

UNIVERSIDAD DE LOS ANDES

BACHELOR THESIS

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# Integer Quantum Hall Effect and Noncommutative Geometry

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*“Elegance should be left to tailors and shoemakers.”*

Ludwig Boltzmann

UNIVERSIDAD DE LOS ANDES

## *Abstract*

Science Faculty

Physics Department

Bachelor

### **Integer Quantum Hall Effect and Noncommutative Geometry**

by Nicolás ESCOBAR VELÁSQUEZ

The most accepted theoretical model that explains the Integer Quantum Hall Effect (IQHE) was proposed by Thouless et al. in the eighties and is usually referred to as the TKNN model. However, as we shall discuss, such an explanation has a number of conceptual difficulties. This document explores an alternative formalism developed by Bellissard et al. to describe the same effect, that builds on and generalizes the TKNN model. We review the basic facts of Quantum Hall Effect as well as the main theoretical tools to describe it. Next, we give a brief overview of the mathematical concepts associated to Noncommutative Geometry, the framework for Bellissard's model. Finally, we discuss how to apply these tools to study the IQHE. We will see that this alternative formalism provides an explicit link between the existence of localization phenomena and the quantization of the Hall conductance.



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*Para ma y pa. Los amo con todo mi corazón. No conozco nadie que se haya dedicado con tanta entrega y tanto cariño a sus hijos. Por todo el amor que me han dado, Dios los bendiga siempre. Y para Camilo, que ya es todo un hombre y que nos hace sentir orgullosos todos los días.*



# Introduction

The goal of this document is to study a model for the Integer Quantum Hall Effect (IQHE) based on the tools of noncommutative geometry. Now, that is an odd goal to pursue. The first solid explanation of the IQHE as the consequence of geometrical considerations was achieved more than fifty years ago. Even the noncommutative description of it was developed in the eighties and nineties and is pretty much complete. So, why should we study a subject like this?

The reason is simple: This study serves as an introduction both to topological phenomena in solid state physics and to Noncommutative Geometry. There is active research in both of those subjects and they have generated impressive results in the last few years. Let us now discuss some of these results.

It turns out that topology plays a major role in solid state physics. On one hand, the appearance of Topological Insulators in the past decade has generated a lot of interest. Roughly speaking, these are materials that behave like insulators in the bulk but have some edge states whose energies close the gap between the valence and electron bands, so they behave like conductors in the surface. These edge states are very robust because their existence is protected by topological mechanisms [11]. The use of topological insulators might lead to a new revolution in the design of memory devices [18] and they might be the key to the physical implementation of Quantum Computation [8], [14].

Topology is also behind the existence and stability of skyrmions [19]. These are spin textures with a nontrivial topological structure that appear in certain materials. The configurations are characterized by a nonzero "skyrmion number", which is a topological invariant analogue to the Chern number. This makes skyrmions stable, since topological invariants only admit discrete values and hence a smooth deformation of the Hamiltonian can't make them disappear. Skyrmions create an emergent electromagnetic field that can couple to conducting electrons, generating several phenomena including the Anomalous Integer Quantum Hall Effect.

All of these applications of topology to solid state physics are based on the same principle as the one it has for IQHE. In all of these cases a gauge field is present and the integral of its curvature is a topological invariant that makes the configuration robust.

Hence, the study of IQHE is a very good introduction to the study of all these phenomena. It also means that any development around the IQHE will have implications in the study of all of these other topological effects.

That is precisely the case for noncommutative geometry. The main topic of this document is Bellissard's treatment of the IQHE using tools from noncommutative geometry [3]. But in recent years, Prodan [21] has shown that those ideas can be used in a much broader context, specifically for the study of topological insulators.

To summarize, far from being a purely academic exercise, the study of IQHE under the light of noncommutative geometry is the first step into the relevant, flourishing field that is the consideration of topological effects in solid state physics.

# Chapter 1

## Electronic Properties of Two Dimensional Systems

As we mentioned in the Introduction, the main topic of this document is the study of a system composed of free, non-interacting electrons constrained to move in a plane. In this chapter, we study some properties and models of this systems. Our main focus is to calculate the electronic properties of this system. Specifically, we want to predict the form of the conductivity and resistivity tensors. This chapter is based on the general references [10], [9] and [16]. For the study of Kubo's Formula we refer the reader to references [20], [7] and [23], where there is a much more detailed description of the subject.

### 1.1 Drude's Model

A simple yet accurate calculation of the resistivity and conductivity tensors can be made in the context of the so-called Drude's Model. This model is classical, but probabilistic. It assumes that the system can be thought of as an electron gas moving across an array of impurities, subject to an external force  $F$ . Each electron collides with an impurity every  $\tau$  units of time, on average. After each collision, the final momentum of the particle is zero, again on average. Over an interval of time  $dt$ , the probability of a collision is  $\frac{dt}{\tau}$ , so the opposite probability is  $(1 - \frac{dt}{\tau})$ . If in the interval  $dt$  there is a collision, the momentum after the interval,  $\vec{p}(t + dt)$  will be zero. If there is no collision,  $\vec{p}(t + dt) = \vec{p}(t) + F dt$ , according to Newton's second law. Hence, the expected value of  $\vec{p}(t + dt)$  is

$$\vec{p}(t + dt) = 0 \cdot \frac{dt}{\tau} + (\vec{p}(t) + \vec{F} dt) \cdot \left(1 - \frac{dt}{\tau}\right). \quad (1.1)$$

Hence, the change of momentum over the interval  $dt$  is

$$d\vec{p} = \vec{p}(t + dt) - \vec{p}(t) = \left( \frac{\vec{p}(t)}{\tau} + F \right) dt. \quad (1.2)$$

We write the current density as  $\vec{j} = ne\vec{v} = \frac{ne\vec{p}}{m}$  and take  $F$  to be the Lorentz force. Furthermore, we take the system to be in equilibrium, so  $\frac{d\vec{p}}{dt}$  must be zero. After some algebra, we can rewrite the equilibrium condition as  $\vec{E} = \rho\vec{j}$ , where

$$\rho = \frac{m}{ne^2\tau} \begin{pmatrix} 1 & \frac{e\tau}{m}B \\ -\frac{e\tau}{m}B & 1 \end{pmatrix} \quad (1.3)$$

The diagonal elements of the tensor correspond to the resistivity of the system along the direction parallel to the electric field, so they are called longitudinal resistivity  $\rho_L$ . Similarly, the off diagonal ones are called transverse, or Hall conductivity,  $\rho_H$ . So, Drude's model predicts that  $\rho_L$  is independent of  $B$ , while  $\rho_H$  is proportional to the magnetic field. Such predictions are correct for room temperatures and weak fields, and the second one was experimentally observed for the first time by Edwin Hall in 1879. Hence the name "classical Hall Effect".

**Remark 1.** Suppose the system has very few impurities, or none at all. Then the scattering time would go to infinity and we would get  $\rho_L = 0$  and  $\rho_H = \frac{B}{ne}$ .

## 1.2 Landau Quantization

Experience tells us that the kind of assumptions and approximations made in Drude's model are not expected to hold for large fields and low temperatures, where quantum effects become important. Hence, we need to develop a quantum mechanical treatment of the system if we want to understand what happens at low temperatures and strong fields. Now, we are going to consider not so strong electric fields, so our approach will be to consider first a system subject only to a strong magnetic field and later on we will treat the electric field as a perturbation.

We start by considering the system to be made of a single electron constrained to an infinite plate located in the  $x - y$  plane, subject only to a magnetic field of magnitude  $B$  oriented in the positive  $z$  direction. We know that in order to incorporate the effect of a magnetic field  $\vec{B} = \vec{\nabla} \times \vec{A}$  in a quantum system, we should replace the usual momentum operator  $\vec{P}$  by

$$\vec{\Pi} = \vec{P} + e\vec{A}. \quad (1.4)$$

In other words, we should use a minimal coupling. However, there is a problem. The vector potential is not uniquely determined, we have gauge freedom. In our case, there are two popular choices: the symmetric gauge

$$\vec{A} = \frac{B}{2}(-y, x, 0) \quad (1.5)$$

and the Landau gauge

$$\vec{A} = B(-y, 0, 0). \quad (1.6)$$

We will see that some elements of our analysis are gauge invariant, but not all of them.

As expected, the spectrum of the Hamiltonian is gauge invariant. Indeed, the Hamiltonian,  $\mathbf{H}$ , can be written as

$$\mathbf{H} = \frac{\mathbf{\Pi}_x^2 + \mathbf{\Pi}_y^2}{2m} \quad (1.7)$$

It can be shown that, regardless of the gauge, we have the commutation relation

$$[\mathbf{\Pi}_x, \mathbf{\Pi}_y] = i \left( -\frac{\hbar^2}{l_B^2} \right) \quad (1.8)$$

where  $l_B = \sqrt{\frac{\hbar}{eB}}$ . Hence,  $\mathbf{\Pi}_x$  and  $\mathbf{\Pi}_y$  are completely analogous to the operators  $P$  and  $X$  of a one dimensional harmonic oscillator. Accordingly, we can define ladder operators

$$\mathbf{a} = \frac{l_B}{\sqrt{2}\hbar} (\mathbf{\Pi}_x - i\mathbf{\Pi}_y) \quad (1.9)$$

$$\mathbf{a}^\dagger = \frac{l_B}{\sqrt{2}\hbar} (\mathbf{\Pi}_x + i\mathbf{\Pi}_y) \quad (1.10)$$

and rewrite

$$\mathbf{H} = \hbar\omega_C \left( \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \right) \quad (1.11)$$

Where  $\omega_C = \frac{eB}{m}$ , the cyclotron frequency. From this last expression it's clear that the energies of the system will be given by

$$E_n = \hbar\omega_C \left( n + \frac{1}{2} \right). \quad (1.12)$$

Although the eigenvalues are the same in both gauges, the natural basis of eigenfunctions associated to each gauge looks rather different.

In the symmetric gauge, we can construct a basis by first defining a "twisted" momentum operator

$$\vec{\tilde{\Pi}} = \vec{\mathbf{P}} - e\vec{\mathbf{A}}. \quad (1.13)$$

It's easy to check that

$$[\tilde{\Pi}_x, \tilde{\Pi}_y] = i \left( -\frac{\hbar^2}{l_B^2} \right) \quad (1.14)$$

So we are motivated to define ladder operators

$$\mathbf{b} = \frac{l_B}{\sqrt{2}\hbar} (\tilde{\Pi}_x - i\tilde{\Pi}_y) \quad (1.15)$$

$$\mathbf{b}^\dagger = \frac{l_B}{\sqrt{2}\hbar} (\tilde{\Pi}_x + i\tilde{\Pi}_y) \quad (1.16)$$

Moreover, the operator  $\tilde{N} = \mathbf{b}^\dagger \mathbf{b}$  commutes with the Hamiltonian, therefore we can find a basis that simultaneously diagonalizes both operators. Naturally, the spectrum of  $\tilde{N}$  is the set of non-negative integers, so we can think of the basis as vectors of the form  $|n, m\rangle$  with the property:

$$\mathbf{H} |n, m\rangle = E_{n,m} |n, m\rangle \quad (1.17)$$

$$\tilde{N} |n, m\rangle = m |n, m\rangle \quad (1.18)$$

It follows that we can follow a procedure completely analogous to that of an harmonic oscillator to find the specific eigenfunctions.

In Landau's gauge, the situation looks rather different. The Hamiltonian is

$$\mathbf{H} = \frac{1}{2m} \left[ (\mathbf{P}_x - eB\mathbf{Y})^2 + \mathbf{P}_y^2 \right], \quad (1.19)$$

so it is clear that  $\mathbf{P}_x$  commutes with the Hamiltonian. We look again for a basis that diagonalizes both  $\mathbf{H}$  and  $\mathbf{P}_x$ . Since we know that the eigenfunctions of  $\mathbf{P}_x$  are plain waves, the eigenvectors must have the form

$$\psi_{n,k} = e^{ikx} \chi_n(y), \quad (1.20)$$

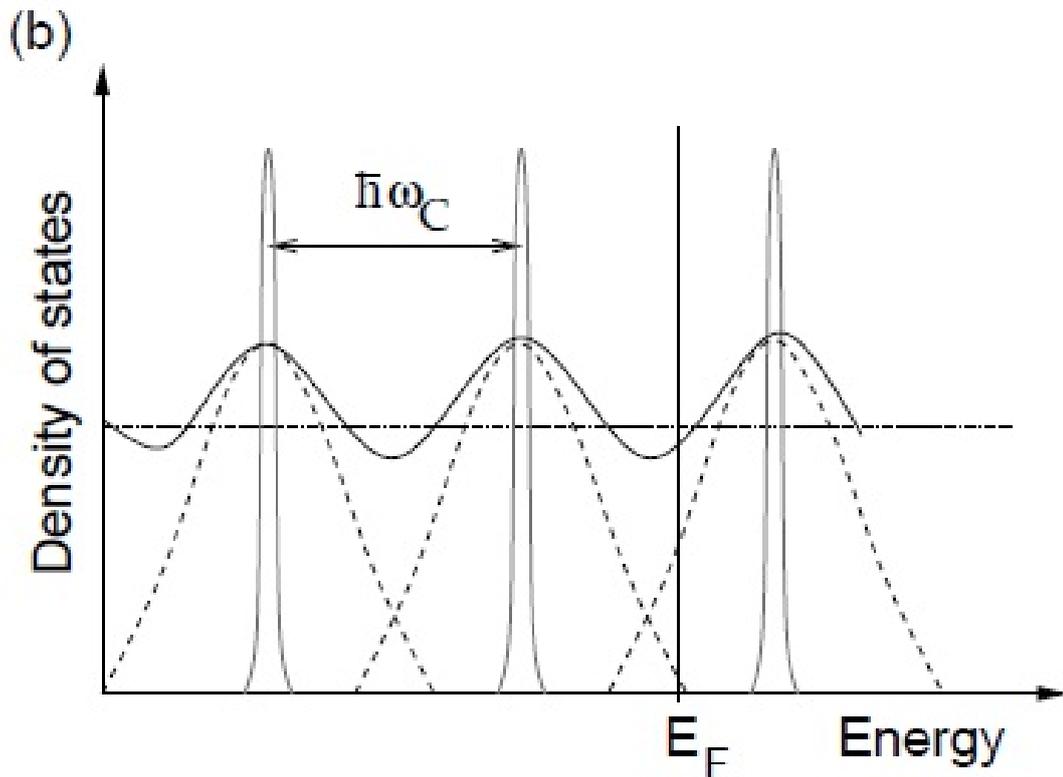


FIGURE 1.1: Schematic representation of the density of states as a function of energy. Taken from [10].

where  $n$  runs over a set of indexes not yet specified. There are some details missing here, they will be completed later.

### 1.3 Shubnikov-de Haas Effect

Although Landau's quantization is a very nice result, we do not expect it to describe accurately what happens in a real material. There, the presence of impurities (among other phenomena) creates a continuum of available energies in the gaps of the Landau levels. Also, due to the finiteness of the sample, we expect the degeneracy of each level to be finite, rather than infinite as predicted in the previous section. Nevertheless, the Landau levels do have a high degeneracy and hence the density of states present local maxima at Landau energies. Figure 1.1 shows schematically the density of states.

Now, as we said in the beginning of the previous section, we don't expect the results of Drude's model to hold at low temperatures. One of the reasons is that in the model we use the total density of charge carriers  $n$ . However, at low temperatures not all

electrons are available to participate in conduction dynamics. Only those whose energy is near the Fermi energy contribute to the conductivity of the material.

This is a consequence of Pauli's exclusion principle. At low temperatures, the distribution function (the probability density that a state with energy between  $E$  and  $E + dE$  is occupied) can be approximated by a step function

$$f(E) = \begin{cases} 1 & E \leq E_F \\ 0 & E > E_F. \end{cases} \quad (1.21)$$

In other words, all states with energies below the Fermi energy are filled, whereas those with energies above  $E_F$  are empty. Now, consider an electron in a state with energy far below  $E_F$ . If we turn on an electric field, the electron cannot be excited by it, because if the energy provided by the field is not enough, the end state of the excitation would be one with energy still below  $E_F$ . But those states are all occupied, so by Pauli's exclusion principle, the excitation cannot happen. In contrast, for an electron close enough to the Fermi energy, the end state will have energy higher than  $E_F$  and will be empty, so the excitation is possible and the electron contributes to the conductivity of the system. It follows from this discussion that the conductivity of the system should be proportional to the density of states  $\rho$ , evaluated at the Fermi energy, that is, it should be proportional to  $\rho(E_F)$ .

Let us now come back to the question of the relation between the conductivity of the system and the magnetic field. The Fermi energy varies very little with the magnetic field, it can be taken to be constant. Now, suppose we start increasing the magnetic field. Then, since  $\omega_C$  is proportional to  $B$ , the Landau levels will start moving towards higher energies and the separation between them will grow. Hence, the graph of the density of states depicted in Figure 1.1 will move to the right and expand and as it does so,  $\rho(E_F)$  will go through a succession of peaks and valleys. In other words,  $\rho(E_F)$  and therefore  $\sigma$  will oscillate with  $B$ . If the magnetic field is strong enough, the separation between the Landau levels will be enough so that the oscillations are experimentally observable. Such oscillations in the longitudinal conductivity were discovered by physicists Lew Wassiljewitsch Schubnikov and Wander Johannes de Haas in 1930 and hence the phenomenon is called Shubnikov - de Haas effect. The effect is shown in Figure 1.2.

A couple of final remarks are in order. First of all, this effect only affects the longitudinal conductivity and resistivity. In the region of  $B$  where Shubnikov-de Haas effects are observed, the Hall resistivity exhibits the same linear behavior as in the classical case. Second and last, although  $\rho(E_F)$  experiences some valleys in this regime, it never vanishes.

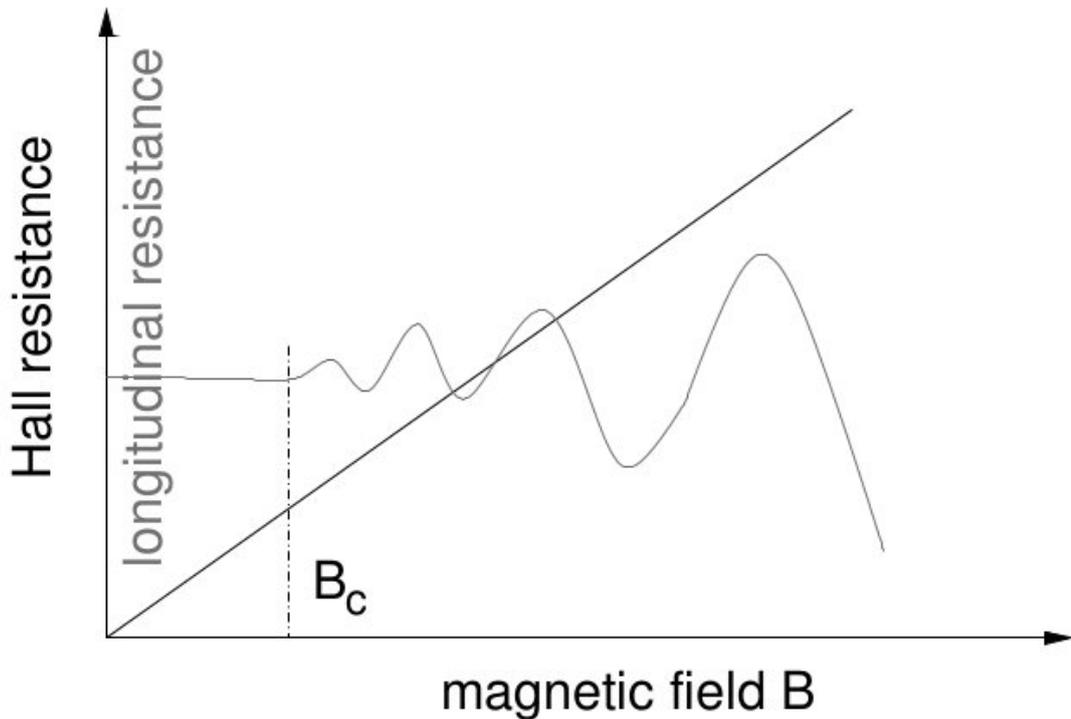


FIGURE 1.2: Longitudinal resistance in Shubnikov - de Haas effect. Taken from [10].

## 1.4 Integer Quantum Hall Effect and Kubo's Formula

What if we keep on increasing the magnetic field? Experimentally, two things have been shown to happen.

The first one has to do with the longitudinal resistivity. Not surprisingly, the oscillations from Shubnikov - de Haas effect continue, but now in the valleys it does get to zero. Also, the peaks are higher.

The second one is much more astonishing and its discovery won Klaus von Klitzing the Nobel Prize in 1985. He observed that, while the overall tendency is still linear, the Hall resistivity has plateaus at a discrete set of values, which are of the form

$$\rho_H = \frac{1}{n} \frac{h}{e^2} \quad n \in \mathbb{N}. \quad (1.22)$$

These plateaus coincide with the values of  $B$  for which  $\rho_L$  vanishes. Moreover, this quantization is incredibly precise (so much that it is used as a standard of resistivity) and it is observed regardless of the geometry of the sample or the details of the material that makes up the sample, such as the presence of impurities. This robustness is perhaps the most important feature of this phenomenon. The situation is depicted in Figure 1.3.

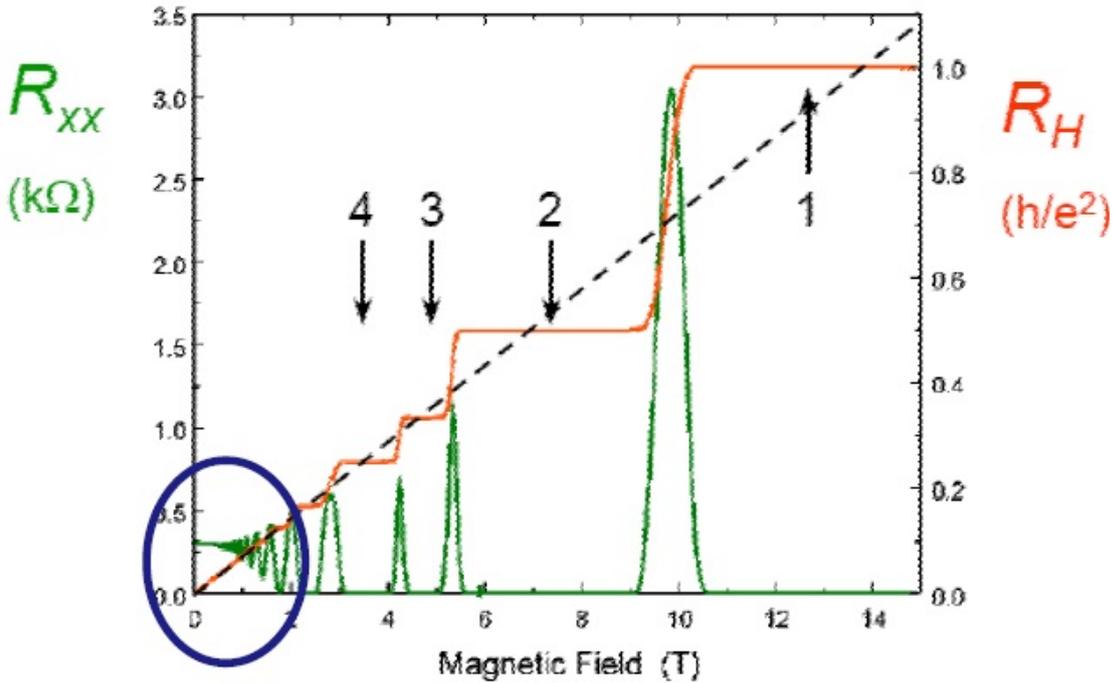


FIGURE 1.3: Longitudinal and Hall resistances in Integer Quantum Hall Effect. Taken from [10].

Now, how could we try to explain this extraordinary behavior? Although the discussion in the previous section provided an accurate qualitative description of the longitudinal resistivity, it does not explain the quantization of the Hall resistivity nor does it provide quantitative predictions for the values of the longitudinal one. So, we need a more detailed treatment of the electrical properties of our system. As we discussed earlier, in such a treatment we can think of the action of the magnetic field as being the most important aspect of the dynamics and consider the electric field as a perturbation, an additional term in the Hamiltonian whose effect on the system we do not completely understand. The framework for such a treatment is provided by Linear Response Theory and summarized in Kubo's formula, which we will discuss in the rest of this section. However, the proof that the quantization of the Hall conductivity follows from Kubo's Formula will be postponed until the next chapter.

### 1.4.1 Linear Response Theory

Suppose we are considering a system made of  $N$  particles described by a Hamiltonian  $\mathbf{H}_0$  (in our case, the part of the Hamiltonian associated to the magnetic field), whose state at equilibrium is given by the density matrix  $\rho_0(t)$ . In the canonical ensemble, we have  $\rho_0 = \frac{1}{Z} e^{-\beta \mathbf{H}_0}$  where  $Z$ , the partition function, is given by  $Z = \text{Tr}(e^{-\beta \mathbf{H}_0})$ . We could equally well consider the grand canonical ensemble, but let us stick with the

canonical ensemble for simplicity. If  $\mathbf{A}$  is some observable with no explicit dependence of time, the expectation value of  $\mathbf{A}$  at equilibrium,  $\langle \mathbf{A} \rangle_0$ , is given by  $Tr(\rho_0 \mathbf{A})$ .

Suppose now that we include an additional term in the Hamiltonian, of the form

$$\mathbf{H}_{int}(t) = F(t)\mathbf{B}, \quad (1.23)$$

where  $F(t) : \mathbb{R} \rightarrow \mathbb{C}$  is an ordinary function and  $\mathbf{B}$  is an observable (in our case, this would correspond to the part of the Hamiltonian associated to the magnetic field). Then the state of the system might not be static any more and it will be represented by some density matrix  $\rho(t)$ . Similarly, the expectation value of  $\mathbf{A}$ ,  $\langle \mathbf{A}(t) \rangle$ , will now depend on time. The goal of this chapter is to determine the relation between the change in the expectation value of  $\mathbf{A}$  due to the presence of  $\mathbf{H}_{int}$ , and the function  $F(t)$ . Let me make this more precise. Define

$$\delta \langle \mathbf{A}(t) \rangle = \langle \mathbf{A}(t) \rangle - \langle \mathbf{A} \rangle_0.$$

We assume  $F(t)$  is relatively weak, so that the ansatz

$$\delta \langle \mathbf{A} \rangle(t) = \int_{-\infty}^{\infty} ds \chi(t-s)F(s)$$

is valid. In other words, we assume the response of the system to the additional term in the Hamiltonian is linear. Then, the goal of this section is to develop some understanding of the function  $\chi$ .

## 1.4.2 Interaction Picture

In order to find  $\chi$ , we need to rewrite our problem in a useful way, making use of the so-called "Interaction Picture" formalism. In a sense, the idea of this formalism is to extract as much information as we can from the original Hamiltonian, which we are assumed to understand well (in our case, we know the spectrum and eigenfunctions of  $\mathbf{H}_0$ , thanks to Landau's quantization) and reducing the problem of time evolution to the additional term in the Hamiltonian,  $\mathbf{H}_{int}$ .

Let us make this precise. Let  $\rho(t)$  be the density matrix that describes the state of the system at time  $t$ , when the system is subject to the full Hamiltonian  $\mathbf{H}(t)$ . We know that the expectation value of an observable  $\mathbf{A}$  is given by

$$\langle \mathbf{A} \rangle(t) = Tr\{\mathbf{A}\rho(t)\}. \quad (1.24)$$

If we add  $\mathbf{I} = e^{-i\mathbf{H}_0} e^{i\mathbf{H}_0}$  before  $\mathbf{A}$  and between  $\mathbf{A}$  and  $\rho(t)$ , we get

$$\langle \mathbf{A} \rangle(t) = Tr\{e^{-i\mathbf{H}_0} e^{i\mathbf{H}_0} \mathbf{A} e^{-i\mathbf{H}_0} e^{i\mathbf{H}_0} \rho(t)\}. \quad (1.25)$$

We can use the cyclic property of the trace to rewrite this as

$$\langle \mathbf{A} \rangle(t) = Tr\{e^{i\mathbf{H}_0} \mathbf{A} e^{-i\mathbf{H}_0} e^{i\mathbf{H}_0} \rho(t) e^{-i\mathbf{H}_0}\}. \quad (1.26)$$

For any operator  $\mathbf{B}$ , let us call  $\mathbf{B}^I(t)$  to the conjugation  $e^{i\mathbf{H}_0 t} \mathbf{B} e^{-i\mathbf{H}_0 t}$ . Then the preceding equation can be written as

$$\langle \mathbf{A} \rangle(t) = Tr\{\mathbf{A}^I(t) \rho^I(t)\}. \quad (1.27)$$

This simple manipulation we have been doing so far becomes powerful when we find an expression for  $\rho^I(t)$ . Indeed, after some simple algebra it can be shown that the traditional Heisenberg form

$$i\hbar \frac{d\rho(t)}{dt} = [\mathbf{H}, \rho(t)] \quad (1.28)$$

is equivalent to

$$i\hbar \frac{d\rho^I(t)}{dt} = [\mathbf{H}_{int}^I(t), \rho^I(t)]. \quad (1.29)$$

Let us take a step back. Recall that, for us,  $\mathbf{H}_{int}$  represents the part of the Hamiltonian associated to the electric field, whose effect on the system should be small compared to that of  $\mathbf{H}_0$ , the part of the Hamiltonian associated to  $B$ . Hence, we can assume  $\rho(t)$  to be  $\rho_0$  plus a small perturbation  $\Delta\rho$ . Naturally,  $\rho^I(t) = \rho_0^I + \Delta\rho^I(t)$ . Since  $\rho_0$  is the density matrix when we restrict ourselves to consider  $\mathbf{H}_0$  rather than the full Hamiltonian,  $\rho_0$  and  $\mathbf{H}_0$  commute, so  $\rho_0 = \rho_0^I$ . Moreover, to first order, we should ignore terms of the form  $\mathbf{H}_{int} \Delta\rho^I$  since they would be quadratic. Then, we end up with

$$i\hbar \frac{d\rho^I(t)}{dt} = i\hbar \frac{d\Delta\rho^I(t)}{dt} = [\mathbf{H}_{int}^I(t), \rho_0] \quad (1.30)$$

which has the formal solution

$$\Delta\rho^I(t) = -\frac{i}{\hbar} \int_{-\infty}^t ds [\mathbf{H}_{int}^I(s), \rho_0] \quad (1.31)$$

assuming that  $\lim_{t \rightarrow -\infty} \Delta\rho^I(t) = 0$ . Replacing this on Equation (1.27), we get

$$\begin{aligned}
\langle \mathbf{A} \rangle(t) &= Tr\{\mathbf{A}^I(t)\rho^I(t)\} \\
&= Tr\{\mathbf{A}^I(t)\rho_0\} + Tr\{\mathbf{A}^I(t)\Delta\rho^I(t)\} \\
&= \langle \mathbf{A} \rangle_0 + \int_{-\infty}^t ds Tr\left\{-\frac{i}{\hbar}\mathbf{A}^I(t) [\mathbf{H}_{int}^I(s), \rho_0]\right\}. \tag{1.32}
\end{aligned}$$

After some algebra, we can infer  $\chi$  from this last expression. We get

$$\chi(t) = -\frac{i}{\hbar}\Theta(t)Tr\{\mathbf{A}^I(t) [\mathbf{B}, \rho_0]\} \tag{1.33}$$

where  $\Theta(t)$  is the Heaviside step function.

When we apply Linear Response Formalism to our case of study, it can be shown that we obtain a conductivity tensor of the form:

$$\sigma_{xy} = \frac{e^2\hbar}{i} \sum_{E_n < E_F < E_m} \frac{\langle n|v_x|m\rangle\langle m|v_y|n\rangle - \langle n|v_y|m\rangle\langle m|v_x|n\rangle}{(E_n - E_m)^2} \tag{1.34}$$

See the references mentioned at the beginning of the chapter for more details.



## Chapter 2

# Vector Bundles, Curvature and Chern Classes

The goal of this chapter is to discuss a geometric concept that lays at the core of our understanding of the Integer Quantum Hall Effect: vector bundles. Roughly speaking, a vector bundle is a construction by which we attach a copy of a fixed vector space to every point of some topological space, called the base space. Sections of a vector bundle are rules that assign to every point of the base space, some vector in the vector space attached to it. Hence, they can be thought of as generalizations of functions.

It is hard to overstate the importance of vector bundles in Physics, they appear everywhere. Perhaps the reason why they are so ubiquitous is that, as we shall see in this chapter, they are the natural framework to discuss covariant derivatives, a notion that plays an important role both in General Relativity and Gauge Theory. However, they are also used in earlier theories in Physics like Classical Mechanics, where configuration spaces in the Hamiltonian formalism are usually cotangent bundles, which are examples of vector bundles. In this document, we use them to describe quantum mechanical systems for which the Hamiltonian depends on a set of parameters.

We will assume that the reader has some familiarity with general topology and differential geometry, just so we can use the language developed there. Since this is a document in Physics, not Mathematics, we will avoid technical proofs. Also, rather than a standard presentation of this topic, this will be a preparation for what is coming in the following chapters. Thus we will avoid some details that are usually discussed in any text on vector bundles, and emphasize other that would be otherwise overlooked. Mostly, we will be following reference [17].

### 2.1 Vector bundles

A vector bundle is made up of three things. First, there is a topological space called *base space* and usually denoted  $X$ . Second, we must have a vector space of finite dimension,  $F$ , called the fiber of the bundle. Third, there is another topological space, the *total*

space, referred to as the  $E$ . This total space is endowed with a projection map  $\pi : E \rightarrow X$  which is surjective. Moreover, all the sets of the form  $\pi^{-1}(x)$  for some  $x \in X$  are vector spaces, all of them isomorphic to  $F$ .

There is one more condition in the definition of a vector bundle. The simplest example of a vector bundle is something of the form  $X \times F$ , where  $X$  is a topological space and  $F$  a vector space. The base space here is  $X$  and the projection map is just projection onto the first factor. Not all vector bundles are trivial, but we demand all of them to be locally trivial. This means that for all  $x \in X$ , there is an open set  $U$  that contains  $x$  for which we can define an homeomorphism  $\phi_U$  between  $U \times F$  and  $\pi^{-1}(U)$ . Such open sets are called *trivializing open sets* and the homeomorphisms are called *trivializations*. Trivializations must "respect the fibers", in the sense that the identity  $\pi(\phi(x, v)) = x$  is satisfied for all  $x \in X$  and every  $v \in \pi^{-1}(x)$ . We denote vector bundles by triples  $(E, X, F)$ , where  $E$  is the total space,  $X$  the basis space and  $F$  is the fiber.

Although the one given in the last paragraph is the formal definition of a vector bundle, in practice it is uncomfortable to build vector bundles using it. Instead, we usually describe them either by using transition functions or projections.

Let us discuss transition functions first. Let  $(E, X, F)$  be a vector bundle and  $U, V \subset X$  be trivializing open sets whose intersection is not empty. Let  $\phi_U : U \times F \rightarrow \pi^{-1}(U)$  and  $\phi_V : V \times F \rightarrow \pi^{-1}(V)$  be the respective trivializations. Then, we have a map  $g_{U,V} : (U \cap V) \times F \rightarrow (U \cap V) \times F$  given by  $\phi_V^{-1} \circ \phi_U$ . Since the trivializations respect the fibers, this map must be of the form  $(x, v) \mapsto (x, g_{U,V}(x)v)$ , where  $g_{U,V}(x)$  is an element in  $\text{hom}(F, F)$ , called the transition function of the bundle.

Transition functions satisfy a special identity called the "cocycle condition". Suppose  $U, V$  and  $W$  are trivializing open sets such that  $U \cap V \cap W \neq \emptyset$ . Then, transitioning from  $U$  to  $W$  directly should be equivalent to transitioning to  $V$  first and then to  $W$ . In other words,

$$g_{U,W} = g_{V,W}g_{U,V}, \quad (2.1)$$

where the transition functions are understood to be restricted to the appropriate subset.

It turns out that transition functions completely characterize vector bundles. In other words, in order to describe completely a vector bundle, it suffices to specify a collection of trivializing sets that cover all of  $X$  and corresponding transition functions for which Equation (2.1) holds, in order to completely describe a vector bundle. See Section 5.3 of [5], for a proof of this assertion.

Perhaps the simplest example of nontrivial vector bundle is Möbius band. It is a vector bundle with base space  $S^1$ , the unit circle, and  $\mathbb{R}$  as a fiber. The simplest way to describe it is using transition functions. The trivializing sets are  $U_S = S^1 - N$  and  $U_N = S^1 - S$ ,

where  $S, N$  refer to the south and north pole respectively. If we think of  $S^1$  as the set  $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ , then  $U_S \cap U_N = U_E \cup U_W$  where  $U_E = \{(x, y) \in S^1 | x > 0\}$  and  $U_W = \{(x, y) \in S^1 | x < 0\}$ . We define the transition functions to be

$$\begin{aligned} \phi_{U_E} : U_E \times \mathbb{R} &\rightarrow U_E \times \mathbb{R} \\ (x, v) &\mapsto (x, v) \end{aligned}$$

and

$$\begin{aligned} \phi_{U_W} : U_W \times \mathbb{R} &\rightarrow U_W \times \mathbb{R} \\ (x, v) &\mapsto (x, -v) \end{aligned}$$

It is hard to convey it in a written text, but these transition functions make precise the notion that a Möbius band is built by pasting two edges of a rectangle in "the wrong way". Anyway, this is an example of how to describe a vector bundle by means of its transition functions.

Let us discuss now how projections can be used to build vector bundles. Consider the manifold

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

To every point in  $S^2$  we can attach the plane in  $\mathbb{R}^3$  made out of vectors that are tangent to it. The resulting space is a vector bundle with a two dimensional fiber, called the *tangent bundle* of  $S^2$  and denoted  $TS^2$ . In other words,  $TS^2 = (E, S^2, \pi)$  where

$$E = \{(\vec{v}_1, \vec{v}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 | \|\vec{v}_1\| = 1, \vec{v}_1 \cdot \vec{v}_2 = 0\}$$

and

$$\begin{aligned} \pi : E &\rightarrow S^2 \\ (\vec{v}_1, \vec{v}_2) &\mapsto \vec{v}_1 \end{aligned}$$

$TS^2$  has the property of being immersed in another vector bundle, namely the trivial bundle over  $S^2$  obtained by just gluing to every point  $x$  in  $S^2$  every vector in  $\mathbb{R}^3$  that begins at  $x$ . Indeed, this larger vector bundle is just  $S^2 \times \mathbb{R}^3$ , so  $TS^2$  is a sub-bundle of a trivial one. This is just an example of a general fact: we can build vector bundles

with base space  $X$  by considering sub-bundles of trivial bundles over  $X$ . That is, by considering first the space  $X \times F$  and then, for every point  $x \in X$  choosing a subspace of  $F$  in a continuous manner. It can be shown that if  $X$  is compact, all vector bundles with  $X$  as base space are built in this form, as shown in [2]. For example, the Möbius band can also be described using projections, as shown in the appendix of [22].

A *section*  $\psi$  of a vector bundle  $(E, X, F)$  is a map  $\psi : X \rightarrow E$  such that  $\pi \circ \psi = id_X$ , where  $\pi$  denotes the projection map of the bundle. As we said at the beginning, a section assigns to every point  $x \in X$  a vector that belongs to the fiber  $\pi^{-1}(x)$ . The set of continuous sections of a vector bundle is denoted  $\Gamma(E)$  and it is a left module over the algebra of continuous functions of the base space,  $C(E)$ . That means we can multiply a section  $\psi$  by a function  $f : X \rightarrow \mathbb{K}$  (where  $K$  is the field of scalars of  $F$ ), according to the rule  $(f\psi)(x) = f(x)\psi(x)$ .

Actually, the structure of  $\Gamma(E)$  as a module over  $C(E)$  reflects the structure of the vector bundle. For instance, if the latter is trivial, the former is trivial too. To be precise, if  $E = X \times F$  then we can choose a basis  $\beta = \{v_1, \dots, v_n\}$  for  $F$  and describe every section in a unique manner as a linear combination

$$\psi(x) = f_1(x)v_1(x) + f_2(x)v_2(x) + \dots + f_n(x)v_n(x) \quad (2.2)$$

where  $v_i(x)$  is the section that assigns  $v_i$  to every  $x \in X$ . This shows that  $\Gamma(E) \cong \bigoplus_{k=1}^n C(X)v_i$ . On the other hand, if the vector bundle is not trivial but a sub-bundle of a trivial bundle, then the same trick will fail as an arbitrary linear combination of elements in a basis of  $F$  might not lay in the appropriate subspace of  $F$ . In that case, we need to consider an additional element. Suppose the sub-bundle is built by attaching the subspace  $F_x \subset F$  to every point  $x \in X$ . Now, let  $P_x : F \rightarrow F_x$  be a projection onto  $F_x$ . In general such projection is not uniquely determined by  $F_x$ , but there is usually a "canonical" choice, we will discuss more about it in the following. Anyway, we can think of sections to the sub-bundle as maps of the form

$$\psi(x) = P_x(f_1(x)v_1(x) + f_2(x)v_2(x) + \dots + f_n(x)v_n(x))$$

the projection  $P_x$  guaranteeing that  $\psi(x)$  belongs to  $F_x$ . Hence,  $\Gamma(E) \cong P \bigoplus_{k=1}^n C(X)v_i$ , where  $P : X \rightarrow \text{hom}(F, F)$  is the assignment that sends  $x$  to  $P_x$ . Notice that  $P^2 = P$ . A module with this structure is called *projective and finitely generated*. If  $X$  is compact, every vector bundle with  $X$  as base space can be seen as a sub-bundle of a trivial one, and hence  $\Gamma(E)$  will always be a projective, finitely generated module over  $C(X)$ . This is the content of the Serre-Swan Theorem [2].

Finally, a word about metrics. A metric on a vector bundle is continuous choice of a

metric for every fiber of the bundle. For instance, in the case of  $TS^2$  there is a natural metric: since elements in the fibers of  $TS^2$  are vectors of  $\mathbb{R}^3$ , we can endow each fiber with the euclidean metric. Metrics provide a preferred choice of projection when describing a vector bundle as a sub-bundle of a trivial one. Indeed, if the trivial bundle is endowed with a metric, the projections  $P_x$  can be chosen to be the orthogonal projections. Vector bundles over compact spaces can always be endowed with metrics.

## 2.2 Connections

Now that we know all we need to know about vector bundles, sections and metrics, we can discuss how to take derivatives of sections. We will see that, although in general there is no preferred way to define what that means, when the vector bundle is thought of as a sub-bundle of a trivial one, which on its turn has a metric, then there is a very natural way to do it.

Let  $(E, X, F)$  be a vector bundle. From now on, we will assume  $E$  and  $X$  are not only topological spaces but also smooth manifolds and that the map  $\pi$  is not just continuous but differentiable. Similarly, sections are now taken to be differentiable functions from  $X$  to  $E$ . A *connection* is a map  $\nabla$  from  $\Gamma(E)$  to  $\Omega^1(X) \otimes \Gamma(E)$  for which the following properties hold:

$$\nabla(r\psi_1 + s\psi_2) = r\nabla(\psi_1) + s\nabla(\psi_2) \quad r, s \in \mathbb{K}, \psi_1, \psi_2 \in \Gamma(E) \quad (2.3)$$

$$\nabla(f\psi) = f\nabla(\psi) + df \otimes \psi \quad f \in C^\infty(X), \psi \in \Gamma(X) \quad (2.4)$$

Connections correspond to covariant derivatives in Physics. To see that, suppose  $U \subset X$  is a trivializing open set. Then if  $\beta = \{v_1, \dots, v_n\}$  is a basis for  $F$ , sections over  $U$  can be decomposed in the same way as in Equation (2.2). Then, applying the properties of connections, we get:

$$\nabla(\psi)(x) = \nabla(f_1v_1(x) + \dots + f_n(x)v_n(x)) \quad (2.5)$$

$$= \nabla(f_1(x)v_1(x)) + \dots + \nabla(f_n(x)v_n(x)) \quad (2.6)$$

$$= f_1(x)\nabla(v_1(x)) + df_1(x) \otimes v_1(x) + \dots + f_n(x)\nabla(v_n(x)) + df_n(x) \otimes v_n(x) \quad (2.7)$$

Now, we can always write

$$\nabla(v_i) = \sum_{j=1}^n A_{ij} \otimes v_j \quad (2.8)$$

for some  $A_{ij} \in \Omega^1(X)$ , so

$$\begin{aligned} \nabla(\psi) &= \sum_{i=1}^n f_i(x) \nabla(v_i)(x) + df_i(x) \otimes v_i(x) \\ &= \sum_{i=1}^n \sum_{j=1}^n f_i(x) A_{ij} v_j(x) + df_i(x) \otimes v_i(x) \\ &= \sum_{i=1}^n \left( \sum_{j=1}^n A_{i,j} f_j(x) + df_i(x) \right) \otimes v_i(x) \end{aligned} \quad (2.9)$$

We usually abbreviate this relation as  $\nabla = d + A$ , where  $A$  is a  $n \times n$  matrix of one-forms called the *connection 1-form*. It is now clear that connections indeed are equivalent to covariant derivatives.

Given a vector field  $V$  over  $X$ , we can couple it with an element of  $\Omega^1(X) \otimes \Gamma(E)$  according to the rule

$$\langle V, \theta \otimes \psi \rangle = \theta(V)\psi$$

obtaining a new section. In particular, we can define

$$\nabla_V \psi = \langle V, \nabla \psi \rangle \quad (2.10)$$

which is a new section that represents the derivative of  $\psi$  along the direction of  $V$ .

Now, suppose  $(E, X, F)$  is a sub-bundle of the trivial bundle  $(X \times H, X, H)$ , obtained by attaching  $F_x \subset H$  to each  $x \in X$ , where  $F_x \cong F$ ,  $F \subset H$  and  $X$  is compact. Let  $\langle, \rangle$  be a positive definite inner product over  $H$  and let  $g$  be the metric on the trivial bundle obtained by the constant choice of  $\langle, \rangle$  for each fiber. For each  $x \in X$ , take  $P_x$  to be the orthogonal projection from  $H$  to  $F_x$  according to  $g$  and let  $P$  the map from  $\Gamma(X \times H)$  to itself defined by the rule  $(P\psi)(x) = P_x\psi(x)$ . In this situation, there is a very natural choice of a connection for the sub-bundle, called the Levi-Civita connection. In order to describe it, notice first that sections of the sub-bundle can be thought of as sections  $\psi$  of the bigger bundle for which  $P\psi = \psi$  holds. According to Equation (2.2), sections of the bigger bundle are of the form  $\psi(x) = \sum_i s_i(x)v_i(x)$ , where the  $v_i$ 's are elements of

a basis for  $H$ . Then, as a definition of a connection, we could consider taking exterior derivatives

$$\nabla \left( \sum_i s_i v_i \right) = \sum_i ds_i \otimes v_i \quad (2.11)$$

Or, in other words,

$$\nabla_X \left( \sum_i s_i v_i \right) = \sum_i X(s_i) v_i \quad (2.12)$$

the ordinary derivative. Although this works as a connection of the trivial bundle, there is no reason for  $\nabla_X \psi(x)$  to belong to  $F_x$ , even if  $\psi(x)$  did, so it doesn't work as a definition of a connection on the sub-bundle. The solution to this problem is simple: we just project the resulting section. Indeed the rule

$$\nabla_X \left( \sum_i s_i v_i \right) (x) = P_x \left( \sum_i X(s_i)(x) v_i(x) \right) \quad (2.13)$$

defines the desired connection on the sub-bundle. In an abbreviated form, for a vector of functions  $\vec{s}$  such that  $P\vec{s} = \vec{s}$ , the connection is given by  $\nabla \vec{s} = Pd\vec{s}$ .

### 2.3 Curvature

Notice that so far, the development of the theory of connections has kept a close analogue to that of General Relativity: he have defined a notion of covariant derivative, which involved the use of the  $A_{i,j}$ 's, which play a similar role to that of Christoffel symbols. We could ask now what is the analogue of the Riemann tensor. In a sense, the Riemann tensor is a second derivative, so we can ask as well what does it mean to take a second covariant derivative.

Let  $(E, X, F)$  be a vector bundle endowed with a connection  $\nabla : \Gamma(E) \rightarrow \Omega(X) \otimes \Gamma(E)$ . Based on  $\nabla$ , we can build a new map  $d^\nabla$  from  $\Omega(X) \otimes \Gamma(E)$  to  $\Omega^2(X) \otimes \Gamma(E)$  by applying a generalized Leibniz rule. We define  $d^\nabla$  by the rule

$$d^\nabla(\theta \otimes \psi) = d\theta \otimes \psi - \theta \wedge \nabla\psi \quad (2.14)$$

We could take the composition  $d^\nabla \circ \nabla$ , which is a map from  $\Gamma(E)$  to  $\Omega^2(X) \otimes \Gamma(E)$ . This map is called the curvature of the connection and is usually denoted  $F^\nabla$ . Curiously, although for connections a version of Leibniz rule applies with respect to multiplication by functions of the base manifold, the curvature is linear with respect to it. Indeed:

$$F^\nabla(f\psi) = d^\nabla(df \otimes \psi + f\nabla\psi) \quad (2.15)$$

$$= d^2f \otimes \psi - df \otimes \nabla\psi + df \otimes \nabla\psi + fd^\nabla\psi \quad (2.16)$$

$$= fF^\nabla(\psi) \quad (2.17)$$

Hence, it makes sense to consider  $F^\nabla$  locally as a  $n \times n$  matrix of 2-forms,  $n$  being the dimension of the fiber. Actually, it is easy to compute a local formula for  $F^\nabla$  in terms of the connection 1-form. We have

$$F^\nabla(v_i) = d^\nabla(\nabla v_i) \quad (2.18)$$

$$= d^\nabla \left( \sum_j A_{ij} v_j \right) \quad (2.19)$$

$$= \sum_j d(A_{ij}) v_j - A_{ij} \nabla(v_j) \quad (2.20)$$

$$= \sum_j d(A_{ij}) v_j - A_{ij} \wedge \left( \sum_k A_{jk} v_k \right) \quad (2.21)$$

$$= \sum_k \left( d(A_{ik}) - \sum_j A_{ij} \wedge A_{jk} \right) \otimes v_k \quad (2.22)$$

This last result is usually written as  $F^\nabla = dA - A \wedge A$ .

Now, what is the curvature of the Levi-Civita connection described in the last section? We could write Equation 2.13 as

$$\nabla \left( \sum_i s_i v_i \right) = P \left( \sum_i ds_i v_i \right) \quad (2.23)$$

$$= \sum_i ds_i P(v_i) \quad (2.24)$$

$$= \sum_i ds_i \sum_j P_{ij} v_j \quad (2.25)$$

$$(2.26)$$

So applying  $d^\nabla$  we get

$$F^\nabla \left( \sum_i s_i v_i \right) = \sum_{i,j} d^\nabla (ds_i \otimes P_{ij} v_j) \quad (2.27)$$

$$= \sum_{i,j} d^2 s_i \otimes P_{ij} v_j - ds_i \nabla (P_{ij} v_j) \quad (2.28)$$

$$= - \sum_{i,j} ds_i \wedge d(P_{ij}) \sum_k P_{jk} v_k \quad (2.29)$$

Before we continue, there are a couple of identities we need. First of all,  $P$  is a projection, so

$$\sum_j P_{ij} P_{jk} = P_{ik} \quad (2.30)$$

Taking differentials on both sides:

$$\sum_j (dP_{ij}) P_{jk} + P_{ij} (dP_{jk}) = dP_{ik} \quad (2.31)$$

and multiplying by  $P_{kl}$  on the right we get

$$\sum_{j,k} (dP_{ij}) P_{jk} P_{kl} + P_{ij} (dP_{jk}) P_{kl} = \sum_k (dP_{ik}) P_{kl} \quad (2.32)$$

$$\sum_j (dP_{ij}) P_{jl} + P_{ij} (dP_{jk}) P_{kl} = \sum_k (dP_{ik}) P_{kl} \quad (2.33)$$

It follows that

$$P_{ij} (dP_{jk}) P_{kl} = 0 \quad (2.34)$$

Also, we know that

$$\sum_i s_i P_{ij} = s_j \quad (2.35)$$

So, taking differentials,

$$\sum_i (ds_i) P_{ij} + s_i (dP_{ij}) = ds_j \quad (2.36)$$

Coming back to our calculation, using (2.36) we get

$$F^\nabla \left( \sum_i s_i v_i \right) = - \sum_{i,j} ds_i \wedge d(P_{ij}) \sum_k P_{jk} v_k \quad (2.37)$$

$$= - \sum_{i,j,k} (ds_i) \wedge (dP_{ij}) P_{jk} v_k \quad (2.38)$$

$$= - \sum_{i,j,k,l} ((ds_l) P_{li} + s_l (dP_{li})) \wedge (dP_{ij}) P_{jk} v_k \quad (2.39)$$

$$= - \sum_{i,j,k,l} (ds_l) \wedge P_{li} (dP_{ij}) P_{jk} v_k + s_l (dP_{li}) \wedge (dP_{ij}) P_{jk} v_k \quad (2.40)$$

$$= - \sum_{i,j,k,l} s_l (dP_{li}) \wedge (dP_{ij}) P_{jk} v_k \quad (2.41)$$

The first summand in the line before the last vanished by Equation (2.34). From the last line, we conclude that the curvature of the Levi-Civita connection can be written as

$$F^\nabla = P dP \wedge dP \quad (2.42)$$

## 2.4 Chern Classes

In the last section, we learned that the curvature of a connection is a matrix of two-forms. Now, on one hand this is only true locally: if we change the trivializing open set, we may get a different description of the curvature. On the other hand, it's hard to infer any information from a matrix directly, we usually look at numbers that are related to the matrix, such as its determinant, or its trace.

It is worth mentioning that if the matrix  $g_{UV}$  is the transition matrix for the intersection of two trivializing open sets  $U$  and  $V$  and  $g_{VU}$  is its inverse, then if  $F_U^\nabla$  and  $F_V^\nabla$  are the matrices that represent the curvature of some connection  $\nabla$  in the respective open sets, the following equation holds:

$$F_V^\nabla = g_{VU} F_U^\nabla g_{UV} \quad (2.43)$$

Equation 2.43 tells us that indeed the matrix that represents the curvature of a connection is not a global construction. However, it also tells us that its trace and its determinant are globally defined objects. Notice that since the entries of the matrix are two-forms rather than numbers, its trace and determinant will also be differential forms. Differential forms obtained in this manner are known as characteristic classes.

Characteristic forms have several remarkable properties that have placed them at the center pure mathematics for the last fifty years or so. One of them is the fact that characteristic classes of a vector bundle are independent of the connection used to build them. This is striking, as its construction was a rather elaborate procedure using the curvature of the connection, and yet its proof is not as complicated as one might expect. Anyway, what that tells us is that characteristic classes convey information about the structure of the vector bundle, rather than the specific notion of covariant derivative we might have chosen.

The characteristic class we will be most interested in is the Chern character. It is defined for vector bundles with complex vector spaces as fibers, according to the equation:

$$Ch(\xi) = \frac{i}{2\pi} Tr \left( (F^\nabla)^{\wedge n} \right) \quad (2.44)$$

where  $\xi = (E, X, F)$  is a vector bundle and  $X$  has dimension  $2n$ . Notice that the Chern character is a differential form of degree  $2n$ . If  $X$  is an orientable manifold, then we can integrate the Chern character to get a number:

$$c = \int_X Ch(\xi) \quad (2.45)$$

This number  $c$  is called the Chern number of the vector bundle and will play a key role in the rest of the document. If  $X$  happens to have dimension 2 and  $\xi$  is a line bundle (that is, its fibers have dimension one), then this formula reduces to

$$c = \frac{i}{2\pi} \int_X F^\nabla \quad (2.46)$$

There is one more property of Chern classes that will be important to us in the future. In order to talk about it, though, we need to discuss the notion of  $K$ -theory.

### 2.4.1 $K$ -theory

Vector bundles can be added. Indeed, let  $\xi = (E, X, F)$  and  $\eta = (\tilde{E}, X, G)$  two vector bundles over the same base space  $X$ . Using both of them, we can build a new vector bundle called the *direct sum of  $\xi$  and  $\eta$*  and denoted  $\xi \oplus \eta$ . We will not get into the details of the construction, but the key point is that the fibers of this new bundle are of the form  $\pi_\xi^{-1}(x) \oplus \pi_\eta^{-1}(x)$ , where  $\pi_\xi$  and  $\pi_\eta$  denote the projection maps of  $\xi$  and  $\eta$  respectively.

Generally speaking, if we are interested in the structure of a vector bundle, in how much the fibers twist or something, there should be no distinction between a vector bundle  $\xi$  and another one obtained by adding a trivial bundle  $\epsilon^n$  to it,  $\xi \oplus \epsilon^n$ . Since the trivial bundle doesn't have any structure, any twisting, then  $\xi$  and  $\xi \oplus \epsilon^n$  should

have the same structure. Hence, we are motivated to consider an equivalence relation on the set of vector bundles over some fixed base space  $X$ , where two vector bundles  $\xi$  and  $\eta$  are considered equivalent if there exist trivial bundles  $\epsilon^n$  and  $\epsilon^m$  such that  $\xi \oplus \epsilon^n \cong \eta \oplus \epsilon^m$ .

Direct sum of vector bundles induces a sum operation on the set of equivalence classes, which is associative and abelian. However, the equivalence classes can't form a group, because sum of vector bundles always increases the dimension of the fibers, so there can't be additive inverses because there is no way the sum of two vector bundles is a vector bundle with 0-dimensional fibers. The situation is similar to that of the positive integers,  $\mathbb{N}$ . Now, the smallest group that contains  $\mathbb{N}$  is  $\mathbb{Z}$  and it is built by adding to  $\mathbb{N}$  another copy of  $\mathbb{N}$  which is presided by a minus sign. A similar construction can be carried out for the equivalence classes of vector bundles. The resulting group is called the  $K$ -theory of  $X$  and it is denoted  $K^0(X)$ .

Why are we interested in  $K$ -theory? Because of some interesting properties of the Chern Character. First of all, if  $\epsilon^n$  is a trivial bundle, then  $Ch(\epsilon^n) = 0$ . This means that the Chern character of a vector bundle only depends on its equivalence class. Also, it is additive:

$$Ch(\xi \oplus \eta) = Ch(\xi) + Ch(\eta) \quad (2.47)$$

So, the Chern character can be seen as a morphism of groups between  $K^0(X)$  and  $H^{2n}(X)$ . We will exploit this point of view in the next chapters.

## 2.5 Geometric Phases

Let us apply the notions developed above to some special physical systems.

Consider a quantum system described by a Hamiltonian  $H$  that depends on a set of parameters  $\nu_1, \nu_2, \dots, \nu_n$ . A very simple example of this situation consists of a single, static spin in the presence of a uniform magnetic field. The Hamiltonian of the system,  $H = -\vec{\mu} \cdot \vec{B}$ , depends on the components of  $\vec{B}$ , which act like parameters of the Hamiltonian. In some cases, the set of all possible values of the parameters can be thought of as a topological space or even a differentiable manifold, let's call it  $M_{par}$ . Indeed, in our simple example, if we fix the magnitude of the magnetic field to be  $B$ , the possible values of the parameters correspond to the possible orientations of the field, which can be thought of as a sphere of radius  $B$ . In those cases, we have a function that assigns an operator in a fixed Hilbert space to every point on  $M_{par}$ . That is, we have

$$\begin{aligned} H : \quad M_{par} &\rightarrow \mathcal{L}(\mathcal{H}) \\ (\nu_1, \nu_2, \dots, \nu_n) &\mapsto H[\nu_1, \nu_2, \dots, \nu_n] \end{aligned} \quad (2.48)$$

Now that we have assigned an element of  $\mathcal{L}(\mathcal{H})$  to every point in  $M_{par}$  (from now on we will think of and denote the elements on  $M_{par}$  as points rather than as tuples of values), it's only natural to consider a construction in which we attach  $\mathcal{H}$  to every point in  $M_{par}$ . That would be reminiscent of a vector bundle. However, vector bundles are usually required to have finite dimensional vector spaces as fibers, and Hilbert spaces in physics are not finite dimensional in general. Moreover, such a fiber bundle would be trivial anyway. As we will see in the following sections, the construction that turns out to be useful is to attach to each point of  $M_{par}$  different one-dimensional subspaces of  $\mathcal{H}$ , according to various criteria. Such construction will give rise to line bundles that will reveal very deep facts about quantum theory in general, and about the conductivity of two-dimensional systems in particular. We will follow references [4] and [24] for this discussion.

### 2.5.1 Berry's Phase

An important example of this construction appears when the function  $H$  has a very important feature: for all values of the parameters  $x \in M_{par}$ , the spectrum of  $H[x]$  has a lowest eigenvalue, which is non-degenerate and isolated. In that case, we denote that fundamental energy by  $\lambda_0(x)$  and we say  $H$  describes a gapped Hamiltonian. Notice that this condition is not that hard to fulfill. For instance, it holds for the one-dimensional harmonic oscillator, for the hydrogen atom, and aside from non-degeneracy, all the other conditions are always satisfied for systems with a finite number of degrees of freedom.

Under such assumption, it is natural to assign to each point  $x \in M_{par}$  the eigenspace associated to  $\lambda_0(x)$ . We call the resulting bundle "Berry's bundle". Now, at first sight, it is hard to see why such construction could be physically relevant. But in fact, it becomes important when we consider the possibility that the parameters of the Hamiltonian are not fixed in time, but instead they are changing. In the example discussed in the introduction to this chapter, this would correspond to a magnetic field whose direction is moving. Such a situation could be modeled as a path in the space of parameters,  $\gamma : [0, 1] \rightarrow M_{par}$ . Moreover, it would give rise to a time-dependent Schrödinger's equation:

$$i\hbar \frac{d\psi(t)}{dt} = H(\gamma(t))\psi(t) \quad (2.49)$$

Now, what if the parameters of the Hamiltonian are changing very, very slowly? In other words, what if  $\|\dot{\gamma}(t)\| \ll 1$  for all  $t \in [0, 1]$ ? If  $\|\dot{\gamma}(t)\|$  were exactly zero,  $\gamma$  would be the constant path  $\gamma(t) = \gamma(0)$  and the Hamiltonian would be independent of time. In that case, if the initial state  $\psi(0)$  was an eigenstate of  $\lambda_0(\gamma(0))$ , then  $\psi(t)$  would also be an eigenstate of the same eigenvalue for all  $t \in [0, 1]$ . So, it would be natural to

expect something similar to happen in the limit when the evolution of the Hamiltonian is very slow. This is indeed the case, and the formal statement of this fact is contained in the next theorem, usually called The Adiabatic Theorem.

**Theorem 1.** *In the notation discussed above, suppose  $H$  describes a gapped Hamiltonian,  $\gamma$  is a curve in the parameter space,  $T$  is a positive constant and  $\psi_T(t)$  is a solution to the Schrödinger equation*

$$i\hbar \frac{d\psi_T(t)}{dt} = H\left(\gamma\left(\frac{t}{T}\right)\right) \psi_T(t) \quad (2.50)$$

*such that  $\psi_T(0)$  is an eigenstate of  $\lambda_0(\gamma(0))$ . Then, in the limit when  $T$  goes to infinity,  $\psi_T(T)$  is an eigenstate of  $H(\gamma(1))$  with eigenvalue  $\lambda_0(\gamma(1))$ .*

We will not discuss the proof of the Adiabatic Theorem here, but it can be found in reference [12]. But we will make three remarks about its statement. First of all, notice that we are not really making any assumptions about  $\gamma$ , in particular, we are not demanding it to be slow. That condition is immediately satisfied by the fact that we are re-scaling its argument.

Second, although the precise statement in the last theorem is about the behavior of the solution of Schrödinger's equation at the end point of  $\gamma$ , we can say something similar about all points along the trajectory. Indeed, suppose  $x = \gamma(t_0)$  for some  $t_0 \in [0, 1]$ . Then, in the limit when  $T$  goes to infinity,  $\psi_T(t_0T)$  is an eigenvector of  $H[x]$  with eigenvalue  $\lambda_0(x)$ . This follows from defining a new path  $\tilde{\gamma}$  by the rule

$$\tilde{\gamma}(t) = \gamma\left(\frac{t}{t_0}\right)$$

and applying the Adiabatic Theorem.

Last and most important, there is an important feature of adiabatic evolution that is central to this thesis. Let  $U$  be an open set of  $M_{par}$  such that Berry's bundle over  $U$  is trivial. Then, there exists a nowhere-vanishing section of the bundle defined on  $U$ , let us denote it  $|n(R)\rangle$ . Let  $\gamma : [0, 1] \rightarrow M_{par}$  be a curve whose image is contained in  $U$ , with initial and endpoints  $x_0$  and  $x_1$  respectively. Suppose a system has initial state  $\psi_T(0) = |n(x_0)\rangle$  and evolves according to the Hamiltonian  $H\left(\gamma\left(\frac{t}{T}\right)\right)$ . Then, in the limit when  $T$  goes to infinity,  $\psi_T(T)$  will be a vector in the same vector subspace as  $|n(x_1)\rangle$ . But this subspace has dimension one, and furthermore, temporal evolution in quantum mechanics preserves norms, so the two vectors will only differ by a global phase,  $\psi_T(T) = e^{i\theta} |n(x_1)\rangle$ . Now, a natural first guess for the angle  $\theta$  would have been

$$\phi = -\frac{1}{\hbar} \int_0^T ds \lambda_0\left(\gamma\left(\frac{s}{T}\right)\right) \quad (2.51)$$

It turns out that this guess is wrong. Indeed,

$$e^{i\theta} = e^{i\theta_B} e^{i\phi}, \quad (2.52)$$

where  $e^{i\theta_B}$  is a nontrivial phase, called Berry's Phase.

We can give a local formula for Berry's phase. If

$$\psi_T(t) = \exp\left(\frac{-i}{\hbar} \int_0^T ds \lambda_0\left(\gamma\left(\frac{s}{T}\right)\right)\right) \exp(i\theta_B(t)) \left| n\left(\gamma\left(\frac{t}{T}\right)\right) \right\rangle \quad (2.53)$$

is to be a solution of 2.50, then by replacing into Schrödinger's equation we get

$$\dot{\theta}_B = i \langle n(\gamma(t)) | \nabla n(\gamma(t)) \rangle \cdot \dot{\gamma}(t) \quad (2.54)$$

which we can integrate to obtain

$$\Delta\theta_B = \oint_C i \langle n(\mathbf{R}) | \nabla n(\mathbf{R}) \rangle \cdot d\mathbf{R} \quad (2.55)$$

where  $\Delta\theta_B$  is the total change of Berry's phase along the closed loop  $C$ . Hence, that change may be nonzero.  $\Delta\theta_B$  can also be calculated using Stokes theorem, to get

$$\Delta\theta_B = \int \int_{\Omega} \nabla \times i \langle n(\mathbf{R}) | \nabla n(\mathbf{R}) \rangle \cdot d\mathbf{S} \quad (2.56)$$

where  $\Omega$  is the region enclosed by  $C$ .

That situation reminds us of something that happens in General Relativity. If you start with some tangent vector over a point in a manifold and take its parallel transport along some closed curve, the final vector may very well be different from the initial one. In fact, following this link, we can read from Equation (2.55) the vector field  $\mathbf{A} = i \langle n(\mathbf{R}) | \nabla n(\mathbf{R}) \rangle$  is actually a connection on Berry's bundle,  $\mathbf{F} = \nabla \times i \langle n(\mathbf{R}) | \nabla n(\mathbf{R}) \rangle$  is its curvature and hence if we take  $\Omega$  to be the entire manifold  $M$ , then  $\Delta\theta_B$  is the Chern number of Berry's bundle.

## 2.5.2 TKNN

A special example of that situation described above is a gas of electrons constrained to a two-dimensional crystal in the presence of a magnetic field perpendicular to the plane of the crystal. As described in detail in references [15] and [25], magnetic translations play the role of the usual translations as long as the flux of the magnetic field through a unit cell is a rational number, in appropriate units. Alternative formulations

of the same idea using projective modules can be found in the Short Communication "Quantum Hall Conductivity and Topological Invariants" of reference [6]. Then, we can use Bloch's theory to construct a Brillouin Zone made of quasimomenta.

On one hand, if we consider a rectangular lattice, these quasimomenta have a geometrical structure, they are a two-torus. On the other hand, the quasimomenta can be thought of as being the parameters of the Hamiltonian of the system. Hence, we can build a Berry's bundle over this two-torus as described in the previous section. But what's more important in this case is that Equation 1.34 can be used to show that  $\sigma_H = \Delta\theta_B$ . By the last remarks in the previous section, this means  $\sigma_H$  is the Chern character of Berry's bundle and hence it is a topological invariant that admits only a discrete set of values.

# Chapter 3

## Noncommutative geometry

### 3.1 Noncommutative spaces

A  $C^*$ -algebra is a Banach algebra over  $\mathbb{C}$  with an involution  $*$ , such that the so-called " $C^*$  property" holds:

$$\|a^*a\| = \|a\|^2. \tag{3.1}$$

The main example of a  $C^*$ -algebra is the algebra of continuous,  $\mathbb{C}$ -valued functions with compact support defined on a locally compact topological space  $X$ , usually denoted  $C_0(X)$ . In that case the involution is given by complex conjugation and the norm is the supremum norm. The  $C^*$  property holds trivially in this case. Notice that this is a commutative algebra with unit, which is given by the constant function 1. It turns out that this is, in a sense, the only example. Indeed, the celebrated Gelfand-Naimark theorem states that every commutative algebra is isomorphic to  $C_0(X)$  for some locally compact topological space  $X$ . Moreover, the isomorphism is a continuous map of Banach algebras and preserves the involution.

Now, if the commutative  $C^*$ -algebras are algebras of functions for some space, what are noncommutative  $C^*$ -algebras? We can think of them as being the algebra of functions of something, though that something is not any topological or algebraic structure we are familiar with. That something is what we call a noncommutative space. Of course, this is not a very precise definition, nor one we could actually use to prove anything. It is just the idea that we should keep in mind when we work in this subject and it will suffice for the scope of this document. Most of the material in this section can be found in [\[13\]](#).

#### 3.1.1 Noncommutative quotients

One of the main reasons that motivated the appearance of the concept of noncommutative space was the study of quotients of topological spaces by the actions of a group.

Indeed, suppose a group  $G$  acts on a space  $X$  in such a way that one of the orbits is dense in  $X$ . The set of orbits might still be fairly rich, yet the topology of the quotient space will be indiscrete. That means we cannot recover the information about the orbits of the action by topological methods, neither by looking directly at the space nor by thinking about the algebra of functions defined on it. However, there is a way to build a  $C^*$ -algebra that still captures this missing information. We will explore this construction in detail in the following, but for now let us just say that it is not a commutative algebra, so it is associated to a noncommutative space. It is worth mentioning that when the quotient space is nice, this algebra is not isomorphic to the algebra of continuous functions on the quotient, but can be shown to contain most of the valuable information encoded in it.

First of all, recall that a *category*  $\mathcal{C}$  is a set of objects,  $Obj(\mathcal{C})$ , together with a set of directed arrows between them, called the morphisms of  $\mathcal{C}$  and denoted  $Hom(\mathcal{C})$ . For each morphism  $f \in Hom(\mathcal{C})$ , we can define its source,  $s(f)$ , as the object where it starts and its target,  $t(f)$ , as the object where it ends. These morphisms are endowed with a composition operation,  $\circ$ , that takes two morphisms  $f_1, f_2$  such that  $t(f_1) = s(f_2)$  and produces a third morphism  $f_3 = f_2 \circ f_1$ , with  $s(f_3) = s(f_1)$  and  $t(f_3) = t(f_2)$ . There are a couple of very natural conditions on  $\mathcal{C}$ , namely, that composition should be an associative operation and that for each  $x \in Obj(\mathcal{C})$ , there should be a morphism  $id_x$  with  $s(id_x) = t(id_x) = x$  and for every morphism  $g$  with  $t(g) = x$ , we should have  $id_x \circ g = g$  and similarly for every morphism  $h$  with  $s(h) = x$ ,  $h \circ id_x = h$ .

A groupoid is a category  $\mathcal{G}$  such that every morphism is invertible. That means that for every  $f \in Hom(\mathcal{G})$ , there exists another morphism  $g \in Hom(\mathcal{G})$  such that  $s(f) = t(g)$ ,  $s(g) = t(f)$ ,  $g \circ f = id_{s(f)}$  and  $f \circ g = id_{s(g)}$ . A groupoid is a generalization of a group. If  $Obj(\mathcal{G})$  is a set with only one point, then  $Hom(\mathcal{G})$  is a group. A topological groupoid is a groupoid where both  $Obj(\mathcal{G})$  and  $Hom(\mathcal{G})$  are topological spaces and the functions  $s : Hom(\mathcal{G}) \rightarrow Obj(\mathcal{C})$ ,  $t : Hom(\mathcal{G}) \rightarrow Obj(\mathcal{C})$  and  $i : Obj(\mathcal{C}) \rightarrow Hom(\mathcal{C})$  (this last one given by  $i(x) = id_x$ ) are all continuous. Also,  $\circ$  should be a continuous map on the subset of  $Hom(\mathcal{C}) \times Hom(\mathcal{C})$  on which it is defined.

Groupoids appear in a number of situations, but the one we are interested in now is when we have an action of a group  $G$  on a set  $X$ . Then there is a very natural groupoid associated to the action, denoted  $\mathcal{G}_{act}$ . Its objects are just the elements of  $X$  and for every  $g \in G$  such that  $g \cdot x = y$ , there is a morphism  $f_g$  with  $s(f_g) = x$  and  $t(f_g) = y$ . If  $G$  is a topological group,  $X$  a topological space and the action of  $G$  on  $X$  is continuous, the groupoid is naturally topological, since in this case the set of morphisms of  $\mathcal{G}_{act}$  can be identified with  $X \times G$ .

Now, there is an algebra naturally associated to every groupoid  $\mathcal{G}$ , called the groupoid algebra and denoted  $C\mathcal{G}$ . As a vector space, is the sum of a copy of  $\mathbb{C}$  for every  $f \in$

$Hom(\mathcal{G})$ :

$$C\mathcal{G} = \bigoplus_{f \in Hom(\mathcal{G})} \mathbb{C}f \quad (3.2)$$

Its multiplicative structure is given by the rule

$$f_1 \cdot f_2 = \begin{cases} f_1 \circ f_2, & t(f_2) = s(f_1) \\ 0, & \text{else} \end{cases} \quad (3.3)$$

Moreover, the algebra has an involution:

$$\left( \sum_{f \in Hom(\mathcal{G})} a_f f \right)^* = \sum_{f \in Hom(\mathcal{G})} \bar{a}_f f^{-1}. \quad (3.4)$$

Although this definition is very intuitive, it is hard to build an appropriate version of it for topological groupoids. But notice we can equivalently think of the groupoid algebra as the set of functions from  $Hom(\mathcal{C})$  to  $\mathbb{C}$  with finite support:

$$C\mathcal{G} = \{F : Hom(\mathcal{C}) \rightarrow \mathbb{C} \text{ with finite support}\}$$

But the multiplication cannot be given by the usual multiplication of functions, since that would make  $C\mathcal{G}$  a commutative algebra, regardless of the structure of  $\mathcal{G}$ . The operation equivalent to the one defined above is the convolution product

$$(H \cdot G)(f) = \sum_{f_1, f_2, f_1 \circ f_2 = f} H(f_1)G(f_2) = \sum_{k \in t^{-1}(s(f))} H(k)G(k^{-1}f) \quad (3.5)$$

The involution in this case is given by

$$H^*(f) = \overline{H(f^{-1})} \quad (3.6)$$

For topological groupoids, we can consider a more general construction. If  $Hom(\mathcal{G})$  is a locally compact space, we can carry on a construction similar to the Haar measure for locally compact groups. In this case, we obtain a "nice" collection of measures  $\mu^x$ , one for every element  $x \in Obj(\mathcal{G})$ , defined on the sets  $t^{-1}(x)$ . Armed with those measures, we can define the groupoid algebra as the set

$$C\mathcal{G} = \{F : Hom(\mathcal{G}) \rightarrow \mathbb{C} \text{ with compact support}\}$$

with its natural structure of vector space, multiplication given by

$$(H \cdot G)(f) = \int_{t^{-1}(s(f))} d\mu^x(k) H(k) G(k^{-1}f) \quad (3.7)$$

and the same involution as in the discrete case.

In the previous section, we said noncommutative spaces are given by noncommutative  $C^*$ -algebras. So far, given a continuous action of a locally compact group on a space, we have defined the associated topological groupoid and its groupoid algebra. So, in order to build a noncommutative space from the action, we could use the groupoid algebra to build a  $C^*$ -algebra. That reduces to give  $C\mathcal{G}$  a norm for which the  $C^*$  property holds. Then we could take the completion of  $C\mathcal{G}$  with respect to that norm and we would be done.

The definition of the norm is associated to a very important notion, that can be defined for any locally compact topological groupoid  $\mathcal{G}$ : the left regular representation of a groupoid algebra. For every  $x \in \text{Obj}(\mathcal{C})$ , consider the Hilbert space  $\mathcal{H}_x = L^2(t^{-1}(x), \mu^x)$ . Then for  $F \in C\mathcal{G}$ ,  $G \in \mathcal{H}_x$ , the convolution product

$$(F \cdot G)(f) = \int_{t^{-1}(s(f))} d\mu^x(k) F(k) G(k^{-1}f) \quad (3.8)$$

defines a representation  $\pi_x : C\mathcal{G} \rightarrow \mathcal{B}(\mathcal{H})$ , where  $\mathcal{B}(\mathcal{H})$  denotes the set of bounded operators on  $\mathcal{H}$ . That is what we call the left regular representation of the groupoid. We can now define a norm of an element  $\phi \in C\mathcal{G}$  by the equation

$$\|\phi\| = \sup_{x \in X} \|\pi_x(\phi)\| \quad (3.9)$$

As we said before, this endows  $C\mathcal{G}$  with a nice norm, so its completion will be the  $C^*$ -algebra we are looking for.

### Information Encoded in $\mathcal{G}_{act}$

Finally, although the construction of the  $C^*$ -algebra associated to a quotient seems to be a fairly natural construction, we haven't discussed exactly what information about the quotient is encoded in this algebra. The answer is simple: most of it. Suppose first that we start with a "nice action", that is, one for which the quotient space is Hausdorff and locally compact. Then all of the information about the orbits of the action can be obtained from the topology of that quotient space. And since, on its turn, the topology of a compact Hausdorff space is completely characterized by its algebra of continuous functions, all of the information about the action can be retrieved from the algebra of

functions on the quotient. In that case, we have:

**Theorem 2.** *Let  $X$  be a locally compact Hausdorff space,  $G$  a locally compact group and suppose we have a continuous action of  $G$  over  $X$  that is free and proper. Then  $C_0(X/G)$  is Morita equivalent to  $CG$ , which is isomorphic to  $C(X) \rtimes G$ .*

Two algebras are Morita equivalent if their respective categories of right modules are equivalent. In other words, if the two algebras have basically the same representations. Although its collection of representations doesn't characterize an algebra, it does characterize its  $K$ -theory. We will discuss the meaning of the  $K$ -theory of an algebra in the following section, but the important point here is that, as we saw in Chapter 2, the Chern number of a vector bundle only depends on its class in  $K$ -theory. Since all we care about in the end is the Chern number, then for all our purposes two Morita equivalent algebras are indistinguishable.

As for the semidirect product, it is defined whenever we have an action  $G$  on an algebra  $A$ , as is the case when we have an action of  $G$  on a space  $X$ , taking  $A$  to be  $C(X)$ , the set of continuous functions on  $X$ . It is an algebra with vector space  $A \otimes CG$  (here  $CG$  is the groupoid algebra of  $G$ , thought of as a groupoid with a single object) and with product

$$(a \otimes g) \cdot (b \otimes h) = a \cdot g(b) \otimes g \cdot h \quad (3.10)$$

Another way to think about  $A \rtimes G$  is that, if  $\pi : G \rightarrow \text{Aut}(A)$  denotes the action, it is the universal algebra generated by  $A$  and  $G$  subject to the relations

$$\pi(g)(a) = g^{-1}ag \quad (3.11)$$

for all  $a \in A$  and all  $g \in G$ .

Now, what happens if the action is not a "nice" one? Then conditions on Theorem 2 are not satisfied and the conclusion doesn't hold. But that is actually a good thing for us. For instance, if we are in the context discussed at the beginning of the section, where we have a dense orbit, then the topology of the quotient is indiscrete and the algebra of functions is just the set of constant functions, so the information about the action can't be recovered from it. On the other hand, the semidirect product still makes sense, and there is no reason for it to become trivial. So it replaces the algebra of functions as our source of information about the quotient.

## 3.2 Constructions Associated to a Noncommutative Space

Most of the constructions that make sense for traditional topological spaces carry over to noncommutative ones or have very close analogs. Specifically, it makes sense to construct the K-theory of a noncommutative space and its dual, K-homology. Also, we can define the Cyclic cohomology of a  $C^*$ -algebra, which plays a similar role to the De Rham cohomology of a space. Finally, just as the traditional Chern character can be thought of as a map from K-theory to de Rham cohomology, we can define a Connes-Chern map from K-homology to Cyclic cohomology.

### 3.2.1 K-theory and K-homology

Let us start by talking about K-theory. In the topological setting, K-theory is built from vector bundles over the space, using a Grothendieck construction. Now, when we think about noncommutative spaces we don't have access to the space itself but to the  $C^*$ -algebra, which is supposed to be the algebra of functions defined on it. So, in order to talk about vector bundles of noncommutative spaces, we need first to formulate the idea of vector bundles for ordinary topological spaces in terms of their algebra of functions. Fortunately, the content of the famous Serre-Swan theorem does just that. It says that for a compact Hausdorff space  $X$ , vector bundles on  $X$  correspond exactly to finitely generated, projective modules on  $C(X)$ . So, given a  $C^*$ -algebra with unit  $A$ , we can define a vector bundle over it as a finitely generated projective module over  $A$ . The collection of such modules is a semigroup with the direct sum as the operation, and the Grothendieck group of that semigroup is what we call the K-theory of  $A$ , denoted as  $K_0(A)$ .

The next construction would be K-homology, which is the dual of K-theory. That is: the K-homology of a  $C^*$ -algebra, which we write as  $K^0(A)$ , is the set of homomorphisms from  $K_0(A)$  to  $\mathbb{Z}$ . Although this is a perfectly good definition, both in the topological and the noncommutative setting, Atiyah proved that we can characterize  $K^0(A)$  in another way that will be key to us later. We can understand the elements of  $K^0(A)$  as *Fredholm modules* on  $A$ .

There are two kinds of Fredholm modules: even and odd ones. An odd Fredholm module over an algebra  $A$  is a pair  $(H, F)$  where  $H$  is a Hilbert space endowed with a representation

$$\pi : A \rightarrow \mathcal{B}(H)$$

( $\mathcal{B}(H)$  denotes the set of bounded operators on  $H$ ) and  $F \in \mathcal{B}(H)$  is a self-adjoint operator such that  $F^2 = I$  and for all  $a \in A$ , the operator  $[F, \pi(a)]$  is a compact operator on  $H$ .

An even Fredholm operator over  $A$  is a triple  $(H, F, G)$  such that  $(H, F)$  is an odd Fredholm module over  $A$  and  $G$  is a bounded self-adjoint operator on  $H$  with the following properties:

$$G^2 = I \quad FG = -GF \quad \pi(a)G = G\pi(a) \quad (3.12)$$

The last equation must hold for all  $a \in A$ .

As we said before, the collection of all Fredholm modules over  $A$  can be identified with the K-homology of  $A$ . Although the proof of this fact is beyond the scope of this book, we can at least define the pairing between a module over  $A$  and a Fredholm module, in order to make this identification plausible. We will do this only for even Fredholm modules, the case that will prove to be useful to us later.

Recall first that a projective module is associated to an idempotent  $e \in M_n(A)$ . On the other hand,  $G$  induces a decomposition  $H = H^+ \oplus H^-$ , where  $H^\pm$  is the eigenspace of  $G$  corresponding to the eigenvalue  $\pm 1$ . Let  $F_e^+$  denote the operator  $eFe$ , which is a map from  $eH^+$  to  $eH^-$ . It can be shown to be a compact operator, so both its kernel and cokernel have finite dimension. An operator  $T$  for which this later condition holds is called a Fredholm operator and we can define its index:

$$\text{ind}(T) = \dim(\ker(T)) - \dim(\text{coker}(T))$$

which is clearly an integer. So, we can define the pairing between an even Fredholm module  $(H, F, G)$  over  $A$  and a finitely generated projective module over  $A$  represented by the idempotent  $e$ , as:

$$\langle (H, F, G), e \rangle = \text{ind}(F_e^+). \quad (3.13)$$

In conclusion, for a  $C^*$ -algebra with unit  $A$ , we can think of finitely generated projective modules of  $A$  as vector bundle over the underlying space (topological or noncommutative) and hence define  $K_0(A)$  as the Grothendieck construction of the semigroup of those modules. On the other hand, we can think of  $K^0(A)$  as the collection of Fredholm modules over  $A$ . For even Fredholm modules, the pairing between K-theory and K-homology is given by the index of a Fredholm operator.

### 3.2.2 Cyclic cohomology and the Connes-Chern maps

The construction that plays the role of de Rham cohomology for noncommutative spaces is cyclic cohomology. In order to define it, we need to consider first Hochschild cohomology. Given a  $C^*$ -algebra  $A$ , define  $C^0(A) = A^*$  and  $C^n(A) = \text{hom}(A^{\otimes(n+1)}, \mathbb{C})$

for  $n \geq 1$ . We can turn this collection of algebras into a co-chain complex, called the Hochschild complex, by defining a boundary map  $\delta : C^n(A) \rightarrow C^{n+1}(A)$  according to

$$\begin{aligned} (\delta f)(a_0, a_1, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^{i+1} f(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &\quad (-1)^{n+1} f(a_{n+1} a_1, \dots, a_n) \end{aligned} \quad n \geq 1$$

It can be easily checked that  $\delta^2 = 0$ , so this is indeed a co-chain complex. The cohomology of the complex  $(C^*(A), \delta)$  is the Hochschild cohomology of  $A$ ,  $HH^*(A)$ .

This co-chain complex admits a very important subcomplex, the complex of cyclic co-chains. An element  $\varphi \in C^n(A)$  is called cyclic if

$$\varphi(a_0, a_2, \dots, a_n) = (-1)^n \varphi(a_n, a_0, \dots, a_{n-1}) \quad (3.14)$$

It is easy to check that if  $\varphi \in C^n$  is a cyclic  $n$ -co-chain, then  $\delta\varphi$  is a cyclic  $(n+1)$ -co-chain, so cyclic co-chains indeed define a subcomplex of Hochschild complex. The cohomology of this subcomplex is called the cyclic cohomology of  $A$ , denoted  $HC^n(A)$ .

Finally, there is a very important map from K-homology to cyclic cohomology, called the Connes-Chern map. As we said previously, the K-homology of an algebra can be identified with the set of Fredholm modules over it. The definition is slightly different for even and odd modules and we will discuss only the one former, since it will be the one useful to us later on. Let  $(H, F, G)$  be an even Fredholm module. Let  $\varphi_{2n}$  be the element of  $C^{2n}(A)$  defined by the equation

$$Ch^{2n}(a_0, a_1, \dots, a_{2n}) = \frac{(-1)^n n!}{2} Tr(GF[F, a_0] \dots [F, a_{2n}]) \quad (3.15)$$

The expression in the last equation can be checked to be cyclic and in the kernel of the map  $\delta$ , so it induces a map from  $K^0(A)$  to  $HC^{2n}(A)$ .

### 3.2.3 Compact and Fredholm operators and Connes index formula

Finally, the most important result of this section is an index formula due to Connes. It relates the value of the pairing between K-theory and K-homology to the Connes-Chern character. We will state it as a theorem for future reference but before we do it, we need to talk a little about Fredholm and compact operators.

Let  $H$  be a separable Hilbert space and  $\mathcal{B}(H)$  the set of bounded operators on it. An operator  $T \in \mathcal{B}(H)$  is said to have finite rank if its range is finite dimensional. The

set of finite rank operators is denoted  $\mathcal{F}(H)$ . Now the set of compact operators on  $H$ ,  $\mathcal{K}(H)$  is defined as the closure of  $\mathcal{F}(H)$  under the operator norm.

Compact operators have a very special spectrum. It is always a discrete set, with no accumulation points other than possibly zero. Every point in the spectrum is an eigenvalue and they all have finite dimensional eigenspaces. Hence, for a compact operator  $T$ , it makes sense to enumerate the eigenvalues of the operator  $|T| = (T^*T)^{\frac{1}{2}}$  as

$$\mu_1(T) \leq \mu_2(T) \leq \dots \leq \mu_n(T) \leq \dots$$

The operator  $T$  is said to be of trace class if

$$\sum_{i=0}^{\infty} \mu_i(T) < \infty \quad (3.16)$$

We can define the trace of a trace class operator. Let  $\{e_n\}_{n=0}^{\infty}$  a basis for  $H$ . If  $T$  is trace class, the sum

$$Tr(T) = \sum_{i=0}^{\infty} \langle Te_i, e_i \rangle \quad (3.17)$$

can be proven to be infinite and independent of the basis. Moreover, if  $\{\lambda_n\}_{n=0}^{\infty}$  are the eigenvalues of  $T$  counted with multiplicity, then we have the usual formula

$$Tr(T) = \sum_{n=0}^{\infty} \lambda_n. \quad (3.18)$$

We are interested in compact operators because Fredholm operators are invertible up to compact operators. That means that if  $P$  is a Fredholm operator, there exists a bounded operator  $S$ , called a *parametrix* of  $P$ , such that both  $I - SP$  and  $I - PS$  are compact operators. Now, since we know they are compact, we could ask whether  $I - SP$  and  $I - PS$  are trace class or not. If they are, we say  $P$  is *summable*. The set of summable operators is denoted  $\mathcal{L}(H)$ .

A compact operator  $T$  is said to be  $p$ -summable if  $|T|^p \in \mathcal{L}(H)$ . The set of  $p$ -summable operators is denoted  $\mathcal{L}^p(H)$  and can be proven to be a double sided ideal of  $\mathcal{B}(H)$ .

Finally, we come back to noncommutative geometry. Recall that an even Fredholm module over the algebra  $A$  is a triple  $(H, F, G)$  where  $H$  is a separable Hilbert space,  $F$  is a bounded operator on  $H$  and we have a representation  $\pi$  from  $A$  to  $\mathcal{B}(H)$  with the property that  $[F, \pi(a)]$  is compact for all  $a \in A$ . An element  $a$  of the algebra  $A$  is said to be  $p$ -summable if  $[F, \pi(a)]$  is  $p$ -summable. For  $p$ -summable elements of  $A$  we have the following remarkable theorem by Connes. Both its hypothesis and its conclusion will

play a key role in our understanding of the Integer Quantum Hall Effect, as we shall see in the next chapter.

**Theorem 3.** *Let  $e$  be an idempotent on  $A$  that is  $p$ -summable and let  $n$  be a positive integer such that  $2n + 1 > p$ . Then*

$$\langle (H, F, G), e \rangle = \frac{1}{n!} Ch^{2n}(e, e, \dots, e) \quad (3.19)$$

## Chapter 4

# The Noncommutative Quantum Hall Effect

### 4.1 Why would we want to revisit the Quantum Hall Effect?

Before we start discussing the noncommutative formalism used to describe the Quantum Hall Effect, it is worth asking ourselves why is it necessary to do it. After all, in Section 2.5 we already studied a mathematical model of this phenomenon that explained the quantization of the Hall conductance and its robustness, by relating the conductivity to a topological invariant.

The fact is that the model we studied before, the TKNN model, has several unpleasant features. First of all, the use of a topological invariant implies that we have a topological space to begin with. In the classical TKNN setting, that space is the First Brillouin Zone, which can be identified with a topological torus. But the notion of a Brillouin space only makes sense when we have a perfectly periodic potential. Of course, that is never the case for realistic materials, specially because of the abundance of impurities in the crystals.

Another defect of the TKNN treatment is that it seems to suggest that the Hall conductivity always assumes quantized values, even near the Landau levels, when that is actually not the case. Moreover, it says nothing about the existence of plateaus in the conductivity. Although it ensures that the values have to be discrete, they very well could change discontinuously, jumping from one value to the other. We would like to have a formalism that ensures the existence of plateaus and gives a precise condition necessary for the quantization of the conductivity. Then we should check that such condition doesn't hold at the Landau levels.

Finally, we would like more physical information about the Quantum Hall Effect. After all, saying that the Hall conductivity is quantized because it's related to the Chern character of a vector bundle is a rather esoteric statement. No physical process is being described as the one responsible for the observed properties of the system, it seems like it's just a mathematical curiosity of the model being used.

## 4.2 The Noncommutative First Brillouin Zone

Let us discuss the first issue: the fact that in the presence of impurities and disorder, there is no longer a clear notion of a Brillouin Space. The idea is to replace it by a noncommutative space. In order to build this space, we need to take into account the fact that the noncommutative treatment of this system should be a generalization of the ideas developed by TKNN. In particular, if the potential happened to be periodic, the noncommutative space should correspond in some way to the First Brillouin zone. So, a good strategy is to start with the Brillouin space and reformulate it in such a way that it can be easily generalized to the noncommutative framework. Recall that the First Brillouin Zone is usually defined as the Wigner-Seitz cell of the reciprocal lattice. This is a very geometrical notion, which is a problem because noncommutative spaces are defined in algebraic terms.

So, we need an algebraic definition of it. It is not hard to come up with a first attempt to do this. Generally speaking, for an arbitrary regular 2D lattice it is possible to give a more algebraic description of the Wigner-Seitz cell. Indeed,  $\mathbb{R}^2$  can be thought to be a group  $\mathbb{R}_{tr}^2$  that acts on  $\mathbb{R}^2$  as a topological space,  $\mathbb{R}_{top}^2$  by translations. We can also think of the regular lattice as being both a subgroup of  $\mathbb{R}_{tr}^2$  isomorphic to  $\mathbb{Z}^2$ , which we denote as  $\mathbb{Z}_{lat, tr}^2$  and as a subspace  $\mathbb{Z}_{lat, top}^2$  of  $\mathbb{R}_{top}^2$ . It is not hard to see that the Wigner-Seitz cell can be built in the following way:

1. Fix a point  $p \in \mathbb{R}_{top}^2$
2. Calculate its orbit  $O_{\mathbb{R}_{tr}^2}(p)$
3. Take the quotient of the orbit by the subspace  $O_{\mathbb{R}_{tr}^2} \cap \mathbb{Z}_{lat, top}^2$

Now, for a perfectly periodic crystal this would be a very silly procedure, since it is clear that it would just generate the quotient  $\mathbb{R}_{top}^2 / \mathbb{Z}_{lat, top}^2$ . At least it is not wrong: it is also easy to see that the Wigner-Seitz cell should be given by that quotient. The virtue of the procedure lies in the fact that it can be imitated, in a way that we are going to make precise, in a more general context.

A "more general context" has a specific meaning here: the presence of a magnetic field. The effect of the magnetic field is that it no longer makes sense to consider translations. Indeed, the reason why usually study them is because the Hamiltonian of a regular lattice in the absence of a magnetic field commutes with some translations, the ones associated to vectors in the regular lattice. But this is no longer the case when  $B \neq 0$ , since now the dynamic part of the Hamiltonian is  $(\mathbf{P} + e\mathbf{A})^2$ , so it is not guaranteed to commute the translations  $e^{i\mathbf{a}\cdot\mathbf{P}}$ . This is not a very serious problem, though: we can now consider the magnetic translations considered in previous sections, which are given by  $T_{\mathbf{a}}^{mag} = e^{i\mathbf{a}\cdot(\mathbf{P}+e\mathbf{A})}$ . Clearly, they commute with the dynamic part of the Hamiltonian.

Notice two things about this magnetic translations. First of all, they cannot be seen as homeomorphisms of  $\mathbb{R}_{top}^2$  as the usual translations, they are just operators on  $\mathcal{H} = L^2(\mathbb{R}^2)$ . Second, they don't give raise to an action of  $\mathbb{R}^2$ , since they don't commute, while  $\mathbb{R}^2$  is abelian. They do give raise to an action of  $\mathbb{R}^2$  on the set of linear operators on  $\mathcal{H}$  by conjugation:

$$\tilde{T}_{\mathbf{a}}^{mag} O = (T_{\mathbf{a}}^{mag})^\dagger O (T_{\mathbf{a}}^{mag}) \quad (4.1)$$

If we wanted to emulate the process stated at the beginning of the section, we already have the action of  $\mathbb{R}^2$ . We need a distinguished point but choosing it is not hard: it should be the Hamiltonian since its conjugations by unitary operators give us information about symmetries of the system. In the second step, however, it is necessary to take the closure of the orbit rather than just the orbit, which we call  $\Omega$ . This is because we want to obtain a compact space. The third step is unnecessary, since that quotient will be taken automatically because if  $H$  commutes with a magnetic translation it will be a fixed point of the action by that translation.

In conclusion, even if there is disorder in the crystal that ruins the periodicity of the potential, it is possible to define a Brillouin Zone, although it will be a noncommutative space. To be precise, it will be the  $C^*$ -algebra built from the groupoid algebra of the groupoid associated to the action of  $\mathbb{R}_{tr}^2$  on  $\Omega$ .

### 4.3 Quantization in the Noncommutative setting

Replacing the classical Brillouin space by a noncommutative analog means that we can study properly the conductivity in aperiodic media. However, there is a serious set back: we need to derive again the quantization of the conductivity in a different way. Indeed, in TKNN, the argument that explained it relayed heavily on the geometrical nature of the Brillouin zone. We needed to build a vector bundle on that space and then evaluate its Chern character. Now that we don't have a traditional space, we need to adapt the argument to this noncommutative setting.

However, this is not a very serious problem. As we saw in the last chapter, all constructions involved in the classical TKNN argument have noncommutative analogs. First of all, vector bundles are replaced by idempotents of the  $C^*$ -algebra that represents the noncommutative space. To be more precise, in Section 2.5 we discussed how Berry's bundle for TKNN,  $E_B \rightarrow X_B$ , is completely characterized by the projection onto the lowest eigenstate,  $P_E$ . In particular, the Chern character can be calculated as  $Ch(E_B) = Tr(P_E dP_E dP_E)$ . Since the projection  $P_E$  also exists in the noncommutative setting, we have our analog of Berry's bundle.

As for the Chern character, we also discussed in the last chapter how it is built in the noncommutative setting. Equation (3.15) tells us that the noncommutative Chern character is indeed a generalization of the commutative one and that it still gives us the right physical information. Notice however that it is a map from  $K$ -homology to cyclic cohomology of the noncommutative space. So we need to choose first an element in the  $K$ -homology of the space, apply the Chern character and only then can we evaluate it on  $P_E$  to get a number.

Let us discuss the choice of the element in  $K$ -homology. This corresponds to selecting a Fredholm module over the  $C^*$ -algebra. We will choose an even one, which should be a triple  $(H, F, G)$ . For our particular case, we discussed in the previous section that there is a natural representation from the  $C^*$ -algebra to the Hilbert space  $L^2(\mathbb{R}^2)$ . Let us take then

$$H = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2) \quad (4.2)$$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.3)$$

It remains to choose the Fredholm operator  $F$ . Since we are working with  $H = L^2(\mathbb{R}^2)$ , we have operators  $X_1$  and  $X_2$ , so it is natural to define

$$u = \frac{X_1 + iX_2}{|X_1 + iX_2|} \quad (4.4)$$

$$F = \begin{pmatrix} 0 & u \\ u^\dagger & 0 \end{pmatrix}. \quad (4.5)$$

Finally, we can discuss the quantization of the Hall conductivity. It is a direct consequence of Theorem 3. Indeed, the right hand side of Equation 3.19 gives us the transverse conductivity of the system, whereas the left hand side of the equation is the index of an operator, which is obviously an integer. But perhaps the most important thing here is not so much the conclusion of the theorem, but rather its hypothesis. Indeed, in order for Equation (3.19) to hold, we need to check first that  $P_E$  is  $p$ -summable for some  $p$ . Far from being a technicality, this is a crucial point. In the next section we will discuss how this condition can be translated in terms of a localization condition.

## 4.4 Disorder, Localization and the Spectrum of the Hamiltonian

Let us take a step back for a moment. At the beginning of this chapter, we discussed how the main motivation to develop an alternative treatment of the Quantum Hall effect was to include the effects that the disorder in the crystal might have on its electrical properties. The presence of disorder will also be related to whether or not  $P$  is summable, but in an indirect way. The precise mechanism involved is an important notion in the study of disordered crystals: Anderson Localization.

### 4.4.1 Anderson Localization

In order to understand Anderson localization, we need to discuss first *discrete lattice models* of crystals. Some features of the behavior of electrons in crystals can be understood by imagining that they move on a discrete, regular lattice, rather than a continuous media. We can think of such a lattice in two dimensions as  $\mathbb{Z}^2$ , and the electrons as being represented by wave functions  $\psi : \mathbb{Z}^2 \rightarrow \mathbb{C}$ . The Hilbert space is then  $\mathcal{H} = L^2(\mathbb{Z}^2)$  and the Hamiltonian can be thought of as an appropriate collection of functions

$$H_{(i,j),(k,l)} : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{C}$$

$$(H\psi)(r, s) = \sum_{(m,n)} H_{(r,s),(m,n)} \psi(m, n)$$

.

The potential can also be thought of a collection of functions  $V_{(i,j),(k,l)} : \mathbb{Z}^2 \times \mathbb{Z}^2 \rightarrow \mathbb{C}$ . For instance, in a first neighbors model we should have the property that the only nonzero entries of  $V$  should be the ones of the form  $V_{(i,j),(i+1,j)}$ ,  $V_{(i,j),(i-1,j)}$ ,  $V_{(i,j),(i,j+1)}$  and  $V_{(i,j),(i,j-1)}$ . The Schrödinger equation is the discrete analogue of the traditional one:

$$i\hbar\dot{\psi}(i, j) = (H\psi)(i, j) = \sum_{(k,r)} E_{(i,j),(k,r)} \psi(k, r) + V_{(i,j),(k,r)} \psi(k, r)$$

This discrete model also admits the construction of a non commutative First Brillouin zone....

Discrete models of crystals have many interesting features that have place them at the center of theoretical studies in Solid State Physics. One of the most important ones is Anderson localization. Indeed, for such a model, Anderson [1] showed an impressive

result. He considered a first neighbors model for which the nonzero entries are exactly one. Only the diagonal entries  $E_{(i,j),(i,j)}$  are nonzero, but they are not traditional functions. Instead, they are random variables, uniformly distributed in  $[-W, W]$ . This randomness is supposed to model the effects of disorder in the crystal. He found that if the initial state of an electron was completely localized at the origin of the lattice, then for any future time  $t$ , the expected value of the distance between the position of the particle at time  $t$  and the origin is bounded by some constant  $C$ . That is:

$$\sum_{(i,j)} |\psi(i,j)|^2 \|(i,j)\| < C \quad (4.6)$$

Anderson localization gives us a qualitative understanding of the vanishing of the longitudinal conductivity. If the wave functions of the electrons are localized, it is impossible for them to carry any current from one border of the material to the other, so the conductivity should vanish. Also, although we will not use it in the future, it is worth mentioning that Anderson localization is a very general phenomenon, not restricted neither in the particular distribution of the  $E_{(i,i)}$ , nor in the particular form of  $V$  (we just need it to decay fast enough with respect to the distance from the origin) or even the dimension of the system.

#### 4.4.2 Other Effects of Disorder

Equation (4.6) is just one way to characterize localization of the wave function, but the same concept can be seen from another three different points of view. First of all, Equation (4.6) is only one of many criteria we could think of to describe the existence of localization. Indeed, another one could be to evaluate the expected value of the mean square average of the displacement after a time interval of  $t$  has passed by, evaluated at a state that was initially completely concentrated at the origin of the lattice  $(0, 0)$ . That is: if  $|(0, 0)\rangle$  denotes this last state, we could calculate

$$\delta X^2 = \lim_{T \rightarrow \infty} \int_0^T \frac{1}{T} \int d\mathbf{P}(\omega) \langle (0, 0) | (X_{t,\omega} - X)^2 | (0, 0) \rangle \quad (4.7)$$

where  $X_{t,\omega}$  denotes the time evolution according to  $H_\omega$ ,  $X_{t,\omega} = \pi_\omega(e^{-iHt})X\pi_\omega(e^{iHt})$ .

Second, another criteria would be to evaluate the return probability of a state originally concentrated at the origin, when time goes to infinity:

$$\xi = \lim_{T \rightarrow \infty} \int_0^T \frac{1}{T} \int d\mathbf{P}(\omega) |\langle (0, 0) | \pi_\omega(e^{iHt}) | (0, 0) \rangle|^2 \quad (4.8)$$

If we are in the presence of diffusion, the wave function would spread all over the crystal, so that probability should vanish. Hence,  $\xi > 0$  is a strong indication of localization.

Third, we saw in Chapter 1 that disorder has an effect on the spectrum of the Hamiltonian of the system. For instance, in the Shubnikov-de Haas effect, the reason why longitudinal conductance doesn't vanish when the Fermi level lays in the gaps between Landau levels is because disorder in the crystal creates a continuum of eigenvalues of the Hamiltonian in those gaps. Since disorder is also responsible for localization, there should be some kind of link between localization and the distribution of eigenvalues of the Hamiltonian near the Fermi energy.

Let us talk first about the structure of the spectrum of the Hamiltonian. Recall that one of the features of IQHE that we want to explain is the existence of plateaus in the Hall conductivity when the Fermi energy lays in a gap between Fermi energies. A plateau in a function is not a punctual property, it is a local one. Hence, the kind of question we should be asking ourselves is not whether some real number  $\lambda$  is or is not an eigenvalue of the Hamiltonian. The right kind of question is this: given an interval  $\Delta \subset \mathbb{R}$ , how many eigenvalues of the Hamiltonian, counting multiplicity, are there in  $\Delta$ ?

A first result in this line is the following:

**Lemma 1.** *Let  $P$  be a projection of the  $C^*$ -algebra. Then,  $\pi_\omega$  is a projection and,  $\mathbf{P}$ -almost surely, its range is either zero or infinite.*

*Proof.* We can calculate the range of a projection as its trace. Since the representation  $\pi$  sends elements of the  $C^*$ -algebra to operators on  $l^2(\mathbb{Z}^2)$ , which has  $|i, j\rangle$  as a basis, we can calculate this trace as

$$\dim(\text{range}(P_\omega)) = \sum_{(i,j)} \langle (i, j) | \pi_\omega(P) | (i, j) \rangle \quad (4.9)$$

Now, since all the sites of the lattice are equivalent we have

$$\langle (i, j) | \pi_\omega(P) | (i, j) \rangle = \langle (0, 0) | \pi_\omega(P) | (0, 0) \rangle \quad (4.10)$$

So,

$$\dim(\text{range}(P_\omega)) = \sum_{(i,j)} \langle (0, 0) | \pi_\omega(P) | (0, 0) \rangle \quad (4.11)$$

This last expression must be either zero or infinite. □

Lemma 1 is telling us that the answer to the question we posed before must be either zero or infinity. Which one of those two is the correct one? The answer to that is related to the return probability  $\xi$  we defined before. Actually, not exactly  $\xi$ , because it doesn't have any dependence of  $\Delta$ , but  $\xi(\Delta)$ , which is defined as the return probability when the time evolution operator is restricted to the part of the spectrum of the Hamiltonian that lays in  $\Delta$ . That is, if  $P_\Delta$  is the projection onto the eigenstates of the Hamiltonian with eigenvalues in  $\Delta$ , we define

$$\xi(\Delta) = \lim_{T \rightarrow \infty} \int_0^T \frac{1}{T} \int d\mathbf{P}(\omega) |\langle (0,0) | \pi_\omega(e^{iHt} P_\Delta) | (0,0) \rangle|^2 \quad (4.12)$$

The relation between the number of eigenvalues of the Hamiltonian in  $\Delta$  and the return probability  $\xi(\Delta)$  is contained in the following Theorem:

**Theorem 4.** *Let  $\Delta \subset \mathbb{R}$  be an open interval. Then the number of eigenvalues of the Hamiltonian in  $\Delta$  is infinite if and only if  $\xi(\Delta) > 0$ .*

*Proof.* The heart of this proof is that the return probability can be rewritten as

$$\xi(\Delta) = \int_\Omega d\mathbf{P}(\omega) \sum_{E \in \sigma(\omega) \cap \Delta} |\psi_{\omega,E}(0)|^4 \quad (4.13)$$

where  $\sigma(\omega)$  is the set of eigenvalues of  $H$  and  $\psi_{\omega,E}$  is an eigenstate of  $\pi_\omega(H)$  with eigenvalue  $E$ . Sadly, the methods used in the proof of this formula are beyond the scope of this document. Nevertheless, the theorem follows directly from equation 4.13. If  $\xi(\Delta) = 0$ , there must be no eigenvalues of  $\pi_\omega(H)$  in  $\Delta$ , because all the terms in the integrand are non negative. On the other hand, if  $\xi(\Delta) > 0$ ,  $\sigma(\omega) \cap \Delta$  can't be empty, and hence must be infinite by Lemma 1.  $\square$

Theorem 4.13 says the criteria related to the spectrum of the Hamiltonian and the return probability are equivalent. On the other hand, we have the mean square displacement, which is the indicator that seems to be more closely related to Anderson's criteria. Inspired by the definition of  $\xi(\Delta)$ , we could define  $l^2(\Delta)$  by to

$$l^2(\Delta) = \lim_{T \rightarrow \infty} \int_0^T \frac{1}{T} \int d\mathbf{P}(\omega) \langle (0,0) | (X_{t,\omega}(\Delta) - X)^2 | (0,0) \rangle \quad (4.14)$$

where  $X_{t,\omega}(\Delta) = \pi_\omega(e^{-iHt} P_\Delta) X \pi_\omega(e^{iHt} P_\Delta)$ . We call  $l^2(\Delta)$  the localization length of  $\Delta$ . The following theorem establishes the fundamental relation between localization and the quantization of Hall conductance we have been looking for:

**Theorem 5.** *Suppose  $\Delta \subset \mathbb{R}$  is an open interval such that  $l^2(\Delta) < \infty$ . Then if  $E$  is not an eigenvalue of  $H$ , the map  $P : \Delta \rightarrow A$  that sends  $E'$  to  $P_{E'}$  is continuous at  $E$ . Moreover,  $P_{E'}$  is 3-summable for all  $E' \in \Delta$ .*

We will not prove Theorem 5, again because it uses techniques beyond the scope of this document. But there are two remarks worth pointing out. On one hand, Theorem 5 implies that for  $E' \in \Delta$ , Theorem 3 will hold for  $P_{E'}$ . But there is more: the continuity of  $P$  implies that the assignment  $E \rightarrow \langle Ch(H, F, G), P_E \rangle$  will also be continuous, so the conductivity will be a continuous function of the Fermi energy. But it is also quantized, and the only continuous functions of a connected set to a discrete one are constants. Hence, the conductivity must have a plateau over  $\Delta$ . In conclusion, we have established that if the localization length of  $\Delta$  is finite, the conductivity is quantized and has a plateau over  $\Delta$ . This is exactly what we wanted to establish, so it marks the end of this document.



# Conclusions

There are three big conclusions that can be drawn from this document.

First of all, using ideas from Noncommutative Geometry we can construct a model of the IQHE that predicts the quantization of the Hall conductance without assuming that the flux of the magnetic field through a unitary cell is rational. As we discussed at the beginning of Chapter 4, such hypothesis is an important drawback of the TKNN model. But the construction of the Noncommutative First Brillouin Zone can be carried out without it. So Bellissard's model is an improvement of the TKNN one from a conceptual point of view.

Second, from a Mathematical Physics point of view, we saw that there exist several criteria that can be used to establish the existence of localization phenomena. Stated in this document are the finiteness of the mean square localization, the non-vanishing of the return probability, the existence of infinite degeneracy on the spectrum of the Hamiltonian and the finiteness of the localization length. Although we do not carry on a complete classification of these criteria, it can be shown that the criteria involving the spectrum of the Hamiltonian and the return probabilities are equivalent.

Finally, one the main results of this document: the finiteness of the localization length near Fermi Energy is a sufficient condition for the quantization of the Hall conductance. We developed all the necessary concepts to state this result in the precise form shown in Theorem 5. However, the techniques used in the proof of this fact are beyond the scope of this document.



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