STUDY OF THE BTZ BLACK HOLE

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Declaration of Authorship

I, ANDRÉS F. VARGAS, declare that this thesis titled, ‘STUDY OF THE BTZ BLACK HOLE’ and the work presented in it are my own. I confirm that:

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■ Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.

■ Where I have consulted the published work of others, this is always clearly attributed.

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■ I have acknowledged all main sources of help.

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Signed:  

Date:  
“It was an All-in-One and One-in-All of limitless being and self not merely a thing of one Space-Time continuum, but allied to the ultimate animating essence of existence’s whole unbounded sweep the last, utter sweep which has no confines and which outreaches fancy and mathematics alike”

H.P Lovecraft, Through The Gates Of The Silver Key. Weird Tales, Vol. 24, No. 1
(July 1934)
Abstract

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**English**: This work explores the lower dimensional solution of $2 + 1$ black hole first proposed by Bañados, Taitelboim and Zanelli. The first chapters deal with the general geometrical and thermodynamic characteristics of the solution. From this the Path Integral formalism is deduced and explained for the gravitational case in order to approach the thermodynamic properties of the black hole in a novel way. Finally two complete examples are made of the gravitational path integral formalism: the Schwarzschild and BTZ black holes.

**Español**: Esta tesis explora la solución de agujero negro propuesta por Bañados, Taitelboim y Zanelli en $2 + 1$ dimensiones. Los primeros capítulos tratan los aspectos geométricos y termodinámicos, desde el enfoque geométrico, de la solución de agujero negro BTZ. De esto se introduce el formalismo de integrales de camino para acercarnos a las características termodinámicas desde otra perspectiva más físicamente completa. Finalmente dos ejemplos de integrales de camino gravitacional son hechos: los agujeros negros de Schwarzschild y el BTZ.
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Physical Constants

Speed of Light \( c = 1 \)

Newtons gravitational constant \( G_N = 1 \)

Boltzmann constant \( k_b = 1 \)

Plank constant (reduced) \( \hbar = 1 \)
Symbols

\[(N^\perp)^2 = N^2\]  Lapse function  unity

\[N^\phi\]  Shift function  length\(^{-1}\)

\[M\]  Mass parameter  unity

\[J\]  Angular Momentum parameter  length

\[r\]  radial distance  length

\[I\]  euclidean action  length
To My Loving Family . . .
Chapter 1

Introduction

1.1 Introduction

General relativity is formulated in the differential geometry language. Where space
time (ST) is portrayed as a four dimensional manifold (3+1). To work in this manifold
we have to know the metric. This in order to formalize geometrical aspects such as:
curvature, distance between points, geodesics, etc... [1]

With this in mind, Einstein field equations are expressions constructed from the metric
(read from left to right [2]) where an astonishing number of twenty variables, ten from
the metric and ten from the energy momentum tensor, are related just by ten equations.
Therefore the theory is born with an intrinsic determination problem, as there are more
variables to solve than equations to use. A common solution to this problem is to impose
restrictions, such as symmetric considerations for example, which guarantee non trivial
solutions to the problem. Historically the first of this solutions was the spherically sym-
mometric Schawrzschild metric that modelled the ST around a spherical body in space. [2]

The mathematical complexity of the theory is a huge challenge to undertake in the di-
mensions previously mentioned. The calculations might become quite cumbersome and
as such the closed solutions are very rare.

Given this reasons there has been a growing interest for gravitation in lower dimensions.
Specially after the seminar works of Deser, Jackiw and ’t Hooft [3, 4]). This works
prove that all the rich physical structures present in the (3+1) theory are also present
in the lower dimensional version of (2+1), where a space component has been removed.
Chapter 1. *Introduction*

It can be imagined as a plane plus a time dimension.

General relativity in (2+1) has become a very popular model for predicting physics given that the calculations are easier but not trivial as it was original thought. A very interesting case of this, which has been recently discovered, is the existence of black holes in this dimensions [5] with the same overall characteristics of their (3+1) counterparts: Mass, Charge and Angular Momentum. Being the black holes the paradigm structure in the intent to unite quantum theory and general relativity, it is a very lucky event that they also exist in this fewer dimensions. With fewer dimensions the calculations are considerable shorter and can become a platform to generalize the fundamental physics there present to the (3+1) case.

The black hole in (2+1) [5] posses very similar properties as the usual ones in higher dimensions. For instance, both posses a very well defined event horizon with the same interpretation. Plus, the thermodynamics are pretentiously similar in (2+1) and (3+1) for example: the entropy is almost copied from one to the other. [?]  

1.2 motivation

The problem of gravitation in (2+1) is one of a very rich nature. In this dimensions the tools to develop and understand the physics there present is a combination of topology and differential geometry. By this reason, sometimes gravity in (2+1) is known as topological gravity [6]. The mathematical structures behind this theory plus the interesting physical predictions made by the theory, clearly state a solid motivation to study this problem.

Leaving the learning motivations aside, it is worth while to note that in three dimensional ST, the geometric structures there present have already been quantized [6]. Giving a first insight to the grand unification of general relativity and quantum mechanics. By this reason it is fundamental to investigate gravitation in (2+1), specially the BTZ black hole as it can en light the physical principles behind the problem of unification.
Chapter 2

Theoretical preliminaries

2.1 Action and Variation

We seek to obtain Einstein equations from a variational principle. This is using the canonical form:

\[ S = \int_M L(\Phi^i, \nabla_\mu \Phi^i) d^n x \]  \hspace{1cm} (2.1)

Given the nature of this theory the natural quantities that would appear in the action must be: a term for the geometry, another for the matter and a possible boundary term. Or mathematically:

\[ S = S_H + S_M + S_B \]  \hspace{1cm} (2.2)

Now from equation (2.1) the next step is to find the proper Lagrangian densities for each action. For the matter it clearly depends on the physical process discussed (electromagnetic, dust matter, perfect fluid, etc...) therefore for the general case it suffices just to bear in mind the term \( S_M \) in its full generality. But the story is different for \( S_H \) and \( S_B \) respectively.

For \( S_H \) accounts for the geometrical aspect of this theory it is natural to suppose a Lagrangian made from a scalar of the geometric theory. A reasonable physical supposition would be that our Lagrangian did not posses higher derivatives of the metric than second order and to be an independent scalar constructed from the metric. Naturally this conditions are full filled by the Ricci scalar. So a suitable choice for \( S_H \) would be:
\[ S_H = \frac{1}{16\pi G} \int \sqrt{-g} R \, d^4x \]  

(2.3)

Where \( g = det(g^{\mu\nu}) \).

This action is known as the Hilbert-Einstein action.

The boundary term \( S_B \) would be an integral over the boundary of the manifold \( (\partial \mathcal{M}) \).

This can be safely introduced into our action by using the Stokes theorem for an arbitrary dimension:

\[ \int_{\partial \mathcal{M}} \eta_{\mu} V^{\mu} \sqrt{\gamma} \, d^{n-1}x = \int_{\mathcal{M}} \nabla_{\mu} V^{\mu} \sqrt{g} \, d^{n}x \]  

(2.4)

Where \( \gamma = det(\gamma^{ij}) \). And \( \gamma_{ij} \) is the induced metric in the boundary.

We could follow a similar heuristic process to find this boundary term as it was done for \( S_H \). But rather than looking for a possible ansatz for this term its far more productive to variate the term \( S_H \).

As it can be clearly anticipated, the variation of \( \delta S_H \) will give an additional boundary term that can not be set to zero. For this will imply using Dirichlet and Neumann conditions, and there is no reason to require both. Therefore a much safer process to deduce Einstein equations from a variational process would be to variate the Einstein-Hilbert action and use the boundary term \((S_B)\) to eliminate any excess term giving the correct field equations. This process of manually gauging the boundary term might not seem elegant but gets the job done. Eventually the correct statement of this term will unlock very interesting physical consequences on its own.

Now well, starting with (2.3), the variation will be on the dynamical field of this theory, the metric \((g^{\mu\nu})\). Therefore \( \delta S_H \) will look like:

\[(16\pi G)\delta S_H = \int (\delta \sqrt{g})g^{\alpha\beta} R_{\alpha\beta} d^4x + \int \sqrt{g}(\delta g^{\alpha\beta})R_{\alpha\beta} d^4x + \int \sqrt{g}g^{\alpha\beta}(\delta R_{\alpha\beta})d^4x \]  

(2.5)
Our goal is to end up with terms of the form: \( \int d^4x \ldots \delta g^{\alpha \beta} \) such as the second term in (2.5). So the two other terms need to be manipulated in order to properly develop a variational principle.

Starting with the first term:

\[
\int (\delta \sqrt{g}) g^{\alpha \beta} R_{\alpha \beta} d^4x
\]

varying the term in parenthesis would look like:

\[
\delta \sqrt{g} = \frac{1}{2 \sqrt{g}} \delta g
\]

So now we have to find the variation of the \( g \) term. Although it can be done by brute force it is tremendously useful to use the following identity:

\[
\ln(|M|) = \text{Tr}(\ln M)
\]  \hspace{1cm} (2.6)

Being \( M \) a square matrix with \( |M| \neq 0 \). This identity gives the variation quite straightforward since it’s just the ”derivative” of itself. Or mathematically:

\[
\frac{\delta |M|}{|M|} = \text{Tr}(M^{-1} \delta M)
\]

So now putting \( M = g_{\alpha \beta} \):

\[
\delta g = g(g^{\alpha \beta} \delta g_{\alpha \beta}) = -g(g_{\alpha \beta} \delta g^{\alpha \beta})
\]  \hspace{1cm} (2.7)

and finally,

\[
\delta \sqrt{g} = \frac{1}{2 \sqrt{g}} \delta g = -\frac{\sqrt{g}}{2} g_{\alpha \beta} \delta g^{\alpha \beta}
\]  \hspace{1cm} (2.8)
Introducing (2.8) into (2.5) gives:

\[
(16\pi G)\delta S_H = \int d^4x\sqrt{g} \left( R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \right) \delta g^{\alpha\beta} + \int d^4x\sqrt{g}g^{\alpha\beta}(\delta R_{\alpha\beta}) \quad (2.9)
\]

In the parenthesis the Einstein tensor is obtained \((G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta})\), tensor which constitutes the left side of the field equations. Given that the right side of the field equation hold the matter term which are contained in the \(S_M\) term, the intuitive guess would be that the third term in (2.9) is in fact a boundary term. This guess is correct however it can not be made zero due to the reasons mentioned previously. Before tackling the solution to the boundary problem, a small proof that the third term is indeed a boundary one will be done.

Recalling the definition of the Ricci tensor:

\[
R_{\alpha\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\beta \Gamma^\lambda_{\alpha\lambda} + \Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{\beta\alpha} - \Gamma^\lambda_{\beta\rho} \Gamma^\rho_{\lambda\alpha} \quad (2.10)
\]

A fairly short analysis of the variation of \(R_{\alpha\beta}\) would reveal that it would depend on partial derivatives of \(\delta \Gamma\) and components of the form: \(\Gamma \delta \Gamma\) -with the correct summing indices of course. Given that \(\delta R_{\alpha\beta}\) is a tensor and \(\delta \Gamma\) is also a tensor then the partial derivative terms can not go alone. Since \(\delta R_{\alpha\beta}\) has partial derivatives of \(\delta \Gamma\) and this is itself a tensor, then a good guess, which is in fact correct, would be:

\[
\delta R_{\alpha\beta} = \nabla_\lambda \Gamma^\lambda_{\alpha\beta} - \nabla_\beta \delta \Gamma^\lambda_{\alpha\lambda} \quad (2.11)
\]

But for completeness of the work it can be shown the terms contracted via covariant derivatives are:

\[
\nabla_\lambda \delta \Gamma^\lambda_{\alpha\beta} = \partial_\lambda \delta \Gamma^\lambda_{\alpha\beta} + \Gamma^\lambda_{\lambda\rho} \delta \Gamma^\rho_{\alpha\beta} - \Gamma^\rho_{\alpha\lambda} \delta \Gamma^\lambda_{\rho\beta} - \Gamma^\rho_{\beta\lambda} \delta \Gamma^\lambda_{\rho\alpha}
\]
And,

\[-\nabla_\beta \delta \Gamma^\lambda_{\alpha \lambda} = -\partial_\beta \delta \Gamma^\lambda_{\alpha \lambda} + \delta \Gamma^\lambda_{\lambda \rho} \Gamma^\rho_{\beta \alpha}\]

Which constituted all terms in the variation of the Ricci tensor. Now making the contraction $g^{\alpha \beta} \delta R_{\alpha \beta}$:

\[g^{\alpha \beta} \delta R_{\alpha \beta} = \nabla_\lambda \left( g^{\alpha \beta} \Gamma^\lambda_{\alpha \beta} \right) - \nabla_\beta \left( g^{\alpha \beta} \delta \Gamma^\lambda_{\lambda \alpha} \right) = \nabla_\lambda \left( g^{\alpha \beta} \Gamma^\lambda_{\alpha \beta} - g^{\alpha \lambda} \Gamma^\beta_{\beta \alpha} \right) (2.12)\]

This explicitly shows that $g^{\alpha \beta} \delta R_{\alpha \beta}$ is a total derivative and as such does not contribute to the dynamical equations but give rise to a boundary term. As we claimed previously.

### 2.1.1 The Gibbons York Hawking term

Until now the variation of the action (2.3) is given by:

\[(16 \pi G) \delta S_H = \int \sqrt{g} d^4 x G_{\alpha \beta} \delta g_{\alpha \beta} + \int \sqrt{g} d^4 x \nabla_\lambda (g^{\lambda \rho} g_{\rho \beta} - g^{\lambda \nu} g_{\nu \beta}) \nabla_\nu \delta g_{\alpha \beta} (2.13)\]

Where the second term is obtained by expanding the gamma terms in (2.12). Now following the notation of [7] we are making the substitution:

\[(\Delta B)^\lambda = (g^{\lambda \alpha} g_{\nu \beta} - g^{\lambda \nu} g_{\alpha \beta}) \nabla_\nu \delta g_{\alpha \beta}\]

With this the second term can be rewritten as:

\[\int \sqrt{g} d^4 x \nabla_\lambda (\Delta B)^\lambda = \epsilon \int_{\Sigma} d^3 y \sqrt{h} N_\lambda (\Delta B)^\lambda (2.14)\]
In the last step the Stoke theorem has been used. Here $N_\lambda$ is the normal vector to the boundary $\Sigma$ of $M$. $h_{\alpha\beta}$ is the induced metric in the boundary and $\epsilon = N^\lambda N_\lambda$. Now with this form \ref{2.14} we can seek a more convenient form and impose Dirichlet or Neumann boundary conditions. Starting with the term $N_\lambda(\Delta B)^\lambda$:

Given

$$g^{\alpha\beta} = h^{\alpha\beta} + \epsilon N^\alpha N^\beta \quad (2.15)$$

then

$$N_\lambda(\Delta B)^\lambda = N^\lambda h^{\alpha\beta} \nabla_\alpha \delta g_{\lambda\beta} - N^\alpha h^{\lambda\beta} \nabla_\alpha \delta g_{\lambda\beta} \quad (2.16)$$

From this equation each term correspond to a possible boundary condition. The first term makes reference to the variation of the metric in the boundary and its tangential derivatives $h^{\alpha\beta} \nabla_\alpha \delta g_{\lambda\beta}$. Meanwhile the second term depends, as well, on the variation of the metric but also to its normal derivative $N^\alpha \nabla_\alpha \delta g_{\lambda\beta}$. With this its possible to impose any of the boundary conditions. However from here on Dirichlet standard boundary conditions are going to be used. I.e:

$$\delta g_{\alpha\beta}|_\Sigma = 0 \quad (2.17)$$

So returning to the variation of \ref{2.3} we have,

$$(16\pi G)S_H = \int \sqrt{g} d^4x G_{\alpha\beta} \delta g^{\alpha\beta} - \epsilon \oint_\Sigma \sqrt{h} d^3y (h^{\lambda\beta} N^\alpha \nabla_\alpha \delta g_{\lambda\beta}) \quad (2.18)$$

The second term on \ref{2.18} is the reason an extra term $(S_B)$ was introduced in the beginning. The previous equation do not reproduce the Einstein equation properly due to the boundary term, so by gauging correctly $\delta S_B$ we can cancel the boundary term and finally get the desired equations. Nevertheless the term $S_B$ is not unique since we are asking it to fulfil the equation $\delta S_B - \epsilon \oint_\Sigma \sqrt{h} d^3y (h^{\lambda\beta} N^\alpha \nabla_\alpha \delta g_{\lambda\beta}) = 0$ given Dirichlet boundary conditions.

A popular choice for this boundary term is the Gibbons Hawking York term,
\[ S_{GHY} = 2\epsilon \oint_{\Sigma} \sqrt{h} d^3y K \]  

(2.19)

Where \( K = h^{\alpha\beta} K_{\alpha\beta} \) and \( K_{\alpha\beta} \) is the extrinsic curvature of \( \Sigma \). This term can be shown to cancel the boundary term born from the variation of \( S_H \) making the action correctly predict the left side of Einstein equation. Finally if we consider the variation of \( S_M \) with respect to the metric then, in general we can expect a term of the form:

\[ \delta S_M = -\frac{1}{2} \int \sqrt{g} d^4x T_{\alpha\beta} \delta g^{\alpha\beta} \]  

(2.20)

Equation that determines the action given some specific energy-momentum tensor or more interestingly vice versa.

With all of this, we can finally write:

\[ S = \frac{1}{16\pi G} \int \sqrt{\bar{g}} d^4x R + 2\epsilon \oint_{\Sigma} \sqrt{h} d^3y K + S_M \]  

(2.21)

Whose variation gives,

\[ \delta S = \int \sqrt{\bar{g}} d^4x \left( \frac{1}{16\pi G} G_{\alpha\beta} - \frac{1}{2} T_{\alpha\beta} \right) \delta g^{\alpha\beta} \]  

(2.22)

Since only the term in parenthesis can be zero, we have finally:

\[ G_{\alpha\beta} = 8\pi G T_{\alpha\beta} \]  

(2.23)

Which is Einstein field equation.
2.2 Weyl Tensor

In this section we want to express the Riemann tensor as the sum of various individual pieces. The decomposition might help to create new terms whose physical interpretation could be easier and far more enriching. The idea is to perform this separation in coordinate invariant elements, such that under coordinate transformations this pieces are mapped unto themselves. In other words, creating irreducible representations of the Lorentz group.

To achieve this goal we have to our disposal taking contractions of the Riemann tensor in order to have information about its traces. All other information, i.e the trace-less elements, will be contained in another tensor, hopefully.

Our first step is to take a contraction on the Riemann tensor. Given the Christoffel connection the only possible contraction is the Ricci tensor. Given by,

$$ R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} $$  \hspace{1cm} (2.24)

All other possible index contractions are either zero or related to this one somehow.

From the Ricci tensor we can create the curvature scalar, or Ricci scalar:

$$ R = R^\mu_{\mu} = g^{\mu\nu} R_{\mu\nu} $$  \hspace{1cm} (2.25)

With the Ricci tensor and scalar all the trace information is stored. All other terms can be stored in the Weyl tensor, which is created by removing all contractions of the Riemann tensor. Namely,

$$ C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)} \left( g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right) + \frac{2}{(n-2)(n-1)} g_{\rho[\mu} g_{\nu]\sigma} R $$  \hspace{1cm} (2.26)
It is clear from this that $C_{\rho\sigma\mu\nu}$ retains all the possible symmetries of the Riemann tensor.

The Weyl tensor has some very intriguing properties such as being invariant to conformal transformations. This means that if one re-scale the metric,

$$g_{\mu\nu}(x) \rightarrow e^{2f(x)}g_{\mu\nu}(x)$$

The tensor $C_{\rho\sigma\mu\nu}$ is unchanged. However,

$$C_{\rho\sigma\mu\nu} \rightarrow e^{2f(x)}C_{\rho\sigma\mu\nu}$$

Note the difference in the index position.

Another fundamental property is that the Weyl tensor is zero (0) if the metric is flat ($\eta_{\mu\nu}(x)$) regardless of the signature. This added to its conformal invariant property, the tensor will be zero for any conformally flat metric. i.e $(g_{\mu\nu}(x) = e^{2f(x)}\eta_{\mu\nu}(x))$

Now well, reorganizing the expression (2.26) it can be shown that,

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - (g_{\rho\mu}P_{\sigma\nu} + P_{\rho\mu}g_{\sigma\nu} - g_{\sigma\mu}P_{\rho\nu} - P_{\sigma\mu}g_{\rho\nu})$$  \hspace{1cm} (2.27)

Where

$$P_{\mu\nu} = \frac{1}{(n - 2)} \left( R_{\mu\nu} - \frac{1}{2(n - 1)}g_{\mu\nu}R \right)$$  \hspace{1cm} (2.28)

And is called the Shouten tensor.

This is very insightful since using Bianchi identities we can express:

$$\nabla^\rho R_{\rho\sigma\mu\nu} = \nabla_\mu R_{\sigma\nu} - \nabla_\nu R_{\sigma\mu}$$

Using the definition for the Weyl tensor given by (2.27) on the previous equation gives,

$$\nabla^\mu C_{\rho\sigma\mu\nu} = (n - 3) [\nabla_\mu P_{\sigma\nu} - \nabla_\nu P_{\sigma\mu}]$$  \hspace{1cm} (2.29)
The tensor in the right is known in literature as the Cotton tensor.

Focusing on the dimensions considered in this work $(2 + 1)$ it is interesting to point out that the Weyl tensor is covariantly conserved. But in fact invoking a symmetric argument on the indices of the Riemann tensor, all the geometric degrees of freedom are inscribed in $(2 + 1)$ on the Ricci Tensor and Scalar [8]. Thus the Weyl tensor is identically zero, since it holds no information.

This can be seen from (2.29) or by inspection in eq. (2.26). But this does not mean all $n = 3$ spaces are conformally flat, since the roll of the Weyl tensor in $(2 + 1)$ is passed unto the Cotton tensor. Since eq (2.29) is trivial for $n = 3$, what it really makes a metric conformally flat, would be the vanishing of the Cotton tensor.

Finally in $n = 3$ from eq. (2.26) the Riemann tensor is totally specified by its traces: $\mathcal{R}_{\mu\nu}, \mathcal{R}$

$$R_{\rho\sigma\mu\nu} = (g_{\rho\mu}R_{\sigma\nu} + g_{\rho\nu}R_{\sigma\mu} - g_{\rho\nu}R_{\sigma\mu}) + \frac{1}{2} (g_{\rho\nu}g_{\sigma\mu} - g_{\rho\mu}g_{\sigma\nu}) \mathcal{R}$$

Which is a very useful equation since it makes the computations significantly easier.

### 2.2.1 The Weyl tensor as the gravitational field propagator

Recall equation (2.26). From here it is straightforward to see that under vacuum considerations, the Weyl tensor is the Ricci tensor:

$$T_{\mu\nu}(x) = 0 \Rightarrow C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$$

This comes from the fact that the Ricci tensor $(\mathcal{R}_{\mu\nu})$ and the Ricci scalar $(\mathcal{R})$ are zero when there is no energy momentum tensor.

Therefore the information of gravitation in vacuum is encoded in the Weyl Tensor. However vacuum solutions are just a finite number of all the possible physical situations. A far more interesting scenario would be to know: how is determined the Weyl tensor by an energy tensor in some region region of space time?

Let's return to equation (2.29):
\[
\n\nabla^\mu C_{\mu\nu\rho\sigma} = (n - 3) (\nabla_\rho P_{\nu\sigma} - \nabla_\sigma P_{\nu\rho})
\]

Where \( P_{\mu\nu} \) is the Schouten tensor given by equation (2.28). Replacing: \( R_{\mu\nu} \) and \( R \) for \( T_{\mu\nu} \) and its trace: \( T_\nu \equiv T \) -using the Einstein n-dimensional equations we get:

\[
R = -\kappa \frac{(n-2)}{2} T \\
R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} T \right)
\]

Inserting this previous equations in the definition (2.29) we obtain for the Weyl tensor:

\[
\nabla^\mu C_{\mu\nu\rho\sigma} = J_{\nu\rho\sigma}
\]

Where:

\[
J_{\mu\rho\sigma} = \kappa \frac{(n-3)}{(n-2)} \left[ \nabla_\rho T_{\nu\sigma} - \nabla_\sigma T_{\nu\rho} - \frac{1}{(n-1)} \left( \nabla_\rho(T g_{\nu\sigma}) - \nabla_\sigma(T g_{\nu\rho}) \right) \right]
\]

With this we can determine the Weyl tensor components uniquely by the source: \( T_{\mu\nu} \). This is reminiscent to Maxwell equations:

\[
\nabla^\mu F_{\mu\nu} = -J_\nu
\]

Where \( F_{\mu\nu} \) is the Maxwell tensor. This gives a fairly intuitive way to understand the propagation of gravity, as we have a gravitational current \( J_{\nu\rho\sigma} \) that produces the gravitational field, manifestly in the Weyl tensor.

### 2.3 Degrees of freedom

In order to find the degrees of freedom of a general \( n \) dimension phase space we are going to invoke a counting argument.

In GR the phase space is characterized by a spatial metric on a constant time hyper surface (\( \mathcal{H} \)). This spatial metric has \( \frac{n(n-1)}{2} \) components and another \( \frac{n(n-1)}{2} \) of their conjugate momenta (time derivatives).
But from Einstein equations we have $n$ constraints and $n$ degrees of freedom that can be eliminated via a clever coordinate choice \[^7\]. With this the total degrees of freedom per space time point is:

$$D_f(n) = \frac{n(n - 1)}{2} + \frac{n(n - 1)}{2} - n - n$$

Which gives,

$$D_f(n) = n(n - 3) \quad (2.35)$$

Now well, if $n = 3$ there are no degrees of freedom within the phase space. As such the geometry is almost completely determined by the constraints. Almost, because although $D_f(n) = 0$; the geometry might still have some global degrees of freedom.

### 2.4 The Newtonian Limit

Given the absence of degrees of freedom in $(2 + 1)$ The Newtonian limit might become very odd. Before tackling the limit problem, it is worthwhile to explain what is meant by Newtonian limit. By this we are considering test particles that move slow compared to the speed of light. This is: \[\frac{v}{c} \ll 1\] and as such any derivative term of the metric which carries a $\frac{v}{c}$ factor can be ignored. In this regime, the energy-momentum tensor takes a fairly simple form as the only non vanishing component is:

$$T_{00} \approx \rho \quad (2.36)$$

Where $\rho$ is the mass density present in the space time considered. This comes from the fact that in a classical regime the mass density works as the cause of gravitational field.

With this clear we can proceed to the problem itself. Let us write the metric as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (2.37)$$

Where $\eta_{\mu\nu}$ is the flat metric (Minkowski) and $h_{\mu\nu}$ is a perturbation. Choosing a gauge in which the field equations take a simpler form we arrive to the expression:

$$-\frac{1}{2} \eta^{\mu\nu} \partial_{\mu} \partial_{\nu} h_{\tau\tau} + \mathcal{O}(h^2) = 8\pi GT_{\tau\tau} \quad (2.38)$$
Chapter 2. *Theoretical preliminaries*

and,

\[ \eta^{\mu\nu} \partial_\mu \overline{h}_{\nu\sigma} = 0 \quad (2.39) \]

where,

\[ \overline{h}_{\sigma\tau} = h_{\sigma\tau} - \frac{1}{2} \eta_{\sigma\tau} \eta^{\mu\nu} h_{\mu\nu} \quad (2.40) \]

However we are interested in obtaining a Newtonian limit that depends on the dimensions of the space time \( n \). To this end we can invert equation (2.40) using the metric \( \eta^{\mu\nu} \) to contract and from the trace obtain \( n \). Mathematically,

\[
\begin{align*}
\overline{h} &= \eta^{\sigma\tau} \overline{h}_{\sigma\tau} \\
&= \eta^{\sigma\tau} h_{\sigma\tau} - \frac{1}{2} (\eta^{\sigma\tau} \eta_{\sigma\tau}) \eta^{\mu\nu} h_{\mu\nu} \\
&= -h \left( \frac{n - 2}{2} \right) 
\end{align*}
\]

Replacing this result in equation (2.40) gives:

\[ h_{\sigma\tau} = \overline{h}_{\sigma\tau} - \frac{1}{n - 2} \eta_{\sigma\tau} \eta^{\mu\nu} \overline{h}_{\mu\nu} \quad (2.44) \]

Equation (2.44) gives the perturbation metric in terms of the dimensions \( n \), just as we desired. From this we can obtain a very rich relationship between the physics and the geometry considered for the problem. Since only the *time-time* component of the energy momentum tensor exist, only this same component of \( \overline{h} \) will exist. Setting this into equation (2.38):

\[
-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \partial_\nu \overline{h}_{00} = 8\pi GT_{00} \\
\approx -\frac{1}{2} \eta^{ij} \partial_i \partial_j \overline{h}_{00} \\
= -\frac{1}{2} \partial_i \partial^i \overline{h}_{00} \\
\approx 8\pi G \rho
\]
The approximation performed in from the first equation to the second consist in ignoring all the time derivative terms, as they are suppressed by the factor: $\frac{v}{c}$. Since the last equation at the Newtonian limit must take the form:

$$\nabla^2 \Phi = 4\pi G \rho$$

Which is the Poisson equation for gravity, being of course $\Phi$ the classical gravitational potential. We can safely conclude that:

$$h_{00} = -4\Phi \quad (2.45)$$

As it was pointed previously the relationship between geometry and physics become apparent with the introduction of $n$. Recalling the geodesic equation, taking the affine parameter $s \mapsto t$ and $u^0 \gg u^i$ in this limit, becomes:

$$\frac{d^2 x^i}{dt^2} - \frac{1}{2} \partial_i h_{00} = 0 \quad (2.46)$$

Combining (2.45) and (2.44):

$$h_{00} = \overline{h}_{00} - \frac{1}{(n-2)} \eta_{00} \eta^{\mu\nu} h_{\mu\nu}$$
$$= -4\Phi + \frac{4}{(n-2)} \Phi$$
$$= -4\Phi \left( \frac{n-3}{n-2} \right)$$

And finally:

$$\frac{d^2 x^i}{dt^2} = -2 \left( \frac{n-3}{n-2} \right) \partial_i \Phi \quad (2.47)$$

This equation defines the force exerted on test particles by a Newtonian potential given the dimension of space time. Although one might expect that in all dimensions a notion of Newtonian force is present, we see that in $n = 2$ we have an infinite acceleration, in $n = 4$ we have the usual Newtonian equations of motion and for $n > 4$ the standard equations can be obtained by rescaling the constant $G$ with respect to $n$. 
However for $n = 3$ there is no force between test particles as the right side of equation (2.47) is zero. This might sound odd, specially in the context of general relativity, because this limit forecasts the notion of mass to the theory. Therefore in $(2+1)$ a common error might be to suppose that there is no concept of mass and with this the idea of gravitation. But as it has been pointed previously $(2+1)$ gravitation is far from trivial, the mass does appear in the theory but arises from a topological setting we will explore later on.
Chapter 3

The BTZ black hole solution

3.1 The BTZ metric

The BTZ black hole metric in \((r, t, \phi)\) coordinates is given by:

\[
\begin{align*}
  ds^2 &= -\left( -M - \Lambda r^2 + \frac{J^2}{4r^2} \right) dt^2 + \frac{dr^2}{\left( -M - \Lambda r^2 + \frac{J^2}{4r^2} \right)} + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2
\end{align*}
\]  

(3.1)

Given the extensive metric it is worthwhile to rewrite:

\[
(N^\perp)^2 = f = \left( -M - \Lambda r^2 + \frac{J^2}{4r^2} \right)
\]

(3.2)

\[
N^\phi = -\frac{J}{2r^2}
\]

(3.3)

With the cosmological constant given by \(\Lambda = -\frac{1}{l^2}\). This variable change is done in order to follow the notation in [6]. \(N^\perp\) is called the lapse function and \(N^\phi\) is the shift function, respectively.

This metric (3.1) is stationary and axially symmetric. Therefore it possesses \(\partial_t\) and \(\partial_\phi\) Killing vectors. Although at the moment it might not be clear, these are the only Killing vectors it possesses, this will be discussed later on.

It is easy to prove (see appendix A) that the metric (3.1) satisfies the vacuum field equations of \((2+1)\) dimensional general relativity.
\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{l^2} g_{\mu\nu} \]  

(3.4)

Although it has not been mentioned explicitly, the constants \( M \) and \( J \) are the mass and angular momentum of the BTZ respectively. The subtleness in mentioning this is born on the fact that in \((2+1)\) there is no Newtonian limit and as such the definition of this concepts are made in a more formal context. In particular, this theory regards the mass without units (see equation \[3.1\]) and the angular momentum therefore only possess distance square units.

There is a theory behind this association of the constants of integration \( M \) and \( J \) with their physical counterparts. This can be done via a modified Komar integral formalism, but sadly this approach is beyond the scope of this work. However at the moment at least it would be productive to make this association sound plausible, as from Chapter 6 the gravitational path integral formalism can en light the meaning of this variables, if they were unknown, by their roles on the thermodynamics of the BTZ. To understand the association we have to remit ourselves to the Einstein action in ADM formalism \[3\] \[4\]:

\[ I = \frac{1}{2\pi} \int_0^T dt \int d^2 x \left[ \pi^{ij} g_{ij} - N^i H - N^i H_i \right] + B, \]  

(3.5)

Here the boundary term \( B \) is introduced to cancel surface integrals in the variation of \( I \) and ensure that the action has genuine extrema, such as the Gibbons Hawking York term discussed previously. Considering variations of the spatial metric that preserve the asymptotic form of the BTZ solution, one finds that \[6\]:

\[ \delta B = T \left[ -\delta M + N^\phi \delta J \right] \]  

(3.6)

From this it is clear that \( M \) is the mass associated with asymptotic translations in the Killing time: \( t \). While \( J \) is the charge associated with rotational invariance, namely: the angular momentum as measured at infinity. This analysis can go further into details of formalization by relating \( M \) and \( J \) to Noether charges given the symmetries of this space time, i.e: associated to the asymptotic Killing vectors \( \partial_\tau \) and \( \partial_\phi \) here present, given that the BTZ space time is asymptotically anti-de Sitter \[4\].

Now well in order to see where the horizons are located, we set: \( R_{tt} = 0 \) and arrive to:
This solution contains a very interesting case. When $J = Ml$, the two horizons coincide $r_+ = r_-$. Which resembles the Kerr solution in (3 + 1) where it is known by the concept of ergosphere. Which as in this case, would be located at:

$$r_{\text{erg}} = M^{1/2}l = \left( r_+^2 + r_-^2 \right)^{1/2}$$

(3.8)

The importance of this radius lies in the region it creates, as for any $r < r_{\text{erg}}$ we would necessarily have $d\phi/d\tau > 0$ -assuming: $(J > 0)$. As such observers in this regions would be dragged by the rotation of the black hole.

Furthermore from equation (3.7) to avoid imaginary values a constraint can be made:

$$|J| \leq Ml$$

(3.9)

between the angular momentum, mass and cosmological constant.

Until now, the discussion has not yet enlightened the BTZ as a real black hole. To prove this the easiest form is to transform the metric (3.1) to Eddington-Finkelstein coordinates using the following coordinate change:

$$d\nu = dt + \frac{dr}{(N^\perp)^2}$$

(3.10)

$$d\tilde{\phi} = d\phi - \frac{N^\phi}{(N^\perp)^2}dr$$

(3.11)

With this the metric takes the form:

$$ds^2 = -(N^\perp)^2d\nu^2 + 2d\nu dr + r^2 \left(d\tilde{\phi} + N^\phi d\nu\right)^2$$

(3.12)

It is now clear that for $r = r_+$ we have a null surface, generated by the geodesics:

$$\frac{d\tilde{\phi}}{d\lambda} + N^\phi(r_+) \frac{d\nu}{d\lambda} = 0$$

(3.13)
Where the affine parameter $\lambda$ is given by: $r(\lambda) = r_+$. 

Equally valid this horizon ($r_+$) is also a Killing horizon, given by the vector:

$$
\chi = \partial_r - N^\phi(r_+) \partial_\phi
$$

Which is normal to this surface. This constructions will be important later on as it is intimately related to the thermodynamics of the black hole as from this the surface gravity can be computed.

The BTZ also allows a Kruskal-like description using the usual coordinate change:

$$
\begin{align*}
u &= \rho(r)e^{-at} \\
v &= \rho(r)e^{at} \\
\frac{d\rho}{dr} &= \frac{a\rho}{(N^\perp)^2}
\end{align*}
$$

Bearing in mind we have to correctly patch the space time. This is done defining two regions, the first ($I$) which goes from $r_- < r < \mathrm{inf}$ and the second ($II$) that covers $0 < r < r_+$. This change would leave the metric in the form:

$$
ds^2_{\pm} = \Omega_{\pm} du dv + r^2 \left( d\tilde{\phi}_{\pm} + N^\phi_{\pm} dt \right)
$$

Where the ($\pm$) sub-indices denote the specific function that cover the first ($I$) or second region ($II$) respectively. The functions $\Omega_{\pm}$ and $\tilde{\phi}_{\pm}$ are quite cumbersome, specially as $r$ and $t$ are defined implicitly through $u$ and $v$. Therefore their specific form will not be here included. For more information see [6].
From this process the maximally extended Penrose-Carter diagram would be:

![Penrose diagram for the BTZ black hole.](image)

The diagram differs with the Kerr black hole case only in the region where $r = \infty$, because at this regions the metric (3.1) has an asymptotically anti-de Sitter behaviour.

Finally it can be anticipated from (3.1) that the BTZ does not posses a curvature singularity at $r = 0$, which may seem a little of for the historical notion of a black hole. Although curvature is constant everywhere in the BTZ, the singularity at $r = 0$ is one in the causal structure. This is deduced from the intent to avoid time-like closed curves. If the killing vector that span geodesics through this point are not demanded to be zero here, they can become of timelike nature and a violation to causality will ensue. To avoid this unwanted situation $r = 0$ must be taken as the ending point for curves that go through it, and therefore 'create' a singularity in the surface determined by this space time point [6].
Chapter 4

Thermodynamics

4.1 Temperature From the Zeroth Law

Given a stationary metric the zeroth law of black holes thermodynamics states that the surface gravity is constant in all the black hole [1]. Linking this to a thermodynamical system in equilibrium the surface gravity will take the role of the system temperature, which of course must be constant. Explicitly if we denote as usual $\kappa$ the surface gravity then:

$$T = \frac{\kappa}{2\pi}$$  \hspace{1cm} (4.1)

where $(2\pi)^{-1}$ is the proportionality constant.

Therefore the calculation of the temperature amounts to the calculation of the surface gravity. Defining this variable as the acceleration needed to hold an object in the black hole horizon as seen from infinity, one can state mathematically the surface gravity as:

$$\nabla_\alpha (\chi^{\beta} \chi_\beta) = -2\kappa \chi_\alpha |_{r+}$$ \hspace{1cm} (4.2)

Where $\chi$ is the killing vector associated to the outer horizon at: $r = r_+$ defined in (3.14). Similar to the case of a black hole with two horizons, the superficial gravity will be:

$$\kappa = \frac{r_+^2 - r_-^2}{r_+ l^2}$$ \hspace{1cm} (4.3)
Therefore inserting the proportionality constant, the Hawking temperature for the black hole would be:

\[ T_H = \frac{r^2_+ - r^2_-}{2\pi r_+ l^2} \]  \hspace{1cm} (4.4)

Where it is clear the temperature will depend on the mass \((M)\), the angular momenta \((J)\) and the cosmological constant parameter \((l^2)\). This result will be deduced later on by performing a Euclidean extension to the metric and demanding periodicity on the corresponding time variable.

### 4.2 Entropy of the BTZ

From the previous results of temperature and linking the charges of the BTZ \((M \kappa\) and \(J\)) to thermodynamic variables a first law can be stated. Given that the surface gravity acts as a temperature and energy is equivalent to mass then the first law would take the form:

\[ dE = \frac{\kappa}{8\pi} dA + \Omega_+ dJ \] \hspace{1cm} (4.5)

Where \(E\) is energy, \(A\) the area of the black hole and all other variables have been previously defined. The factor \((8\pi)^{-1}\) has been added for consistency in the geometrical units been used. Note that the angular velocity \(\Omega_+\) acts as a chemical potential to the black hole.

Performing a Legendre transform on equation (4.5) of the form:

\[ dF = dE - \frac{1}{8\pi} d(\kappa A) \]

\[ dF = \frac{\kappa}{8\pi} dA + \Omega_+ dJ - \frac{A}{8\pi} d\kappa - \frac{\kappa}{8\pi} dA \]

\[ dF = -\frac{A}{8\pi} d\kappa + \Omega_+ dJ. \]

We end with a thermodynamical potential similar to the free Helmholtz energy. Now replacing equation (4.1) on the previous, a more insightful expression is obtained:

\[ dF = -\frac{A}{4} dT + \Omega_+ dJ \] \hspace{1cm} (4.6)
Comparing to the Helmholtz free energy, it can be read:

\[ S_{BH} = \frac{A}{4} \quad (4.7) \]

Where \( S_{BH} \) is the entropy of Bekenstein-Hawking after their discovers. Now for the specific case of the BTZ, the area can be evaluated as:

\[ A = \int_0^{2\pi} \sqrt{|g(r_+)|} d\phi \quad (4.8) \]

Where \(|g(r_+)|\) is the determinant of the BTZ metric evaluated on the radius of the outer horizon. The calculation of \( g \) yields:

\[
\begin{align*}
g &= (M - \frac{r^2}{l^2}) \left( \frac{r^2}{N^2} \right) - \frac{j^2}{4} \\
g &= \frac{r^2}{N^2} \left( M - \frac{r^2}{l^2} - \frac{j^2}{4r^2} \right) \\
g &= \frac{r^2}{N^2} N^2 \\
g &= r^2.
\end{align*}
\]

Therefore:

\[ |g| = r^2 \quad (4.9) \]

Resuming the calculation of the surface area \( A \):

\[
\begin{align*}
A &= \int_0^{2\pi} \sqrt{|g(r_+)|} d\phi \\
A &= \int_0^{2\pi} \sqrt{r_+^2} d\phi \\
A &= r_+ \int_0^{2\pi} d\phi \\
A &= 2\pi r_+.
\end{align*}
\]

Interesting enough the surface area is equivalent to the perimeter of the BTZ.
With this the entropy of the BTZ would be:

\[ S = \frac{2\pi r_+}{4} = \frac{C(r_+)}{4} \tag{4.10} \]

Where \( C(r_+) \) is the circumference at a radius \( r = r_+ \).

Remarkable the entropy in this black hole goes with the boundary, i.e: with a surface of dimension \( n - 1 \), similar to their \( 3 + 1 \) counter parts that go with the surface area.

Although this is the process to find the thermodynamic variables via geometrical arguments it is not the only reasonable process. All of this properties can be investigated using the \textit{path integral} formalism. This process will be done in the next chapter.
Chapter 5

Path Integral Formalism

In this chapter a brief detour from General Relativity will ensue. In order to introduce the Path Integral formalism, from its original roots, which concern Quantum Mechanics. Following this introduction the linkage to General Relativity will be made via the thermal partition function. Much of the information of this section is based on [9], [10], [11]

5.1 Path Integrals in Quantum Mechanics

Given the central importance of the Hamiltonian in QM, one can ask the role Lagrangian mechanics formalism can have in this theory. To answer this, first a simple question must be stated:

If I know a particle is at position: $x$ at a time $t = 0$, what is the probability amplitude to find it on a later time $t = T$ at another position: $x'$?

To answer this lets assume for simplicity a complete orthonormal set for the position eigenstate: $|x\rangle$, i.e:

$$\hat{X}|x\rangle = x|x\rangle, \langle x'|x\rangle = \delta(x' - x), \int dx|x\rangle\langle x| = 1$$

And a one dimensional Hamiltonian:

$$H = \frac{\hat{P}^2}{2m} + V(\hat{X})$$
Chapter 5. Path Integral Formalism

Now well, we want:

\[ \Gamma(x', x) = \langle \Psi_f | \Psi(T) \rangle = \langle x'| e^{-iHT} | x \rangle. \]

However if the time \( T \) is divided into many intervals: \( \Delta T = \frac{T}{N} \) where \( N \) is the number of this intervals, the previous can be written as:

\[ \Gamma(x', x) = \langle x'| \prod_{j=0}^{N-1} (e^{-iHT\Delta T})^j | x \rangle \]  

(5.1)

In order to make this partition exact the limit \( N \to \infty \) must be taken, or equivalently \( \Delta T \to 0 \). Labelling \( |x_j\rangle \) the intermediate \( j \)- position state then the previous can be written using the closure relation as:

\[ \Gamma(x', x) = \lim_{\Delta T \to 0} \int dx_1...dx_{N-1} \langle x'| e^{-iHT\Delta T} | x_{N-1} \rangle ... \langle x_1| e^{-iHT\Delta T} | x \rangle \]

or:

\[ \Gamma(x', x) = \lim_{\Delta T \to 0} \int dx_1...dx_{N-1} D_{N,N-1}D_{N-1,N-2}...D_{2,1}D_{1,0} \]  

(5.2)

Where,

\[ D_{j,i} \equiv \langle x_j| e^{-iHT\Delta T} | x_i \rangle \]  

(5.3)

To keep track of the original states: \( j = N \) is \( |x'\rangle \) and \( j = 0 \) is \( |x\rangle \). It is an interesting event that \( x \) and \( x' \) are not integration variables. The matrix element \( D_{j,i} \) can be understood as the propagator from a position \( x_i \) to the position \( x_j \). Therefore, the absence of the initial and final positions are related to the fact we take all possible paths that connect this positions.

Now well for a general propagator \([5.3]\) the time interval is made by construction very small \( (\Delta T \leq 1) \) so it can be expanded as:

\[ D_{j,i} = \langle x_j| (1 - i\Delta TH) | x_i \rangle = \langle x_j| x_i \rangle - i\Delta T \langle x_j| H | x_i \rangle. \]
Using the orthogonality:

\[ D_{j,i} = \delta(x_j - x_i) - i\Delta T \langle x_j \mid \left( \frac{\hat{p}^2}{2m} + V(\hat{X}) \right) \mid x_i \rangle \]  \hspace{1cm} (5.4)

Since the Dirac delta can be expanded in Fourier modes as:

\[ \delta(x_j - x_i) = \int \frac{dp_i}{(2\pi)} e^{ip_i(x_j - x_i)} \]  \hspace{1cm} (5.5)

It would be enriching to expand the second term in (5.4) into a similar form. This can be done by using a closure relation of the momenta states (\mid p_i \rangle). Explicitly:

\[ i\Delta T \langle x_j \mid \left( \frac{\hat{p}^2}{2m} + V(\hat{X}) \right) \mid x_i \rangle = i\Delta T \int \frac{dp_i}{(2\pi)} \left( \langle x_j \mid \frac{\hat{p}^2}{2m} \mid p_i \rangle + \langle x_j \mid V(\hat{X}) \mid p_i \rangle \right) (\langle p_i \mid x_i \rangle) \]

Since:

\[ \langle x \mid p \rangle = e^{ipx} \]  \hspace{1cm} (5.6)

Then the previous can be finally written as:

\[ i\Delta T \int \frac{dp_i}{(2\pi)} \left( \frac{p_i^2}{2m} + V(x_j) \right) e^{ip_i(x_j - x_i)} \]  \hspace{1cm} (5.7)

Joining everything:

\[ D_{j,i} = \frac{dp_i}{(2\pi)} e^{ip_i(x_j - x_i)} \left[ \mathbb{I}_d - i\Delta T \left( \frac{p_i^2}{2m} + V(x_j) \right) + \mathcal{O}(\Delta T^2) \right] \]  \hspace{1cm} (5.8)

Regrouping the above terms into the Taylor series of the exponential function the propagator can be expressed in the form:

\[ D_{j,i} = \int \frac{dp_i}{(2\pi)} e^{ip_i(x_j - x_i)} e^{-i\Delta T H(x_j, p_i)} \]

Joining the arguments of the exponentials and factorizing the term \( i\Delta T \) gives us:
\[ i\Delta T \left( \frac{p_i(x_j - x_i)}{\Delta T} - H(x_j, p_i) \right) \]  
(5.9)

In particular the index \( j \) in the propagator is always of the form \( i + 1 \). Setting in \( j = i + 1 \) and using the limit of \( \Delta T \to 0 \), the propagator becomes:

\[ \mathcal{D}_{i+1,i} = \int \frac{dp_i}{(2\pi)^{\frac{1}{2}}} e^{i\Delta T(p_i\dot{x} - H(x_{i+1}, p_i))} \]  
(5.10)

Taking the product of all the propagators, yield:

\[ \prod_{i=0}^{N-1} \mathcal{D}_{i+1,i} = \int \prod_{i=0}^{N-1} \frac{dp_i}{(2\pi)^{\frac{1}{2}}} e^{i\Delta T \left( \sum_{i=0}^{N-1} (p_i\dot{x} - H(x_{i+1}, p_i)) \right)} \]  
(5.11)

Since

\[ H = \sum_{i=1}^{N-1} (p_i\dot{x}_i - L) \Rightarrow L = \sum_{i=1}^{N-1} (p_i\dot{x}_i - H) \]

Exploiting the limit \( \Delta T \to 0 \) the Riemann’s integral definition can be achieved:

\[ \lim_{\Delta T \to 0} \sum_{i=0}^{N-1} \Delta T L(x_{i+1}, \dot{x}_i) = \lim_{N \to \infty} \sum_{i=0}^{N-1} \Delta T L(x_{i+1}, \dot{x}_i) \]

\[ = \int_0^T dt L(x, \dot{x}, t). \]

Notice that since the summation is made over dummies indices when the continuous limit is taken \( i + 1 \to i \) as such \( (x_{i+1}, \dot{x}_i) \to (\dot{x}, x) \). Continuing The last line on the previous equation is the definition of the action \(^1\):

\[ I = \int_0^T dt L(x, \dot{x}, t) \]  
(5.12)

Caution must be taken in the notation here used for \( x, \dot{x} \). In fact this variables are vectors with all \( i \)-components of position and velocity respectively. From this using the previous result in equation \( [5.2] \), the probability amplitude becomes:

\(^1\)From here on the action is denoted by \( I \) to avoid confusion with entropy \( S \)
Γ(x′, x) = \int \prod_{i=1}^{N-1} dx_i \int \prod_{i=1}^{N-1} d p_i \frac{p_i}{(2\pi)} e^{iI(x, \dot{x}, t)} \tag{5.13}

Using the notation:

\mathbb{D}Y = \prod_{i=1}^{N-1} Y_i \tag{5.14}

Here an abuse of notation will be made as \( \frac{p}{2\pi} \to p \), in the end the overall \( 2\pi \) factors can be used to rescale. With this finally the path integral formula is achieved:

\langle \Psi_f | \Psi(T) \rangle = \int \mathbb{D}p \mathbb{D}x e^{iI(x, \dot{x}, t)} \tag{5.15}

This equation is known as the phase-space path integral \cite{9}. The integral is taken over all possible velocities (\( \dot{x}(t) \)) and position (\( x(t) \)) the particle an reach in the time: \( T \) between \( t_0 = 0 \) and \( t_f = T \).

From this expression it is clearly stated how to go from a state \( \psi_1 \to \psi_2 \), however it is complicated to evaluate as it depends on all possible (\( \dot{x}(t), x(t) \)). Furthermore it is intimately linked to the topology of the configuration space spanned by \( \dot{x}(t) \) and \( x(t) \).

This characteristic of the path integral will be explored in the following chapters as this particular trait truly makes this theory remarkable agreeable with General Relativity.
5.2 Topological Considerations and Consequences

In the previous section the path integral was derived as the propagator of a state unto another. If we perform a Wick rotation on $t$, i.e. $t \rightarrow i\tau$ with $\tau$ the new time variable then (5.15) takes the form:

$$\Gamma_E = \int_{(\tau=0)=\phi_1}^{(\tau=\beta)=\phi_2} \mathcal{D}\phi e^{-I_E[\phi]}$$

This is the Euclidean path integral. Where $\beta$ correspond to the Wick rotation of $T$ and to generalize the results of the past section, $x$ has been replaced by a field.

An Euclidean path integral defines a transition amplitude by evolving the field in imaginary time under the action of the operator: $\exp(-\beta H)$. With this definition: $\phi_1$ and $\phi_2$ are boundary conditions defined on the space at Euclid time $\tau = 0$ and $\tau = \beta$ respectively. We can depict this in a plane by Figure 5.1:

![Figure 5.1: Visualization from the topological perspective of a path integral. $\beta$ defines the length of integration in the plane.](image)

With this the Euclidean path integral is done over a strip: $\mathbb{R}^{d-1} \times \mathbb{R}$ where $\mathbb{R}^{d-1}$ are the constant $\tau$ space-like hyper surfaces and $\mathbb{R}$ is the interval of integral defined by $\tau = \beta$. In this sense a path integral is performed over the space and restricted by the topology of this.

The prescription of:

$$\Gamma(\phi_1, \phi_2) = \langle \phi_2 | e^{-\beta H} | \phi_1 \rangle$$

(5.17)
Can be exploited to define a state: $|\Psi\rangle$ by cutting the Euclidean manifold, i.e: not defining the boundary condition at $\tau = \beta$. With this a quantum state can be defined simply as:

$$|\Psi\rangle = \int_{(\tau=0)=\phi_1}^{(\tau=\beta)=\phi} D\phi e^{-I_{E}[\phi]}$$  \hspace{1cm} (5.18)

Or graphically as

![Graphical representation of the path integral formalism.](5.2)

**Figure 5.2:** A quantum state can be created using the path integral formalism as a space with one of the boundary conditions unspecified

### 5.2.1 Density Matrix and Thermal Partition Function

The possibility to define a state as a manifold with a missing final boundary condition gives the chance to ask: what if both boundary conditions are unspecified? Taking the definition of formally, then this path integral is equal to:

$$e^{-\beta H} = \int_{\phi(\tau=0)=\phi_1}^{\phi(\tau=\beta)=\phi_f} D\phi e^{-I_{E}[\phi]}$$ \hspace{1cm} (5.19)

Equation (5.19) takes a bra state ($\langle \phi_f |$) and a ket state ($| \phi_i \rangle$) and produce a complex number. The operator $exp(-\beta H)$ consist in the matrix form of the energy density in an canonical ensemble. As such it is known as the **Density Matrix**. In this definition caution must be taken as this Matrix is not normalized.

Naming $\rho$ the density matrix then for some unspecified initial and final states:

$$\rho = \int_{\phi_i}^{\phi_f} D\phi e^{-I_{E}[\phi]}$$ \hspace{1cm} (5.20)

Possessing a density matrix in the path integral formalism opens the door for all thermodynamic properties to be explored as (5.20) can be linked to the **thermal partition function** labelled: $Z$, by:
\[ Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_\phi \langle \phi | e^{-\beta H} | \phi \rangle \] (5.21)

The previous definition [5.21] can be represented as a path integral over a surface that posses periodic boundary conditions. Given that \( \beta \) is the integration length, it can be naturally understood as the period of the space boundary conditions. The fact the Wick period \( \beta \) can be interpreted as a constant in the system will be related to its role of inverse temperature in the system, seen as a thermodynamic ensemble.

Continuing with the topological interpretation of the path integral, for example if the initial boundary condition \( \phi \) is over a circle \( S^1 \) the integration with length \( \beta \) creates a cylinder \( (\mathbb{R} \times S^1) \). Given the final condition \( \phi_f \) is the same as the initial: \( \phi \) the cylinder can be thought as having its extrema glued together to form a torus \( (S^1 \times S^1) \), this is done by the action of the Tr() in [5.21].

Graphically it can be seen as [11]:

![Visualization of a thermal partition function from the topological perspective of the path integral formalism. Notice how \( \beta \) works as a periodic time parameter for the boundary conditions.](image)

Like this many other integrals over a given manifold \( \mathcal{M} \) can be turned into a periodic manifold with a given topology. In fact, this topological thinking can be used to deduce the thermodynamics of compact and orientable manifolds, of course under certain restrictions [12].

In the next chapter all the previous information will be transcribed to the language of General Relativity. From this the thermodynamics of Black Holes will become available from the topological approach and in the process have a solid foot hold physics-wise.
Chapter 6

Gravitational Path Integral

6.1 Gravitational Path Integral

In the previous chapter the Path Integral was defined over a fixed space-time manifold $\mathcal{M}$ and integrated over vector fields $(x, p, \phi, ...)$ defined over this manifold.

In this sense, to talk about a gravitational path integral is to integrate the geometry itself [11]. Since the dynamical fields in a space-time manifold $\mathcal{M}$ from the perspective of general relativity are the metric: $g_{\mu\nu}$ and the matter fields: $\phi_{\mu\nu}$ it is reasonable to propose them as integration variables.

Therefore in general a good ansatz would be to write for the gravitational path integral:

$$\langle g_2, \phi_2 | g_1, \phi_1 \rangle = \int Dg D\phi e^{-I_E[g, \phi]}$$

(6.1)

Where $\langle g_2, \phi_2 | g_1, \phi_1 \rangle$ represents the probability amplitude to start in a state with metric and matter field $g_1, \phi_1$ respectively and to end in a state with $g_2$ and $\phi_2$.

$I_E[g, \phi]$ is the action associated to the space time as defined in 2.21 but with Euclidean signature. Explicitly:

$$I_E[g, \phi] = -\frac{1}{16\pi} \int_\mathcal{M} \sqrt{g}(R + ...) - \frac{1}{8\pi} \int_{\partial \mathcal{M}} \sqrt{h}(K + ...) + (\text{Matter terms})(\phi)$$

(6.2)

Here $R + ...$ is the Ricci scalar and other possible invariants of the space time that can go into the action, for example a cosmological constant. Since the space time need
boundary conditions to have a true maxima when doing the variation (See Chapter 2) then a boundary term must be added. From this $K + \ldots$ is the extrinsic curvature and further terms that can be added into the boundary $\partial M$. Finally the Matter terms contain the information of $\phi$.

The meaning of the path integral depends on the boundary conditions imposed. In (6.1) both states, the initial and final are their respective conditions on the manifold at a difference of Wick time $\tau = \beta$. However a manifold $\mathcal{M}$ can not have two metrics as it is globally defined independent of the chosen coordinates. Since this same argument is valid for the matter fields then the left side of (6.1) can be changed to:

$$
\langle g_2, \phi_2 | g_1, \phi \rangle = \langle g, \phi | e^{-\beta H} | g, \phi \rangle
$$

The factor $e^{-\beta H}$ comes from the fact the action is Euclidean and as such the states evolve under this operator.

Equation (6.3) is a very strong result because this is exactly (5.21) in the case of gravity. With this the famous result proposed by Hawking [13] is achieved:

$$
\mathcal{Z}(\beta) = \int Dg D\phi e^{-I_E[g,\phi]}
$$

Although it may sound circular the periodic boundary conditions that is forced by the uniqueness of the metric can only be achieved by demanding the Wick time $\tau$ to be periodic, now in a time $\beta$. Or more concisely to identify:

$$
\tau \sim \tau + \beta
$$

With this the problem of finding a path integral for gravity is in essence solved. This initial hype is ended through a thorough examination of equation (6.4) as in fact it is ill defined mathematically. In a first instance, the Euclidean action has no lower bound [11] and as such the exponential can diverge gruesomely. In a second instance their is no proper definition of how to integrate over the geometry, this is to take: $\int Dg D\phi(...)$.

With this two problems the correct evaluation of the path integral seems rather cumbersome. Never the less since the integration is done over the metrics that yield an extrema on the action: $I_E$. Given the weight $Dg$ in the integral the one that produce the greater extrema most contribute the most to the integral. This motivates the use of
the classical approximation which consist in expanding around a classical saddlepoint of the equations of motion [11], in this case the Einstein Field Equations:

\[ Z(\beta) \approx \exp \left( -I_E[\bar{g}, \bar{\phi}] + I^{(1)}_E + ... \right) \]  

(6.6)

The first term in parenthesis is the leading term of the action which is evaluated in the classical solution \( \bar{g}, \bar{\phi} \). Since the boundary conditions can only be specified in a manifold when \( r \to \infty \) [11], to evaluate the Euclidean action the manifold must be sliced into hyper surfaces of constant \( r_0 \) and take the limit \( r_0 \to \infty \) after.

The factors \( I^{(1)}_E + ... \) can be thought as terms with loops as in QFT and by this same reason are quite hard to calculate. Still some have been calculated, in fact for the BTZ is possible to get the first correction to the thermal partition function. Sadly this correction comes from a very different approach than the present here. It uses the Chern-Simons formalism rather than the direct usage of (6.4), for more information see: [14].

### 6.2 Thermodynamics from the Gravitational Path Integral

In this section (6.4) will be approximated by the classical approximation 6.6. From this the thermodynamics behind the gravitational path integral will be explored.

The thermal partition function corresponds to a canonical ensemble. Specifically it posses a constant temperature \( (\beta)^{-1} \) which makes the system to be in thermal equilibrium. Having clear the relationship between the geometry and the thermodynamics, embodied in the role this period plays both in the path integral formalism and in the thermodynamic system, all other consequences are straight forward.

From this, the thermal partition function: \( Z \) can be used to get the Helmholtz free energy \( F \) since:

\[ F = -\frac{1}{\beta} \log(Z) = E - ST \]  

(6.7)

This potential relates the total energy \( E \) to how much useful energy can be extracted from it by subtracting the amount of energy that will be lost to entropy \( ST \).

Rewriting (6.10) as:

\[ \log(Z) = S - \beta E \]
Immediately yields how to solve for the entropy and total energy, namely:

\[ S = (1 - \beta \partial_\beta) \log(Z) \]  
(6.8)

and

\[ E = -\partial_\beta \log(Z) \]  
(6.9)

This two equations in the case of Black Holes will correctly predict the results of Chapter 4, specifically the area law and the black hole mass respectively.

**Particular Case: BTZ Black Hole**

The previous discussion in the particular case of Black Holes can be extended to an angular potential \( \Omega \) and electrical potential \( Q \). Since this two act as potentials they can be added as energy that is being absorbed by the thermodynamic process.

In particular the general BTZ \[5\] \[15\] is rotating. From this the Helmholtz free energy will be modified to:

\[ F = -\frac{1}{\beta} \log(Z) = E - ST - \Omega J \]  
(6.10)

Where \( \Omega J \) is the energy lost due to rotation.

Interesting enough, here \( \Omega J \) is playing the role of chemical potential. This is normally a form of potential energy *molecules* can absorb or release during a process. In the BTZ Black Hole the existence of the rotating term creates a potential energy that is absorbed by the *radiant modes* that are responsible of Hawking radiation, of course after the BTZ has been coupled to a scalar field which manifest itself through the modes. This can phenomena is called *super-radiant modes* since for certain conditions the modes can become highly excited or energetic. This same characteristic is present in the Kerr black hole solution and is hypothesized to be the cause of some *quasars* super radiance. Sadly this topic is out of the scoop of this work but for more information see Chapter 5.4 of \[11\] or equation (5.9) of \[6\].
6.3 Applications

In this section two examples of how to evaluate the Euclidean action and the corresponding thermodynamics will be done. The first example is the Schwarzschild black hole followed by the one of main interest in this work: the BTZ black hole.

6.3.1 The Schwarzschild Black Hole

To calculate the thermodynamic properties from the path integral formalism of the Schwarzschild black hole the recipe explained before will be used. The first step is to calculate the Euclidean action $I_E[\hat{g}]$ where $\hat{g}$ is the metric of the black hole and the matter fields have been discarded as there are none present in the solution. For future reference, to keep the notation clean $\hat{g} = g$ where it is clear $g$ is the Schwarzschild metric.

Now well the Euclidean action is:

$$I_E[g] = -\frac{1}{16\pi} \int_M d^4x \sqrt{\tilde{g}} R - \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{\tilde{h}} K$$

Where:

$$ds_E^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Here $\Omega^2$ is the solid angle.

Since the Euclidean signature only covers the space time from the outer horizon $r_+ = 2M$ the previous metric (6.12) can be thought as a coordinate polar system with $\tau$ the angle. From this the Wick time must be periodic or else a conical singularity will appear. Following this line of thought and expanding the metric near the horizon has $r = r_+ + \delta$ where $\delta \ll 1$ then it can be recasted to:

$$ds_E^2 = \frac{\delta}{r_+} d\chi^2 + \frac{r_+}{\delta} d\delta^2$$

Here the solid angle is untouched as it does not depend on $\delta$ or $\tau$. Taking a change of coordinate: $\rho = 2\sqrt{r_+}\delta$ and $\chi = \frac{\tau}{2r_+}$ [10], then:

$$ds_E^2 = \rho^2 d\chi^2 + d\rho^2$$
Clearly to make the angle $\chi$ periodic its maximal value can be $2\pi$ which makes $\beta = \tau(2\pi) = 4\pi r_+$ from the change of variable. With this:

$$\tau \sim \tau + 4\pi r_+ \tag{6.15}$$

Now we can split the Euclidean Schwarzschild manifold into hyper surfaces of constant $\tau$ where this variable goes from 0 to $\beta$. Recalling (6.11) the first term is zero since for a vacuum solution $R = 0$ this reduces considerably the calculations. From this we are left to calculate the Gibbons Hawking York term:

$$I_E[g] = -\frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{\gamma} K \tag{6.16}$$

The extrinsic curvature scalar $K$ can be defined by the extrinsic curvature as:

$$K = h^{ab} K_{ab} = h_{\alpha}^{i} \nabla_{i} N_{\beta} \tag{6.17}$$

Here $h$ is the induced metric on the hyper surface, $\nabla$ is the gradient in spheric coordinates and $N$ is the normal vector to the boundary surface.

Calculating the normal vector $N_{\mu}$ under the condition: $S = r - r_0$ which specifies constant radial surfaces:

$$N_{\mu} = \frac{\partial_{\mu} S}{\sqrt{g^{\alpha \beta} \partial_{\alpha} S \partial_{\beta} S}} = \frac{1}{\sqrt{g^{rr}}} \delta_r$$

Then

$$N_r = \frac{1}{\sqrt{g^{rr}}} \delta_r \tag{6.18}$$

Since the induced metric $h$ is defined over each hyper surface it can be thought as a projector from the 4-D space into 3-D. Mathematically this can be written as:

$$h_{ij} = \delta_{ij} - N_i N_j \tag{6.19}$$
Inserting (6.41) and (6.18) into (6.17), gives:

\[ K = \nabla_r N^r \]  

(6.20)

Calculating:

\[ K = \nabla_r N^r = \frac{1}{r^2} \partial_r \left( r^2 \sqrt{1 - \frac{2M}{r}} \right) = \frac{2}{r} \sqrt{1 - \frac{2M}{r}} + \frac{M}{r^2 \sqrt{1 - \frac{2M}{r}}} \]

With this:

\[ K = \frac{2}{r} \sqrt{1 - \frac{2M}{r}} + \frac{M}{r^2 \sqrt{1 - \frac{2M}{r}}} \]  

(6.21)

In (6.16) \( h \) is the determinant of the induced metric. Since it is diagonal the determinant is the multiplication of the diagonal entries. then:

\[ \sqrt{h} = \sqrt{h_{rr} h_{\theta\theta} h_{\phi\phi}} = \sqrt{1 - \frac{2M}{r} - r^2 \sin(\theta)} \]  

(6.22)

Joining everything together:

\[ I_{GHY}(r_0) = -\frac{1}{8\pi} \int_{\partial M} \sqrt{\hat{h}} K |_{r = r_0} \]

\[ = -\frac{1}{8\pi} \int_0^\beta d\tau \int_0^{2\pi} d\phi \int_0^\pi d\theta \left( \sqrt{1 - \frac{2M}{r_0} r^2 \sin(\theta)} \right) \left( \frac{2}{r_0} \sqrt{1 - \frac{2M}{r_0}} + \frac{M}{r_0^2 \sqrt{1 - \frac{2M}{r_0}}} \right) \]

\[ = \frac{1}{8\pi} \left[ 4\pi \beta \sqrt{1 - \frac{2M}{r_0} r^2} \left( \frac{2}{r_0} \sqrt{1 - \frac{2M}{r_0}} + \frac{M}{r^2 \sqrt{1 - \frac{2M}{r_0}}} \right) \right] . \]

Taking as previously stated the limit \( r_0 \to \infty \) gives:
\[ I_{GHY}(r_0) = \frac{3}{2} \beta M - \beta r_0 \]  

(6.23)

Notice that this term is divergent since in this limit \( r_0 \) tends to infinity. Although this is a classical model still this kind of problems tend to appear. The solution to this divergences consists in subtracting a counter term to \( I_{GHY} \) that does not affect the dynamical equations. The physical solution is to subtract the action associated to the metric of Minkowski:

\[ I_{K0} = \frac{1}{2\pi} \int \sqrt{h} K_0 \]  

(6.24)

Where \( K_0 \) is the extrinsic curvature of Minkowski metric. It can be easily proven that,

\[ K_0 = \frac{2}{r} \]  

(6.25)

Inserting (6.25) into (6.24) and adding this to equation (6.23) gives after integrating the Minkowski term and replacing \( r = r_0 \):

\[
\begin{align*}
I_E &= I_{GHY} + I_{K0} \\
&= \frac{3}{2} \beta M - \beta r_0 + \beta \sqrt{1 - \frac{2M}{r_0} \frac{2r_0^2}{2}} \\
&= \frac{3}{2} \beta M - \beta r_0 + \beta r_0 \left( 1 - \frac{M}{r_0} + ... \right) \\
&= \frac{3}{2} \beta M - \beta r_0 + \beta r_0 - \beta M \\
&= \frac{\beta M}{2}.
\end{align*}
\]

Using (6.15) then finally the whole action (GHY and counter term) can be written as:

\[ I_E = \frac{\beta^2}{16\pi} \]  

(6.26)

At this moment all the hard work is done. To get the thermal partition function at first order is simply to take the exponential of (6.26), which gives:

\[ Z(\beta) = \exp\left(-\frac{\beta^2}{16\pi}\right) \]  

(6.27)
Taking the derivatives in (6.9) and (6.8) gives for the energy and entropy respectively:

\[ E = -\partial_\beta \log(Z(\beta)) = M \]  \hspace{1cm} (6.28)

and

\[ S = (1 - \beta \partial_\beta) \log(Z(\beta)) = 4\pi M^2 = \frac{A}{4} \]  \hspace{1cm} (6.29)

Equations that are correctly predicted as they are the energy mass equivalence and the Area law.

It is evident that the hardest part of this approach is to evaluate the action and in need come with a counter term that correctly cancels the divergences. This term depends both in the dimensions and the space that is being treated, as it will be seen in the next example for AdS\(_3\) the term needed to cancel the divergences will be completely different.
6.3.2 The BTZ Black Hole

Here the example for the BTZ will be done. Using the metric form (A.1) a foliation will be made in constant radial hyper surfaces. This is to consider the restriction (constrain): \( S^r = r - r_0 \).

Since the metric must have Euclidean signature the correct Wick rotation is to take:

\[
\begin{align*}
t &= \imath \tau \\
J &= \imath \Gamma.
\end{align*}
\]

This comes natural due to the coupling of \( t \) and \( \phi \) coordinates. With this the BTZ metric in euclidean signature is:

\[
ds^2_E = N_E^{-2} dr^2 + \left[ N_E^2 d\tau^2 + r^2 (d\phi + \omega d\tau)^2 \right]_{r=r_0}
\]

Where \([...]_{r=r_0}\) refers to the foliation on constant radial surfaces in the variables: \( \tau \) and \( \phi \). Here:

\[
\begin{align*}
N_E^2 &= -M + \frac{r^2}{l^2} - \frac{\Gamma}{4r^2} \\
\omega &= -\frac{\Gamma}{2r^2}
\end{align*}
\]

Or more succinctly (3.2) and (3.3) in Euclidean signature.

In euclidean signature the Action in \( AdS_3 \) is:

\[
I_E[g] = -\frac{1}{16\pi} \int_M d^3 x \sqrt{g} \left( R + \frac{2}{l^2} \right) - \frac{1}{8\pi} \int_{\partial M} d^2 x \sqrt{h} K
\]

Here the bulk term which involves the Ricci scalar does not vanish as in the BTZ:

\[
R = -\frac{6}{l^2}
\]

Therefore the integration of this term must be cautious. Since the black hole is in Euclidean signature, the metric does not cover anything behind the first horizon: \( r_+ \).
but does extend to infinity. So this term must be integrated from the boundary of the horizon to some $r_0$ which will be taken to infinity ($r_0 \rightarrow \infty$) when all contributions to the action have been calculated, given that they may diverge. Now continuing with the evaluation of the first term:

$$I_{E,1}[g] = -\frac{1}{16\pi} \int_M d^3x \sqrt{g} \left( R + \frac{2}{l^2} \right)$$

(6.36)

Since $R = -\frac{6}{l^2}$ the only factor to be calculated is $g$, the determinant of the euclidean BTZ metric. Using (6.32) to write the metric gives:

$$g_{\mu\nu}^E = \begin{pmatrix} r^2 \omega^2 + N_E^2 & 0 & r^2 \omega \\ 0 & N_E^{-2} & 0 \\ r^2 \omega & 0 & r^2 \end{pmatrix}$$

(6.37)

It can be easily calculated the determinant since (6.37) is a $3 \times 3$ matrix, in this case it yields:

$$g = r^2 \left( 1 + \frac{r^2 \omega^2}{N_E^2} (1 - \omega) \right)$$

(6.38)

With this (6.36) becomes:

$$I_{E,1}[g] = \frac{1}{4l^2\pi} \int_0^{2\pi} d\tau \int_0^\beta d\sigma \int_{r_+}^{r_0} dr \sqrt{r^2 \left( 1 + \frac{r^2 \omega^2}{N_E^2} (1 - \omega) \right)}$$

$$= \frac{\beta}{2l^2} \left[ \int_{r_+}^{r_0} dr \sqrt{r^2 \left( 1 + \frac{r^2 \omega^2}{N_E^2} (1 - \omega) \right)} \right]$$

$$\approx \frac{\beta}{2l^2} \left[ \int_{r_+}^{r_0} dr \left( r + \frac{\Gamma l^2}{r^3} \right) \right]$$

$$= \frac{\beta}{2l^2} \left( \frac{r^2}{2} - \frac{\Gamma l^2}{2r^2} \right) \Big|_{r_+}^{r_0}.$$  

In the third line, given the limit $r_0 \rightarrow \infty$ the integrand was approximated to save a cumbersome calculation. Evaluating the integral gives:

$$I_{E,1}[g] = \frac{\beta}{4l^2} \left[ (r_0^2 - r_+^2) + \Gamma l^2 \left( \frac{1}{r_+} - \frac{1}{r_0} \right) \right]$$

(6.39)
Now working the second term of the Euclidean integral (6.35):

\[ I_{E,2}[h] = -\frac{1}{8\pi} \int_{\partial M} d^2x \sqrt{h} K \]  

(6.40)

First to calculate the determinant of the induced metric in the hyper surface the induced metric must be calculated. To this end the equation:

\[ h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \]

is used. Following the previous gives for the induced metric \( h_{ij} \):

\[ h_{ij} = \begin{pmatrix} N_E^2 + r^2 \omega^2 & r^2 \omega \\ r^2 \omega & r^2 \end{pmatrix} \]  

(6.41)

The determinant of this matrix \( h \) is readily obtained and is equal to:

\[ h = r^2 N_E^2 \]  

(6.42)

It is an interesting event that the determinant is independent of the rotational components of space time: \( h_{t\phi} \) and \( h_{\phi t} \), however the rotational information that is carried in \( \Gamma \) is still stored in \( N_E^2 \).

In order to calculate the extrinsic curvature scalar: \( K \) must be obtained. Repeating the process already described for the Schwarzschild black hole gives for the normal component to the hyper surface:

\[ N^r = N_E = \sqrt{-M + \frac{r^2}{l^2} - \frac{\Gamma^2}{4r^2}} \]  

(6.43)

and using

\[ K = \frac{1}{r} \partial_r (r N^r) \]

to calculate the extrinsic curvature scalar gives finally:

\[ K = \frac{N_E}{r} + \frac{1}{N_E} \left( \frac{r}{l^2} + r \omega^2 \right) \]  

(6.44)
Inserting this to (6.40) and integrating:

\[
I_{E,2} = -\frac{1}{8\pi} \left[ \int_0^{2\pi} d\phi \int_0^\beta d\tau r_0 N_E \left( \frac{N_E}{r} \frac{1}{N_E} \left( \frac{r}{\ell^2} + r_\omega^2 \right) \right) \right] = -\frac{\beta}{4} \left[ -M + \frac{r_0^2}{\ell^2} - \frac{\Gamma^2}{4r_0^2} + \frac{r_\omega^2}{\ell^2} + \frac{\Gamma^2}{4r_0^2} \right].
\]

and thus,

\[
I_{E,2} = \frac{\beta M}{4} - \frac{\beta r_0^2}{2\ell^2}.
\]  

(6.45)

Joining (6.39) and (6.45) gives for the action:

\[
I_E[g] = \frac{\beta}{4\ell^2} \left[ (r_0^2 - r_+^2) + \Gamma^2 \left( \frac{1}{r_+} - \frac{1}{r_0} \right) \right] + \frac{\beta M}{4} - \frac{\beta r_0^2}{4\ell^2}.
\]  

(6.46)

Note that if the limit \( r \to \infty \) is taken the euclidean action for the BTZ diverges as: \( r_0^2 \).

In order to solve this problem we can subtract the vacuum energy of the hyper surfaces, mathematically:

\[
I_E[h] = \frac{1}{8\pi l} \int_{\partial \mathcal{M}} \sqrt{h}
\]  

(6.47)

the factor of 1/l is chosen to get the correct counter term to cancel the divergences when taking the limit. The evaluation of this integral is straight forward and is equal to:

\[
I_E[h] = \frac{\beta r_0^2}{4\ell^2}.
\]  

(6.48)

Now applying this counter term to (6.46) gives:

\[
I_{E,\text{tot}}[g] = \frac{\beta}{4\ell^2} \left[ (r_0^2 - r_+^2) + \Gamma^2 \left( \frac{1}{r_+} - \frac{1}{r_0} \right) \right] + \frac{\beta M}{4} - \frac{\beta r_0^2}{4\ell^2}.
\]

\[
= -\frac{\beta r_+^2}{4\ell^2} + \frac{\beta M}{4} + \frac{\beta \Gamma}{4} \left( \frac{1}{r_+} - \frac{1}{r_0} \right) - \frac{\beta r_0^2}{4\ell^2}.
\]

\[
= -\frac{\beta r_+^2}{4\ell^2} + \frac{\beta M}{4} + \frac{\beta \Gamma}{4} \left( \frac{1}{r_+} - \frac{1}{r_0} \right).
\]
Taking now the limit $r \to \infty$ since there are no divergent terms reveal the real action for the BTZ black hole:

$$I_E = \beta \frac{M}{4} - \beta \left( \frac{r_+^2}{4l^2} - \frac{\Gamma}{4r_+^2} \right)$$ 

(6.49)

However this is not exactly the most appealing form given the difficulty to understand each term. In the desire to leave this action somewhat similar to the expected form (6.10) the identities:

$$M = \frac{r_+^2 - r_-^2}{l^2}$$ 

(6.50)

$$J = \frac{2r_+ r_-}{l}$$ 

(6.51)

are introduced. Also as in the case of the Schwarzschild black hole, a identification of $\tau$ as an angle obliges the periodicity of this in some time, that is by construction: $\beta$.

To find the value of the period $\beta$ first it most be noted that the Euclidean BTZ metric (6.32) is a positive-definite metric of constant negative curvature, and the spacetime is therefore locally isometric to hyperbolic 3-space $\mathbb{H}^3$ [6]. This isometry is manifest in trying to represent the metric as:

$$ds_E^2 = \frac{l^2}{z^2} (dx^2 + dy^2 + dz^2)$$ 

(6.52)

The coordinate transformation that takes (6.32) to (6.52), as it can be checked by direct calculations is:

$$x = \left( \frac{r_+^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \cos \left( \frac{r_+}{l^2} \phi + \frac{|r_-|}{l} \phi \right) \exp \left[ \frac{r_+}{l} \phi - \frac{|r_-|}{l^2} \tau \right]$$ 

(6.53)

$$y = \left( \frac{r_+^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \sin \left( \frac{r_+}{l^2} \phi + \frac{|r_-|}{l} \phi \right) \exp \left[ \frac{r_+}{l} \phi - \frac{|r_-|}{l^2} \tau \right]$$ 

(6.54)

$$z = \left( \frac{r_+^2 - r_-^2}{r^2 - r_+^2} \right)^{1/2} \exp \left[ \frac{r_+}{l} \phi - \frac{|r_-|}{l^2} \tau \right].$$ 

(6.55)

Since the metric is only defined from $r = r_+$ then to avoid a conical singularity at the $z$ coordinate in this value, a periodic boundary condition must be set to the trigonometric
functions and exponentials present in the definitions of: \( x \) and \( y \). Writing the set of linear equations:

\[
\frac{r_+}{l^2} \beta + \frac{|r_-|}{l} \Phi = 2\pi \\
\frac{r_+}{l} \Phi - \frac{|r_-|}{l^2} \beta = 0.
\]

Where \( \beta \) is the period in the \( \tau \)-cycle and \( \Phi \) the period in the \( \phi \)-cycle. This can be solved into:

\[
\beta = \frac{2\pi r_+ l^2}{r_+^2 - r_-^2} \quad (6.56)
\]

and,

\[
\Phi = \frac{2\pi r_- l}{r_+^2 - r_-^2} \quad (6.57)
\]

As promised in Chapter 4, the Hawking Temperature \( T_H \) is deduced by continuing analytically the BTZ solution to Euclidean signature. Note that as it would be expected:

\[
\beta = \frac{1}{T_H} \quad (6.58)
\]

This means our canonical ensemble is at some constant temperature \( (T_H) \) as predicted by the Zeroth law of black hole thermodynamics.

Resuming the present calculation, by using the value of \( \beta \) and the identities (6.50), (6.51) the action takes the form:

\[
I_E[\hat{g}_{BTZ}] = \frac{2\pi r_+}{4} - \beta (M - \Omega J) \quad (6.59)
\]

Where \( \Omega = \frac{r_-}{r_+} \), is the angular velocity of the BTZ black hole. Caution must be taken since (6.59) takes as variable the original angular momentum \( J \) and not the Wick angular momenta \( \Gamma \). This comes from the identification of \( \Omega = \frac{\Phi}{\beta} \) and since for euclidean metric the inner horizon \( |r_-| \) corresponds to \(-ir_-\), where \( r_- \) is the original inner horizon of the BTZ described in (3.7). Now well this action can be readily worked in a thermodynamic setting by taking the logarithm of the partition function, using the classical approach, from this:
\[ Z_{\text{BTZ}} = e^{I_E[\delta_{\text{BTZ}}]} \] (6.60)

taking the logarithm:

\[ \log(Z_{\text{BTZ}}) = \frac{2\pi r_+}{4} - \beta(M - \Omega J) \] (6.61)

From this the energy will be:

\[ E = -\partial_\beta(\log(Z_{\text{BTZ}})) = M - \Omega J \] (6.62)

and the entropy:

\[ S = (1 - \beta \partial_\beta)(\log(Z_{\text{BTZ}})) = \frac{2\pi r_+}{4} \] (6.63)

We see the entropy is the same that was derived in Chapter 4: (4.10). The path integral formalism permit us to derive a new quantity not founded by the geometrical approach which is the energy \( E \). Contrary to the expected the energy of the black hole is not only the rest mass \( M \) of it. This as pointed earlier is because the introduction of a chemical potential \( \Omega \) absorbs some of the energy withhold in the resting mass and turns it into potential energy that can be released in super-radiant modes, if an auxiliary scalar field is considered.

With this the examples of the gravitation path integral are over. Successfully this formulation predicts the same results and extends them in a very physically viable sense. However it is important to remember its limitation as in fact the complete evaluation of them are still in dubious terrain.
Chapter 7

Conclusions

7.1 Conclusions

The BTZ black hole is a highly exciting physical model which contains all the physical rich substance that their higher dimension counter-part possesses. Although it is born in a virtually structure less manifold (AdS$_3$) the correct identifications of this manifold patches covert it into a space-time very much like the (3 + 1)-dimensional Kerr black hole\cite{AdS3}.

The rich structure of the geometry has been studied and condensed into the most important features it posses in this work. Clearly most of it must be hold in the model at hand, since far flung conclusions are not necessarily equivalent in others yet the avoidance of mathematical complications makes the endeavour of calculation and physical interpretation more transparent.

Going aside of the geometry most of the work has been dedicated to the exploration of the gravitational path method with the BTZ black hole as background. The real importance of the formalism is stated obvious from the results obtained from the thermodynamic approach here presented. As a pedagogical tool, the complete evaluation of the path integral in both the Schwarzschild and BTZ black holes comes at hand for future references in this particular field as many sources avoid the explicit calculations, some that are not always trivial. As stated repeated times previously, the hype of the formalism may seem obscure when faced to the real mathematical problems the correct evaluation of the integral have, specifically with out the help of the classical approximation. But still the results are valid and worthwhile enough to research on this approach as it may bring light to the topic of quantum gravity unification.
Finally most of the thematic here developed works better for pedagogical reasons. Clearly the world we live in is not \((2 + 1)\) but the results here obtained with out the nasty complications a higher dimensions bring truly reflect the abundant physics that lies on the model. Perhaps enough to moderately draw conclusions about similar characteristics higher dimensional models may posses. Given the nature of the BTZ there is still many problems to solve. But the resemblance to \((3 + 1)\), in particular the Kerr black hole, is so uncanny it is not unimaginable to see the possible future impact the study of this black hole system may have in future physics

### 7.2 Final Remarks

Many of the thematics here developed are just the tip of many rich topics that can be investigated. In a personal perspective, the question for path integrals, at least in this dimensions, is open to future work and reflection.

A question that has been intentionally avoided in this work is the true nature of the thermal partition function. Understood as a thermodynamic ensemble, the partition function speaks about the *statistical* nature of the BTZ. Therefore what are the micro-states that are being counted when taking this interpretation? Although at the moment most answers to this seem rather complicated or strange, it is a good starting point for further research given the central importance this answer may hold for quantum gravity.
Appendix A

Geometric Aspects and BTZ Validity

In this Appendix we will derive the geometric aspects of the BTZ solution: metric, Christoffel symbols, Riemann tensor, Ricci tensor, scalar and so on. From this an explicit check of the validity of the BTZ metric as a solution of the Einstein field equation.

Using the line element (3.1) we can write the metric in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} r^2(N^\phi)^2 - (N^\perp)^2 & 0 & r^2(N^\phi) \\ 0 & (N^\perp)^{-2} & 0 \\ r^2(N^\phi) & 0 & r^2 \end{pmatrix}$$ \hspace{1cm} (A.1)

Where $N^\perp$ and $N^\phi$ are defined in (3.2) and (3.3) respectively.

The contravariant metric $g^{\mu\nu}$ can be calculated by inverting (A.1). Doing this we arrive to:

$$g^{\mu\nu} = \begin{pmatrix} -(N^\perp)^2 & 0 & \frac{N^\phi}{(N^\perp)^2} \\ 0 & N^2 & 0 \\ \frac{N^\phi}{(N^\perp)^2} & 0 & -\frac{(r^2(N^\phi)^2 - (N^\perp)^2)}{r^2(N^\perp)^2} \end{pmatrix}$$ \hspace{1cm} (A.2)

For simplicity here after $N^\perp = N$ to reduce the notation.
Christoffel symbols

In order to proceed in finding the geometric elements here purposed the Christoffel symbols for this metric must be calculated. As always, the generating formula is:

\[ \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left( \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\alpha g_{\mu\nu} \right) \]  \hspace{1cm} (A.3)

Given that \( g_{\mu\nu} \) has crossed terms the number of Christoffel symbols is considerable. However the symmetry of the metric and the reduced dimensions contribute significantly to the task of calculating them. Given the quantity of symbols to find, only an example case will be done explicitly, all others will only be quoted.

**Case:** \( \alpha = t \), \( \mu = t \) and \( \nu = r \)

The Christoffel equation has the form:

\[ \Gamma^t_{tr} = \frac{1}{2} g^{t\sigma} \left( \partial_r g_{\sigma t} + \partial_t g_{\sigma r} - \partial_t g_{tr} \right) \]

Since the metric is static, both time derivatives are zero. Which only leaves the terms, after expanding the sum:

\[ \Gamma^t_{tr} = \frac{1}{2} g^{tt} \partial_r g_{tt} + \frac{1}{2} g^{t\phi} \partial_r g_{t\phi} \]

\[ = \frac{1}{2} \left( - \frac{1}{N^2} \right) \partial_r \left( - \frac{J^2}{4r^2} + M - \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right) + \frac{1}{2} \left( \frac{N_{\phi}}{N^2} \right)^2 \partial_r \left( - \frac{J}{2} \right) \]

\[ = - \frac{1}{2N^2} \partial_r \left( M - \frac{r^2}{l^2} \right) \]

\[ = \frac{1}{N^2} \left( \frac{r}{l^2} \right). \]

All other Christoffel symbols are calculated similar to this example.
Taking into account the symmetry of the Christoffel symbols on $\mu$ and $\nu$, the full list is:

\[
\begin{align*}
\Gamma_{tr}^t &= \frac{r}{N^2 l^2} \\
\Gamma_{r\phi}^t &= -\frac{J}{r N^2} \\
\Gamma_{tt}^r &= \frac{r N^2}{l^2} \\
\Gamma_{rr}^r &= -\frac{1}{r N^2} \left( \frac{r^2}{l^2} + r^2 (N\phi)^2 \right) \\
\Gamma_{r\phi}^\phi &= -r N^2 \\
\Gamma_{tr}^\phi &= -\frac{J}{2 r N^2 l^2} \\
\Gamma_{r\phi}^{\phi r} &= -\frac{1}{r N^2} \left( r^2 (N\phi)^2 - N^2 \right). 
\end{align*}
\]

It is worthwhile to remember that $\Lambda = -\frac{1}{l^2}$, where $\Lambda$ is the cosmological constant of this theory. However, it is easier to work with $1/l^2$ as the signs are easier to manage.

From the table above, the construction of the other geometrical elements can be undertaken.

**Riemann tensor, Ricci tensor and Ricci scalar**

Now the discussion will turn to the calculation of the Riemann tensor, Ricci tensor and scalar in order to see that the BTZ metric is in fact solution to the Einstein field equations.

To start the equation that determines the Riemann tensor is:

\[
R^\mu_{\nu \rho \sigma} = \partial_\rho \Gamma^\mu_{\nu \sigma} - \partial_\sigma \Gamma^\mu_{\nu \rho} + \Gamma^\mu_{\alpha \rho} \Gamma^\alpha_{\nu \sigma} - \Gamma^\mu_{\alpha \sigma} \Gamma^\alpha_{\nu \rho} \quad (A.4)
\]

Although the Riemann tensor contain a substantial number of components, most of them are zero. This phenomena arises due to the dimension we are working on. Recalling the counting argument used in Chapter 2 section 3 a similar one can be stated in this case. Since the Riemann tensor is anti symmetric on permuting individual indices, i.e: $(\mu \leftrightarrow \nu)$ or $(\rho \leftrightarrow \sigma)$ but is symmetric on permuting pairs of indices: $(\mu \nu) \leftrightarrow (\rho \sigma)$ the number of independent components given $n$ dimensions is:

\[
D(n)_{\text{Riemann}} = \frac{n^2 (n^2 - 1)}{12} \quad (A.5)
\]
Which in our specific case \((2+1)\) amounts to a total of \(D(3)_{\text{Riemann}} = 6\). Invoking the original argument of Chapter 2 section 3, a tensor such as the Ricci can only possess:

\[
D(n)_{\text{Ricci}} = \frac{n(n+1)}{2} \tag{A.6}
\]

independent components. As an expected result, given the vanishing of the Weyl tensor, the Ricci tensor has \(D(3)_{\text{Ricci}} = 6\) independent components. Therefore in this dimensions:

\[
D(3)_{\text{Riemann}} = D(3)_{\text{Ricci}}
\]

This is a very powerful result as it reduces considerable the possible components of the Riemann tensor, in fact they must be the same components at least of the ones present in the Ricci tensor as it would be expected by equation \((2.30)\).

With this at hand it is reasonable to start the other way around and find first the Ricci scalar and from it construct the other geometric entities. Given the Einstein field equations with cosmological constant:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = 0 \tag{A.7}
\]

We can take the contraction of \((A.7)\) with \(g^{\mu\nu}\) and find:

\[
R - \frac{3}{2} R - \frac{3}{l^2} = 0
\]

\[
R = -\frac{6}{l^2}.
\]

As such our space time has constant curvature, dictated by the Ricci scalar:

\[
R = -\frac{6}{l^2} \tag{A.8}
\]

It is intriguing that the curvature does not depend any how on the Mass or Angular momentum of the black hole. This is due to the the absence of degrees of freedom in the \((2+1)\) theory \([6]\).

The existence of a constant curvature in the BTZ space time is a huge milestone for calculating effort as it permits the Riemann to be expressed totally as:
Appendix A. Geometric Aspects and BTZ Validity

\[ R_{\mu\nu\rho\sigma} = \frac{R}{6} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \]  
\[ \text{(A.9)} \]

This is a general relationship for space times of constant curvature (\textbf{Dinv aqui}). The generalization is given by \(6 \rightarrow n(n - 1)\) where \(n\) is the dimension of the space time.

Tracing this equation by \(g^{\rho\mu}\) gives:

\[
R^\rho_{\nu\rho\sigma} = g^{\rho\mu} R_{\mu\nu\rho\sigma} \\
= \frac{R}{6} (g^{\rho\mu} g_{\mu\rho} g_{\nu\sigma} - g^{\rho\mu} g_{\mu\sigma} g_{\nu\rho}) \\
= \frac{R}{6} (3g_{\nu\sigma} - g_{\rho\sigma} g_{\nu\rho}) \\
= \frac{R}{6} (3g_{\mu\sigma} - g_{\sigma\nu}) \\
= \frac{R}{3} g_{\nu\sigma}.
\]

Interchanging the last indices to \(\nu \rightarrow \mu\) and \(\sigma \rightarrow \nu\) and inserting the value of \(R\) \([\text{A.8}]\) we find for the Ricci tensor \(R_{\mu\nu} = g^{\rho\mu} R_{\mu\rho\nu}\):

\[
R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu} \quad \text{(A.10)}
\]

This result will come at hand when validating the BTZ as a solution of the Einstein field equations. Furthermore, given the equation \([2.30]\) it is straight forward to find the Riemann tensor. Here for completeness only the result will be cited:

\[
\begin{align*}
R_{ttrt} &= \frac{1}{N^2 l^2} \left( r^2 (N^\phi)^2 - N^2 \right) \\
R_{ttrt} &= 0 \\
R_{trrr} &= -\frac{r^2 N^\phi}{N^2} \\
R_{trt\phi} &= \left( \frac{r N^\phi}{l} \right)^2 \\
R_{t\phi\phi} &= 0 \\
R_{r\phi\phi} &= -\left( \frac{r}{N l} \right)^2.
\end{align*}
\]
BTZ Validation

Following the results obtained previously the validation of the BTZ solution can be successfully done. Consider Einstein field equations:

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = 0 \]

Now replacing (A.10) and (A.8) on this equation gives:

\[
\left( -\frac{2}{l^2} g_{\mu\nu} \right) - \frac{1}{2} \left( -\frac{6}{l^2} \right) g_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} = 0
\]

\[
\left( \frac{3}{l^2} - \frac{3}{l^2} \right) g_{\mu\nu} = 0
\]

\[ 0 = 0. \]

With this the solution is consistent with the field equations.
Bibliography


