Spontaneous Symmetry Breakdown
An algebraic Approach

by

Pedro Fernando Atencio Gómez

Submitted to the Pedro Fernando Atencio Gómez
Department of Physics
in partial fulfillment of the requirements for the degree of

Bachelor of Science in Physics

at

UNIVERSIDAD DE LOS ANDES

November 2013

Author ....
Pedro Fernando Atencio Gómez
Department of Physics
November 21, 2013

Certified by ....
Andrés Reyes Der.rer.nat.
Associate Professor
Thesis Supervisor

Spontaneous Symmetry Breakdown
An algebraic Approach

by

Pedro Fernando Atencio Gómez

Submitted to the Pedro Fernando Atencio Gómez
Department of Physics
on November 21, 2013, in partial fulfillment of the
requirements for the degree of
Bachelor of Science in Physics

Abstract

Spontaneous Symmetry Breakdown is a concept which explains some abrupt changes that can occur to a system. The phenomena exposes itself as the possibility of finding inequivalent representations, each representation corresponding to a different realization of the same physical system. This inequivalent representations can only appear on Quantum Mechanics with infinite degrees of freedom. So here is the puzzle: We have a theory with finite degrees of freedom that cannot describe the behavior of the system and an infinite idealization that does take account of this new behavior but cannot possibly correspond to reality given the system is in fact finite. In the present work we present an account of these issues based on the representation theory of a C^*-algebra and illustrate the phenomenon of Spontaneous Symmetry Breaking through explicit numerical computation.

Thesis Supervisor: Andrés Reyes Der.rer.nat.
Title: Associate Professor

Acknowledgments

I would like to thank my advisor Andrés Reyes who taught me a new approach to physics and a fair share of aspects about the rather particular construction of this fascinating universe. To professor Alonso Botero, whose teachings were an invaluable part of my formation and was always willing to mediate some philosophical problems I have had with this thing called physics. To my dear friends who were always willing to listen to many of my nonsense when the subject was not at all clear to me. To professor Monika Winklmeier who spend precious time and effort revising this work. To my mother who allowed, helped, encourage, backed, subsidize, nursed a dream of mine come true.

-To my mother und meine Großmutter. ***(Requiescat in pace.)***

The secret to life is to have a task, something you devote your entire life to,
something you bring everything to, every minute of the day for your whole life.
And the most important thing is
- it must be something you cannot possibly do.
(Henry Moore)

THIS PAGE INTENTIONALLY LEFT BLANK

Contents

Chapter 1

Preliminaries

In the following sections we will give two different ways to describe Quantum Mechanics. One way is related to the usual description one learns as an undergraduate student which is the Hilbert Space picture, the other is an algebraic description which will allow us to expose the concept of Spontaneous Symmetry Breakdown. The prerequisites for the reading of this text are up to Quantum Mechanics I and Statistical Physics, both to the undergraduate level.

1.1 Looking for Directions

The present work is designed to be a "tour" through the various aspects that Spontaneous Symmetry Breakdown involves. However it is not the intention of this work to give rigorous proofs of the theorems and propositions related to the subject. The mathematical entusiast is encouraged to look for more appropiate accounts of the theorems in the references exposed on the bibliographical remarks at the end of each chapter. It is within the aims of this work to motivate the subject physically and mathematically hand in hand, thus every effort is made to give a good account of how to state all the physical concepts involved in a correct mathematical language.

The summary of the text goes as follows:

- Give a mathematical background with the main results involving theorems, propositions, constructions and definitions needed to state in appropriate terms the concept of Spontaneous Symmetry Breaking while physically motivating the formal statements.

- Give an account of the desirable properties the representations must have to be considered a physically relevant one.
- State formally the concept of a symmetry both at the representation level and at the algebraic level in order to define the concept of Spontaneous Symmetry Breakdown in an adequate manner.
- Go on to show an example illustrating the concept of Spontaneous Symmetry Breakdown demonstrating where the mathematical background shows up and noting its importance. All of this is done by working out the finite and infinite dimensional setup for an Ising chain with a transverse field, a system which explicitly exhibits Spontaneous Symmetry Breakdown.
- Direct the reader's attention towards the philosophical importance of Spontaneous Symmetry Breakdown and the enigma it posses.
- Give a solution to this dispute and conciliate the two problematic points of view on subject.

I hope the present work is a fine recollection of the account on the subject exposed in a coherent manner. It is organized in such a way that permits the reader to follow a clear line of reasoning avoiding the inconvenience of looking for references which are more oriented towards the mathematical or physical side. The objective was to create an understandable text with some sort of equilibrium such that it was not a logical treaty nor a loose account based completely on intuition.

1.2 Quantum Mechanics in Hilbert Spaces

1.2.1 Hilbert Space

This is the mathematical framework used to formalize Quantum Mechanics. The person responsible for realizing that this was the underlying structure of Quantum Mechanics was John von Neumann [1].

To the question: *What is exactly a Hilbert space?*

Cauchy Sequence: Let $(V, \| \cdot \|)$ be a normed vector space, a sequence (f_n) in V is called a Cauchy sequence if for all $\epsilon > 0$ exists $N \in \mathbb{N}$ such that for all $n, m > N \quad \|f_n - f_m\| < \epsilon$.

Note that every convergent sequence is a Cauchy sequence but not every Cauchy sequence converges to a point in V.

Banach Space: It is a normed vector space $(V, \| \cdot \|)$ in which every Cauchy sequence converges to an element in the vector space with respect to the metric induced by the norm.

Hilbert Space: It is an inner product vector space $(V, \langle \, , \rangle)$ which is a Banach space with respect to the norm induced by the inner product.

Note that every closed subspace of a Hilbert space is a Hilbert space itself.
A ***state*** is a normalized vector in the Hilbert space. With this interpretation in mind the condition for the vector space to be a normed one allow us to have a notion of distance between the elements, i.e. we are able to say if two states are close or far from each other.

1.2.2 Operators on a Hilbert Space

Linear Operator: A linear operator on a Hilbert space is a linear map $a : \mathcal{H} \to \mathcal{H}$.

It would seem desirable to require that for a sequence: (f_n) such that $\lim_{n\to\infty} f_n = f$ this implicates: $\lim_{n\to\infty} af_n = af$. As it turn out this is the definition of a ***continuous operator***, so this condition is not so strange.

As it turns out a **linear operator on a Hilbert space is bounded if and only if it is continuous** [1]. Hence asking for continuous operators is equivalent to asking for bounded operators. And the requirement for continuous operators is a more physically motivated request.

Adjoint: For a Hilbert space $\mathcal{H}$ and a bounded operator a acting on $\mathcal{H}$ the adjoint of a is uniquely determined by: $\langle af, g\rangle = \langle f, a^*g\rangle$ for all f, g

To the question: *Why should the observables be self-adjoint operators?*

Proposition 1.2.1. *An linear operator a acting on a complex vector space is self-adjoint if and only if $\langle f, af \rangle \equiv \langle f \,|\, a \,|\, f \rangle \in \mathbb{R}$ for all f, g.*

In Quantum Mechanics the Hilbert spaces where we work are complex vector spaces and the measurements are going to have a correspondence with self-adjoint linear operators, since the measurements are going to be finite real quantities, we see that observables should be bounded self-adjoint operators acting on a Hilbert space whose normalized elements are going to have an interpretation in terms of the states of the system.

What about the spectrum of a?
For a finite dimensional Hilbert space the spectral theorem of linear algebra tells us that each self-adjoint linear operator a admits a spectral decomposition i.e. there is an orthonormal basis of eigenvectors each corresponding to a real eigenvalue, the set of eigenvalues is called the spectrum. In general the spectrum of a is defined by:

$$\sigma(a) = \{\lambda \in \mathbb{C} \;|\; det(a - \lambda 1) = 0\}$$

Compact Operators: A bounded operator of $\mathcal{H}$ will be called compact if and only if it is the norm-limit of a sequence of finite rank operators.

Trace-class Operator: A compact operator will be called trace-class if and only if its trace-norm: $\|a\|_1 = \sum_k \sqrt{\mu_k}$ is finite, where μ_k are the eigenvalues of a^*a.

If a is a trace-class operator, then its trace is absolutely convergent since each μ_k is positive and $\|a\|_1$ is finite.

Density Matrix: An operator ρ will be called a density matrix if ρ is a positive trace-class operator with $\mathrm{Tr}(\rho) = 1$ (hence $\|\rho\|_1 = 1$). Or in an equivalent manner if $\rho = \sum_i \lambda_i p_i$ strongly with $\dim(p_i) < \infty$, $0 < \lambda_i \leq 1 \;\; \forall i$ and $\sum_i \lambda_i = 1$. Where p_i is the projection corresponding to λ_i.

What can we do in the laboratory?

Well, for starters we have a quantum system (state) and all the information we could possibly extract from it can only be aquired using measurements (observables), so any knowledge about the system is obtained by means of a pairing betweeen the measurement aparatus and the system itself [1], i.e. in our mathematical theory we need two kinds of objects: states and operators, and the measurements will correspond to a pairing: $\mathrm{Tr}(a\rho)$ wich will be a finite real number since it is going to be associated with an outcome of the measurement. Following this line of thought, we can check that all of the requirements have a reason to exist. This is in order to ilustrate that the mathematical setting of the theory is not a bunch of crazy mathematical requirements. There are in fact multiple reasons why this Hilbert space structure is appropriate when describing the quantum world.

1.3 Quantum Mechanics in the Algebraic Picture

In this section the reader should keep in mind the Heisenberg interpretation of Quantum Mechanics where the observables evolve in time and the state does not. So a quantum system would be described by some set of observables with a time evolution defined on the set and an operation (inner product or trace) that get numbers out of performing pairings of observables with states.

The algebraic formalism used in this section to describe Quantum Mechanics has the essence of the Heisenberg interpretation where we have a reference vector and the time evolution is defined on the set of observables. As mentioned before, the insight concerning the fact that a quantum system can and should be caracterized by its algebra of observables is essentially due to von Neumann, this insight is founded

[1] von-Neumann was the one who realized this was a more appropriate way of characterize a quantum system. Since all Hilbert spaces of the same dimension are isomorphic, we cannot caracterize a quantum system by saying its Hilbert space is $L^2(\mathbb{R}^3)$, but by properties and conditions of the observables.

on the fact that while describing an infinite quantum system [2] the usual notion of state would hardly hold. Since the system is infinite, it would prove difficult to write an explicit expression of the state, however, since the observables are going to be local operations on the state, this seems like a more appropriate way of making the description. Another path is to take a reality based argument. Say you have two physical systems, if no measurements are performed then for all we know both systems are equal, it is only by performing measurements on each system that we can differentiate them [2]. Moreover, for all intents and purposes we can interpret each of the physical systems as the distribution of the outcomes of each measurement aparatus, this kind of reasoning leads to the conclusion that physical operations on the system should play a fundamental role in the description of the system.

1.3.1 C*-algebras

This is the formalism required to study infinite quantum systems, which also applies to ususal Quantum Mechanics.

What is a C*-algebra *exactly?*

Banach Algebra: Is a Banach space A with a product $A \times A \to A$, $(a, b) \mapsto ab$ sucha that A becomes an algebra. In addition it must satisfy: $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

Involution: Is an operation $* : A \to A$ such that:

$$(a^*)^* = a, \qquad (ab)^* = b^* a^*, \qquad (\lambda a)^* = \overline{\lambda} a^*, \quad \forall a, b \in A\ ,\ \lambda \in \mathbb{C}.$$

An algebra with an involution operation is called a *-algebra.

Unital Algebra: An algera A with a unity 1 is called a unital algebra.

C*-algebra: Is a Banach algebra with an involution operation in which the additional property $\|a^* a\| = \|a\|^2$ holds.

[2]Infinite Quantum Mechanics refers to Quantum Mechanics with infinite degrees of freedom.

Let A, B be a C*-algebra and $T : A \to B$ a map. Then T is called a homomorphism is it preserves the algebraic structures, i.e. if for all $a_1, a_2 \in A$ and $\lambda \in \mathbb{C}$ $T(a_1 + \lambda a_2) = T(a_1) + \lambda T(a_2)$.

1.3.2 States

Now that we have defined the structure of what is going to eventually play the role of the observables of our physical system, the next natural question is:

What is going to play the role of states in this algebraic formalism?

State: A state on a unital C*-algebra will be a positive normalized linear functional:

$$\omega : A \to \mathbb{C}, \qquad \omega(a^*a) \geq 0, \qquad \omega(1) = 1.$$

From this definition we see that every convex combination of states is also a state, meaning that the set of states is a convex set as expected since this implies that mixed states are going to belong to such set. If this is going to be the definition of state then:

How can we differenciate a pure state from a mixed state in this algebraic formalism?

Pure State: A state ω will be called pure if and only if the decomposition $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ for $\lambda \in (0,1)$ implies $\omega_1 = \omega_2$, i.e. ω cannot be decomposed as the convex sum of two other states. Otherwise it will be called mixed.

Lastly, we need to answer: *What is the spectrum of a?*

Spectrum: The spectrum of $a \in A$ will be the following set:

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \quad \textit{has no inverse in } A\}.$$

we remark that the set of all bounded operators on a Hilbert space, $\mathcal{B}(\mathcal{H})$ is a unital C*-algebra under the supremum norm.

Representation: A representation of a C*-algebra on a Hilbert space $\mathcal{H}$ is a *-homomorphism $\pi : A \to \mathcal{B}(\mathcal{H})$. Since we are talking about a homomorphism between C*-algebras, this will be a mapping preserving the sum, product and involution [2].

Cyclic Representation: A representation is called cyclic if exists $\Omega \in \mathcal{H}$ such that $\mathcal{H} = \overline{\pi(\mathcal{A})\Omega}$. Ω is called a cyclic vector.

1.4 The GNS Construction

Now, to the question: *How to reconcile these two ways of describing a quantum system?*

The construction due to Gelfand-Naimark-Segal is responsible for recovering the usual representation setup of Quantum Mechanics starting from the structure provided by this new formalism and establishing the bridge between the two different ways of describing Quantum Mechanics.

Theorem 1.4.1 (*cf.* [2] pag. 34)**.** *For any state ω over a unital C*-algebra A there exists a triple composed of a Hilbert space, a representation and a cyclic vector: $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ such that:*

$$\omega(a) = \langle \Omega_\omega \,|\, \pi_\omega(a) \,|\, \Omega_\omega \rangle \quad \forall\ a \in A$$

The triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is called the GNS representation. The idea of the construction goes like this:

- Define a sesquilinear form as: $(a, b) := \omega(a^* b)$ and realize that it may fail to be positive definite since $N_\omega = \{a \mid \omega(a^* a) = 0\}$ can have nonzero vectors.

- Take the quotient space A/N_ω and end up with an inner product space with a cyclic vector where: $\Omega_\omega = [1]$, $\pi_\omega(a)\Omega_\omega = [a] \equiv \Omega_{[a]}$ and:

$$\left\langle \Omega_{[a]} \,\middle|\, \Omega_{[b]} \right\rangle = \langle \pi_\omega(a)\Omega_\omega \,|\, \pi_\omega(b)\Omega_\omega \rangle = \omega(a^* b)$$

- This will be a pre-Hilbert space, so it is necessary to take the completion of it in the inner product norm.

A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is called ***irreducible*** if and only if the only invariant closed subspaces of $\mathcal{H}$ under $\pi(A)$ are the trivial subspace and $\mathcal{H}$ itself.

Theorem 1.4.2 (*cf.* [1])**.** *The GNS representation is irreducible if and only if ω is pure.*

1.5 A Few More Aspects

1.5.1 About Irreducibility

Proposition 1.5.1 (*cf.* [1])**.** *The next conditions on a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ are equivalent:*

- *π is irreducible*
- *Every nonzero vector in $\mathcal{H}$ is cyclic.*
- *$\pi(\mathcal{A})' = \mathbb{C}1$ where $\pi(\mathcal{A})'$ is the set of all bounded linear operators that commute with $\pi(\mathcal{A})$*

Also it is helpful to keep in mind that **every C*-algebra admits an injective representation** $\pi : A \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ [1]. The importance of the injectivity relies on the fact that the elements of the C*-algebra are going to be physical operations. As such it would be inconvenient to have different elements of the algebra mapped to a same element in $\mathcal{B}(\mathcal{H})$ since this would have as consequence the interpretation that different physical operations would give the same information about the system.

1.5.2 About the Dynamics

In this formalism, we have already said, we would like for the elements of the C*-algebra to play the role of the physical operations, and to have a state that would allow us to perform pairings and get some numbers out of the theory.

Let us address the question involving *What produces the dynamics in the C*-algebra A.*

Dynamics: Let $Aut(A)$ be the group of automorphisms in A. Then a dynamics on A will be given by a continuos group homomorphism $\alpha : \mathbb{R} \to Aut(A)$ such that $t \mapsto \alpha_t$. Such continuity requirement will prove important in the section regarding physically relevant representations discussed in the next chapter.

1.5.3 About the Ground state

For $\beta \in Aut(A)$, ω a state let us define a new state as $\beta^*\omega(a) \equiv \omega(\beta(a))$ for all $a \in A$. In addition, for $\beta \in Aut(A)$ and $\pi : A \to \mathcal{B}(\mathcal{H})$ we say that β is ***implemented*** in π (the representation) by the unitary transformation $u : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, if $\pi(\beta(a)) = u\pi(a)u^*$ for all $a \in A$. Before the definition of ground state, a couple of theorems:

Theorem 1.5.2 (*cf.* [1])**.** *Let $\beta \in Aut(A)$ and $\beta^*\omega = \omega$ i.e. $\omega(\beta(a)) = \omega(a)$ for all $a \in A$. Then there exists a unitary operator $u : \mathcal{H}_\omega \to \mathcal{H}_\omega$ implementing β such that: $u\Omega_\omega = \Omega_\omega$.*
In particular, for a continuous group homomorphism $\alpha : G \to Aut(A) \mid \alpha_g \equiv \alpha(g)$ such that $\alpha_g^*\omega = \omega$ for all $g \in G$ one obtains a family of unitaries $u_g : \mathcal{H}_\omega \to \mathcal{H}_\omega$ satisfying:

$$u_g\Omega_\omega = \Omega_\omega, \qquad \pi_\omega(\alpha_g(a)) = u_g\pi_\omega(a)u_g^*.$$

and such family of unitaries is a continuous representation of G on $\mathcal{H}_\omega$.

Theorem 1.5.3 (Stone's theorem (*cf.* [1]))**.** *Let $u : \mathbb{R} \to U(\mathcal{H}) \mid u_t \equiv u(t)$ be a strongly continuous group homomorphism. Define (a possibly unbounded operator) h such that: $h : D(h) \to \mathcal{H} \mid \; h\psi := i\lim_{t\to 0} \frac{u_t - 1}{t}\psi$ and $D(h)$ consists of all $\psi \in \mathcal{H}$ such that the previous limit exists. Then $D(h)$ is dense in $\mathcal{H}$ and h is self-adjoint.*

Once defined the dynamics on A we will say a state ω is a **ground state** of (A, α_t) if and only if:

- ω is time-independent, i.e. $\alpha_t^*\omega(a) \equiv \omega(\alpha_t(a)) = \omega(a)$.

- The generator h_ω of the continuous unitary representation $t \mapsto u_t$ has a positive spectrum, where: $u_t \pi(a) u_t^* = \pi(\alpha_t(a))$.

The first point in the definition of the ground state puts us in the particular case of the *Theorem* 1.5.2 This assures the existence of a family of unitary transformations in the representation space corresponding to the GNS construction. The second item in the definition refers to the generator which is to say the operator defined via *Stone's theorem* and it guarantees that the spectrum $\sigma(h_\omega)$ of such self-adjoint operator is positive having as a consequence boundedness from below.

For an alternative more algebraic definition of a ground state which coincides at the representation level with the previous definition we invoke the next proposition.

Proposition 1.5.4 (*cf.* [1]). *ω is a ground state if and only if $-i\omega(a^*\delta(a)) \geq 0$ for all $a \in D(\delta)$ where δ is the algebraic derivation associated to time evolution:* $\delta(a) = \lim_{t \to 0} \frac{\alpha_t(a) - a}{t}$

The previous two definitions of ground states coincide in the finite dimensional case, however in the infinite dimensional case the definition that holds is the one associated to *Proposition* 1.5.4.

Bibliographical Remarks: Throughout the extention of this chapter we made extensive use of references [2], [1] due to Araki and Landsman, where all these results and various proofs can be found.

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 2

Finite/Infinite Quantum Mechanics & Physically Relevant Representations

For a quantum system the corresponding C*-algebra is going to be interpreted in terms of the set of physical operations. The job is to find representations of this algebra into the bounded operators of some Hilbert space since these representations are the ones which are going to be understood as a physical realization of the system.

2.1 Finite Quantum Mechanics

First let me be clear what I am refering to when talking about Quantum Mechanics with finite degrees of freedom or the less accurate term "finite Quantum Mechanics". This means usual Quantum Mechanics where we have $2n$ operators $\{q_1, ..., q_n, p_1, ..., p_n\}$ acting on a Hilbert space $\mathcal{H}$ and satisfying canonical conmutation relations. In the light of this new formalism, the C*-algebra of interest is the Weyl algebra formed by taking as symplectic vector space the Heisenberg algebra $\mathfrak{h}_n$, i.e. the algebra generated by the n $q's$ and the n $p's$ endowed with the standard symplectic form on $\mathbb{R}^{2n}$ [3].

Now we have to look for representations of such an algebra. As it turns out, the variety of representations one can find for the finite case (still refering to degrees of freedom) is kind of limited in scope because all possible representations are unitarily equivalent and equivalent to the Schrödinger representation. This is not at all a trivial fact and its veracity is assured by the Stone-von Neumann uniqueness theorem. In particular, all representations are equivalent to the GNS representation. This allows us to appreciate the importance of the GNS construction and permits us to retrieve the usual notion of Quantum Mechanics without having to worry about additional representations that might arise (at least for the finite case). It is also an important matter because, as you can easily convince yourself, the task of finding representations is not easy, even less so in the infinite case, and having an explicit construction guarantees that at least one such representation can be found.

2.2 Infinite Quantum Mechanics

In this section we are interested in finding out which will be the C*-algebra of observables for the case of infinitely extended quantum systems. Next we make use of physical considerations in order to state properties the algebra of observables must possess.

Local Structure: To each open bounded region in space-time $\mathcal{O}$ we can associate the C^*-algebra of all bounded operators in such region $\mathcal{A}(\mathcal{O})$. This is the algebra generated by the localized canonical variables:

$$a(f) = \int dx\ \psi(x)\bar{f}(x), \qquad a(g) = \int dx\ \psi^*(x)g(x).$$

Were f and g are test functions with compact support in $\mathcal{O}$, $\psi(x)$ are the quantum field solutions of the equations of motion. That is, $\mathcal{A}(\mathcal{O})$ is the algebra generated by the $a(f)$'s, which correspond to the "smearing" of every test function with compact support in $\mathcal{O}$ with the field [**?**].

Isotony Property: The mapping between a region and its algebra of observables must fulfill: If $\mathcal{O}_1 \subseteq \mathcal{O}_2,\ \ then\ \ \mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$.

Causality: If $\mathcal{O}_1, \mathcal{O}_2$ are causally complete regions[1] such that $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$, i.e. algebras of disjoint causally complete regions must commute.

Finally the algebra of observables will be the norm closure of: $\mathcal{A}_L = \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$ i.e. $\mathcal{A} = \overline{\mathcal{A}_L}$. This will give rise to the smallest C*-algebra with satisfactory localization properties [5] [4].

Now, a natural requirement for $\mathcal{A}$ to satisfy is that the algebra at infinity, i.e. the generalized intersection of the localized algebras $\mathcal{A}(\mathcal{O})$, should be the algebra generated by the unital element. This should be true since physical operations are localized in a region of space-time which have as a result that the only physical operation that can be performed in all space-time must be the identity. This requirement defines a property on $\mathcal{A}$ called ***cluster property***.

Theorem 2.2.1 (*cf.* [1])**.** *A state ω has a trivial algebra at infinity if and only if for all $a \in \mathcal{A}$, $\epsilon > 0$ exists Λ' such that: $\|\omega(ab) - \omega(a)\omega(b)\| \leq \epsilon$ for all $b \in \mathcal{A}(\Lambda')$ with Λ' a space region [1].*

Now, by adding to $\mathcal{A}$ a automorphism corresponding to space translations $\alpha_x : \mathcal{A}(\Lambda) \to \mathcal{A}(\Lambda + x)$, a new property arises called ***asymptotic abelianess property*** which states that: $\lim_{x\to\infty}[\pi_\omega(a), \pi_\omega(\alpha_x(b))] = 0$.

This are some properties the *quasi local algebra* $\mathcal{A}$ must fulfill, distinct authors give preference to diferent properties all of which are physically motivated, these correspond to Landsman [1]. These relevant physical properties help to reduce the set of possible representations that would be of physical interest. Such considerations are necessary since we are dealing with Quantum Mechanics with infinite degrees of freedom, and in this case the representations are not all unitarily equivalent one example being the Ising chain. Moreover, they are not all going to be of physical relevance so we need some properties that every physically relevant representation must satisfy in order to restrict the search.

[1] A region is causally complete if its double causal complement is the region itself, where the causal complement is the set of points that are space-like to all points in the region.

2.3 Physically Relevant Representations

We have seen the very famous GNS construction which allows us to construct a representation starting with a state. Since in infinite dimensional Quantum Mechanics we might find inequivalent representation these are some properties any physically relevant representation must fulfill. These properties are the ones given precedence by Strocchi [4].

Regarding the Existence of Energy and Momentum

The space and time translations in the representation must be implemented by strongly continuos groups of unitary operators. This ensures, thanks to Stone's theorem, the existence of the generators of such unitary groups (P and H) who will be understood in terms of momentum and energy respectively and will correspond to self-adjoint operators in the representation space $\mathcal{H}_\pi$.

Regarding the Stability of H

We will also require for H to have a spectrum $\sigma(H)$ which is bounded from below. This ensures the fact that under small perturbations the system does not collapse to lower and lower energy states.

Regarding the Ground State

The $\min(\sigma(H))$ must be an non-degenerate eigenvalue of H, the corresponding eigenvector will be called the ground state Ψ_o and **it must be** a *cyclic vector* i.e. $\mathcal{H}_\pi = \overline{\pi(\mathcal{A})\Psi_o}$ and the *unique translationally invariant state* in $\mathcal{H}_\pi$.

The previous definition of ground state coincides in finite Quantum Mechanics with the other two. As previously said the definition of ground state in infinite Quantum Mechanics is the one given by *Proposition* 1.5.4.

About the representations: At this point a lot of information has been quickly shoved down our eyes and the connections can get a bit fuzzy so let me make a remark. The general idea is one has this C*-algebra structure but the physical realizations of a system (the thing one sees in real life while working in the laboratory) does not correspond to this kind of structure but to the representations of it. At this point it should be clear that since what is going to have a direct physical interpretation are the representations, the restrictions are going to be imposed on the representations as well, this is why we require some properties about the representations in order to be considered physically relevant. However, a very handy representation that can be explicitly constructed comes to play (the GNS representation), so a great deal of theorems appear connecting certain aspects regarding the C*-algebra and the GNS construction basically stating the conditions under which such representation will have all the desirable properties of a physical representation (this is why theorems at the algebraic level make their appearance). One more reason why these theorems appear is this: in the finite dimensional case (degrees of freedom wise speaking) all irreducible representations are unitarily equivalent, so one would like to have an irreducible GNS construction in order to be able to safely say that such representation is unitarily equivalent to any other. On the other hand for infinite Quantum Mechanics (Quantum Mechanics with infinite degrees of freedom) the search for representations can be very difficult, so, if by imposing restrictions on the C*-algebra one can assure that there is going to be at least one representation that does the job (the GNS representation) it would be a wonderful world. Moreover, the whole problem can be reduced to searching GNS representations induced by different pure states (pure states because we want the corresponding GNS representations to be irreducible). Lastly, one would like the representations to be irreducible because irreducibility "closes" somehow the physical world represented, meaning there are no closed subworlds, i.e. no invariant subspaces.

Bibliographical Remarks: Concepts presented in this chapter used at length reference [4] for the concept of physically relevant representations, and [5] for some conditions the representation of an infinitely extended quantum system should satisfy.

THIS PAGE INTENTIONALLY LEFT BLANK

Chapter 3

Symmetry & Spontaneous Symmetry Breakdown

This chapter is devoted to establish the difference between a symmetry at the algebraic level and a symmetry at the representation level. This difference is the essence of Spontaneous Symmetry Breakdown.

3.1 Symmetry and its Breakdown

Theorem 3.1.1. *For any mapping* $\mathbf{T} : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}'$ *(projective space) such that:*

$$|(\mathbf{T}\hat{\phi}, \mathbf{T}\hat{\psi})| = |(\hat{\phi}, \hat{\psi})|$$

There exists a mapping $U : \mathcal{H} \to \mathcal{H}'$ *such that:* $U\phi \in \mathbf{T}\hat{\phi}$ *if* $\phi \in \hat{\phi}$ *with* U *either unitary or antiunitary.[6]*

This important theorem due to Wigner elucidates the concept of symmetry at the representation level in Quantum Mechanics, i.e. it states the type of operations that can be performed on the Hilbert space that have no effect on the transition probabilities, which are either unitary or antiunitary. In consequence such operations do not change the information we can possibly gather about the system since this is the information we can extract about the system. This is to be expected since unitary or antiunitary transformations are isometries with respect to the inner product.

Before starting to examine the concept of symmetry breaking we must first talk about what is a symmetry at the algebraic level and how is it related to a Wigner symmetry (symmetry at the representation level).

What is a symmetry of the system in our new formalism of C-algebras?*

For a $C*$-algebra A, a symmetry β on A, called an ***algebraic symmetry***, is a *-automorphism, i.e. $\beta \in \mathrm{Aut}(\mathcal{A})$. As you can see this is quite natural since any automorphism will leave the $C*$-algebra structure invariant and as such will not change the physical information (because as von Neumann's insight suggested, a system is properly described by the physical operations one can perform on the system), meaning we are still describing the same system.

How do we connect this algebraic symmetry with the usual Wigner symmetry?

Before stating the connection and finally stating what is going to be understood for Spontaneous Symmetry Breakdown, keep in mind that the problem of studying representations can be reduced to the study of GNS representations (further ahead we will see how). In addition, we look for states to be pure because there is an equivalence between the "purity" of a state and the irreducibility of the representation and we want irreducible representations because they are "closed worlds". The idea behind the concept of Spontaneous Symmetry Breakdown is to see if this physical realizations, this closed worlds, are equivalent. As we saw previously, any automorphism β will define a new state $\beta^*\omega$ in A, so the problem at hand will be to check if the representations of $(\mathcal{H}_\omega, \pi_\omega)$ and $(\mathcal{H}_{\beta^*\omega}, \pi_{\beta^*\omega})$ are unitarily equivalent.

A system will show **Spontaneous Symmetry Breakdown** if for an algebraic symmetry $\beta \in Aut(A)$ and a pure state ω the GNS representations $(\mathcal{H}_\omega, \pi_\omega)$ and $(\mathcal{H}_{\beta^*\omega}, \pi_{\beta^*\omega})$ are not unitarily equivalent, i.e. if the symmetry fails to be implemented at the representation level, or in more mathematical terms:

$$\nexists\, U : \mathcal{H}_\omega \to \mathcal{H}_{\beta^*\omega} \textit{ such that } U\pi_\omega(a)U^{-1} = \pi_{\beta^*\omega}(a)\ \ \forall a \in \mathcal{A}.$$

If on the contrary such a unitary operator does exist, then we say that the representations are **equivalent**, that the phase $(\mathcal{H}_\omega, \pi_\omega)$ is β**-symmetric** or that the symmetry is **unbroken.**

When referring to Spontaneous Symmetry Breakdown we are referring to a symmetry at the algebraic level that fails to be implemented at the representation level. It might seem a little curious that it is defined in the GNS representations but as we can see this is very general. For now let us announce the next theorem which makes life a bit easier:

Theorem 3.1.2 (cf. [1])**.** *An automorphism $\beta \in Aut(A)$ can be implemented in $(\mathcal{H}_\omega, \pi_\omega)$ if and only if $(\mathcal{H}_\omega, \pi_\omega)$ and $(\mathcal{H}_{\beta^*\omega}, \pi_{\beta^*\omega})$ are equivalent.*

With this additional information we can state in an equivalent manner that if: $\exists\, U : \mathcal{H}_\omega \to \mathcal{H}_\omega$ such that $U\pi_\omega(a)U^{-1} = \pi_\omega(\beta(a))$ for all $a \in A$ the phase $(\mathcal{H}_\omega, \pi_\omega)$ is β**-symmetric**.

Let us talk a bit about: *Why the GNS representation is such a general thing?*

Here is what happens: for any physically relevant representation of a $C*$-algebra we can find $(\mathcal{H}_\pi, \pi)$ with reference cyclic vector Ψ we can define the algebraic state ω by $\omega(a) = \langle \Psi \,|\, \pi(a) \,|\, \Psi \rangle$ and the GNS representation $(\mathcal{H}_\omega, \pi_\omega)$ associated to ω is going to be equivalent to π. This means the whole problem of finding representations of the algebra is equivalent to constructing the GNS representations of the state space. In addition we want the representations to be irreducible in order to be able to associate the concept of “closed world” to them. Under this interpretation Spontaneous Symmetry Breakdown can be understood as the impossibility of connecting two phases by means of a unitary transformation, as a result they cannot be equivalent. Physically speaking they are different realizations of the same physical system. For another interpretation, let me introduce a new concept.

Folium: [7] For an pure algebraic state ω we will consider the representation π_ω and say the folium $\mathcal{F}_\omega$ will consist of all algebraic states induced by states in $\mathcal{H}_\omega = \overline{\pi_\omega(a)\Omega_\omega}$. Considering the representation is irreducible since ω is pure, and taking a look at the second item in *Proposition* 1.5.1 we have that for all $\omega_1, \omega_2 \in \mathcal{F}_\omega \implies \mathcal{F}_{\omega_1} = \mathcal{F}_{\omega_2}$. In consequence we have that for two irreducible GNS representations π_ω and π_ϕ they will be either unitarily equivalent or $\mathcal{F}_\omega \cap \mathcal{F}_\phi = \emptyset$ [7]. Under this perception of the matter at hand the problem is to find *How many different folia are there?*. The interesting thing about this approach is that it is a question formulated over some sets whose elements are within the scope of the algebraic formalism.

3.2 Stone-von Neumann Theorem

The Stone-von Neumann uniqueness theorem refers to the unitary equivalence of all possible representations of a given quantum system with finite degrees of freedom. This is a remarkable result with important historical relevance which gave a positive answer to the question concerning the equivalence of matrix and wave quantum mechanics as descriptions of the same physical reality [8].

Preliminaries of the theorem:

- Pontryagin dual: Let L be a locally compact abelian group, then the Pontryagin dual of L is the group of continuous homomorphims from L to T (the circle group), and it is denoted by $\widehat{L}$.
- Isomorphism between $\mathbb{R}^n$ and $\widehat{\mathbb{R}^n}$: Let $y \in \mathbb{R}^n$ and $\chi_y \in \widehat{\mathbb{R}^n}$ the association: $y \mapsto \chi_y$ such thath $\chi_y(x) = x \cdot y + Z$ for all $x \in \mathbb{R}^n$ is an isomorphism.
- Translation and Modulation operators: These operators are defined on $L^2(\mathbb{R}^n)$ on the following manner:

$$T_x f(u) := f(u - x), \quad M_y f(u) \equiv M_{\chi_y} f(u) := e^{2i\pi\chi_y(u)} f(u).$$

These operators satisfy commutation relations: $[T_x, M_y] = e^{-2i\pi\chi_y(u)} 1_{L^2(\mathbb{R}^n)}$, which are the commutation relations of the Weyl operators.

- Let $H := \{e^{2i\pi t}T_xM_{\chi_y} : e^{2i\pi t} \in T, x \in \mathbb{R}^n, \chi_y \in \hat{\mathbb{R}^n}\}$

Theorem 3.2.1. *(1) The Hilbert space $L^2(\mathbb{R}^n)$ has no non-trivial proper closed subspace invariant under H.*
(2) Let $\mathcal{H}$ be a Hilbert space and $\rho : H \to U(\mathcal{H})$ a continuos group homomorphism such that $\rho(e^{2i\pi t}) = e^{2i\pi t}1_{\mathcal{H}}$, then, there exists an orthogonal sum decomposition of Hilbert spaces: $\mathcal{H} = \bigoplus_\alpha \mathcal{H}^\alpha$ such that for each α, there exists, up to scaling, a unique isometry $W^\alpha : L^2(\mathbb{R}^n) \to \mathcal{H}^\alpha$ satisfying $W^\alpha(hf) = \rho(h)W^\alpha(f)$ for all $h \in H, f \in L^2(\mathbb{R}^n)$.

Proof. Let $W_k = e^{i\pi xy}T_xM_y \ \forall k = (x,y) \in \mathbb{R}^{2n}$. These unitary operators are called the Weyl operators, and satisfy the relations: $(W_k)^* = W_{-k} \quad W_kW_l = e^{-i\pi\omega(k,l)}W_{k+l}$ where $\omega(k,l) = \omega((x,y),(u,v)) = yu - xv$ is the standard symplectic form on $\mathbb{R}^{2n}$.
For $f, g \in \mathcal{H}$ let us define the Fourier-Wigner transform as:

$$\begin{aligned} V(f,g)(k) := \langle W_kf \,|\, g\rangle &= \int_{\mathbb{R}^n} e^{2i\pi y(u-x/2)}f(u-x)\overline{g(u)}du \\ &= \int_{\mathbb{R}^n} e^{2i\pi y\cdot u}f(u+x/2)\overline{g(u-x/2)}du \end{aligned}$$

Let $F(x,y) = f(u)\overline{g(x)}$. In this case, we can see $V(F) := V(f,g)$ as the Fourier transform with respect to one variable after performing the change of variables: $(u,x) \mapsto (u+x/2, u-x/2)$, whose Jacobian is equal to one, i.e, we can interpret it as an isometry of $L^2(\mathbb{R}^{2n})$. As a consequence:

$$\langle V(f,g) \,|\, V(\phi,\psi)\rangle = \langle f \,|\, \phi\rangle \left\langle \overline{g} \,\middle|\, \overline{\psi}\right\rangle = \langle f \,|\, \phi\rangle \langle \psi \,|\, g\rangle \qquad .$$

Then, the proof of part (1) goes like this:
Let M be a non-trivial closed subspace of $L^2(\mathbb{R}^n)$ invariant under H. Then:
$\exists f \neq 0 \in M$, let $g \in M^\perp \to g \perp W_kf \ \forall k \in \mathbb{R}^{2n} \to V(f,g) = 0 \to$
$\|V(f,g)\| = \|f\|\|g\| = 0 \to \|g\| = 0 \to M = L^2(\mathbb{R}^n)$. So H is irreducible in $L^2(\mathbb{R}^n)$.

For the second part of the theorem we need to proof a couple of lemmas first. Let $\Phi \in L^1(R^{2n})$. Then:

$$\langle \rho(W_\Phi)v \,|\, w\rangle := \textstyle\int_{\mathbb{R}^{2n}} \Phi(k) \langle \rho(W_k)v \,|\, w\rangle \, dk \qquad \forall v, w \in \mathcal{H}$$

gives rise to a well defined bounded operator on $\mathcal{H}$, as follows:

$$\rho(W_\Phi) : \mathcal{H} \mapsto \mathcal{H} \quad \text{such that:} \quad \rho(W_\Phi) = \int_{\mathbb{R}^{2n}} \Phi(k)\rho(W_k)dk$$

Lemma 3.2.2. *Let $\mathcal{H} \neq 0$, and $\Phi \in L^1(\mathbb{R}^{2n})$. If $\rho(W_\Phi) = 0$, then $\Phi = 0$ almost everywhere.*

Proof. $\forall v, w \in \mathcal{H}$

$$\begin{aligned} 0 &= \langle \rho(W_k)\rho(W_\Phi)\rho(W_{-k})v \,|\, w\rangle = \textstyle\int \Phi(l) \langle \rho(W_k)\rho(W_l)\rho(W_{-k})v \,|\, w\rangle \, dl \\ &= \textstyle\int \Phi(l)e^{-i\pi\omega(l,-k)}e^{-i\pi\omega(k,l-k)} \langle \rho(W_l)v \,|\, w\rangle \, dl = \int \Phi(l)e^{-2i\pi\omega(k,l)} \langle \rho(W_l)v \,|\, w\rangle \, dl \end{aligned}$$

Given this is true for all v, w it is allowed to choose $w = \rho(W_l)v$ with v a unit vector. Then, by taking $k = l \implies \Phi(l) = 0$ for almost all of $\mathbb{R}^{2n}$. □

Let $\rho(W_\Phi) \equiv W_\Phi$ when refering to the canonical representation.

Lemma 3.2.3. *Let $\Phi \in L^1(\mathbb{R}^{2n})$, $f, g \in L^2(\mathbb{R}^n)$, and $\Phi = \overline{V(\phi,\psi)}$. Then $W_\Phi f = \langle f \,|\, \phi\rangle \, \psi$.*

Proof.

$$\langle \rho(W_\Phi)f \,|\, g\rangle = \int_{\mathbb{R}^{2n}} \overline{V(\phi,\psi)}(k) \langle W_k f \,|\, g\rangle \, dk = \langle V(f,g) \,|\, V(\phi,\psi)\rangle = \langle f, \,|\, \phi\rangle \langle \psi \,|\, g\rangle$$

Now, for $\phi = \psi$ unit vector, $W_\Phi f$ will be the orthogonal projection of f on ϕ and in consequence: $W_\Phi W_\Phi = W_\Phi$, and using the canonical representation in the definition of $\rho(W_\Phi)$ we get: $(W_\Phi)^* = W_\Phi$. □

Now we have the necessary machinery. The proof of part (2) goes as follows: Consider $N = \text{range}\{\rho(W_\Phi)\} \subseteq \mathcal{H}$ and let v_α be an orthonormal basis of N. Let $\mathcal{H}^\alpha := \overline{\{\rho(W_k)v_\alpha : k \in \mathbb{R}^{2n}\}}$. Then, each $\mathcal{H}^\alpha$ is invariant under $\rho(H)$. In addition $\forall v, w \in N$ we have: $\rho(W_\Phi)v = v$, $\rho(W_\Phi)w = w$ since $\rho(W_\Phi)$ is idempotent and v, w

are in the range of $\rho(W_\Phi)$. Using this facts we have:

$$\langle \rho(W_k)v \,|\, \rho(W_l)w \rangle = \langle \rho(W_k)\rho(W_\Phi)v \,|\, \rho(W_l)\rho(W_\Phi)w \rangle$$
$$= e^{-i\pi\omega(w,l)} \langle \rho(W_\Phi)\rho(W_{k-l})\rho(W_\Phi)v \,|\, w \rangle.$$

By using *Lemma*3.2.3 $(W_\Phi W_k W_\Phi)f = \langle f \,|\, \phi \rangle \langle W_k\phi \,|\, \phi \rangle \phi = \langle W_k\phi \,|\, \phi \rangle W_\Phi f$
$= W_{\langle W_k\phi \,|\, \phi \rangle \Phi} f$, so we will have:

$$\langle \rho(W_k)v \,|\, \rho(W_l)w \rangle = e^{-i\pi\omega(w,l)} \langle \rho(W_\Phi W_{k-l} W_\Phi)v \,|\, w \rangle = e^{-i\pi\omega(w,l)} \left\langle \rho(W_{\langle W_{k-l}\phi \,|\, \phi \rangle \Phi})v \,\middle|\, w \right\rangle$$
$$= e^{-i\pi\omega(w,l)} \langle W_{k-l}\phi \,|\, \phi \rangle \langle \rho(W_\Phi)v \,|\, w \rangle = e^{-i\pi\omega(w,l)} \langle W_{k-l}\phi \,|\, \phi \rangle \langle v \,|\, w \rangle.$$

Let $v = v_\alpha, w = v_\beta$ with $\alpha \neq \beta$. Then from the previous expresion we can see that $\mathcal{H}^\alpha \perp \mathcal{H}^\beta$. In consequence: $N = \bigoplus_\alpha \mathcal{H}^\alpha$ is an invariant orthogonal sum of Hilbert spaces, as is: $M \equiv (\bigoplus_\alpha \mathcal{H}^\alpha)^\perp$ then: $\rho(W_\Phi)M \subseteq M \cap N = \emptyset$, using *Lemma 5.0.4.*, $M = 0$ since $\Phi \neq 0$, which implies $\mathcal{H} = \bigoplus_\alpha \mathcal{H}^\alpha$ and $W^\alpha(W_k\phi) = \rho(W_k)v_\alpha$ is a unique covariant isometry $W^\alpha : L^2(\mathbb{R}^n) \to \mathcal{H}^\alpha$ up to a scale factor [9]. □

Next the importance of this result is explained. All of the observables are going to be some sort of combinations of operators P_i and Q_i (that is to say the $C*$-algebra of interest in Quantum Mechanics with finite degrees of freedom is the Weyl algebra formed by taking as symplectic vector space the Heisenberg algebra). However all we know about these operators is that they must fulfill commutation relations of the form $[Q, P] = i\hbar$, so we look for a representation of this operators on a Hilbert space $\mathcal{H}$. Now, *What if it would be possible to find inequivalent representations?*. In that case depending on the representation used, the physical predictions of a quantum system would be different, so we would run into trouble. Luckily Stone-von Neumann theorem ensures that all representations are going to be equivalent, so we are on safe ground and can proceed to choose a particular representation, i.e. the Schrödinger representation and calculate. As we can realize this is something quite nontrivial and indeed a remarkable result by Stone and von Neumann. The fermionic analog of this theorem is the Jordan-Wigner uniqueness theorem.

3.3 Summary

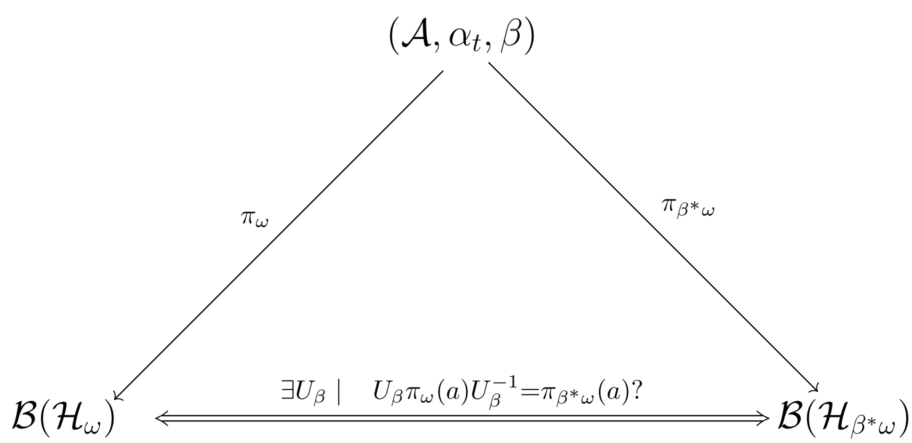
As we saw, for a quantum system with finite degrees of freedom there is a theorem called the *Stone-von Neumann uniqueness theorem* (for the bosonic case, CCR). The fermionic analog of the theorem being the *Jordan-Wigner uniqueness theorem* (for the fermionic case, CAR). In any case the theorem states that all representations are unitarily equivalent, so there cannot be Spontaneous Symmetry Breaking in finite Quantum Mechanics. If there were it would mean that the representations π_ω and $\pi_{\beta^*\omega}$ are inequivalent in contradiction to the statement of the theorem. However, this theorem no longer holds in infinite Quantum Mechanics, so the phenomenon concerning Spontaneous Symmetry Breaking appears. It should be clear from this paragraph that the phenomena of Spontaneous Symmetry Breaking is one which has as a fundamental requirement for its appearance to deal with quantum systems involving infinite degrees of freedom.

When dealing with a quantum system (finite or infinite dimensional) one way to encompass such a system into a mathematical structure is by stating the set of physical operations. This set has the structure of a $C*$-algebra. Next we look for states on this structure (positive normalized linear functionals) in order to construct representations (via GNS construction) which are going to be interpreted as a phase of the system (i.e. a physical realization). With the notion of representation we recover the "usual setup" of Quantum Mechanics having somehow the "essence" of the Heisenberg picture (this is the formalism used to describe entirely an quantum system). This is a realization of a system. For instance, talking about the infinite Ising model does not correspond to a particular physical realization. It is only when we specify the representation that it makes sense, this is so because there can be inequivalent representations so the model can belong to different closed worlds [7]. We will work these examples in the next chapter, but in order to get a sense of direction, here is a layout of the problem:

- **Ising Model with a Transverse Field:** For the n-site system there is only one ground state and with respect to a 180° rotation around the z-axis the phase is going to be symmetric. However, in the infinite case, i.e. when the chain is infinitely extended in both directions, there are two inequivalent representations and Spontaneous Symmetry Breakdown will make its appearance.

- **Ising Model:** In this case for the n-site system will have two ground states (all spins up or all spins down). Each representation is unitarily equivalent to the other, but in the infinite case there are infinite ground states each giving rise to a representation which can be classified in two groups. All representations in a given group being equivalent between each other but inequivalent between different groups.

This is not the end of the problem, actually it gets a bit more weird and interesting. If we take the appropriate limit of the ground state in the finite case, it does not converge to either of the ground states in the infinite case. As a matter of fact, it converges to a mixed state. In case the concept of Spontaneous Symmetry Breakdown was not clear enough, the next diagram should do.



Bibliographical Remarks: For the concept of a symmetry at the representation level on Quantum Mechanics reference [6] was essential where Wigner's theorem regarding the implementation of symmetries appears while for the concept of Spontaneous Symmetry Breakdown [4] was widely consulted. The text found on [9] was essential for the writing of section 3.2 and the summarizing tour at the end was highly motivaded by the discussion held at [8].

Chapter 4

Ising Model & Spontaneous Symmetry Breakdown

In this chapter we are going to exemplify the concept of Spontaneous Symmetry Breaking.

4.1 n site Ising Chain with Transverse Field

In this section we consider the n site Ising chain with open boundary conditions. Let us make a statement involving the relevant structures for this particular case:

$C*$-algebra: $\mathcal{A} = \bigotimes^{n} M_2(\mathbb{C})$

Time Evolution: $\alpha_t(a) = U_t^\dagger a U_t$ where $U_t = e^{iH't}$ for $H' = -\sum_{i=1}^{n-1} \sigma_i^z \sigma_{i+1}^z - \lambda \sum_{i=1}^{n} \sigma_i^x$

Symmetry: The algebraic symmetry is given by: $\beta(a) = U_\beta a U_\beta^*$ where $U_\beta = \bigotimes^{n} \sigma^x$

By looking at the definitions of α_t and β, we easily see that these are homomorphisms. Clearly α_{-t} is the inverse of α_t, and $\beta^2 = 1$, hece both α_t and β are automorphisms. At this point we have defined the algebraic structure, so the next step is to look for physically relevant representations, for instance, let $\mathcal{H} = \bigotimes^{n} \mathbb{C}^2$ and π be the trivial representation on $\mathcal{B}(\mathcal{H})$.

We want to show that this trivial representation is physically relevant. Clearly H' is the generator of the group $\{U_t\}$ and its spectrum is bounded since H' is finite dimensional. We want to check there is a non degenerate ground state in order to check that the trivial representation is in fact a physically relevant one. Let's get to work then.

Solving the problem for $H' = -\sum_{i=1}^{n-1} \sigma_i^z \sigma_{i+1}^z - \lambda \sum_{i=1}^{n} \sigma_i^x$ is equivalent to solving the problem for $H = -\sum_{i=1}^{n-1} \sigma_i^x \sigma_{i+1}^x - \lambda \sum_{i=1}^{n} \sigma_i^z$. This is true since H' and H are related by a unitary transformation which consists of a 180° rotation around the x-axis and the determinant is invariant under unitary transformations so the spectrum of both operators will be the same, since this is so, the Hamiltonian that we are going to use is H.

So, the problem consists on finding the spectrum of H [10]:

$$H = -\sum_{i=1}^{N-1} \sigma_i^x \sigma_{i+1}^x - \lambda \sum_{i=1}^{N} \sigma_i^z. \tag{4.1}$$

The first thing we are going to do is to introduce fermionic operators (Jordan-Wigner transformation):

$$c_i^\dagger := \prod_{j<i} \left(\sigma_j^z\right) \sigma_i^+ \quad c_i := \prod_{j<i} \left(\sigma_j^z\right) \sigma_i^-. \tag{4.2}$$

Now, it is not difficult to see that σ_i^x and σ_i^z in terms of these new operators are:

$$\sigma_i^x = \prod_{j<i} \sigma_j^z (c_i^\dagger + c_i), \quad \sigma_i^z = [c_i^\dagger, c_i] = 2c_i^\dagger c_i - 1. \tag{4.3}$$

In terms of these new operators the Hamiltonian looks a bit less atractive but it will prove useful:

$$H = \lambda N - 2\lambda \sum_{i=1}^{N} c_i^\dagger c_i + \sum_{i=1}^{N-1} c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + c_i^\dagger c_{i+1}^\dagger - c_i c_{i+1}. \tag{4.4}$$

Ok, so far so good. What we want to do next is to arrive at an expression for the Hamiltonian that looks something like this:

$$H = \lambda N + \sum_{i,j} c_i^\dagger A_{ij} c_j + \frac{1}{2} \sum_{i,j} B_{ij} (c_i^\dagger c_j^\dagger - c_i c_j). \tag{4.5}$$

Where the sum of i, j goes from one to N. It turns out that it is possible to do this. The elements of the matrices A and B can be obtained by arranging the i^{th} term of the sum and the matrices that do the job are:

$$A = \begin{pmatrix} -2\lambda & 1 & & \mathbf{0} \\ 1 & -2\lambda & \ddots & \\ & \ddots & \ddots & 1 \\ \mathbf{0} & & 1 & -2\lambda \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ \mathbf{0} & & -1 & 0 \end{pmatrix}.$$

The next step is to perform a Bogoliubov transformation of the variables, i.e. we want the Hamiltonian to be in terms of new fermionic operators such that these new operators are linear combinations of the ones we already have: $\eta_q = \sum_k g_{qk} c_k + h_{qk} c_k^\dagger$, with the hope that the Hamiltionian gets into a diagonal form:

$$H = \sum_q \Lambda_q \eta_q^\dagger \eta_q \tag{4.6}$$

+ cst From Lieb-Schultz-Mattis [10] we know the explicit form of the constant:

$$H = \sum_q \Lambda_q \eta_q^\dagger \eta_q - \frac{1}{2} \sum_q \Lambda_q \tag{4.7}$$

For such an ansatz we want the new operators to be fermionic operators so we will require that:

$$\{\eta_q, \eta_{q'}^\dagger\} = \delta_{q,q'}, \quad \{\eta_q, \eta_{q'}\} = \{\eta_q^\dagger, \eta_{q'}^\dagger\} = 0. \tag{4.8}$$

Now, these three requirements transform into two requirements that the $g's$ and $h's$ must fulfill (the last two relations give rise to the same condition):

$$\sum_k g_{qk}g_{q'k} + f_{qk}f_{q'k} = \delta_{qq'} \iff GG^T + FF^T = 1 \tag{4.9}$$

$$\sum_k g_{qk}f_{q'k} + f_{qk}g_{q'k} = 0 \iff GF^T + FG^T = 0 \tag{4.10}$$

Moreover, if η_q really does diagonalize the Hamiltonian (as we want to) then, by using the anticommutation relations that the new operators must also fulfill, we have $[\eta_q, H] = \Lambda_q \eta_q$. By doing this calculation and equating to both sides the coefficients accompanying the c_i's and the $c_i^\dagger$'s, we get as a result the next two equations:

$$\Lambda_q g_{qi} = \sum_j g_{qj}A_{ij} + f_{qj}B_{ij}, \tag{4.11}$$

$$\Lambda_q h_{qi} = \sum_j -f_{qj}A_{ij} - g_{qj}B_{ij}. \tag{4.12}$$

Let $\phi_{qi} = g_{qi} + f_{qi}$, $\psi_{qi} = g_{qi} - f_{qi}$. Then equations (4.11) (4.12) are equivalent to:

$$\Lambda_q \phi_{qi} = \sum_j A_{ij}\psi_{qj} - B_{ij}\psi_{qj} \iff (A - B)\psi_q = \Lambda_q \phi_q \tag{4.13}$$

$$\Lambda_q \psi_{qi} = \sum_j A_{ij}\phi_{qj} + B_{ij}\phi_{qj} \iff (A + B)\phi_q = \Lambda_q \psi_q \tag{4.14}$$

Multiplying (4.11) by $(A + B)$ and (4.12) by $(A - B)$ we get:

$$(A + B)(A - B)\psi_q = \Lambda_q^2 \psi_q, \tag{4.15}$$

$$(A - B)(A + B)\phi_q = \Lambda_q^2 \phi_q. \tag{4.16}$$

As we can see, the problem has been reduced to diagonalize $(A + B)(A - B)$ or $(A-B)(A+B)$ because, as it turns out, both matrices will have the same eigenvalues. Note the following: $A_{ij} = -2\lambda\delta_{i,j} + \delta_{i-1,j} + \delta_{i+1,j}$; $B_{ij} = \delta_{i+1,j} - \delta_{i-1,j}$. Clearly if one of the two indices i or j in the Kronecker delta are not in the range $[1, N]$, then such an element is zero.

$$
(A+B) = \begin{pmatrix} -2\lambda & 2 & & \text{\Large 0} \\ 0 & -2\lambda & \ddots & \\ & \ddots & \ddots & 2 \\ \text{\Large 0} & & 0 & -2\lambda \end{pmatrix}, \quad (A-B) = \begin{pmatrix} -2\lambda & 0 & & \text{\Large 0} \\ 2 & -2\lambda & \ddots & \\ & \ddots & \ddots & 0 \\ \text{\Large 0} & & 2 & -2\lambda \end{pmatrix}.
$$

Now we proceed to multiply the two matrices. A direct calculation shows:

$$
(A+B)(A-B) = \begin{pmatrix} 4(\lambda^2+1) & -4\lambda & & \text{\Large 0} \\ -4\lambda & 4(\lambda^2+1) & \ddots & \\ & \ddots & \ddots & -4\lambda \\ \text{\Large 0} & & -4\lambda & 4\lambda^2 \end{pmatrix}
$$

This is the matrix whose eigenvalues we need to find in order to solve equation (4.15). The resulting eigenvalues Λ_q are called elementary excitations. Given that we are working with finite dimensions the spectrum has a lower bound which corresponds to the smallest eigenvalue, if it is not degenerate, then the trivial representation is indeed physically relevant. Let us stop for a moment and realize that we have reduced a problem of finding the 2^N eigenvalues of H acting on $\mathcal{H} = \bigotimes^{n} \mathbb{C}^2$ to a problem of finding the N eigenvalues of the previous matrix.

The next question comes up: *How are they related?*

This is the Fock space interpetation of the matter, so the energies are going to correspond to any sum of the elementary excitations Λ_q, one quickly checks that indeed the numbers match since the size of the power set of N elements is 2^N. To find the elementary exitations Λ_q we are going to proceed by using numerical calculations since we have reduced the size of the problem and in consequence the computational time.

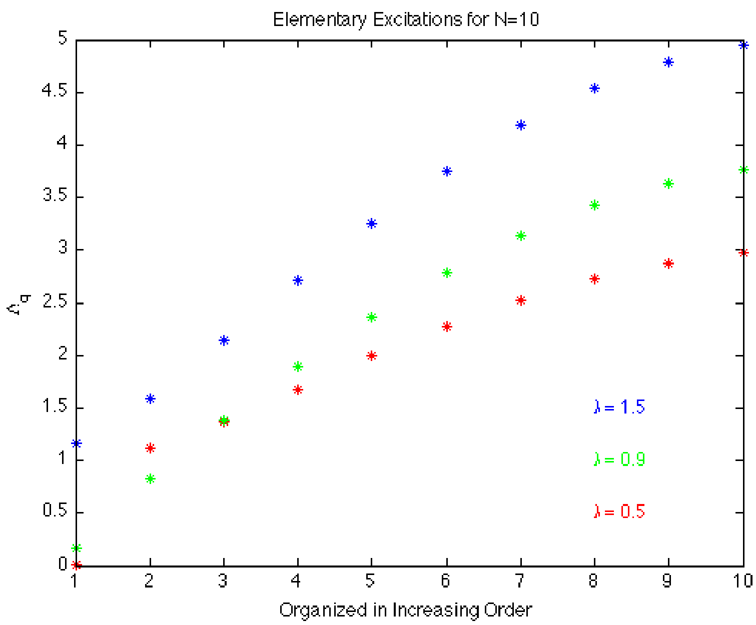


Figure 4-1: Dispersion Relation

Figure 4.1 corresponds to the plot of the elementary excitations Λ_q that appear explicitly in the Hamiltonian corresponding to equation (4.7). A particular feature we can see from the graph is that the first elementary excitation increases as λ does.

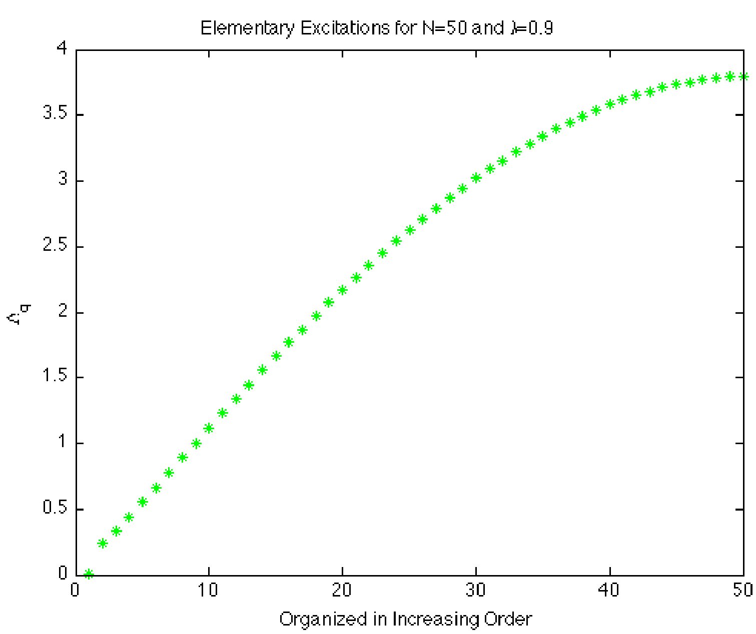


Figure 4-2: Dispersion Relation for 50 sites and $\lambda = 0.9$.

Figure 4.1 shows in addition how the first elementary excitation decreases when the number of sites N grows for the region of parameters: $0 < \lambda \leq 1$, since the gap ΔE_{01} corresponds to the first elementary exitation. The compound behavior of the first elementary excitation as a variable of λ and N is summarized as follows: For a fixed $0 < \lambda \leq 1$ there exists a sufficiently large N such that the first elementary excitation (which is also the gap between the ground state and the first excited state) is almost zero, and the required N is bigger for a greater value of λ, Figure 4.1 shows the behavior of the gap as N increases for different values of λ confirming among other things this assertion.

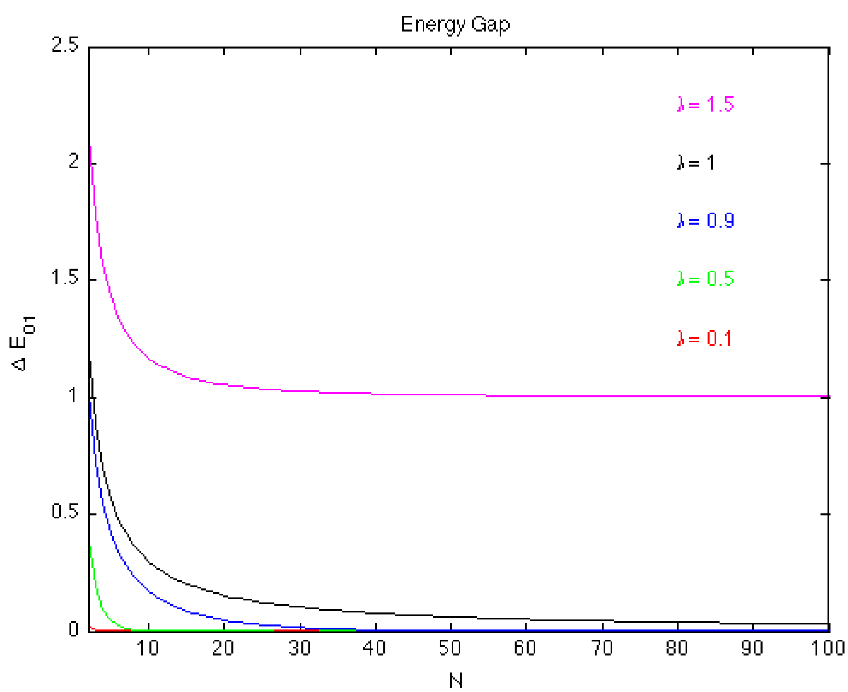


Figure 4-3: Energy gap as a function of the number of sites.

Another important fact that comes up from Figure (4.1) is how the gap ΔE_{01} approaches zero at an exponential rate when $0 < \lambda \leq 1$. However this kind of behavior does not occur when $\lambda > 1$. In this case no matter how great the value of N can get, energy gap is never zero. We can also see that indeed a greater N is required for the gap to be almost zero as λ increases. For instance, while for $\lambda = 0.5$ ten sites might be sufficient for the gap to be practically zero, for $\lambda = 0.9$ about 40 sites are needed.

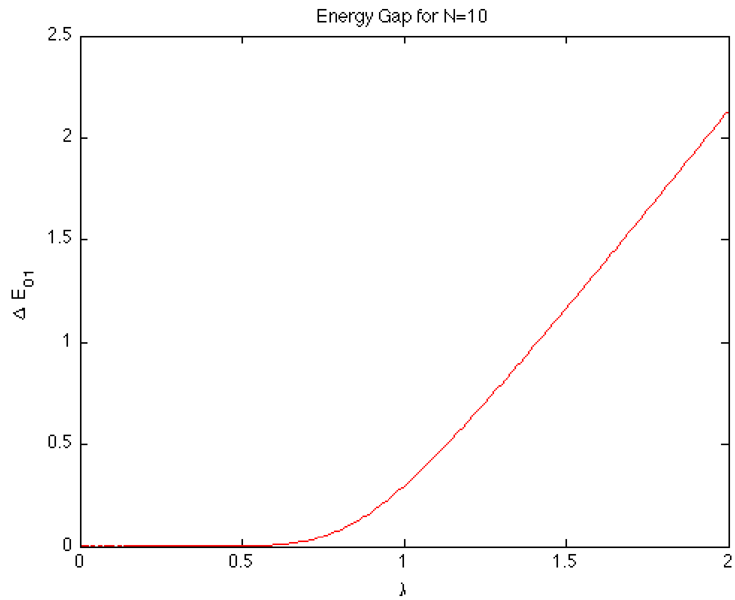


Figure 4-4: Energy gap as a function of λ for 10 sites.

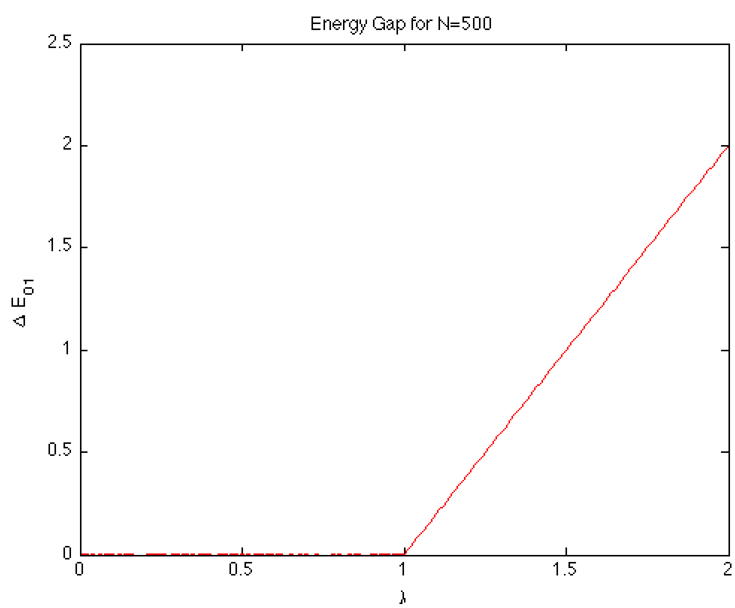


Figure 4-5: Energy gap as a function of λ for 500 sites.

Figures 4.1 and 4.1 show the closing of the gap as N increases for the range of parameters $0 \leq \lambda \leq 1$ and how the analicity gets lost around the value $\lambda = 1$ when

N is sufficiently large. It also exhibits the expected result $\Delta E_{01} = 2|1-\lambda|$ for $\lambda \geq 1$.

For $\left|\Psi^0_{1/N}\right\rangle$ the ground state and $\left|\Psi^1_{1/N}\right\rangle$ the first excited state let us show by means of a simple calculation that the transition probability between the states

$$\left|\Psi^{\pm}_{1/N}\right\rangle = \frac{\left|\Psi^0_{1/N}\right\rangle \pm \left|\Psi^1_{1/N}\right\rangle}{\sqrt{2}} \tag{4.17}$$

Goes to zero as $N \to \infty$. Taking into account the results shown on Figure 4.1, the energy difference between $\left|\Psi^0_{1/N}\right\rangle$ and $\left|\Psi^1_{1/N}\right\rangle$ goes to zero as $N \to \infty$, (for $0 < \lambda < 1$ which is our region of interest), so it is possible to think of the two previous combinations as asymptotical eigenstates of H with eigenvalue E_o:

$$\lim_{N\to\infty} \frac{H}{E_{o,N}} \left|\Psi^{\pm}_{1/N}\right\rangle = \left|\Psi^{\pm}_{1/N}\right\rangle. \tag{4.18}$$

With the actual calculation being:

$$\begin{aligned}\frac{H}{E_{o,N}} \left|\Psi^{\pm}_{1/N}\right\rangle &= \frac{1}{E_{o,N}} H \left\{ \frac{\left|\Psi^0_{1/N}\right\rangle \pm \left|\Psi^1_{1/N}\right\rangle}{\sqrt{2}} \right\} = \frac{1}{\sqrt{2}} \left\{ \left|\Psi^0_{1/N}\right\rangle \pm \left(1 + \frac{\Delta E_{01,N}}{E_{o,N}}\right) \left|\Psi^1_{1/N}\right\rangle \right\} \\ &= \left|\Psi^{\pm}_{1/N}\right\rangle + \left(\frac{\Delta E}{\sqrt{2} E_{o,N}}\right) \left|\Psi^1_{1/N}\right\rangle \approx \left|\Psi^{\pm}_{1/N}\right\rangle\end{aligned} \tag{4.19}$$

Meaning the error of the approximation is almost zero for sufficiently large N. Moreover, using this asymptotic approximation, the transition probability between the $\left|\Psi^{\pm}_{1/N}\right\rangle$ states is going to be:

$$P_{+-} = \left| \left\langle \Psi^-_{1/N} \middle| \Psi^+_{1/N} \right\rangle \right|^2 = \frac{\left| \frac{2\Delta E}{E_o} + \frac{\Delta E^2}{E_o^2} \right|^2}{4} \approx \left(\frac{\Delta E}{E_o}\right)^2 \tag{4.20}$$

Meaning that $P_{+-} \to 0$ as $N \to \infty$. All of this will be important in the following chapter when trying to reconcile Spontaneous Symmetry Breakdown with finite dimensional Quantum Mechanics. For the moment let us focus on what we have done. The reduction process and numerical procedures performed have allowed us to show by explicit calculation that the spectrum of the Hamiltonian has an minimum, that the minimum is a non-degenerate eigenvalue of the Hamiltonian and the Fock state representation shows that the corresponding eigenvector is a cyclic vector, i.e. the trivial representation is in fact a physically relevant representation.

4.2 Infinite Dimensional Ising Chain with Transverse Field

In this section we will mainly discuss some results that apply to the ground states of the XY model for the infinite dimensional case. In particular we will be interested in the region of parameters ($|\lambda| < 1$, $\gamma = 1$, $J = 1/2$) locating us in the region of interest (Ising with transverse field) for posterior comparison. The partial results enunciated in this section are due to Araki and Matsui [13] and will allow us to exemplify the concept of symmetry breaking with respect to a symmetry consisting of 180° rotation around the z axis. Let's start.

The algebra we are dealing with here is a uniformly hiperfinite algebra (UHF for short) meaning that it is isomorphic to the tensor product of matrix algebras, and it is: $\mathcal{U} = \text{gen}\{\sigma_\alpha^j\} \mid \alpha = (x, y, z),\ \ j \in \mathbb{Z}$, now, in order to define time evolution we first define the local Hamiltonian:

$$H(a,b) = -J\big\{\sum_{j=a}^{b-1}(1+\gamma)\sigma_x^j\sigma_x^{j+1} + (1-\gamma)\sigma_y^j\sigma_y^{j+1} + 2\lambda\sum_{j=a}^{b}\sigma_z^j\big\}. \tag{4.21}$$

Once defined the local Hamiltonian we proceed to define time evolution in $\mathcal{U}$ as

$$\alpha_t(A) \equiv \lim_{N\to\infty} e^{itH(-N,N)}Ae^{-itH(-N,N)}\ \ \forall A \in \mathcal{U}. \tag{4.22}$$

That this operator is well defined is shown in ref. [14], and the generator of time evolution has as a core the subalgebra $\mathcal{U}_o \subseteq \mathcal{U}$ consisting of local observables, that is to say: for each $A \in \mathcal{U}_o \quad \delta(A) = \dot{A} \equiv i\,[H(-N,N), A]$ is independent of N for a suficiently large N. In consequence a ground state is defined by the property:

$$-i\omega(A^*\delta(A)) \geq 0 \ \forall A \in \mathcal{U}_o. \tag{4.23}$$

This implies $\omega(\alpha_t(A)) = \alpha(A) \ \forall A \in \mathcal{U}$. This is interesting since (keeping in mind we are looking for the GNS representation to be physically relevant) this assures via *Theorem* 1.5.2 that time evolution is going to be unitarily implemented by a family of unitaries in $\mathcal{H}_\omega$. Now let us introduce some results due to Araki and Matsui. First it is worth noting that, when refering to extremal ground states they are refering to states that cannot be decomposed as the convex sum of two other states. In such sense they are extremal, and according to the definition of pure states in chapter 1, extremal ground states are pure ground states.

Theorem 4.2.1. *The number of extremal ground states is:*

(α) *1 for* $|\lambda| \geq 1$ *or* $(|\lambda| < 1, \gamma = 0)$

(β) *2 for* $(|\lambda| < 1, \gamma \neq 0)$ *and* $(\lambda, \gamma) \neq (0, \pm 1)$

(γ) ∞ *for* $(\lambda, \gamma) = (0, \pm 1)$

For the Ising case, i.e. (γ)*, there are 2 extremal ground states which are the continuation of the 2 extremal ground states in* (β)*. Additionally there are 2 other irreducible representations in which any vector of an infinite dimensional subspace of the representation space gives rise to an extremal ground state.*

Note that we are in fact interested in GNS representations of pure ground states since *Theorem*1.4.2 states the representation $(\mathcal{H}_\omega, \pi_\omega)$ will be irreducible if and only if the state is pure, and irreducibility is indeed a desirable property responsible for the concept of "closed world" to make sense.

Now, there is paragraph related to the ergodicity of the previous ground states. For an extremal ground state ω, consider the perturbed state:

$$\omega_B(A) \equiv \omega(B^*AB)/\omega(B^*B) \ \mid B \in \mathcal{U}. \tag{4.24}$$

We will say ω is *ergodic under time evolution* if

$$\lim_{t\to\infty} \omega_B(\alpha_t(A)) = \omega(A) \;\; A, B \in \mathcal{U}. \tag{4.25}$$

As we can realize this property is related to the stability of the state ω. A state satisfying this property is asymptotically in time impervious to such perturbations, i.e. when sufficient time has passed, the state is going to be very close to the original unperturbed extremal ground state. The next theorem is related to this fact.

Theorem 4.2.2. *The unique ground state in* (α) *is ergodic under time evolution while any other ground state in* (β) *or* (γ) *fails to fulfill ergodicity.*

Now we are going to revise how the concept of a Jordan-Wigner transformation plays its role in the infinite dimensional case.

Jordan-Wigner Transformation

The idea of the transform goes like this: We enlarge $\mathcal{U}$ by adding an element T that plays the role of multiplication by σ_z^j for all $j \leq 0$ like we did in the n-site case in order to define the fermionic operators c_j and $c_j^\dagger$. Then we define the automorphism corresponding to the symmetry related to a 180° rotation around the z-axis. After that, new fermionic operators which are linear combinations of the c_j's and $c_j^\dagger$'s are produced. This is done in order for the time evolution to take a simpler form which allowed the authors of ref. [13] to get some of the results stated in here. Some of the theorems to come make use of an operator that comes out of this transformation so we make a pause to talk about the Jordan-Wigner Transformation.

The first thing to be done is to enlarge the algebra by adding a new element called T with the following set of rules:

$$T^2 = 1, \quad T^* = T, \quad TA = \Theta_-(A)T \qquad \forall A \in U. \tag{4.26}$$

where Θ_- is a selfinverse automorphism defined by:

$$\Theta_-(A) = \lim_{N\to\infty} \left(\prod_{j=0}^{-N} \sigma_z^j\right) A \left(\prod_{j=0}^{-N} \sigma_z^j\right). \tag{4.27}$$

Thus, this automorphism included in the definition of T corresponds to a 180° rotation around the z-axis for the left half of the chain and has an effect on the generators of $\mathcal{U}$ in the following way:

$$\Theta_-(\sigma_x^j) = -\sigma_x^j \text{ if } j \leq 0, \qquad \Theta_-(\sigma_y^j) = -\sigma_y^j \ if \ j \leq 0. \tag{4.28}$$

Otherwise, i.e. if we are referring to $j \geq 1$ or σ_z^j for any j, it leaves the element invariant. With the addition of this new element, the enlarged algebra $\hat{\mathcal{U}}$ is decomposed as a direct sum:

$$\hat{\mathcal{U}} = \mathcal{U} + \mathcal{U}T. \tag{4.29}$$

We can extend the automorphism Θ_- in order to act in $\hat{\mathcal{U}}$ in the following manner:

$$\Theta_-(A_1 + A_2T) = \Theta_-(A_1) + \Theta_-(A_2)T \quad \forall A_1, A_2 \in \mathcal{U}. \tag{4.30}$$

Now we have enlarged the algebra we can introduce fermionic operators:

$$c_j^\dagger := TS_j(\sigma_x^j + \sigma_y^j)/2, \quad c_j := TS_j(\sigma_x^j + \sigma_y^j)/2, \tag{4.31}$$

$$S_j := \begin{cases} \sigma_z^1 \ldots \sigma_z^{j-1}, & j \geq 2 \\ 1, & j = 1 \\ \sigma_z^0 \ldots \sigma_z^j, & j \leq 0. \end{cases} \tag{4.32}$$

As previously said these are fermionic operators satisfying CAR relations. The C^*-subalgebra of $\hat{\mathcal{U}}$ generated by this operators will be called $\mathcal{U}^{car}$. The 180° rotation symmetry around the z-axis is going to be enforced (at the algebraic level) by the following automorphism in $\hat{\mathcal{U}}$:

$$\Theta(\sigma_x^j) = \sigma_x^j\,, \quad \Theta(\sigma_y^j) = \sigma_y^j\,, \quad \Theta(\sigma_z^j) = \sigma_z^j\,, \quad \Theta(T) = T. \tag{4.33}$$

Note that since $\forall A \in \hat{\mathcal{U}}$ it is possible to write such element as $A = A_+ + A_-$ with $A_\pm = (A \pm \Theta(A))/2$, it is possible to decompose $\mathcal{U}^{CAR}$ and $\mathcal{U}$ as sets in their even and odd parts each set being Θ-invariant. The algebras resulting from the decompositions are related by:

$$\mathcal{U}_+ = \mathcal{U}_+^{CAR}, \quad \mathcal{U}_- = \mathcal{U}_-^{CAR} T \tag{4.34}$$

The next thing to be done is to produce new fermionic operators which will be linear combinations of the previous ones, i.e. c_j's and $c_j^\dagger$'s.

$$c^\dagger(f) = \sum_{j\in\mathbb{Z}} c_j^\dagger f_j \ , \quad c(f) = \sum_{j\in\mathbb{Z}} c_j f_j \quad | \ f = (f_j) \in l^2(\mathbb{Z}) \tag{4.35}$$

With these definitions we look for new fermionic operators of the form:

$$B(h) = c^\dagger(f) + c(g) \text{ such that } h = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{4.36}$$

It is worth mentioning a few relations these new operators satisfy:

$$[B(h_1), B(h_2)] = (h_1, h_2) \ , \quad B(h)^\dagger = B(\Gamma h) \tag{4.37}$$

$$\Gamma \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}. \tag{4.38}$$

Finally, in the basis consisting of the $B's$ the operator of time evolution acts like this:

$$\alpha_t(B(h)) = B(e^{2JiKt}h), \tag{4.39}$$

with

$$K = \begin{pmatrix} U + U^* - 2\lambda & \lambda(U - U^*) \\ -\lambda(U - U^*) & -(U + U^* - 2\lambda) \end{pmatrix}, \qquad (Uf)_j = f_{j+1}, \ (U^* f)_j = f_{j-1}. \tag{4.40}$$

Now let us continue stating a couple more theorems, bare with me, we are almost there. Let $L = 2JK$, as we can see this is a self-adjoint operator. In addition, by writing $\Gamma = (conjugation) \circ \sigma_x$ it is not difficult to check by explicit computation that $\Gamma L = -L\Gamma$. L is going to be the generator of time evolution in the basis of the

$B's$, so its interpretation is going to be associated to the energy.

Theorem 4.2.3. *Let* $\alpha_t(B(h)) = B(e^{iLt}h)$ *for* $L^* = L$, $\Gamma L = -L\Gamma$.
There exists a unique ground state of $(\mathcal{U}^{car}, \alpha_t)$ *if and only if 0 is not an eigenvalue of* L. *The unique ground state is the Fock state* ω_{E_+} *for which* $B(h)$ *with* h *satisfying* $E_+h = h$ *is a creation operator and* $B(h)$ *with* h *satisfying* $E_-h = h$ *is an annihilation operator, where* $E_\pm$ *are the spectral projections of* L *on* $(0, \infty)$ *and* $(-\infty, 0)$ *respectively.*

In the next theorem E_o and E_e are the projections to the eigenspaces corresponding to the eigenvalues 0 and e respectively.

Theorem 4.2.4. *Let* $\alpha_t(B(h)) = B(e^{iLt}h)$ *for* $L^* = L$, $\Gamma L = -L\Gamma$.

(1) *There exists a unique ground state of* $(\mathcal{U}_+^{CAR}, \alpha_t)$ *if and only if one of the following (mutually exclusive) conditions is satisfied:*

(α) $E_o = 0$ *and the infimum of the positive part of the spectrum of* L *is not an eigenvalue of* L.

(β) $\dim(E_o)=1$.

In both cases the unique ground state in $(\mathcal{U}_+^{CAR}, \alpha_t)$ *is the restriction of any ground state in* $(\mathcal{U}^{CAR}, \alpha_t)$.

(2) *If* $E_o = 0$ *and the infimum of the positive part of the spectrum of* L, e *is an eigenvalue of* L *with the eigenprojection* E_e, *then an extremal ground state* ω *of* $(\mathcal{U}_+^{CAR}, \alpha_t)$ *is either the restriction of* ω_{E_+} *to* $\mathcal{U}_+^{CAR}$ *or the state defined by*

$$\omega_h(A) = \omega_{E_+}(B(h)^* A B(h))/(h, h), \tag{4.41}$$

where h *is any vector satisfying* $E_e h = h$. *The cyclic representations associated to* ω_h *are equivalent and disjoint from the representation associated to* ω_{E_+}. *Any other state on* $\mathcal{U}_+^{car}$ *will be a convex sum of such extremal ground states.*

Keep in mind that these two last theorems refer to matters on $\mathcal{U}^{CAR}$ or $\mathcal{U}_+^{CAR}$ and not to the state of affairs in $\mathcal{U}$ (the algebra we are interested in). Lemma aside the next propositions will refer to matters in $\mathcal{U}$.

Lemma 4.2.5. **(1)** *For* $(\lambda,\gamma)\neq(0,\pm1)$, *K has an absolutely continuous spectrum.*
(2) *For* $(\lambda,\gamma)=(0,\pm1)$, $\sigma(K)=\{2,-2\}$ *i.e. K/2 is self-adjoint unitary.*

Proposition 4.2.6. **(1)** *For* $(\lambda,\gamma)\neq(0,\pm1)$ *a* Θ*-invariant ground state of* $(\mathcal{U},\alpha_t)$ *is unique and it is given by*

$$\bar{\omega}_{E_+}(A_+ + A_-)\equiv\omega_{E_+}(A_+),\quad A_\pm\in\mathcal{U}_\pm. \tag{4.42}$$

(2) *For* $(\lambda,\gamma)=(0,\pm1)$ *all* Θ*-invariant ground states of* $(\mathcal{U},\alpha_t)$ *are given by*

$$\bar{\omega}(A_+ + A_-)\equiv\alpha_o\omega_{E_+}(A_+)+\sum_j\alpha_j\omega_{h_j}(A_+),\quad A_\pm\in\mathcal{U}_\pm. \tag{4.43}$$

where $\alpha_j\geq0$, $\sum\alpha_j=1$ *and* h_j*'s are mutually orthogonal.*

Note that the left side of equation (4.42) or (4.43) defines (in any case, unique or not) a Θ-invariant state in $(\mathcal{U},\alpha_t)$ while using a state on $(\mathcal{U}^{car},\alpha_t)$ on the right hand side of the equation.

Proposition 4.2.7. **(1)** *For* $|\lambda|\geq1$ *or* $(|\lambda|<1,\gamma=0)$, *the unique* Θ*-invariant ground state* $\bar{\omega}_{E_+}$ *is pure and it is the unique extremal ground state in* $(\mathcal{U},\alpha_t)$.
(2) *For* $(|\lambda|<1,\gamma\neq0)$ *and* $(\lambda,\gamma)\neq(0,\pm1)$, *the unique* Θ*-invariant ground state* $\bar{\omega}_{E_+}$ *is an average of the two pure states* $\omega_\pm$ *that exhaust extremal ground states of* $(\mathcal{U},\alpha_t)$. *The cyclic representations associated to* $\omega_\pm$ *are mutually disjoint and* Θ *interchanges* ω_+ *and* ω_- *i.e.* $\omega_\pm(\Theta(A))=\omega_\mp(A)\ \forall A\in\mathcal{U}$.
(3) *For* $(\lambda,\gamma)=(0,\pm1)$ *the* Θ*-invariant ground states* $\bar{\omega}_{E_+}$ *and* $\bar{\omega}_h$ *are the averages of the extremal states* $\omega_\pm$ *and* $\omega_{h\pm}$ *respectively, which exhaust extremal ground states of* $(\mathcal{U},\alpha_t)$. *Cyclic representations associated to* $\omega_\pm$ *are mutually disjoint and disjoint from those associated to* $\omega_{h\pm}$. *Cyclic representations associated to* $\omega_{h\pm}$ *for different h are all equivalent among* ω_{h+} *and* ω_{h-} *but disjoint between* ω_{h+} *and* ω_{h-}.

In section *4.1.* we saw that the phase for the n-site case was β-symmetric (the symmetry being implemented by a unitary operator in the trivial representation) and physically relevant. We were promised the symmetry was going to break down in the infinite dimensional case for the range of parameters ($|\lambda| < 1, \gamma = 1, J = 1/2$). Ok, time to cash in.

Now that we have seen all these impressive results, it is time to organize them and see if they indeed state what is expected. It goes like this: *Lemma* 4.2.5 says for the region we are interested in, the spectrum of K is absolutely continuous so $\sigma_d(K) = \emptyset$. In consequence there are no eigenvalues. If there are no eigenvalues, 0 cannot be an eigenvalue of K, so, via *Theorem* 4.2.3 we know there is a unique ground state of $(\mathcal{U}^{CAR}, \alpha_t)$ and it is the Fock state ω_{E_+}. Moreover, $E_o = 0$ and the infimum of the positive part of the spectrum of K is not an eigenvalue of K since, as we have previously noted, $\sigma_d(K) = \emptyset$. This allows us to use the results of *Theorem 4.2.4. (1,α)* and state that $(\mathcal{U}_+^{CAR}, \alpha_t)$ has a unique ground state and it is the restriction of any ground state in $(\mathcal{U}^{car}, \alpha_t)$. Since there is only one ground state in $(\mathcal{U}^{car}, \alpha_t)$, the unique ground state of $(\mathcal{U}_+^{car}, \alpha_t)$ is the restriction of ω_{E_+}. On the other hand, *Proposition* 4.2.6 dictates there is a unique Θ-invariant ground state of $(\mathcal{U}, \alpha_t)$ and it is given by $\bar{\omega}_{E_+}(A_+ + A_-) = \omega_{E_+}(A_+)$. Now, since in particular this is a state, it will be given by a convex combination of the two extremal ground states that *Theorem 6.2.1* assures there are, and, as it turns out, *Proposition 4.2.7 (2)* says that $\bar{\omega}_{E_+}$ is the average of the two extremal ground states $\omega_\pm$, so the cyclic representation associated to ω_+ and ω_- are not going to be β-symmetric but disjoint, i.e. inequivalent, and the symmetry is going to be spontaneously broken. It is worth mentioning that we were always focused on the region of parameters ($|\lambda| < 1, \gamma = 1$) and that the cyclic representations associated to ω_+ and ω_- are physically relevant since they are pure ground states. As such, the representations are going to be irreducible with a cyclic ground state in the usual representation sense and time evolution being implemented by a family of unitary operators.

4.3 Infinite Dimensional Ising Chain Without Magnetic Field

In case the contrast between the previous two sections illustrating the concept of Spontaneous Symmetry Breakdown was not entirely clear, here is another example in which we compare section *4.1.* with $\lambda = 0$ with the present one [7]. Consider an infinitely extended version of the original algebra.

C*-algebra: $\mathcal{A} = \bigotimes^{\mathbb{Z}} M_2(\mathbb{C})$,

Time Evolution: $\alpha_t(a) = U_t^\dagger a U_t \;:\; U_t = e^{iHt}$ for $H = -\sum_{\mathbb{Z}} \sigma_i^z \sigma_{i+1}^z$,

Symmetry: The algebraic symmetry is given by: $\beta(a) = U_\beta a U_\beta^*$ with $U_\beta = \bigotimes^{\mathbb{Z}} \sigma^x$.

The C^*-algebra $\mathcal{A}$ can also be considered as:

$$\mathcal{A} = \text{gen}\{\sigma_\alpha^j\} \text{ such that } \alpha = (x, y, z),\;\; j \in \mathbb{Z}. \tag{4.44}$$

The algebraic symmetry U_β will correspond to a 180° rotation around the x-axis. The symmetry is completely determined by its action on the basis:

$$U_\beta(\sigma_x^j) = \sigma_x^j, \quad U_\beta(\sigma_y^j) = -\sigma_y^j, \quad U_\beta(\sigma_z^j) = -\sigma_z^j. \tag{4.45}$$

The idea is to find a representation of $\mathcal{A}$ on some Hilbert space. We know such representation exists since every $C*$-algebra A admits an injective representation for some Hilbert space $\mathcal{H}$.

Let us consider the following set:

$$S^{(+)} := \{(s_k)_{k\in\mathbb{Z}} \text{ such that } s_k = \pm 1 \;\; \forall k \in \mathbb{Z} \;\text{ differing from } s_k^+ \text{ in only finitely many entries}\} \tag{4.46}$$

where s_k^+ is the sequence with all entries being $+1$. Now let us define the Hilbert space.

$$\mathcal{H}^{(+)} := \{f : S^{(+)} \to \mathbb{C} \mid \sum_{s\in S^{(+)}} |f(s)|^2 < \infty\}. \tag{4.47}$$

The former Hilbert space $\mathcal{H}^{(+)} \subseteq l^2(\mathbb{Z})$ has ususal inner product.

$$\langle f \mid g \rangle^{(+)} = \sum_{s \in S^{(+)}} \bar{f}(s) g(s) \; \forall f, g \in l^2(\mathbb{Z}) \tag{4.48}$$

In addition it has as orthonormal basis [15].

$$\mathcal{H}^{(+)} = \text{span}\{\phi_s^+ \mid \; \phi_s^+(s') = \delta_{ss'} \quad \forall s, s' \in S^{(+)}\} \tag{4.49}$$

Note that there is a natural isomorphism between the elements of the basis and the elements of $S^{(+)}$. Now that we have defined the Hilbert space, we are going to look for representations in $\mathcal{B}(\mathcal{H}^{(+)})$.

$$\pi(\sigma_\alpha^k)\phi_s^+ \equiv \begin{cases} \phi_{\theta_k s}^+ & \alpha = x, \\ i s_k \phi_{\theta_k s}^+ & \alpha = y, \\ s_k \phi_s^+ & \alpha = z. \end{cases} \tag{4.50}$$

One can check that in fact this is a representation with θ_k being an operator that flips s_k, i.e. the k^{th} element of the sequence s. In addition this representation is irreducible [16], so the Hilbert space $\mathcal{H}^{(+)}$ will describe a closed phase and every vector will be a cyclic vector (*Proposition 1.5.1*). Moreover, there is a ground state which is the one corresponding to $\phi_{(s_k)^+}^+$ and every other state in the basis can be reached from it by means of a physical operation corresponding to fliping finitely many entries of $(s_k)^+$, so it corresponds to a monomial in the operators $\{\pi(\sigma_\alpha^k)\}$. In consequence the representation is physically relevant (note that for infinitely extended systems the requirement of finite energy does not hold since it is in fact too restrictive, instead the energy density must be finite). Let us for a moment focus our attention to the expectation value of a particular observable in one of the basis elements, since such elements are in one to one correspondence with elements in $S^{(+)}$ (elements differing from $(s_k)^+$ in finitely many entries) it is safe to asume that for the expectation value of the following operator (the magnetization in the positive z-direction) the result will be +1:

$$m_z^+ = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \pi(\sigma_z^k) \tag{4.51}$$

The result on any basis element $\{\phi_s^+\}$ will be:

$$\left\langle \phi_s^+ \,\middle|\, m_z^+ \,\middle|\, \phi_s^+ \right\rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} s_k = +1. \tag{4.52}$$

Good, now we have established this construction and the expectation value of this particular observable in any basis element let us undergo the same construction and procedures but taking $(S^{(-)}, \mathcal{H}^-)$. In this new setup the basis elements are in one to one correspondence to $S^{(-)}$, that is, elements that differ from $(s_k)^-$ in finitely many elements. However the representation of $\mathcal{A}$ does not change, as a result

$$\left\langle \phi_s^- \,\middle|\, m_z^+ \,\middle|\, \phi_s^- \right\rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} s_k = -1. \tag{4.53}$$

Ok let's get down to business. Suppose there were a unitary operator rendering the two representations $(\mathcal{H}^+, \pi)$ and $(\mathcal{H}^-, \pi)$ unitarily equivalent, then the following would be true:

$$U\pi(a)U^\dagger = \pi(a) \quad \forall a \in \mathcal{A} \quad \text{and} \quad U\phi_{(s_k)^+} = \phi_{(s_k)^-}. \tag{4.54}$$

In particular it would also be true that

$$\left\langle \phi_s^+ \,\middle|\, m_z^+ \,\middle|\, \phi_s^+ \right\rangle = \left\langle U\phi_s^+ \,\middle|\, U m_z^+ U^\dagger \,\middle|\, U\phi_s^+ \right\rangle = \left\langle \phi_s^- \,\middle|\, m_z^+ \,\middle|\, \phi_s^- \right\rangle \tag{4.55}$$

But by means of explicit construction we have seen that the left side of the equation is +1 while the right side of the equation is -1. We have arrived at a contradiction, thus we can conclude that no such unitary transformation U exists. In consequence the symmetry U_β is spontaneously broken.

Bibliographical Remarks: The main references of this chapter in order of exposition were [10], [13], [7].

Chapter 5

Philosophy & Conclusions

Well, we have come to the end of this short discussion. In this final chapter we will be discussing the subject in a very general but precise way asking the questions that made this subject interesting and trying to motivate how the setup discussed during the document is a proposal of a solution. The main intention is to dissipate any left over mystique, much of the elements or ideas exposed were highly motivated by the paper *Spontaneous Symmetry Breaking in Quantum Systems: Emergence or Reduction?* [19].

Physically speaking, what was the problem we had?

Suppose you have a quantum system in its ground state (think of a spin chain) and you start to increase the size of the system (by adding one spin after the other). Every time you increase it, the system gets a little bit disturbed, the disturbance passes and it reorganizes in its lower energy state. Suddenly, the system undergoes an abrupt change and it is permanent (all spins point up or down). Moreover, if you repeat this experiment, sometimes the change leaves your system in one state and some times in another, i.e. there are several posibilities for the "outcome system" to adopt, there is no telling, what will it do. Up until now it seems like an interesting quantum feature but nothing to worry about. So with the best intentions and utter confidence we rely on Quantum Mechanics to predict this ***emergent feature***, we inmerse ourselves in calculations use all we know about Quantum Mechanics and quite rapidly recall a fantastic result due to Stone-von Neumann (for the bosonic case) or Jordan-Wigner (for the fermionic case). It is a uniqueness theorem in any case, stating that all representations are equivalent to each other for finite degrees of freedom. Well, clearly, we are dealing with finite systems, it might be of the order of 10^{23} particules but all and all finite, so we say, ok let's look for the representation that describes one of the outcomes after the abrupt change, that representation should be able to explain in fact all possible outcomes since there is only one (all other being unitarily equivalent to it), so we count our particles, see how many fermions and how many bosons we have, throw in some $a's$, $a^{\dagger}{'s}$ satisfying CAR relations and some $b's$, $b^{\dagger}{'s}$ satisfying CCR relations, and with a little help construct in a proper manner our representation, we hope that this will predict this emergent feature, this brusque behavior, as it turns out no matter what we do it does not, it is not capable to predict this particular nature of the system.

What went wrong?

That is the natural question one would ask, after all we are dealing with a quantum system and we are using the correct theory to deal with such kind of physics.

How can we be sure that indeed the theory is unable to describe this behavior? Since the uniqueness theorems assure us that all representations are equivalent (for finite degrees of freedom as the system at hand is) then once we have chosen a representation this should describe all the possible outcomes of the system, moreover, since the representation is irreducible (a natural requirement when performing a description of a quantum system)[1] then any vector on the represenation is a cyclic vector meaning that one could connect two possible outcomes by means of applying the correct sequence of operators (physical operations) on one of the "outcome states", but one goes to the laboratory and it is not possible to change one state to the other by means of applying any of these sequences [2]. Actually each "outcome system" is

[1]After all we want the characterization of the system to be "closed" i.e. that all of the physical operations that can be possibly done on the system and all of the physics that comes out from it be within the scope of the theory, from a logical point of view one would say it would be desirable to have a theory which is complete with respect to the physics one is trying to make sense of.

[2]This is stricly true if the system is indeed infinite since to change one ground state to the other one would require a non-local physical operation and we can only act on the system locally.

described by a cyclic ground state and the representations associated to them are inequivalent in the physical sense, there is no isometry from one representation to the other. That is the problem.

What can be concluded from the problem, what happened?
It happened that the theory that we have at hand is not capable of describing a particular quantum phenomena, in this sense the theory is not complete.

What do we do now?
Ok, clearly we have a problem so let's try to find a solution, the first endeavor we take is to look for representations in Hilbert spaces which have infinite degrees of freedom since we know any attempt of finding inequivalent irreducible representations for finite degrees of freedom will fail, this particular direction is muddy territory so we take with us some useful guidelines in our quest, that is to say some expectations the representation must meet in order to be considered physically relevant, like having an irreducible representation, for the representations of some very basic operators to exist (like the one associated to the energy i.e. the Hamiltonian) and for the energy density to be finite (even in an infinite system this seems like a good requirement), it so happens that we succeed in this enterprise and find some representations, now we have something to hold on to, infinite degrees of freedom representations, furthermore, in this type of representations it is possible to find inequivalent ones, the uniqueness theorems do not longer hold, so we have a chance of explaining the pletora of outcomes of our experiment, it seems like we are on track but the road is long.

Let's slow down a little bit for just a second. *What is so particular about this problem? Why the imperative necessity of a* ***higher theory****?*
As we have seen the theory at hand does not predict this type of conduct, as I have said we physicists are always longing our theories to be complete, this phenomena seem to advocate Mills view:

"the phenomena of life, which result from the juxtaposition of those parts in a certain manner, bear no analogy to any of the effects which would be produced by the action of the component substances considered as mere physical agents. To whatever degree we might imagine our knowledge of the properties of the several ingredients of a living body to be extended and perfected, it is certain that no mere summing up of the separate actions of those elements will ever amount to the action of the living body itself."[3]

This contrasts with our problem in the sense that we have a quantum theory that is made of fundamental constituents and this new quality that arises cannot be explained in terms of this fundamental parts, in that sense it seems the phenomena is a thing in itself, to quote Hempel & Oppenheim [17]:

"this not merely in the psychological sense of being unexpected, but in the theoretical sense of being unexplainable, or unpredictable"

Ok, let's go back to the realm of physics and logical reasoning, let me introduce a brief definition due to Silberstein [18], we will say a theory H:

"bears predictive/explanatory emergence with respect to some lower-level theory L if L cannot replace H, if H cannot be derived from L [i.e., L cannot reductively explain H], or if L cannot be shown to be isomorphic to H."

Now we can formulate the next question: *What are we looking for?*
We are looking as we always do for a general framework (Algebraic Formulation) in which both the theory H (infinite degrees of freedom Quantum Mechanics) that can explain the ***emergent phenomena*** and the theory L (finite degrees of freedom Quantum Mechanics) which is the one that seems appropiate to study this kind of problems (since the systems are finite in extention) come to existence. This is why we introduce all of the C^*-algebra formalism and the whole structure that comes with it, because this is our hope for a more complete theory that is an equivalent formulation for the usual setup but that is capable of predicting novel behavior. With this goal in mind we encounter some theorems and propositions and a very helpful GNS construction that combined guarantee very desirable properties of the representation like:

- Pureness if and only if Irreducibility.

[3]Citation taken from [19]

- $\alpha \in Aut(\mathcal{A})$ gets implemented in $\mathcal{H}_\omega$ if $\alpha^*\omega = \omega$.
- Ground State satisfies $\alpha_t^*\omega = \omega$ and in consequence α_t seen as a symmetry i.e. α_t as an element of $Aut(\mathcal{A})$ gets implemented in $\mathcal{H}_\omega$.
- Existence of Hamiltonian operator if time evolution is implemented by a continuous group homomorphism.

And with this theorems regarding implementations of symmetries (elements of $Aut(\mathcal{A})$) we can say that two representations are equivalent if there exists an isometry between them (this can be seen as a change of basis for the finite dimensional case since any invertible matrix can be interpreted as such) if such condition is not satisfied one arrives at a formal definition of Spontaneous Symmetry Breaking, one says that a system shows Spontaneous Symmetry Breaking if for an algebraic symmetry $\beta \in Aut(\mathcal{A})$ there is no isometry between $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, and $(\mathcal{H}_{\beta^*\omega}, \pi_{\beta^*\omega}, \Omega_{\beta^*\omega})$ which is equivalent to saying that β does not get implemented in $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ or that the representations are unitarily inequivalent, in this way we can say the two systems are different since for the same measurement we get a different distribution of outcomes depending on which system we are measuring in, and the distribution of outcomes of the measurements is exactly what characterizes a system [2]. In this sense we have disjoint phases or disjoint "closed worlds" .

Note that the problem at hand is very puzzling, we have a system of finite dimension and the theory that should describe it does not, however by taking the idealization corresponding to considering the system as infinite this novel abrupt behavior is explained but clearly the system is not infinite in extension. The example we gave in chapter 4 showed that for the finite case we have only one representation so there cannot be disjoint phases because this would mean we have two inequivalent representations and we know this cannot be the case, however by taking the idealization consisting on thinking of the system as indeed infinite we found two inequivalent representations that broke a rotation symmetry (180° around the z-axis or the x-axis depending on the case, that is: H or H'), i.e. there was no isometry between the representations meaning that by performing measurements of the same observable for both systems the results would be distinct so they would be distinguishable and in consequence different, in fact we gave an example of a particular observable whose outcome would be different depending on the representation we were using to calculate its value (the magnetization along the positive z-direction).

Even though it is a conundrum that we require an explicit idealization (infinite degrees of freedom) to be able to give account of this behavior it is indeed quite remarkable that feature can be explained within this context because this allows (in the sheet of paper) the system to end up in different phases for a single algebra. It would be nice though if there were some way of connecting these two setups (the finite and infinite one) and see one as a natural occurrence of some sort of limiting process taken on the other, this leads to the next concept [19]:

"we speak of asymptotic emergence if the following conditions are satisfied:

1. A higher-level theory H is a limiting case of some lower-level theory L.
2. Theory H is well defined and understood by itself.
3. Theory H has emergent features that cannot be explained by L."

In this way one would say Classical Mechanics is the $\hbar \to 0$ limit of Quantum Mechanics, or that Thermodynamics is the $N \to \infty$ limit of Quantum Statistical Mechanics. In our particular example we would like to see if the infinite dimensional representations can be taken to be the $N \to \infty$ limit of the finite representation. As it turns out this can be done, and the way to do it is considering the set of values for the parameter N as a manifold and as the fibers each of the C^*-algebras or States Spaces depending on the element one is interested in i.e. it depends if you are interested in defining the limit of an element of the algebra or in defining the limit of a state. In *pro* of clarity, for the case explained in chapter 4 (considering one increases the size of the chain by adding one element to right and one to the left i.e. by two at a time):

Manifold: $I := \{0\} \cup 1/2\mathbb{N}^*$,

Algebra Fiber: $\mathcal{U}_{1/N} := \bigotimes^{N} M_2(\mathbb{C}) \quad \mathcal{U}_o := \bigotimes^{\mathbb{Z}} M_2(\mathbb{C})$,

State Fiber: $\mathcal{S}_{1/N} := \mathcal{S}_N$, where $\mathcal{S}_N$ refers to the states of the algebra $\mathcal{U}_{1/N}$, then, in order to define one element in $\mathcal{U}_o$ or S_o to be the limit of an element in $\mathcal{U}_N$ or $\mathcal{S}_N$ one makes use of continuous sections in the respective total space. Raggio and Werner have done some work on in this direction [20] and at this point we follow their instructions also found in [19], in order to define a continuous section of $\mathcal{U}$.

A couple of definitions first:

- We will say that a sequence $(a) \equiv (a_N)_{N \in 2\mathbb{N}^*}$ is ***local*** if there exists an M such that $a_N = \iota_{MN}(a_M)\ \forall N \geq M$ where $\iota_{MN} : \mathcal{U}_{1/M} \hookrightarrow \mathcal{U}_{1/N}$ is the inclusion map from $\mathcal{U}_{1/M}$ to $\mathcal{U}_{1/N}$, that is to say that for all N bigger than some M the N^{th} term of the sequence (a) is the tensor product of a_M with the necessary amount of unit matrices to make it an element of $\mathcal{U}_{1/N}$.

- We will say a sequence (a) is ***quasi-local*** if $\forall \epsilon > 0\ \exists M, (a') : \|a_N - a'_N\| < \epsilon$ $\forall N > M$ with (a') a local sequence.

Now we want to introduce an equivalence relation and redefine $\mathcal{U}_o$ with the aim of defining a section (note that every local sequence is quasi-local).

- Introduce an equivalence relation in the set of *quasi-local* sequences in the following way: $a \sim a'$ if and only if $\lim_{N \to \infty} \|a_N - a'_N\| = 0$

- Denote $\mathcal{U}_o$ as the set of all equivalence clases of quasi-local sequences: $a_\infty \equiv [a]$, this form a C^*-algebra under pointwise operations (N^{th} term of one sequence with N^{th} term of the other) and norm $\|a_\infty\| = \lim_{N \to \infty} \|a_N\|$.

Once we have made this considerations we define a cross section of $\mathcal{U}$ to be:

$$\sigma(x) = \begin{cases} a_N, & x = 1/N \mid N \in 2\mathbb{N}^*, \\ a_\infty, & x = 0. \end{cases} \tag{5.1}$$

In this way many quasi-local sequences define a cross-section that ends up in the same element, with the additional observation that we are dealing with equivalence classes and since every equivalence relation induces a partition on a set, we are somehow splitting the elements of $\mathcal{U}_{1/N}$ into disjoint sets.

For the section corresponding to the states, consider a state $\omega_o \in \mathcal{S}_o$, by restriction this defines a state $\omega_{1/N} \in \mathcal{S}_{1/N}$ since $\bigotimes^{N} M_2(\mathbb{C}) \subset \bigotimes^{\mathbb{Z}} M_2(\mathbb{C})$ and the ensuing field of states (ω) is continuous. Conversely, if one has a continuous field of states $(\omega') \in \mathcal{S}$ it must be assymptotically equal to one of the above in the sense that the field (ω) defined by $\omega_o = \omega'_o$ satisfies: $\lim_{N\to\infty} \|\omega_{1/N}(a_N) - \omega'_{1/N}(a_N)\| = 0$ for any quasi-local sequence (a).

These states the fact that there is a good way of expressing the concept of limit of a state on our model using continuous cross-sections, the natural question would be to ask:

What does the ground state of the finite dimensional case converges to?

Well, it converges to a mixed state, this is to be expected from a mathematical point of view since the limiting process conserves the property of a state being symmetry invariant [19] and as we have seen from the results of Araki and Matsui the only Θ-invariant state for the region of parameters we were concerned with is a mixed state corresponding to the average of the two extremal ground states, so another immediate question would be:

If the ground state of the lower theory L does not converge to any of the ground states in the higher theory H, then which state does?

These states do:

$$\left|\Psi^{\pm}_{1/N}\right\rangle = \frac{\left|\Psi^{0}_{1/N}\right\rangle \pm \left|\Psi^{1}_{1/N}\right\rangle}{\sqrt{2}} \tag{5.2}$$

And we saw in chapter 4 that these states are going to be asymptotically ground states since the gap is going to decrease exponentially with N for the region of parameters we are interested in, so each of them is going to be asymptotically a reference vector and as such is expected to give rise to a representation, moreover, it was also shown in the short calculation at the end of *section 4.1* that the transition probability between one and the other decreased exponentially meaning going from one to the other is going to be highly unlikely when N is sufficiently large, so it is possible to consider this representations to be asymptoticaly disjoint, this is somehow encoded on the fact that the representations coming out of the limits of these states on the higher theory H are going to give rise to inequivalent representations.

Check out this quasi-magical statement [19]: since the system is going to be coupled with an enviroment this is usually going to introduce asymmetric terms in the Hamiltonian and for almost any perturbation, that is, any perturbation that does not decrease so fast as $N \to \infty$ as ΔE_{01} does, the perturbation is going to disturb the ground state and is going to do this job more effectively as $N \to \infty$, moreover, for a great variety regarding the type of asymmetric perturbation introduced on the Hamiltonian, the perturbed ground state is going to converge to $\left|\Psi^{+}_{1/N}\right\rangle$ or $\left|\Psi^{-}_{1/N}\right\rangle$ and in consequence in the limiting process is going to converge to one of the two ground states $\omega_{\pm}$.

This assertion made in [19] will help us sleep at night since the Spontaneous Symmetry Breaking behavior that H exhibits is not completely disconnected from L, however the requirement for the existence of the H theory in order for the formal concept of Spontaneous Symmetry Breaking to show up remains, since it is only in this context that Spontaneous Symmetry Breaking can be clearly stated and appears as a feature of the theory in the strict sense, so you can see we started with a problem involving an emergent behavior L could not explain, found a theory H that could, encompassed them in one single C^{*}-algebra approach and finally connected the two instances the approach produced by means of a limiting process involving fibre bundles using the very special fact that the gap decreases exponentially with N and that almost any perturbation to the ground state is going to induce this Spontaneous Symmetry Breaking phenomena by changing the ground state $\Psi^{0}_{1/N}$ to $\left|\Psi^{\pm}_{1/N}\right\rangle$ [19]. In this new algebraic framework a system is described by a C^{*}-algebra structure (of physical operations) but the physical realizations of the system correspond to representations of it (with good properties) and in here the novel behavior makes its appearance. Well that's that. Hope you enjoyed it.

THIS PAGE INTENTIONALLY LEFT BLANK

Appendix A

Matlab Code

```
clear all %THIS IS THE CODE RELATED TO THE DISPERSION RELATION
clc
N=10;
lambda=[0.5,0.9,1.5];
elEx=zeros(N,length(lambda));
for i=1:length(lambda)
    clearvars M
    M=zeros(N);
      for k=1:N
        if k==1
        M(k,k)=4*(lambda(i)^2+1);
        M(k,k+1)=-4*lambda(i);
        elseif k==N
        M(k,k)=4*lambda(i)^2;
        M(k,k-1)=-4*lambda(i);
        else
        M(k,k+1)=-4*lambda(i);
        M(k,k)=4*(lambda(i)^2+1);
        M(k,k-1)=-4*lambda(i);
        end
      end
    elEx(:,i)=sort(sqrt(eig(M)));
```

```
end
plot(1:length(elEx(:,1)),elEx(:,1),'*r')
hold on
plot(1:length(elEx(:,2)),elEx(:,2),'*g')
hold on
plot(1:length(elEx(:,3)),elEx(:,3),'*b')
title('Elementary Excitations for N=10')
ylabel('\Lambda_q')
xlabel('Organized in Increasing Order')
text(8, 0.5, '\lambda = 0.5', 'Color', 'r')
text(8, 1, '\lambda = 0.9', 'Color', 'g')
text(8, 1.5, '\lambda = 1.5', 'Color', 'b')
```

```
clear all %THIS IS THE CODE RELATED TO THE EXPONENTIAL GAP
clc
N=2:200;
lambda=[0.1,0.3,0.9,1,5];
gap=zeros(length(lambda),length(N));
for j=1:length(lambda)
    for i=1:length(N)
        clearvars M
        clearvars elEx
        M=zeros(N(i));
        for k=1:N(i)
            if k==1
            M(k,k)=4*(lambda(j)^2+1);
            M(k,k+1)=-4*lambda(j);
            elseif k==N(i)
            M(k,k)=4*lambda(j)^2;
            M(k,k-1)=-4*lambda(j);
            else
            M(k,k+1)=-4*lambda(j);
            M(k,k)=4*(lambda(j)^2+1);
            M(k,k-1)=-4*lambda(j);
            end
```

```
        end
        elEx=sort(sqrt(eig(M)));
        gap(j,i)=elEx(1);
    end
end
plot(log(N),log(gap(1,:)),'r')
hold on
plot(log(N),log(gap(2,:)),'g')
hold on
plot(log(N),log(gap(3,:)),'b')
hold on
plot(log(N),log(gap(4,:)),'k')
hold on
plot(log(N),log(gap(5,:)),'m')
title('Energy Gap')
ylabel('\Delta E_{01}')
xlabel('N')
text(80, 1.25, '\lambda = 0.1', 'Color', 'r')
text(80, 1.5, '\lambda = 0.5', 'Color', 'g')
text(80, 1.75, '\lambda = 0.9', 'Color', 'b')
text(80, 2, '\lambda = 1', 'Color', 'k')
text(80, 2.25, '\lambda = 1.5', 'Color', 'm')
```

```
clear all %THIS IS THE ONE RESPONSIBLE FOR THE ENERGY GAP AT FIXED N
clc
N=10;
lambda=0:0.01:2;
gapJ=zeros(1,length(lambda));
for j=1:length(lambda)
        clearvars M
        clearvars elEx
        M=zeros(N);
        for k=1:N
            if k==1
            M(k,k)=4*(lambda(j)^2+1);
```

```
            M(k,k+1)=-4*lambda(j);
            elseif k==N
            M(k,k)=4*lambda(j)^2;
            M(k,k-1)=-4*lambda(j);
            else
            M(k,k+1)=-4*lambda(j);
            M(k,k)=4*(lambda(j)^2+1);
            M(k,k-1)=-4*lambda(j);
            end
        end
        elEx=sort(sqrt(eig(M)));
        gapJ(j)=elEx(1);
end
plot(lambda,gapJ,'r')
title('Energy Gap for N=10')
ylabel('\Delta E_{01}')
xlabel('\lambda')
```

Bibliography

[1] Landsman N.P. *Lecture Notes on Spontaneous Symmetry Breaking and the Higgs Mechanism.* $http://www.math.ru.nl/\ landsman/SSB.pdf$ (2012).

[2] Araki H. *Mathematical Theory of Quantum Fields.* (Oxford Science Publications, 8-27, 1999).

[3] Woit P. *Topics in Representation Theory: The Heisenberg Algebra* (2013). $www.math.columbia.edu/\ woit/notes20.pdf$

[4] Strocchi F. *Symmetry Breaking.* (Springer, 2008).

[5] Haag R. *Local Quantum Physics: Fields, Particles, Algebras.* (Springer, 105-108, 1996).

[6] Bargmann V. (1964). *Note on Wigner's Theorem on Symmetry Operations.* Journal of Mathematical Physics, Volume 5, Number 7.

[7] Schwarz G. *Philosophical Aspects of Spontaneous Symmetry Breaking* (2012). $http://philsci-archive.pitt.edu/9303/1/SSB.pdf$

[8] Rosenberg J. *A selective history of the Stone-von Neumann Theorem.* (2003).$http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.79.2084\ \&\ rep=rep1\ \&\ type=pdf$

[9] Prasad A. *An easy proof of the Stone-von Neumann-Mackey Theorem.*. Contemporary Mathematics. (2009). $http://arxiv.org/pdf/0912.0574.pdf$

[10] Lieb E. & Schultz T. & Mattis D. *Two Soluble Models of an Antiferromagnetic Chain*. Annals of Physics: 16, 407-466. (1961).

[11] Reyes A. Personal Communication. Group of Theoretical Physics. Universidad de Los Andes. Bogotá. (2013).

[12] Pfeuty P. *The One-Dimensional Ising Model with a Transverse Field*. Annals of Physics: 57, 79-90. (1997).

[13] Araki H. & Matsui T. (1985). *Ground States of the XY Model*. Communications in Mathematical Physics. 101, 213-245.

[14] Bratteli O. & Robinson D. *Operator Algebras and Quantum Statistical Mechanics 2*. Second Edition. (Springer, 1997).

[15] Sewell G. *Quantum Mechanics and its Emergent Macrophysics*. (Princeton University Press, 2002).

[16] Liu G. & Emch G. *Explaining Quantum Spontaneous Symmetry Breaking*. (2005). $http://philsci-archive.pitt.edu/1778/1/SSBqm1.mss.le.3.04.pdf$

[17] Hempel C. & Oppenheim P. *On the idea of emergence*. Eds. Bedau, M.A. & Humphreys. 6180. Cambridge (Mass.): (MIT Press, 2008).

[18] Silberstein, M. *Reduction, Emergence and Explanation*, in The Blackwell Guide to the Philosophy of Science (eds P. Machamer and M. Silberstein), Blackwell Publishers Ltd, Oxford, UK. (2008)

[19] Landsman N.P. *Espontaneous Symmetry Breaking in Quantum Systems: Emergence or Reduction?* (2013). $http://arxiv.org/abs/1305.4473$

[20] Raggio G.A. & Werner R.F. (1989). *Quantum statistical mechanics of general mean field systems*. Helvetica Physica Acta 62, 9801003.