On Gauging Symmetry of 2D Topological Quantum Field Theories

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In the context of topological quantum field theory, quantum evolutions are described by a functor from the category of cobordisms $\mathcal{B}ord$ to the category of vector spaces $\mathcal{V}ec$. This construction is usually known as a topological quantum field theory (TQFT). It has been shown that 2-TQFT's is in bijective correspondance with the set of Frobenius algebras [1] [2]. We will use this fact to characterize the gauging symmetries of 2-TQFT's. To tackle this problem, we will define the structure of a group-crossed Frobenius algebra which will lead us to the concept of orbifolding. We will find that every crossed Frobenius algebra has an associated Frobenius algebra, and therefore an associated TQFT. Then the symmetries of the TQFT are described by the subalgebra of invariants of the crossed Frobenius algebra. In this project, we will give the restrictions necessary to characterize the orbifoldings and the gaugings of a TQFT. The full characterization will be completed in future works.

En el contexto de la teoría cuántica de campos topológica, la evolución en sistemas cuánticos es descrita usualmente mediante un functor que va de la categoría de cobordismos $\mathcal{C}ob$ a la categoría de espacios vectoriales $\mathcal{V}ec$. Esta construcción recibe el nombre de *Topological quantum field theory* (TQFT). Es conocido que el conjunto de TQFT's en 2 dimensiones se encuentra en correspondencia biyectiva con el conjunto de álgebras de Frobenius. Usando este hecho, pretendemos caracterizar las simetrías de las 2-TQFT's. Para esto, se va a definir una estructura de G-algebra cruzada de Frobenius la cual llamaremos un *orbifolding*. Como se verá durante el proyecto, toda álgebra cruzada de Frobenius tiene un álgebra de Frobenius asociada, y por lo tanto se encuentra asociada a una TQFT. Las simetrías de la TQFT quedan entonces descritas por el subálgebra de G-invariantes del álgebra cruzada. De esto nuestro objetivo final es caracterizar los *orbifoldings* asociados a una TQFT. En este trabajo, nos limitaremos a dar las restricciones algebráicas necesarias para la caracterización de estos *orbifoldings*. El trabajo se completará en futuros proyectos.

Acknowledgements

I would like to thank all the little people who made this possible. My mother, my brother, my friends, my colleagues...

I am especially grateful to my advisor César and my co-advisor Manuel who helped me in difficult moments to complete this project. In addition, I want to thank my brother Diego, who inspired me to follow an academic career. To all of you, thank you very much.

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Introduction

Recently, the boom of topological quantum computing has renewed the interest for the so-called Topological Quantum Field Theories (TQFT's). In quantum mechanics, all the physical information of a system is contained in a state that lives in a vector space associated to a manifold. That this theory is topological means that the evolution of quantum states only depends on the topology of the associated manifolds. It is simple to see how this idea can be better expressed in the language of category theory. Indeed, the relation between quantum mechanics and category theory was explained in the famous quote by Edward Nelson: "First quantization is a mystery, but second quantization is a functor".

The mathematical structure used to represent these interactions is known as a Topological Quantum Field Theory (TQFT) [3]. These are functors from the category of oriented topological spaces to the category of complex vector spaces $\mathcal{V}ec$. The main idea is that each manifold M is assigned to a vector space $\mathcal{T}_{qft}(M)$ that belongs to the space of states of the system . This functor should also satisfy the axiom of multiplicativity under disjoint unions ($\mathcal{T}(M \sqcup M') = \mathcal{T}(M) \otimes \mathcal{T}(M')$), meaning that the quantum states belonging to disjoint manifolds are independent.

However, a complete quantum theory should also include a concept of time-evolution. This is achieved using the idea of cobordism to describe the time-evolution between two manifolds. More precisely, if we have two n-dimensional manifolds M and M', then an $n+1$-cobordism between them will be an $n+1$-manifold N such that its boundary is $\partial N = M \sqcup M'$. We can now assign to each cobordism an operator invariant $\tau : \mathcal{T}_{qft}(M) \to \mathcal{T}_{qft}(M')$ which will give us a relationship between the space of states of M and M' through time-evolution.

In this project we are only interested in working with $1+1$-TQFT's which are known to be classified by the category of Frobenius algebras [1] [4]. We intend to characterize the gauging symmetries in a $1 + 1$-TQFT. To do this, we will define the concept of a group-crossed Frobenius algebra [5] [6], which is a group-Frobenius algebra graded by another group.

It will be simple to verify that any group-crossed Frobenius algebra will be naturally associated to a simple Frobenius algebra. However, it is not straightforward that for any Frobenius algebra $\hat{A}$ and any symmetry group G there is a G-crossed Frobenius algebra A associated to $\hat{A}$. In case of existence we will call A an orbifolding of $\hat{A}$. Then the gauging of $\hat{A}$ will be defined as the algebra of G-invariants of A . In this new language, the problem of gauging symmetries of a 2-TQFT reduces to characterizing the crossed Frobenius algebras associated to a particular Frobenius algebra $\hat{A}$. In this project, we will proof a reduced version of the whole characterization problem for the case when the graded Frobenius algebra is strongly graded [7].

Chapter 1

Preliminaries

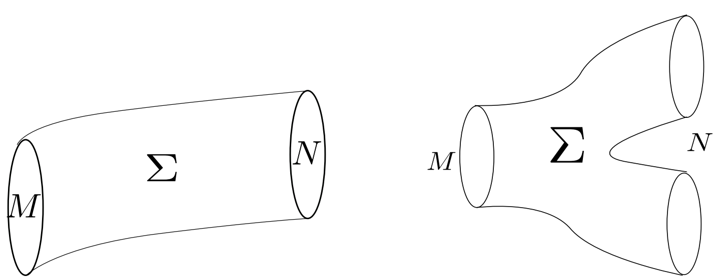
1.1 Cobordisms

Definition 1.1.1. Let M and N be two compact n-manifolds without boundary. An $n+1-$*cobordism* (Σ) from M to N is a compact $(n+1)$-manifold with boundary $(\partial\Sigma)$ such that:

$$\partial\Sigma \simeq M \sqcup N,$$

where $\simeq$ is the disjoint union between both manifolds.

In figures 1.1.a) and 1.1.b) we illustrate to 2-cobordisms from two different pairs of 1-dimensional manifolds.



(a) $M = \mathbb{S}^1, N = \mathbb{S}^1$ (b) $M = \mathbb{S}^1, \quad N = \mathbb{S}^1 \sqcup \mathbb{S}^1$

Figure 1.1: a) Represents a cobordism between two circles. b) Is a cobordism between the circle $\mathbb{S}^1$ and the disjoint union of two circles $\mathbb{S}^1 \sqcup \mathbb{S}^1$.

If M and N are both oriented manifolds, we induce in Σ an orientation such that M will preserve its initial orientation while N will take the opposite orientation. In this situation we say that Σ is an oriented cobodirsm from M to N and we denote it by $M \overset{\Sigma}{\Longrightarrow} N$. From now on we will work only with oriented cobordisms.

Since we are going to work in topological quantum field theories (TQFT's), our theory should be invariant under smooth deformations of cobordisms. To establish this equality we are going to introduce a concept of topological equivalence:

Definition 1.1.2. Given two oriented cobordisms $M \overset{\Sigma_1}{\Longrightarrow} N$ and $M' \overset{\Sigma_2}{\Longrightarrow} N'$. We say Σ_1 and Σ_2 are equivalent if there is an orientation preserving diffeomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that $\phi(M) = M'$ and $\phi(N) = N'$.

For example the cobordisms in figure 1.2 are equivalents since they can be smoothly deformed to each other preserving the orientation. These cobordisms are also equivalent to the cobordism presented in figure 1.1.a) but not to the one in figure 1.1.b) since $\mathbb{S}^1$ is not diffeomorphic to $\mathbb{S}^1 \sqcup \mathbb{S}^1$.

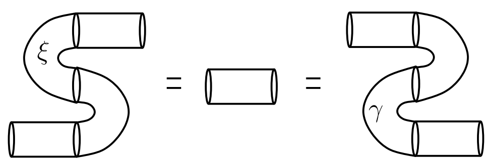


Figure 1.2:

Now, if $M_1 \overset{\Sigma_1}{\Longrightarrow} M_2$ and $M_2 \overset{\Sigma_2}{\Longrightarrow} M_3$ are two cobordisms, there is an associated cobordism from M_1 to M_3 given by the composition of these two cobordisms ($\Sigma_2 \circ \Sigma_1$) (See figure 1.3).

This operation is compatible with topological equivalence in the sense that, if Σ_1 is equivalent to Σ_1' and Σ_2 is equivalent to Σ_2' we get that $\Sigma_2 \circ \Sigma_1$ is equivalent to $\Sigma_2' \circ \Sigma_1'$. We can use this fact to define a category of cobordisms:

Definition 1.1.3. The category of oriented $(n+1)$-cobordisms ($\mathcal{B}ord_{n+1}$) is the category such that:

- The objects of $\mathcal{B}ord_{n+1}$ are oriented topologically non-equivalent n-manifolds without boundary.

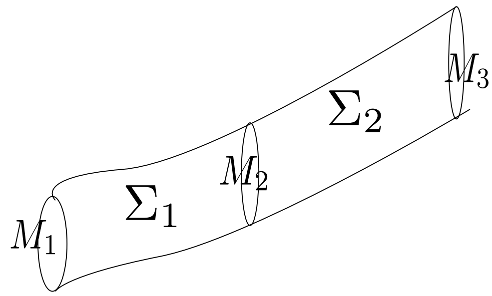
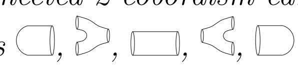


Figure 1.3: $\Sigma_2 \circ \Sigma_1$

- The morphisms of $\mathcal{B}ord_{n+1}$ are non-equivalent oriented $(n+1)$-cobordisms between these manifolds.

In this project, we are only interested in the category of 2-cobordisms $\mathcal{B}ord_2$. The objects of this category are 1 dimensional compact manifolds without boundary, which under topological equivalence can only take the form of a disjoint union of circles $(\sqcup_{i=1}^{n}\mathbb{S}^1)$. Additionally, the morphisms are 2D compact oriented cobordisms. The following lemma completes the classification of $\mathcal{B}ord_2$:

Theorem 1.1.1. ***[1, Lemma 1.4.17]*** *Every connected 2-cobordism can be obtained by the composition and disjoint union of the generators* *.*

A consequence of this is that we can understand every 2-cobordism by its decomposition on the basic units described by Theorem 1.1.1. This fact will be crucial to understand the TQFT generated by $\mathcal{B}ord_2$, which will be defined in the following section.

1.2 Topological Quantum Field Theories (TQFT)

Definition 1.2.1. A $(n+1)$-TQFT is a functor $\mathcal{T}_{qft}$ from the category of $(n+1)$-cobordisms $(\mathcal{B}ord_{n+1})$ to the category of vector spaces over a field k $\mathcal{V}ec_k$:

$$\mathcal{T}_{qft} : (\mathcal{B}ord_{n+1}, \sqcup) \longrightarrow (\mathcal{V}ec, \otimes)\,,$$

with $\mathcal{T}_{qft}(M_1 \sqcup M_2) = \mathcal{T}_{qft}(M_1) \otimes \mathcal{T}_{qft}(M_2)$.

The functorial definition of a TQFT has the following interesting consequences:

- First of all, a $(n+1)$-TQFT is a mapping that associates every compact n-manifold without boundary to a vector space, and every oriented $n+1$-cobordism to an homomorfism between the corresponding vector spaces.

- Since $\mathcal{B}ord_{n+1}$ is defined as the category of *non-equivalent* cobordisms. If two cobordisms are equivalent they will be mapped to the exact same homomorphism. For instance, the cobordism in figure 1.1.a) is mapped through $\mathcal{T}_{qft}$ to the identity.

- If $M_1 \overset{\Sigma_1}{\Longrightarrow} M_2$ and $M_2 \overset{\Sigma_2}{\Longrightarrow} M_3$ are two cobordisms as in figure 1.1, the composition $M_1 \overset{\Sigma_2 \circ \Sigma_1}{\Longrightarrow} M_3$ defines an homomorphism between the corresponding vector spaces by:

$$\begin{aligned} &\mathcal{T}_{qft}(\Sigma_2 \circ \Sigma_1) : \mathcal{T}_{qft}(M_1) \to \mathcal{T}_{qft}(M_3) \\ &\mathcal{T}_{qft}(\Sigma_2 \circ \Sigma_1) = \mathcal{T}_{qft}(\Sigma_2) \circ \mathcal{T}_{qft}(\Sigma_1) \end{aligned} \tag{1.1}$$

- The disjoint union of two n-manifolds is mapped to the tensor product of the corresponding vector spaces ($\mathcal{T}_{qft}(M_1 \sqcup M_2) = \mathcal{T}_{qft}(M_1) \otimes \mathcal{T}_{qft}(M_2)$). In 2-TQFT's, since the possible 1-manifolds take the form $\bigsqcup_{i=1}^{n} \mathbb{S}^1$, the corresponding vector spaces are $\mathcal{T}_{qft}(\mathbb{S}^1)^{\otimes n}$.

Example 1.2.1. The simplest example of a TQFT is the trivial functor that maps every n-manifold in $\mathcal{B}ord_{n+1}$ to the field k, and every $(n+1)$-cobordism to the identity. This functor satisfies the condition $\mathcal{T}_{qft}(M_1 \sqcup M_2) = \mathcal{T}_{qft}(M_1) \otimes \mathcal{T}_{qft}(M_2)$ since $\mathcal{T}_{qft}(M_1) = \mathcal{T}_{qft}(M_2) = k = k \otimes k$.

A much more interesting example of a TQFT will be given in section 1.4.

1.3 Frobenius algebras

First of all, we recall the basic concepts of algebra and coalgera.

Definition 1.3.1. An *associative algebra* A over a field k is a vector space endowed with two maps: A multiplication $\mu : A \otimes A \to A$ and a unit map $\eta : k \to A$, such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id} \otimes \mu} & A \otimes A \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \otimes \mathrm{id}} & A \otimes A & \xleftarrow{\mathrm{id} \otimes \eta} & A \otimes k \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & A & & \end{array}$$

A coalgebra instead, is simply defined as a structure that satisfies the exact opposite relations:

Definition 1.3.2. A *coalgebra* A over a field k is a vector space dotted with two maps: A comultiplication $\delta : A \to A \otimes A$ and a counit $\epsilon : A \to k$, such that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\delta} & A \otimes A \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \mathrm{id} \otimes \delta} \\ A \otimes A & \xrightarrow{\delta \otimes \mathrm{id}} & A \otimes A \otimes A \end{array} \qquad \begin{array}{ccccc} k \otimes A & \xleftarrow{\epsilon \otimes \mathrm{id}} & A \otimes A & \xrightarrow{\mathrm{id} \otimes \epsilon} & A \otimes k \\ & \nwarrow & \uparrow{\scriptstyle \delta} & \nearrow & \\ & & A & & \end{array}$$

From now on, it will be implicit that any algebra or coalgebra A will be a vector space over the field k.

Definition 1.3.3. Let A be an associative algebra. A *pairing* over A is a linear map $\langle \cdot | \cdot \rangle : A \otimes A \to k$ that commutes with the multiplication of the algebra ($\langle ab|c \rangle = \langle a|bc \rangle$).

This pairing is said to be non-degenerate if $\langle a|b \rangle = 0 \;\; \forall b \in A$ implies that $a = 0$. In matrix notation, this fact is equivalent to say that the matrix form associated to $\langle \cdot | \cdot \rangle$ has non-zero determinant in any basis. The pairing $\langle \cdot | \cdot \rangle$ is symmetric if $\langle a|b \rangle = \langle b|a \rangle$. There is an equivalent definition of non-degeneracy that will be useful in further sections:

Theorem 1.3.1. *Let A be a finite algebra with a pairing $\langle \cdot | \cdot \rangle$. Then the following statements are equivalent:*

1. *The pairing $\langle \cdot | \cdot \rangle$ is non-degenerate.*
2. *There exists a copairing $\gamma : k \to A \otimes A$ such that the following diagram commutes:*

$$\begin{array}{ccc} A \cong k \otimes A & \xrightarrow{\gamma \otimes \mathrm{id}} & A \otimes A \otimes A \\ & \searrow^{\mathrm{id}_A} & \downarrow{\scriptstyle \mathrm{id} \otimes \langle \cdot | \cdot \rangle} \\ & & k \otimes A \cong A \end{array}$$

Proof. [1, Lemma 2.1.12] □

Using these concepts we can now give a first definition of a Frobenius algebra:

Definition 1.3.4. A *Frobenius algebra* is an associative, unital finite dimensional algebra endowed with a non-degenerate pairing $\langle \cdot | \cdot \rangle : A \otimes A \to k$.

If the pairing $\langle \cdot | \cdot \rangle$ is symmetric, the symmetric Frobenius algebra has a natural trace $\epsilon : A \to k$ given by the relation $\epsilon(a) = \langle a | 1_A \rangle$. Reciprocally, a trace ϵ induces a symmetric pairing by $\langle a | b \rangle = \epsilon(ab)$. We conclude that a symmetric Frobenius algebra can be defined in terms of the trace.

Definition 1.3.5. A symmetric *Frobenius algebra* (A, ϵ) is an associative, unital finite dimensional $k-$algebra dotted with a non-degenerate trace $\epsilon : A \to k$.

Where a pairing ϵ is said to be non-degenerate if the its associated pairing is non-degenerate.

Example 1.3.2. Take $\mathbb{C}$ as an algebra over $\mathbb{R}$ and pick the trace $\epsilon = Re : \mathbb{C} \to \mathbb{R}$, the real part operator. Then $(\mathbb{C}, Re)$ is a Frobenius algebra. We can readily check that Re is non-degenerate by cheking that for any complex $z \neq 0$, $Re(z(\frac{1}{z})) = 1 \neq 0$.

Example 1.3.3. Let k be the field, $A = M_k(n \times n)$ be the algebra of $n \times n$ matrices over k and ϵ be the natural trace on A. Then (A, ϵ) is a Frobenius algebra. Indeed, it is well known that if $\epsilon(ab) = tr(ab) = 0 \ \ \forall b \in A$ then $a = 0$ which implies that the trace is non-degenerate as we wanted.

However the most common definition of Frobenius algebra is given in terms of its coalgebra structure. In this case the trace ϵ will take the place of the counit and it is possible to show that the comultiplication $\delta : A \to A \otimes A$ is uniquely defined for this structure. Reciprocally, any associative algebra with a compatible coalgebra structure defines a Frobenius algebra. We remark this result in the following theorem.

Theorem 1.3.4. *Let A be an associative algebra with unit η and multiplication μ. Then the following statements are equivalent:*

1. *A is a Frobenius algebra with trace ϵ.*

2. *A is a coalgebra with counit ϵ and comultiplication $\delta : A \to A \otimes A$ such that the following diagram commutes:*

$$\begin{array}{ccc} A \cong k \otimes A & \xrightarrow{(\delta \circ \eta)\otimes \mathrm{id}} & A \otimes A \otimes A \\ & \searrow^{\mathrm{id}_A} & \downarrow{\scriptstyle \mathrm{id} \otimes (\epsilon \circ \mu)} \\ & & k \otimes A \cong A \end{array}$$

Remark 1.3.5. The commutative diagram in the statement (2) basically implies the notion of compatibility between the algebra and the coalgebra structures. This condition is equivalent to the condition of non-degeneracy of the trace from Theorem 1.3.1. Note that function $\delta \circ \eta : k \to A$ defines the coparing γ on A. While $\epsilon \circ \mu$ is simply the pairing $\langle \cdot | \cdot \rangle$ of the algebra.

Proof. The proof of this theorem will be left for the following section. □

We conclude this section by defining the category of Frobenius algebras $\mathcal{F}rob$ as follows:

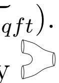
Definition 1.3.6. The category of *Frobenius algebras* ($\mathcal{F}rob$) is the category such that:

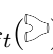
1. The objects are Frobenius algebras (A, ϵ) .
2. Morphisms f between two Frobenius algebras (A_1, ϵ_1) and (A_2, ϵ_2) are trace compatible homomorphisms, i.e. $\epsilon_1(a) = \epsilon_2(f(a))$.

1.4 Frobenius algebras and 2-TQFT's

In the case of 2 dimensions, Theorem 1.1.1 permits a full characterization of possible 2-TQFT's. The main result is remarked in the following theorem:

Theorem 1.4.1. *There is a bijective correspondance between the set of equivalence classes of commutative Frobenius algebras and 2-TQFT's.*

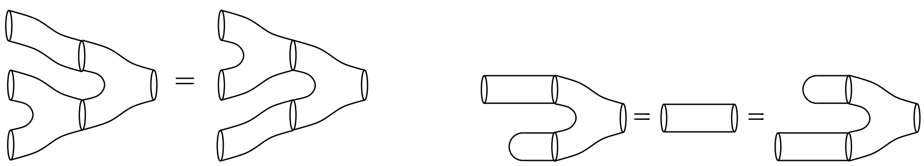
Proof. We will follow the proofs from [1] [4]. Lets start with a particular 2-TQFT ($\mathcal{T}_{qft}$). Call A the vector space given by $\mathcal{T}_{qft}(\mathbb{S}^1)$. The cobordism from $\mathbb{S}^1 \sqcup \mathbb{S}^1$ to $\mathbb{S}^1$ given by  is mapped through $\mathcal{T}_{qft}$ to an homomorphism

$\mu = \mathcal{T}_{qft}($  $) : A \otimes A \to A.$ (1.2)

Let $\mu = \mathcal{T}_{qft}($$)$ be the multiplication in A. The unit morphism instead is given by the cobordism :

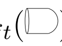
$\eta = \mathcal{T}_{qft}($$) : k \to A.$ (1.3)

Now since the following cobordisms are equivalent

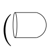
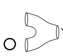
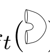


they should be mapped to the same homomorphism through the functor $\mathcal{T}_{qft}$. It follows that the associativity and the unit diagrams from definition 1.3.1 are satisfied. Therefore A has the structure of an algebra with multiplication μ and unit η.

Furthermore, the cobordism  induces a trace/counit ϵ by

$\epsilon = \mathcal{T}_{qft}($$) : A \to k,$ (1.4)

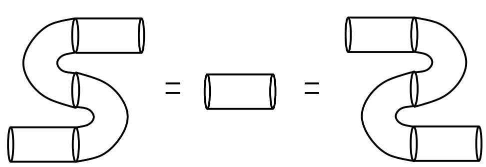
which also induces a pairing $\langle\cdot|\cdot\rangle$ by the composition

$\langle\cdot|\cdot\rangle = \epsilon \circ \mu = \mathcal{T}_{qft}($ $\circ$ $) = \mathcal{T}_{qft}($$)$ (1.5)

In addition, the following cobordism is mapped to a copairing γ by

$\gamma = \mathcal{T}_{qft}($$) : k \to A \otimes A.$ (1.6)

We still need to show that ϵ is non-degenerate to proof that A if Frobenius. Take a look to the following cobordisms:

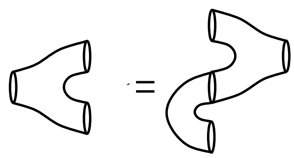


Since they are topologically equivalent, they map through $\mathcal{T}_{qft}$ to the same homomorphism. A careful look to this identity will reveal the following commutative diagram:

$$\begin{array}{ccc} A \cong k \otimes A & \xrightarrow{\gamma \otimes \mathrm{id}} & A \otimes A \otimes A \\ & \searrow^{\mathrm{id}_A} & \downarrow{\scriptstyle \mathrm{id} \otimes \langle \cdot | \cdot \rangle} \\ & & k \otimes A \cong A \end{array}$$

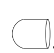
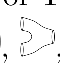
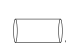
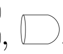
This is exactly the condition of non-degeneracy from Theorem 1.3.1. Therefore A is a Frobenius algebra.

To proof the converse take a Frobenius algebra $(A, \langle \cdot | \cdot \rangle)$. The associated $\mathcal{T}_{qft}$ will be the functor such that the equations ((1.2)-(1.6)) are satisfied. We also need to define an homomorphism for $\mathcal{T}_{qft}$() which is assigned using the following topological equivalence



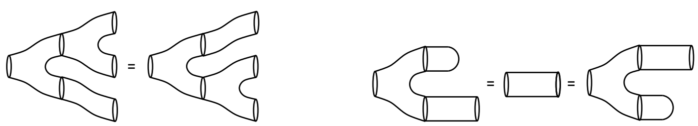
Thus we obtain that $\Delta := \mathcal{T}_{qft}$() is the homomorphism such that the following diagram commutes :

$$\begin{array}{ccc} A \cong A \otimes k & \xrightarrow{\mathrm{id} \otimes \gamma} & A \otimes A \otimes A \\ & \searrow^{\Delta} & \downarrow{\scriptstyle \mu \otimes \mathrm{id}} \\ & & A \otimes A \end{array}$$

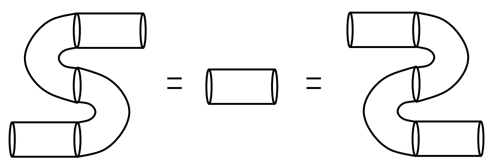
The $\mathcal{T}_{qft}$ of any other cobordism is automatically defined by means of Theorem 1.1.1. This is because the $\mathcal{T}_{qft}$ is already defined for the basic components , , , , . This map is then expanded through all the cobordisms via composition of these basic components. The only missing problem is to proof topological invariance. This fact is not trivial at all. In principle, if two equivalent cobordisms had distinct decompositions, the could be sent by $\mathcal{T}_{qft}$ to different homomorphism. The general idea to proof this is to show that topological invariance is preserved under small changes in the basic components of the cobordism. Then this idea is extended by induction to conclude that any two equivalent cobordisms will be mapped to the same homomorphism. For more details about this part we reference to [4, Lemma 4.3.3]. This argument completes the proof. □

Remark 1.4.2. Theorem 1.4.1 implies that any 2-TQFT is characterized by a commutative Frobenius algebra. In particular any of these algebras as in example 1.3.2 gives a non-trivial example of a TQFT. Moreover, this theorem also establishes some sort of dictionary between algebra and topology. This implies that we can use topological equivalence to proof algebraic identities as we did in the proof of Theorem 1.4.1. For instance we are going to proof Theorem 1.3.4 using this new language.

Proof. [**Theorem 1.3.4**] Suppose we start with a Frobenius algebra with trace ϵ (A, ϵ). Let $\mathcal{T}_{qft}$ be the associated TQFT. A comultiplication Δ is automatically defined by $\Delta = \mathcal{T}_{qft}($$)$, any other homomorphism is defined as in the proof of Theorem 1.4.1. Then the topological equivalence of the cobordisms



implies the commutativity of both diagrams in definition 1.3.2. Additionally the equivalence



implies the compatibility of between the algebra and coalgebra structures in Theorem 1.3.4. Thus A is also a coalgebra.

To proof the other side start with an algebra with a compatible coalgebra structure. Then the counit ϵ defines a trace in the algebra and the non-degeneracy of the trace is obtained from the compatibility between the algebra and coalgebra structures. Therefore (A, ϵ) is a Frobenius algebra.

□

This theorem completes the mathematical preeliminaries about Frobenius Algebras. In the following chapter we are going to generalize this definition to G-graded algebras in order to formulate the problem of orbifolding.

Chapter 2

Group Frobenius Crossed Algebras

In physics, the symmetries of a system can be characterized taking the algebra of invariants under a certain group action over the original space. In the previous chapter we saw that the 2-TQFT's can be characterized by the set of Frobenius algebras over a field k. Therefore to characterize the symmetries over a 2-TQFT we will first need to generalize the definition of Frobenius algebra to include the grading and the action of a particular group onto the Frobenius structure. This type of construction is presented in [5] and receives the name of *group-crossed Frobenius algebra*.

2.1 X-graded Algebras

Before generalizing the notion of Frobenius algebra, we recall the basic concepts of X-graded algebras. We will follow [5, Chapter II] for this part.

Definition 2.1.1. Let X be a group. An *X-graded algebra* A is an associative algebra endowed with a decomposition

$$A = \bigoplus_{x \in X} A_x \tag{2.1}$$

such that the multiplication of the algebra respects the grading structure, i.e. $A_x A_{x'} \subseteq A_{xx'}$ and $1_A \in A_1$.

From now on we will reefer to X as a group and A as the graded algebra over X.

Notation: An homogeneous element in A is an element that belongs to one of the homogeneous components (A_x). We will be denote an homogeneous element in A_x by a_x, i.e. $a_x \in A_x$.

Example 2.1.1. Consider the group algebra $k[X]$. This algebra is naturally graded by

$$k[X] = \bigoplus_{x \in X} k[u_x].$$

where u_x is the element in $k[X]$ representing x. Clearly we have that $k[u_x]k[u'_x] \subseteq k[u_{xx'}]$.

Example 2.1.2. Let $\{\gamma(x,h) \in k^*\}_{x,h \in X}$ be a normalized 2-cocycle to the group of invertible elements in k (k^*), i.e $\gamma(g,1) = \gamma(1,h) = 1$ and

$$\gamma(f,x)\gamma(fx,h) = \gamma(x,h)\gamma(f,xh) \quad f,x,h \in G. \tag{2.2}$$

We will define the twisted X-graded algebra A^γ as follows. The homogeneous components of A^γ are given by $A^\gamma_x = ku_x$ $(k[X] = \bigoplus_{x \in X} ku_x)$. The multiplication is twisted by the cocycle γ as $u_x u_h = \gamma(x,h)u_{xh}$ and extended by linearity to the entire algebra. It is simple to check that this is an X-graded algebra and the element u_1 is a unit since $u_1 u_x = \gamma(1,x)u_x = u_x \ \ \forall x \in X$.

Note that in general a graded algebra is only required to satisfy $A_x A_{x'} \subseteq A_{xx'}$, but not $A_x A_{x'} \cong A_{xx'}$. However, we can see that examples 2.1.1 and 2.1.2 satisfy this property since $k[u_x]k[u'_x] = k[u_{xx'}]$. We can obtain an stronger version of a graded algebra imposing the condition $A_x A_{x'} = A_{xx'}$. This lead us to the following definition:

Definition 2.1.2. We say that an X-graded algebra A is *strongly graded* if $A_x A'_x = A_{xx'} \ \ \forall x, x' \in X$.

In general the theory that we will develop for X-Frobenius algebras will work for any X-graded algebra. However, most of the examples will be given for strongly graded algebras. Indeed, this will be an important assumption for the last part of the project.

2.2 X-Frobenius algebra

We are interested in adding a Frobenius structure to an X-algebra. Therefore, we need to define a pairing $\langle \cdot | \cdot \rangle$ over an X-algebra A compatible with the X-grading. This concept of compatibility is included in the following definition.

Definition 2.2.1. An X-*pairing* is a pairing $\langle\cdot|\cdot\rangle$ (see definition 1.3.3) over an X-graded algebra A such that $\langle A_x|A_{x'}\rangle = 0$ if $xx' \neq 1$.

Once again, we say the X-pairing $\langle\cdot|\cdot\rangle$ is non-degenerate if $\langle a|b\rangle = 0 \quad \forall b \in A$ implies $a = 0$.

Embedding this new pairing into the X-algebra structure we will obtain what is called an X-Frobenius algebra.

Definition 2.2.2. An X-Frobenius algebra $(A, \langle\cdot|\cdot\rangle)$ is an X-graded algebra A endowed with a non-degenerate X-pairing.

Example 2.2.1. To obtain an example of an X-Frobenius algebra we just need to endow the X-algebra $k[X]$ (see example 2.1.1) with an appropriate pairing. Lets take the pairing defined by $\langle u_x|u_h\rangle = \delta(xh)$, where δ is the function such that

$$\delta(x) := \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1. \end{cases} \tag{2.3}$$

This pairing is extended by linearity to the whole algebra. From the definition of δ function it is straightforward that the pairing $\langle\cdot|\cdot\rangle$ is indeed non-degenerate and satisfies the property given in definition 2.2.1. Therefore $(k[X], \langle\cdot|\cdot\rangle)$ is an X-Frobenius algebra.

It is important at this point to note that if $(A, \langle\cdot|\cdot\rangle)$ is an X-Frobenius algebra, then the space A_1 is a subalgebra. This is clear since A_1 is closed under multiplication. In addition, A_1 is endowed with the pairing $\langle\cdot|\cdot\rangle$ restricted to the space $A_1 \otimes A_1$ ($\langle\cdot|\cdot\rangle|_{A_1\otimes A_1}$). This pairing preserves the property of non-degeneracy. Note that the properties of A_1 mentioned above are just the properties of a Frobenius algebra. Therefore we have the following lemma:

Lemma 2.2.2. *Let $(A, \langle\cdot|\cdot\rangle)$ be an X-Frobenius algebra. Then $(A_1, \langle\cdot|\cdot\rangle|_{A_1\otimes A_1})$ is a Frobenius algebra.*

Proof. □

Example 2.2.3. In example 2.1.2, we consider an X-pairing given by $\langle u_x|u_h\rangle = \gamma(x,h)\delta(xh)$. To cheek the pairing properties we only need to consider the case when $h = g^{-1}$. We can check that the pairing is symmetric from equation (2.2) since

$$\langle u_x|u_{x^{-1}}\rangle = \gamma(x, x^{-1}) = \frac{\gamma(x, x^{-1}x)\gamma(x^{-1}, x)}{\gamma(xx^{-1}, x)} = \gamma(x^{-1}, x) = \langle u_{x^{-1}}|u_x\rangle,$$

for any $x \in X$. We also check that this pairing commutes with the multiplication

$$\langle ab|1\rangle = \langle \sum_{x\in X} \lambda_x u_x \sum_{h\in X} \lambda'_h u_h |1\rangle = \sum_{x,h\in X} \lambda_x \lambda'_h \gamma(x,h)\langle u_{xh}|1\rangle$$
$$= \sum_{x,h\in X} \lambda_x \lambda'_h \gamma(x,h)\delta(xh) = \langle a|b\rangle.$$

From this, $\langle\cdot|\cdot\rangle$ is an X-pairing and $(A^\gamma, \langle\cdot|\cdot\rangle)$ is an X-Frobenius algebra. As stated by lemma 2.2.2, $A_1 = k[u_1]$ is the trivial Frobenius algebra with the restricted pairing $\langle k_1u_1|k_2u_1\rangle = k_1k_2\gamma(1,1) = k_1k_2$.

We want now to create an example of an X-graded Frobenius algebra A with a non-trivial induced Frobenius algebra A_1, i.e. $A_1 \ncong k$.

Example 2.2.4. Let $(\hat{A}, \langle\cdot|\cdot\rangle_{\hat{A}})$ be a Frobenius algebra over k. Let $\gamma : X \times X \to \hat{A}*$ be a 2-cocycle satisfying the same equations of in example 2.1.2. Define A^γ as the X-graded Frobenius algebra with grading given by

$$A_x = \hat{A}u_x.$$

The multiplication of two elements in the base is $(\hat{a}u_x)(\hat{b}u_h) = \hat{a}\hat{b}\gamma(x,h)u_{xh}$ and is extended by linearity. The X-pairing is defined similarly by the following identity:

$$\langle \hat{a}u_x|\hat{b}u_h\rangle = \langle\hat{a}|\hat{b}\rangle_{\hat{A}}\gamma(x,h)\delta(xh).$$

It is simple to check that this pairing satisfies again the properties of symmetry and commutativity with multiplication. Therefore $(A^\gamma, \langle\cdot|\cdot\rangle)$ is an X-graded Frobenius algebra. Note that in this case the Frobenius algebra $(A_1, \langle\cdot|\cdot\rangle|_{A_1\otimes A_1}) \cong (\hat{A}, \langle\cdot|\cdot\rangle_{\hat{A}})$.

Last example shows implicitly that Lemma 2.2.2 is true for its converse. That is, for any arbitrary Frobenius algebra $(\hat{A}, \langle\cdot|\cdot\rangle_{\hat{A}})$ there exists an X-graded Frobenius algebra $(A^\gamma, \langle\cdot|\cdot\rangle)$ such that the induced Frobenius algebra $(A_1, \langle\cdot|\cdot\rangle|_{A_1\otimes A_1}) \cong (\hat{A}, \langle\cdot|\cdot\rangle_{\hat{A}})$. This is a first aproximation to the concept of orbifolding which will be explained in the next chapter.

2.3 Crossed X-algebras

Working with X-algebras brings some difficulties with the notion of commutativity. Imagine for example that we have two elements a_x, a_h in different homogeneus components of an

X-graded algebra A. Note that $a_x a_h \in A_{xh}$ and $a_h a_x \in A_{hx}$. Thus a_x and a_h can commute only if $xh = hx$. This condition is too restrictive. To solve this problem we are going to define a crossed structure over the X-Frobenius algebra which will solve the problem of commutativity.

Definition 2.3.1. Take a 4-tuple $(A, X, \phi, \langle\cdot|\cdot\rangle)$, where A is an X-algebra , $\phi : X \times A \to A$ is an action denoted by $\phi(x, a) = {}^x a$, and $\langle\cdot|\cdot\rangle$ is the X-pairing. Then we say that $(A, X, \phi, \langle\cdot|\cdot\rangle)$ is an X-*Frobenius crossed algebra* if the following axioms are satisfied $\forall x \in X$:

1. ${}^x(\cdot) : A \to A$ is an algebra automorphism such that ${}^x A_{x'} \subseteq A_{xx'x^{-1}} \;\; \forall x' \in X$.
2. $({}^x b)a_x = a_x b \;\; \forall a_x \in A_x, b \in A$.
3. ${}^x a_x = a_x \;\; \forall a_x \in A_x$.
4. $\langle {}^x a | {}^x b \rangle = \langle a|b \rangle \;\; \forall a, b \in A$.

Remark 2.3.1. Axiom (2) defines a notion of commutativity in graded-algebras. It also implies that the homogeneous component of the identity A_1 is central in A, i.e. $ba_1 = a_1 b \;\; \forall b \in A$. From axioms $(2), (3)$ we obtain that every homogeneous component is commutative (i.e. $a_x b_x = b_x a_x \;\; \forall a_x, b_x \in A_x, \forall x \in X$). In addition, we can readily check using similar arguments, that Lemma 2.2.2 is still valid in this case. This implies that A_1 is a central commutative Frobenius algebra. On the other hand, axioms (1),(4) are notions of compatibility between the action ϕ and the pairing $\langle\cdot|\cdot\rangle$ with the grading of A respectively. In particular we can check from (1) that the algebra A_1 is close under the action of X (i.e. ${}^x A_1 \subseteq A_1$).

Theorem 2.3.2. *Let $(A, X, \phi, \langle\cdot|\cdot\rangle)$ be an X-crossed Frobenius algebra. Then the algebra of X-invariants A^X together with the pairing $\langle\cdot|\cdot\rangle|_{A^X \otimes A^X}$ is a commutative Frobenius algebra.*

Proof. This theorem is a particular case of Theorem 2.3.6. □

Note that in an X-crossed Frobenius algebra the grading group is the same group that acts on the algebra. We can obtain a less-restrictive definition if we suppose that these two groups are different. To do this, we first need to introduce the concept of crossed module.

Definition 2.3.2. A *crossed module* is a triple (G, X, ∂), where G and X are groups with G acting on X (${}^g x := g \cdot x$) and $\partial : X \to G$ is a group homomorphism satisfying:

- ${}^{\partial x}x' = xx'x^{-1}$
- $\partial({}^{g}x) = g\partial(x)g^{-1}$

Example 2.3.3. Let X be a normal subgroup of G ($X \trianglelefteq G$) and let G act by conjugation over X (i.e. ${}^{g}x = gxg^{-1}$).Since X is normal this conjugation is well defined. Take $\partial = i$, the inclusion of X in G. Then both conditions of theorem 2.3.2 are trivially satisfied by definition.

Example 2.3.4. Let $\partial : X \to G$ be a surjective homomorphism such that $ker(\partial)$ is central (i.e. $ker(\partial) \subseteq Z(X)$, where $Z(X)$ is the center of X). Let $\phi : G \to X$ be a section satisfying $\partial \circ \phi = \mathrm{id}_G$. Suppose G acts on X by ${}^{g}x = \phi(g)x\phi(g)^{-1}$. Then (G, X, ∂) is a crossed module:

- Note that $\partial(\phi(\partial(x))) = \partial(x)$. Then $x^{-1}\phi(\partial(x)) \in ker(\partial) \subseteq Z(X)$ which implies that

  $$\begin{aligned} {}^{\partial x}x' =& \phi(\partial(x))x'\phi(\partial(x))^{-1} \\ =& xx^{-1}\phi(\partial(x))x'\phi(\partial(x))^{-1}xx^{-1} = xx'x^{-1}, \end{aligned}$$

  since $x^{-1}\phi(\partial(x))$ commutes with x'.
- $\partial({}^{g}x) = \partial(\phi(g)x\phi(g)^{-1}) = g\partial(x)g^{-1}$

We can use crossed modules to extend the definition of crossed Frobenius algebras to a case where the group that acts on A (G) is different to the grading group (X).

Definition 2.3.3. Take an structure with the form $(A, G, X, \partial, \phi, \langle\cdot|\cdot\rangle)$, where (G, X, ∂) is a crossed module, A is an X-graded algebra, ϕ is an action of G over A (${}^{g}a$), and $\langle\cdot|\cdot\rangle$ is an X-paring . Then we say that $(A, G, X, \partial, \phi, \langle\cdot|\cdot\rangle)$ is a G, X-*Frobenius crossed algebra* if the following axioms are satisfied:

1. ${}^{g}(A_x) \subseteq A_{{}^{g}x} \;\; \forall x \in X, g \in G$.
2. $({}^{\partial(x)}b)a_x = a_x b$.
3. ${}^{\partial x}a_x = a_x \;\; \forall a_x \in A_x$.
4. $\langle {}^{g}a|{}^{g}b\rangle = \langle a|b\rangle \;\; \forall a, b \in A$.

Remark 2.3.5. This structure generalizes the concept of G-Frobenius crossed algebra. Axioms $(1) - (4)$ are simply the extensions of axioms $(1) - (4)$ in definition 2.3.1. Again, the homogeneous component of the identity A_1 is a central commutative Frobenius algebra closed under the action of G.

Theorem 2.3.6. *Let $(A, G, X, \partial, \phi, \langle\cdot|\cdot\rangle)$ be a G, X-crossed Frobenius algebra. Then the algebra of G-invariants A^G together with the pairing $\langle\cdot|\cdot\rangle|_{A^G\otimes A^G}$ is a commutative Frobenius algebra.*

Proof. Note that A^G is an algebra. Axiom (2) implies thas A^G is commutative since $ba_x = ({}^{\partial x}b)a_x = a_x b \quad \forall b \in A^G$. It remains to show that the pairing $\langle\cdot|\cdot\rangle|_{A^G\otimes A^G}$ is non-degenerate. Take $a \in A^G$. Since $\langle\cdot|\cdot\rangle$ is non-degenerate, there exists $b \in A$ such that $\langle a|b\rangle \neq 0$. If $b \in A^G$ the problem is over, so we are going to assume that $b \notin A^G$. Define $\hat{b} := \sum_{g\in G} {}^g b$. Then

$$ {}^{g'}\hat{b} = \sum_{g\in G} {}^{g'}({}^g b) = \sum_{g\in G} {}^{g'g}b = \hat{b}. \tag{2.4}$$

Therefore, $\hat{b} \in A^G$ and

$$\langle a|\hat{b}\rangle = \sum_{g\in G}\langle a|{}^g b\rangle = \sum_{g\in G}\langle {}^g a|{}^g b\rangle = \sum_{g\in G}\langle a|b\rangle = |G|\langle a|b\rangle \neq 0. \tag{2.5}$$

□

We will give examples of (G, X)-crossed Frobenius algebras in the following chapter.

2.4 Group-Crossed Frobenius Example

We want now to construct non-trivial examples of a (G, X)-crossed Frobenius algebra. For this, we need to recall the following facts:

- In example 2.2.4 we observed that for any Frobenius algebra $\hat{A}$ the group algebra A^γ, graded by $A^\gamma_x = \hat{A}u_x$ and with multiplication twisted by cocycle $\gamma : X \times X \to \hat{A}^*$ (i.e. $u_x u_{x'} = \gamma(x, x')u_{xx'}$), is an X-graded Frobenius algebra with homogeneous component $A^\gamma_1 \cong \hat{A}$.
- From remark 2.3.1, note that $\hat{A}$ must be commutative, otherwise A^γ will not satisfy the axioms of a crossed Frobenius algebra.
- To endow the previous algebra A^γ with a crossed-Frobenius structure we are going to suppose that G acts trivially on $\hat{A}$. Then we extend this action to the algebra A^γ with a morphism $\epsilon : G \times X \to A^*$ by ${}^g(\hat{a}u_x) = \hat{a}\epsilon(g, x)u_{{}^g x}$. This ϵ must satisfy certain properties that are included in the following theorem.

Theorem 2.4.1. *Let $\hat{A}$ be a commutative Frobenius algebra, (G, X, ∂) be a crossed module, $\gamma : X \times X \to \hat{A}^*$ be a two-cocycle and A^γ be the X-graded Frobenius algebra defined in example 2.2.4. Let G act trivially on $\hat{A}$ and extend the G-action to A^γ by ${}^g(\hat{a}u_x) = \epsilon(g, x)({}^g\hat{a})u_{({}^gx)}$, where $\epsilon : G \times X \to A^*$ is a morphism that satisfies the following properties:*

- $\epsilon(fg, x) = \epsilon(f, {}^g x)\epsilon(g, x)$.
- $\epsilon(\partial(x), h) = \frac{\gamma(x,h)}{\gamma(xhx^{-1},x)}$.
- $\gamma(x, x^{-1}) = \epsilon(g, x)\epsilon(g, x^{-1})\gamma({}^gx, {}^g x^{-1})$.

Then A^γ with action induced by ϵ is a (G, X)-crossed Frobenius algebra, and is denoted $A^{\gamma,\epsilon}$.

Proof. First of all, note that the first property is equivalent to say that ϵ is an action, i.e. the identity ${}^{fg}u_x = {}^f({}^gu_x)$ is satisfied. Now it is sufficient to check that the axioms of a (G, X)-crossed Frobenius algebra are satisfied:

1. ${}^g(A^{\gamma,\epsilon}_x) \subseteq A^{\gamma,\epsilon}_{{}^gx}$: By definition ${}^g(u_x) = \epsilon(g, x)u_{({}^gx)} \in A^{\gamma,\epsilon}_{{}^gx}$.
2. $({}^{\partial(x)}b)a_x = a_xb$: Note

$$\begin{aligned}({}^{\partial(x)}u_h)u_x =&\epsilon(\partial(x), h)u_{xhx^{-1}}u_x\\ =&\frac{\gamma(x, h)}{\gamma(xhx^{-1}, x)}(\gamma(xhx^{-1}, x)u_{xh}) = u_xu_h\end{aligned}$$

   Since $\hat{A}$ is G-invariant, this result is extended by $\hat{A}$-linearity to all $A^{\gamma,\epsilon}$.

3. ${}^{\partial x}a_x = a_x$: This is clear since ${}^{\partial x}u_x = \epsilon(x, x)u_x = u_x$.
4. $\langle {}^ga|{}^gb\rangle = \langle a|b\rangle$: For elements in the base

$$\begin{aligned}\langle {}^g(\hat{a}u_x)|{}^g(\hat{b}gu_h)\rangle =&\epsilon(g, x)\epsilon(g, h)\langle \hat{a}|\hat{b}\rangle\langle u_{({}^gx)}|u_{({}^gh)}\rangle\\ =&\epsilon(g, x)\epsilon(g, h)\langle \hat{a}|\hat{b}\rangle\gamma({}^gx, {}^g h)\delta(xh)\\ =&\langle \hat{a}|\hat{b}\rangle\epsilon(g, x)\epsilon(g, x^{-1})\gamma({}^gx, {}^g x^{-1})\delta(xh)\\ =&\langle \hat{a}|\hat{b}\rangle\gamma(x, x^{-1})\delta(xh)\\ =&\langle \hat{a}u_x|\hat{b}u_h\rangle\end{aligned}$$

   This result is extended by linearity to the whole algebra.

□

Remark 2.4.2. The properties of ϵ were chosen such that $A^{\gamma,\epsilon}$ satisfies the definition of crossed-Frobenius algebra. Hence, if one of this properties is not full-filled, we could conclude that $A^{\gamma,\epsilon}$ is not a crossed Frobenius algebra. On the other hand, if we replace $g = \partial(f)$, with $f \in X$, in the third property. We obtain

$$\begin{aligned}\gamma(x, x^{-1}) =&\epsilon(\partial f, x)\epsilon(\partial f, x^{-1})\gamma(fxf^{-1}, fxf^{-1}) \\ =&\frac{\gamma(f,x)}{\gamma(fxf^{-1}, f)}\frac{\gamma(f,x^{-1})}{\gamma(fx^{-1}f^{-1}, f)}\gamma(fxf^{-1}, fxf^{-1}).\end{aligned} \tag{2.6}$$

Last equation imposes a new restriction to cocycle γ that must be satisfied if $A^{\gamma,\epsilon}$ is a (G, X)-crossed Frobenius algebra.

Since the previous example is a crossed Frobenius algebra, Lemma 2.2.2 and Theorem 2.3.6 guarantee that the homogeneous component of the identity $A_1^{\gamma,\epsilon}$ and the algebra of G-invariants $(A^{\gamma,\epsilon})^G$ are both commutative Frobenius algebras. We can readily check that $A_1^{\gamma,\epsilon}$ is the algebra $\hat{A}$. The algebra of G-invariants $(A^{\gamma,\epsilon})^G$ is much more interesting. Note that if an element $a = \sum_x \hat{a}_x u_x \in (A^{\gamma,\epsilon})^G$, with $\hat{a}_x \in \hat{A} \ \ \forall x \in X$, then

$$^g a = \sum_{x\in X} \hat{a}_x \epsilon(g, x) u_{(^g x)} = \sum_{x\in X} \hat{a}_x u_x = a$$

for all $g \in G$. Matching the coefficients of u_x for each x we obtain

$$\hat{a}_x = \hat{a}_{{}^{(g^{-1})}x}\epsilon(g, {}^{(g^{-1})}x).$$

And using the first property of ϵ we get

$$\epsilon(g, x)\hat{a}_x = \hat{a}_{{}^{(g^{-1})}x}.$$

Therefore, for any x' in the orbit of x under the action of G $\mathcal{O}rb_G(x)$ (i.e. $\exists g \in G| \ {}^g x = x'$), the coefficient $\hat{a}_{x'}$ is completely determined by $\hat{a}_x$. This implies that $(A^{\gamma,\epsilon})^G$ is the algebra of elements with the form

$$\sum_{x\in X/G} \hat{a}_x \left(\sum_{g\in G} \epsilon(g^{-1}, x) u_{^g x} \right),$$

where the $\hat{a}_x$ are free coefficients in $\hat{A}$. We conclude that $(A^{\gamma,\epsilon})^G$ is isomorphic to the group algebra generated by the orbits of X ($\hat{A}[X/G]$), with a γ, ϵ-twisted multiplication induced by the multiplication of the generators of the form $\{\sum_{g\in G} \epsilon(g^{-1}, x) u_{^g x}\}_{x\in X/G}$. We remark this result in the following lemma:

Lemma 2.4.3. *$(A^{\gamma,\epsilon})^G$ is isomorphic to $\hat{A}[X/G]$ with the γ,ϵ-twisted multiplication induced by the multiplication of the algebra generators*

$$\{\sum_{g\in G}\epsilon(g^{-1},x)u_{^g x}\}_{x\in X/G}.$$

Theorem 2.4.1 gives a whole family of examples of (G,X)-crossed Frobenius algebras, starting from a Frobenius algebra $\hat{A}$. This fact completes this chapter about group-Frobenius crossed algebras. In the following section we will show how to use these concepts to characterize the gauging symmetries of a 2-TQFT.

Chapter 3

Orbifolding and Gauging

In the previous chapter we showed that if A is a (G,X)-crossed Frobenius algebra, then the homogeneous component of the identity A_1 and the algebra of G-invariants A^G are both commutative Frobenius algebras. We are now interested in the inverse problem. Suppose we start with a commutative Frobenius algebra $(\hat{A}, \langle\cdot|\cdot\rangle)$ and a crossed module (G, X, ∂), with G acting on A, such that the pairing is G-invariant ($\langle ga|gb\rangle = \langle a|b\rangle$). We want to know if it is possible to find a (G,X)-crossed Frobenius algebra A such that $\hat{A} \cong A_1$ as G-algebras. In addition, in case of existence, we would like to characterize the set of crossed algebras that satisfy these conditions. Motivated by these questions, we introduce the following definitions.

Definition 3.0.1. [6, Definition 2.1.4] Let $\hat{A}$ be a Frobenius algebra. Suppose there exists a crossed Frobenius structure A that satisfies the properties above . Then we are going to call A an *orbifolding* of $\hat{A}$ and the algebra of G-invariants A^G will be the *gauging* of $\hat{A}$ associated to A.

Notation: Let $\hat{A}$ be a Frobenius algebra. We denote $\mathcal{O}rbs(\hat{A}, G, X)$ the set of isomorphism classes of orbifoldings of $\hat{A}$. The set of gaugings of $\hat{A}$ will be denoted by $\mathcal{G}auge(\hat{A}, G, X)$.

Example 3.0.1. In the case where the action of G over $\hat{A}$ is trivial, the orbifolding problem is equivalent to find a X-graded Frobenius algebra such that $A_1 \cong \hat{A}$. In example 2.2.4 the set of algebras $A^{\gamma,\epsilon}$ gives an answer to the orbifolding problem. In this case, since the action is trivial, the gauging of $\hat{A}$ associated to $A^{\gamma,\epsilon}$ coincides exactly with $A^{\gamma,\epsilon}$. Note that even in this case where the action is trivial, the orbifoldings of $\hat{A}$ are a very complete set of crossed-Frobenius algebras .

In other words we want to characterize the sets $\mathcal{O}rbs(\hat{A}, G, X)$ and $\mathcal{G}auge(\hat{A}, G, X)$ for any commutative Frobenius algebra A with a G-pairing. To accomplish a full characterization of these sets in the most general case is too complicated. Hence we are going to restrict to the case where the graded algebra in the crossed structure is strongly graded (see Definition 2.1.2). The properties of strongly graded algebras that simplify this characterization will be discussed in the following section.

3.1 Central Strongly Graded Algebras and Picard groups

Definition 3.1.1. A strongly graded algebra $A = \oplus_{g \in G} A_g$ (see Definition 2.1.2) is called *central* if $A_1 \subset Z(A)$.

Note that if A is a group-drossed Frobenius algebra as in our case, the homogeneous component of the identity A_1 is always central as we obseved in Remark 2.3.1. Therefore we only need to develop the theory for central strongly graded Frobenius algebras.

We can characterize these algebras using Picard groups. To see this we first need the following definitions:

Definition 3.1.2. Let R be a commutative algebra and let M be an R-module. We say that M is invertible if there exists an R-module N such that

$$M \otimes_R N \cong N \otimes_R M \cong R,$$

as R-modules.

Definition 3.1.3. Let R be a commutative algebra. The *Picard group* of R (denoted by $\mathcal{P}ic(R)$) is the set of isomorphism classes of invertible R-modules with product induced by $\otimes_R$.

Thus for elements $[M], [N] \in \mathcal{P}ic(R)$ the product is defined by $[M][N] := [M \otimes N]$. The group identity is the trivial module R, and every element has an inverse by definition of $\mathcal{P}ic(R)$.

Proposition 3.1.1. *Let R be a finite dimensional commutative algebra. Then $\mathcal{P}ic(R)$ is trivial.*

Proof. Since R is finite dimensional, R is an artinian algebra. Then by [8, Theorem 8.7], R is a finite product of commutative Artinian local rings. Since for local rings the Picard group is trivial and the Picard group of a product of rings is the product of the associated Picard group, we have that R have trivial Picard group. □

The relationship between strongly graded algebras and the Picard group then arises from the following facts. In section 2.3 we showed that in a crossed Frobenius algebra A the homogeneous component of the identity A_1 is central in A. Moreover, since $A_1A_x = A_x$ and the elements of A_1 commute with the elements of A_x, we can conclude that A_x is an A_1-module for all $x \in X$. Thus, the multiplication in A induces A_1-module homomorphisms

$$\begin{aligned} \Phi_{x,y} : A_x \otimes_{A_1} A_y &\to A_{xy} \\ a_x \otimes_{A_1} a_y &\to a_x a_y, \end{aligned} \tag{3.1}$$

for every $x, y \in X$. In the case where A is strongly graded, the homomorphisms $\Phi_{x,y}$ are surjective, since $\Phi_{x,y}(A_x \otimes_{A_1} A_y) = A_xA_y = A_{xy}$. Actually, it is possible to show that $\Phi_{x,y}$ is an isomorphism $\forall x, y \in X$ [9, Corollary 3.1.2] . In particular $A_x \otimes_{A_1} A_{x^{-1}} \cong A_1$, so that A_x is an invertible A_1-module for all $x \in X$. Hence $A_x \in \mathcal{P}ic(A_1)$ and since $A_x \otimes_{A_1} A_y \cong A_{xy}$, we conclude there is an homomorphism $\phi : X \to \mathcal{P}ic(A_1)$ given by $\phi(x) = A_x$.

3.2 The Orbifolding of a Commutative Frobenius Algebra $\hat{A}$

To solve the orbifolding problem, lets start with a commutative Frobenius algebra $\hat{A}$ and a crossed module (G, X, ∂) with G acting on $\hat{A}$ by Frobenius algebra automorphisms. Since $\hat{A}$ is Frobenius, we know in particular that $\hat{A}$ is finite (see Definition 1.3.4). It follows from proposition 3.1.1 that $\mathcal{P}ic(\hat{A})$ is trivial. Therefore, $\phi : X \to \mathcal{P}ic(\hat{A})$ is the trivial homomorphism, i.e. $\phi(x) = \hat{A} \quad \forall x \in X$. Hence, if an strongly-graded algebra A satisfies that $A_1 = \hat{A}$, then A is graded by $A_x = \hat{A} \quad \forall x \in X$. In addition, since $A_x \otimes_{\hat{A}} A_y \cong \hat{A} \otimes_{\hat{A}} \hat{A} \cong \hat{A}$, then the $\hat{A}$-module isomorphisms $\Phi_{x,y} : A_x \otimes_{A_1} A_y \to A_{xy}$ are completely determined by the element where $u_x \otimes_{\hat{A}} u_y$ is sent in $A_{xy} \cong \hat{A}$. We will denote these elements by $\gamma(x,y) := \Phi(u_x \otimes_{\hat{A}} u_y)$, where $\gamma : X \otimes X \to A^*$. Note that $\gamma(x,y)$ must be a unit of A, otherwise Φ would not be an isomorphism.

The multiplication on A is defined over homogeneous components by $a_x a_y := \Phi_{x,y}(a_x, a_y)$, and then extended by linearity. Now, since A is an algebra, its multiplication has to satisfy the properties of associativity and unity from definition 1.3.1. In this context, the associative diagram is

$$\begin{array}{ccc} A_x \otimes_{\hat{A}} A_y \otimes_{\hat{A}} A_z & \xrightarrow{\Phi_{x,y} \otimes_{\hat{A}} \mathrm{id}} & A_{xy} \otimes_{\hat{A}} A_z \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes_{\hat{A}} \Phi_{y,z}} & & \Big\downarrow {\scriptstyle \Phi_{xy,z}} \\ A_x \otimes_{\hat{A}} A_{yz} & \xrightarrow{\Phi_{x,yz}} & A_{xyz} \end{array},$$

which is equivalent to the equation $\Phi_{x,yz} \circ (\mathrm{id} \otimes_{\hat{A}} \Phi_{y,z}) = \Phi_{xy,z} \circ (\mathrm{id} \otimes_{\hat{A}} \Phi_{x,y})$. The unitary condition is $\Phi_{1,x} = \mathrm{id}_x = \Phi_{x,1}$

Evaluating both equations at $u_x \otimes_{\hat{A}} u_y \otimes_{\hat{A}} u_z$ we obtain

$$\gamma(y,z)\gamma(x,yz) = \gamma(x,y)\gamma(xy,z) \quad \forall x,y,z \in X \tag{3.2}$$

and $\gamma(1,x) = \gamma(x,1) = 1$. These restrictions are equivalent to say that γ is a normalized cocycle (see Equation (2.1.2)). Then the multiplication in A is twisted by cocycle γ. We conclude that any strongly graded algebra A such that $A_1 = \hat{A}$ has the form of A^γ as defined in example 2.2.4.

On the other hand, we extend the G-action on $\hat{A}$ to A^γ by ${}^g(\hat{a}u_x) = {}^g\hat{a}\,{}^g u_x \;\; \forall x \in X$. Then the action of G over A_x is completely determined by the single element where u_x is sent. We will denote this element by $\epsilon(g,x)u_{{}^g x} := {}^g u_x$ with $\epsilon(g,x) \in \hat{A}^*$ (Otherwise the action of g would not be an automorphism of $\hat{A}$). This ϵ must satisfy the same properties given in Theorem 2.4.1 to conclude that $A^{\gamma,\epsilon}$ is an orbifolding of $\hat{A}$.

In the case where G acts trivially on $\hat{A}$, Theorem 2.4.1 shows that $A^{\gamma,\epsilon}$ is indeed an orbilfolding. In any other case, we can proof that the second property of crossed Frobenius algebras $(({}^{\partial(x)}b)a_x = a_x b)$ is not satisfied:

Proof. Following the same steps of the proof of Theorem 2.4.1 suppose $({}^{\partial(x)}\hat{a}u_h)a_x = a_x\hat{a}u_h$. Then

$$\begin{aligned}({}^{\partial(x)}\hat{a}u_h)u_x &= {}^{g}\hat{a}\epsilon(\partial(x), h)u_{xhx^{-1}}u_x \\ &= {}^{g}\hat{a}\frac{\gamma(x,h)}{\gamma(xhx^{-1}, x)}(\gamma(xhx^{-1}, x)u_{xh}) \\ &= {}^{g}\hat{a}u_x u_h = a_x\hat{a}u_h.\end{aligned}$$

Since $\hat{a} \in Z(A)$, we conclude that ${}^{g}\hat{a} = \hat{a} \quad \forall g \in F, \hat{a} \in \hat{A}$. Therefore, the second property is satisfied if and only if G-acts trivially on $\hat{A}$. □

The remark the final results in the following theorem.

Theorem 3.2.1. *Let $\hat{A}$ be a commutative Frobenius algebra and (G, X, ∂) be a crossed module with G acting on $\hat{A}$. Suppose the orbifoldings of $\hat{A}$ are restricted to strongly graded algebras. Then $\mathcal{O}rbs(A, G, X) = \{A^{\gamma,\epsilon} | A^{\gamma,\epsilon}$ satisfies the same properties of Theorem 2.4.1$\}$ if G acts trivially on A. Otherwise, $\mathcal{O}rbs(A, G, X) = \emptyset$.*

Proof. □

We conclude this chapter characterizing the gauging of $\hat{A}$. In the case where $\hat{A}$ is G-invariant lemma 2.4.3 implies that $\mathcal{G}auge(A, G, X) = \{\hat{A}[X/G] \quad |\hat{A}[X/G]$ has the γ, ϵ-twisted multiplication $\}$. Otherwise $\mathcal{G}auge(A, G, X) = \emptyset$.

Conclusions

Using the relation between 2-TQFT's and Frobenius algebras (see section 1.4), we were able to characterize the orbifoldings of a 2-TQFT in the case where the crossed Frobenius algebra is strongly graded (see section 3.2):

- Supposing that G acts trivially on the initial Frobenius algebra $\hat{A}$, we found that all the possible orbifoldings of $\hat{A}$ are of the form $A^{\gamma,\epsilon}$ as it was defined in Theorem 2.4.1. Therefore, we can obtain $\mathcal{O}rbs(\hat{A}, G, X)$ by characterizing the set of cocycles which satisfy equation (2.6)

$$\begin{aligned}\gamma(x, x^{-1}) =& \epsilon(\partial f, x)\epsilon(\partial f, x^{-1})\gamma(fxf^{-1}, fxf^{-1}) \\ =& \frac{\gamma(f, x)}{\gamma(fxf^{-1}, f)} \frac{\gamma(f, x^{-1})}{\gamma(fx^{-1}f^{-1}, f)} \gamma(fxf^{-1}, fxf^{-1}),\end{aligned}$$

  and the set of ϵ-morphisms satisfying the properties of Theorem 2.4.1.

  In addition, given an orbifolding $A^{\gamma,\epsilon}$ of $\hat{A}$ we obtained that the corresponding gauging is $\hat{A}[X/G]$ with the γ, ϵ-twisted multiplication.

- In the case where the G-action is not trivial, we showed that the set of orbifoldings of $\hat{A}$ is empty. This only implies that our assumptions are still too restrictive, but in principle there could exist orbifoldings for non-strongly graded algebras.

For future projects, it remains to complete the gauging characterization in the case where the algebras are not assumed to be strongly graded.

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