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Orientation in Topological K-Theory

Author:
Nicolás ESCOBAR VELÁSQUEZ

Supervisor:
J. Andrés ÁNGEL CÁRDENAS, Ph.D.

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"Physical laws should have mathematical beauty."

P.A.M. Dirac

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To ma and pa, the best parents someone like me could have possibly asked for. I love you.

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Abstract

The subject of this document is the interaction between topological K-theory and Clifford Algebras in the specific context of K-orientations. Both K-theory and Clifford Algebras are very interesting topics by themselves. The former has been used, for instance, to solve the problem of the maximum number of independent vector fields that a sphere admits. More recently, topological K-theory has came up in the discussion of topological order in solid state physics. On the other hand, Clifford Algebras are deeply rooted in quantum mechanics through Dirac's equation and the notion of spin. However, what's more astonishing about them is not their individual virtues, but the fact that they are deeply connected. As we shall see, Atiyah-Bott-Shapiro's construction establishes a powerful link between them, which we are going to explode to talk about K-orientations.

This text is divided in three chapters. The first one discusses the basic definitions and results related to Clifford Algebras. The twisted adjoint representation, the groups Pin and $Spin$ and their complex analogues are introduced. We carry on a complete classification of a specific class of Clifford algebras, their irreducible representations and their graded modules. Finally, we discuss the graded rings A_* and A_*^c.

The second chapter is on vector bundles. We study them from the beginning, giving all the basic definitions and describing ways to build new vector bundles from old ones. Also, we give a survey of results which will be used later on.

The last chapter is on K-theory. Again, all the basic definitions and results are given. Specifically, we show the existence of long exact sequences and Mayer-Vietoris sequences. We show how to construct classes in K-theory from exact sequences of vector bundles and finally apply all of these results into the subject of K-orientations.

Chapter 1

Clifford Algebras and their Representations

In this chapter, we study Clifford Algebras in detail. We start with the definition and basic properties of these algebras and then we discuss the twisted adjoint representation which gives rise to the Pin and Spin subgroups. Next, we study the classification and periodicity of Clifford algebras and the structure of their representations. We will be following references [5] and [2].

1.1 Definitions and periodicity

1.1.1 Quadratic forms

Definition 1. *A quadratic form q on $\mathbb{R}^n$ is a function $q : \mathbb{R}^n \to \mathbb{R}$ of the form*

$$q(v) = v^T A v$$

for some symmetric matrix $A \in M_{n\times n}(\mathbb{R})$.

Associated to any quadratic form there is a bi-linear one $q(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$q(v, w) = v^T A w$$

Since A is symmetric, $q(v, w)$ is symmetric too. Also, $q(v, v) = q(v)$. Moreover, $q(v, w)$ can be described as the polarization of q:

$$q(v, w) = \frac{1}{2}\left(q(v + w) - q(v) - q(w)\right)$$

Indeed,

$$\begin{aligned} q(v,w) &= \frac{1}{2}\left[(v+w)^T A(v+w) - v^T Av - w^T Aw\right] \\ &= \frac{1}{2}\left[v^T Av + v^T Aw + w^T Av + w^T Aw - v^T Av - w^T Aw\right] \\ &= \frac{1}{2}\left[v^T Aw + w^T Av\right] = v^T Aw \end{aligned}$$

Two vectors $v, w \in \mathbb{R}^n$ are called q-orthogonal if $q(v,w) = 0$. Now, it is a well known fact from Linear Algebra that, since A is symmetric, it can be diagonalized. In other words, there exists a basis $\beta = \{v_i\}_{i=1,\dots,n}$ for V such that v_i and v_j are q-orthogonal if $i \neq j$.

We say q is non-degenerate if there exists a q-orthogonal basis $\beta = \{v_i\}_{i=1,\dots,n}$ such that $q(v_i) \neq 0$ for $i = 1, \dots, n$. From now on, we will always assume q is non-degenerate. In that case, we can further replace v_i by $\frac{v_i}{\sqrt{q(v_i)}}$ or $\frac{v_i}{\sqrt{-q(v_i)}}$, whichever makes sense, in order to get a basis of q-orthogonal vectors for which $q(v_i)$ is either 1 or -1. This discussion shows that for all practical purposes, we can always assume A is diagonal with entries 1 or -1. We call a basis like β a q-orthonormal basis.

1.1.2 Clifford Algebras

Definition 2. *Inside the tensor algebra of $V = \mathbb{R}^n$, $\mathscr{T}V$, let I_q be the two-sided ideal generated by elements of the form $v \otimes v + 1 \cdot q(v)$. The Clifford Algebra associated to V and q is the quotient algebra*

$$Cl(V,q) = \mathscr{T}V / I_q$$

In a sense, Definition 2 reminds us of the definition of the exterior algebra. The difference is that for Clifford Algebras $v \cdot v = -q(v)$, whereas for the exterior algebra $v \cdot v = 0$. Although this difference changes the multiplicative structure, as the following theorem shows, the vector space structure remains the same.

Theorem 1. *The algebras $Cl(V,q)$ and $\bigwedge^* V$ are canonically isomorphic as vector spaces.*

Proof. Consider the map T from $\mathscr{T}V$ to $Cl(V,q)$ given by

$$T(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\deg \sigma} v_{\sigma(1)} \cdot \cdots \cdot v_{\sigma(k)} \tag{1.1}$$

First of all, T factors as $\mathscr{T}V \xrightarrow{A} \mathscr{T}V \xrightarrow{\pi} Cl\,(V,q)$ where A is the map from $\mathscr{T}V$ to itself defined by the equation

$$A\,(v_1 \otimes \cdots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\deg \sigma}\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \tag{1.2}$$

and π is just the canonical projection. Now, $A^2 = A$. Moreover, when we restrict A to the k-th tensor power of V, $\mathscr{T}^kV$, the image of $A|_{\mathscr{T}^kV}$ is isomorphic to the k-th exterior product of V, $\bigwedge^k V$. In fact, if i denotes this isomorphism, the diagram

$$\begin{array}{ccc} \mathscr{T}^kV & \xrightarrow{\pi_A} & \bigwedge^k V \\ {\scriptstyle A} \searrow & & \swarrow {\scriptstyle i} \\ & Im\,(A) & \end{array}$$

where π_A is just the restriction of the canonical projection from $\mathscr{T}V$ to $\bigwedge^* V$. So, we can think of T as factoring through $\mathscr{T}V \xrightarrow{\pi_A} \bigwedge^* V \xrightarrow{\pi \circ i} Cl\,(V,q)$.

Let us show that $\pi \circ i$ is injective. Suppose $\theta \in \bigwedge^* V$ is in the kernel. Since the projection π_A is surjective, $\theta = \pi_A\,(\Theta)$ for some $\Theta \in \mathscr{T}^kV$. Then Θ is in the kernel of $\pi \circ i \circ \pi_A = \pi \circ A$. By definition of $Cl\,(V,q)$ we must have

$$A\,(\Theta) = \sum_{i \in I} \alpha_i \otimes (v_i \otimes v_i + q\,(v_i)) \otimes \beta_i \tag{1.3}$$

for some finite set of indexes I, some tensors $\alpha_i, \beta_i \in \mathscr{T}V$ which we can consider pure and some vectors $v_i \in V$. Let J be the subset of I consisting of those indexes j for which $\deg \alpha_j + \deg \beta_j$ is maximal. Clearly the forms of higher degree in $A\,(\Theta)$ are precisely $\sum_{j \in J} \alpha_j \otimes v_j \otimes v_j \otimes \beta_j$. Let's call that degree n. Now, from 1.3 it follows that

$$\begin{aligned} A^2\,(\Theta) &= A\left(\sum_{i \in I} \alpha_i \otimes (v_i \otimes v_i + q\,(v_i)) \otimes \beta_i\right) \\ &= \sum_{i \in I} A\,(\alpha_i \otimes v_i \otimes v_i \otimes \beta_i) + \sum_{i \in I} q\,(v_i)\, A\,(\alpha_i \otimes \beta_i) \\ &= \sum_{i \in I} q\,(v_i)\, A\,(\alpha_i \otimes \beta_i) \end{aligned}$$

since evidently $A\,(\alpha_i \otimes v_i \otimes v_i \otimes \beta_i) = 0$. But $A^2 = A$, so

$$A^2\,(\Theta) = A\,(\Theta) = \sum_{i \in I} \alpha_i \otimes (v_i \otimes v_i + q\,(v_i)) \otimes \beta_i$$

Which can be truth only if $\sum_{i \in I} \alpha_i \otimes v_i \otimes v_i \otimes \beta_i = 0$. These terms must vanish degree by degree, so in particular we must have $\sum_{j \in J} \alpha_j \otimes v_j \otimes v_j \otimes \beta_j = 0$. So, $A\,(\Theta)$ must be made of form of degree lesser than n.

Our hypothesis is $A\,(\Theta) \in I_q$ so that it is in the kernel of π. But $\sum_{i \in I} \alpha_i \otimes v_i \otimes v_i \otimes \beta_i = 0$

so $\sum_{i\in I} q(v_i)\,\alpha_i \otimes \beta_i$, which is made of forms of degree lesser than n, must belong to I_q. Then, we must have

$$\sum_{i\in I} q(v_i)\,\alpha_i \otimes \beta_i = \sum_{i\in \hat{I}} \hat{\alpha}_i \otimes (\hat{v}_i \otimes \hat{v}_i + q(\hat{v}_i)) \otimes \hat{\beta}_i$$

for some new $\hat{I}$, $\hat{\alpha}_i$, $\hat{\beta}_i$ and $\hat{v}_i$. We can now repeat the same argument to conclude that $A(\Theta)$ is of degree lesser than $n-1$. If we keep repeating this process we conclude that $A(\Theta) = \theta = 0$. Thus, $\pi \circ i$ is injective.

Now, let us show that $\pi \circ i$ is surjective. Notice two things. First, π_A restricted to the copy of V inside $\mathscr{T}V$ is just the identity, so the injectivity of $\pi \circ i$ means that we have a copy of V inside $Cl(V,q)$. Second, this copy actually generates $Cl(V,q)$. Indeed, every element $\phi \in \mathscr{T}V$ can be written as

$$\phi = \sum_{i\in I} v_{i,1} \otimes v_{i,2} \otimes \cdots \otimes v_{i,n_i}$$

and π is surjective and a homomorphism of algebras so for every element $\varphi \in Cl(V,q)$ there exists a ϕ such that $\varphi = \pi(\phi)$. So we must have

$$\varphi = \sum_{i\in I} v_{i,1} \cdot v_{i,2} \cdot \cdots \cdot v_{i,n_i}$$

where $\cdot$ denotes multiplication in $Cl(V,q)$.

The discussion on the preceding paragraph shows that any sub-algebra of $Cl(V,q)$ that contains the copy of V inside $Cl(V,q)$ must be the entire $Cl(V,q)$. But the image under $\pi \circ i$ of $\bigwedge^* V$ is one such sub-algebra. We conclude that $\pi \circ i$ is onto. □

Usually, when we have a construction like the one in Definition 2, there is an associated universal property. In the proof of Theorem 1, we mentioned that there is a copy of V inside $Cl(V,q)$, so it makes sense to talk about extending maps from V to $Cl(V,q)$.

Proposition 1. *Let A be a unital algebra over $\mathbb{R}$ and let f be a linear map from $f : V \to A$ such that*

$$f(v) \cdot f(v) = -1 \cdot q(v)$$

Then there exists a unique algebra homomorphism $\hat{f} : Cl(V,q) \to A$ such that $\hat{f}\big|_V = f$. Moreover, $Cl(V,q)$ is the unique algebra with that property.

Proof. Since f is a linear map from V into an algebra, it extends in a unique way to an homomorphism of algebras $\tilde{f} : \mathscr{T}V \to A$. Clearly, I_q is in the kernel of this map, so $\tilde{f}$ induces a map $\hat{f} : Cl(V,q) \to A$.

As we mentioned in the proof of Theorem 1, every element $\varphi \in Cl(V, q)$ must be of the form

$$\varphi = \sum_{i \in I} v_{i,1} \cdot v_{i.2} \cdot \cdots \cdot v_{i,n_i}$$

where $\cdot$ denotes multiplication in $Cl(V, q)$. Hence, if $\hat{f}$ is any homomorphism of algebras from $Cl(V, q)$ into A, we must have

$$\begin{aligned} \hat{f}(\varphi) &= \hat{f}\left(\sum_{i \in I} v_{i,1} \cdot v_{i.2} \cdot \cdots \cdot v_{i,n_i}\right) \\ &= \sum_{i \in I} \hat{f}(v_{i,1} \cdot v_{i.2} \cdot \cdots \cdot v_{i,n_i}) \\ &= \sum_{i \in I} \hat{f}(v_{i,1}) \cdot \hat{f}(v_{i,2}) \cdot \cdots \cdot \hat{f}(v_{i,n_i}) \\ &= \sum_{i \in I} f(v_{i,1}) \cdot f(v_{i,2}) \cdot \cdots \cdot f(v_{i,n_i}) \end{aligned}$$

which shows $\hat{f}$ is completely determined by f and so must be unique.

That $Cl(V, q)$ is the unique algebra with this property follows from the traditional arguments. □

Corollary 1. *Let $T : V \to V$ be a linear map such that $q(T(v)) = q(v)$ for all $v \in V$. Then, T extends uniquely to an algebra homomorphism $\hat{T} : Cl(V, q) \to Cl(V, q)$.*

Proof. If $i : V \hookrightarrow Cl(V, q)$ denotes the inclusion of V into $Cl(V, q)$, then $i \circ T$ is a linear map from V into $Cl(V, q)$. Also,

$$(i \circ T(v)) \cdot (i \circ T(v)) = v \cdot v = -q(v)$$

so by Proposition 1, $i \circ T$ extends uniquely to a map $\hat{T} : Cl(V, q) \to Cl(V, q)$. □

Corollary 2. *Let $T, S : V \to V$ be two linear maps that preserve the quadratic form. By Corollary 1, $T \circ S$ extends to $T \hat{\circ} S : Cl(V, q) \to Cl(V, q)$. We have $T \hat{\circ} S = \hat{T} \circ \hat{S}$.*

Proof. The map $\hat{T} \circ \hat{S}$ is an algebra homomorphism from $Cl(V, q)$ into itself that restricts to $T \circ V$ on the copy of V in $Cl(V, q)$, so by the uniqueness of the extension in Proposition 1, the result follows. □

Remark 1. *We mentioned at the end of the previous section that we can choose a basis $\beta = \{e_i\}_{1 \leq i \leq n}$ for V made of q-orthonormal vectors. For that basis the following identity holds:*

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij} q(e_i)$$

Indeed, it is obvious for $i = j$ and for $i \neq j$

$$\begin{aligned} 0 &= q(e_i, e_j) = q(e_i + e_j) - q(e_i) - q(e_j) \\ &= -(e_i + e_j) \cdot (e_i + e_j) + e_i \cdot e_i + e_j \cdot e_j \\ &= -e_i \cdot e_i - e_j \cdot e_j - e_i \cdot e_j - e_j \cdot e_i + e_i \cdot e_i + e_j \cdot e_j \\ &= -e_i \cdot e_j - e_j \cdot e_i \end{aligned}$$

Graded Structure Consider the map $\alpha : V \to V$ that consists of multiplying by -1. Since $q(-v) = q(v)$, it preserves the quadratic form, so by Corollary 1 it extends to an automorphism $\hat{\alpha} : Cl(V, q) \to Cl(V, q)$. Now, $\alpha^2 = Id_V$. By Corollary 2, $\hat{\alpha}^2 = Id_{Cl(V,q)}$. So, $\hat{\alpha}$ is diagonalizable and has eigenvalues 1 and -1, with eigenspaces $Cl^0(V, q)$ and $Cl^1(V, q)$ respectively. The map α induces a decomposition of $Cl(V, q)$ as a vector space:

$$Cl(V, q) = Cl^0(V, q) \oplus Cl^1(V, q)$$

Additionally, since $\hat{\alpha}$ is an algebra homomorphism, it easy to check that $Cl^i(V, q) \cdot Cl^j(V, q) \subset Cl^{i+j}(V, q)$ where the sum of the indexes is taken mod 2. Note further that only $Cl^0(V, q)$ is a sub-algebra. Also, $Cl^0(V, q)$ can be though of as those elements in $Cl(V, q)$ generated exclusively by sums of forms each of which is a product of an even number of elements on V, whereas $Cl^1(V, q)$ is the same but for an odd number of vectors. For that reason, $Cl^0(V, q)$ is called the even part of $Cl(V, q)$, while $Cl^1(V, q)$ is referred to as the odd part.

1.2 Representations and modules of Clifford Algebras

1.2.1 The twisted adjoint representation

Some of the most important concepts related to Clifford Algebras, such as the groups Pin and $Spin$, have to do with the twisted adjoint representation, so let's study it now. Let $Cl^\times$ the group of invertible elements in $Cl(V, q)$.

Definition 3. *The twisted adjoint representation of $Cl^\times$ is the map $\tilde{Ad} : Cl^\times \to End(Cl(V, q))$ given by the following rule: if $\varphi \in Cl^\times$ and $\theta \in Cl(V, q)$, $\tilde{Ad}_\varphi(\theta) = \hat{\alpha}(\varphi)\theta\varphi^{-1}$.*

One initial remark is that for $v, w \in V$, $q(v) \neq 0$, we can give a more explicit formula for $\tilde{Ad}_v(w)$. Indeed, since $\alpha(v) = -v$ for all $v \in V$, we have

$$\begin{aligned} \tilde{Ad}_v(w) &= -v \cdot w \cdot \frac{v}{-q(v)} = -[(v + w) \cdot (v + w) - v \cdot v - w \cdot w - w \cdot v] \cdot \frac{v}{-q(v)} \\ &= -[-2q(v, w) - w \cdot v] \cdot \frac{v}{-q(v)} = -\frac{2q(v, w)}{q(v)} v + w \end{aligned}$$

where $q(v,w) = (v+w)\cdot(v+w) - v^2 - w^2$ is the polarization of q. From this formula, we can say a couple of things. We put the first one as a Remark for future reference:
Remark 2. *$\tilde{Ad}_v(w)$ is the reflection with respect to a plane q-orthogonal to v, multiplied by a minus sign.*

Second, it's clear that $\tilde{Ad}_v(V) \subset V$. Moreover, the quadratic form is also preserved:

$$\begin{aligned} q\left(\tilde{Ad}_v(w)\right) &= q\left(w - \frac{2q(v,w)}{q(v)}v\right) = -\left(w - \frac{2q(v,w)}{q(v)}v\right)\cdot\left(w - \frac{2q(v,w)}{q(v)}v\right) \\ &= -\left[w^2 - \frac{2q(v,w)}{q(v)}(v\cdot w + w\cdot v) + \left(\frac{2q(v,w)}{q(v)}\right)^2 v^2\right] \\ &= -\left[w^2 + \frac{(2q(v,w))^2}{q(v)} + \left(\frac{2q(v,w)}{q(v)}\right)^2 v^2\right] = q(w) \end{aligned}$$

Let us call $P(V,q)$ the subgroup of elements $\varphi \in Cl^\times(V,q)$ such that $\tilde{Ad}_\varphi(V) \subset V$. Then $\tilde{Ad}$ becomes a representation of $P(V,q)$ into $Aut(V)$. The previous calculation shows vectors $v \in V$ such that $q(v) \neq 0$ belong to $P(V,q)$.

Later on, the fact stated in the next proposition about the twisted adjoint representation will turn out to be very important. Before discussing it, it should be mentioned that $\mathbb{R}^\times \subset P(V,q)$. Indeed, for any $r \in \mathbb{R}^\times$ and any $v \in V$ such that $q(v) \neq 0$, $r = (rv)\cdot\left(\frac{v}{-q(v)}\right)$.

Proposition 2. *The kernel of $\tilde{Ad} : P(V,q) \to Aut(V)$ is $\mathbb{R}^\times$.*

Proof. Suppose $\varphi \in \ker \tilde{Ad}$. That means $\tilde{Ad}_\varphi = Id_V$ or in other words that $\alpha(\varphi)v\varphi^{-1} = v$ for all $v \in V$. Equivalently $\alpha(\varphi)v = v\varphi$. Now, as we discussed in the previous section, φ can be written in a unique way as $\varphi = \varphi_0 + \varphi_1$, where $\varphi_i \in Cl^i(V,q)$. Hence

$$\alpha(\varphi) = \alpha(\varphi_0) + \alpha(\varphi_1) = \varphi_0 - \varphi_1$$

that means

$$\varphi_0 v - \varphi_1 v = v\varphi_0 + v\varphi_1$$

Equating the even and odd parts, we conclude $\varphi_0 v = v\varphi_0$ and $-\varphi_1 v = v\varphi_1$.

As the closing remarks of the previous section showed, we can choose a basis $\beta = \{e_i\}_{1\leq i\leq n}$ for V made of q-orthonormal vectors. In the proof of Proposition 1, we mentioned that every element of $Cl(V,q)$, in particular φ_0 and φ_1, can be written as

$$\begin{aligned} \varphi_0 &= \sum_{i\in I} v^0_{i,1}\cdot v^0_{i.2}\cdot\dots\cdot v^0_{i,n_i} \\ \varphi_1 &= \sum_{i\in I} v^1_{i,1}\cdot v^1_{i.2}\cdot\dots\cdot v^1_{i,n_i} \end{aligned}$$

for some $v^j_{i,k} \in V$. By further expanding each vector in terms of the basis β, we can rewrite

$$\begin{aligned} \varphi_0 &= a_0 + e_1 a_1 \\ \varphi_1 &= \tilde{a}_0 + e_1 \tilde{a}_1 \end{aligned}$$

where a_0, a_1, $\tilde{a}_0$ and $\tilde{a}_1$ are pure elements of $Cl\,(V,q)$ and e_1 doesn't appear in any of them. Notice that a_0, $\tilde{a}_0$ must be even forms, whereas a_1, $\tilde{a}_1$ must be odd, so in view of Remark 1, e_1 commutes with a_0 and $\tilde{a}_0$ but anti-commutes with a_1 and $\tilde{a}_1$. Now, we have $\varphi_0 v = v\varphi_0$, so if we take $v = e_1$:

$$\begin{aligned} (a_0 + e_1 a_1)\, e_1 &= e_1\,(a_0 + e_1 a_1) \\ a_0 e_1 + e_1 a_1 e_1 &= e_1 a_0 + e_1^2 a_1 \\ e_1 a_0 - e_1^2 a_1 &= e_1 a_0 + e_1^2 a_1 \end{aligned}$$

We deduce $a_1 = 0$. That means e_1 doesn't appear in φ_0. We can repeat the same process for e_2, e_3, etc, deducing every time that e_i doesn't appear in φ_0. So, we must have $\varphi_0 \in \mathbb{F}$. Using the same kind of argument, it also follows that $\varphi_1 \in \mathbb{R}$. Hence, $\varphi \in \mathbb{R}$. Since $\varphi \in \tilde{Ad} \subset P\,(V,q) \subset Cl^{\times}\,(V,q)$, $\varphi \neq 0$, so $\varphi \in \mathbb{R}^{\times}$. □

The subgroups Pin, Pin^c, $Spin$ and $Spin^c$.

On the tensor algebra of V, $\mathscr{T}V$, we can define an operation of transposition given by

$$\begin{aligned} *^t : \quad \mathscr{T}V &\to \mathscr{T}V \\ v_1 \otimes v_2 \otimes \cdots \otimes v_n &\to v_n \otimes \cdots \otimes v_2 \otimes v_1 \end{aligned}$$

This operation preserves $\mathscr{I}_q$, so it descends to $Cl\,(V,q)$. Also, it is an anti-automorphism, since clearly $(\alpha\beta)^t = \beta^t \alpha^t$ for all α and β in $Cl\,(V,q)$.

For any $\varphi \in Cl\,(V,q)$, let $N\,(\varphi) = \varphi \cdot \alpha\left(\varphi^t\right)$. We have the following pair of results about N:

Proposition 3. *If $\varphi \in P\,(V,q)$, $N\,(\varphi) \in \mathbb{R}^{\times}$.*

Proof. For any $v \in V$ $\alpha\,(\varphi)\, v\varphi^{-1} \in V$, so

$$\left[\alpha\,(\varphi)\, v\varphi^{-1}\right]^t = \alpha\,(\varphi)\, v\varphi^{-1}.$$

On the other hand

$$\begin{aligned} \left[\alpha\,(\varphi)\, v\varphi^{-1}\right]^t &= \left(\varphi^{-1}\right)^t v\alpha\,(\varphi)^t \\ &= \left(\varphi^t\right)^{-1} v\alpha\left(\varphi^t\right), \end{aligned}$$

since it's easy to see that $\alpha(\varphi)^t = \alpha(\varphi^t)$ and $(\varphi^{-1})^t = (\varphi^t)^{-1}$. Comparing both expressions, we get

$$\begin{aligned} \alpha(\varphi)v\varphi^{-1} &= (\varphi^t)^{-1} v\alpha(\varphi^t) \\ \varphi^t\alpha(\varphi)v\varphi^{-1}\alpha\left((\varphi^{-1})^t\right) &= v \\ \alpha(\alpha(\varphi^t)\varphi)v\varphi^{-1}\alpha\left((\varphi^{-1})^t\right) &= v. \end{aligned} \tag{1.4}$$

Finally, if we knew $\alpha(\varphi^t)\varphi \in P(V,q)$, we could conclude the proof by noticing that the left hand side of equation 1.4 is $\tilde{Ad}_{\alpha(\varphi^t)\varphi}v$. So, what 1.4 is telling us is that $\alpha(\varphi^t)\varphi$ is in the kernel of $\tilde{Ad}$ and by Proposition 2, the result would follow. But this is easy to check. First of all, $\varphi^t \in P(V,q)$. Indeed, for any $v \in V$,

$$\begin{aligned} \alpha(\varphi^t)v(\varphi^t)^{-1} &= \alpha(\varphi)^t v(\varphi^{-1})^t = [\varphi^{-1}v\alpha(\varphi)]^t = -[\alpha(\alpha(\varphi^{-1})v\varphi)]^t \\ &= -\left[\alpha\left(\tilde{Ad}_{\varphi^{-1}}v\right)\right]^t. \end{aligned}$$

But $\varphi^{-1} \in P(V,q)$ because $P(V,q)$ is a subgroup. Since both α and $*^t$ preserve the vector space, $-\left[\alpha\left(\tilde{Ad}_{\varphi^{-1}}v\right)\right]^t \in V$. We conclude that $\varphi^t \in P(V,q)$. On the other hand, for any $\theta \in P(V,q)$, $\alpha(\theta) \in P(V,q)$ because

$$\alpha(\alpha(\theta))v\alpha(\theta)^{-1} = -\alpha\left(\alpha(\theta)v\theta^{-1}\right) \in V$$

In conclusion, $\varphi^t \in P(V,q)$ so $\alpha(\varphi^t) \in P(V,q)$ and hence $\alpha(\varphi^t)\varphi \in P(V,q)$. □

Proposition 4. *N is an homomorphism of groups from $P(V,q)$ to $\mathbb{R}^\times$. Also, $N(\alpha(x)) = N(x)$.*

Proof. Let $\theta, \varphi \in P(V,q)$. Then

$$N(\theta\varphi) = \theta\varphi\alpha\left((\theta\varphi)^t\right) = \theta\varphi\alpha\left(\varphi^t\theta^t\right) = \theta\varphi\alpha\left(\varphi^t\right)\alpha\left(\theta^t\right)$$

Now, by Proposition 3 $\varphi\alpha(\varphi^t) \in \mathbb{R}^\times$, so it commutes with every element of $Cl(V,q)$, in particular with $\alpha(\theta^t)$. So

$$N(\theta\varphi) = \theta\alpha(\theta^t)\varphi\alpha(\varphi^t) = N(\theta)N(\varphi^t).$$

On the other hand,

$$N(\alpha(x)) = \alpha(x)\alpha\left([\alpha(x)]^t\right) = \alpha\left(x\alpha(x^t)\right) = \alpha(N(x)) = N(x).$$

□

Proposition 5. *For all* $\varphi \in P(V,q)$ *and* $v \in V$, $q\left(\tilde{Ad}_{\varphi} v\right) = q(v)$. *In other words,* $\tilde{Ad}$ *maps* $P(V,q)$ *into* $O(V,q)$.

Proof. Notice that for every $v \in V$, $N(v) = \alpha(v)\, v^t = -vv^t = q(v)$, so

$$q\left(\tilde{Ad}_{\varphi} v\right) = N\left(\tilde{Ad}_{\varphi} v\right) = N\left(\alpha(\varphi)\, v\varphi^{-1}\right) = N(\alpha(\varphi))\, N\left(\varphi^{-1}\right) N(v).$$

Since $N(\alpha(\varphi)) = N(\varphi)$, $q\left(\tilde{Ad}_{\varphi} v\right) = N(v) = q(v)$. □

The group $Pin(V,q)$ is defined as the kernel of N. The group $Spin(V,q)$ is defined as $Pin(V,q) \cap Cl^0(V,q)$. Now, in Proposition 5 we proved that $\tilde{Ad}$ sends $P(V,q)$ to $O(V,q)$. We might wonder how $\tilde{Ad}$ looks like when restricted to $Pin(V,q)$. The answer is in the following Proposition.

Proposition 6. *The following sequences are exact:*

$$\begin{aligned} 0 &\to \{1,-1\} \to Pin(V,q) \overset{\tilde{Ad}}{\to} O(V,q) \to 0 \\ 0 &\to \{1,-1\} \to Spin(V,q) \overset{\tilde{Ad}}{\to} SO(V,q) \to 0 \end{aligned}$$

Proof. First of all, let us show that $\tilde{Ad}$ restricted to $Pin(V,q)$ is surjective onto $O(V,q)$. Notice that every vector $v \in V$ with $q(v) = 1$ belongs to $Pin(V,q)$, since $N(v) = q(v)$ for $v \in V$. As it was proved in Remark 2, for those elements $\tilde{Ad}_v$ is just a reflection with respect to a q-orthogonal plane. A classical theorem due to Dieudonné and Cartan (See Theorem 7.2.1 in [3]) states that $O(V,q)$ is generated by such reflections, so the result follows. Additionally, $SO(V,q)$ must be the subgroup of $O(V,q)$ generated by an even number of such reflections, because each one of them has determinant -1. So $\tilde{Ad}$ restricted to $Spin(V,q)$ is surjective onto $SO(V,q)$.

The fact that the kernel of $\tilde{Ad}$ is $\{1,-1\}$ follows from Proposition 2. □

Let us consider now the complexified algebras $Cl(V,q) \otimes \mathbb{C}$. We can extend the definitions of α and $*^t$:

$$\begin{aligned} \alpha(\varphi \otimes z) &= \alpha(\varphi) \otimes z \\ (\varphi \otimes z)^t &= \varphi^t \otimes \bar{z}. \end{aligned}$$

Let's call $P^c(V,q)$ the subgroup of invertible elements φ in $Cl(V,q) \otimes \mathbb{C}$ such that for every $v \in V$, $\alpha(\varphi)\, v\varphi^{-1} \in V$. Then the entire discussion from Remark 2 up to Proposition 5 follows through changing $\mathbb{C}$ by $\mathbb{R}$. We define Pin^c as the kernel of $N : P^c(V,q) \to \mathbb{C}^{\times}$ and $Spin^c \subset Pin$ as the pre-image under $\tilde{Ad}$ of $SO(V,q)$. Then the analogous of Proposition 6 is:

Proposition 7. *The following sequences are exact:*

$$\begin{aligned} 0 &\rightarrow U(1) \rightarrow Pin^c(V,q) \overset{\tilde{Ad}}{\rightarrow} O(V,q) \rightarrow 0 \\ 0 &\rightarrow U(1) \rightarrow Spin^c(V,q) \overset{\tilde{Ad}}{\rightarrow} SO(V,q) \rightarrow 0 \end{aligned}$$

1.2.2 Clifford Modules

Classification of Cl

As we shall see later on, it turns out that most applications of the concepts we've been developing so far have to do with modules over Clifford Algebras, rather than with the algebras themselves. That is the case for Bott periodicity and also for orientations in K-theory. This may be partly because, as we will show in the following, we can give an explicit description of the irreducible representations of these algebras.

Before we get to the main discussion, a couple of comments are in order. First, for $V = \mathbb{R}^n$ and $r, s \in \mathbb{N}$ such that $r+s = n$, we denote by $q_{r,s}$ the quadratic form associated to the matrix

$$\begin{pmatrix} I_{r\times r} & 0 \\ 0 & -I_{s\times s} \end{pmatrix}$$

and we refer to $Cl(\mathbb{R}^n, q_{r,s})$ as $Cl_{r,s}$.

A second remark has to do with describing maps from $Cl(V,q)$ to some other algebra. According with Proposition 1, it suffices to define a linear map T from V into the algebra in such a way that $T(v) \cdot T(v) = -q(v)$. It follows from Remark 1 that for a q-orthogonal basis $\beta = \{e_i\}_{i=1,\dots n}$, a necessary condition would then be

$$T(e_i)T(e_j) + T(e_j)T(e_i) = -2\delta_{ij}q(e_i)$$

since, as we saw in Remark 1, this condition follows from $(e_i + e_j)\cdot(e_i + e_j) = -q(e_i + e_j)$. But it turns out this is also a sufficient condition. Indeed, for $v = \sum \lambda_i e_i$:

$$\begin{aligned} T(v)T(v) &= T\left(\sum_i \lambda_i e_i\right) T\left(\sum_j \lambda_j e_i\right) \\ &= \sum_i \lambda_i^2 T(e_i)T(e_i) + 2\sum_{i\neq j} T(e_i)T(e_j) + T(e_j)T(e_i) \\ &= -\sum_i \lambda_i^2 q(e_i) = -q(v). \end{aligned}$$

Finally, a word about tensor products. For the following discussion we will consider the tensor product of two Clifford Algebras $Cl_{r,s}$ and $Cl_{n,m}$ as an algebra, with multiplication given by

$$(\phi_1 \otimes \phi_2) \cdot (\varphi_1 \otimes \varphi_2) = \phi_1 \cdot \varphi_1 \otimes \phi_2 \varphi_2.$$

Having made this comments, we can get into the study of the algebras $Cl_{r,s}$ and its representations, specially the cases $Cl_{n,0}$. We will start by establishing isomorphims between these algebras and classical matrix algebras, using an algorithm that is, in a sense, recursive. The main idea of this algorithm is contained in the following Theorem.

Theorem 2. *There exist the following algebra isomorphisms:*

1. $Cl_{r+2,0} \cong Cl_{0,r} \otimes C_{2,0}$
2. $Cl_{0,r+2} \cong Cl_{r,0} \otimes Cl_{0,2}$
3. $Cl_{r+1,s+1} \cong Cl_{r.s} \otimes Cl_{1,1}$

Proof. We will show the first isomorphism only, the other two are completely analogous. Let $\{e_i\}_{i=1,\ldots,r}$ be the standard basis for $Cl_{0,r}$, $\{e'_i\}_{i=1,2}$ be the corresponding basis for $Cl_{2,0}$ and $\{e''_i\}_{i=1,\ldots,r}$ the one for $Cl_{r+2,0}$. Consider the map

$$\begin{aligned} f : \mathbb{R}^{r+2} &\to Cl_{0,r} \otimes Cl_{2,0} \\ e''_i &\mapsto \begin{cases} e_i \otimes e'_1 \cdot e'_2 & i \leq r \\ 1 \otimes e'_{i-r} & i = r+1, r+2. \end{cases} \end{aligned}$$

For $i, j \leq r$, we have

$$\begin{aligned} f\left(e''_i\right) \cdot f\left(e''_j\right) + f\left(e''_j\right) \cdot f\left(e''_i\right) &= \left(e_i \otimes e'_1 \cdot e'_2\right) \cdot \left(e_j \otimes e'_1 \cdot e'_2\right) + \left(e_j \otimes e'_1 \cdot e'_2\right)\left(e_i \otimes e'_1 \cdot e'_2\right) \\ &= \left(e_i \cdot e_j \otimes e'_1 \cdot e'_2 \cdot e'_1 \cdot e'_2\right) + \left(e_j \cdot e_i \otimes e'_1 \cdot e'_2 \cdot e'_1 \cdot e'_2\right) \\ &= -\left(e_i \cdot e_j + e_j \cdot e_i\right) \otimes 1 = 2\delta_{ij} \end{aligned}$$

The other cases are checked similarly. By the universal property, f induces a map $\hat{f} : Cl_{r+2,0} \to Cl_{0,r} \otimes C_{2,0}$.

Let us show that $\hat{f}$ is onto. It's easy to check that the set $\left\{e_i \otimes e'_j\right\}_{i,j}$ generates $Cl_{0,r} \otimes Cl_{2,0}$. But, for $i \leq r$, we have

$$\begin{aligned} \hat{f}\left(e''_i \cdot e_{r+1}\right) &= \hat{f}\left(e''_i\right)\hat{f}\left(e_{r+1}\right) = \left(e_i \otimes e'_1 \cdot e'_2\right) \cdot \left(1 \otimes e'_1\right) = -e_i \otimes e'_2 \\ \hat{f}\left(e''_i \cdot e_{r+2}\right) &= \hat{f}\left(e''_i\right)\hat{f}\left(e_{r+2}\right) = \left(e_i \otimes e'_1 \cdot e'_2\right) \cdot \left(1 \otimes e'_2\right) = -e_i \otimes e'_1 \end{aligned}$$

which means that $\left\{e_i \otimes e'_j\right\}_{i,j}$ is in the image of $\hat{f}$. Hence $\hat{f}$ must be onto.

By dimension count, $\hat{f}$ must be an isomorphism. □

Corollary 3. *We have the following algebra isomorphisms:*

1. $Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0}$
2. $Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8}$

Proof. We have

$$\begin{aligned} Cl_{n+8,0} &\cong Cl_{0,n+6} \otimes Cl_{2,0} \cong Cl_{n+4,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \cong Cl_{0,n+2} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \\ &\cong Cl_{n,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \cong Cl_{n,0} \otimes Cl_{8,0} \end{aligned}$$

And similar for the second one. □

Theorem 2 and Corollary 3 are almost everything we need to compute any $Cl_{n,0}$ or $Cl_{0,n}$. First of all, it tells us that it suffices to study the first 8 ones, since the structure is the same from then on. Moreover, Theorem 2 can be used repeatedly to express any of these 8 Clifford Algebras as a tensor product of $Cl_{1,0}$, $Cl_{0,1}$, $Cl_{2,0}$ and $Cl_{0,2}$. All that's left is for us to do is "calculating" these last five algebras and describing some rules to compute tensor products. We have:

- The algebra $Cl_{1,0}$ is generated by two elements, 1 and e. Moreover, $e \cdot e = -q(e) = -1$. So $Cl_{1,0} \cong \mathbb{C}$.
- The algebra $Cl_{2,0}$ has a basis with three elements: 1, $i = e_1$, $j = e_2$ and $k = e_1 e_2$. It's easy to verify that $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$ and $ki = j$. So, $Cl_{2,0} \cong \mathbb{H}$.
- The algebra $Cl_{0,2}$ has generators e_1 and e_2 for which the relation

  $$e_i e_j + e_j e_i = 2\delta_{ij} \tag{1.5}$$

  holds. Let us define a map $T : \mathbb{R}^2 \to M_2(\mathbb{R})$ given by sending

  $$e_1 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  We can check directly that this definition preserves the relation 1.5, so it extends to a map $\hat{T} : \mathbb{R}^2 \to M_2(\mathbb{R})$. Since

  $$\hat{T}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{T}(e_1 e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the matrices

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

are a basis for $M_2(\mathbb{R})$, $\hat{T}$ is an isomorphism. So $Cl_{0,2} \cong M_2(\mathbb{R})$.

The case $Cl_{0,1}$ is slightly more complicated. On $\mathbb{R}^2$, we can define a multiplication according to

$$(a, b) \cdot (c, d) = (ac, bd).$$

We call $\mathbb{R} \oplus \mathbb{R}$ the algebra that has $\mathbb{R}^2$ as a vector space, endowed with this multiplication. Now, let $T : \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ the map that sends e, a unit element that generates $\mathbb{R}$, to $(1, -1)$. Since $(1, -1) \cdot (1, -1) = (1, 1)$, the unit of the algebra, T extends to $\hat{T} : Cl_{0,1} \to \mathbb{R} \oplus \mathbb{R}$. But $(1, 1)$ and $(1, -1)$ are a basis for $\mathbb{R} \oplus \mathbb{R}$, so $\hat{T}$ is an isomorphism. A completely analogous definition can be made for any other algebra instead of $\mathbb{R}$.

Now that we've discussed the "irreducible" cases, we move on to the rules for calculating tensor products:

Theorem 3. *For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{K}$, there exist the following real algebra isomorphisms.*

1. $M_n(\mathbb{R}) \otimes_{\mathbb{R}} M_m(\mathbb{R}) \cong M_{nm}(\mathbb{R})$
2. $M_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{F} \cong M_n(\mathbb{F})$
3. $\mathbb{C} \oplus \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$
4. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$
5. $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$

Proof. We can reinterpret the first isomorphism in the form

$$\hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) \otimes_{\mathbb{R}} \hom_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^m) \cong \hom_{\mathbb{R}}(\mathbb{R}^n \otimes \mathbb{R}^m, \mathbb{R}^n \otimes \mathbb{R}^m).$$

Let us prove this last statement. We have a bi-linear map TP from $\hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) \times \hom_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^m)$ into $\hom_{\mathbb{R}}(\mathbb{R}^n \otimes \mathbb{R}^m, \mathbb{R}^n \otimes \mathbb{R}^m)$ that takes a pair (T, S) into the map $T \otimes S$. The map $T \otimes S \in \hom_{\mathbb{R}}(\mathbb{R}^n \otimes \mathbb{R}^m, \mathbb{R}^n \otimes \mathbb{R}^m)$ is defined by the rule $(T \otimes S)(v \otimes w) = T(v) \otimes S(w)$. Hence, TP induces a linear map $\tilde{TP}$ from $\hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) \otimes_{\mathbb{R}} \hom_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^m)$ into $\hom_{\mathbb{R}}(\mathbb{R}^n \otimes \mathbb{R}^m, \mathbb{R}^n \otimes \mathbb{R}^m)$. We can check that this maps preserves the multiplicative structure of $\hom_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) \otimes_{\mathbb{R}} \hom_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^m)$: given two pairs (T, S) and (Q, R), for any $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, we have

$$\begin{aligned} [(T, S) \cdot (Q, R)](v \otimes w) &= (TQ, SR)(v \otimes w) = TQ(v) \otimes SR(w) \\ &= (T \otimes S)(Q(v) \otimes R(w)) \\ &= (T \otimes S)(Q \otimes R)(v \otimes w). \end{aligned}$$

n	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_{16}(\mathbb{R})$

TABLE 1.1: $Cl_{n,0}$ and $Cl_{0,n}$ expressed as matrix algebras.

Thus, $\tilde{TP}$ is actually a map of algebras.

Let us show now that it is onto. If $e_1,\ldots,e_n$ and $u_1,\ldots,u_m$ are basis for $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, the set $\{e_i\otimes u_j\}_{i=1,\ldots n;j=1,\ldots m}$ is a basis for $\mathbb{R}^n\otimes\mathbb{R}^m$. For $i,k=1,\ldots.n$, $j,l=1,\ldots,m$ the maps $T^{k,l}_{i,j}:\mathbb{R}^n\otimes\mathbb{R}^m\to\mathbb{R}^n\otimes\mathbb{R}^m$ defined by the rule $T^{k,l}_{i,j}\left(e_{i'}\otimes u_{j'}\right)=\delta_{i,i'}\delta_{j,j'}e_k\otimes u_l$ are a basis for $\hom_{\mathbb{R}}\left(\mathbb{R}^n\otimes\mathbb{R}^m,\mathbb{R}^n\otimes\mathbb{R}^m\right)$. Let us call R^k_i the linear map from $\mathbb{R}^n$ into itself defined by $R^k_i(e_s)=\delta_{i,s}e_s$, and analogously the map S^l_j belongs to $\hom_{\mathbb{R}}(\mathbb{R}^m,\mathbb{R}^m)$. Clearly $T^{k,l}_{i,j}=R^k_i\otimes S^l_j$, so $T^{k,l}_{i,j}$ is in the image of $\tilde{TP}$. This is enough to show $\tilde{TP}$ is onto. By dimension count, it must be an isomorphism.

Let's move on to the second isomorphism. We have a bi-linear mar from $M_n(\mathbb{R})\times\mathbb{F}$ into $M_n(\mathbb{F})$ given by sending the pair $\left((a_{ij})_{i,j},k\right)$ to the matrix $(ka_{ij})_{i,j}$. It induces a linear map from $M_n(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{F}$ into $M_n(\mathbb{F})$ which clearly preserves the multiplicative structure. Now, if $\beta=\{e_i\}$ is a basis for $\mathbb{F}$ as a vector space over $\mathbb{R}$, the matrices $A^k_{i',j'}=\left(e_k\delta_{i',i}\delta_{j',j}\right)_{i,j}$ are a basis for $M_n(\mathbb{F})$. But clearly $A^k_{i',j'}$ is the image of $\left(\left(\delta_{i',i}\delta_{j',j}\right)_{i,j}\otimes e_k\right)$. So, the induced map is onto. By dimension count, it is an isomorphism.

The third isomorphism is given by the assignments

$$\begin{aligned}(1,0) &\mapsto \frac{1}{2}(1\otimes 1+i\otimes i)\\ (0,1) &\mapsto \frac{1}{2}(1\otimes 1-i\otimes i).\end{aligned}$$

For the fourth isomorphism, we can see $\mathbb{C}^2$ as $\mathbb{H}$. So, for $z\in\mathbb{C}$ and $h\in\mathbb{H}$, we can define a map from $\mathbb{C}^2$ into $\mathbb{C}^2$ by the rule $v\mapsto zv\bar{h}$. This is $\mathbb{C}$-linear, so it establishes a bi-linear map from $\mathbb{C}\times\mathbb{H}$ to $M_2(\mathbb{C})$.

For the fifth, we see $\mathbb{R}^4$ as $\mathbb{H}$ and repeat the same process: given $h,g\in\mathbb{H}$, we can define a map $\mathbb{R}^4$ into itself by the rule $v\mapsto hv\bar{g}$. This is $\mathbb{R}$-linear, so it establishes a bi-linear map from $\mathbb{H}\times\mathbb{H}$ to $M_4(\mathbb{R})$. Basic linear algebra shows that these last three maps are indeed isomorphisms. □

Example 1. *Let us "calculate"* $Cl_{6,0}$*:*

$$\begin{aligned}Cl_{6,0} &\cong Cl_{0,4}\otimes Cl_{2,0}\cong Cl_{2,0}\otimes Cl_{0,2}\otimes Cl_{2,0}\cong\mathbb{H}\otimes M_2(\mathbb{R})\otimes\mathbb{H}\\ &\cong M_4(\mathbb{R})\otimes M_2(\mathbb{R})\cong M_8(\mathbb{R})\end{aligned}$$

Proceeding similarly, we get Table 1.1.

n	1	2
$\mathbb{C}l_n$	$\mathbb{C}\oplus\mathbb{C}$	$M_2(\mathbb{C})$

TABLE 1.2: Complex Clifford algebras.

The Complexified Clifford Algebras Table 1.1 is essentially everything we wanted for $\mathbb{F}=\mathbb{R}$. Let us now consider the case $\mathbb{F}=\mathbb{C}$. That is, let us consider $Cl_{n,0}\otimes\mathbb{C}$ and $Cl_{0,n}\otimes\mathbb{C}$ for some n, with some basis $\beta=\{e_1,\ldots,e_n\}$. The second case corresponds to $q(e_i)=-1$ for all $i=1,\ldots,n$. But then we could build a new basis $\hat{\beta}=\{\hat{e}_1,\ldots,\hat{e}_n\}$ with $\hat{e}_i=ie_i$. Since $q(\hat{e}_i)=-(q(e_i))=1$, we see that $Cl_{n,0}\otimes\mathbb{C}$ and $Cl_{0,n}\otimes\mathbb{C}$ must be isomorphic. We call the common algebra $\mathbb{C}l_n$.

Most of the arguments follow through:

Corollary 4. $\mathbb{C}l_{n+2}\cong\mathbb{C}l_n\otimes_{\mathbb{R}}\mathbb{C}l_2$

Proof. According to Theorem 2, $Cl_{n+2,0}\cong Cl_{0,n}\otimes_{\mathbb{R}}Cl_{2,0}$. Taking tensor products with $\mathbb{C}$ we get

$$\begin{aligned} Cl_{n,0}\otimes_{\mathbb{R}}\mathbb{C} &\cong Cl_{0,n}\otimes_{\mathbb{R}}\mathbb{C}\otimes_{\mathbb{R}}Cl_{2,0}\otimes_{\mathbb{R}}\mathbb{C} \\ \mathbb{C}l_{n+2} &\cong \mathbb{C}l_n\otimes_{\mathbb{R}}\mathbb{C}l_2. \end{aligned}$$

□

Notice that Corollary 4 plays the role of both Theorem 2 and Corollary 3: at the same time it tells us how to compute higher dimensional complex Clifford Algebras and it suggests that it suffices to consider $\mathbb{C}l_1$ and $\mathbb{C}l_2$. Using Proposition 1.1 and Theorem 3 we get Table 1.2.

Irreducible Modules For purposes that will become clear later on, we're interested in the irreducible representations of Clifford Algebras. Let us say a couple of words about representations before we get into the specific subject that interests us.

Definition 4. *Let A be a real algebra. A representation of A is a map of algebras R from A into the set of linear maps of some real vector space E into itself. We say E is an A-module.*

Two representations $R:A\to\hom_{\mathbb{F}}(E,E)$, $S:A\to\hom_{\mathbb{F}}\left(\hat{E},\hat{E}\right)$ are said to be equivalent if there exists a linear isomorphism L between E and $\hat{E}$ such that for every $\varphi\in A$ and every $v\in E$,

$$R[\varphi](v)=L^{-1}\circ S[\varphi]\circ L(v).$$

This definition of equivalence defines an equivalence relation on the set of representations of A.

n	1	2	3	4	5	6	7	8
I_n	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$

TABLE 1.3: Groups I_n

n	1	2
J_n	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$

TABLE 1.4: Groups J_n

A representation $R : A \to \hom_{\mathbb{F}}(E, E)$ is said to be irreducible if there isn't any subspace $E_0 \subset E$ such that for every $\varphi \in A$, $R[\varphi](E_0) \subset E_0$.

Roughly speaking, we want to know how many equivalence classes of irreducible representations does an algebra $Cl_{n,0}$ have. We encode this information in the following definition:

Definition 5. *For every positive integer n, I_n is the free group generated by the equivalence classes of irreducible representations of $Cl_{n,0}$. J_n is the analogue for $\mathbb{C}l_n$.*

The key fact to calculate the groups I_n and J_n is contained in the following theorem:

Theorem 4. *Let $\mathbb{F} = \mathbb{R}$,$\mathbb{C}$ or $\mathbb{H}$. Then $M_n(\mathbb{F})$ has a unique equivalence class of irreducible representations, given by ρ, the natural representation over $\mathbb{F}^n$. On the other hand, $M_n(\mathbb{F}) \oplus M_n(\mathbb{F})$ has two equivalence classes, associated to$\rho_1(\varphi_1 \oplus \varphi_2) = \rho(\varphi_1)$ and $\rho_2(\varphi_1 \oplus \varphi_2) = \rho(\varphi_2)$.*

Proof. See [4]. □

Tables 1.3 and 1.4 follow directly from Theorem 4 and Tables 1.1 and 1.2.

The Groups A_k

Although we have achieved a characterization of Clifford Algebras and their representations, for reasons that will become clear later on, we are interested in a particular class of representations: graded ones. This will be the subject of this subsection.

Definition 6. *Let V_1 and V_2 be two real vector spaces with quadratic forms q_1, q_2. From their Clifford Algebras $Cl(V_1, q_1)$ and $Cl(V_2, q_2)$, we can build a third algebra called their Twisted Tensor Product and denoted $Cl(V_1, q_1)\hat{\otimes}Cl(V_2, q_2)$. The vector space is given by $Cl(V_1, q_1) \otimes Cl(V_2, q_2)$. For pure forms $\phi_1, \phi_2 \in Cl(V_1, q_1)$ and $\varphi_1, \varphi_2 \in Cl(V_2, q_2)$, we* *define their product as*

$$(\phi_1 \otimes \varphi_1) \cdot (\phi_2 \otimes \varphi_2) = (-1)^{\deg \phi_2} \phi_1 \phi_2 \otimes \varphi_1 \varphi_2$$

where $\deg \phi_2 = i$ *if* $\phi_2 \in Cl^i (V_1, q_1)$.

The importance of twisted tensor products lies in the next Proposition. Before we state it, let us make a small remark. Keeping the notation of Definition 6, we can define a quadratic form $q_1 \oplus q_2$ on $V_1 \oplus V_2$ simply by declaring V_1 and V_2 to be $q_1 \oplus q_2$-orthogonal. That is, for $v_1 \in V_1$, $v_2 \in V_2$, we define

$$(q_1 \oplus q_2)(v_1 + v_2) = q_1 (v_1) \oplus q_2 (v_2)$$

Proposition 8. $Cl (V_1 \oplus V_2, q_1 \oplus q_2) \cong Cl (V_1, q_1) \hat{\otimes} Cl (V_2, q_2)$

Proof. Consider the map

$$\begin{array}{rccc} T: & V_1 \oplus V_2 & \to & Cl (V_1, q_1) \hat{\otimes} Cl (V_2, q_2) \\ & v_1 \oplus v_2 & \to & v_1 \hat{\otimes} 1 + 1 \hat{\otimes} v_2. \end{array}$$

We have

$$\begin{aligned} T (v_1 \oplus v_2) \cdot T (v_1 \oplus v_2) &= \left(v_1 \hat{\otimes} 1 + 1 \hat{\otimes} v_2\right) \cdot \left(v_1 \hat{\otimes} 1 + 1 \hat{\otimes} v_2\right) \\ &= v_1 \cdot v_1 \hat{\otimes} 1 + v_1 \hat{\otimes} v_2 - v_1 \hat{\otimes} v_2 + 1 \hat{\otimes} v_2 \cdot v_2 \\ &= -q (v_1) - q (v_2) = -(q_1 \oplus q_2)(v_1 \oplus v_2). \end{aligned}$$

That means T extends to a map $\hat{T}$ from $Cl (V_1 \oplus V_2, q_1 \oplus q_2)$ to $Cl (V_1, q_1) \hat{\otimes} Cl (V_2, q_2)$. Let us show that it's surjective. It's easy to see that elements of the form $v_1 \hat{\otimes} v_2$ generate $Cl (V_1, q_1) \hat{\otimes} Cl (V_2, q_2)$, so it suffices to show that they are in the algebra generated by the image of T. But that's easy:

$$v_1 \hat{\otimes} v_2 = \left(v_1 \hat{\otimes} 1 + 1 \hat{\otimes} 0\right) \cdot \left(1 \hat{\otimes} 0 + 1 \hat{\otimes} v_2\right) = T (v_1 \oplus 0) \cdot T (0 \oplus v_2).$$

□

Proposition 8 tells us that twisted tensor products are the correct way to build Clifford algebras in higher dimensions using those in lower dimensions. In the following, we will say a few words about how modules of higher dimensional Clifford Algebras interact with modules of lower dimensional ones.

Definition 7. *Let* V *a vector space with a quadratic form* q. *A graded module of* $Cl (V, q)$ *is a* $Cl (V, q)$*-module* E *that admits a vector space decomposition* $E = E^0 \oplus E^1$ *such that if* $\varphi \in Cl^i (V, q)$ *and* $v \in E^j$, $\varphi \cdot v \in E^{i+j \mod 2}$.

k	1	2	3	4	5	6	7	8
M_k	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$

TABLE 1.5: Groups M_k

k	1	2
N_k	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$

TABLE 1.6: Groups N_k

We would like to give a classification of graded modules similar to the one developed in the last section. The key fact in this process is described in the following Propositions:
Proposition 9. *There is a one-to-one correspondence between graded modules of $Cl(V,q)$ and ungraded modules of $Cl^0(V,q)$.*

Proof. Let E be an ungraded $Cl^0(V,q)$-module. Then $\hat{E} = Cl(V,q) \times_{Cl^0(V,q)} E$ is a graded $Cl(V,q)$-module with $\hat{E}_0 = E$ and $\hat{E}_1 = Cl^1(V,q) \times_{Cl^0(V,q)} E$. On the other hand, if $F = F_0 \oplus F_1$ is a graded module, F_0 is an ungraded $Cl^0(V,q)$-module. The assignments $E \mapsto \hat{E}$ and $F \mapsto F_0$ are inverses. □

Proposition 10. $Cl_{r,s} \cong Cl^0_{r+1,s}$

Proof. Let $\beta = \{e_1, e_2, \ldots, e_{r+s+1}\}$ be the canonical basis of $\mathbb{R}^{r+s+1}$. Then

$$\hat{\beta} = \{e_1, e_2, \ldots, \hat{e}_{r+1}, \ldots, e_{r+s+1}\}$$

(the hat denotes omission of that vector) is a basis for $\mathbb{R}^{r+s}$. Consider the map

$$f : \mathbb{R}^{r+s} \to Cl^0_{r+1,s}$$

defined on $\hat{\beta}$ by the rule $f(e_i) = e_{r+1}e_i$ for $i = 1, \ldots, r+s+1$, $i \neq r+1$. Then,

$$\begin{aligned} f(e_i)f(e_j) + f(e_j)f(e_i) &= e_{r+1}e_ie_{r+1}e_j + e_{r+1}e_je_{r+1}e_i \\ &= -e_{r+1}e_{r+1}e_ie_j - e_{r+1}e_{r+1}e_je_i = e_ie_j + e_je_i \end{aligned}$$

so there is an extension $\hat{f} : Cl_{r,s} \to Cl^0_{r+1,s}$. The map $\hat{f}$ sends basis to basis, so it is an isomorphism. □

Let M_k denote the free group generated by the irreducible graded modules of $Cl_{k,0}$ and N_k denote the analogous for $\mathbb{C}l_k$. Then Tables 1.3 and 1.4, together with Propositions 9 and 9 give us Tables 1.5 and 1.6.

Finally, we can make the definition that will be of most use for us later on. Consider the inclusion $i : \mathbb{R}^n \to \mathbb{R}^{n+1}$. If $\mathbb{R}^n$ is endowed with the quadratic form $q_{n,0}$ and $\mathbb{R}^{n+1}$ has q_{n+1}, this inclusion preserves the quadratic form so it extends to a map $\hat{i} : Cl_{n,0} \to Cl_{n+1,0}$. Hence, its pullback gives a function $i^* : M_{n+1} \to M_n$. We set $A_n \equiv M_n / i^* (M_{n+1})$. Similarly, we define $A_k^c \equiv N_k / i^* (N_{k+1})$. Now, all the information we've gathered so far about Clifford modules allows us to compute almost at once all the groups A_k and A_k^c. The only problematic cases are A_{4k} and A_{2k}^c.

Proposition 11. *$A_{4k} \cong \mathbb{Z}$, $A_{2k}^c \cong \mathbb{Z}$.*

Proof. From Table 1.3 we see that $M_{4k} \cong \mathbb{Z} \oplus \mathbb{Z}$. In fact, Table 1.1 and Theorem 4 tell us that the two generators come from the two in-equivalent representations of $M_n(\mathbb{H}) \oplus M_n(\mathbb{H})$ or $M_n(\mathbb{R}) \oplus M_n(\mathbb{R})$, call them x and y. Now consider the map $M_n(\mathbb{H}) \oplus M_n(\mathbb{H}) \to M_n(\mathbb{H}) \oplus M_n(\mathbb{H})$ that exchanges the order of the summands, ie., sends (φ_1, φ_2) into (φ_2, φ_1). It follows from Theorem 4 that the pullback of this map flips the generators of M_{4k}. But $M_{4k+1} \cong \mathbb{Z}$, so the image under i^* of the generator of M_{4k+1}, z, must be left invariant. Hence, by counting dimensions,we must have $z = x + y$ and hence $A_{4k} = \mathbb{Z}$. The complex case is completely analogous. □

Chapter 2

Vector Bundles

In this chapter, we will study vector bundles in detail. We will discuss the basic definitions and properties, as well as a series of results that will be useful when we start considering K-theory in the next chapter. We will be mostly interested in complex vector bundles over compact Hausdorff spaces.

2.1 Basic definitions

Definition 8. *A vector bundle is a 4-tuple $\xi = (E, X, F, \pi)$ where E and X are topological spaces called the total and base space of the bundle respectively, π is a continuous surjective map called the projection of the bundle and F is a (real or complex) finite dimensional vector space referred to as the fiber of the bundle. Two conditions must hold for the triple:*

- *For each $x \in X$, $\pi^{-1}(x)$ must have a vector space structure, linearly isomorphic to F.*
- *Local triviality: For each $x \in X$ there exists an open set of X, U, that contains x and a map*

  $$h_U : U \times F \to \pi^{-1}(U)$$

  that is an homeomorphism onto its image, for each $\tilde{x} \in U$ defines a linear isomorphism $h_U|_{proj_1^{-1}(x)} : F \to \pi^{-1}(x)$ and makes the diagram

  $$\begin{array}{ccc} U \times F & \xrightarrow{h_U} & \pi^{-1}(U) \\ {}_{proj_1} \searrow & & \swarrow_{\pi} \\ & U & \end{array}$$

  commute.

A couple (U, h_U) is called a local trivialization of ξ, U is called a trivializing open set and h_U a trivializing map. A vector bundle is called trivial if there exists a local trivialization such that the trivializing open set is X.

Example 2. *Consider the real projective space $\mathbb{R}P^n$. Let E be the subspace of $\mathbb{R}P^n \times \mathbb{R}^n$ given by*

$$E = \{([\vec{v}], \vec{w}) \mid \exists \lambda \in \mathbb{R} : \vec{w} = \lambda \vec{v}\}$$

and $X = \mathbb{R}P^n$. Projection onto the first factor gives us a continuous, surjective mapping $\pi : E \to X$. From the definition of E it's clear that the pre-images of π are one-dimensional subspaces of $\mathbb{R}^n$.

For the trivializations, let

$$U_i = \{[a_0, a_1, \ldots, a_n] \in \mathbb{R}P^n | a_i \neq 0\} \quad i = 0, 1, \ldots, n$$

Each element on U_i can be written in a unique way in the form $[a_0, \ldots, 1, \ldots a_n]$ where 1 is in the i-th position. The map

$$\begin{array}{cccc} h_{U_i}: & U_i \times \mathbb{R} & \to & \pi^{-1}(U_i) \\ & ([a_0, \ldots, 1, \ldots a_n], t) & \mapsto & (ta_0, \ldots t, \ldots ta_n) \end{array}$$

gives us the required homeomorphisms. Thus, $\xi = (E, X, F, \pi)$ is a vector bundle.

Now that we know how vector bundles are defined, let us consider how they connect to each other.

Definition 9. *Let $\xi = (E, X, F, \pi)$ and $\eta = \left(\hat{E}, \hat{X}, \hat{F}, \hat{\pi}\right)$ be two vector bundles such that F and $\hat{F}$ have the same field of scalars. An homomorphism of vector bundles, denoted $\left(f, \tilde{f}\right)$: $\xi \to \eta$, is a pair of continuous maps $\left(f, \hat{f}\right)$ where $\hat{f} : E \to \hat{E}$, $f : X \to \hat{X}$, such that the diagram*

$$\begin{array}{cccc} & E & \stackrel{\hat{f}}{\to} & \hat{E} \\ \pi & \downarrow & & \downarrow & \hat{\pi} \\ & X & \stackrel{f}{\to} & \hat{X} \end{array}$$

commutes and $\hat{f}\Big|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \hat{\pi}^{-1}(f(x))$ is a linear map for every $x \in X$.

Given a third vector bundle $\zeta = \left(\tilde{E}, \tilde{X}, \tilde{F}, \tilde{\pi}\right)$ (where $\tilde{F}$ again has the same field of scalars as F and $\hat{F}$) and a vector bundle homomorphism $(g, \hat{g}) : \eta \to \zeta$ (where $g : \hat{X} \to \tilde{X}$ and $\hat{g} : \hat{E} \to \tilde{E}$), the composition of $\left(f, \tilde{f}\right)$ and $(g, \hat{g})$ is the vector bundle homomorphism $(g, \hat{g}) \circ \left(f, \tilde{f}\right) : \xi \to \zeta$ given to the pair $\left(g \circ f, \hat{g} \circ \hat{f}\right)$.

The identity homomorphism of ξ, Id_ξ, is the vector bundle homomorphism $Id_\xi : \xi \to \xi$ associated to the pair (Id_X, Id_E).

An isomorphism of vector bundles between ξ and η is an homeomphism of vector bundles f : $\xi \to \eta$ for which there exists another homomorphism of vector bundles $g : \eta \to \xi$ such that

$f \circ \eta = Id_\eta$ and $\eta \circ \xi = Id_\xi$.

Example 3. *In the context of Example 2, consider the trivial bundle η given by taking $\hat{E} = \mathbb{R}P^n \times \mathbb{R}^n$, $X = \mathbb{R}P^n$, $\hat{F} = \mathbb{R}^n$ and $\hat{\pi} = proj_1$. As can be directly checked, the pair $(Id_{\mathbb{R}P^n}, i)$, where $i : E \hookrightarrow \hat{E}$ is just the inclusion, induces an homeomorphism of vector bundles $\hat{\imath} : \xi \to \eta$.*

If all the other elements in the definition of vector bundle are sufficiently clear from the context, it is a common abuse of notation to refer to the total space as a vector bundle. For instance, in Example 3 we may say that $\eta = \mathbb{R}P^n \times \mathbb{R}^n$. Consequently, it is also common to refer to an homomorphism of vector bundles with the same symbol as the one used for the associated map between the total spaces and to talk about them indistinctively.

Remark 3. *Perhaps the hardest part of checking that a pair $\left(f, \tilde{f}\right)$ is indeed a vector bundle homomorphism is verifying continuity. There is, however, an useful criterion to do this. Fix $x \in X$. Let (U_x, h_{U_x}) and $\left(V_{f(x)}, h_{V_{f(x)}}\right)$ be local trivializations around x and $f(x)$ respectively. We can assume without loss of generality that the image of $f \circ h_{U_x}$ is contained in the image of $h_{V_{f(x)}}$ (if it doesn't we can intersect U_x with $\pi\left(f^{-1}\left(Im\left(h_{V_{f(x)}}\right)\right)\right)$ and restrict h_{U_x}). Then $\alpha_x^f : h_{V_{f(x)}}^{-1} \circ f \circ h_{U_x}$is a map from $U_x \times F$ to $V_{f(x)} \times \hat{F}$. If for every $x \in X$ we can choose U_x and $V_{f(x)}$ such that α_x^f is continuous, then f must be continuous, because continuity is a local property and both h_{U_x} and $h_{V_{f(x)}}^{-1}$ are local homeomorphisms.*

Last but not least, we have the notion of a section of vector bundles, which is critical in many applications of these notions:

Definition 10. *Let $\xi = (E, X, F, \pi)$ be a vector bundle. A section is a map $\psi : X \to E$ such that $\pi \circ \psi = Id_X$. The set of sections of ξ is denoted by $\Gamma(\xi)$. Since each fiber $\pi^{-1}(x)$ is a vector space, given two sections ψ_1 and ψ_2 we can sum them to get a new section according to $(\psi_1 + \psi_2)(x) = \psi_1(x) + \psi_2(x)$. Also, we can define a multiplication by the field of scalars of F, according to $(r\psi)(x) = r\psi(x)$. Hence $\Gamma(\xi)$ is a vector space over the same field of scalars as F.*

2.2 Construction of bundles

This section is devoted to explain some techniques to construct new vector bundles from preexisting ones. So, throughout this section, let

$$\xi = (E, X, F, \pi) \qquad \eta = \left(\hat{E}, X, \hat{F}, \hat{\pi}\right)$$

be vector bundles over the same topological space space.

The pullback of a vector bundle Let Y be a topological space and $f : Y \to X$ a continuous function. We will construct a new vector bundle with Y as base space using this information, called the pullback of ξ by f, or $f^*(\xi)$. First, define $\tilde{E} \subset Y \times E$ as

$$\tilde{E} = Y \times_{(f,\pi)} E = \{(y, v) \,|\, f(y) = \pi(v)\}$$

the so called fiber product. Projection onto the first factor gives us the required map to Y, $\tilde{\pi}$. The fibers of this projection are also the fibers of the vector bundle ξ, so they are endowed with a vector space structure isomorphic to F. Finally, for the trivializations, fix $y \in Y$. We know that there exists an open set $U_x \subset X$ around x and a local homeomorphism $h_{U_x} : U_x \times F \to \pi^{-1}(U_x)$. Then

$$\begin{aligned} \tilde{\pi}^{-1}\left(f^{-1}(U_x)\right) &= f^{-1}(U_x) \times_{(f,\pi)} E \\ &= f^{-1}(U_x) \times_{(f,\pi)} \pi^{-1}(U_x) \\ &\cong f^{-1}(U_x) \times_{(f,\pi)} U_x \times F \\ &\cong f^{-1}(U_x) \times F \end{aligned}$$

Since $f^{-1}(U_x) \times_{(f,\pi)} U_x$ is just $f^{-1}(U_x)$. Thus we have the required local triviality and we can conclude that $f^*(\xi)$ is indeed a vector bundle.

Notice that the map $proj_2 : \tilde{E} \to E$ makes the diagram

$$\begin{array}{ccccc} & \tilde{E} & \xrightarrow{proj_2} & E & \\ \tilde{\pi} & \downarrow & & \downarrow & \pi \\ & Y & \xrightarrow{f} & X & \end{array}$$

commute and it's obviously a linear homomorphism on each fiber, so the pair $(f, proj_2)$ defines an homomorphism of vector bundles from $f^*(\xi)$ to ξ.

Duals Recall that $\pi^{-1}(x)$ is a vector space for every $x \in X$. As such, it has a dual vector space $\left[\pi^{-1}(x)\right]^*$. Define

$$E_{dual} = \bigsqcup_{x \in X} \left[\pi^{-1}(x)\right]^*$$

Now every $v \in E_{dual}$ belongs to one and only one $\left[\pi^{-1}(x)\right]^*$, so there is a natural assignment $\pi_{dual} : E_{dual} \to X$ although we cannot discuss continuity yet, since we haven't defined a topology on E_{dual}.

In order to do just that, recall that we have a local trivialization (U, h_U) around every $x \in X$. Moreover, for each $\hat{x} \in U$, $h|_{proj_1^{-1}(\hat{x})}$ is a linear isomorphism from F to $\pi^{-1}(\hat{x})$.

Hence, $\left(h|_{proj_1^{-1}(\hat{x})}\right)^*$ is a linear isomorphism from $\left[\pi^{-1}(x)\right]^*$ to F^*. Repeating this process over each $\hat{x} \in U$, we get a function

$$h_U^* : \bigsqcup_{x \in U} \left[\pi^{-1}(x)\right]^* \to U \times F^*$$

that is bijective because each $\left(h|_{proj_1^{-1}(\hat{x})}\right)^*$ is a linear isomorphism. Now, consider $V \subset E_{dual}$. Given a local trivialization (U, h_U), V generates a subset of $U \times F^*$ by $V_U = h_U^* \left(V \cap \pi_{dual}^{-1}(U)\right)$. We declare V to be open if and only if each one of these V_U is open. It's easy to check that this is indeed a topology on E_{dual} and that it makes π_{dual} continuous. Moreover the pairs $\left(U, (h_U^*)^{-1}\right)$ are local trivializations of E_{dual}. This makes the 4-tuple $(E_{dual}, X, F^*, \pi_{dual})$ is a vector bundle. It is known as the *dual bundle to* ξ and denoted by ξ^*.

Tensor products and the bundle of isomorphisms An important step in the definition of the dual bundle was to define a topology for E_{dual}. We can imitate this process to form two additional vector bundles, the bundle of homomorphisms $Hom(\xi, \eta)$ and the tensor product of ξ and η, $\xi \otimes \eta$.

First, define

$$E_{Hom(\xi,\eta)} = \bigsqcup_{x \in X} Hom\left(\pi^{-1}(x), \hat{\pi}^{-1}(x)\right)$$

Again there is a natural assignment $\pi_{Hom(\xi,\eta)}$ from $E_{Hom(\xi,\eta)}$ to X. For each $x \in X$ there are local trivializations around x, $\left(U_x^\xi, h_{U_x}^\xi\right)$ and $\left(U_x^\eta, h_{U_x}^\eta\right)$, for ξ and η respectively. For every $\tilde{x} \in U_x^\xi \cap U_x^\eta$ and every element $\alpha_x \in Hom\left(\pi^{-1}(x), \hat{\pi}^{-1}(x)\right)$, there is a unique linear map $\tilde{\alpha}_x$ that makes the diagram

$$\begin{array}{ccccc} & \pi^{-1}(\tilde{x}) & \xrightarrow{\tilde{\alpha}_{\tilde{x}}} & \hat{\pi}^{-1}(\tilde{x}) & \\ \left(h_{U_x}^\xi\Big|_{proj^{-1}(\tilde{x})}\right) & \uparrow & & \uparrow & \left(h_{U_x}^\eta\big|_{proj^{-1}(\tilde{x})}\right) \\ & F & \xrightarrow{\alpha_{\tilde{x}}} & \hat{F} & \end{array}$$

commute. The assignment $\tilde{*}_x : Hom\left(F, \hat{F}\right) \to Hom\left(\pi^{-1}(x), \hat{\pi}^{-1}(x)\right)$ that sends α_x to $\tilde{\alpha}_x$ is clearly a linear isomorphism. This allows us to define a function

$$\tilde{*}_{U_x^\xi \cap U_x^\eta} : \left(U_x^\xi \cap U_x^\eta\right) \times Hom\left(F, \hat{F}\right) \to \pi_{Hom(\xi,\eta)}^{-1}\left(U_x^\xi \cap U_x^\eta\right)$$

We can use these functions to define a topology on $E_{Hom(\xi,\eta)}$ in a manner completely analogous to what we did for the dual vector bundle. Again it's easy to verify that all the conditions are given to make the tuple $\left(E_{Hom(\xi,\eta)}, X, Hom\left(F, \hat{F}\right), \pi_{Hom(\xi,\eta)}\right)$ a vector bundle. This is the bundle of homomorphisms, $Hom(\xi, \eta)$.

Remark 4. *A small remark should be made about the sections of the bundle* $Hom\,(\xi,\eta)$*. It follows from the preceding discussion that an element* $s \in \Gamma\,(Hom\,(\xi,\eta))$ *is a map that assigns to every point in* $x \in X$ *an homomorphism* $L_x : \pi^{-1}(x) \to \hat{\pi}^{-1}(x)$ *in a continuous fashion. But according to Definition 9, this is exactly what a homomorphism of vector bundles should do. In other words, every homomorphism of vector bundles between* ξ *and* η *can be though of as a section of the bundle* $Hom\,(\xi,\eta)$ *and viceversa.*

The construction of $\xi \otimes \eta$ is completely analogous.

2.3 Basic properties of vector bundles

Bijective homomorphisms are invertible In contrast to what happens in Point Set Topology, where a continuous function may be both surjective and injective but its "set theoretical" inverse may not be continuous, bijective homomorphisms of vector bundles are invertible.

Proposition 12. *Let* $\xi = (E, X, F, \pi)$ *and* $\eta = \left(\hat{E}, X, \hat{F}, \hat{\pi}\right)$ *be two vector bundles over the same base space such that* F *and* $\hat{F}$ *have the same field of scalars. Let* s *be an element of* $\Gamma\,(Hom\,(\xi,\eta))$ *such that* $s\,(x)$ *is invertible for every* $x \in X$*. Then the assignment* $s^{-1} : X \to Hom\,(\xi,\eta)$ *given by* $s^{-1}(x) = (s\,(x))^{-1}$ *is continuous. In fact, it is also an element of* $\Gamma\,(Hom\,(\xi,\eta))$*.*

Proof. Let

$$E_{GL(\xi,\eta)} = \bigsqcup_{x\in X} GL\left(\pi^{-1}(x), \hat{\pi}^{-1}(x)\right)$$

endowed with the topology inherited as a subspace of $E_{Hom(\xi,\eta)}$. Keeping the notation from last section, the trivializations $\tilde{*}_{U_x^\xi \cap U_x^\eta}$ can be restricted to give local homeomorphisms

$$\tilde{*}_{U_x^\xi \cap U_x^\eta} : U_x^\xi \cap U_x^\eta \times GL\left(F, \hat{F}\right) \to E_{GL(\xi,\eta)} \cap \pi^{-1}_{Hom(\xi,\eta)}\left(U_x^\xi \cap U_x^\eta\right)$$

Which are in fact group isomorphisms over each fiber. Now, consider the map $inv : E_{GL(\xi,\eta)} \to E_{GL(\xi,\eta)}$ that sends sends T to T^{-1}. The same idea as the one used in Remark 3 shows that in order to check continuity of inv it suffices to check the continuity of each of the maps $\alpha_x = \left(\tilde{*}_{U_x^\xi \cap U_x^\eta}\right)^{-1} \circ inv \circ \left(\tilde{*}_{U_x^\xi \cap U_x^\eta}\right)$. But $\alpha_x(x, S) = \left(x, S^{-1}\right)$. Since inversion is a continuous function in the Lie Group $GL\left(F, \hat{F}\right)$, these α_x are all continuous so inv is indeed continuous. Finally, $s^{-1} = inv \circ s$, so it is continuous. It's obviously a section, so $s^{-1} \in \Gamma\,(Hom\,(\xi,\eta))$. □

Corollary 5. *Let $\xi = (E, X, F, \pi)$ and $\eta = \left(\hat{E}, \hat{X}, \hat{F}, \hat{\pi}\right)$ be two vector bundles over the same base space such that F and $\hat{F}$ have the same field of scalars. Then if $\left(f, \tilde{f}\right) : \xi \to \eta$ is a vector bundle homomorphism such that f is an homeomorphism of the base spaces and that $\tilde{f}\big|_{\pi^{-1}(x)}$ is invertible for every $x \in X$, then $\left(f, \tilde{f}\right)$ is an isomorphism of vector bundles.*

Proof. Under the hypothesis, it's clear that $\tilde{f}$ must be bijective. Moreover, it's "set theoretical" inverse, $\tilde{f}^{-1}$ makes the diagram

$$\begin{array}{ccc} \hat{E} & \overset{\tilde{f}^{-1}}{\longrightarrow} & E \\ {}_{\hat{\pi}}\searrow & & \swarrow_{\pi} \\ & X & \end{array}$$

commute and its restriction to the fibers is given by $\left(\tilde{f}\big|_{\pi^{-1}(x)}\right)^{-1}$, which is a linear transformation. So, all that remains to be shown is that $\tilde{f}^{-1}$ is indeed continuous. But this is clear now, since, as stated in Remark 4, $\left(f, \tilde{f}\right)$ can be thought of as a section of the bundle $Hom\,(\xi, \eta)$ and by Lemma 12, its inverse is also continuous. □

The co-cycle theorem In the context of Definition 8, let (U, h_U), (V, h_V) be two local trivializations and suppose $U \cap V \neq \emptyset$. Then the map $g_{U,V} = h_V^{-1}\big|_{\pi^{-1}(U\cap V)} \circ h_U\big|_{(U\cap V)\times F}$ is an homeomorphism from $(U \cap V) \times F$ onto itself that respects the fibers (that is, for every $x \in U \cap V$ and $v \in F$, $(x, v) \in F$ is sent to $(x, \tilde{v})$ for some other $\tilde{v} \in F$) and it's a linear automorphism on each one (that is, the map that sends v to $\tilde{v}$ is automorphism of F). The functions $g_{U,V}$ are called transition functions. For instance, in Example 2, consider $x \in U_0 \cap U_1$. Then a pair $([1, a_1, \ldots, a_n], t)$ is sent by h_{U_0} to $([1, a_1, \ldots, a_n], (t, ta_1, \ldots, ta_n)) \in \mathbb{R}P^n$. Now $(t, ta_1, \ldots, ta_n) = ta_1\left(\frac{1}{a_1}, 1, \ldots, \frac{a_n}{a_1}\right)$, so $([1, a_1, \ldots, a_n], (t, ta_1, \ldots, ta_n))$ is sent by $h_{U_1}^{-1}$ to $([1, a_1, \ldots, a_n], ta_1)$. We conclude then that g_{U_0,U_1} consists of sending pairs $([1, a_1, \ldots, a_n], t)$ into pairs $([1, a_1, \ldots, a_n], ta_1)$.

The vector bundle provides then an open cover of the base space $\{U_\alpha\}_{\alpha\in\Lambda}$ and a collection of maps $g_{U_\alpha,U_\beta} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$. It can be easily verified that a co-cycle relation holds for this maps:

$$g_{U_\alpha,U_\gamma}\big|_{U_\alpha\cap U_\beta\cap U_\gamma} = g_{U_\beta,U_\gamma}\big|_{U_\alpha\cap U_\beta\cap U_\gamma} \circ g_{U_\alpha,U_\beta}\big|_{U_\alpha\cap U_\beta\cap U_\gamma}$$

A very important fact about vector bundles is that you can actually recover the entire bundle just from the open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and the collection of maps g_{U_α,U_β}. Indeed, let us define a total space $\hat{E}$ in the following three steps. First by forming the disjoint union of all local products

$$\tilde{E} = \bigsqcup_{\alpha\in\Lambda} U_\alpha \times F$$

Second, we define on $\tilde{E}$ the minimal equivalence relation such that if $x \in U_\alpha \cap U_\beta$, the pairs $(x, v) \in U_\alpha \times F$ and $(x, \tilde{v}) \in U_\beta \times F$ are related if $g_{U_\alpha, U_\beta}(x, v) = (x, \tilde{v})$. Finally, call $\hat{E}$ the quotient of $\tilde{E}$ by this relation. Since the points that are being identified share the same first coordinate, the projection on the first coordinate is still well defined. Moreover, if ι_α denotes the inclusion of $U_\alpha \times F$ in $\tilde{E}$ and π is the quotient map $\pi : \tilde{E} \to \hat{E}$, then $\pi \circ \iota_\alpha$ are the trivializations of $\hat{E}$. All conditions are satisfied then, so we have a vector bundle. It can be easily verified that this new bundle is isomorphic to the bundle we started with.

The preceding discussion showed that we can recover the entire bundle just from the open cover associated to it and the transition functions. Moreover, we can interpret this result as a way to construct new bundles. Indeed, given a base space X, all we have to do is define an open cover $\{U_\alpha\}_\alpha$ of X and transition functions $g_{U_\alpha, U_\beta} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ that are the identity on the first component and such that the co-cycle condition holds. The construction of $\tilde{E}$ and $\hat{E}$ can be carried out in exactly the same way as before, so as to get a vector bundle. This observation is called *The Co-cycle Theorem*.

Example 4. *Consider $X = S^1 \subset \mathbb{R}^2$, and the open cover $U_S = S^1 \setminus \{(0,1)\}$, $U_N = S^1 \setminus \{(0,-1)\}$. Define the transition functions as*

$$g_{U_S, U_N}((x,y), v) = \begin{cases} ((x,y), v) & x < 0 \\ ((x,y), -v) & x > 0 \end{cases}$$

and $g_{U_N, U_S} = (g_{U_S, U_N})^{-1}$. The co-cycle condition holds immediately, since there are only two sets in the open cover. The preceding discussion shows that this information defines a vector bundle, which is a very famous one: the Möbius band.

Extension of sections defined on a closed subset For the remainder of this section, we will add an additional hypothesis: we will consider all base spaces to be compact and Hausdorff.

Lemma 1. *Let $\xi = (E, X, F, \pi)$ be a vector bundle, Y a closed subset of X and $s : Y \to E$ a section defined only on Y. Then there exists an open set U containing Y and a section defined on U, $t : U \to E$ such that $t|_Y = s$.*

Proof. For each $x \in Y$, there exists a local trivialization (U_x, h_{U_x}) around x. There, $s \circ h_{U_x}$ is a F-valued function defined in the closed set $Y \cap U_x$. By Tietze's Extension Theorem, we can extend $s \circ h_{U_x}$ to a function $t_x : U_x \to F$. Since Y is also compact, we can choose a finite sub-collection $\{U_{x_i}\}_{i=1,\ldots,n}$ that covers Y and a subordinated partition of unity $\{p_i\}_{i=1,\ldots,n}$. Then $U = \bigcup_{i=1}^n U_i$ and $t = \sum_{i=1}^n p_i t_i$ are the required open set and section. □

Proposition 13. *Let* $\xi = (E, X, F, \pi)$ *and* $\eta = \left(\hat{E}, \hat{X}, \tilde{F}, \hat{\pi}\right)$ *be two vector bundles. Let* Y *be a closed subset of* X *and suppose we have an isomorphism of vector bundles over* Y*,* $s : \pi^{-1}(Y) \to \hat{\pi}^{-1}(Y)$*. Then* s *can be extended to an open set containing* Y*.*

Proof. We can think of s as a section of the bundle $Hom(\xi, \eta)$ defined over Y. By Lemma 1, it can be extended to an open set U containing Y. Since $\det s$ is a continuous function that doesn't vanish over Y, it must take nonzero values over some open subset of U that contains Y. □

Corollary 6. *Let* $\xi = (E, X, F, \pi)$ *be a vector bundle,* Y *a topological space and suppose we have two homotopic functions* $f_0, f_1 : Y \to X$*. Then,* $f_0^*(\xi) \cong f_1^*(\xi)$*.*

Proof. Let $F : Y \times I \to X$ be an homotopy between f_0 and f_1, and $f_t = F \circ i_t$ where $i_t : Y \to Y \times I$ is given by $i_t(y) = F(y, t)$. We will show that, as a function of $t \in I$, the isomorphism class of $f_t^*(\xi)$ is locally constant. Since I is connected, that would be enough to show the result.

Fix $t \in I$. Define $F_t : Y \times I \to X$ by the rule $F_t(y, t) = f_t(y)$. We have two vector bundles over $Y \times I$, namely $F^*(\xi)$ and $F_t^*(\xi)$. Now, these two bundles are isomorphic over $Y \times \{t\}$, which is a closed subset of $Y \times I$. By Proposition 13 they must be isomorphic over some open set containing $Y \times \{t\}$. By the Tube Lemma, such open set must contain a subset of the form $Y \times \delta$ for some sub-interval δ around t that has nonzero length. This means that $f_{t'}^*(\xi) \cong f_t^*(\xi)$ for all $t' \in \delta$, as we wanted to show. □

Corollary 7. *Every vector bundle over a contractible topological space is trivial.*

Proof. This follows from Corollary 6, since for a contractible set the identity is homotopic to a constant map and every vector bundle over a one-point set is trivial. □

Theorem 5. *Every vector bundle over a paracompact Hausdorff space admits a riemannian metric.*

Proof. Let X be a topological space with such properties and ξ a vector bundle over it. Then, there exists a locally finite open cover $\{U_\alpha\}_\alpha$, made of trivializing open sets of ξ. Over each one of those open sets we can define trivially a riemannian metric for ξ

g_α. The space also admits a partition of unity $\{\rho_\alpha\}_\alpha$ subordinated to the cover $\{U_\alpha\}_\alpha$. Then the metric $g = \sum_\alpha \rho_\alpha g_\alpha$ is a riemannian metric for ξ. □

Theorem 6. *Every vector bundle ξ over a compact space X has a direct complement.*

Proof. Let $\{U_i\}_{i=1,\ldots,r}$ be a finite open cover of X made of trivializing open sets of ξ and $\{\rho_i\}_{i=1,\ldots,n}$ a subordinated partition of unity. Let $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ be the trivializing maps associated to the cover, $proj_2 : U_i \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection onto the second factor and $f_i = proj_2 \circ h_i$ the composition of these last two maps. Finally, consider $S : \xi \to B \times \mathbb{R}^{rn}$ given by

$$S(v) = (\pi(v), \rho_1(\pi(v)) f_1(v), \ldots, \rho_r(\pi(v)) f_r(v)).$$

Then S is a fiberwise injection of ξ into the trivial bundle of dimension n over X, ϵ^{nr}. Now, we can endow ϵ^{nr} with the usual metric, and at each point $x \in X$ we can take the orthogonal complement of the image of ξ under S, to form a vector bundle. Clearly, it would be a direct complement of ξ. □

Chapter 3

Topological K-Theory

This is the main chapter of this document. The main idea here is to associate to a topological space a group built from the collection of vector bundles over that space. We will study how to do this precisely and some properties of this construction. Then main result here is the calculation of the K-theory of a point thanks to Atiyah-Bott-Shapiro's construction and a generalization of that calculation for the Thom complex of real, even dimensional, complex vector spaces over compact Hausdorff spaces.

3.1 Basic Notions

3.1.1 Definitions

Let X be a compact topological space and let $Vect_{\mathbb{R}}(X)$ and $Vect_{\mathbb{C}}(X)$ be the set of isomorphism classes of real and complex vector bundles over X respectively. There is a natural operation on both sets given by direct sum, which is an associative, commutative operation with a zero element: the zero vector bundle. For a general space, the cancellation property doesn't necessarily hold for this operation. Indeed, consider $X = S^2$, τ its tangent bundle, N its normal bundle and ϵ^3 its trivial bundle of range three. As can be easily checked, $N \cong \epsilon^1$, the trivial line bundle. Also, $\tau \oplus N \cong \epsilon^3$, so $\tau \oplus \epsilon^1 \cong \epsilon^3$. Naturally, $\epsilon^2 \oplus \epsilon^1 \cong \epsilon^3$. Yet, τ is not isomorphic to ϵ^2 as it is well known. $Vect_{\mathbb{F}}(X)$ is what we call an abelian semigroup: a set with an abelian, associative operation with a neutral element. As the next proposition shows, this algebraic structure can always be adapted to create a group.

Proposition 14. *Let A be an abelian semigroup with sum $+$ and neutral element 0. Then there exists an abelian group $K(A)$ and a map of semigroups $i : A \to K(A)$ such that if G is an abelian group and f is a map of semigroups $f : A \to G$, there exists a unique map of groups $\hat{f} : K(A) \to G$ such that $f = \hat{f} \circ i$. Moreover, the pair $(K(A), i)$ is the only one with this property.*

Proof. We will construct $K(A)$ and i directly. On $A \times A$ define an equivalence relation according to

$$(a,b) \sim (c,d) \iff \exists e \in A : \; a+d+e = c+b+e.$$

Naturally, $A \times A$ has an operation defined by coordinate-wise sum, which descends to the set of equivalence classes $K(A) \equiv A \times A / \sim$. But on $K(A)$, the operation has inverses, besides the other properties it inherits. Indeed, for any $a, b \in A$,

$$[(a,b)] + [(b,a)] = [(a+b, a+b)] = [(0,0)].$$

So $K(A)$ is actually a group. Let $i : A \to K(A)$ be the map defined by $i(a) = [(a,0)]$. If G is an abelian group and $f : A \to G$ is a map of semigroups, then f extends to a map $\tilde{f} : A \times A \to G$ by sending (a,b) into $f(a) - f(b)$. Clearly if $(a,b) \sim (c,d)$, $\tilde{f}((a,b)) = \tilde{f}((c,d))$ so $\tilde{f}$ descends to a map $\hat{f} : K(A) \to G$. It's also clear that $f = \hat{f} \circ i$ and since the images of the pairs $[(a,0)]$ determine $\tilde{f}$ and hence $\hat{f}$ completely, the extension is unique. The fact that the pair $(K(A), i)$ is unique follows from the traditional arguments. □

We will be working mostly with complex vector bundles. We define $K(X)$ to be $K(Vect_{\mathbb{C}}(X))$ and if ξ is a vector bundle, we denote $i(\xi)$ by $[\xi]$. The construction in the proof of Proposition 14 shows very clearly that if ζ and η are two vector bundles, then $[\xi] = [\eta]$ if and only if there exists a third vector bundle ζ such that $\xi \oplus \zeta \cong \eta \oplus \zeta$. In that case, we say ξ and η are stably equivalent Also, it was clear in the proof that $[(a,0)] = -[(0,a)]$, the elements of $K(A)$ can be written as $i(a) - i(b)$. So, in our case, generic elements of $K(X)$ are of the form $[\xi] - [\eta]$ for some vector bundles ξ and η.

Now, according to Theorem 6, vector bundles over X admit direct complements. So in the previous paragraph, if $\xi \oplus \zeta \cong \eta \oplus \zeta$ we can add on both sides a vector bundle ρ such that $\zeta \oplus \rho \cong \epsilon^n$ for some n. Then $\xi \oplus \epsilon^n \cong \eta \oplus \epsilon^n$. In other words, if two vector bundles are stably equivalent, we may assume that we can add the same trivial bundle to both of them and get isomorphic vector bundles. Similarly, generic elements of $K(X)$ can be assumed to be of the form $[\xi] - \epsilon^n$ for some n.

If X happens to be a pointed space with distinguished point x_0, the pullback of the inclusion $i : \{x_0\} \hookrightarrow X$ gives us a map from $K(X)$ to $K(x_0)$. We call the kernel of that map $\tilde{K}(X)$. Now, the collapsing map that sends all of X to x_0 makes the sequence

$$0 \to \tilde{K}(X) \hookrightarrow K(X) \xrightarrow{i^*} K(x_0) \to 0$$

split, so $K(X) \cong \tilde{K}(X) \oplus K(x_0)$. It's easy to see that the elements of $\tilde{K}(X)$ are pairs $[\xi] - [\epsilon^n]$ in $K(X)$ where n is exactly the range of ξ. Finally, for any space X (but mostly for not pointed spaces) we call X^+ the space made of adding an isolated point to X, $X^+ = X \sqcup \{pt\}$. It is a pointed space with distinguished point pt and $\tilde{K}(X^+) \cong$

$K(X)$. Finally, for a compact pair (X,Y), we define $K(X,Y)$ to be $\tilde{K}(X/Y)$, where the distinguished point of X/Y is Y/Y. If $Y=\emptyset$, $K(X,Y)$ is just $K(X)$.

In order to define K-theory groups of higher indexes, we need to talk about a previous concept first. Let X and Y be two pointed spaces with distinguished points x_0 and y_0 respectively. Their smash product, $X \wedge Y$, is the space

$$X \wedge Y = X \times Y/\left(X \times \{y_0\} \cup \{x_0\} \times Y\right).$$

It's not hard to see that for any non-negative integers, $S^n \wedge S^m \cong S^{n+m}$. For a pointed space X, the reduced n-th suspension of X is the space $S^n X \equiv S^n \wedge X$.

Definition 11. *Let X be a pointed space with distinguished point x_0, Y a subspace of X that contains x_0 and Z some other space, not necessarily pointed. Then:*

- $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$
- $K^{-n}(X,Y) = \tilde{K}^{-n}(X/Y) = \tilde{K}^{-n}(S^n(X/Y))$
- $K^{-n}(Z) = \tilde{K}^{-n}(Z^+)$.

3.1.2 Important facts

Lemma 2. *Let X be a compact space and let Y be a closed contractible subset of X. If π denotes the canonical map from X to X/Y, then the map $\pi^* : K(X/Y) \to K(X)$ is an isomorphism.*

Proof. We will prove this Lemma by constructing a map $\theta : Vect_n(X) \to Vect_n(X/Y)$ that is a two sided inverse for $\pi^* : Vect_n(X/Y) \to Vect_n(X)$.

Let ξ be a vector bundle over X with total space E and projection map π_ξ. Since Y is contractible, Corollary 7 tells us that $\xi|_Y$ is trivial. That means that there is an isomorphism of vector bundles, α, from $\xi|_Y$ to the trivial bundle $Y \times \mathbb{F}^n$, where n is the range of ξ. Then $\tilde{\alpha} = proj_2 \circ \alpha$ gives us a map from $\xi|_Y$ to $\mathbb{F}^n$. Let us define now an equivalence relation on E, the total space pf ξ, as follows: for any two elements v_1 and v_2 on E, $v_1 \sim v_2$ if $v_1 = v_2$, or if both $\pi_\xi(v_1)$ and $\pi_\xi(v_2)$ belong to Y and $\tilde{\alpha}(v_1) = \tilde{\alpha}(v_2)$. In other words, $\sim$ identifies the horizontal stripes of $\xi|_Y$ according to α, so to speak. Let $E_\alpha = E/\sim$.

Now, let us show that we can build a vector bundle over X/Y using E_α. Since $\sim$ identifies points over Y, π_ξ descends to a map $\pi_\alpha : E_\alpha \to X/Y$. For any $x \in X \backslash Y$, $\pi_\alpha^{-1}(x)$ is just $\pi_\xi^{-1}(x)$, so it has a vector space structure. The same local trivialization for E around x works as a local trivialization of E_α, although we may have to restrict the domain so as to exclude Y, which is not a problem, since Y is closed. So the only thing that is yet to be verified is that we have a local trivialization of E_α around the point Y in X/Y. But, by Proposition 13, we can extend the isomorphism α to an isomorphism

β from $\xi|_U$ to $U \times \mathbb{F}^n$, where U is some open set that contains Y. Taking the quotient by $\sim$ amounts to turning β into an isomorphism $\beta_\sim$ from $\pi_\alpha^{-1}(U/Y)$ to $(U/Y) \times \mathbb{F}^n$. Since U/Y is open in X/Y, $\beta_\sim$is the local trivialization we were looking for. So $\xi_\alpha = (E_\alpha, X, \mathbb{F}^n, \pi_\alpha)$ is a vector bundle over X/Y and it's not hard to check that $\pi^*(\xi_\alpha) = \xi$.

Finally, we claim that the structure of ξ_α is actually independent of the particular trivialization chosen. In other words, that all the ξ_α's are isomorphic to one another. This follows from Proposition the fact that all possible trivializations α are homotopic to each other. □

Corollary 8. *Let X be a pointed space. Then $K(SX) \cong K(CX/X)$.*

Proof. If x_0 is the distinguished point of X, SX is obtained from CX/X by collapsing $I \times x_0$, which is a closed contractile subset of CX/X. Hence, by Lemma 2, the projection $\pi : CX/X \to SX$ induces an isomorphism $\pi^* : K(SX) \to K(CX/X)$. □

Proposition 15. *Let (X, Y) be a pair of pointed compact spaces. If $i : Y \to X$ denotes the inclusion and $\pi : X \to X/Y$ denotes the projection, then the sequence*

$$\tilde{K}(X/Y) \xrightarrow{\pi^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(Y) \tag{3.1}$$

is exact.

Proof. The map $\pi \circ i$ is constant, so $i^* \circ \pi^*(\eta) = \epsilon_Y^n$ for any η in $Vect_n(X/Y)$. Then $i^* \circ \pi^* \left([\eta] - \left[\epsilon_{X/Y}^n\right]\right) = [\epsilon_Y^n] - [\epsilon_Y^n] = 0$. In other words, $i^* \circ \pi^* = 0$, or $Im(\pi^*) \subset Ker(i^*)$. On the other hand, suppose $[\xi] - [\epsilon_X^n] \in Ker(i^*)$. That means

$$i^*(\xi) \oplus \epsilon_Y^k = i^*\left(\xi \oplus \epsilon_X^k\right) \cong \epsilon_Y^{n+k} \tag{3.2}$$

Doing the same thing as in Lemma 2, Equation 3.2 tells us that we can build a vector bundle η over X/Y such that $\pi^*(\eta) \cong \xi \oplus \epsilon_X^k$, which means $\pi^*\left([\eta] - \left[\epsilon_X^{n+k}\right]\right) = [\xi] - [\epsilon_X^n]$. This shows that $Ker(i^*) \subset Im(\pi^*)$. □

Let X be a compact space and Y some closed subspace of X. Then, by taking $X = X^+$ and $Y = Y^+$, we get an exact sequence

$$K(X, Y) \xrightarrow{\pi^*} K(X) \xrightarrow{i^*} K(Y) \tag{3.3}$$

On the other hand, we can replace the first term on the left of Equation 3.1 by $\tilde{K}(X \cup CY)$. Indeed, we can obtain X/Y from $X \cup CY$ by collapsing CY, which is a closed contractible, so by Lemma 2, the projection $\pi_C : X \cup CY \to X/Y$ induces an isomorphism in $\tilde{K}$. Moreover, if $i_C : X \to X \cup CY$ denotes the inclusion, the diagram

$$\begin{array}{ccc} X & \overset{i_C}{\to} & X \cup CY \\ & {\scriptstyle \pi} \searrow & \downarrow \pi_C \\ & & X/Y \end{array}$$

commutes, so the associated diagram

$$\begin{array}{ccc} \tilde{K}\left(X\right) & \overset{i_C^*}{\leftarrow} & \tilde{K}\left(X \cup CY\right) \\ & {\scriptstyle \pi^*} \nwarrow & \uparrow \pi_C^* \\ & & \tilde{K}\left(X/Y\right) \end{array}$$

also commutes. Hence, the sequence

$$\tilde{K}\left(X \cup CY\right) \overset{i_C^*}{\to} \tilde{K}\left(X\right) \overset{i^*}{\to} \tilde{K}\left(Y\right) \tag{3.4}$$

is also exact. Now, we could iterate this process to obtain

$$\tilde{K}\left(X \cup C_3\left(X \cup C_1 Y\right) \cup C_2 X \cup C_1 Y\right) \to \tilde{K}\left(X \cup C_2 X \cup C_1 Y\right) \to \tilde{K}\left(X \cup CY\right) \to \tilde{K}\left(X\right) \tag{3.5}$$

This ideas are useful in establishing the following result:

Theorem 7. *Let (X, Y) be a pair of compact pointed spaces. Then there is an exact sequence*

$$\cdots \to \tilde{K}^{-n}\left(X/Y\right) \overset{\pi^*}{\to} \tilde{K}^{-n}\left(X\right) \overset{i^*}{\to} \tilde{K}^{-n}\left(Y\right) \to \dots$$
$$\cdots \to \tilde{K}^{-1}\left(X\right) \overset{i^*}{\to} \tilde{K}^{-1}\left(Y\right) \overset{\theta}{\to} \tilde{K}\left(X/Y\right) \overset{\pi^*}{\to} \tilde{K}\left(X\right) \overset{i^*}{\to} \tilde{K}\left(Y\right)$$

Proof. First of all, it suffices to show the exactness of

$$\tilde{K}^{-1}\left(X/Y\right) \overset{\pi^*}{\to} \tilde{K}^{-1}\left(X\right) \overset{i^*}{\to} \tilde{K}^{-1}\left(Y\right) \overset{\theta}{\to} \tilde{K}\left(X/Y\right) \overset{\pi^*}{\to} \tilde{K}\left(X\right) \overset{i^*}{\to} \tilde{K}\left(Y\right)$$

since the rest of the sequence can be obtained by replacing X by $S^n X$ and Y by $S^n Y$. Furthermore, the sub-sequence

$$\tilde{K}^{-1}\left(X/Y\right) \overset{\pi^*}{\to} \tilde{K}^{-1}\left(X\right) \overset{i^*}{\to} \tilde{K}^{-1}\left(Y\right)$$

is just 3.1 with $X = SX$ and $Y = SY$, since $S\left(X/Y\right) \cong SX/SY$. So, the proof reduces to show that the sequence

$$\tilde{K}^{-1}\left(X\right) \overset{i^*}{\to} \tilde{K}^{-1}\left(Y\right) \overset{\theta}{\to} \tilde{K}\left(X/Y\right) \overset{\pi^*}{\to} \tilde{K}\left(X\right)$$

is exact (of course, in the process we have to define θ).

Actually, what we intent to show is that this sequence is equivalent to 3.5. The details are in [1]. □

Corollary 9. *If Y is a retract of X, then $K^{-n}(X) \cong K^{-n}(X,Y) \oplus K^{-n}(Y)$.*

Proof. Let $f : X \to A$ be the retraction. Then the fact that $i^* \circ f^* = Id_{K(Y)}$ implies the map $K(X) \xrightarrow{i^*} K(Y)$ is surjective. Similarly, if $f_n : S^n X^+ \to S^n Y^+$ are the maps induced by the retraction, the fact that $i^* \circ f_n^* = Id_{K^n(Y)}$ implies that $K^n(X) \xrightarrow{i^*} K^n(Y)$ is also surjective for all n. Then, by the exactness in Theorem 7, all the maps $\tilde{K}^{-n-1}(Y) \xrightarrow{\theta_n} \tilde{K}^{-n}(X/Y)$ must be trivial. This makes every short sequence $K^n(X,Y) \xrightarrow{\pi^*} K^n(X) \xrightarrow{i^*} K^n(Y)$ a short exact split sequence

$$0 \to K^n(X,Y) \xrightarrow{\pi^*} K^n(X) \xrightarrow{i^*} K^n(Y) \to 0.$$

The result follows immediately. □

Corollary 10. *If X and Y are pointed spaces,*

$$\tilde{K}^{-n}(X \times Y) \cong \tilde{K}^{-n}(X \wedge Y) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y).$$

Proof. This Corollary follows from a double application of Corollary 9. First, X is a retract of $X \times Y$, and the quotient is $X \times Y / X \times \{y_0\}$, so

$$\tilde{K}^{-n}(X \times Y) \cong \tilde{K}^{-n}(X \times Y / X \times \{y_0\}) \oplus \tilde{K}^{-n}(X).$$

Second, Y is a retract of $X \times Y / X \times \{y_0\}$, with quotient $X \wedge Y$, so

$$\tilde{K}^{-n}(X \times Y / X \times \{y_0\}) \cong \tilde{K}^{-n}(X \wedge Y) \oplus \tilde{K}^{-n}(Y).$$

The result follows immediately. □

The importance of Corollary 10 lies in the fact that it allows us to define a pairing that turns $K^*(X) = \sum_n K^{-n}(X)$ into a graded ring. Let X and Y be two pointed spaces and let ξ and η be vector bundles over X and Y respectively. Then $\pi_X^*(\xi) \otimes \pi_Y^*(\eta)$ is a vector bundle over $X \times Y$. Moreover, the assignment $(\xi,\eta) \mapsto \pi_X^*(\xi) \otimes \pi_Y^*(\eta)$ is bi-linear, so it induces a map $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \times Y)$. Actually, the image of this map is in the kernel of both $i_X : \tilde{K}(X \times Y) \to \tilde{K}(X \times \{y_o\})$ and $i_X : \tilde{K}(X \times Y) \to \tilde{K}(Y \times \{x_o\})$, so by Corollary 10 it must be $\tilde{K}(X \wedge Y)$. Replacing X by $S^n X$ and Y by $S^m Y$, we get a pairing

$$\tilde{K}^{-n}(X) \otimes \tilde{K}^{-m}(Y) \to \tilde{K}^{-m-n}(X \wedge Y). \tag{3.6}$$

Furthermore, by taking $X = X^+$ and $Y = Y^+$, we get

$$K^{-n}(X) \otimes K^{-m}(Y) \to K^{-m-n}(X \times Y). \tag{3.7}$$

The following Theorem is one of the most important mathematical results of the twentieth century and it will be key in developing our understanding of orientations in K-theory. It is also fundamental to see K-theory as a cohomology theory. Sadly, its proof is outside the scope of this document. We limit ourselves to state it in the way we are going to use it further along.

Theorem 8. *Let X be a compact space and consider the pairing $\tilde{K}(S^2) \otimes K^{-n}(Y) \to K^{-n-2}(Y)$ obtained by setting $Y = Y^+$ on 3.6. Let H be the Hopf bundle on $S^2 = \mathbb{C}P^1$ and $b = [H] - \epsilon^1$ the associated class in $\tilde{K}(S^2)$. Then, the map*

$$\begin{aligned} \mu_b : K^{-n}(Y) &\to K^{-n-2}(Y) \\ a &\mapsto b \cdot a \end{aligned}$$

is an isomorphism.

Proof. See [1]. □

As a conclusion of this section, let us now present a couple of technical results that will be used in the future.

Lemma 3. *Suppose we have the following diagram:*

$$\begin{array}{ccccccccccccc} \cdots \to & C''_{n-1} & \overset{\delta_{n-1}}{\to} & C'_n & \overset{p_n}{\to} & C_n & \overset{i_n}{\to} & C''_n & \overset{\delta_n}{\to} & C'_{n+1} & \to \cdots \\ & f''_{n-1}\downarrow & & f'_n\downarrow & & f_n\downarrow & & f''_n\downarrow & & f'_{n+1}\downarrow & \\ \cdots \to & D''_{n-1} & \underset{\delta'_{n-1}}{\to} & D'_n & \underset{q_n}{\to} & D_n & \underset{j_n}{\to} & D''_n & \underset{\delta'_n}{\to} & D'_{n+1} & \to \cdots \end{array}$$

where the C_n's and D_n's are abelian groups , the horizontal sequences are exact and the maps f'_n are isomorphisms. Then the sequence

$$\cdots \to C_n \overset{(i_n, f_n)}{\longrightarrow} C''_n \oplus D_n \overset{f''_n - j_n}{\longrightarrow} D''_n \overset{\Delta_n}{\longrightarrow} C_{n-1} \to \cdots$$

$\left(\textit{where } \Delta_n = p_{n+1} \circ \left(f'_{n+1}\right)^{-1} \circ \delta'_n\right)$ *is exact.*

Proof. This is a long but simple exercise in diagram chasing. □

Theorem 9. *Let X be a compact space and let A and B be two closed subspaces of X such that $X = A \cup B$. Then there is a long exact sequence of the form*

$$\dots K^{n-1}(A \cap B) \xrightarrow{\delta} K^{-n}(X) \xrightarrow{(i_A^*, i_B^*)} K^n(A) \oplus K^n(B) \to K^n(A \cap B) \to K^{n+1}(X) \dots$$

where the map from $K^n(A) \oplus K^n(B)$ to $K^n(A \cap B)$ sends (α, β) to $j_A^*(\alpha) - j_B^*(\beta)$, j_A and j_B being the inclusions of $A \cap B$ in A and B respectively.

Proof. In Lemma 3, take $C_n'' = K^{-n}(A)$, $C_n = K^{-n}(X)$, $C_n' = K^{-n}(X, A)$, $D_n'' = K^{-n}(A \cap B)$, $D_n = K^{-n}(B)$ and $D_n' = K^{-n}(B, A \cap B)$. Notice that $B/A \cap B$ and X/A are homeomorphic, so the maps f_n' are indeed isomorphisms. □

3.2 K-Orientations

Definition 12. *Let X be a compact space and $\pi : V \to X$ a real vector bundle over X, with dimension n and endowed with some riemannian metric. Let $B(V)$ and $S(V)$ be the associated disc and sphere bundles. The space $X^V \equiv B(V)/S(V)$ is called the Thom complex of V. Moreover, the bundle V is called K-orientable if there exists a class $\mu_V \in \tilde{K}(X^V)$ such that $\tilde{K}^*(X^V)$ is a free $K^*(X)$-module with generator μ_V. The class μ_V is called a Thom class.*

The goal of this section is two show that real vector bundles of even dimension are K-orientable. We will do so by constructing explicitly the class μ_V mentioned in Definition 12. But in order to get there, we have to go through a series of steps first.

Step 1: Constructing classes in $\tilde{K}^*(X^V)$ from exact sequences of vector bundles In the setting of Definition 12, let E_1 and E_0 be two vector bundles over $B(V)$, and suppose there exists an isomorphism $\sigma : E_1|_{S(V)} \to E_0|_{S(V)}$. We want to show that from this information, we can define a class $\mu \in \tilde{K}^*(X^V)$.

Let A be the topological space made from two disjoint copies $B(V)_1$ and $B(V)_2$ of the space $B(V)$, but identifying the subspaces $S(V)_1$ and $S(V)_2$ into a common subspace $S(V)$. One one hand, the inclusion $\phi : (B(V)_1, S(V)) \hookrightarrow (A, B(V)_2)$ gives an isomorphism ϕ^* between $K(A, B(V)_2)$ and $K(B(V)_1, S(V))$. So, it suffices to construct a class in this last group. On the other hand, Proposition 15 gives us the exact sequence

$$K(A, B(V)_2) \to K(A) \to K(B(V)_2).$$

Now consider the following two vector bundles on A. First, the bundle F obtained by setting E_1 over $B(V)_1$, E_2 over $B(V)_2$ and using σ as a transition function on $S(V)$. Second, the vector bundle $F_2 = \pi^*(E_2)$. Clearly, the two bundles coincide over $B(V)_2$, so the class $F - F_2$ goes to zero on $K(B(V)_2)$. Then, by exactness, it comes from from a class G on $K(A, B(V)_2)$. But then $\phi^*(G)$ is a class on $K(B(V)_1, S(V))$.

Step 2: Building an exact sequence like the one on Step 1. Let P be a principal $Spin^c_{2n}$-bundle over X . From Proposition 11 we know A^c_{2k} has a unique generator, F, which is a irreducible graded module on $\mathbb{C}l_{2n}$.

In particular, F is a $Spin^c_{2n}$-module so let $E = E_0 \oplus E_1 = P \times_{Spin^c_{2n}} F$ be the associated bundle over X. Now consider the two bundles $\pi^*(E_0)$ and $\pi^*(E_1)$ over $B(V)$. Since the elements on $B(V)$ can be seen as vectors in $\mathbb{R}^{2n}$, which is a subspace of $\mathbb{C}l_{2n}$ it makes sense to define over each $v \in S(V)$ the map $\sigma : \pi^*(E_0)|_v \to \pi^*(E_1)|_v$ that sends w to $v \cdot w$ for every $w \in \pi^*(E_0)|_v$. This establishes an isomorphism of bundles $\pi^*(E_0)$ and $\pi^*(E_1)$ defined on $S(V)$.

So far, Steps 1 and 2 have allowed us to define a class μ in $K(X^V)$. We need to show now that indeed $\tilde{K}(X^V)$ is a free module over $K(X)$. This is what the following three steps are for.

Step 3: X^V is locally a suspension of X Let C be a closed subset of X such that V is trivial over C. Then $\tilde{K}(B(V|_C)/S(V|_C)) \cong K^{-2n}(C)$. Indeed, it's easy to see that

$$B(V|_C)/S(V|_C) \cong C^+ \times S^{2n} / \left(C^+ \times \{pt\} \cup S^{2n} \times \{+\}\right) = S^{2n} \wedge U^+$$

Hence, taking $\tilde{K}$ on both sides, we obtain the result.

Notice that we could apply Step 1, 2 and 3 to $X = pt$. For each $n \in \mathbb{N}$, we would get a class μ_n in $\tilde{K}(S^{2n})$.

Theorem 10. *The classes $\mu_n \in \tilde{K}(S^{2n})$ are generators of $\tilde{K}(S^{2n})$.*

Proof. See [2]. □

Step 4: μ is locally Thom. Notice that over each point, the construction in the past three steps restricts exactly to the construction in Theorem 10. So multiplication by μ is equivalent to multiplication by the generator of $\tilde{K}(S^{2n})$ in the setting

$$\tilde{K}(S^{2n}) \otimes K^{-m}(C) \to K^{-m-2n}(C)$$

But this is an isomorphism for all n, by Theorem 8. Hence μ is locally a Thom class.

Step 5: Constructing a global isomorphism. Let X be covered by a finite number of closed sets $C_1, C_2, \ldots, C_N$ that trivialize V. We want to show that multiplication by μ gives us an isomorphism on $C_1 \cup C_2 \cup \cdots \cup C_N$. We show this by induction. For

$n = 1$, this is obvious. Suppose we already know it for $C_1 \cup C_2 \cup \cdots \cup C_{N-1} = D_{N-1}$. Then multiplication by an appropriate restriction of μ gives us a morphism of exact sequences

$$\begin{array}{ccccccc} \cdots & K^{-i}(D_{N-1} \cup C_N) & \to & K^{-i}(D_{N-1}) \oplus K^{-i}(C_N) & \to & K^{-i}(D_{N-1} \cap C_N) & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots & K^{-i}\left(E|_{D_{N-1} \cup C_N}\right) & \to & K^{-i}\left(E|_{D_{N-1}}\right) \oplus K^{-i}(E|_{C_N}) & \to & K^{-i}\left(E|_{D_{N-1} \cap C_N}\right) & \cdots \end{array}$$

The result follows now from the induction hypothesis and the 5-lemma.

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