separation of the electromagnetic energy field tensor into free field and coulomb field was made possible because as we will see both terms of the integrand of equation (111) are total differentials. Note here the similarities with section 4.2, where we separated the external force and the self force to analyze separately the effect of each one on the particles motion, here the free fields will be the responsible for the external force and the Coulomb field will be the ones producing the self force.

Now our next step is based on a very important cornerstone of classical dynamics, that is the momentum and energy conservation of the closed systems, which for our case is the system composed by the particle and the radiation fields, whose 4-momentums are $p^\mu = mv^\mu$ and P^μ , where as we said m corresponds to the effective mass, the momentum conservation law reads:

$$dp^\mu + dP^\mu = 0 \tag{115}$$

So combining this equation with equation ((114)) we obtain the equation for the total 4-force experienced by the particle:

$$ma^\mu = \overline{F}^\mu \equiv \frac{e}{c}\overline{F}^{\mu\nu} v_\nu \tag{116}$$

Now in order to finish our way to the LAD equation we must get involved with some algebra to obtain an expression of the four vector $\overline{F}^\mu$ in terms of the dynamical variables, that is the position and velocity of the particle. The procedure will be again to incorporate the expression for the fields produced by the particle given by the Maxwell equations in four vector form in the force law just obtained. First of all we note that the expression equations (17,19) for the retarded and advanced fields may be written as:

$$F^{\mu\nu}_{\substack{ret\\adv}}(x) = \pm \frac{e}{\rho}\frac{d}{d\tau}\left(\frac{v^\mu R^\nu - v^\nu R^\mu}{\rho}\right) \tag{117}$$

Now since we are interested in the interaction between the free field and the particle, we are interested in the interchanges of momentum and energy in the vicinity of the particle , thus using the same definitions for u^μ, ρ and R^μ as in section (2) we consider the behavior of the advanced and retarded potentials in in the limit $\rho \to 0$. For convenience we place the particle at $z^\mu(\tau_0) = z(0)$, then we expand the retarded and advanced positions around $z^\mu(0)$ in order to analyze the behavior $F^{\mu\nu}$ around the particle. Again this is done in view that the retarded and advanced time differ from the particles proper time in a time of the order of τ_e. Then the retarded position corresponding to the observation point x is

$$z^\mu(-\tau) = z^\mu - \tau v^\mu + \frac{\tau^2}{2}a^\mu - \frac{\tau^3}{6}\dot{a}^\mu + \cdots\cdot \tag{118}$$

a similar expansion can be made for the advanced position with $z(+\tau)$, so we find that the potentials may be written as

$$\begin{aligned} F^{\mu\nu}_{\substack{ret\\adv}}(x) &= \pm\frac{2e}{\rho(\mp\tau)}\frac{d}{d(\mp\tau)}\left(\frac{v^\mu(\mp\tau)R^\nu(\mp\tau) - v^\nu(\mp\tau)R^\mu(\mp\tau)}{\rho(\mp\tau)}\right) \\ &= \frac{-2e}{\rho(\mp\tau)}\frac{d}{d\tau}\left(\frac{v^\mu(\mp\tau)R^\nu(\mp\tau) - v^\nu(\mp\tau)R^\mu(\mp\tau)}{\rho(\mp\tau)}\right). \end{aligned}$$

The advanced and retarded 4-velocities are expressed as

$$v^\mu(\mp\tau) = v^\mu \mp \tau a^\mu + \frac{\tau^2}{2}\dot{a}^\mu + \cdots\cdots \tag{119}$$

now combining the expression for $z(\mp\tau)$ and $R^\mu(\tau) = \rho u^\mu$ we find that:

$$R^\mu(\mp\tau) = x^\mu - z^\mu(\mp\tau) = \rho u^\mu - \frac{\tau^2}{2}a^\mu \pm \frac{\tau^3}{6}\dot{a}^\mu + \cdots\cdot \tag{120}$$

Now we multiply the expressions for $v^\mu(\mp\tau)$ and $R^\mu(\mp\tau)$, and using the fact that $\rho_{\substack{ret\\adv}} = \mp v_\mu R^\mu_{\substack{ret\\adv}}/c$ we find an expression for $\rho(\mp\tau)$ to third order

$$\rho(\mp\tau) = \mp v_\mu(\mp\tau)R^\mu(\mp\tau) = \tau(1 + \rho a_\mu) \mp \frac{\rho\tau^2}{2}\dot{a}_u + \frac{\tau^3}{6}a^2 \tag{121}$$

where we have defined $a_u \equiv a_\lambda u^\lambda$, $\dot{a}_u \equiv \dot{a}_\lambda u^\lambda$, that is the component of the four vector in the u^λ direction since u is unitary. Also we used that $a^2 = a_\lambda a^\lambda$.In what follows be aware of not confounding the subindex μ with the subindex u .

Since R^μ is a light-like vector $R_\mu R^\mu = 0$, from this we have that:

$$\begin{aligned} R_\mu(\mp\tau)R^\mu(\mp\tau) &= \rho^2 u^\mu u_\mu \mp 2\tau\rho u_\mu v^\mu - \tau^2\rho a_\mu u^\mu \mp \frac{\tau^3\rho}{3}\dot{a}^\mu u_\mu \\ &+ \tau^2 v_\mu v^\mu \mp \tau^3 v_\mu a^\mu - \frac{\tau^4}{3}v_\mu\dot{a}^\mu + \frac{\tau^4}{4}a^2 = 0 \end{aligned} \tag{122}$$

In order to get ρ from the last equation we use the fact that $v^\mu v_\mu = -c^2 = -1$which implies $a^\mu v_\mu = 0$ and $-\dot{a}^\mu v_\mu = a^2$ so we get that

$$\rho^2 = \tau^2\left(1 + \rho a_u \mp \frac{\rho\tau}{3}\dot{a}_u + \frac{\tau^2}{12}a^2\right) \tag{123}$$

Now in the next steps we will use a series of approximations which are based in the fact that we are looking in the surroundings of the particle. The first one is to note that to first order $\rho = c\tau = \tau$ so that

$$\rho^2 = \tau^2\left[1 + \rho a_u + \frac{\rho^2}{3}\left(\frac{a^2}{4} \mp \dot{a}_u)\right)\right] \tag{124}$$

Now in the next steps we will use the binomial expansion

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots\cdot \tag{125}$$

and then suprime the terms which include τ in an order higher than 2. Thus we get

$$\left(\frac{\tau}{\rho}\right)^2 = 1 - \rho a_u + \rho^2 a_u^2 + \rho^2 a_u^2 - \frac{\rho^2}{3}(\frac{a^2}{4} \mp \dot{a}_u) \tag{126}$$

$$= \frac{1}{1+\rho a_u}\left[1 - \frac{\rho^2}{3}\left(\frac{a^2}{4} \mp \dot{a}_u\right)\right] \tag{127}$$

$$\frac{\tau}{\rho} = \frac{1 - [(\rho^2/6)(a^2/4 \mp \dot{a}_u)]}{\sqrt{1+\rho a_u}} \tag{128}$$

Now factorizing τ from equation (122) and again using the binomial expansion to invert this equation we find that

$$\frac{1}{\rho(\mp\tau)} = \frac{1}{\tau}\frac{1}{\rho a_u}(1 \pm \frac{1}{2}\rho\tau\dot{a}_u - \frac{1}{6}\tau^2 a^2) \tag{129}$$

For the next step we will use a notation rather estrange but very useful in simplifying : We define:

$$a^{[\mu}b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu \tag{130}$$

it has the following associative rule (take it as a definition)

$$(a^{[\mu} + b^{\nu[})c^{\lambda]} = a^{[\mu}c^{\lambda]} + b^{[\nu}c^{\lambda]} \tag{131}$$

in this notation $F^{\mu\nu}$ reads

$$F^{\mu\nu}_{\substack{ret\\adv}}(x) = \frac{-e}{\rho(\mp\tau)}\frac{d}{d\tau}\left(\frac{v^{[\mu}(\mp\tau)R^{\nu]}(\mp\tau)}{\rho(\mp\tau)}\right) \tag{132}$$

Now with the above equations we calculate this tensor, using the expansions for $R^\mu(\mp\tau)$ and $v^\mu(\mp\tau)$ we get that(have on mind relation (131))

$$\frac{v^{[\mu}(\mp\tau)R^{\nu]}(\mp\tau)}{\rho(\mp\tau)} = \frac{1}{\rho(\mp\tau)}\left(v^{[\mu} \mp \tau a^{[\mu} + \frac{\tau^2}{2}\dot{a}^{[\mu}\right)\left(\rho u^{\nu]} - \frac{\tau^2}{2}a^{\nu]} \pm \frac{\tau^3}{6}\dot{a}^{\nu]}\right) \tag{133}$$

so ignoring terms of order higher than second and inserting the expression for $\frac{1}{\rho(\mp\tau)}$ we get that

$$\begin{aligned} \frac{v^{[\mu}(\mp\tau)R^{\nu]}(\mp\tau)}{\rho(\mp\tau)} = \frac{1}{1+\rho a_u} \quad (& \quad \frac{\rho}{\tau}v^{[\mu}u^{\nu]} \mp \rho a^{[\mu}u^{\nu]} + \frac{\tau}{2}v^{[\mu}a^{\nu]} \pm \frac{1}{2}\rho^2\dot{a}_u v^{[\mu}u^{\nu]} \\ - & \quad \frac{1}{6}\rho\tau a^2 v^{[\mu}u^{\nu]} \pm \frac{\tau^2}{3}\dot{a}^{[\mu}v^{\nu]} + \frac{\rho\tau}{2}\dot{a}^{[\mu}u^{\nu]}) \end{aligned} \tag{134}$$

Differentiating we get

$$\frac{d}{d\tau}\frac{v^{[\mu}(\mp\tau)R^{\nu]}(\mp\tau)}{\rho(\mp\tau)} = \frac{1}{1+\rho a_u}(\frac{-\rho}{\tau^2}v^{[\mu}u^{\nu]} + \frac{1}{2}v^{[\mu}a^{\nu]} - \frac{1}{6}\rho a^2 v^{[\mu}u^{\nu]} \pm \frac{2\tau}{3}\dot{a}^{[\mu}v^{\nu]} + \frac{\rho}{2})$$

inserting this in the field strength tensor we get that it is:

$$F^{\mu\nu}_{\substack{ret\\adv}}(x) = \frac{2e}{(1+\rho a_u)^2}\left(\frac{-\rho}{\tau^3}v^{[\mu}u^{\nu]} + \frac{1}{2\tau}v^{[\mu}a^{\nu]} \mp \frac{\rho^2}{2\tau^2}\dot{a}_u v^{[\mu}u^{\nu]} \pm \frac{2}{3}\dot{a}^{[\mu}v^{\nu]} + \frac{\rho}{2\tau}\dot{a}^{[\mu}u^{\nu]}\right) \tag{135}$$

Now we can use the expression that we derived above (eq (129)) to replace τ by ρ and obtain

$$F^{\mu\nu}_{\substack{ret\\adv}}(x) = \frac{2e}{\sqrt{1+\rho a_u}}\left(\frac{1}{\rho^2}v^{[\mu}u^{\nu]} - \frac{1}{2\rho}v^{[\mu}a^{\nu]} + \frac{1}{2}a_u v^{[\mu}a^{\nu]} + \frac{a^2}{8}v^{[\mu}u^{\nu]} - \frac{1}{2}\dot{a}^{[\mu}u^{\nu]} \mp \frac{2}{3}\dot{a}^{[\mu}v^{\nu]}\right) \tag{136}$$

So what is more interesting about the last equation is that in the limit $\rho \to 0$ the advanced and retarded potentials differ only by the last term:

$$F^{\mu\nu}_{-}(z) \equiv \frac{1}{2}(F^{\mu\nu}_{ret} - F^{\mu\nu}_{adv}(z)) = \frac{2e}{3c^4}\dot{a}^{[\mu}v^{\nu]} \tag{137}$$

Now we define the quantity Γ as:

$$\begin{aligned}\Gamma^{\mu} \equiv \overline{F}^{\mu} - F^{\mu}_{in} \equiv \frac{e}{c}(\overline{F}^{\mu\nu} - F^{\mu\nu}_{in})v_{\nu} &= \frac{e}{c}\frac{1}{2}(F^{\mu\nu}_{out} - F^{\mu\nu}_{in})v_{\nu} = \frac{e}{c}F^{\mu\nu}_{-}v_{\nu} \\ &= -\frac{2e^2}{3c^5}(\dot{a}^{\mu}v^{\nu}v_{\nu} - v^{\mu}a^{\lambda}a_{\lambda}).\end{aligned} \tag{138}$$

So we found that $\overline{F}^{\mu} \equiv F^{\mu}_{in} + \Gamma^{\mu}$ so inserting this in the equation (68) we arrive at the famous Lorenz-Dirac-Equation for the case that no external field is present:

$$ma^{\mu} = \frac{e}{c}F^{\mu\nu}_{in}v_{\nu} + \frac{2}{3}\frac{e^2}{c^3}(\dot{a}^{\mu} - v^{\mu}a^{\lambda}a_{\lambda}). \tag{139}$$

For the case that an external field is present the conservation of momentum law reads:

$$dp^{\mu} + dP^{\mu} = F^{\mu}_{ext}d\tau \tag{140}$$

This comes not from any electromagnetic law but rather from Newton's law, thus the force equation becomes

For the case that an external field is present the force law reads:

$$ma^{\mu} = \overline{F}^{\mu} - F^{\mu}_{ext}$$

which for the case that the incoming free field is zero ($F^{\mu\nu}_{in} = 0$)gives

$$ma^{\mu} = F^{\mu}_{ext} + \frac{2}{3}\frac{e^2}{c^3}(\dot{a}^{\mu} - v^{\mu}a^{\lambda}a_{\lambda}) \tag{141}$$

Which is the famous Abraham Lorentz Dirac equation for the motion of a charged particle. Note that in its derivation we never used a form factor or anything like that, instead we integrated around a tube of infinitesimal radius ϵ around the particles position to calculate the effect of the fields, therefore this equation also corresponds to the motion of a point particle model .As we can see is a third order equation thus as we will see it exhibits the problems of runanway solutions and preacceleration already mentioned.

5.1 Proof of equation (111)

First of all the three dimensional differential of the surface is:

$$d^3\sigma^\mu = u^\mu d^3\sigma \tag{142}$$

where $d^3\sigma$ must have the following form:

$$d^3\sigma = \alpha d\tau\rho^2 d\Omega \tag{143}$$

where $\rho^2 d\Omega$ is the surface element of a solid sphere α a constant which we will determine and $\alpha d\tau$ is a time-like vector parallel to v^μ. In order to find this α consider a point x^μ on the surface of the tube. If the interval of time $\tau_2 - \tau_1$ is very small, any vector from the particles position $z(\tau)$ to any point of the tube $x^\mu - z^\mu = \rho$ will be perpendicular to v^μ .

$$\frac{d}{d\tau}[(x^\mu - z^\mu)v_\mu] = 0 \tag{144}$$

Then

$$(\frac{dx^\mu}{d\tau} - \frac{dz^\mu}{d\tau})v_\mu + \rho u^\mu a_\mu = 0 \tag{145}$$

So we get

$$v_\mu dx^\mu = -(1 + \rho a_\mu)d\tau \tag{146}$$

Hence we find that $\alpha = 1 + pa_\mu$

$$d^3\sigma = (1 + pa_\mu)d\tau\rho^2 d\Omega \tag{147}$$

Then our integral takes the form

$$-\int_1^2 \Theta^{\mu\nu}_{elm} d\sigma^3_\nu = -\int_{\tau_1}^{\tau_2} d\tau \int d\Omega\Theta^{\mu\nu}u_\nu\rho^2(1 + \rho a_\mu) \tag{148}$$

So now we must insert in this equation the value of $\Theta^{\mu\nu}$ in terms of $F^{\mu\nu}$ as given by equation (21) and use the solution for $F^{\mu\nu}$ we get that:

$$\begin{aligned} F^{\mu\nu} = F^{\mu\nu}_{in} + F^{\mu\nu}_{ret} = F^{\mu\nu}_{in} + F^{\mu\nu}_{+} + F^{\mu\nu}_{-} &= F^{\mu\nu}_{in} + F^{\mu\nu}_{+} + \frac{1}{2}(F^{\mu\nu}_{ret} - F^{\mu\nu}_{adv}) \\ &= \overline{F}^{\mu\nu} + F^{\mu\nu}_{+} \end{aligned}$$

Therefore

$$\Theta^{\mu\nu}_{elm}u_\nu = \frac{1}{4\pi}(F^{\mu\alpha}F^{\nu}_{\alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})u_\nu \tag{149}$$
$$= \frac{1}{4\pi}((\overline{F}^{\mu\alpha} + F^{\mu\alpha}_{+})(\overline{F}^{\nu}_{\alpha} + F^{\nu}_{\alpha+}) + \frac{1}{4}\eta^{\mu\nu}(\overline{F}_{\alpha\beta} + F_{\alpha\beta+})(\overline{F}^{\alpha\beta} + F^{\alpha\beta}_{+}))u_\nu$$

Clearly $\overline{F}^{\mu\nu}$ will be provided from the asymptotic condition but $F^{\mu\nu}_{+}$ must be calculated from the equation (135) and the definition $F^{\mu\nu}_{+} \equiv \frac{1}{2}(F^{\mu\nu}_{ret} + F^{\mu\nu}_{adv})$. Since we are working in the limit $\rho \to 0$ we will only keep terms which include $\frac{1}{\rho^n}$ in order $(n \geq 2)$ because the other terms will be neglible compared to these. Thus no term which includes only $\overline{F}^{\mu\nu}$ will survive. Now lets calculate the result from the terms including only $F^{\mu\nu}_{+}$.

first

$$F^{\mu\nu}_{+}(x) = \frac{2e}{\sqrt{1+\rho a_u}}\left(\frac{1}{\rho^2}v^{[\mu}u^{\nu]} - \frac{1}{2\rho}v^{[\mu}a^{\nu]} + \frac{1}{2}a_u v^{[\mu}a^{\nu]} + \frac{a^2}{8}v^{[\mu}u^{\nu]} - \frac{1}{2}\dot{a}^{[\mu}u^{\nu]}\right) \tag{150}$$

This calculation is greatly simplified by the facts $u_\lambda v^\lambda = 0$, $a_\lambda v^\lambda = 0$ and $a_\lambda a^\lambda = \dot{a}_\lambda v^\lambda$

$$F^{\mu\alpha}_{+}F^{\nu}_{\alpha+}u_\nu = \frac{e^2}{1+\rho a_u}\left(\frac{1}{\rho^2}v^{[\mu}u^{\alpha]} - \frac{1}{2\rho}v^{[\mu}a^{\alpha]} + \frac{1}{2}a_u v^{[\mu}a^{\alpha]} + \frac{a^2}{8}v^{[\mu}u^{\alpha]} - \frac{1}{2}\dot{a}^{[\mu}u^{\alpha]}\right)$$
$$* \left(\frac{1}{\rho^2}v_{[\alpha}u^{\nu]} - \frac{1}{2\rho}v_{[\alpha}a^{\nu]} + \frac{1}{2}a_u v_{[\alpha}a^{\nu]} + \frac{a^2}{8}v_{[\alpha}u^{\nu]} - \frac{1}{2}\dot{a}_{[\alpha}u^{\nu]}\right)u_\nu \tag{151}$$
$$= \frac{e^2}{1+\rho a_u}(\frac{1}{\rho^4}u^\mu - \frac{1}{2\rho^3}u^\mu a_u + \frac{1}{2\rho^2}u^\mu a_u + \frac{a^2}{8\rho^2}u^\mu - \frac{1}{2\rho^2}v^\mu \dot{a}_u + \frac{1}{2\rho^2}v^\mu \dot{a}_u$$
$$+ \frac{1}{2\rho^2}u^\mu a^2 - \frac{1}{2\rho^3}a^\mu + \frac{1}{4\rho^2}a^\mu a_u - \frac{1}{\rho^2}a^\mu a_u + \frac{1}{8\rho^2}u^\mu a^2 - \frac{1}{2\rho^2}u^\mu a^2)$$
$$= \frac{e^2}{1+\rho a_u}((\frac{1}{\rho^4} + \frac{1}{4\rho^2}u^\mu a^2)u^\mu - (\frac{1}{2\rho^3} - \frac{3}{4\rho^2}a_u)a^\mu)$$

The contribution from the crossed terms is $(e/4\pi\rho^2)\overline{F}^{\mu\nu}v_\nu$. We also need to calculate $F^{\alpha\beta}_{+}F_{\alpha\beta+}$ but as we will see in a minute it will vanish a cause of its direct dependence of u^μ, and it will only help us to get the correct result for $\Theta^{\mu\nu}_{elm}u_\nu$ which turns out to be :

$$\Theta^{\mu\nu}_{elm}u_\nu = \frac{e/4\pi}{1+\rho a_u}\left(e(\frac{1}{2\rho^4} - \frac{1}{2\rho^2}a^2)u^\mu - e(\frac{1}{2\rho^3} - \frac{3}{4\rho^2}a_u)a^\mu + \frac{1}{\rho^2}\overline{F}^{\mu\nu}v_\nu\right) \tag{152}$$

Now we insert this in equation (148) but we must remember that $\alpha\, d\tau$ was a time-like vector parallel to v^μ, consequently the things in the integral that involve u^μ or a_u will cancel, then we have that

$$-\int_1^2 \Theta^{\mu\nu}_{elm}d\sigma^3_\nu = \int_{\tau_1}^{\tau_2} d\tau \int d\Omega \frac{1}{1+\rho a_u}\left(-\frac{e^2/4\pi}{2\rho^3}a^\mu + \frac{e}{4\pi\rho^2}\overline{F}^{\mu\nu}v_\nu\right)\rho^2(1+\rho a_\mu) \tag{153}$$
$$= \frac{e^2}{4\pi}\int_1^2\int \frac{d\Omega}{d\rho}a^\mu d\tau - \frac{1}{4\pi}\int_1^2\int \overline{F}^\mu d\Omega d\tau$$

but looking carefully to the first term we see that since $a^\mu = dv^\mu/d\tau$ we may write this term as

$$\int_1^2 \frac{d}{d\tau}\left(\frac{e^2}{2\rho}v^\mu\right)d\tau = \frac{dP^\mu_{coul}}{d\tau}, \tag{154}$$

where we write P^μ_{coul} because we identify this expression with the 4-momentum carried along by the coulomb field produced by the charged particle.Thus if we replace this in the preceding equation we have proven the equation :

$$-\int_1^2 \Theta^{\mu\nu}_{elm} d\sigma_\nu = \int_1^2 \left(\frac{dP^\mu_{coul}}{d\tau} - \overline{F}^\mu\right)d\tau. \tag{155}$$

where the integration over $d\Omega$ has been trivially performed. So with this expression the demonstration that the effect of the self fields on the motion of a particle (in the point particle model) is is given by the LAD equation is fully completed and we may proceed to try to eliminate the flaws of the equation.

6 Incorporation of the asymptotic conditions to the LAD equation.

Now in order to obtain a true equation of motion it is necessary to incorporate the asymptotic condition to the LAD equation.

We define:

$$K^\mu(\tau) \equiv F^\mu_{in} + F^\mu_{ext} - \frac{1}{c^2}\Re v^\nu \tag{156}$$

where $\Re$ is the total radiation rate derived in section 2 and given by equation (39).

Then the LAD equation may be written as:

$$m(a^\mu - \tau_e \dot{a}^\mu) = K^\mu. \tag{157}$$

where $\tau_e = 2e^2/mc^3$ as already defined. This equation is easy to integrate. Multiplying by $e^{\frac{-\tau}{\tau_e}}$ we get that:

$$-\frac{d}{d\tau}(e^{\frac{-\tau}{\tau_e}}a^\mu(\tau)) = \frac{1}{m\tau_e}e^{\frac{-\tau}{\tau_e}}K^\mu. \tag{158}$$

now we integrate this equation from τ to ∞ and we get

$$a^\mu(\tau)e^{\frac{-\tau}{\tau_e}} - \lim_{\tau'\to\infty} a^\mu(\tau')e^{\frac{-\tau'}{\tau_e}} = \frac{1}{m\tau_e}\int_\tau^\infty e^{\frac{-\tau'}{\tau_e}}K^\mu(\tau')d\tau'. \tag{159}$$

By the asymptotic condition we know that :

$$\lim_{\tau' \to \infty} a^{\mu}(\tau') e^{\frac{-\tau'}{\tau_e}} = 0 \tag{160}$$

so we get that

$$a^{\mu}(\tau) = \frac{e^{\frac{\tau}{\tau_e}}}{m\tau_e} \int_{\tau}^{\infty} e^{\frac{-\tau'}{\tau_e}} K^{\mu}(\tau') d\tau'. \tag{161}$$

but the asymptotic condition also implies that

$$\lim_{|\tau| \to \infty} \left[\frac{e^{\frac{\tau}{\tau_e}}}{m\tau_e} \int_{\tau}^{\infty} e^{\frac{-\tau'}{\tau_e}} K^{\mu}(\tau') d\tau' \right] = 0 \tag{162}$$

thus we get the following integral equation for the motion of the particle

$$a^{\mu}(\tau) = \int_{\tau}^{\infty} e^{\frac{\tau-\tau'}{\tau_e}} (\frac{1}{m\tau_e}(F^{\mu}_{in} + F^{\mu}_{ext}) - \frac{1}{c^2} a^{\lambda}(\tau') a_{\lambda}(\tau') v^{\nu}(\tau')) d\tau'. \tag{163}$$

Now performing another integration in τ' from τ to ∞ we get the integral equation for the momentum:

$$p^{\mu}(\tau) = p^{\mu}_{in} + \frac{1}{\tau_e} \int_{-\infty}^{\tau} e^{\frac{\tau'}{\tau_e}} d\tau' \int_{\tau'}^{\infty} e^{\frac{\tau''}{\tau_e}} K^{\mu}(\tau'') dt'' \tag{164}$$

$$p^{\mu}(\tau) = p^{\mu}_{out} + \frac{1}{\tau_e} \int_{\tau}^{\infty} e^{\frac{\tau'}{\tau_e}} d\tau' \int_{\tau'}^{\infty} e^{\frac{\tau''}{\tau_e}} K^{\mu}(\tau'') dt'' \tag{165}$$

with

$$\begin{aligned} p^{\mu}_{in} &= m v^{\mu}_{in} = \lim_{|\tau| \to -\infty} m v^{\mu}(\tau) \\ p^{\mu}_{out} &= m v^{\mu}_{out} = \lim_{|\tau| \to +\infty} m v^{\mu}(\tau) \end{aligned} \tag{166}$$

so the former equations full fills the asymptotic conditions, in fact these conditions are necessary for the existence of the momentum p^{μ} thus these equations are truly the equation of motion of the point particle, even though they don't determine completely the motion of the particle.

To make a comparison with Newton's equations we introduce the next variable :

$$\alpha \equiv \frac{\tau' - \tau}{\tau_e} \tag{167}$$

then equation (161) becomes

$$ma^\mu(\tau) = \int_0^\infty K^\mu(\tau + \alpha\tau_e)e^{-\alpha}d\alpha. \tag{168}$$

This equation with the asymptotic condition is equivalent to the equations (164) and (165), but it largely differ from the relativistic force equation:

$$ma^\mu(\tau) = \frac{e}{c}\overline{F}^{\mu\nu}(z)v_\nu(\tau) \tag{169}$$

One of the most concerning difference is that the force on the relativistic Newton equation depends only upon z^μ and v^μ whereas K^μ depends on z^μ, v^μ and a^μ, this means that in the integrodifferential equation (168) the acceleration at certain time τ depends on the forces in future times up to a time$\tau + \tau_e$, in other words in order to describe the dynamics of the system at time τ one has to specify its position $z^\mu(t)$ and velocities $v^\mu(t)$ in future times up to $t = \tau + \tau_e$, which clearly constitutes a violation of the principle of causality during the short interval of time τ_e. So in conclusion the motion has a kind of memory function which remembers the conditions of the dynamics for times of the order of τ_e.

Looking at equation equation (168) it is easy to realize that K^μ is responsible for the acceleration of the particle. The parameter $\tau_0 = \frac{2e^2}{3mc^3}$ appearing in this equations is a very small parameter (for electrons is $0,62 \times 10^{-23}s$) thus we can make the approximation

$$ma^\mu(\tau) = \int_0^\infty K^\mu(\tau + \alpha\tau_e)e^{-\alpha}d\alpha \simeq \int_0^\infty K^\mu(\tau)e^{-\alpha}d\alpha = K^\mu(\tau) \tag{170}$$

Thus under this approximation the Lorentz-Dirac equation takes the form

$$ma^\mu(\tau) = F_{in}^{\mu\nu} + F_{ext}^{\mu\nu} - \frac{1}{c^2}\Re v^\mu \tag{171}$$

which differ from the Lorentz-Dirac equation only by the term

$$\frac{2}{3}\frac{e^2}{c^3}\dot{a}^\mu \tag{172}$$

which is called the Schott term. It is the presence of this term on the LAD equation which causes that $a^\mu(\tau)$ depends not only on the force at τ but also at times τ_e later. But anyways without it, that is in the approximation made above equation (170), there still a violation of causality, just that it is of very small magnitude. In equation (168) we can see that the acceleration at τ depends on the effective force present at the slightly different future time $\tau + \xi\tau_e$ (with ξ of the order of 1). This means that if we have a particle at rest but then suddenly at time τ_i we turn on a force, the particle would feel the force at some time $\tau_i - \tau_e$ before the force was turned on. But to detect experimentally some effect that includes an interval of time of the order of $0,6 \times 10^{-23}s$ would required an apparatus of such an accuracy that its description would go into the quantum mechanical domain. If we include the Schott term we would have to also make a measurement with the same accuracy, thus the same reasoning applies. In conclusion the LAD equation in principle implies violation of causality but over a such small time that there is little hope to ever observe experimentally such effects, this is the reason why most of the electromagnetism is developed without accounting for radiation reaction effects.

But pre-acceleration is not the only undesirable feature present in the covariant version of LAD equation, so much as its covariant counterpart it also presents runaway solutions. Note that when there is no external force present the radiation reaction term in LAD equation (141) stills survives, which implies that even though there is no external force acting on the particle it is radiating which is a very mischievous situation because it is widely accepted that radiation only occurs for particles under the influence of an external force as we already saw in section 2. Also in this same point it is illogical that a particle losing energy through radiation starts to accelerate without any external system transmits energy to it. Therefore we conclude that both the covariant and the noncovariant version of LAD present the very unsatisfactory features of preacceleration and runaway solutions which makes necessary to perform some changes on this equations in order to understand the real effects of the radiation back reaction force on the dynamic of the particle.

7 Landau Lifshitz equation.

In the third volume of the famous collection of books in theoretical physics written by Landau and Lifshitz there is a strange derivation of the non covariant LAD equation (61) in which the effect of the retarded potentials of a system of point particles on the dynamics of these particles was analyzed by a method very similar to the one employed in section 4.2. At the end in order to determine the effects of the self fields on a single particle he makes the limit in which the whole system is composed of a single particle and arrives again at the LAD equation. Now we are going to present the line of reasoning which leads from the LAD equation to the second order equation known as the LL equation.

For a point charge no radiation can be emitted, without the presence of an external force.

The radiation reaction equation is a correction of order τ_e to the Lorentz force law:

$$m_0 \dot{v}^\mu = F^\mu_{ext}$$

Thus we may replace these expression in the LAD equation and obtain an equation with an error of the order of τ_e:

$$m_0 \dot{v}^\mu = F^\mu_{ext} + \frac{2}{3}\frac{e^2}{m_0}(\eta^{\mu\alpha} + v^\mu v^\alpha)\frac{d}{d\tau}F^{ext}_\alpha \tag{173}$$

where $\eta^{\mu\nu}$ is the metric as already defined

$$m_0\dot{v}^\mu = eF^{\mu\alpha}v_\alpha + \frac{2}{3}\frac{e^3}{m_0}(\eta^{\mu\alpha} + v^\mu v^\alpha)\frac{d}{d\tau}F^{ext}_\alpha$$

Now we write:

$$\frac{d}{d\tau} = \frac{\partial x^\beta}{\partial\tau}\frac{d}{\partial x_\beta}$$

The external force is assumed to be the Lorentz force acting on the particle.

$$\begin{aligned}
\eta^{\mu\alpha}\frac{d}{d\tau}F^{ext}_\alpha &= \frac{dx^\beta}{d\tau}\frac{\partial}{\partial x_\beta}eF^\alpha_{ext} = v^\beta\partial_\beta(eF^{\mu\alpha}v_\alpha) \\
&= ev^\beta(v^\beta\partial_\beta F^{\mu\alpha})v_\alpha + eF^{\mu\alpha}v^\beta\partial_\beta v_\alpha \\
&= ev^\beta(v^\beta\partial_\beta F^{\mu\alpha})v_\alpha + \frac{e}{m_0}F^{\mu\alpha}F_{\alpha\beta}v^\beta
\end{aligned}$$

so the equation of motion is:

$$m_0\dot{v}^\mu = eF^{\mu\alpha}v_\alpha + \frac{2}{3}\frac{e^3}{m_0}(v^\beta\partial_\beta F^{\beta\alpha}v_\alpha + \frac{e}{m_0}F^{\mu\alpha}F_{\alpha\beta}v^\beta + \frac{e}{m_0}v^\mu v_\alpha F_{\alpha\beta}F^{\beta\gamma}v^\gamma) \tag{174}$$

which is the same equation Landau and Lifshitz wrote 40 years before the publication of Rohrlich, Spohn and O'Connell papers. But as Landau and Lifshitz point out due to the approximation made, this equation is only valid if the damping force (that is the radiation reaction) is small with the force exerted for an external field. For example for a periodic motion with period (ω) for the external force to be smaller than the damping force it is required that:

$$\omega \ll \frac{mc^3}{e^2} = \tau_e^{-1} \tag{175}$$

In the following section we intend to clarify under what conditions is this equation appropriate to describe the motion of charged particles and to demonstrate that in the classical limit it is indeed the correct equation of motion for the particles.

8 Spohn's method

As we have seen the major problem to the solution proposed by Abraham, Lorentz and Dirac to the radiation reaction problem is that in some cases it lead us to unstable solutions for the motion of the particle in which the particle begins to self accelerate to infinity. But this happens only under certain initial conditions for the dynamical variables of the particle.

Therefore the work of Spohn is focused on demonstrating that under a certain set of initial conditions the LAD equation leads to the already mentioned Landau-Lifshitz equation with an error of the order of a small parameter ε which happens to be of the same order of the error obtained deriving LAD, and that indeed these initial conditions correspond to the physically acceptable ones. Spohn point of view is called geometrical because his main point is that the physical acceptable solutions describe a submanifold $\mathcal{C}_\varepsilon$ of $\mathbb{R}^3 \times \mathbb{V}$ where $\mathbb{R}^3$ is the position space and $\mathbb{V}$ is the velocity space, that is the tangent space of $\mathbb{R}^3$. He demonstrates that if some dynamical system lies in $\mathcal{C}_\varepsilon$ it will remain there. The general method he uses is called singular perturbation theory because he begins by finding the solutions to the Lorentz force equation without radiation reaction and then he studies how the manifold formed by this solutions deform when the radiation corrections are included.

8.1 Setting of the problem

We consider a charged particle with position $q(t) \in \mathbb{R}^3$, with velocity $v(t) \in \mathbb{V}^3 = \{\vec{v} : |\vec{v}| < c\}$ and with charge distribution ρ such that:

$$e = \int d^3x \rho(x) \tag{176}$$

which we assume to be smooth, that is $\rho \in \mathcal{C}^\infty$. As the particle moves it is coupled to an electromagnetic field $\vec{E}(x)$ and $\vec{B}(x)$ which is in part due to an external electromagnetic field and in part to a self electromagnetic field. Therefore we have the phase space $\mathcal{M} = \mathbb{R}^3 \times \mathbb{V}$ for the dynamical variables of the particle and the more general phase space $Y = (\vec{E}(x), \vec{B}(x), \vec{q}, \vec{v})$ which includes the fields, so that the fields, the velocity and the position constitute the complete set of the dynamical variables of the system. The object formed by the moving particle and its co moving field it's call by Spohn the charge soliton, (for unknown reasons to me) and it is denoted by $S_{q,v}(t) = (\vec{E}_{\vec{v}}(\vec{x} - \vec{q}), \vec{B}_{\vec{v}}(\vec{x} - \vec{q}), \vec{q}, \vec{v})$, the subindex q, v represents where is the center of the soliton a what is its velocity.$S_{q,v}(t)$ represents a set of physical values for the dynamical variables hence it represents also a set of solutions to the Maxwell's equations for the fields produced by the particle.

The Lagrangian of the system looks like:

$$\begin{aligned} L &= -m_{\mathrm{b}}(1 - \dot{\vec{q}}^2)^{1/2} - (\phi_{\mathrm{ex}} + \phi - \dot{\vec{q}} \cdot \vec{A}_{\mathrm{ex}} - \dot{\vec{q}} \cdot \vec{A}) * \rho(\vec{q}) \\ &\quad + \frac{1}{2} \int d^3x \big[(\nabla\phi + \partial_t \vec{A})^2 - (\nabla \times \vec{A})^2\big] . \end{aligned} \tag{177}$$

where the asterisk $*$ denotes convolution and with m_b being the bare mass and ϕ and $\vec{A}$ the potentials which produce the fields. This Langrangian determines the energy of the system :

$$\mathcal{E}(\vec{E}, \vec{B}, \vec{q}, \vec{v}) = m_{\mathrm{b}}\gamma(\vec{v}) + e\phi_{\mathrm{ex}} * \rho\,(\vec{q}) + \frac{1}{2} \int d^3x \big(\vec{E}(\vec{x})^2 + \vec{B}(\vec{x})^2\big) \tag{178}$$

which is conserved as the particle moves. This is because although the particle losses energy through radiation it is transferred to the fields who's energy is also counted in the Lagrangian. The mechanical momentum of the particle is given by:

$$m_{\mathrm{b}}\gamma\vec{v} \tag{179}$$

and the momentum of the fields is:

$$\mathcal{P}_f = \int d^3x\big(\vec{E}(\vec{x}) \times \vec{B}(\vec{x})\big)\,. \tag{180}$$

So adding this two momentums we have the total momentum:

$$\mathcal{P} = m_{\mathrm{b}}\gamma\vec{v} + \mathcal{P}_f \tag{181}$$

which by the *Nöether* theorem is a conserved quantity (the Lagrangian is invariant under spatial translations). So now if we set the momentum $\mathcal{P}$ fixed and try to minimize the energy, we would obtain a set of solutions for the Maxwell equations (183)

$$S_{\vec{q},\vec{v}}(t) = (\vec{E}_{\vec{v}}(\vec{x} - \vec{q} - \vec{v}t),\ \vec{B}_{\vec{v}}(\vec{x} - \vec{q} - \vec{v}t), \vec{q} + \vec{v}t, \vec{v}) \tag{182}$$

which would represent a particle moving at constant velocity $\vec{v}$. Although this seems unuseful it will be quite important later on. It is important to note that in all the analysis of this section the velocity of light will have value equal to one, that is $c = 1$.

8.2 Coupling of Maxwell and Lorentz force equations for the Abraham model.

The self force problem arises when one tries to understand the effect of the radiation produced by a particle on its own movement. Since radiation is described by Maxwell equations and the motion of a particle is described by the Lorentz force Law an obvious beginning step is to introduce Maxwell in Lorentz as we already did noncovariantly in section (4.2).

In Fourier space the first two Maxwell equations read:

$$\begin{aligned} \partial_t\hat{\vec{B}}(\vec{k},t) &= -i\vec{k} \times \hat{\vec{E}}(\vec{k},t)\,, \\ \partial_t\hat{\vec{E}}(\vec{k},t) &= i\vec{k} \times \hat{\vec{B}}(\vec{k},t) - \hat{\vec{j}}(\vec{k},t) \end{aligned} \tag{183}$$

and the other two are:

$$\begin{aligned} i\vec{k} \cdot \hat{\vec{E}}(\vec{k},t) &= \hat{\rho}(\vec{k},t) \\ i\vec{k} \cdot \hat{\vec{B}}(\vec{k},t) &= 0 \end{aligned} \tag{184}$$

The solution to this equation when there is no currents or charge density (that is the homogeneous case is:)

$$\hat{\vec{E}}(\vec{k},t) = \hat{\vec{E}}(\vec{k},0)\cos|\vec{k}|t + (1-\cos|\vec{k}|t)\frac{1}{|\vec{k}|^2}[\vec{k}\cdot\hat{\vec{E}}(\vec{k},0)]\vec{k} + (\frac{1}{|\vec{k}|}\sin|\vec{k}|t)i\vec{k}\times\hat{\vec{B}}(\vec{k},0) \tag{185}$$

$$\hat{\vec{B}}(\vec{k},t) = \hat{\vec{B}}(\vec{k},0)\cos|\vec{k}|t + (1-\cos|\vec{k}|t)\frac{1}{|\vec{k}|^2}[\vec{k}\cdot\hat{\vec{B}}(\vec{k},0)]\vec{k} - (\frac{1}{|\vec{k}|}\sin|\vec{k}|t)i\vec{k}\times\hat{\vec{E}}(\vec{k},0) \tag{186}$$

where $\vec{E}(\vec{k},0)$ and $\vec{B}(\vec{k},0)$ are the initial values of the fields.

Now in order to solve the equations for the electromagnetic fields we insert the last result in equation (183) and use the constraints given by the Maxwell's equations eq (184), so

$$\begin{aligned} i\vec{k}\times\hat{\vec{E}}(\vec{k},t) &= i\vec{k}\times\hat{\vec{E}}(\vec{k},0)\cos|\vec{k}|t - (\frac{1}{|\vec{k}|}\sin|\vec{k}|t)\vec{k}\times(\vec{k}\times\hat{\vec{B}}(\vec{k},0)) \\ i\vec{k}\times\hat{\vec{B}}(\vec{k},t) &= i\vec{k}\times\hat{\vec{B}}(\vec{k},0)\cos|\vec{k}|t + (\frac{1}{|\vec{k}|}\sin|\vec{k}|t)\vec{k}\times(\vec{k}\times\hat{\vec{E}}(\vec{k},0)) \end{aligned} \tag{187}$$

So performing the triple cross product and imposing the constraints we get that:

$$i\vec{k}\times\hat{\vec{E}}(\vec{k},t) = i\vec{k}\times\hat{\vec{E}}(\vec{k},0)\cos|\vec{k}|t + (|\vec{k}|\sin|\vec{k}|t)\hat{\vec{B}}(\vec{k},0) \tag{188}$$

$$i\vec{k}\times\hat{\vec{B}}(\vec{k},t) = i\vec{k}\times\hat{\vec{B}}(\vec{k},0)\cos|\vec{k}|t - (\frac{1}{|\vec{k}|}\sin|\vec{k}|t)(i\hat{\rho}\vec{k} + |\vec{k}|^2\vec{E}) \tag{189}$$

So finally we insert this equations in (183) and after performing the integration on the time we obtain the expression for the fields.

$$\begin{aligned} \hat{\vec{E}}(\vec{k},t) &= (\cos|\vec{k}|t)\hat{\vec{E}}(\vec{k},0) + (|\vec{k}|^{-1}\sin|\vec{k}|t)i\vec{k}\times\hat{\vec{B}}(\vec{k},0) \\ &+ \int_0^t ds\big(-(|\vec{k}|^{-1}\sin|\vec{k}|(t-s))i\vec{k}\hat{\rho}(\vec{k},s) - (\cos|\vec{k}|(t-s))\hat{\vec{j}}(\vec{k},s)\big) \end{aligned} \tag{190}$$

$$\begin{aligned} \hat{\vec{B}}(\vec{k},t) &= (\cos|\vec{k}|t)\hat{\vec{B}}(\vec{k},0) - (|\vec{k}|^{-1}\sin|\vec{k}|t)i\vec{k}\times\hat{\vec{E}}(\vec{k},0) \\ &+ \int_0^t ds(|\vec{k}|^{-1}\sin|\vec{k}|(t-s))i\vec{k}\times\hat{\vec{j}}(\vec{k},s) \end{aligned} \tag{191}$$

where the initial terms account for the fields radiated until time t and the terms in the integral account for the retarded fields.

Now the Lorentz force Law reads:

$$\vec{F} = \int \rho(x)(\vec{E} + \frac{1}{c}\vec{J}(x) \times \vec{B})d^3x \tag{192}$$

where we integrate over the volume of the particle.

Now in this equations the source of the electromagnetic field is not specified, therefore we may decompose them in external fields and self fields , where the last ones describe the radiation coming from the particle. Therefore the force spreads in two terms:

$$\vec{F} = \vec{F}_{\text{self}} + \vec{F}_{ext} \tag{193}$$

where

$$\vec{F}_{\text{self}} = \int \rho(x)(\vec{E}_{self} + \frac{1}{c}\vec{J}(x) \times \vec{B}_{self})d^3x. \tag{194}$$

The next step is to introduce the expression for the fields produced by the particle found in eq (190) in the Lorentz force law, in order to do this we will use the next expression for the current:

$$\vec{J}(x', t) = \rho(x', t)\vec{v}(t) \tag{195}$$

Now since the particle travels at velocities much smaller than that of the fields, the particle will never interact with the initial fields (see section 6), so we may set $\hat{\vec{E}}(\vec{k}, 0) = 0$ and $\hat{\vec{B}}(\vec{k}, 0) = 0$ in equation (190), so the only fields which interact with the particle are the retarded ones.

So combining eq (190), (193) and (195) we get the following initial expression for the self force in Fourier space :

$$\begin{aligned} \vec{F}_{self}(t) &= \int_0^t ds\varepsilon \int d^3k|\hat{\rho}(\varepsilon\vec{k})|^2 e^{-i\vec{k}\cdot(\vec{q}(t)-\vec{q}(s))} \\ &* \Big((|\vec{k}|^{-1}\sin|\vec{k}|(t-s))i\vec{k} - (\cos|\vec{k}|(t-s))\vec{v}(s) \\ &- (|\vec{k}|^{-1}\sin|\vec{k}|(t-s))\vec{v}(t) \times (i\vec{k} \times \vec{v}(s))\Big) \end{aligned} \tag{196}$$

where we have introduced the dimensionless scaling parameter ε, of which we will talk now.

8.3 The scaling parameter

We have already seen in Rohrlich (section 6) that in order for the LAD equation to describe the motion of the particle at certain time t, it is necessary to know the position and velocity of the particle for future times of the order of $\tau_e = 2e^2/3mc^3$. Although it is a violation of causality this has a good point, that is that the violation of causality happens during an interval of time of the order of $\tau_e =$ which happens to be a very small number. Therefore $\tau_e =$ sets a time scale and a length scale $R_{\rho=\tau_e=r/c}$ for which the radiation reaction effects are relevant. Normally electromagnetic fields used in labs do not change appreciably in this scale of time, thus it is reasonable to suppose that the potentials which produce the fields vary very slowly in this scale of time. This is because a field that would change appreciably in this scale of time would need to have a frequency of the order of $10^{23}Hz$ which is far beyond the usual frequencies used in earth labs and also far beyond the scope of study of classical electrodynamics. So in order to keep track of the different scales involved in the motion of the particle it is useful to introduce a dimensionless scaling parameter ε.

The scale parameter is a mathematical tool which serves to keep track of the different space times scales which arises when coupling the dynamics of the system of the charged particle and its surrounding fields with the external fields. This is because the particle and the external fields may be moving in very different space-time scales. Thus to visualize clearly this point it is necessary that the scale parameter be introdunced in the potentials which produce the external fields in the following manner:

$$\phi_{ex}(\varepsilon\vec{x}), \qquad \vec{A}_{ex}(\varepsilon\vec{x}) \tag{197}$$

which in the limit $\varepsilon \to 0$ varies slowly. It is important that the energy scale be fixed otherwise the scaling parameter won't have any sense. For example if a particle is moving in circles due to a magnetic field $(0, 0, B_0)$, it comes from a potential of the form $\vec{A} = (0, xB_0, 0)$ (in the apropiate gauge).So since this potential is linear in x the introduction of ε sets the field strength to εB_0, and in the limit $\varepsilon \to 0$ this is a very small magnetic field. Then what we do to obtain ε is to set a reference field B_0 and then we compare it with the magnetic field of the system that we are dealing with. So this means that in order for ε to be a meaningful parameter it is necessary to state first the physical situation of the system, which includes the mass, the charge of the particle and the magnitude of the external fields.

To gain better understanding let's fixed some time scales:

- **Microscopic scale**: in this scale $t = \mathcal{O}(t_\rho)$, $\vec{q} = (R_\rho)$, the particle moves in straight line and the electromagnetic field is basically Couloumbic.
- **Macroscopic scale**: in this scale $t = \mathcal{O}(\varepsilon^{-1}t_\rho)$ and $\vec{q} = \varepsilon^{-1}(R_\rho)$. In this scale the motion of the particle is subject to external forces and the potentials are allowed to vary. Here the mechanical energy is almost conserved, this is because the particle losses energy trough radiation in an amount proportional to ε.
- **Friction scale**: In this long time scale energy looses through radiation become important, thus it is necessary to make some corrections to the effective Hamiltonian. Since the external forces are proportional to ε and the radiation emmited is proportional to

$\dot{\vec{v}}^2$ the energy lost is proportional to ε^2 compared to the microscopic scale. But since this occurs for times of the order of $\varepsilon^{-1}t_\rho$ the friction actually modify the motion of the system to orders of ε. In longer scales of time of order $\varepsilon^{-2}t_\rho$ the radiation reaction results in correction to the Hamiltonian of order $\mathcal{O}(1)$.

So now we give a brief outline of the method that will be used in order to understand the behavior of the dynamics of the particle in the limit $\varepsilon \to 0$. Lets suppose the particle is initially at $\vec{q}^0$ with velocity $\vec{v}^0$. Since before time $t = 0$ the potentials have not been turned on ,the initial fields must be Coulombic which translates into the condition that

$$Y(0) = S_{q^0,v^0}(0) \tag{198}$$

Now we define the six dimensional manifold $S = \{S_{q,v}, \vec{q} \in \mathbb{R}^3, \vec{v} \in \mathbb{V}\}$ which is a submanifold of the phase space $\mathcal{M}$. If the initial data lies on S and if there are no external forces then $\vec{q}$ and $\vec{v}$ will remain on S along all its path. The idea is that if a slowly varying potential is turned on, the solutions will be in a manifold ε close to S. This distance will be measured using energy differences and perhaps the following metric:

$$d(Y_1, Y_2) = \|\vec{E}_1 - \vec{E}_2\| + ||\vec{B}_1 - \vec{B}_2|| + |\vec{q}_1 - \vec{q}_2| + |\gamma(\vec{v}_1)\vec{v}_1 - \gamma(\vec{v}_2)\vec{v}_2| \,. \tag{199}$$

So in order to perform a comparison between quantities in the macroscopic scale and in the micrcoscopic scale, we want to transform the Maxwell equations and the dynamical variables to the macroscopic scale, this is done by setting :

$$t' = \varepsilon t, \qquad \vec{x}' = \varepsilon \vec{x} \tag{200}$$

so the velocity and the position take the form

$$\vec{q}'(t') = \varepsilon \vec{q}(t) \ , \qquad \vec{v}'(t') = \frac{d}{dt'}\vec{q}' = \vec{v}(t) \,, \tag{201}$$

so we denote:

$$\vec{q}^\varepsilon(t) = \varepsilon \vec{q}(\varepsilon^{-1}t) \ , \ \vec{v}^\varepsilon(t) = \vec{v}(\varepsilon^{-1}t). \tag{202}$$

From the insertion of the scale parameter in the external potentials eq (197), we get that the Lorentz force reads:

$$\frac{d}{dt}(m\gamma\vec{t}) = \int (\rho(x - q(t))[\varepsilon \vec{E}_{ex}(\varepsilon x) + \vec{E}_{self}(x,t) + \vec{v}(t) \times (\vec{B}_{ex}(\varepsilon x) + \vec{B}_{self})]d^3x. \tag{203}$$

where

$$\vec{E}_{ex} = -\nabla \phi_{ex}, \quad \vec{B} = \nabla \times \vec{A}_{ex} \tag{204}$$

There are some quantities which are independent of the scale like the total charge of particle and the total energy of the system. Therefore the charge density is :

$$\rho_\varepsilon(\vec{x}) = \varepsilon^{-3}\,\rho(\varepsilon^{-1}\vec{x})\,, \tag{205}$$

which leaves the charge ε independent $\int d^3x\,\rho_\varepsilon(\vec{x}) = e$. The total energy in the macroscopic time scale is

$$\mathcal{E}_{\rm mac} = m_{\rm b}\gamma(\vec{v}) + \phi_{\rm ex} * \rho_\varepsilon(\vec{q}) + \frac{1}{2}\int d^3x\left(\vec{E}(\vec{x})^2 + \vec{B}(\vec{x})^2\right) \tag{206}$$

which in order to be ε independent requires that the fields be:

$$\vec{E}'(\vec{x}',t') = \varepsilon^{-3/2}\,\vec{E}(\vec{x},t)\ ,\ \vec{B}'(\vec{x}',t') = \varepsilon^{-3/2}\,\vec{B}(\vec{x},t)\,. \tag{207}$$

finally in this time scale the Maxwell equations and Lorentz force Law reads:

$$\begin{aligned}
\partial_t\vec{B}(\vec{x},t) &= -\nabla\times\vec{E}(\vec{x},t)\,,\\
\partial_t\vec{E}(\vec{x},t) &= \nabla\times\vec{B}(\vec{x},t) - \sqrt{\varepsilon}\rho_\varepsilon(\vec{x}-\vec{q}^{\varepsilon}(t))\vec{v}^{\varepsilon}(t)\,,\\
\frac{d}{dt}\left(m_{\rm b}\gamma\vec{v}^{\varepsilon}(t)\right) &= \vec{E}_{\rm ex} * \rho_\varepsilon(\vec{q}^{\varepsilon}(t)) + \vec{v}^{\varepsilon}(t)\times\vec{B}_{\rm ex} * \rho_\varepsilon(\vec{q}^{\varepsilon}(t))\\
&\quad + \int d^3x\,\sqrt{\varepsilon}\,\rho_\varepsilon(\vec{x}-\vec{q}^{\varepsilon}(t))\,[\vec{E}(\vec{x},t) + \vec{v}^{\varepsilon}(t)\times\vec{B}(\vec{x},t)]
\end{aligned} \tag{208}$$

Having define all the equations and dynamical variables in the macroscopic time scale we are in position to continue our analysis.

8.4 Expansion of the force

The velocity of the particle is naturally bounded from above by the special theory of relativity, but this theory does not tell us anything about higher derivatives but let's see that the acceleration must also be bounded. The fact that the electron and other charged particles have a well established rest mass makes that the energy of the system of the moving particle and its surrounding fields is bounded from below by the rest energy of the particle.In order that this be true it is necessary that the bare mass of the particle is not negative or one could infinitely take of mass to the particle but it would compensate with the mass coming from the energy of the field to give a positive effective mass, but as we will show in section 9. Now since the particle lost of radiation is proportional to $\dot{\vec{v}}^2$ the acceleration must also be bounded for if the particle could be infinitely accelerated in a certain moment by a finite amount of time it would emit infinite radiation which would contradict the lower bound for the energy of the particle. Also this implies that $\ddot{\vec{v}}$ must also be bounded to comply with this bound. Indeed Spohn gets to demonstrate by similar arguments the following theorem:

Theorem 1 For a moving charged particle with the Abraham model, with smooth charge density $\rho(x) \in \mathcal{C}^\infty$, under the influence of smooth external fields $\phi_{ex} \in \mathcal{C}^\infty$ and $\vec{A}_{ex} \in \mathcal{C}^\infty$, there exist constants $\bar{e}$ and C depending on the initial conditions, such that in the microscopic time scale we have that:

$$\sup_t |\vec{v}(t)| \leq \overline{v} < 1, \ \sup_t \left|\left(\frac{d}{dt}\right)^n \vec{v}(t)\right| \leq C\,\varepsilon^n, \ n = 1, 2, 3\,, \tag{209}$$

provided the charge is such that $e < \bar{e}$.

So in particular we know that

$$\sup_{t\in\mathbb{R}} |\ddot{\vec{q}}^\varepsilon(t)| \leq C, \ \sup_{t\in\mathbb{R}} |\dddot{\vec{q}}^\varepsilon(t)| \leq C, \ \sup_{t\in\mathbb{R}} |\ddddot{\vec{q}}^\varepsilon(t)| \leq C \tag{210}$$

so as ε becomes smaller so does C. This smallnes sugest that we could make a taylor expansion in the equation for the self lorentz force eq (196). To this purpose we set $\vec{v}^\varepsilon(t) = \vec{v}$ and $t - s = t'$. Then we have the expansions:

$$\vec{v}^\varepsilon(s) = \vec{v}^\varepsilon(t-t) = \vec{v} - \varepsilon\dot{\vec{v}}t + \frac{1}{2}\varepsilon^2\ddot{\vec{v}}t^2 + \mathcal{O}(t^3) \tag{211}$$

$$\begin{aligned} e^{-i\vec{k}(\vec{(q)}^\varepsilon(t)-\vec{(q)}^\varepsilon(s))} &= e^{-i\vec{k}(\vec{(q)}^\varepsilon(t)-\vec{(q)}^\varepsilon(t-t))} && (212)\\ &= e^{-i(\vec{k}\cdot\vec{v})t}\Bigg(1 + \frac{1}{2}i\varepsilon t^2(\vec{k}\cdot\dot{\vec{v}}) - \frac{1}{6}i\varepsilon^2 t^3(\vec{k}\cdot\ddot{\vec{v}}) \\ &- \frac{1}{2}(\frac{1}{2}\varepsilon t^2(\vec{k}\cdot\dot{\vec{v}}) - \frac{1}{6}\varepsilon^2 t^3(\vec{k}\cdot\ddot{\vec{v}}))^2 + \mathcal{O}(t^3)\Bigg) && (213) \end{aligned}$$

so inserting this expansions in (196) and also introducing the macroscopic time scaled quantities $s' = \varepsilon^{-1}s$, $\vec{k}' = \varepsilon\vec{k}$ we get that:

$$\begin{aligned} \vec{F}^\varepsilon_{self}(t) &= \int_0^t ds\varepsilon \int d^3k|\hat{\rho}(\varepsilon\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t} && (214)\\ &* \Bigg((|\vec{k}|^{-1}\sin|\vec{k}|t)i\vec{k} - \cos|\vec{k}|(t)(\vec{v} - \varepsilon\dot{\vec{v}}t + \frac{1}{2}\varepsilon^2\ddot{\vec{v}}t^2) \\ &- (|\vec{k}|^{-1}\sin|\vec{k}|t)\vec{v}\times[i\vec{k}\times(\vec{v} - \varepsilon\dot{\vec{v}}t + \frac{1}{2}\varepsilon^2\ddot{\vec{v}}t^2)] \\ &+ (\frac{1}{2}i\varepsilon t^2(\vec{k}\cdot\dot{\vec{v}}))\Big((|\vec{k}|^{-1}\sin|\vec{k}|t)i\vec{k} - \cos|\vec{k}|(t)(\vec{v} - \varepsilon\dot{\vec{v}}t) \\ &- (|\vec{k}|^{-1}\sin|\vec{k}|t)\vec{v}\times[i\vec{k}\times(\vec{v} - \varepsilon\dot{\vec{v}}t)]\Big) \\ &- (\frac{1}{6}i\varepsilon^2 t^3(\vec{k}\cdot\ddot{\vec{v}}) + \frac{1}{8}\varepsilon^2 t^4(\vec{k}\cdot\dot{\vec{v}})^2)\Big((|\vec{k}|^{-1}\sin|\vec{k}|t)i\vec{k} - \cos|\vec{k}|(t)\vec{v} \\ &- (|\vec{k}|^{-1}\sin|\vec{k}|t)\vec{v}\times(i\vec{k}\times\vec{v})\Big)\Bigg) + \mathcal{O}(\varepsilon^3) \end{aligned}$$

The terms of lower order vanish therefore canceling these and performing the the triple cross product we get that:

$$\begin{aligned}
\vec{F}^{\varepsilon}_{self}(t) &= \int_0^t ds\varepsilon^{-1}\int d^3k|\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t} \\
&* \Bigg[\cos|\vec{k}|(t)(\varepsilon\dot{\vec{v}}t - \frac{1}{2}\varepsilon^2\ddot{\vec{v}}t^2) \\
&+ (|\vec{k}|^{-1}\sin|\vec{k}|t)i\varepsilon t\left((\vec{v}\cdot\dot{\vec{v}})\vec{k} - (\vec{v}\cdot\vec{k})\dot{\vec{v}}\right) \\
&+ (|\vec{k}|^{-1}\sin|\vec{k}|t)i\frac{1}{2}\varepsilon^2 t^2\left((\vec{v}\cdot\vec{k})\ddot{\vec{v}} - (\vec{v}\cdot\ddot{\vec{v}})\vec{k}\right) \\
&+ (\frac{1}{2}i\varepsilon t^2(\vec{k}\cdot\dot{\vec{v}}))\Big((|\vec{k}|^{-1}\sin|\vec{k}|t)i\vec{k} - \cos|\vec{k}|(t)(\vec{v} - \varepsilon\dot{\vec{v}}t) \\
&+ i(|\vec{k}|^{-1}\sin|\vec{k}|t)((\vec{v}\cdot\vec{k})\vec{v} - (\vec{v}\cdot\vec{v})\vec{k}) \\
&+ i\varepsilon t(|\vec{k}|^{-1}\sin|\vec{k}|t)((\vec{v}\cdot\dot{\vec{v}})\vec{k} - (\vec{v}\cdot\vec{k})\dot{\vec{v}}))\Big) \\
&- (\frac{1}{6}i\varepsilon^2 t^3(\vec{k}\cdot\ddot{\vec{v}}) + \frac{1}{8}\varepsilon^2 t^4(\vec{k}\cdot\dot{\vec{v}})^2)\Big((|\vec{k}|^{-1}\sin|\vec{k}|t)i\vec{k} - \cos|\vec{k}|(t)\vec{v} \\
&- i(|\vec{k}|^{-1}\sin|\vec{k}|t)((\vec{v}\cdot\vec{k})\vec{v} - (\vec{v}\cdot\vec{v})\vec{k})\Big)\Bigg] + \mathcal{O}(\varepsilon^3)
\end{aligned} \tag{215}$$

now we factorize terms of the different orders of ε and we get that:

$$\begin{aligned}
\vec{F}^{\varepsilon}_{self}(t) &= \int_0^t ds\varepsilon^{-1}\int d^3k|\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t} \\
&* \Bigg[(|\vec{k}|^{-1}\sin|\vec{k}|t)i\varepsilon\Bigg(t\left((\vec{v}\cdot\dot{\vec{v}})\vec{k} - (\vec{v}\cdot\vec{k})\dot{\vec{v}}\right) \\
&+ \frac{1}{2}it^2\left(i(\vec{k}\cdot\dot{\vec{v}}))\vec{k} + i(\vec{k}\cdot\dot{\vec{v}}))(\vec{v}\cdot\vec{k})\vec{v} - i(\vec{k}\cdot\dot{\vec{v}}))(\vec{v}\cdot\vec{v})\vec{k}\right)\Bigg) \\
&+ t\varepsilon(\cos|\vec{k}|(t))(\dot{\vec{v}} + \frac{t}{2}\vec{v}(\dot{\vec{v}}\cdot\vec{k})) \\
&+ \varepsilon^2(|\vec{k}|^{-1}\sin|\vec{k}|t)i\Bigg[\frac{t^2}{2}\left((\vec{v}\cdot\vec{k})\ddot{\vec{v}} - (\vec{v}\cdot\ddot{\vec{v}})\vec{k}\right) \\
&+ \frac{t^3}{2}\left(i(\vec{v}\cdot\dot{\vec{v}})(\vec{k}\cdot\dot{\vec{v}})\vec{k} - i(\vec{v}\cdot\vec{k})(\vec{k}\cdot\dot{\vec{v}})\dot{\vec{v}}\right) \\
&- \frac{t^3}{6}\left((\vec{k}\cdot\ddot{\vec{v}})i\vec{k} - (\vec{k}\cdot\ddot{\vec{v}})(\vec{v}\cdot\vec{k})i\vec{v} + (\vec{k}\cdot\ddot{\vec{v}})(\vec{v}\cdot\vec{v})i\vec{k}\right) \\
&- \frac{t^4}{8}\left((\vec{k}\cdot\dot{\vec{v}})^2 i\vec{k} - (\vec{k}\cdot\dot{\vec{v}})^2(\vec{v}\cdot\vec{k})i\vec{v} + (\vec{k}\cdot\dot{\vec{v}})^2(\vec{v}\cdot\vec{v})i\vec{k}\right)\Bigg] \\
&+ \varepsilon^2\cos|\vec{k}(t)|\left(-\frac{t^2}{2}\ddot{\vec{v}} - \frac{t^3}{6}(3i((\vec{k}\cdot\dot{\vec{v}})\dot{\vec{v}} + i(\vec{k}\cdot\ddot{\vec{v}})\vec{v})\right)\Bigg]
\end{aligned} \tag{216}$$

Now we know the following property of the Fourier transform:

$$\mathfrak{F}\{\nabla f(\vec{x})\} = -i\vec{k}\mathfrak{F}\{f(\vec{x})\} \tag{217}$$

so from this we get that the last equation reads:

$$
\begin{aligned}
\vec{F}^{\varepsilon}_{self}(t) &= \int d^3k|\hat{\rho}(\vec{k})|^2 && (218)\\
&* \Bigg[\Big(-(\vec{v}\cdot\dot{\vec{v}})\nabla_v + \dot{\vec{v}}(\vec{v}\cdot\nabla_v)\Big)\int_0^{\epsilon^{-1}t} dt e^{i(\vec{k}\cdot\vec{v})t}(|\vec{k}|^{-1}\sin|\vec{k}|t)\\
&+ \left(\dot{\vec{v}} + \frac{1}{2}\vec{v}(\dot{\vec{v}}\cdot\nabla_v)\right)\int_0^{\epsilon^{-1}t} dt\, t e^{i(\vec{k}\cdot\vec{v})t}(\cos|\vec{k}|t)\\
&+ \varepsilon\left(\frac{1}{2}\Big[-(|\vec{v}|^2-1)(\dot{\vec{v}}\cdot\nabla_v)\nabla_v + \vec{v}(\vec{v}\cdot\nabla_v)(\dot{\vec{v}}\cdot\nabla_v) + (\vec{v}\cdot\ddot{\vec{v}})\nabla_v\right.\\
&- \ddot{\vec{v}}(\vec{v}\cdot\nabla_v)\Big] + \frac{1}{6}\Big[(|\vec{v}|^2-1)(\ddot{\vec{v}}\cdot\nabla_v)\nabla_v - \vec{v}(\vec{v}\cdot\nabla_v)(\ddot{\vec{v}}\cdot\nabla_v)\\
&+ 3(\vec{v}\cdot\dot{\vec{v}})(\dot{\vec{v}}\cdot\nabla_v)\nabla_v - 3\dot{\vec{v}}(\vec{v}\cdot\dot{\vec{v}})(\dot{\vec{v}}\cdot\nabla_v)\Big] + \frac{1}{8}\Big[(|\vec{v}|^2-1)(\dot{\vec{v}}\cdot\nabla_v)^2\nabla_v\\
&- \left.\vec{v}(\vec{v}\cdot\nabla_v)^2\Big]\right)\int_0^{\epsilon^{-1}t} dt\, t e^{i(\vec{k}\vec{v})t}(|\vec{k}|^{-1}\sin|\vec{k}|t)\\
&+ \epsilon\left(-\ddot{\vec{v}} - \frac{1}{6}[\vec{v}(\dddot{\vec{v}}\cdot\nabla_v) + 3\dot{\vec{v}}(\ddot{\vec{v}}\cdot\nabla_v)]\right)\\
&\quad \int_0^{\epsilon^{-1}t} dt\, t^2 e^{i(\vec{k}\vec{v})t}(\cos|\vec{k}|t)\Bigg] + \mathcal{O}(\varepsilon^2)
\end{aligned}
$$

where

$$\nabla_v = \hat{i}\frac{\partial}{\partial v_x} + \hat{j}\frac{\partial}{\partial v_y} + \hat{k}\frac{\partial}{\partial v_z} + \qquad (219)$$

Now we want to evaluate the integrals in the limit $\varepsilon \to 0$, so in order to do this for the terms with sines we do

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \int_0^{\epsilon^{-1}t} dt \int d^3k|\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t}(|\vec{k}|^{-1}\sin|\vec{k}|t)t^p && (220)\\
&= \int_0^{\infty} dt \int d^3k \int d^3x\rho(\vec{x})e^{-i\vec{k}\cdot\vec{x}} \int d^3y\rho(\vec{y})e^{i\vec{k}\cdot\vec{y}}e^{-i(\vec{k}\cdot\vec{v})t}(|\vec{k}|^{-1}\sin|\vec{k}|t)t^p && (221)\\
&= \int_0^{\infty} dt \int d^3x \int d^3y \int d^3k\rho(\vec{x})\rho(\vec{y})e^{-i\vec{k}\cdot(\vec{v}t+\vec{x}-\vec{y})}(|\vec{k}|^{-1}\sin|\vec{k}|t)t^p && (222)
\end{aligned}
$$

now we set $d^3k = |\vec{k}|^2 dk\, d\phi\, \sin\theta_k d\theta_k$ also we do the following change of variable $\vec{b} = \vec{x}+\vec{v}t-\vec{y}$ so with this $e^{-i\vec{k}\cdot(\vec{v}t+\vec{x}-\vec{y})} = e^{-i|\vec{k}||\vec{b}|\cos\theta_k}$, then our integral takes the following form

$$
\begin{aligned}
= \int_0^{\infty} dt \int d^3x \int d^3y \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} |\vec{k}|^2 dk\, d\phi\, \sin\theta_k d\theta_k \qquad &\rho(\vec{x})\rho(\vec{y})e^{-i|\vec{k}||\vec{b}|\cos\theta_k}\\
&*(|\vec{k}|^{-1}\sin|\vec{k}|t)t^p
\end{aligned}
$$

performing the angular integrals we get that

$$= \int_0^\infty dt \int d^3x \int d^3y \int_0^\infty |\vec{k}|^2 dk \rho(\vec{x})\rho(\vec{y}) \frac{2\pi}{i|\vec{k}||\vec{b}|} \left(e^{i|\vec{k}||\vec{b}|} - e^{-i|\vec{k}||\vec{b}|}\right) (|\vec{k}|^{-1} \sin |\vec{k}|t) t^p$$
$$= \int_0^\infty dt \int d^3x \int d^3y \int_0^\infty \frac{4\pi|\vec{k}|}{|\vec{b}|} dk \rho(\vec{x})\rho(\vec{y})(\sin |\vec{k}||\vec{b}|)(|\vec{k}|^{-1} \sin |\vec{k}|t) t^p$$
$$= \int_0^\infty dt \int d^3x \int d^3y \int_0^\infty \frac{4\pi}{|\vec{b}|} dk \rho(\vec{x})\rho(\vec{y})(\sin |\vec{k}||\vec{b}|)(\sin |\vec{k}|t) t^p$$
$$= \int_0^\infty dt \int d^3x \int d^3y \frac{1}{4\pi|\vec{b}|} \rho(\vec{x})\rho(\vec{y})\delta(|\vec{b}| - t) t^p$$
$$= \int_0^\infty dt \int d^3x \int d^3y \frac{1}{4\pi t} \rho(\vec{x})\rho(\vec{y})\delta(|\vec{x} + \vec{v}t - \vec{y}| - t) t^p$$

Therefore we get that

$$\lim_{\varepsilon\to 0} \int_0^{\epsilon^{-1}t} dt \int d^3k |\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t} (|\vec{k}|^{-1} \sin |\vec{k}|t) t \tag{223}$$
$$= \int_0^\infty dt \int d^3x \int d^3y \frac{1}{4\pi t} \rho(\vec{x})\rho(\vec{y})\delta(|\vec{x} + \vec{v}t - \vec{y}| - t) t$$
$$= \int_0^\infty dt \int d^3x \int d^3y \frac{1}{4\pi} \rho(\vec{x})\rho(\vec{y})\delta(|\vec{x} + \vec{v}t - \vec{y}| - t)$$
$$= \int d^3x \int d^3y \frac{\gamma^2}{4\pi} \rho(\vec{x})\rho(\vec{y})$$

Also we get that for $p = 0$

$$\lim_{\varepsilon\to 0} \int_0^{\epsilon^{-1}t} dt \int d^3k |\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t} (|\vec{k}|^{-1} \sin |\vec{k}|t) \tag{224}$$
$$= \int_0^\infty dt \int d^3x \int d^3y \frac{1}{4\pi t} \rho(\vec{x})\rho(\vec{y})\delta(|\vec{x} + \vec{v}t - \vec{y}| - t)$$
$$= \int d^3k |\hat{\rho}(\vec{k})|^2 [\vec{k}^2 - (\vec{k}\cdot\vec{v})^2]^{-1} \tag{225}$$

Now in a similar manner to the above expressions we evaluate the expressions with cosines using $(|\vec{k}|^{-1} \cos |\vec{k}|t) = \frac{d}{dt}(|\vec{k}|^{-1} \sin |\vec{k}|t)$ and then performing a partial integration:

$$\lim_{\varepsilon\to 0} \int_0^{\epsilon^{-1}t} d\tau \int d^3k |\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})\tau} \tau^{1+p} \frac{d}{d\tau}(|\vec{k}|^{-1} \sin |\vec{k}|\tau) \tag{226}$$
$$= \int_0^\infty dt \int d^3k |\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t} t^{1+p} \frac{d}{dt}(|\vec{k}|^{-1} \sin |\vec{k}|t) + \tag{227}$$
$$= \int_0^\infty dt \int d^3k |\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t} t^p (-i\vec{k}\cdot\vec{v}t - 1 - p)(|\vec{k}|^{-1} \sin |\vec{k}|t) \tag{228}$$

where the boudary term has vanished, because of the limits of integration. Now again using the property of the fourier transmor already mentioned eq (217) we get that:

$$= -((\vec{v}\cdot\nabla_v)+1+p)\int_0^\infty dt\int d^3k|\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})t}t^p(|\vec{k}|^{-1}\sin|\vec{k}|t) \tag{229}$$

$$= -((\vec{v}\cdot\nabla_v)+1+p)\int_0^\infty dt\int d^3x\int d^3y\frac{1}{4\pi t}\rho(\vec{x})\rho(\vec{y})\delta(|\vec{x}+\vec{v}t-\vec{y}|-t)t^p$$

So using this we get that for $p=1$:

$$\lim_{\varepsilon\to 0}\int_0^{\epsilon^{-1}t} dt\int d^3k|\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})\tau}\tau(|\vec{k}|^{-1}\cos|\vec{k}|\tau) \tag{230}$$

$$= -((\vec{v}\cdot\nabla_v)+2)\int_0^\infty dt\int d^3x\int d^3y\frac{1}{4\pi}\rho(\vec{x})\rho(\vec{y})\delta(|\vec{x}+\vec{v}t-\vec{y}|-t)$$

$$= -\int d^3x\int d^3y\frac{2\gamma^2}{4\pi}\rho(\vec{x})\rho(\vec{y})$$

and for $p=0$

$$\lim_{\varepsilon\to 0}\int_0^{\epsilon^{-1}t} dt\int d^3k|\hat{\rho}(\vec{k})|^2 e^{-i(\vec{k}\cdot\vec{v})\tau}(|\vec{k}|^{-1}\cos|\vec{k}|\tau) \tag{231}$$

$$= -((\vec{v}\cdot\nabla_v)+1)\int_0^\infty dt\int d^3x\int d^3y\frac{1}{4t\pi}\rho(\vec{x})\rho(\vec{y})\delta(|\vec{x}+\vec{v}t-\vec{y}|-t)$$

$$= -((\vec{v}\cdot\nabla_v)+1)\int d^3x\int d^3y\frac{1}{4|\vec{x}+\vec{v}t-\vec{y}|\pi}\rho(\vec{x})\rho(\vec{y})$$

$$= -((\vec{v}\cdot\nabla_v)+1)\int d^3x\int d^3y\frac{1}{4|\vec{x}+\vec{v}t-\vec{y}|\pi}\rho(\vec{x})\rho(\vec{y})$$

$$= \int d^3k|\hat{\rho}(\vec{k})|^2(|\vec{k}|^2+(\vec{k}\cdot\vec{v})^2)[\vec{k}^2-(\vec{k}\cdot\vec{v})^2]^{-2} \tag{232}$$

Now with these integrals we replace in the expression for the self force eq (218) and we get that:

$$\begin{aligned}
\vec{F}_{self}(t) &= \\
&* \left[\Big(-(\vec{v}\cdot\dot{\vec{v}})\nabla_v+\dot{\vec{v}}(\vec{v}\cdot\nabla_v)\Big)\int d^3k|\hat{\rho}(\vec{k})|^2[\vec{k}^2-(\vec{k}\cdot\vec{v})^2]^{-1}\right. \qquad (233)\\
&+ \left(\dot{\vec{v}}+\frac{1}{2}\vec{v}(\dot{\vec{v}}\cdot\nabla_v)\right)\int d^3k|\hat{\rho}(\vec{k})|^2(|\vec{k}|^2+(\vec{k}\cdot\vec{v})^2)[\vec{k}^2-(\vec{k}\cdot\vec{v})^2]^{-2}\\
&+ \left(\frac{1}{2}\Big[-(|\vec{v}|^2-1)(\dot{\vec{v}}\cdot\nabla_v)\nabla_v+\vec{v}(\vec{v}\cdot\nabla_v)(\dot{\vec{v}}\cdot\nabla_v)+(\vec{v}\cdot\ddot{\vec{v}})\nabla_v\right.\\
&- \left.\ddot{\vec{v}}(\vec{v}\cdot\nabla_v)\Big]+\frac{1}{6}\Big[(|\vec{v}|^2-1)(\ddot{\vec{v}}\cdot\nabla_v)\nabla_v-\vec{v}(\vec{v}\cdot\nabla_v)(\ddot{\vec{v}}\cdot\nabla_v)\right.\\
&+ \left.3(\vec{v}\cdot\dot{\vec{v}})(\dot{\vec{v}}\cdot\nabla_v)\nabla_v-3\dot{\vec{v}}(\vec{v}\cdot\dot{\vec{v}})(\dot{\vec{v}}\cdot\nabla_v)\Big]+\frac{1}{8}\Big[(|\vec{v}|^2-1)(\dot{\vec{v}}\cdot\nabla_v)^2\nabla_v\right.\\
&- \left.\vec{v}(\vec{v}\cdot\nabla_v)^2\Big]\right)\int d^3k|\hat{\rho}(\vec{k})|^2[\vec{k}^2-(\vec{k}\cdot\vec{v})^2]^{-1}\\
&+ \epsilon\left(-\dddot{\vec{v}}-\frac{1}{6}[\vec{v}(\dddot{\vec{v}}\cdot\nabla_v)+3\dot{\vec{v}}(\ddot{\vec{v}}\cdot\nabla_v)]\right)\\
&* \left.\int d^3x\int d^3y\frac{-2\gamma^2}{4\pi}\rho(\vec{x})\rho(\vec{y})\right]+\mathcal{O}(^2)
\end{aligned}$$

So finally going again to the real space and collecting all terms we get that:

$$\vec{F}^{\varepsilon}_{self}(t) = -m_f(v)\dot{\vec{v}} + \varepsilon\frac{e^2}{6\pi}\left[\gamma^4(\vec{v}\cdot\ddot{\vec{v}})\vec{v} + 3\gamma^6(\vec{v}\cdot\dot{\vec{v}})^2\vec{v} + 3\gamma^4(\vec{v}\cdot\dot{\vec{v}})\dot{\vec{v}} + \gamma^2\ddot{\vec{v}}\right] + \mathcal{O}(\varepsilon^2) \quad (234)$$

where the first accounts for the effects of the electromagnetic fields in the change of total momentum and the part proportional to ε is the radiation reaction term. We see that it contains the undesirable $\ddot{\vec{v}}$ terms, so a priori we suspect that it suffer from the same problems as LAD. So including again the external fields and omitting the $\mathcal{O}(\varepsilon^2)$ terms we get that the equation of motion is:

$$\begin{aligned} m(\vec{v})\dot{\vec{v}} &= e\big(\vec{E}_{\text{ex}}(\vec{q}) + \vec{v}\times\vec{B}_{\text{ex}}(\vec{q})\big) \\ &+\varepsilon(e^2/6\pi)\left[\gamma^4(\vec{v}\cdot\ddot{\vec{v}})\vec{v} + 3\gamma^6(\vec{v}\cdot\dot{\vec{v}})^2\vec{v} + 3\gamma^4(\vec{v}\cdot\dot{\vec{v}})\dot{\vec{v}} + \gamma^2\ddot{\vec{v}}\right]. \end{aligned} \quad (235)$$

8.5 Illustration of the singular perturbation method

The singular perturbation method that we will use to obtain the solution to the self force problem resembles to the perturbation theory used in quantum mechanics. The idea is to obtain the results for the motion of the particle under an external force which due to the smallness of the parameter ε contributes much more than the radiation reaction force. Therefore the effects of the radiation reaction may be treated as a perturbation. First let's illustrate the method with a general mathematical example which will turn out to have the same form as the equations of motion of the particle. Consider the set of coupled equations:

$$\dot{x} = f(x,y), \qquad \varepsilon\dot{y} = y - h(x) \quad (236)$$

where $\{h, f \in \mathcal{C}^\infty\}$,are bounded and $(x,y)\in\mathbb{R}^2$.

In order to understand the solutions to this equations for $\varepsilon \to 0$ lets first assume that $\varepsilon = 0$ (this would correspond to the unperturbed case). In this case

$$y = h(x) \qquad \Longrightarrow \qquad \dot{x} = f(x, h(x)) \quad (237)$$

which means that the x-y plane has been reduced to the curve $y = h(x)$, so we have reduced the space $\mathbb{R}^2$ to a one dimensional submanifold of $\mathbb{R}^2$which we call the critical manifold to zero-th order in ε and we denote it by $\mathcal{C}_0$. This submanifold defined by $\{y = h(x), x \in \mathbb{R} = \mathcal{C}_0\}$. Now to observe how the solutions may vary when $\varepsilon \to 0$ we analyze the equations in the slow time scale which is $\tau = \varepsilon^{-1}t$. We denote differentiation with respect to τ with a prime '. Then in this scale the equation (236) takes the following form:

$$\begin{aligned} \varepsilon\dot{x} &= \varepsilon f(x,y) \quad \Rightarrow \quad x' = \varepsilon f(x,y) \\ \varepsilon\dot{y} &= \frac{\dot{y}}{\varepsilon^{-1}} = y - h(x) \quad \Rightarrow \quad y' = y - h(x) \end{aligned} \quad (238)$$

So in this scale when $\varepsilon \to 0$ we have that $x' = 0$ hence $x = x_0$ for some constant x_0. Therefore $y' = y - h(x_0)$ with solution

$$y = (y_0 - h(x_0))e^t + h(x_0) \quad (239)$$

Therefore solutions in this time scale corresponds to exclusively runaway ones. So we see that in the limit $\varepsilon \to 0$ the equations (236) have two possible solutions, which behave very differently and depend on which time scale we are looking in. Thus we may say that the set of solutions of equations (236) possesses an eigenvalue λ and that in the limit $\varepsilon \to 0$ they are of the form:

$$\begin{aligned} y &= (y_0 - h(x_0))e^{\lambda t} + h(x_0) \\ \dot{x} &= (\lambda - 1)f(x, h(x)) \end{aligned} \tag{240}$$

with $\lambda = 0$ corresponding to the microscopic scale solution, which is stable, and with $\lambda = 1$ corresponding to the solution in the low time scale which happens run away. The space generated by the solutions with eigenvalue $\lambda = 0$ is called the center manifold (at $\varepsilon = 0$) and stills being denoted by $\mathcal{C}_0$. Now the main result from singular perturbation theory that we are going to use here is the fact that for some very small non zero value of ε there exist a submanifold $\mathcal{C}_\varepsilon$ diffeomorphic to $\mathcal{C}_0$. This submanifold $\mathcal{C}_\varepsilon$ is invariant to the solutions flow of equations (236) and is very close to $\mathcal{C}_0$. This means that if some solution begins on $\mathcal{C}_0$ it will pass rapidly to $\mathcal{C}_\varepsilon$ where it will remain forever. But for solutions which are initially slightly of $\mathcal{C}_\varepsilon$ the solution will rapidly diverge from $\mathcal{C}_\varepsilon$ therefore it will runaway. Then $\mathcal{C}_\varepsilon$ is a deformation of order ε of $\mathcal{C}_0$, therefore $\mathcal{C}_\varepsilon$ is also of the form $\{y = h_\varepsilon(x), x \in \mathbb{R}\}$. So according to (236) x evolves as

$$\dot{x} = f(x, h_\varepsilon(x))$$

Now $h_\varepsilon(x)$ must be similar to $h(x)$ with modification of order ε^n, so it must be a smooth function on ε therefore it should have a representation in terms of orders of ε like this

$$h_\varepsilon(x) = \sum_{j=0}^{m} \varepsilon^j \, h_j(x) + \mathcal{O}(\varepsilon^{m+1}) = h_0(x) + \varepsilon h_1(x) + \varepsilon^2 h_2(x) + \cdots \tag{241}$$

Now we have that $y = h(x)_\varepsilon$, so we make a Taylor expansion of equation (241) around $h(x)$

$$\dot{x} = f(x, h_\varepsilon(x)) = f(x, h(x)) + \partial_y f(x, h(x))(h_\varepsilon(x) - h(x)) + \cdots \tag{242}$$

also we have that

$$y' = \frac{d}{d\tau} h(x)_\varepsilon = \frac{dx}{d\tau}\frac{\partial}{\partial x} h(x)_\varepsilon = x' \partial_x h(x)_\varepsilon \tag{243}$$

so combining this with equation (238) we get that

$$\varepsilon \partial_x h(x)_\varepsilon f(x, h_\epsilon(x)) = h_\epsilon(x) - h(x) \tag{244}$$

So introducing the expansion for $h_\epsilon(x)$ we get that

$$\varepsilon \partial_x (h_0(x) + \varepsilon h_1(x)) f(x, h(x)) = h_0(x) + \varepsilon h_1(x) - h(x) \tag{245}$$

So comparing terms of order ε^0 we get that

$$h_\varepsilon(x) = h(x) \tag{246}$$

to order order ε we get that

$$h_1(x) = h'(x)\, f(x, h(x)) \tag{247}$$

So to first order in ε the evolution of x (eq (241)) is given by

$$\dot{x} = f(x, h_\varepsilon(x)) = f(x, h(x)) + \partial_y f(x, h(x)) h'(x)\, f(x, h(x)) \tag{248}$$

and these equation determine the manifold $\mathcal{C}_\varepsilon$.

8.6 The true equation of motion.

Now lets see that the equations of motion for the electron (235) have the same form as the example we just did for the singular perturbation theory eq (236). In order to do this we must make the certain definitions:

First we set set

$$(\vec{x}_1, \vec{x}_2) = \vec{x} = (\vec{q}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{V} \;\text{ and }\; \vec{y} = \dot{\vec{v}} \in \mathbb{R}^3 \tag{249}$$

with this we define the function $\vec{f}(\vec{x}, \vec{y})$ as follows

$$\vec{f}(\vec{x}, \vec{y}) = (\vec{x}_2\,, \vec{y}) \tag{250}$$

We also define the following matrix operator $\kappa(\vec{v}) = \mathbb{1} + \gamma^2\, |\vec{v}.\rangle\langle\vec{v}|$ with inverse matrix $\kappa(\vec{v})^{-1} = \mathbb{1} - |\vec{v}\rangle\langle\vec{v}|$. This operator in a while will be very useful, if applied over any vector $\vec{w}$, it has the following effect:

$$\kappa(\vec{v})^{-1}\vec{w} = \mathbb{1}\vec{w} - |\vec{v}\rangle\langle\vec{v}|\vec{w} = \vec{w} - (\vec{v}\cdot\vec{w})\vec{v} \tag{251}$$

Finally we define

$$\begin{aligned} \vec{g}(\vec{x}, \vec{y}, \varepsilon) \;&=\; \gamma^{-2}\kappa(\vec{x}_2)^{-1}\big((6\pi/e^2)\,[m(\vec{x}_2)\vec{y} - \vec{F}_{\rm ex}(\vec{x})] \\ &\quad -\varepsilon\,[3\gamma^6(\vec{x}_2\cdot\vec{y})^2\,\vec{x}_2 + 3\gamma^4\,(\vec{x}_2\cdot\vec{y})\vec{y}]\big) \end{aligned} \tag{252}$$

Finally for the sake of simplicity we define

$$\vec{A} \;=\; \varepsilon(e^2/6\pi)\left[3\gamma^6(\vec{v}\cdot\dot{\vec{v}})^2\vec{v} + 3\gamma^4(\vec{v}\cdot\dot{\vec{v}})\dot{\vec{v}}\right] \tag{253}$$

Then the equation of motion takes the form

$$m\dot{\vec{v}} = \vec{F}_{\rm ex} + \vec{A} + \varepsilon(e^2/6\pi)(\gamma^2\ddot{\vec{v}} + \gamma^4(\vec{v}\cdot\ddot{\vec{v}})\vec{v}) \tag{254}$$

then

$$\varepsilon(\gamma^2\ddot{\vec{v}} + \gamma^4(\vec{v}\cdot\ddot{\vec{v}})\vec{v}) = (6\pi/e^2)(m\dot{\vec{v}} - \vec{F}_{\text{ex}} - \vec{A})$$

therefore

$$\varepsilon(\gamma^2\ddot{\vec{v}}(\mathbb{1} + \gamma^2\,|\vec{v}.\rangle\langle\vec{v}|) = (6\pi/e^2)(m\dot{\vec{v}} - \vec{F}_{\text{ex}} - \vec{A})$$

multiplying both sides of the equation by $\gamma^{-2}\kappa(\vec{v})^{-1}$ we get that

$$\varepsilon\ddot{\vec{v}} = \gamma^{-2}(6\pi/e^2)\kappa(\vec{v})^{-1}(m\dot{\vec{v}} - \vec{F}_{\text{ex}} - \vec{A}) = \vec{g}(\vec{v}, \dot{\vec{v}}, \varepsilon)$$

or equivalently in terms of the new set of variable defined above we have that

$$\varepsilon\dot{\vec{y}} = \vec{g}(\vec{x}, \vec{y}, \varepsilon) \tag{255}$$

also from equations (249) and (250)we have that

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{y})$$

so we have a similar set of equations as the one for the example above. For $\varepsilon = 0$ we have that:

$$\vec{g}(\vec{x}, \vec{y}, 0) = 0 \implies \vec{y} = h(x)$$

where we have settled $\vec{h}(\vec{x}) = m^{-1}\,\vec{F}_{\text{ex}}(\vec{x})$. So we have found our critical manifold $\mathcal{C}_0$ for $\varepsilon = 0$ it is defined by

$$\mathcal{C}_0 = \{(\vec{x}, \vec{h}(\vec{x})),\ \vec{x} \in \mathbb{R}^3 \times \mathbb{V}\} = \{(\vec{q}, \vec{v}, \dot{\vec{v}}) : m(\vec{v})\dot{\vec{v}} = \vec{F}_{\text{ex}}(\vec{q}, \vec{v})\}\,, \tag{256}$$

So we happily find that our center manifold corresponds to the solutions to the Lorentz force law under external fields. Now similarly to what we did in the example lets find out the behavior in the macroscopic scale by setting $\tau = \varepsilon^{-1}t$ and again denoting derivative with respect to τ by primes. Then

$$\vec{y}' = \vec{g}(\vec{x}, \vec{y}, \varepsilon)\ \ \vec{x}' = \varepsilon\vec{f}(\vec{x}, \vec{y}) \tag{257}$$

then making the limit $\varepsilon \to 0$ we get

$$\begin{aligned} \vec{x}\,' &= (\vec{x}_1', \vec{x}_2') = 0 \\ \vec{y}\,' &= \gamma^{-2}\kappa(\vec{x}_2)^{-1}\left((6\pi/e^2)\,[m(\vec{x}_2)\vec{y} - \vec{F}_{\text{ex}}(\vec{x})]\right. \end{aligned} \tag{258}$$

hence

$$\begin{aligned}(\vec{x}_1, \vec{x}_2) &= (\vec{x}_{1_0}, \vec{x}_{2_0}) = \vec{x}_0 \\ \vec{y}' &= \gamma^{-2}\kappa(\vec{x}_2)^{-1}\left((6\pi/e^2)\left[m\vec{y} - \vec{F}_{\mathrm{ex}}(\vec{x_0})\right]\right.\end{aligned} \tag{259}$$

hence the eigenvalue equation for the components of the last equation looks like:

$$\lambda = \gamma^{-2}\kappa(\vec{x}_2)^{-1}\left((6\pi/e^2)\left[m - F_{\mathrm{ex}}(\vec{x_0})\right]\right. \tag{260}$$

where F_{ex} is not a vector but a component. So the solution in components looks like

$$y = [m - \vec{F}_{\mathrm{ex}}(\vec{x_0})]e^{\lambda\tau} + y_0 \tag{261}$$

Therefore this is the second solution for the limit $\varepsilon \to 0$. But this solution has a small problem, that is that its eigenvalue is dominated by the term γ^{-2} which goes to zero as $|\vec{v}| \to 1$. Therefore this solution is not always repulsive or uniformly hyperbolic because of this discontinuity. This is very mischievous because this is required in order to apply the singular perturbation theory (see Singular Perturbation Theory by R. S. Johnson page 22 or Sakamoto page 45). So what we are going to do is to construct a $\vec{g}_\delta$ such that it is the same as $\vec{g}$ in the subspace $\mathbb{R}^3 \times \{\vec{v}, |\vec{v}| \leq 1 - \delta\}$ for some $0 < \delta$. This means that the solution will actually be the same as long as the velocity is such that $|\vec{v}| < 1 - \delta$. So now we are in position to apply the singular perturbation theory as long as $\vec{v}$ remain under the desire values, that is we want to find a $\delta = \delta(\bar{v})$ such that for $|\vec{v}(0)| \leq \bar{v} < 1$ for some $\bar{v}$, it holds for all times that $|\vec{v}(t)| \leq 1 - \delta$. What we want to demonstrate with this is that for the condition $|\vec{v}|(t) \leq< 1$ to hold for all times t it is mandatory for the dynamical variables to be in the center manifold, and since due to special relativity this is a compelling condition for the system to have any physical sense, we will conclude that only those solutions belonging to the center manifold will have any sense. This will be demonstrated by using some energy considerations. From the equation for the Lagrangian of the system (177) we see that the main contribution in the equation of motion for the charged particle may be derived from an effective Lagrangian:

$$L_{\mathrm{eff}}(\vec{q}, \dot{\vec{q}}) = T(\dot{\vec{q}}) - e\big(\phi_{\mathrm{ex}}(\vec{q}) - \dot{\vec{q}} \cdot A_{\mathrm{ex}}(\vec{q})\big)\,, \tag{262}$$

This is the same as (177) just that without the purely electromagnetic contribution. From this lagrangian we derived the conserved energy:

$$E_{\mathrm{eff}}\big(\vec{p} - eA_{\mathrm{ex}}(\vec{q})\big) + e\phi_{\mathrm{ex}}(\vec{q})\,. \tag{263}$$

and also the Hamiltonian:

$$H(\vec{q}, \vec{v}) = E_s(\vec{v}) + e\phi_{\mathrm{ex}}(\vec{q})\,, \tag{264}$$

where E_s is the energy minimizer for a given $\vec{v}$.

This effective energy is conserved because the Lagrangian and the Hamiltonian don't account for radiation losses. So in order to account for this dissipation of energy we subtract from the Hamiltonian the Schot term and obtain the following energy function:

$$G_\varepsilon(\vec{q},\vec{v},\dot{\vec{v}}) = H(\vec{q},\vec{v}) - \varepsilon\,(e^2/6\pi)\,\gamma^4(\vec{v}\cdot\dot{\vec{v}})\,. \tag{265}$$

So that the lost of energy by unit time is given by :

$$\frac{d}{dt}\,G_\varepsilon\,(\vec{q},\vec{v},\dot{\vec{v}}) = -\varepsilon\,(e^2/6\pi)\left[\gamma^4\dot{\vec{v}}^2 + \gamma^6(\vec{v}\cdot\dot{\vec{v}})^2\right]. \tag{266}$$

therefore we see that this energy function is decreasing in time. Since the energy of the particle possesses a lower bound established by its rest energy, the Schot term must be bounded on time that is :

$$\int_0^\infty dt\left[\gamma^4\,\dot{\vec{v}}(t)^2 + \gamma^6\,(\vec{v}(t)\cdot\dot{\vec{v}}(t))^2\right] < \infty \tag{267}$$

this again implies the asymptotic condition:

$$\lim_{t\to\infty}\,\dot{\vec{v}}(t) = 0\,. \tag{268}$$

So now let's see how does the dynamics evolves. The initial velocity is such that $|\vec{v}(0)| \le \overline{v} < 1$ therefore it is in the center manifold and thus $\vec{g}_\delta = \vec{g}$. So this dynamics must be governed by a critical manifold $\dot{\vec{v}} = \vec{h}_\varepsilon(\vec{q},\vec{v})$ and as we have seen this acceleration must be bounded in this critical manifold therefore, we have that $|\vec{h}_\varepsilon(\vec{q},\vec{v})| \le c_1 = c_1(\delta)$, for some constant c_1 depending on δ.

Now since this motion begins on $\mathcal{C}_\varepsilon$ it should remain there for some time and then at some critical time τ it will reach the boundary where $|\vec{v}| = 1-\delta$ and from then on $\vec{g}_\delta \ne \vec{g}$.We choose δ such that $\overline{v} \le 1-2\delta$.

Now since the energy function $G_\varepsilon\,(\vec{q}(t),\vec{v}(t),\vec{h}_\varepsilon(t))$ is a decreasing function, we have the following inequality:

$$\begin{aligned} G_\varepsilon\,(\vec{q}(t),\vec{v}(t),\vec{h}_\varepsilon(t)) &\le G_\varepsilon(0) = H(\vec{q}(0),\vec{v}(0)) - \varepsilon(e^2/6\pi)(\vec{v}(0)\cdot\vec{h}_\varepsilon(0)) \\ &\le E_s(\overline{v}) + e\phi_{\rm ex}\,(\vec{q}(0)) + \varepsilon c_1\,. \end{aligned} \tag{269}$$

Now since at τ the particle has already radiated some energy we have that

$$\begin{aligned} E_s(\vec{v}(\tau)) + e\overline{\phi} &\le H(\vec{q}(\tau),\vec{v}(\tau)) = G_\varepsilon(\tau) + \varepsilon(e^2/6\pi)\,\gamma^4\,(\vec{v}(\tau)\cdot\vec{h}_\varepsilon(\tau)) \\ &\le E_s(\overline{v}) + e\phi_{\rm ex}\,(\vec{q}(0)) + 2\varepsilon c_1 \end{aligned} \tag{270}$$

replacing $|\vec{v}| = 1-\delta$ and $\overline{v} \le 1-2\delta$ it follows that

$$E_s(1-\delta) \le E_s(1-2\delta) + e\,(\phi_{\rm ex}\,(\vec{q}(0)) - \overline{\phi}) + 2\varepsilon c_1\,. \tag{271}$$

For small δ: $E_s(1-\delta) \cong 1/\sqrt{\delta}$, which implies

$$\frac{1}{\sqrt{\delta}} \leq c_2 + 4\,\varepsilon c_1 \tag{272}$$

with $c_2 = 2e\,(\phi(\vec{q}(0)) - \overline{\phi})$. We choose now δ so small that $1/\sqrt{\delta} \geq c_2 + 1$ and then ε so small that $4\varepsilon c_1 < 1$. Then we have that

$$c_2 + 4\,\varepsilon c_1 < \frac{1}{\sqrt{\delta}} \tag{273}$$

which is a contradiction to equation (272). So by assuming that $\tau < \infty$ we arrived at a contradiction, so we conclude that $\tau = \infty$ which means that the solution will always remain on $\mathcal{C}_\varepsilon$. It needs to be remarked the fact that in this proof it was necessary that ε was a very small parameter.

Now that we have proven that the physically meaningful solutions to the equation of motion are always on $\mathcal{C}_\varepsilon$, lets find out how does this manifold looks like. Analogous to what was done in the mathematical example we have that for ε very close to zero we have that the critical manifold is of the form

$$\dot{\vec{v}} = \vec{y} = \vec{h}_\varepsilon(\vec{q}, \vec{v}) \tag{274}$$

Since the equations of motion for our system has exactly the same form as the equations from the example above it can be treated in a very analogous way, therefore $\vec{h}_\varepsilon$ depends smoothly on ε and again it must have a representation in terms of powers of ε whose first terms are:

$$\vec{h}_\varepsilon(\vec{q}, \vec{v}) = h_0(\vec{q}, \vec{v}) + \varepsilon \vec{h}_1(\vec{q}, \vec{v}) + \varepsilon^2 \vec{h}_2(\vec{q}, \vec{v}). \tag{275}$$

So deriving eq (274)

$$\dot{\vec{y}} = \frac{d}{dt}\vec{h}_\varepsilon(\vec{q}, \vec{v}) = \nabla_q \vec{h}_\varepsilon(\vec{q}, \vec{v})\vec{v} + \nabla_v \vec{h}_\varepsilon(\vec{q}, \vec{v})\vec{y} \tag{276}$$

where ∇_q means differentiation with respect to the components of q. Then from equation (254) we get that

$$\begin{aligned} \varepsilon(\nabla_q \vec{h}_\varepsilon(\vec{q}, \vec{v})\vec{v} + \nabla_v \vec{h}_\varepsilon(\vec{q}, \vec{v})\vec{y}) \;&=\; \gamma^{-2}\kappa(\vec{x}_2)^{-1}\left((6\pi/e^2)\,[m(\vec{x}_2)\vec{y} - \vec{F}_{\text{ex}}(\vec{x})]\right. \\ &\qquad \left. -\,\varepsilon\,[3\gamma^6(\vec{x}_2\cdot\vec{y})^2\,\vec{x}_2 + 3\gamma^4\,(\vec{x}_2\cdot\vec{y})\vec{y}]\right) \end{aligned} \tag{277}$$

So replacing the expansion for $\vec{y}$ for terms up to $\mathcal{O}(\varepsilon)$ we get that:

$$\begin{aligned} \varepsilon(\nabla_q(\vec{h}_0 + \varepsilon\vec{h}_1)\vec{v} + \nabla_v(\vec{h}_0 + \varepsilon\vec{h}_1)\vec{y}) \;&=\; \gamma^{-2}\kappa(\vec{x}_2)^{-1}\left((6\pi/e^2)\,[m(\vec{x}_2)(\vec{h}_0 + \varepsilon\vec{h}_1) - \vec{F}_{\text{ex}}(\vec{x})]\right. \\ &\qquad -\varepsilon\,[3\gamma^6(\vec{x}_2\cdot(\vec{h}_0 + \varepsilon\vec{h}_1))^2\,\vec{x}_2 \\ &\qquad \left. +\,3\gamma^4\,(\vec{x}_2\cdot(\vec{h}_0 + \varepsilon\vec{h}_1))(\vec{h}_0 + \varepsilon\vec{h}_1)]\right) \end{aligned} \tag{278}$$

Comparing terms of order zero we obtain that

$$m\vec{h}_0(\vec{q},\vec{v}) = m\vec{h}(\vec{q},\vec{v}) = \vec{F}_{\rm ex} \tag{279}$$

as was expected. Comparing terms of order ε

$$\begin{aligned}\nabla_q\vec{h}_0\vec{v} + \nabla_v\vec{h}_0\vec{y} = \gamma^{-2}\kappa(\vec{x}_2)^{-1}\,(6\pi/e^2)m\vec{h}_1 \quad &+ \quad \gamma^{-2}\kappa(\vec{x}_2)^{-1}[3\gamma^6(\vec{x}_2\cdot(\vec{h}_0))^2\,\vec{x}_2 \\ &\quad +3\gamma^4\,(\vec{x}_2\cdot\vec{h}_0)\vec{h}_0]\end{aligned}$$

Solving for $\vec{h}_1$:

$$\begin{aligned}m\vec{h}_1 = \gamma^2\kappa(\vec{x}_2)\,(e^2/6\pi)(\nabla_q\vec{h}_0\vec{v} + \nabla_v\vec{h}_0\vec{h}_0) \quad &- \quad (e^2/6\pi)[3\gamma^6(\vec{x}_2\cdot(\vec{h}_0))^2\,\vec{x}_2 \\ &\quad +3\gamma^4\,(\vec{x}_2\cdot\vec{h}_0)\vec{h}_0]\end{aligned}$$

so with this we replace in the expansion for $\dot{\vec{v}} = \vec{y} = \vec{h}_\varepsilon$ and we obtain the equation of motion:

$$\begin{aligned}m\dot{\vec{v}} = \vec{F}_{\rm ex} \quad &+ \quad \gamma^2\kappa(\vec{v})\,(e^2/6\pi)((\nabla_q\cdot\vec{F})_{\rm ex}\vec{v} + (\nabla_v\cdot\vec{F}_{\rm ex})\vec{F}_{\rm ex}) \\ &- \quad (e^2/6\pi)[3\gamma^6(\vec{v}\cdot\vec{F}_{\rm ex})^2\,\vec{v} + 3\gamma^4\,(\vec{v}\cdot\vec{F}_{\rm ex})\vec{F}_{\rm ex}]\end{aligned} \tag{280}$$

or equvivalently

$$m\dot{\vec{v}} = \vec{F}_{\rm ex} + \gamma^2\kappa(\vec{v})\,(e^2/6\pi)\left(\frac{d}{dt}\vec{F}_{\rm ex} - [3\gamma^6(\vec{v}\cdot\vec{F}_{\rm ex})^2\,\vec{v} + 3\gamma^4\,(\vec{v}\cdot\vec{F}_{\rm ex})\vec{F}_{\rm ex}]\right)$$

It is seen that this is a second order equation therefore it won't present preacceleration solutions and as explained before it wont runaway. Now the preceding analysis was done using as a starting point equation (235) which is not exactly the LAD equation, but it differs from it in a γ^2 term, which is a accounts for a proper relativistic kinetic energy, which is reflected in a change of the mass. The difference between this two equations arises because equation (235) was derived using the Abraham model whereas the LAD equation is calculated using the point particle model . So since this two equations share exactly the same structure thus their center manifold should look very alike. So now let's apply the singular perturbation method to LAD equation. Note that in order to apply the singular perturbation method to the LAD equation it is necessary to discard the point particle model because singular perturbation requires that **Theorem 1** in section 8.4 applies and one of the fundamental assumptions to prove it is that the charge density should be smooth, therefore if we apply singular perturbation theory to LAD we are demanding that the charge distribution should be extended.

The LAD equation in covariant form (we introduce again the c)reads:

$$ma^\mu = F^\mu_{ext} + \frac{2}{3}\frac{e^2}{c^3}(\ddot{v}^\mu - v^\mu\dot{v}^\lambda\dot{v}_\lambda). \tag{281}$$

Lets find out how it looks like in three vectors form. In order to do this reduction we will need the expression which relates us the four vectors with three vector which are

$$
\begin{aligned}
v^\mu &= (\gamma c; \gamma \vec{v}) \\
\dot{v}^\mu &= (\frac{\gamma^4}{c}\vec{v}\cdot\dot{\vec{v}}; \gamma^2\dot{\vec{v}} + \frac{\gamma^4}{c^2}\vec{v}\cdot\dot{\vec{v}}\,\vec{v}) \\
\ddot{v}^\mu &= (\ddot{v}^0; \gamma^3\ddot{\vec{v}} + \frac{3\gamma^5}{c^2}\vec{v}\cdot\dot{\vec{v}}\,\dot{\vec{v}} + \frac{1}{c}\ddot{v}^0\vec{v}) \\
\ddot{v}^0 &= \frac{\gamma^5}{c}(\vec{v}\cdot\ddot{\vec{v}} + \dot{\vec{v}}\cdot\dot{\vec{v}}) + 4\frac{\gamma^7}{c^3}(\vec{v}\cdot\dot{\vec{v}})
\end{aligned}
$$

Now replacing this expressions in the LAD equation we make the reduction and we get that:

$$
\begin{aligned}
m(\gamma^2\dot{\vec{v}} + \frac{\gamma^4}{c^2}(\vec{v}\cdot\dot{\vec{v}})\,\vec{v}) &= \gamma\vec{F}_{ext} + \frac{2}{3}\frac{e^2}{c^3}\Big[\gamma^3\ddot{\vec{v}} + \frac{3\gamma^5}{c^2}(\vec{v}\cdot\dot{\vec{v}})\,\dot{\vec{v}} + \frac{\vec{v}}{c}\big(\frac{\gamma^5}{c}(\vec{v}\cdot\ddot{\vec{v}} + \dot{\vec{v}}\cdot\dot{\vec{v}}) + 4\frac{\gamma^7}{c^3}\vec{v}\cdot\dot{\vec{v}}\big) \\
&- \frac{\vec{v}}{c^2}[(\frac{\gamma^4}{c}\vec{v}\cdot\dot{\vec{v}})^2 - (\gamma^2\dot{\vec{v}} + \frac{\gamma^4}{c^2}(\vec{v}\cdot\dot{\vec{v}})\,\vec{v})^2]\Big]
\end{aligned}
$$

This expression is greatly simplified if we introduce the already defined $\kappa(\vec{v})$. So simplifying we get that:

$$
m_0\,\gamma\,\kappa(\vec{v})\dot{\vec{v}} = \vec{F}_{\text{ext}} + \frac{2}{3}\frac{e^2}{c^3}\,\gamma^2\kappa(\vec{v})\,[\ddot{\vec{v}} + 3\gamma^2\,c^{-2}\,(\vec{v}\cdot\dot{\vec{v}})\,\dot{\vec{v}}]\,, \tag{282}
$$

No in order to apply SPT we again write the LAD equation (282) in singular perturbathion theory form:

$$
\dot{\vec{x}} = \vec{f}(\vec{x},\vec{y}), \qquad \varepsilon\dot{\vec{y}} = \vec{g}(\vec{x},\vec{y},\varepsilon) \tag{283}
$$

also we define:

$$
\vec{f}(\vec{x},\vec{y}) = (\vec{x}_2, \vec{y}) \tag{284}
$$

$$
\begin{aligned}
\vec{g}(\vec{x},\vec{y},\varepsilon) &= (6\pi c^3/e^2)\big(m_0\,\gamma^{-1}\vec{y} - e\,\gamma^{-2}\kappa(\vec{x}_2)^{-1}(\vec{E}(\vec{x}_1) + c^{-1}\vec{x}_2\times\vec{B}(\vec{x}_1))\big) \\
&\quad -3\varepsilon\gamma^2c^{-2}(\vec{x}_2\cdot\vec{y})\vec{y}\,.
\end{aligned} \tag{285}
$$

which after aplying the singular perturbation methos procedure in exactly the same manner as we already did, and fixing the parameter $\varepsilon = 1$ we obtain that:

$$
\begin{aligned}
\dot{\vec{q}} &= \dot{\vec{v}}\,, \\
m_0\,\gamma\,\kappa(\vec{v})\dot{\vec{v}} &= e(\vec{E} + c^{-1}\vec{v}\times\vec{B}) \\
&\quad +\frac{2}{3}\frac{e^2}{c^3}\Big[\frac{e}{m_0}\,\gamma\,(\vec{v}\cdot\nabla)(\vec{E} + c^{-1}\vec{v}\times\vec{B}) + \big(\frac{e}{m_0}\big)^2\,c^{-1}\Big((\vec{E}\times\vec{B}) \\
&\quad +c^{-1}(\vec{v}\cdot\vec{E})\vec{E} + c^{-1}(\vec{v}\cdot\vec{B})\vec{B} + \big(-\vec{E}^2 - \vec{B}^2 \\
&\quad +c^{-2}(\vec{v}\cdot\vec{E})^2 + c^{-2}(\vec{v}\cdot\vec{B})^2 + 2c^{-1}\vec{v}\cdot(\vec{E}\times\vec{B})\big)\gamma^2c^{-1}\vec{v}\Big)\Big]\,.
\end{aligned} \tag{286}
$$

but this may be written as

$$m\frac{d}{dt}(\gamma\vec{v}) = \vec{F} + \tau_e\left[\gamma\frac{d\vec{F}}{dt} - \frac{\gamma^3}{c^2}\left(\frac{d\vec{v}}{dt}\times(\vec{v}\times\vec{F})\right)\right] \tag{287}$$

where the fields are calculated from the aproximation:

$$\frac{d\vec{v}}{dt} = \vec{F} \tag{288}$$

Which is also the equation at which Ford and O'Conell arrive, which actually is the already mentioned LL equation in noncovariant form. Under physically acceptable initial conditions the solution to this equations will be stable, that is to say that C_ε is invariant under the solution flow of this equation, therefore since this equation avoids the problems inputed lo LAD we conclude that it is the correct equation of motion for a charged particle under the Abraham model for the particle.

9 Stochastic treatment of the self force

The problem of radiation reaction arises a cause of the energy lost in form of radiation during the motion of a particle under acceleration. But where does this energy goes?. Clearly once it leaves the particle this energy becomes part of the electromagnetic field which surrounds the particle. This transference of energy from the particle to the field implies that the motion of the particle will be damped, therefore its dynamics resembles in some how the friction problems in classical mechanics or the dissipative systems in statical mechanics. It is in this context that G.W Ford's stochastic approach for the problem of radiation reaction arises. Ford and O'Conell treat the problem of the radiation reaction by treating the radiation field as a heat bath in which a quantum dipole oscillator is embedded, then the radiation field acts as a dissipative system to which the quantum particle transfers energy and momentum as it moves. So the procedure that we'll follow to expose the stochastic treatment of the self force will be like this. First we will make a brief introduction to the Langevin equation which describes the dynamics of many nonconservative systems, then we will find out some of the memory function included in the Langevin equation, from this we will use the Hamiltonian of a quantum dipole oscillator embedded in heat bath to obtain the explicit form of the memory function for our case and with this we make connection with the Langevin equation to use it as the equation of motion of the charged particle.

9.1 The Langevin equation

In a dissipative one dimensional problem the equation which describes the motion of a particle interacting with a dissipative system is the Langevin equation:

$$m\ddot{x}(t) + \int_{-\infty}^{t} dt'\mu(t-t')\dot{x}(t') + \frac{dV(x)}{dx} = f(t)\,, \tag{289}$$

where m is the mass of the particle and x is its position, *f(t)* is a fluctuating random force whose mean value is zero, $V(x)$ is the potential of the system and $\mu(t)$ its called the memory function. The potential we will use will have the form of an oscillator

$$V(x) = \frac{1}{2} + Kx^2$$

Here the interaction of the particle with its surrounding fields is contained in the term $\int_{-\infty}^{t} dt' \mu(t-t')\dot{x}(t')$ which includes the so called memory function $\mu(t)$ which behaves as a complex distribution.

This term has very interesting properties which contain the dynamics of the motion of the particle and which will be studied in the following sections.

9.2 Properties of the memory function

The first remarkably property of $\mu(t)$ is the so called reality condition, stated as:

$$\tilde{\mu}(\omega + \imath 0^{+}) = \tilde{\mu}(-\omega + \imath 0^{+})^{*} \tag{290}$$

with the asterisk denoting complex conjugation the tilde over the μ denoting that it is in Fourier space. Let's see how this condition arises. If we apply Fourier transform to the Langevin equation we get that

$$-m\omega^2 \tilde{x}(\omega) - \imath\omega\tilde{\mu}(\omega)\tilde{x}(\omega) + K\tilde{x}(\omega) = \tilde{F}(\omega) \tag{291}$$

Then this equation takes the form:

$$\tilde{x}(\omega) = \alpha(\omega)\tilde{F}(\omega) \tag{292}$$

with α given by

$$\alpha(\omega) = [-m\omega^2 - \imath\omega\tilde{\mu}(\omega) + K]^{-1} \tag{293}$$

Thus since $x(t)$ is an hermitian real valued operator this implies that its Fourier transform must satisfy the reality condition $\tilde{x}(\omega) = \tilde{x}(-\omega)*$ thus in general $\alpha(\omega)$ must satisfy this condition which implies that $\mu(\omega)$ must also satisfy the reality condition (290).

The second interesting property of the memory function is that as we approach the real axis from above the real part of the function is positive, that is to say:

$$\mathbf{Re}[\tilde{\mu}(\omega + \imath 0^{+})] \geq 0, \qquad -\infty < \omega < \infty \tag{294}$$

The beauty of this condition is that it is a direct consequence of the second law of thermodynamics applied to our system of the dipole oscillator coupled to the radiation field.

The second law of thermodynamics states in one of it's form's that "it is impossible to construct an engine whose net effect on a system is to extract heat from the system and convert it into mechanical work". This in turn implies that if we have an isolated reservoir and we apply some force on it the net work done by the force on the reservoir must be positive. Therefore considering the system of the particle coupled to it's radiation field as a heat reservoir and a incident external field which exerts a force $f(t)$ in the system we know that the work on the particle and it's field must be positive. This work is given by:

$$W = \int_{-\infty}^{\infty} dt f(t) \langle v(t) \rangle \qquad > 0 \tag{295}$$

Now we may express the work in terms of the Fourier transform of $f(t)$ and $v(t)$ by making use of the Parseval's theorem:

$$W = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{f}(\omega) \langle \tilde{v}(-\omega) \rangle \quad > 0 \tag{296}$$

where we have used mean values for the velocity because that due to the random motion of particles in stochastic systems the second law of thermodynamics refers to mean values not to specific values of the dynamical variables of the particles.

Now in order to obtain an expression for $\tilde{f}(\omega)$ we go back to the Langevin equation.

$$m\dot{v}(t) + \int_{-\infty}^{t} dt' \mu(t-t') \langle v(t') \rangle = f(t) \tag{297}$$

Now from this equation we can obtain $\tilde{f}(\omega)$, we just need to apply Fourier transform, but in order to do it we need to apply the convolution theorem on the second term.

The convolution theorem says that if $f(t)$ and $g(t)$ are two integrable functions and if $h(t)$ is its convolution then

$$h(t) = \int_{-\infty}^{\infty} f(t-t') g(t') dt' \tag{298}$$

therefore the Fourier transform of $h(t)$ is the product of the Fourier transforms of $f(t)$ and g(t) that is

$$\tilde{h}(\omega) = \tilde{f}(\omega) \cdot \tilde{g}(\omega) \tag{299}$$

Then applying this theorem to the second term of equation (297) we get that the Fourier transform of this equation is :

$$-i\omega m \langle \tilde{v}(\omega) \rangle + \tilde{\mu}(\omega + 0^{+}) \tilde{v}(\omega) \rangle = \tilde{f}(\omega) \tag{300}$$

Thus inserting this expression into the expression for the work (equation (296)) we get that

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega[-i\omega m + \tilde{\mu}(\omega + 0^{+})]\langle\tilde{v}(\omega)\rangle\langle\tilde{v}(-\omega)\rangle \tag{301}$$

Now since $\langle v(t)\rangle$ is a real function it's Fourier transform satisfies the reality condition

$$\tilde{v}(-\omega) = \tilde{v}(\omega)* \tag{302}$$

where the asterisk denotes complex conjugation.

Thus we get that

$$\begin{aligned} W &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega[-i\omega m + \tilde{\mu}(\omega + 0^{+})]\langle\tilde{v}(\omega)\rangle^{2} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega Re[\tilde{\mu}(\omega + 0^{+})]\langle\tilde{v}(\omega)\rangle^{2} \end{aligned} \tag{303}$$

here the first term vanished because it is an odd function. We left only the real part of the $\tilde{\mu}(\omega + 0^{+})$ because that due to the reality condition (equation (290))) we see that the real part of $\tilde{\mu}(\omega + 0^{+})$ is an even function and its imaginary part is an odd function thus only the odd part vanishes when we perform the integration. Now since the work is a positive quantity, so it is $\langle\tilde{v}(\omega)\rangle^{2}$ we conclude that $Re[\tilde{\mu}(\omega + 0^{+})]\langle$ must also be positive. This is a very important result which we'll be very useful later to ensure causality in the equation of motion of the electron.

9.3 From Langevin equation to the Quantum dipole oscillator.

Now in order to apply the Langevin equation to a specific system it is necessary to obtain an explicit expression for the memory function. It is in this function that the information and dynamics of the system is contained therefore it must be contained somehow in the Hamiltonian. For the case in which we are working of a radiating dipole coupled to a radiation field the Hamiltonian is found to be (Appendix 3):

$$H = \frac{1}{2m}[\mathbf{p} + \frac{e}{c}\mathbf{A}]^{2} + V(\mathbf{r}) + \sum_{\mathbf{k},s}\hbar\omega_{k}(a^{\dagger}_{\mathbf{k},s}a_{\mathbf{k},s} + \frac{1}{2}) \tag{304}$$

Where $\mathbf{A}$ is the vector potential given by

$$\mathbf{A} = \sum_{k,s}\left[\frac{2\pi\hbar c^{2}}{\omega_{k}\mathbf{V}}\right]^{\frac{1}{2}} f_{k}\hat{\mathbf{e}}_{\mathbf{k}s}(a_{\mathbf{k},s} + a^{\dagger}_{\mathbf{k}s}) \tag{305}$$

with f_k being the electron form factor and $\hat{\mathbf{e}}_{\mathbf{k}s}$ the polarization vector.

The form factor is the tree dimensional fourier transform of the charge density:

$$\rho(\vec{x}) = \frac{e}{(2\pi)^3} \int d^3k f(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \tag{306}$$

Now in appendix 2 it is shown that this Hamiltonian is the three dimensional generalization of another Hamiltonian which is equivalent by a pair of canonical transformations to the Hamiltonian of an independent oscillator model (IO model). It is important to note that the following analysis is of Quantum Mechanical origin therefore we will be talking of operators not coordinates.

In the independent oscillator model a quantum particle moves in a system composed of a large number of heat bath particles which are coupled to it by spring like forces. Therefore the Hamiltonian for this system is given by:

$$H = \frac{p^2}{2m} + \mathbf{V}(x) + \sum_j [\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 (q_j - x)^2] \tag{307}$$

where the first two terms account for the energy of the particle under the potential $V(x)$ and the terms in the summation account for the kinetic energy of the heat bath particles and their interaction with the moving one. Now applying the Heisenberg equations to the Hamiltonian we obtain that:

$$\dot{x} = \frac{1}{i\hbar}[x, H] = \frac{1}{i\hbar}[x, \frac{p^2}{2m}] = \frac{2p}{i\hbar 2m}[x, p] = \frac{p}{m} \tag{308}$$

$$\begin{aligned} \dot{p} &= \frac{1}{i\hbar}[p, H] = \frac{1}{i\hbar}[p, V(x) + \sum_j \frac{1}{2} m_j \omega_j^2 (q_j - x)^2] \\ &= -\frac{dV(x)}{dx} + \sum_j m_j \omega_j^2 (q_j - x) \end{aligned}$$

so eliminating the momentum from these equations we get that:

$$\dot{p} = m\ddot{x} = -\frac{dV(x)}{dx} + \sum_j m_j \omega_j^2 (q_j - x) \tag{309}$$

And for the heat bath particles the Heisenberg equations give:

$$\dot{q}_j = \frac{1}{i\hbar}[q_j, H] = \frac{p_j}{m_j}, \tag{310}$$

$$\dot{p}_j = \frac{1}{i\hbar}[p_j, H] = -m_j \omega_j^2 (q_j - x). \tag{311}$$

Again we eliminate the momentums and we get that:

$$\dot{p}_j = \ddot{q}_j = -m_j\omega_j^2(q_j - x). \tag{312}$$

This is a non homogeneous differential equation. If the values of $q_j(t)$ and $\dot{q}_j(t)$ at $t = 0$ are $q_j(0) = q_j$ and $\dot{q}_j(0) = p_j/(m_j\omega_j)$ where q_j and p_j are time-independent operators which satisfy the usual commutation rule $[q_j, p_j] = i\hbar$ then the solution to the homogeneous equation is:

$$q_j^h = q_j \cos(w_j t) + p_j \frac{\sin(w_j t)}{m_j\omega_j} \tag{313}$$

The general solution of this equation then takes the form:

$$q_j(t) = q_j^h(t) + x(t) - \int_{\infty}^{t} dt' \cos[\omega_j(t - t')]\dot{x}(t') \tag{314}$$

The fact that in this solution we integrate from time $-\infty$ to time t t in the retarded solution, confirms the nonmarkovian nature of the system, that is the motion of the particle depends on the past history of the system, which is due to the fact that the electromagnetic field has a time dependence because not only the electromagnetic fields affects the particle but the particle affects the electromagnetic field. This might be seen clearer if we insert equation (314) into (309) to obtain the resulting equation of motion for the particle:

$$m\ddot{x} + \frac{dV(x)}{dx} = \sum_j m_j\omega_j^2 q_j^h(t) - \sum_j m_j\omega_j^2 \int_{\infty}^{t} dt' \cos[\omega_j(t - t')]\dot{x}(t') \tag{315}$$

It is observed in this equation that the the particle abbeys a Lagevin type equation of motion, with the last term being the dissipative force term. Therefore comparing equations (315) and (289) we might identify the memory function which turns out to be:

$$\mu(t) = \sum_j m_j\omega_j^2 \cos(\omega_j t)\Theta(t) \tag{316}$$

with $\Theta(t)$ a Heaviside function accounting for time causality as already mentioned. The other terms in the equations are, $V'(x)$ which is the responsible for the force caused by the external field and $\sum_j m_j\omega_j^2 q_j^h(t)$ whose mean value is zero (because the mean value of the sine and cosine functions is zero) therefore this term is identified as the random fluctuating force $F(t)$.

Now in order to perform the complete determination of $\mu(t)$ let's find its Fourier transform $\tilde{\mu}(z)$:

$$\begin{aligned}
\tilde{\mu}(z) &= \int_0^\infty e^{izt}\sum_j m_j\omega_j^2\cos(\omega_j t)\Theta(t)dt = \sum_j m_j\omega_j^2\frac{iz}{z^2-\omega_j^2} \\
&= \frac{i}{2}\sum_j m_j\omega_j^2[\frac{1}{z-\omega_j}+\frac{1}{z+\omega_j}]
\end{aligned} \tag{317}$$

Now as we know from above we are not interested in the whole function $\tilde{\mu}(z)$ but in the real part of it near the real axis $\mathbf{Re}[\tilde{\mu}(\omega + i0^+)]$ to obtain this we use the formula:

$$\frac{1}{x+i0^+} = P(\frac{1}{x}) - i\pi\delta(x) \tag{318}$$

so we get that

$$\mathbf{Re}[\tilde{\mu}(\omega + i0^+)] = \frac{\pi}{2}\sum_j m_j\omega_j^2[\delta(\omega-\omega_j)+\delta(\omega+\omega_j)] \tag{319}$$

Now since this memory function was obtained for the Hamiltonian of the IO model which is equivalent to a form of our Hamiltonian for the blackbody radiation field but in one dimension (Appendix 2) it is necessary to make certain modifications. First we need to multiply by a factor $\frac{2}{3}$ which accounts for the two spatial components of $\mathbf{k}$ that contributes two the coupling (there's one component that does not contribute because the transversality condition). Secondly to make explicit the fact that we are working with a extended model of the electron, we use a form factor f defined by the formula:

$$m_k = \frac{4\pi e^2 f^2}{\omega_k^2 V}, \tag{320}$$

therefore we obtain for the blackbody radiation system:

$$\mathbf{Re}[\tilde{\mu}(\omega + i0^+)] = \frac{\pi}{3}\sum_k m_k\omega_k^2[\delta(\omega-\omega_k)+\delta(\omega+\omega_k)] \tag{321}$$

$$= \frac{4\pi^2 e^2}{3V}\sum_k f_k^2[\delta(\omega-\omega_k)+\delta(\omega+\omega_k)] \tag{322}$$

Now working in a large volume of space we may pass the variable $\mathbf{k}$ from discrete to continous space

$$\sum_k \rightarrow \frac{V}{(2\pi)^3}\int d\mathbf{k} \tag{323}$$

so the spectral distribution becomes

$$\mathbf{Re}[\tilde{\mu}(\omega + i0^{+})] = \frac{e^2}{6\pi}\int d\mathbf{k} f_k{}^2[\delta(\omega - \omega_k)] = \frac{2e^2\omega^2}{3c^3} f_k^2 \tag{324}$$

Now it is necessary to choose a convenient form factor. Since the shape of the electron is not an observable, the form factor which in some manner characterizes the shape of the charge can't be measured, thus it is in some way arbitrary, but necessary to assure that the self energy of the particle be finite.The only real requirement that we impose on it is that it must be different from one $f \neq 1$ because as already mentioned this leads to a charge density in form a Dirac delta which corresponds to a point particle and we would end up again with the LAD equation . Therefore we want that the equation of motion of the particle won't be affected by the specific form of the form factor.

Hence we use the following form factor first introduced by Feynman

$$f^2 = \frac{\Omega^2}{\omega^2 + \Omega^2}. \tag{325}$$

In order to give physical meaning to this form factor note that it is very close to the unity up to some cutoff frequency Ω and then it falls out to zero. So if we think on De Broglie frequencies and wave lengths we realize that the cutoff frequencies determines in some manner the particles wavelength which going classically gives the particles radius. This form factor was first introduced by Feynman in one of his famous papers on Quantum Electrodynamics, so once again we somehow rely in the results of some higher order theory and then perform the limit to the classical case.

Now with this form factor, we can make use of the Stieltjes inversion theorem to find the representation of the Fourier transform of the memory function in the upper half plane. The Stieltjes inversion theorem states that the representation of the most general positive function in the upper half plane is of the following form:

$$\tilde{\mu}(z) = -icz + \frac{2iz}{\pi}\int_0^\infty d\omega \frac{\mathbf{Re}[\tilde{\mu}(\omega + i0^{+})]}{z^2 - \omega^2}, \tag{326}$$

where c is a positive constant which for our case is the speed of light.

$$\mathbf{Re}[\tilde{\mu}(\omega + i0^{+})] = \frac{2e^2\omega^2}{3c^3}\frac{\Omega^2}{\omega^2 + \Omega^2} \tag{327}$$

$$\begin{aligned} \tilde{\mu}(z) &= -icz + \frac{2iz}{\pi}\frac{2e^2\Omega^2}{3c^3}\int_0^\infty d\omega \frac{\omega^2}{(\omega^2 + \Omega^2)(z^2 - \omega^2)} \\ &= -icz - \frac{2iz}{\pi}\frac{e^2\Omega^2}{3c^3}\frac{\pi}{\Omega - iz} \\ &= \frac{2e^2\Omega^2}{3c^3}\frac{z}{z + i\Omega} \end{aligned} \tag{328}$$

Finally we do the inverse Fourier transform for $\tilde{\mu}(z)$ to obtain $\mu(t)$, hence

$$\mu(t) = \frac{2e^2\Omega^2}{3c^3}[2\delta(t) - \Omega e^{-\Omega t}] = M\Omega^2\tau_e[2\delta(t) - \Omega e^{-\Omega t}] \tag{329}$$

where we have called $M = 2e^2/3c^3$ because it corresponds to the observed value for the mass of the electron. Therefore M is the effective mass of the electron .A widely used result from Quantum electrodynamics states that its relation with the bare mass of the electron is given by:

$$M = m + \frac{2e^2\Omega}{3c^3} = m + \tau_e\Omega M \tag{330}$$

(compare with section 4.2)

Then replacing this form factor in the Langevin equation we get

$$m\ddot{x}(t) + M\Omega^2\tau_e^2\dot{x}(t) - M\Omega^3\tau_e\int_{-\infty}^{t} dt' e^{-\Omega(t-t')}\dot{x}(t') + V'(x) = F(t) + f(t) \tag{331}$$

Now in order to solve this equation we multiply by the factor $e^{-\Omega t}\frac{d}{dt} \times e^{\Omega t}$ and we obtain the following operators equation :

$$\begin{aligned} e^{-\Omega t}\frac{d}{dt}[e^{\Omega t}F(t) + e^{\Omega t}f(t)] &= e^{-\Omega t}\frac{d}{dt}\big[e^{\Omega t}m\ddot{x}(t) \\ &+ e^{\Omega t}M\Omega^2\tau_e^2\dot{x}(t) - M\Omega^3\tau_e\int_{-\infty}^{t} dt' e^{\Omega t}e^{-\Omega(t-t')}\dot{x}(t') + V'(x)\big] \end{aligned} \tag{332}$$

Now performing the operations we get that

$$\begin{aligned} \Omega F(t) + \dot{F}(t) + \Omega f t + \dot{f}(t) &= \Omega m\ddot{x}(t) + m\dddot{x}(t) + M\Omega^3\tau_e^2\dot{x}(t) + M\Omega^2\tau_e^2\ddot{x}(t) \\ &- M\Omega^3\tau_e\left(x(t')e^{\Omega t'}\Big|_{-\infty}^{t} - \int_{-\infty}^{t} dt' e^{\Omega t'}x(t')\right) \\ &+ \Omega V'(x) + \dot{V}'(x) \end{aligned}$$

Now the terms coming from the integral (the ones in the second lane) should vanish because we can set the initial position wherever we want, so dividing by Ω we obtain that :

$$F(t)_{\text{eff}} + f(t)_{\text{eff}} - V'(x)_{\text{eff}} = m\ddot{x}(t) + \frac{m}{\Omega}\dddot{x}(t) + M\Omega^2\tau_e^2\dot{x}(t) + M\Omega\tau_e^2\ddot{x}(t)$$

where

$$f(t)_{\text{eff}} \equiv f(t) + \frac{\dot{f}(t)}{\omega}$$

similarly with V_{eff} and $F(t)_{\text{eff}}$. So dividing by Ω and using the expression for the renormalized mass (330) we find that:

$$M(\Omega - \tau_e^{-1})\dddot{x}(t) + M\ddot{x}(t) = F(t)_{\text{eff}} + f(t)_{\text{eff}} - V(x)_{\text{eff}} \tag{333}$$

Thus since this is a operators equation in order to go to the classical case we take mean values in which case the random force vanishes, so if we have a zero external potential , we get the following classical equation:

$$M(\Omega^{-1} - \tau_e)\dddot{\vec{x}}(t) + M\ddot{\vec{x}}(t) = \vec{f}(t) + \frac{\dot{\vec{f}}(t)}{\omega} \tag{334}$$

which is a third order equation, but we will see that it might be reduced under certain approximations to a second order equation.

9.4 Avoiding runaway solutions

Now we have seen that runaway solutions appear as a conceptual problem in the LAD equation because when one solves it for the case of no external field there appears a solution which predicts the particle will accelerate to infinity, but we will now see that time causality assures that this will be rather unphysical situation. So lets first solve equation (334) for the no external force case. The solutions are

$$\ddot{x}(t) = 0 \qquad or \qquad \ddot{x}(t) = e^{t/(\Omega^{-1}-\tau_e)}\ddot{x}(0) \tag{335}$$

Clearly the first solution does not run away, so lets see that actually the second one neither does. From the renormalization relation (330) we have that

$$\Omega\frac{M}{m} = (\Omega^{-1} - \tau_e)^{-1} \tag{336}$$

Thus the second solution reads:

$$\ddot{x}(t) = e^{-\Omega\frac{M}{m}t}\ddot{x}(0) \tag{337}$$

It is seen that the runaway solutions for this equation occurs for $m \leq 0$, otherwise the acceleration will begin to decrease until constant movement. But now we will see that $m \leq 0$ is not posible. This will be based on the results of 9.1 which stated that the memory function must be a positive function.

If we apply Fourier Transform to the Langevin equation (297) this equation takes the form:

$$\tilde{x}(\omega) = \alpha(\omega)\tilde{F}(\omega) \tag{338}$$

with α given by

$$\alpha(\omega) = [-m\omega^2 - \imath\omega\tilde{\mu}(\omega) + K]^{-1} \tag{339}$$

Now if we insert in this equation the expression found for $\omega\tilde{\mu}$ (eq)we get that:

$$\alpha(\omega) = \frac{\omega + i\Omega}{-m\omega^3 - \imath M\Omega\omega^2 + K(\omega + i\Omega)} \tag{340}$$

The taking the roots of the denominator in this expression we find that it should be factorisable to yield an expression of the form:

$$\alpha(\omega) = \frac{\omega + i\Omega}{-m(\omega + i\Omega')(\omega_0^2 - \omega^2 - i\sigma\omega)} \tag{341}$$

Then comparing this equations we obtain that

$$\frac{1}{\Omega} = \frac{1}{\Omega'} + \tau_e.$$

$$\omega_0 = (K/M)^{1/2}, \qquad \sigma = 2e^2\omega_0^2/3Mc^3$$

Due to the fact that $\mu(\omega)$ is a positive function the poles of $\alpha(\omega)$ (equation (339)) lie in the lower half plane of the complex plane, therefore from equation (341) we see that $\frac{1}{\Omega'} \geq 0$ which implies $\Omega^{-1} \geq \tau_e$ which from the expression for the renormalized mass (equation (330)) implies that $m \geq 0$. Thus by imposing causality we proved that $\omega\tilde{\mu}$ is a positive function and this combined with the the Langevin equation implies that the bare mass of the particle is a positive quantity and that therefore runaway solutions are not physically possible. It is worth mentioning that if we do the limit $\lim_{\Omega\to\infty} f_k$ and replace this form factor in the Langevin equation we arrive at the LAD equation. This happens because this limit corresponds to the point electron. But as we have seen we have imposed an upper limit to Ω which is $\Omega_e = \tau_e^{-1}$ therefore the point electron is not allowed in our model. So we see that we have put an upper bound to the cutoff frequency which might be interpreted classically that the electron radius has a lower bound.

9.5 Second order equation

Note that the results that we have obtained until now have not yet putted any value to the cutoff frequency, except the condition that it be no bigger than τ_e^{-1}. This very good because the setting of an exact value for this frequency would need insight into the structure features of the particle and therefore there would not be possible to solve the problem by a classical approach. Moreover we want that the equation of motion of the electron to be independent of *Omega* so lets see how much is this equation affected by this value. Since as we have said the cutoff factor is closely related to the radius of the electron its value should be near τ_e^{-1}

thus the third order term in equation (334) should be very small compared to the other terms. Now the term dependent on the time derivative of the force is of order $\omega\Omega^{-1}$. So for fields with frequencis much smaller than Ω this term is also very small compared to the other ones, so a first aproximation to the dynamics of the particle is:

$$M\ddot{\vec{x}}(t) = \vec{f}(t)$$

But if we work in a high frequency (but under the established limit) to first order the derivative of this equation constitutes an very well approximation to the term $\dddot{x}$, thus replacing this in equation (334) we get the second order equation :

$$\dot{\vec{f}}(\Omega^{-1} - \tau_e)(t) + M\ddot{\vec{x}}(t) = f(t) + \frac{\dot{\vec{f}}(t)}{\omega}$$

thus

$$M\ddot{\vec{x}}(t) = \vec{f}(t) + \tau_e\dot{\vec{f}}(t) \qquad (342)$$

We see that have obtained a second order equation which does not present the classical problems of the LAD equation and which is independent of the cutoff frequency. But we have derived it under the requirement that Ω be smaller but of the order of ($\Omega_e = \tau_e^{-1}$) and that $\omega \ll Omega$ which might be seem an arbitrary requirement. But from the anlysis above and some Quantum electrodynamical results it is found that indeed Ω is of the order of τ_e^{-1} . Also the requirement that the frequency of the fields be smaller than $Omega$ is very plausible because a field of this order would have a frequency about $10^{24}Hz$ which besides being far away from the attainable fields obtained in labs its study is rather of the domain of Quantum Field theory. Therefore we see that for the classical case the Landau Lifshitz equation is the right equation of motion for the electron.

Now lets perform the relativistic generalization. In four vectors the Lorentz force Law looks like

$$Ma^\mu = \frac{e}{c}F^{\mu\lambda}v_\lambda \qquad (343)$$

so looking at (342) we conclude that the radiation terms adds to the Lorentz force an extra term F^μ_{self} which must be linear in τ_e. Looking at the procedure that we used to derive (342) we see that F^μ_{self} must depend on the derivative with respect to the proper time of the external force. Also the four vector must satisfy the identity $F^{mu}_{\text{self}}v_\mu = 0$ which is valid for all forces. Then a four vector which acomplishes this properties and where added to the Lorentz force law reduces to (342) in the particles rest frame is given by :

$$F^\mu_{\text{self}} = \tau_e\frac{e}{c}\left(\frac{d}{d\tau}F^{\mu\lambda}v_\lambda - \frac{1}{c^2}v^\mu v_\alpha\frac{d}{d\tau}(F^{\alpha\lambda}v_\lambda)\right) \qquad (344)$$

in a fixed reference frame this equation takes the form

$$m\frac{d}{dt}(\gamma\vec{v}) = \vec{F} + \tau_e\left[\gamma\frac{d\vec{F}}{dt} - \frac{\gamma^3}{c^2}\left(\frac{d\vec{v}}{dt}\times(\vec{v}\times\vec{F})\right)\right] \tag{345}$$

and in the particle rest frame clearly reduces to (342). Now using the aproximation

$$\frac{dv^\mu}{d\tau} = \frac{e}{mc^2}F^{\mu\lambda}v_\lambda \tag{346}$$

we get that:

$$m_0\dot{v}^\mu = eF^{\mu\alpha}v_\alpha + \frac{2}{3}\frac{e^3}{m_0}(v^\beta\partial_\beta F^{\beta\alpha}v_\alpha + \frac{e}{m_0}F^{\mu\alpha}F_{\alpha\beta}v^\beta + \frac{e}{m_0}v^\mu v_\alpha F_{\alpha\beta}F^{\beta\gamma}v^\gamma)$$

which again is the Relativistic Landau Lifshitz equation . So we see that actually Ford and O'Conell have not derive a new equation but they have clarified why is that this equation actually correspond to the equation of motion of a charged particle in the classical case.

10 Effects of the radiation reaction on the particle's spin

My next is to continue analyzing the dynamics of charged particles but this time my purpose is to include the spin effects, these effects are in origin quantum mechanical but we will see that in principle there in nothing that prevent us to make a semi classical analysis. Our beginning point will be the equation which describes the motion of the spin, called the BMT equation.

In electrodynamics the magnetic moment $\vec{\mu}$ of a charged particle is a quantity proportional to its angular momentum $\vec{L}$ which measures how much magnetism it produces, that is how the particle movement and position is affected by external magnetic fields and how the particle magnetic fields interact with it. The effect of an external magnetic field on a charged particle is to produce a torque give by

$$\vec{\tau} = \vec{\mu}\times\vec{B}' \tag{347}$$

where the magnetic moment $\vec{\mu}$ is related to the angular momentum by

$$\vec{\mu} = g\frac{e}{2mc}\vec{L} \tag{348}$$

where g is the famous Land? factor. Since at this stage we are only considering the intrinsec angular momentum, that is the spin of the particle we have that

$$\vec{\mu} = g\frac{e}{2mc}\vec{s}. \tag{349}$$

Now since a torque is the time derivative of the angular momentum from (347) (349) we obtain the time rate of change for the spin:

$$\frac{d\vec{s}}{dt'} = g\frac{e}{2mc}\vec{s} \times \vec{B}' \tag{350}$$

with $\vec{s}$ being the spin of the particle in its rest frame and primes also denoting quantities in the rest frame. Now in order to obtain a covariant version of (350) it is necessary to first generalize the spin to a 4-vector S^{α} which in the particles rest frame should be the same as the vector $\vec{s}$ with zero time component, this is complied by imposing the following constraint

$$U_{\alpha}S^{\alpha} = 0 \tag{351}$$

which implies that in a reference frame in which the particle's velocity is $c\vec{\beta}$ and the spin is $\vec{S}$, the time component of the spin is related to the space components by

$$S_0 = \vec{\beta} \cdot \vec{S}. \tag{352}$$

According to the Lorentz transformation Laws we have that :

$$\vec{s} = \vec{S} - \frac{\gamma}{\gamma+1}(\vec{\beta} \cdot \vec{S})\vec{\beta} \tag{353}$$

$$\vec{S} = \vec{s} + \frac{\gamma^2}{\gamma+1}(\vec{\beta} \cdot \vec{s})\vec{\beta}$$

$$S_0 = \gamma(\vec{\beta} \cdot \vec{s})$$

Now looking at (350) we see that it contains linearly the external magnetic field $\vec{B}$ and the spin $\vec{s}$, that suggest that its generalization must be linear in the external field $F^{\alpha\beta}$ and in the spin 4-vector $S^{\alpha\beta}$ and to ensure covariance it may include the 4-velocity U^{α} and its time derivative. The most general formula fulfilling these requirements is:

$$\frac{\vec{S}^{\alpha}}{d\tau} = A_1 F^{\alpha\beta}S_{\beta} + \frac{A_2}{c^2}(S_{\lambda}F^{\lambda\beta}U_{beta})U^{\alpha} + \frac{A_3}{c^2}(S_{\beta}\frac{dU^{beta}}{d\tau})U^{\alpha} \tag{354}$$

where A_1, A_2 and A_3 are constants to be determined from the previous considerations. Now differentiating the constraint equation $S_{\alpha}S^{\beta} = 0$ with respect to τ we get that :

$$S_\alpha \frac{dU^\alpha}{d\tau} + U_\alpha \frac{dS^\alpha}{d\tau} = 0, \tag{355}$$

then multiplying (354) by U_α, using $U_\alpha U^\alpha = -c^2$ and inserting all this in the last equation we get that

$$(A_1 - A_2) U_\alpha F^{\alpha\beta} S_\beta + (1 + A_3) S_\alpha \frac{dU^\alpha}{d\tau} = 0 \tag{356}$$

Now let's perform a reduction of this equation to the particles rest frame. In this frame $S^0 = 0$ so the last equation reads:

$$(A_1 - A_2) c F^{0\beta} S_\beta + (1 + A_3) \vec{s} \cdot \frac{d\vec{U}}{d\tau} = 0 \tag{357}$$

but since in this reference frame $\vec{s} \cdot \frac{d\vec{U}}{d\tau} = 0$ we can conclude that $A_1 = A_2$, therefore we insert this in equation (356) and conclude that $A_3 = -1$. Now in order to to obtain an specific value for A_1 we compare the x-component of equation (354) in the particle's rest frame with the x-component of equation (350)(there is nothing special about the x-component, the y and z components give the same result).

So from equation (354) we get that:

$$\frac{dS^1}{d\tau} = A_1 F^{1\beta} S_\beta = A_1 (s_y B_z - s_z B_y) \tag{358}$$

where the last two terms are not present because they involve the particles velocity which is zero. The last term was obtained by looking at the electromagnetic field-strength tensor. Now the x-component of (350) is

$$\left(\frac{d\vec{s}}{dt'} \right)_x = \frac{ge}{2mc} (s_y B_z - s_z B_y) \tag{359}$$

so comparing we conclude that $A_1 = \frac{ge}{2mc}$.

Now that we know the values of the constants we have a covariant expression for the movement of the spin:

$$\frac{dS^\alpha}{d\tau} = \frac{ge}{2mc} \left[F^{\alpha\beta} S_\beta + \frac{1}{c^2} (S_\lambda F^{\lambda\beta} U_\beta) U^\alpha \right] - \frac{1}{c^2} (S_\beta \frac{dU^\beta}{d\tau}) U^\alpha \tag{360}$$

Now if we assume that the particle is under Lorentz forces (without radiation corrections) the last term in this equation may be replaced by:

$$\frac{dU^\beta}{d\tau} = \frac{e}{mc} S_\lambda F^{\lambda\alpha} U_\alpha \tag{361}$$

and with this we obtain the famous Bargman-Michel-Telegdy equation

$$\frac{dS^\alpha}{d\tau} = \frac{e}{mc}\left[\frac{g}{2}F^{\alpha\beta}S_\beta + \frac{1}{c^2}\left(\frac{g}{2}-1\right)(S_\lambda F^{\lambda\beta}U_\beta)U^\alpha\right]. \tag{362}$$

Now if we want to include radiation reaction corrections, the obvious path to take is to replace the covariant Lorentz force law by the Landau-Lifshitz equation:

$$m\dot{U}^\beta = eF^{\beta\nu}U_\nu + \frac{2}{3}\frac{e^3}{m}\left(U^\mu\partial_\mu F^{\beta\nu}U_\nu + \frac{e}{m}F^{\beta\nu}F_{\nu\mu}U^\mu + \frac{e}{m}U^\beta U^\nu F_{\nu\mu}F^{\mu\gamma}U_\gamma\right) \tag{363}$$

so we replace this expression in the last term of equation (360)

$$\begin{aligned}\frac{dS^\alpha}{d\tau} &= \frac{e}{mc}\left[\frac{g}{2}F^{\alpha\beta}S_\beta + \frac{1}{c^2}(\frac{g}{2}-1)(S_\lambda F^{\lambda\beta}U_\beta)U^\alpha\right] - \frac{U^\alpha}{mc^2}\left[\frac{2}{3}\frac{e^3}{mc^3}S_\lambda U^\beta(\partial_\beta F^{\lambda\mu})U_\mu\right.\\ &- \left.\frac{2e^4}{3m^2c^5}\left(S_\lambda F^{\lambda\mu}F_{\mu\beta}U^\beta + S_\lambda U^\lambda U^\alpha F_{\alpha\beta}F^{\beta\gamma}U_\gamma\right)\right]\end{aligned} \tag{364}$$

But this equation is nonlinear in the electromagnetic field $F^{\alpha\beta}$ which is a contradiction to one of our assumptions that we used to get to equation (360), but that does not lesses validity to this equation because it actually reduces to (350) in the reference frame, so what this equation means is that radiation reaction effects also affects the spin of the particle by making it precess.

11 Conclusions

Historically the first equation taking into account the radiation back reaction in the dynamics of the charged particle was the one derived by Abraham and Lorentz at the beginning of the XX century, but as we saw in presented some flaws mainly originated y the fact that the equation was derived using the point particle model which we demonstrated in section (9) to correspond to a unphysical situation thus implying unphysical conclusions. For many years many improving attempts were made to improve this equation and overcome its problems (the literature is indeed vast), the most remarkable one is the relativistic generalization of the equation presented by Paul Dirac (equation (139)). Although this equation was covariant and completely agreed with the general relativity postulates it did not alleviated at all the flaws present in the Abraham Lorentz equation. Later on with the advent of Quantum Electrodynamics the problem of the self force was completely understood in a quantum level therefore the problem of the radiation back reaction in the classical level was forgotten for some years. Then at the beginning of the 50's Lev Landau and Evgeny Lifshitz on the third volume of their famous collection of lectures in theoretical physics presented an equation which accounted for the radiation back reaction effects and was not third order in time as the LAD equation, therefore it did not present any flaws or inconsistencies. However this equation was also forgotten for many years mainly because it used some approximations which seemed very coarse and whose physical implications were not well understood. I was not until the mid 80's that under a new approach Ford and O'Connell rederived the Landau-Lifshitz equation and demonstrated that in the classical scale (in contrast with the quantum mechanic scale) it correctly describes the equation of motion of a charged particle taking into account radiation back reaction effects. One decade later Herbert Spohn also arrived at the LL equation but using a completely different approach based on singular perturbation theory. The main difference between the two approaches is that Ford and O'Connell begin their studies relying in higher order theories, therefore he uses some results from Quantum Electrodynamics and Quantum mechanics and then he makes the reduction to the classical limit, whereas Spohn really begins with the classical electro dynamical equations and then under his geometric point of view shows that the physical situations concerning the classical limit are described by the LL equation. The fact that they both arrived at the same equation of motion for the electron despite the deep differences in their analysis methods is not surprising because they actually begin with the same energy function. In appendix 3 it is shown that the Hamiltonian from which Ford and O'Conell obtain the memory function used in the Langevin equation is the quantized version of the Hamiltonian coming from the Lagrangian used by Spohn to derive his equations of motion. We may say that neither Spohn or Ford and O'Connell derived a new equation for solving the radiation back reaction problem but they did settled the basis to demonstrate that the Landau Lifshitz equation is the right equation of motion of the charged particle in the classical case and that terms of higher order in their approximations would lead to time and space scale which would be clearly out of the range of the classical domain. The fact that the Landau Lifshitz equation includes a term dependent on the derivative of the force is very important because when we apply the LAD equation to certain specific systems it predicts certain results which were thought not to be possible to describe by means of a classical theory. An example of this is the case of an electron orbiting a proton in which if we use the Landau Lifshitz equation to describe the motion of the particle there appears some force terms which may account for some of the Lamb Shift corrections.

A simple but fundamental application of the LL equation was is the equation of motion for the expectation value of the spin, which we developed in the preceeding chapter. It is well known that the famous BMT equation is only valid for weak fields being linear in $F^{\mu\nu}$. However we found out that for almost homogeneous fields when the LL equation is used in the BMT equation instead of the Lorentz force law higher derivatives of $F^{\mu\nu}$ with respect to time appear on the BMT equation (as was expected). In this way we showed that the LL equation indeed induces part of the missing terms in the equation of the space time motion of the particle.

12 Appendix.

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12.1 Appendix 1: Brief review of electromagnetism

The Maxwell's equations in vectorial form are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_o} \ (Gauss's law). \tag{365}$$

$$\nabla \cdot \vec{B} = 0 \ (No magnetic monopoles). \tag{366}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ (Faraday's\ Law). \tag{367}$$

$$\nabla \times \vec{B} = \mu_o \vec{J} + \frac{\partial \vec{E}}{\partial t} \ (Ampere - Maxwell\ Law). \tag{368}$$

Since$\nabla \cdot \vec{B} = 0$ there exist some vectorial quantity $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$. So replacing in (iii) we obtain that

$$\nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0 \tag{369}$$

With this we can write

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \tag{370}$$

for some scalar potential ϕ. Then if we introduce this potential in (i) we get

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}. \tag{371}$$

Now we use the next identity in (iv)

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \tag{372}$$

and we obtain that

$$\nabla^2 \vec{A} - \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla(\nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = -\mu_o \vec{J}) \tag{373}$$

these equations are gauge invariant, that is their invariant under transformations of the form $\vec{A} \rightarrow \vec{A} + \nabla \lambda \ \phi \rightarrow \phi - \frac{\partial \lambda}{\partial t}$ for any function λ.

We choose the Lorentz gauge

$$\nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = 0 \tag{374}$$

And we get that:

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = \Box \phi = -\frac{\rho}{\epsilon_o} \tag{375}$$

In Gaussian units we obtain:

$$\Box \phi = -4\pi\rho \tag{376}$$

Also we have:

$$\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \Box \vec{A} = -\mu_o \vec{J} \tag{377}$$

So in gaussian units we obtain:

$$\Box \vec{A} = -4\pi \vec{J} \tag{378}$$

So we define the next 4-vector $A^\mu = (\phi, \vec{A})$ and $J^\mu = (\rho, \vec{J})$
and we obtain the wave equation:

$$\Box A^\mu = -4\pi J\mu \tag{379}$$

with the current conservation Law

$$\partial_\alpha J^\alpha = 0 \tag{380}$$

So there are four homogeneous wave equations, who's solution is well known (Jackson, p243).

$$A^\mu(\vec{x}, t) = \int_\Omega \frac{J^\mu(\vec{x}', t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} d^3x' \tag{381}$$

The integral is evaluated in retarded time for ensuring causality.

Now for obtaining the fields from the potentials one uses equations (134, 140 142) and from it one also might obtain its components, for example

$$E_x = -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} = -(\partial^0 A^1 - \partial^1 A^0)$$

$$b_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)$$

and if we do this we would notice that there is certain symmetry which suggest that a the components of the electric and magnetic field are part of a symmetric second rank tensor. In fact

$$F^{\alpha\beta} = -(\partial^\alpha A^\beta - \partial^\beta A^\alpha)$$

12.2 Appendix 2: Independent Oscillator model and Blackbody radiation Field

Consider the following Hamiltonian:

$$H_v = \frac{1}{2m}\left[p + \sum_j m_j\omega_j q_j\right]^2 + V(x) + \sum_j \left[\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2 q_j^2\right] \tag{382}$$

This Hamiltonian is called the velocity coupling model (VC model) and it describes a quantum particle surrounded by many heath bath particles coupled to the first one by momentum interactions. If the external potential is zero $V(x) = 0$ the hamiltonian is invariant under space translations therefore preserves the momentum. But actually it is the same as the Hamiltonian for the IO model (sec 9.3) just that in a new set of momentum coordinates, that is the IO model and the VC model differ from each other by a point transformation.

Consider the following unitary transformation operator

$$U = e^{-\frac{i}{\hbar}x\sum_j m_j\omega_j q_j} = 1 - \frac{i}{\hbar}x\sum_j m_j\omega_j q_j - \frac{x^2}{2\hbar^2}\left(\sum_j m_j\omega_j q_j\right)^2 + \frac{ix^3}{6\hbar^3}\left(\sum_j m_j\omega_j q_j\right)^3 \cdots \tag{383}$$

so lets aply the following transformations to the coordinates:

$$p \to U^\dagger p U, \qquad x \to x \tag{384}$$

with

$$U^\dagger p U = \mathbb{1}p\mathbb{1} + \frac{i}{\hbar}\sum_j m_j\omega_j q_j[xp - px] = p - \sum_j m_j\omega_j q_j \tag{385}$$

where in the las expansion we just did the first three terms because higher order terms cancel with each other. In the same way:

$$p_j \to p_j - m_j\omega_j q_j, \qquad q_j \to q_j \tag{386}$$

Also we aply the transformation to the Hamiltonian H_v

$$H_v \to U^\dagger H_v U \tag{387}$$

which after a lenghty but not complicated algebra gives:

$$H_v \to U^\dagger H_v U = \frac{p^2}{2m} + V(x) + \sum_j \left[\frac{1}{2m_j}(p_j + m_j\omega_j x)^2 + \frac{1}{2}m_j\omega_j^2 q_j^2\right] \tag{388}$$

Now we perform another point transformation but this time on the heat bath coordinates:

$$U_i = e^{-\frac{i\pi}{2\hbar}\sum_j\left(\frac{1}{2m_j\omega_j}p_j^2+\frac{1}{2}m_j\omega_j q_j^2\right)} = 1 - \frac{i\pi}{2\hbar}\sum_j\left(\frac{1}{2m_j\omega_j}p_j^2 + \frac{1}{2}m_j\omega_j q_j^2\right)\cdots. \tag{389}$$

therefore

$$q_j \rightarrow U_i^\dagger q_j U = \frac{i}{2\hbar}\frac{1}{2m_j\omega_j}(q_j p_j^2 - p_j^2 q_j) = -\frac{1}{m_j\omega_j}p_j \tag{390}$$

and in the same way

$$p_j \rightarrow U_i^\dagger p_j U = m_j\omega_j q_j \tag{391}$$

which gives the following transformation for the Hamiltonian (388)

$$H_v \rightarrow U^\dagger H_v U = \frac{p^2}{2m} + V(x) + \sum_j\left[\frac{p_j}{2m_j} + \frac{1}{2}m_j\omega_j^2(q_j - x)^2\right] \tag{392}$$

but this is exactly the Hamiltonian for the IO model postulated in section 9.3. So since we can go from the Hamiltonian of the IO model to the VC model by point transformations the Heisenberg equations for both systems of coordinates should be the same. Now as we said before the VC Hamiltonian is just the one dimensional version for the Black Body radiation Hamiltonian, therefore also the dynamics of both Hamiltonians must be the same. The Hamiltonian for the Blackbody radiation is:

$$H = \frac{1}{2m}[\vec{p} + \frac{e}{c}\vec{A}]^2 + V(\vec{r}) + \sum_{\vec{k},s}\hbar\omega_k(a^\dagger_{\vec{k},s}a_{\vec{k},s} + \frac{1}{2}) \tag{393}$$

with the vector potential given by

$$\vec{A} = \sum_{k,s}\left[\frac{2\pi\hbar c^2}{\omega_k\vec{V}}\right]^{\frac{1}{2}} f_k\hat{\vec{e}}_{\vec{k}s}(a_{\vec{k},s} + a^\dagger_{\vec{k}s}) \tag{394}$$

so if we write

$$a_{\vec{k},s} = \frac{m_k\omega_k\vec{q}_{\vec{k},s} + i\vec{p}_{\vec{k},s}}{\sqrt{2m_k\hbar\omega_k}} \tag{395}$$

the Hamiltonian (393) takes the following form :

$$H_{QED} = \frac{1}{2m}\left[\vec{p} + \sum_{\vec{k},s}m_k\omega_k q_{q,s}\hat{\vec{e}}_{k,s}\right]^2 + V(\vec{r}) + \sum_{\vec{k},s}\left[\frac{1}{2m_k}p^2_{\vec{k},s} + \frac{1}{2}m_k\omega_k^2 q^2_{\vec{k},s}\right]^2 \tag{396}$$

which clearly has the same form as the VC Hamiltonian (392) just that in vectorial form. Obviously when passing from the IO model Hamiltonian to its three dimensional generalization (the black body radiation field Hamiltonian) some minor changes should be made, but they are carried out in section 9.3.

12.3 Appendix 3: Quantum Hamiltonian and classical Lagrangian for the charged particle.

As we have seen the Lagragian of a charged particle coupled to a electromagnetic field is given by:

$$\begin{aligned} L &= -m_{\rm b}(1-\dot{\vec{q}}^2)^{1/2} - (+\phi - \dot{\vec{q}}\cdot\vec{A}) * \rho(\vec{q}) \\ &\quad + \frac{1}{2}\int d^3x \left[(\nabla\phi + \partial_t\vec{A})^2 - (\nabla\times\vec{A})^2\right]. \end{aligned} \tag{397}$$

where again the asterix denotes convolution, and the potential take account for both the self and external fields. Now we use the well known relation between the Lagrangian and Hamiltonian:

$$H = \sum_i \eta_i p_\eta - L \tag{398}$$

where η_i represents all the variables. This calculation is easily carried out and gives the following Hamiltonian:

$$\begin{aligned} H &= \frac{1}{m}\vec{p}\cdot\left[\vec{p} - \frac{e}{c}\vec{A}(r)\right] + \int \frac{d^3r}{4\pi c}\left[\dot{\vec{A}}(\vec{r})\cdot\left(\frac{1}{c}\dot{\vec{A}} + \nabla\phi\right) - \frac{E^2 - B^2}{8\pi}\right] \\ &\quad + e\phi(\vec{r}) - \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2 - \frac{e}{mc}\vec{A}(\vec{r})\cdot\left[\vec{p} - \frac{e}{c}\vec{A}(r)\right] \\ &= \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(r)\right]^2 + e\phi(\vec{r}) + \int \frac{d^3r}{8\pi}\left[E^2 + B^2 - 2\nabla\phi\cdot(\frac{1}{c} + \nabla\phi)\right] \end{aligned} \tag{399}$$

Now if we use the Coulonb gauge $\nabla\vec{A} = 0$ the term in the integral which includes the cross terms between the potentials vanish after an integration by parts. Also we have that:

$$\int \frac{d^3r}{4\pi}\nabla\phi\cdot\nabla\phi = \int \frac{d^3r}{8\pi}\phi\nabla^2\phi = \int d^3r\phi(\vec{r})\rho(\vec{r}) = e\phi \tag{400}$$

So this term cancels with the third one in the integral. Therefore the Hamiltonian takes the following form:

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(r)\right]^2 + \int \frac{d^3r}{8\pi}\left[E^2 + B^2\right] \tag{401}$$

Now we will perform the quatization of the field by using the second quantization formalism. In order to this we must have in mind that the electric and magnetic fields are alway produced by charge source and do not exist by them self alone. Then let's write the electric field energy density in terms of the potentials:

$$\begin{aligned} \int \frac{d^3r}{8\pi}E^2 &= \int \frac{d^3r}{8\pi}\left[\frac{1}{c^2}\dot{\vec{A}}^2 + (\nabla\phi)^2 + \frac{2}{c}\nabla\phi\cdot\dot{\vec{A}}\right] \\ &= \frac{1}{2}\sum_j \frac{ee_j}{r_j} + \int \frac{d^3r}{8\pi c^2}\dot{\vec{A}}^2 \end{aligned} \tag{402}$$

where the summation is over all the charges producing the fields . We know tath the electric field is the conjugate momentum density of the pontetial vector, so they obbey the following conmutation relation

$$\left[\vec{A}_\alpha(\vec{r},t), \frac{-E_\beta}{4\pi}(\vec{r'},t)\right] = i\delta_{\alpha,\beta}\delta(\vec{r}-\vec{r'}) \tag{403}$$

In order to satisfy this conmutation relation we express the vector potential in terms of the creation and anihilation operators as follows:

$$\vec{A}_\alpha = \sum_{k,s}\left[\frac{2\pi\hbar c^2}{\omega_k V}\right]^{\frac{1}{2}} \vec{\xi}_{\vec{k},s} e^{\vec{k}\cdot\vec{r}}(a_{\vec{k},s}e^{-i\omega_{\vec{k}}t} + a^\dagger_{\vec{k}s}e^{i\omega_{\vec{k}}t}) \tag{404}$$

where $\vec{\xi}_{\vec{k},s}$ is the unit polarization vector, which gives the direction of the two tranvese modes which in the summation are represented by the letter s The time derivative of the vector potential is :

$$\dot{\vec{A}}_\alpha(\vec{r},t) = -i\sum_{k,s}\left[\frac{2\pi\hbar c^2}{V}\right]^{\frac{1}{2}} \vec{\xi}_{\vec{k},s} e^{\vec{k}\cdot\vec{r}}(a_{\vec{k},s}e^{-i\omega_{\vec{k}}t} - a^\dagger_{\vec{k}s}e^{i\omega_{\vec{k}}t}) \tag{405}$$

Now the electric field is given in terms of the potentials by:

$$\vec{E}_\alpha = -\frac{1}{c}\dot{\vec{A}}_\alpha - \nabla\phi \tag{406}$$

So since the scalar potential is not expressed in terms of the creation and anihilation operators it does not influence the conmutatot (403), so this conmutation relation must be given by

$$\left[\vec{A}_\alpha(\vec{r},t), \dot{\vec{A}}_\alpha(\vec{r'},t)\right] = 4\pi c i\delta_{\alpha,\beta}\delta(\vec{r}-\vec{r'}) \tag{407}$$

and this is exactly what we get when we replace the expressions for the vector potential and its derivative (eq (404) and (405)) in the conmutator.

Finally we write the energy density of the electric and magnetic fields present in the last term of the Hamiltonian in terms of the vector potentials or its representation in terms of anihilation and creation operators and after some algebraic manipulations and having in mind that the photon energy is $\omega_k = ck$ we get that the Hamiltonian takes the following form:

$$H = \frac{1}{2m}[\vec{p} + \frac{e}{c}\vec{A}]^2 + V(\vec{r}) + \sum_{\vec{k},s}\hbar\omega_k(a^\dagger_{\vec{k},s}a_{\vec{k},s} + \frac{1}{2}) \tag{408}$$

where we have identified $\frac{1}{2}\sum_{j}\frac{ee_j}{r_j}$ with a potential energy $V(\vec{r})$. And this is the Hamiltonian used by Ford and O'Connell to derive the dynamics of the particle. It is clear that the heath bath of which they talk about is conformed by the electromagnetic field which is constituted by the quantized photons which we identify with the heat bath particle referred in section 9. We see that the energy functios used by Spohn and Ford and O'Connell are the same just that the latters used a quantized version therefore their procedure relies on results from a higher level theorie which is quantum mechanics.

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