NONEXPONENTIAL NUCLEAR QUANTUM DECAY. AT LARGE TIMES

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CHAPTER 1

Introduction

The description of unstable systems and each one of its characteristics has been one of the most important issues that physics has taken care of. The unstable systems, precisely, are common in nature, thereby, the importance that their analysis would have is getting higher over the time. From the creation of resonances to the analysis of radioactive processes, the laws of decaying that each one of them must satisfy have been documented and studied in order to analyze the implications of these phenomena in different fields. As a matter of fact, until 1930, it was believed, according to the works of Weisskopf [1] and Wigner[2] and others, that the description that fits the behavior of the decay processes would be according with the laws that the newtonian mechanics obeys, based on the studies of Pierre and Marie Curie of the radioactive phenomena . The experimental results that were obtained at that time, from different kind of phenomena, radioactive processes, nuclear metastable systems, led to some hypotheses that the radioactive systems would follow[3]:

1. Radioactive nucleus have a certain probability of undergoing decay process
2. The probability of undergoing a decay process doesn't depend on the past history of the individual decay nucleus

The classical description of the decay processes agreed with the hypotheses mentioned above. According to these, the variation of number of radioactive nuclei in time must be proportional to the number of radioactive nuclei at a specific time, so that doesn't depend on the past history of the system and the environment surrounding it. The number of radioactive nuclei is decreasing in time, because of the decaying processes that the radioactive nuclei suffer. Mathematically:

$$
\begin{aligned}
dN &= -N\Gamma dt && (1.1)\\
\frac{dN}{dt} &= -\Gamma \\
N(t) &= N_0 \exp(-\Gamma t) \\
P(t) &= \exp(-\Gamma t) && (1.2)
\end{aligned}
$$

Where $\tau = \frac{1}{\Gamma}$ is defined as the lifetime of radioactive nucleus, the mean time the sample of the radioactive nuclei has decreased by half, and $P(t)$ is defined as the survival probability. The survival probbility is defined as the probability that the physical situation described by the initial wavefunction $\left|\Psi_0\right\rangle$ does not change after the system evolves in time; or, the probability that the system would be in the same initial state after a time t.

In [1, 2] the survival probability was described as an exponential behavior in time, like it was shown in the relations above. It was a phenomenological approach to the problem of unstable systems, as no effort was made to understand the mechanism which is responsible for the decay; thereby, it didn't depend on the form of the Hamiltonian and its dependence on time. This hypothesis was confronted by the experimentalists and experts in radioactive phenomena, and it agreed so well with the results. Therefore, no one ever had a single doubt about the survival probability and its behavior in time, as the domain of the classical mechanics in the observations was reasonable in the macroscopic world.

However, the rising of the quantum mechanics and the relevance it took in the 1920's led to a reformulation of the metastable and unstable system analysis.In 1928, R W Gurney, E U Condon and G Gamow [5], separately, gave the first theoretical description of the decay mechanism, based on the quantum tunnel effect. It was one of the first quantum mechanical descriptions of a metastable system. Meanwhile, Breit and Wigner[4] began to think about the interaction that let the decay occurs; from the perspective and relation between the decay phenomena and the energy dependence of the involved variables in the problem, Fock and Krylov[6] claimed that the exponential decay could not be theoretically accepted; one of the most transcendental results that they got, the Fock Krylov theorem will be one of the most important topics that this work will consider, in order to find the necessary results that complete the description of the survival probability at large times, as it will be showed later in this work.

Later, Khalfin [7], in one of the most important results in the development of the unstable system analysis confirmed the existence of non exponential decay on the basis of a mathematical theorem given by Paley and Wiener[8]; he also showed that in quantum mechanics, the exponential law is only an approximation. This important result will be explained later, as it is the corner stone that led to a comprehension of non exponential decay in the nature. Other remarkable works in this field could be the one that Hellund[9] presented in 1953, on the basis of nonexponential law at long time behavior; or Namiki and Mugiyabashi [10], who claimed about a distinction in time for three step behavior in the survival probability: A Gaussian law at short times, the exponential law at intermediate times, and a power law at longer times.

Some works explored the relation between the S matrix elements and the statistical mechanics, taking the decay process as a Markovian process, trying to get the master equation, and finding the virial coefficients. Others led the way to find the analytic properties of the S matrix terms, and thereby the possible poles that could correspond to its form, as Munachata, Kawaguchi and Goto[11], who claimed that the exponential behavior stems from simple poles located on the second Riemannian sheet of the analytic expression of the relevant S matrix elements. In order to analyze the behavior of the unstable systems, the survival probability must be redefined, as the concept of wavefunction takes a huge importance. From a definite state $|\Psi\rangle$, the survival probability is defined from the fact that the physical situation described by this wavefunction doesn't change during the time interval $(0,t)$;it corresponds to the probability to find an undecayed state after a time t, after the production of the unstable state.

According to the quantum mechanical precepts, the evolution of the system is given by the Schrödinger equation:

$$\hat{H}|\Psi\rangle = i\hbar\frac{\partial|\Psi\rangle}{\partial t} \tag{1.3}$$

or the evolution operator:

$$|\Psi(t)\rangle = \exp(\frac{-i\hat{H}t}{\hbar})|\Psi_0\rangle \tag{1.4}$$

The survival probability is defined by:

$$P(t) = |A(t)|^2 \tag{1.5}$$

Where $A(t)$ is called survival amplitude:

$$A(t) = \langle\Psi_0|\exp(-\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle \tag{1.6}$$

As $\hat{H}$ is the hamiltonian that governs the dynamical evolution of the quantum system.

The rate of change of the survival probability defined classically at time equal zero is different from the one that the quantum mechanics has defined from the relation (1.5):

$$\left.\frac{dP(t)}{dt}\right|_{t=0} = -\Gamma\exp(-\Gamma t)\Big|_{t=0} = -\Gamma \tag{1.7}$$

Meanwhile, in the quantum mechanical case:

$$
\begin{aligned}
\left.\frac{dP(t)}{dt}\right|_{t=0} &= \frac{d}{dt}\left|\langle\Psi_0|\exp(-\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle\right|^2 \\
\left.\frac{dP(t)}{dt}\right|_{t=0} &= \frac{d}{dt}\Big(\langle\Psi_0|\exp(-\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle\langle\Psi_0|\exp(\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle\Big) \\
\left.\frac{dP(t)}{dt}\right|_{t=0} &= \langle\Psi_0|(-i\hat{H})\exp(-\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle\langle\Psi_0|\exp(\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle \\
&+\langle\Psi_0|\exp(-\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle\langle\Psi_0|(i\hat{H})\exp(\frac{i\hat{H}t}{\hbar})|\Psi_0\rangle \\
\left.\frac{dP(t)}{dt}\right|_{t=0} &= -i\langle\Psi_0|\hat{H}|\Psi_0\rangle\langle\Psi_0|\Psi_0\rangle \\
&+i\langle\Psi_0|\hat{H}|\Psi_0\rangle\langle\Psi_0|\Psi_0\rangle \\
\left.\frac{dP(t)}{dt}\right|_{t=0} &= 0
\end{aligned}
\tag{1.8}
$$

Where $\big|\Psi(t=0)\big\rangle = \big|\Psi_0\big\rangle$.

Therefore, there is a really huge difference between the two definitions of the survival probability, corresponding to the same physical situation. So, it can be deduced that the exponential behavior that the survival probability shows is only an approximation. Another demonstration about the theoretical description of the decay process and the incompleteness of the exponential behavior to describe it for all times was given by Erzak, (1969)[19] :
If there is an unstable wavefunction that describes the physical situation of the state, as it was mentioned before, its evolution is ruled by time evolution operator, so, its evolution can be expressed in terms of the survival amplitude in order to find the expression of the wavefunction at the time t:

$$
\begin{aligned}
\big|\Psi(t)\big\rangle &= \exp(\frac{-i\hat{H}t}{\hbar})\big|\Psi_0\big\rangle \\
\big|\Psi(t)\big\rangle &= A(t)\big|\Psi_0\big\rangle + \big|\psi(t)\big\rangle
\end{aligned}
\tag{1.9}
$$

That $\big|\psi\big\rangle$ wavefunction must satisfy that $\big\langle\Psi_0\big|\psi(t)\big\rangle = 0$,for all t a condition of orthogonality between the initial state and the physical state described by the $\big|\psi(t)\big\rangle$ wavefunction, the wavefunction that leads to the decay in the evolution of time.
If the evolution operator, at a different time t' is applied on the relation (1.9):

$$
\exp(\frac{-i\hat{H}t'}{\hbar})\exp(\frac{-i\hat{H}t}{\hbar})\big|\Psi_0\big\rangle = A(t)\exp(\frac{-i\hat{H}t'}{\hbar})\big|\Psi_0\big\rangle + \exp(\frac{-i\hat{H}t'}{\hbar})\big|\psi(t)\big\rangle
$$

But according to the relation (1.9), the evolution operator is related with the survival amplitude, it could be replaced, so, for a time t':

$$\big|\Psi(t')\big\rangle = A(t')\big|\Psi_0\big\rangle + \big|\psi(t')\big\rangle$$

Therefore:

$$\exp(\frac{-i\hat{H}(t+t')}{\hbar})\big|\Psi_0\big\rangle = A(t)A(t')\big|\Psi_0\big\rangle + A(t)\big|\psi(t')\big\rangle + \exp(\frac{-i\hat{H}(t')}{\hbar})\big|\psi(t)\big\rangle$$

But, for a particular time $t+t'$:

$$\big|\Psi(t+t')\big\rangle = A(t+t')\big|\Psi_0\big\rangle + \big|\psi(t+t')\big\rangle$$

therefore :

$$A(t+t')\big|\Psi_0\big\rangle + \big|\psi(t+t')\big\rangle = A(t)A(t')\big|\Psi_0\big\rangle + A(t)\big|\psi(t')\big\rangle + \exp(\frac{-i\hat{H}(t')}{\hbar})\big|\psi(t)\big\rangle$$

taking the inner product respect to the initial wavefunction $\big|\Psi_0\big\rangle$, i.e. multiplying by its bra $\big\langle\Psi_0\big|$:

$$A(t+t') = A(t)A(t') + \big\langle\Psi_0\big|\exp(\frac{-i\hat{H}t'}{\hbar})\big|\psi(t)\big\rangle \tag{1.10}$$

If the second term of the relation (1.10) would be identically zero, the expression of the survival amplitude would take an exponential decreasing behavior, so it would be guaranteed that the survival probability would take that behavior from the survival amplitude. But this term isn't identically zero for all cases: only takes a zero value when the initial state is an eigenstate of the nonperturbed hamiltonian(it means the part of the hamiltonian without the interaction term that causes the decay). As a matter of fact this last term is impossible to go to zero for an unstable state as an initial one, as Williams(1971)[12], Fonda and Ghirardi(1972)[13] and Sinha(1972)[14]proved, since the spectrum of the total hamiltonian doesn't cover the entire real axis. This last term reflects the history of the physical system, depending on the behavior of the hamiltonian that leads to the decay. In that sense, it introduces some sort of "memory" in the description of the decay process.

But, under this statement, one question arises: What happens with the survival probability at long and at short times, then? What is the theoretical behavior that the survival amplitude obeys when long times compared to the mean lifetime are taken into account? And what happens precisely on very small intervals of time? These kind of questions led to different sorts of investigations in order to find the real answer about the survival probability.

1.1 Survival probability at short times

As the survival probability must be related with the survival amplitude, and it doesn't show an exponential behavior for all times, it can be inferred that the survival probability has a special behavior when it is analyzed both at large and short times . As the survival amplitude must be convergent as long as t goes to ∞, Riemann and Lebesgue claimed that the survival amplitude as $t \to \infty$ must be gone to zero[15]:

$$lim_{t\to\infty} A(t) = 0 \tag{1.11}$$

Some of the works at short times in this field assumed that the survival probability has a gaussian behavior [16], the survival probability behaves according to the hamiltonian that induces the decay, and, therefore, the variance in the energy spectrum.

At small times, the decay slows down, so, an expansion of the evolution operator could be done, in order to find an analytic expression for the survival probability:

$$\begin{aligned} P(t) &\simeq \langle \Psi_0 | 1 - it\hat{H} - \frac{t^2\hat{H}^2}{2} + \cdots + \frac{(-i\hat{H}t)^n}{\hbar^n} | \Psi_0 \rangle \\ &\langle \Psi_0 | 1 + it\hat{H} - \frac{t^2\hat{H}^2}{2} + \cdots + \frac{(i\hat{H}t)^n}{\hbar^n} | \Psi_0 \rangle \\ P(t) &\simeq 1 + \langle \Psi_0 | \hat{H} | \Psi_0 \rangle^2 t^2 - \langle \Psi_0 | \hat{H}^2 | \Psi_0 \rangle t^2 \\ P(t) &\simeq 1 - (\Delta_\Psi \hat{H})^2 t^2 \end{aligned} \tag{1.12}$$

with $(\Delta_\Psi \hat{H})^2$ the variance of the expected value of the hamiltonian. As the quantum system reaches stability, the variance of the hamiltonian goes to zero, and the survival probability goes to one. As the energy and the time are conjugated variables, according to the Heisenberg's uncertainty principle, a measurement where the variance of the energy is zero(the most accurate measurement it could be developed) implies necessarily that it would take an infinity time. So, a stable system has an infinity lifetime.

It can be inferred, then, that if the eigenstates of the hamiltonian are normalized, and all the moments of $\hat{H}$ are finite [16], the survival probability shows a gaussian behavior:

$$P(t) \simeq \exp\left(-\frac{t^2}{(\Delta_\Psi \hat{H})^2}\right) \tag{1.13}$$

As the quantum mechanical descriptions demand the initial state to describe the evolution of the system in time, it is necessary to say that the initial state could be or not one eigenstate of the hamiltonian; therefore, it could be a metastable (resonance) or an eigenstate of the unperturbed hamiltonian $\hat{H}_0$ (stable). If the energy spectrum given by the hamiltonian is not finite, and the initial state of the system is not normalizable, the system could evolve different, and the survival probability would be different from the one given by (1.13).

As $\hat{H}$ has a continuous spectrum, a state can be spanned in terms of the eigenstates of the hamiltonian; besides, these eigenstates must follow the completeness relation; so an unstable initial state can be spanned in terms of these eigenstates; as the hamiltonian has a continuum spectrum(energy spectrum), and an operator $\hat{a}$ commutes with the hamiltonian, $[\hat{a}, \hat{H}] = 0$, the completeness relation is given by:

$$\int dEda \big|\Psi_{E,a}\big\rangle\big\langle\Psi_{E,a}\big| = \mathbb{I} \tag{1.14}$$

The initial state $\big|\Psi_0\big\rangle$ can be spanned in terms of the eigenstates of the hamiltonian, the basis of the Hilbert space:

$$\big|\Psi\big\rangle = \int\int dEda \big|\Psi_{E,a}\big\rangle\big\langle\Psi_{E,a}\big|\Psi\big\rangle$$

as the survival amplitude is defined by the relation (1.6), the wavefunction $\big|\Psi_0\big\rangle$ can be spanned in terms of the basis of eigenstates of the hamiltonian operator:

$$A(t) = \int\int\int\int dEdE'dada' \big\langle\Psi_{E',a'}\big| \exp(-\frac{i\hat{H}t}{\hbar})\big|\Psi_{E,a}\big\rangle\Big(\big\langle\Psi_0\big|\Psi_{E',a'}\big\rangle\Big)^*\big\langle\Psi_0\big|\Psi_{E,a}\big\rangle$$

as the elements of the basis of the *Hilbert* space are orthonormals:

$$A(t) = \int\int dE\, dE'\, da\, da' \exp(-\frac{i\,E\,t}{\hbar})\delta(E-E')\delta(a-a')\Big(\big\langle\Psi_0\big|\Psi_{E',a'}\big\rangle\Big)^*\big\langle\Psi_0\big|\Psi_{E,a}\big\rangle$$

thereby:

$$A(t) = \int\int\int\int dE\, da \exp(-\frac{i\,E\,t}{\hbar})\Big|\big\langle\Psi_0\big|\Psi_{E,a}\big\rangle\Big|^2 \tag{1.15}$$

As the integral goes from $-\infty$ to ∞, it can be shown that the survival amplitude corresponds to the Fourier transform of a function called **"density of states"**, defined as :

$$\omega(E) = \int da\Big|\big\langle\Psi_0\big|\Psi_{E,a}\big\rangle\Big|^2 \tag{1.16}$$

This last statement corresponds to the Fock Krylov theorem, given by Fock and Krylov[6] through their work on unstable systems.

Theorem 1.1 (Fock Krylov theorem) *The survival amplitude is defined as the Fourier transform of the energy density*

So, a question arises naturally: what is the physical meaning of the density of states $\omega(E)$? Well, it can be defined as follows:

> "The spectral function $\omega(E)$ is a probability density to find the eigenstates of the unperturbed hamiltonian in the initial unstable state, or the continuum probability density of states in the unstable state(resonance)." [18]

As the energy spectrum is physically bounded by below in a minimum amount of energy as a physical limit for the process to succeed, called "Threshold energy"[1] (there couldn't be infinite negative energies by definition), the density of states must be restricted in its domain, so:

$$\omega(E) = \begin{cases} 0 & \text{If } -\infty < E < E_{minimum}, \\ \omega(E) & \text{If } E_{minimum} \leq E < \infty \end{cases} \tag{1.17}$$

Therefore, in order to find an analytical expression for the survival probability, it is indeed necessary to express the form and properties that the density of states must follow; there are many properties and characteristics that define the density of states; some of them are going to be developed and mentioned in the second chapter of the present work. The importance and relevance that the density of states has is huge, due to the fact that its behavior is related directly with the survival amplitude, according to the theorem ((1.1)). Some of the different forms and expressions associated with the $\omega(E)$ function will be detailed in the second chapter.

1.2 Intermediate time behavior of $P(t)$

If that is the behavior of the survival probability at short times, the natural question that arises is: what happens after the survival probability takes the gaussian form? Would it continue showing this behavior throughout time? As a matter of fact, some authors have worked on a possible boundary between the gaussian behavior explained in the last section and the exponential law that the unstable systems follow according to the laws that describe the macroscopic world. Some authors also believed, as it will be shown in the following chapters, that the nonexponential behavior at short and large times could be considered as a result of the transformation of the non hermiticity of the evolution operator, redefined as there would be a singularity in the wavefunction as t approaches to zero, due to the impossibility to normalize the wavefunction.

One of the topics concerned at this point was precisely the interval corresponding to the behavior exposed by the survival probability, and its duration; Peres[21] found an analytic expression to determine the boundary between the nonexponential behavior and the exponential one. He said that there would be a parameter, T, depending on the decay width and the support of the energy measure, that characterizes the difference between the two relations for the survival probability. The work of Peres and others let bring a parameter to identify the distinction between the classical description of the

[1]The concept of threshold energy is one of the most important concepts in scattering theory; according to the scattering theory, the threshold energy corresponds to the minimal amount of kinetical energy that a particle must have in order to produce an endothermic reaction that satisfies the kinematical conservation laws; in classical scattering theory, it is related with the sum of the masses of the particle and the target in the collision process. Throughout the development of this work, the definition of the threshold energy will take a higher relevance, and it will be emphasized later

decay phenomena, and the quantum description that led to the nonexponential decay relations.

In the so called "intermediate times", the exponential term that expresses the classical description according to the relation (1.1) is dominating over the possible contribution that leads to the nonexponential decay. Only with times compared with the mean life time , this exponential behavior determines completely the survival probability; in particular, when the nonexponential contribution is comparable to the exponential decay law that delimits the behavior of the survival probability $P(t)$ in the intermediate times, there can be an oscillation effect present in the analytic form of the survival probability $P(t)$. Bogdanowitz, Pindor and Rackza [17] found a relation that lets determine the critical time where the transition between the nonexponential behavior and the exponential decay law can be observed, in metastable systems characterized by the survival probability $P(t)$:

$$\frac{1}{2}\Gamma t_c e^{-\frac{\Gamma t_c}{2}} = \frac{1}{\pi}\Big(\frac{\Gamma}{2(E_r - E_{\text{threshold}})}\Big)^2 \tag{1.18}$$

The solution of the relation (1.18) is related with the behavior of the exponential function; in particular, one solution would be near zero, and another in the large temporal domain; but physically, the solution near zero is meaningful, since the critical time t_c is the transition time between the exponential and the power law at large times. Hence this critical time, must be large in order to see appreciable changes in the survival probability behavior; that is the reason why at a macroscopic scale, most of the unstable states follow an exponential decay law: As the scale of time gets bigger compared with the mean life time, the nonexponential decay effects begin to be taken into account. So, in order to start watching the possible decay processes with a nonexponential survival probability, it is necessary to reach a large scale of energy, or very close to the threshold, corresponding to very short and large times respectively.

1.3 Power law behavior of $P(t)$ at large times

As it was mentioned before in a previous section, Khalfin[7]and Erzac[19] demonstrated that the exponential behavior of the survival probability for all unstable systems couldn't be satisfied along the time. As a matter of fact, in 1957, Khalfin used a fundamental theorem on Fourier transforms, the Payley Wiener theorem[8], to find out the possible properties of the survival probability at different times:

Theorem 1.2 (Payley Wiener theorem) *If the energy density function, defined in the range $-\infty < t < \infty$ vanishes identically for $E < E_{minimum}$, then its Fourier transforms necessarily satisfies:*

$$\int_{-\infty}^{\infty} \frac{\Big|\ln\big|A(t)\big|\Big|}{1+t^2}\, dt \;<\; \infty \tag{1.19}$$

Khalfin claimed that in order to have convergence in the integral described by the relation (1.19), it was necessary that the numerator of the term inside the integral must be approximated to a potential term:

$$\Big|\log\big|A(t)\big|\Big| \underset{t\to\infty}{\approx} Bt^{2-p} \qquad \text{with } p > 1 \tag{1.20}$$

Besides, Fock and Krylov[6] pointed out that the probability amplitude must go to zero as the time goes to infinity and the natural logarithm of the survival amplitude is negative; therefore, the survival probability takes the form:

$$P(t) \underset{t\to\infty}{\approx} \exp(-ct^q) \qquad q < 1, c > 0 \tag{1.21}$$

So, according to Khalfin's analysis, the asymptotic form that the survival probability takes is greater than the one that the exponential behavior of the classical treatment demands, as the energy spectrum is bounded from below, and the wavefunction is normalizable. Khalfin, in his studies, adopted a Lorentzian form for the energy density $\omega(E)$[7] , that will be discussed later, that guided to a simple pole expression for the density of states, and assuming that the width of that distribution was a constant independent of the energy spectrum, he found the power behavior that the survival probability follows:

$$P(t) \underset{t\to\infty}{\approx} \frac{\hbar^2\Gamma^2}{4\pi^2((E_{minimum})^2 + \frac{\Gamma^2}{4})t^2} \tag{1.22}$$

Khalfin demonstrated that the behavior that the survival probability follows at large times must be a power law which was different from the postulates that the classical theory demanded. But, other works along time found different types of power law behavior according to the hypothesis that were considered and affected the energy distribution and the energy density, as it will be mentioned later in the development of the present work. So, apart from a literature survey and comparison between different approaches to the power law is also to find a realistic example of the long time power law decay of a resonance. The proximity to the threshold energy starts to get relevance as the density of states is analyzed, and its properties, as it will be shown later, specially for long times:

> The degree of violation of exponential decay for long times, depends on the proximity of the energy of the decaying state to the threshold of the continuous spectrum, regardless of the form of the density of states[20]

In conclusion the development of the theory that leads to a complete understanding of the phenomena involved in the unstable or metastable states evolution in time has passed through many different stages, and there are a few things that remain as questions without answers. The quantum mechanical view guided to a different theoretical approach of the phenomena, but there are some contradictory hypothesis in the analysis that would be commented in this project. To guarantee a better understanding of the problem to be solved, each one of the most transcendental developments and supports will be discussed, in order to give a complete perspective of the problem to analyze. From some ideas taken from the statistical mechanics(like the Beth Uhlenbeck theorem that will be explained later and its implications) and some considerations about the properties of the density of states, one of the goals will be to express the formal behavior that the survival amplitude would follow according to the considerations that Fock and Krylov, Khalfin assumed and claimed, whose implications were explained throughout this chapter. Once this behavior has been analyzed near the threshold energy, the next step would be take into account some processes that give the opportunity to study the creation of metastable systems and its evolution in time, like the excited states of nuclei. As it will be developed in the following chapters, there are many properties that the density of states follows, and some of them will be used in order to find a survival probability that expresses the behavior of the involved variables in each one of the processes very well.

CHAPTER 2

Findings in literature and controversies related with nonexponential decay

According to the exposition of the main problem described in the previous chapter, the controversy that led on the nonexponential behavior of the survival probability at large and short times arose from the uncertainty of the experimental results that were taken in order to observe and analyze the theoretical hypothesis that the quantum perspective gave in the unsolved problem of metastable systems. As some restrictions and boundary conditions that depends on the nature of the system and characterize its behavior in time were introduced , some ideas were postulated, in order to clarify the time evolution of metastable systems, and their decay processes.

But, in order to analyze and determine the evolution of some of the resonant systems that are going to be discussed in the present work, it is necessary to analyze the different kinds of behavior that the survival probability would take in the domain of time. Some authors have investigated the domain of short times, finding some interesting explanations about the so called "Quantum Zeno Effect", that would not be discussed since the aim of this work is precisely the description of the decay processes at large times.

Meanwhile, other physicists have tried to find an analytic expression that leads to a better understanding of the survival probability and extrapolate some information about its behavior at short and long times, in order to find out the physical implications that the considered hypothesis lead us to think about the generalities of the decay process. Some of them have found different kinds of nonexponential behavior(explicitly power law behavior at long times), that contrast with each other, depending on the considerations that each one of their models propose. In this chapter, some of the most transcendental models that guide to a complete understanding of the behavior of the survival probability at long times would be discussed, and, therefore, their discrepancies and similarities, before the model that it is going to be used in this work would be introduced.

2.1 Different approaches to power law behavior at large times

As it was observed by Khalfin [7], the description of nonexponential behavior of the survival probability became more and more relevant to analyze and study, even though there wasn't a unique opinion that led to a common point of view. As the experimental development was trying to describe the resonance behavior at long and small times, compared to the lifetime of the system, the exponential decay was getting stronger as the unique possibility, due to the limitations of the experimental measurements. Some authors, in the development of their works, talked about the nonobservability of the nonexponential decay in some processes like the decay of a proton, or some stable particles, with a large lifetime. Some of the reasons that those experimental data couldn't find an appropriate long time behavior as the theory demands is the uncertainty in the domain: At what time would be found the long time behavior of the survival probability ? As it was mentioned in the previous chapter, the boundary between the nonexponential and exponential decay depends on some of the characteristics of the system. So, in order to determine the "critical time" where the behavior of the survival probability changes, it would be required to characterize the unstable system, the resonance, for example, and analyze its characteristics. As different authors treated the metastable system with a different formalism, they made some assumptions that led to discrepancies in the description of the survival probability as it would be shown.

One of the most important developments in the topic of the unstable systems was given by ***Khalfin*** [7], from the Payley Wiener theorem (1.19); the description of the nonexponential decay in the decay processes guided to many physicists to try to find an analytic expression for the survival probability in order to determine the possible law that the metastable system would behave in the long time spectrum. On the important conditions to be considered is that the wavefunction of an unstable state cannot be an eigenfunction of the Hamiltonian; only stable states can have exact energy eigenvalues, thus leading to a constant survival probability, namely $P(t) = 1$. Besides, as it was mentioned before, the energy spectrum would be bounded from below, including a minimum energy that, according to kinematical considerations, corresponds to the minimum energy needed to guarantee the development of the reaction, called *Threshold energy*. Anyway, Khalfin [7], among others, demonstrated that if there wouldn't be a lower value in the spectrum, the exponential behavior of the survival probability would be satisfied for all times. And, according to the Fock Krylov theorem ((1.1)), the survival amplitude is the *Fourier* transform of the density of states $\omega(E)$, the best way to find an analytic expression of the survival probability is, precisely , to characterize the density of states according to the conditions mentioned above.

2.1.1 Green's function method

Nevertheless, there exist some methods in order to obtain an analytic expression of the survival probability; as a matter of fact, according to the regular solutions of the *Schrödinger* equation (1.3), it can be expressed in terms of the Green functions corresponding to the differential equation, which act like propagators that determine the temporal evolution of the wavefunction, subject to the total hamiltonian that corresponds to the system. The Green function related with the temporal evolution of the wavefunction must behave according to:

$$\big(-\frac{\partial^2}{\partial r} + V(r)\big)g(r, r', t) = i\frac{\partial}{\partial t}g(r, r', t) \tag{2.1}$$

with the initial condition:

$$g(r, r', 0) = \delta(r - r')$$

Moshinsky, Garcia Calderón and Mateos found an alternative form to find an analytic expression for the survival probability, according to the properties of the Green function and what are called the regular solutions [1] of the Schrödinger equation [25]. The regular solutions must accomplish the boundary condition at $r = 0$:

$$\psi(k, 0) = 0 \tag{2.2}$$

$$\frac{\partial \psi_{regular}(k, r)}{\partial r}\Big|_{r=0} = 1 \tag{2.3}$$

$$\tag{2.4}$$

Meanwhile, as the regular solution $\psi_{regular}(k, r)$ expresses the condition over the regular point $r = 0$, it is necessary to define a solution of the radial equation of the Schrödinger equation that takes the boundary condition to the irregular point, at $r \to \infty$; so, it can be defined two irregular solutions of the Schrödinger equation $f_\pm(k, r)$, which are solutions of the radial equation, with the corresponding boundary condition[26]:

$$\lim_{r\to\infty} e^{\mp ikr} f_\pm(k, r) = 1 \tag{2.5}$$

The irregular solutions $f_\pm(k, r)$ are defined so that $f_+(k, r)$ is well defined in the upper half of the k plane(imaginary part of k is greater than 0), and $f_-(k, r)$ is well defined in the lower(imaginary part of k is less than 0). In particular, it can be deduced the

[1] The regular solutions of the Schrödinger equation are linearly independent solutions of the differential equation so that they vanish in a singular regular point. In this case, the regular point is 0; therefore the regular solutions must be built from boundary conditions in the regular point.

analytic behavior of $f_{\pm}(k,r)$ for all k, according to the behavior of the potential, that induces the decay. For all $k > 0$, $f_-(k,r) = f_+^*(k,r)$, extending the analyticity of the irregular solutions over the entire k plane. Now, as the irregular solutions are linearly independent solutions of the Schrödinger equation, the regular solution can be expressed as a linear combination of the irregular solutions. As a matter of fact, the linear coefficients in the expansion of the regular solution in terms of the irregular solutions can be found, as the wronskian[2] of the irregular solutions $W[f_+(k,r), f_-(k,r)]$is a constant, independent of r; so, the coefficients of the expansion of the regular solution can be found explicitly from the wronskians of the irregular solutions:
Let it be $\psi_{regular}(r,k)$ the regular solution of the Schrödinger equation:

$$\psi_{regular}(r,k) = af_+(r,k) + bf_-(r,k) \tag{2.6}$$

with

$$b = \frac{W[f_+(r,k), \psi_{regular}(r,k)]}{W[f_+(r,k), f_-(r,k)]}$$
$$a = \frac{W[f_-(r,k), \psi_{regular}(r,k)]}{W[f_+(r,k), f_-(r,k)]}$$

As $f_+(r,k)$ and $f_-(r,k)$ are linearly independent solutions of the Schrödinger equation, its wronskian is constant [26]:

$$\psi_{regular}(r,k) = \frac{W[f_-(r,k), \psi_{regular}(r,k)]f_+(r,k) - W[f_+(r,k), \psi_{regular}(r,k)]f_-(r,k)}{2ik} \tag{2.7}$$

where $W[f_+(r,k), f_-(r,k)] = -2ik$, independent of r.

Resonances from zeros of the Jost function in the complex momentum plane

Along the literature it is defined a function called the Jost function $\mathcal{F}(k)$, from the wronskian of one of the irregular solutions of the radial equation and the regular solution[26]; so:

$$\mathcal{F}(k) = W[f_+(k,r), \psi_{regular}(k,r)] \tag{2.8}$$

The Jost function is an analytic function of k regular in the first half of the k complex plane, with a continuous derivative. As $|k| \to \infty$, the Jost function goes to one, but that limit depends on the strength of the potential.

As the Jost function is related with the irregular solution in the first half plane of the k complex plane, it would be related with the definition of the irregular solution of

[2] The Wronskian of two functions $f(r)$ and $g(r)$ is defined by: $W[f,g] = f(r)g'(r) - f'(r)g(r)$; if $f(r)$ and $g(r)$ are solutions of the same second order differential equation, the wronskian is independent of the variable

the radial equation in the second half plane, also; so, the analyticity of the Jost function can be extended to the second half plane of the k complex plane:

$$\mathcal{F}^*(k) = \mathcal{F}_-(k) \tag{2.9}$$

where:

$$\mathcal{F}_-(k) = W[f_-(k,r), \psi_{regular}(k,r)]$$

Therefore, from the Jost function and the irregular solutions of the Schrödinger equation, the outgoing Green function $G^+(r,r',t)$, that acts as a propagator of outgoing waves, subject to the hamiltonian that causes the decay, can be found; in particular, the outgoing Green function describes the propagation characteristics of the state vectors in time. If the Laplace transform $\mathcal{G}(r,r';s) = \int_0^\infty g(r,r';t)e^{-st}dt$ is applied over the equation (2.1), and introducing $k = \sqrt{is}$:

$$\Big(\frac{\partial^2}{\partial r^2} + k^2 - V(r)\Big)\mathcal{G}(r,r';s) = -i\delta(r-r') \tag{2.10}$$

as:

$$\mathcal{G}(r,r';s) = -iG^+(r,r';k)$$

So, it can be found the differential equation that the outgoing Green function must follow:

$$\Big(\frac{\partial^2}{\partial r^2} + k^2 - V(r)\Big)G^+(r,r';s) = \delta(r-r') \tag{2.11}$$

The outgoing Green function is continuous at $r = r'$, but, according to the definition of a Green function, its derivative has a discontinuity of 1[26]:

$$\frac{\partial G^+(r,r',k)}{\partial r}\Big|_{r=r'-0}^{r=r`+0} = 1 \tag{2.12}$$

Besides, the outgoing Green function must be regular at $r = 0$, and contains only outgoing waves at $r \to \infty$; so, the analytic form of the outgoing Green function corresponds to :

$$G^+(r,r',k) = \begin{cases} \psi_{regular}(k,r)h_1 r & \text{for } r < r' \\ f_+(k,r)h_2(r) & \text{for } r > r' \end{cases}$$

where:

$$h_1(r) = -\frac{f_+(k,r)}{\mathcal{F}(k)}$$

and

$$h_1(r) = -\frac{\psi_{regular}(k,r)}{\mathcal{F}(k)}$$

Therefore, in order to find the poles of the outgoing Green function $G^{+}(r, r', t)$ is necessary to find the zeros of the Jost function; in the lower half of the complex k plane, the zeros of the Jost function on the imaginary k axis are related to the bound states produced by the potential $V(r)$; meanwhile, the number of zeros of $\mathcal{F}(k)$ in the upper half of the k plane must be finite, in order to fulfill the regularity of $\mathcal{F}$ for $Imk > 0$; besides, the zeros of $\mathcal{F}(k)$ in the upper half of the complex plane must be necessarily simple. Since energy is a doubled value function, there exist two Riemanian sheets of the complex energy surface; the plane corresponding to $Imk > 0$ is labeled as the "physical" sheet, and that corresponding to $Imk < 0$ is labeled as the "unphysical" one. It is relevant to say that all the potentials considered in the model don't have attractive part, so they don't have any kind of bound states in the upper half of the k plane (poles in the "physical sheet"). Gamow[5] found that all resonant states must follow the radial equation; all states must be regular in the origin and, as the considered potential is localized(so that it vanishes as $r \to \infty$), a boundary condition(outgoing boundary condition) was defined in order to guarantee that the eigenvalues of k would be complex. Resonances (unstable states) are found on the fourth quadrant in the lower half of the complex plane, so that the energy eigenvalues would be $E_n = \epsilon_n - i\frac{\Gamma}{2}$ with Γ and ϵ_n greater than zero), lying in the so called"unphysical" sheet:

$$\frac{d^2}{dr^2} + \big(k^2 - V(r)\big)u_n(r) = 0 \tag{2.13}$$

with the boundary conditions:

$$u_n(0) = 0 \tag{2.14}$$

$$\frac{du_n}{dr}\Big|_{r=R_-} = -ik_n u_n(R) \tag{2.15}$$

As at $r > R$, the resonant state acts as outgoing waves, with no incident particle, so it is proportional to the complex exponential function that delimits the behavior of the outgoing waves.

As a matter of fact, the energy complex eigenvalues corresponds to the poles of the outgoing Green function $G^{+}(r, r', r)$, that follows the equation (2.15), as they correspond to the zeros of the Jost function in the upper half of the k plane; so, the resonant states, related with the energy complex eigenvalues, are the residues of the outgoing Green function calculated on the energy eigenvalues mentioned above[27]:

$$Res[G^{+}, k_n] = \frac{u_n(r)\ u_n(r')}{2k_n} \tag{2.16}$$

Where k_n is one of the poles of the outgoing Green function. Even though the **S** matrix and the outgoing Green function are inversely proportional to the Jost function $\mathcal{F}(k)$, the behavior of the poles of the two of them aren't necessary the same, as the evaluation of the terms of the **S** matrix corresponds to energy dependent eigenfunction, which for complex energies(for example the energies corresponding to the resonant poles of the $G^+(r, r', t)$ function) aren't normalizable. The **S** matrix is related with the **T** matrix, so, it is related with the propagator(in this case the outgoing Green function); although, the analytic form that leads to the **T** matrix depends on the behavior of the potential. That's the main reason why the behavior of all the poles of the **S** matrix doesn't act exactly as the poles of the outgoing Green function; as a matter of fact, some of the simple poles of the outgoing Green function doesn't correspond to simple poles of the **S** matrix, even though they are related with the zeros of the Jost function. It is necessary to say that as the resonant states are the residues of the outgoing Green function according to (2.16), they need to be normalized; so the normalization of the resonant states is given by:

$$\left(\int_0^R u_n^2(r)dr\right) + \frac{iu_n^2(R)}{2k_n} = 1 \tag{2.17}$$

A contour in the k plane is taken, like the one showed in the figure (Fig.(2.1)), with a large circle, whose radius will go to infinity eventually, and small circles surrounding the poles of the outgoing Green function in the k plane, as well as the point where the function given by $\frac{G^+(r,r',z)}{z-k}$ isn't analytic at all. As a matter of fact, the outgoing Green function is analytic almost everywhere in the plane, except on the spectrum of the hamiltonian where it has discrete poles corresponding to the bound states, and a branch cut(arising from the scattering states, extending over all possible energies from zero to infinity, where the boundary between the "physical" and "unphysical" sheets is given, so that $Imk = 0$). So, according to the Cauchy theorem, from the complex variable calculus, this integral calculated on the contour in the Fig. ((2.1)) must vanish:[3]

$$\oint_C \frac{G^+(r, r', z)}{z - k} dz = 0 \tag{2.18}$$

So, it would be needed to find the analytic expression of the outgoing Green function, in order to determine its poles, and the residues to obtain the resonant states. From the definition of the outgoing Green function in terms of the Jost function, García - Calderón, Moshinsky and Mateos found the behavior of the outgoing Green function in the upper half of the complex k plane,as they denoted that if the contour shown in the Fig. (2.1) is followed to calculate the integral (2.18), and the integral along the big circle with radius k is determined, as long as the radius of the circle would go to infinity, the potential can be disregarded because the outgoing Green function acts like a propagator of free waves in the future, and the considered potential is localized(so

[3]The Cauchy- Goursat theorem stands that as C is a simple, closed contour, and $f(z)$ would be an analytic function in the interior of C , the integral $\oint_C f(z)\, dz$ will vanish[28]

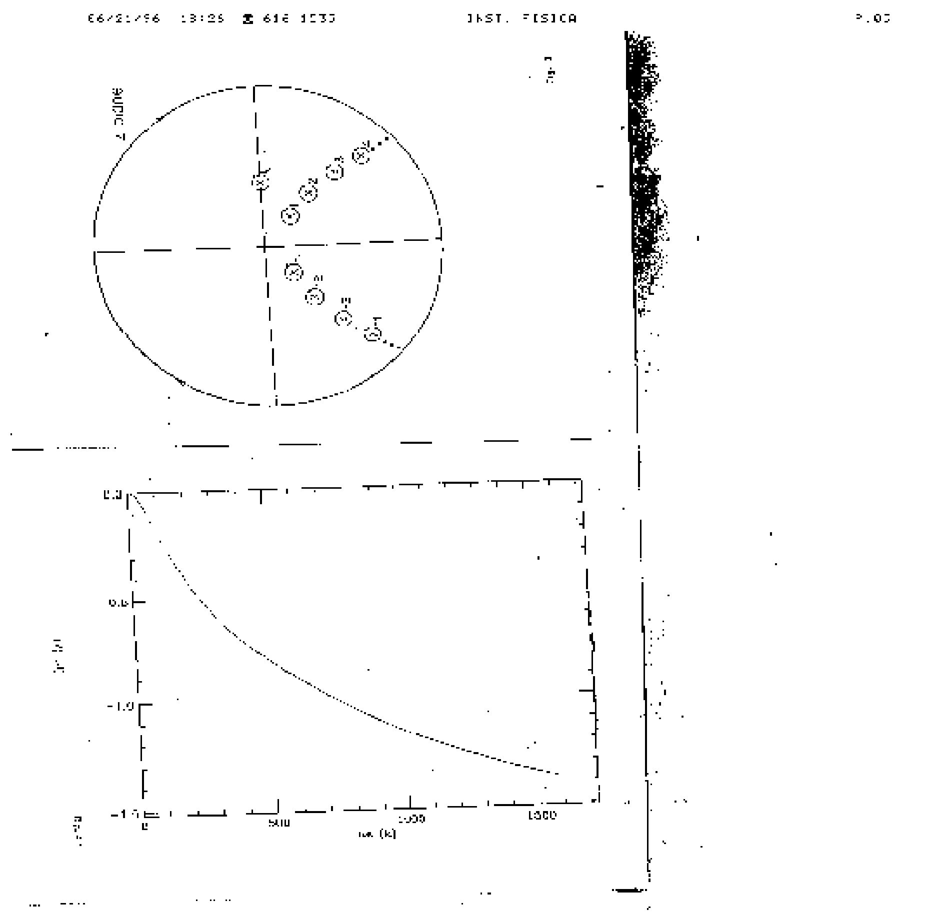

Figure 2.1: Considered contour to find the behavior of the outgoing Green function taken from [25](up). Description of the survival probability for this system(down)

that, it vanishes after some R different from zero)[25]; so, the solution of the equation would take this form :

$$G^{+}(r,r',k) = \begin{cases} e^{iz(r+r')} - e^{iz(r'-r)} & \text{If } 0 \leq r \leq r' \leq R, \\ e^{iz(r+r')} - e^{iz(r-r')} & \text{If } 0 \leq r' \leq r \leq R, \end{cases} \tag{2.19}$$

Meanwhile, when the outgoing Green function behavior is analyzed in the second half of the k complex plane, nothing can be assure specifically, due to the behavior of the poles in that domain; nevertheless, García Calderón and Berrondo[29], using an appropriate form of the Born approximation to find an analytic form of the outgoing Green function, found that the considered Green function vanishes exponentially in the "unphysical" sheet, as long as $(r, r') < R$. That's because of the behavior of the irregular solution of the Schrödinger equation, and the Jost function associated with one of them: As long as k would be real, the Jost function can't be zero, in order to guarantee that the regular solution wouldn't be zero, giving the trivial solution; as $Imk < 0$ the zeros of the Jost function can't be related with the bound states, as the poles of the **S** matrix in the physical sheet corresponds to the bound states so, the possible contributions to the integral given by (2.18) corresponds to the circles around the poles of the Green function and the explicit pole given when $z = k$ in the imaginary axis, as it can be shown in the Fig.((2.1)). According to the behavior of the outgoing Green function given by (2.19), as long as the outgoing condition is satisfied, the behavior of the outgoing Green function is similar to the superposition of two different waves, with the same wavenumber. The poles of the outgoing Green function overlaps at large energies, but at small energies, can be isolated. As a matter of fact, as it was mentioned before, according to the properties of the irregular solutions of the radial equation, and from time reversal considerations, for all poles in the fourth quadrant of the k plane there exists a relation with the poles in the third quadrant of the plane. The importance of the Moshinsky, Mateos and García Calderón model is bigger as all the potentials considered must be localized on a region of the space, so that, they vanishes after some distance; as the considered potential is localized, the Jost function is analytic over the complex plane,because of the behavior of the irregular solutions of the Schrödinger equation, as long as they would be entire for localized potentials[31]; so that, $\mathcal{F}(k)$ and $\mathcal{F}(-k)$ would be analytic, but not identically zero at the same time; the poles of the **S** matrix are disposed symmetrically respect to the imaginary k axis in the lower plane of the complex plane(the unphysical sheet). So:

$$k_{-p} = k_p^* \tag{2.20}$$

With k_{-p} representing the possible poles in the third quadrant, and k_p the poles in the fourth quadrant[30].

As it was mentioned before, all the poles of the **S** matrix, in the first upper half of the k plane $Imk > 0$(Riemanian physical sheet) are considered as bound states, and the irregular solutions of the Schrödinger equation behave exponentially decreasing for large r; nevertheless it is important to notice that the number of poles of the Green function, in the upper half of the k plane must be finite, according to the behavior of the potential, and its convergence. In the model described by Moshinsky, Mateos and García Calderón, the potential is localized over a region of the space. As the resonant states corresponds to residues of the outgoing Green function in the poles, an expansion of the Green function can be obtain in terms of the resonant states, as long as the normalization condition and the definition of the resonant states can be considered (2.16),(2.17). Therefore, the outgoing Green function can be spanned in terms of the resonant states as the following relation[27], as the only contributions to the integral (2.18) are given by the circles that enclose the poles:

$$G^+(r,r',k) = \sum_{n=-\infty}^{\infty} \frac{u_n(r)u_n(r')}{2k_n(k-k_n)} \qquad (r,r') < R \tag{2.21}$$

The resonant states $u_n(r)$ are solutions of the radial equation; therefore, some relations characterize the resonant states in different regions of the space, according to the behavior of the outgoing Green function, finding relations that express the behavior of the resonant states:

$$\sum_{n=-\infty}^{\infty} \frac{u_n(r)u_n(r')}{k_n} = 0 \qquad (r,r') < R \tag{2.22}$$

also:

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} u_n(r)u_n(r') = \delta(r-r') \qquad (r,r') < R \qquad (r,r') < R \tag{2.23}$$

On the expansion of the outgoing Green function given by (2.21), a partial fractions method can be introduced, in order to get an analytic expression of the outgoing Green function, so that, it can be simplified according to the relations of the resonant states mentioned above(2.22); so, as $\frac{1}{k_n(k-k_n)} = -\frac{1}{k_nk} + \frac{1}{k(k-k_n)}$:

$$G^+(r,r',k) = \sum_{n=-\infty}^{\infty} \left(-\frac{u_n(r)u_n(r')}{2kk_n} + \frac{u_n(r)u_n(r')}{2k(k-k_n)} \right) \qquad (r,r') < R$$

according to the relation (2.22), the sum of resonant states, for all $r \neq r'$ over the corresponding pole of the outgoing Green function must be zero; therefore, the first term of the expansion vanishes, and the outgoing Green function is defined by:

$$G^+(r,r',k) = \sum_{n=-\infty}^{\infty} \frac{1}{2k} \frac{u_n(r)u_n(r')}{(k-k_n)} \qquad (r,r') < R \tag{2.24}$$

This is the analytic expression of the outgoing Green function in terms of the resonant states and its poles.

Temporal evolution of the unstable system

Once the outgoing Green function is determined in terms of the resonant states, it is necessary to determine the time dependent Green function, in order to analyze the temporal evolution of the system, and so, determine the wavefunction $\Psi(t)$. With the analytic description of the wavefunction, it would be simple to find the survival probability, according to the relation (1.6). It is necessary to expand the wavefunction in terms of the resonant states; but in order to do that, it is relevant to find the temporal evolution of the wave function, so that, the system can be characterize in a time t; therefore, the need to find an analytic expression of the temporal dependent Green function, given by (2.1), is sustained. As it was mentioned before, the outgoing Green function is related with the Laplace transform of the temporal Green function that gives the temporal evolution of the system; so, in order to find the temporal Green function, it is relevant to do an inverse Laplace transform. According to the complex variable theory, the inverse Laplace Transform is given by the Bromwich integral formula [28][4]:

> Let it be F(s) an analytic function in the semiplane $Re\ s \leq 0$. Suppose that there are three positive constants, $m, R_0, \text{and} k$, as $|F(s) \leq \frac{m}{|s|^k}$ when $|s| < R_0$. Then, there exists a function $f(t)$, whose Laplace transform is $F(s)$; this function is given by:
>
> $$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}ds \tag{2.25}$$
>
> The integration is taken along the line $Re\ s = a$, or along any other contour that can be obtained from that line, applying the trajectory independence principle

As it was mentioned earlier, if it is assumed that $k = \sqrt{is}$, and the integral is taken over the contour given by the Fig. ((2.1)), the inverse Laplace transform of the outgoing Green function $G^+(r, r', k)$ can be calculated. But the contour shown in the Fig.((2.1)) can be deforming, in order to use the analytical properties that the outgoing Green function has in the corresponding domain. So, the line integral must be calculated over a new contour, since there is an exponential term that decrease on the first half of the imaginary plane; as there is no bound state related to the potential, there can't be a single pole of the Green function, and the **S** matrix in the physical sheet, so, the outgoing Green function is analytic in the first half corresponding to $Im\ k > 0$; this new contour corresponds to a Bromwich contour built according to the convergence of the exponential term of the integral: So, the Laplace inverse transform takes this form:

$$g(r, r', t) = \frac{1}{2\pi i}\oint_{C'} G^+(r, r', k)e^{-ik^2t}(2k)dk \tag{2.26}$$

where $s = -ik^2$ and C' is the deformed Bromwich contour given in the Fig. ((2.2)); as the exponential term converges in the first half of the plane, the integral over the hyperbolic contour goes to zero as long as $k \to \infty$; meanwhile, the integral over the first contour, the quarter circle where $\pi \leq \theta \leq \frac{\pi}{2}$ vanishes, as the outgoing Green function vanishes exponentially in the first half of the plane. So, the growth of the denominator is greater than the numerator as the radius of the circulus goes to infinity; therefore,

[4] p. 453

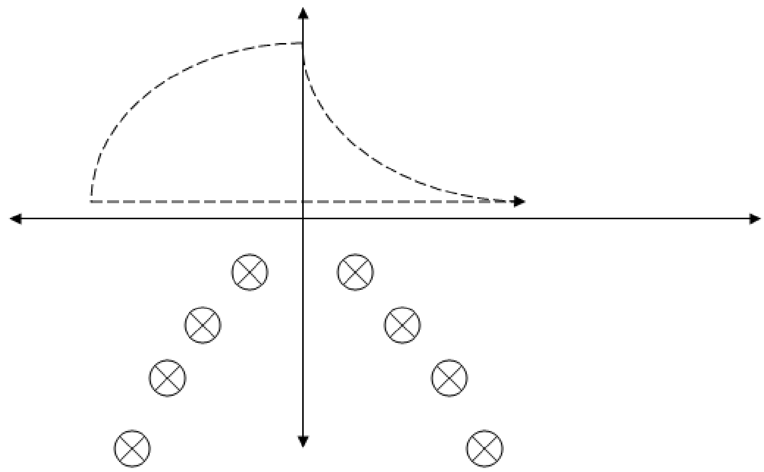

Figure 2.2: Bromwich contour obtained by deforming the previous contour, taken from [25]; the poles are located below the real axis

the only contribution to calculate the integral over the Bromwich contour ((2.25)) is given by the real axis:

$$g(r, r', t) = \int_{-\infty}^{\infty} G^{+}(r, r'; k) e^{-ik^2 t} 2k dk \tag{2.27}$$

As the outgoing Green function is spanned in terms of the resonant states (2.24), the temporal Green function $g(r, r', t)$ which expresses the temporal evolution of the system after the decaying process, can be spanned in terms of the resonant states also:

$$g(r, r', t) = \frac{i}{2\pi} \sum_{n=-\infty}^{\infty} u(r) u(r') \int_{-\infty}^{\infty} \frac{e^{-ik^2 t}}{k - k_n} dk \tag{2.28}$$

So, in order to define the temporal evolution of the system, it is necessary to determine the integral $\int_{-\infty}^{\infty} \frac{e^{-ik^2 t}}{k-k_n} dk$, that depends on time; this integral has been developed throughout some problems on transient effects in Quantum mechanics; this expression is known as the Moshinsky function, and it is defined as:

$$\mathcal{M}(k_n, t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik^2 t}}{k - k_n} dk = -w(\sqrt{-it} k_n) \tag{2.29}$$

where $w(z)$ is the error function, related with Erfc(z) according to [24][5]:

$$w(z) = e^{-z^2}\text{Erfc}(-iz)$$

where:

$$\text{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt \tag{2.30}$$

The Erfc(z) function corresponds to the complementary error function; its utility is relevant in different kind of problems of physics and mathematics, due to the integral definition of the function, as the integral of a gaussian distribution, and due to the failure of the error function as it approaches to huge values; so the temporal Green function of the system is given by:

$$g(r, r', t) = \sum_{n=-\infty}^{\infty} u_n(r)u_n(r')\mathcal{M}(k_n, t) \qquad \text{For } r, r' < R \tag{2.31}$$

As the temporal evolution of the wavefunction lets define the state of the system in a time t, the wavefunction in that particular time can be defined as:

$$\psi(t) = \int_0^R \psi_0(r')g(r, r', t)dr' \tag{2.32}$$

Survival amplitude and survival probability

As the temporal Green function is defined by (2.31), there's a coefficient that can be defined, for simplicity, that resumes the expansion of the wavefunction $\psi(r, t)$ in terms of the resonant states $u_n(r)$; the wavefunction, therefore, can be shown as an expansion in terms of the orthogonal resonant states, as the outgoing Green function and the temporal Green function have been spanned:

$$\psi(r, t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} C_n\mathcal{M}(k_n, t)u_n(r) \qquad r < R \tag{2.33}$$

where:

$$C_n = \int_0^R u_n(r')\psi_0(r')dr'$$

and:

$$\bar{C}_n = \int_0^R u_n(r')\psi_0^*(r')dr'$$

[5]page 297

As the resonant states $u_n(r)$ have some interesting properties,(2.22) and (2.23), the coefficients C_n and $\bar{C_n}$ will have analog properties as the resonant states are related to them(multiplying by $\psi_0(r)$ and $\psi_0^*(r')$ and integrating over the range of the potential,0 to R):

$$\sum_{n=-\infty}^{\infty} \frac{C_n\bar{C_n}}{k_n} = 0 \tag{2.34}$$

and

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} C_n\bar{C_n} = 1 \tag{2.35}$$

Meanwhile, it is relevant to say that the wavefunction $\psi(r,t)$, that corresponds to the state of the system in the time t, must be normalized, as the expansion in terms of the resonant states are normalized according to the relation (2.17).

Once the wavefunction of the system has been determined, the survival amplitude can be calculated, according to the relation (1.6); so, the survival amplitude can be also spanned in terms of the resonant states, and the coefficients C_n and $\bar{C_n}$; so, replacing (2.33) in (1.6), an analytic expression for the survival amplitude can be obtained:

$$A(t) = \sum_{n=-\infty}^{\infty} C_n\bar{C_n}\mathcal{M}(k_n,t) \tag{2.36}$$

According to the relation (1.5), an analytic expression of the survival probability can be found, replacing in that equation the relation given by ; so:

$$P(t) = \big(\sum_{n=-\infty}^{\infty} C_n\bar{C_n}\mathcal{M}(k_n,t)\big)^*\big(\sum_{l=-\infty}^{\infty} C_l\bar{C_l}\mathcal{M}(k_l,t)\big)$$

in other way:

$$P(t) = \big(\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} C_n C_l^*\bar{C_n}\bar{C_l}^*\mathcal{M}(k_n,t)\mathcal{M}^*(k_l,t)\big) \tag{2.37}$$

as the C_n and $\bar{C_n}$ coefficients are related with the initial wavefunction $\psi_0(r)$:

$$\bar{C_l}^*\bar{C_n} = \int_0^R\int_0^R \psi_0(r)\psi_0^*(r')u_l^*(r)u_n(r')drdr'$$

The initial wavefunction must be normalized, so:

$$\bar{C_l}\bar{C_n}^* = \int_0^R\int_0^R \delta(r-r')u_l^*(r)u_n(r')drdr'$$

therefore:

$$I_{nl} = \int_0^R u_l^*(r) u_n(r) dr$$

and:

$$P(t) = \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} C_n C_l^* I_{nl} \mathcal{M}(k_n, t) \mathcal{M}^*(k_l, t) \tag{2.38}$$

The main conclusion that can be taken from the expression above is that the survival probability depends on the behavior of the term $\mathcal{M}(k_n, t)\mathcal{M}^*(k_l, t)$ for n, l from $-\infty$ to ∞. These terms express the temporal behavior of the survival probability $P(t)$. The main difficulty is given by the behavior of the poles; the poles of the outgoing Green function in different quadrants affect the behavior of the Moshinsky function $\mathcal{M}(k_n, t)$), as the symmetry of the poles respect to the imaginary axis in the plane gives that $k_p = -k_{-p}^*$, as it was mentioned before, so the temporal behavior of the survival probability can be separated in tow contributions: One from the poles in the third quadrant and one from the poles in the fourth quadrant. The Moshinsky function $\mathcal{M}(k_n, t)$ can be written in the fourth quadrant as [27]:

$$\mathcal{M}(k_n, t) = e^{-ik_n^2 t} - \mathcal{M}(-k_n, t) \tag{2.39}$$

So, the survival amplitude $A(t)$ takes this form:

$$A(t) = \sum_{n=1}^{\infty} C_n \bar{C}_n \mathcal{M}(k_n, t) + \sum_{n=-\infty}^{0} C_n \bar{C}_n \mathcal{M}(k_n, t)$$

making a change of variables, the limits of the second sumatory can be rewritten:

$$A(t) = \sum_{n=1}^{\infty} C_n \bar{C}_n \mathcal{M}(k_n, t) + \sum_{n=0}^{\infty} C_{-n} \bar{C_{-n}} \mathcal{M}(k_{-n}, t)$$

As there is a symmetry relation between the poles of the outgoing Green function in the third and the fourth quadrant,mentioned before, and, by that, there's a relation between the coefficients C_n and $\bar{C}_n$ related with the poles k_n, as $C_{-n}\bar{C_{-n}} = C_n^* \bar{C}_n^*$; besides, replacing a relation that lets analyze the behavior of the Moshinsky function related with the poles of the third quadrant of the plane, given by (2.39) :

$$A(t) = \sum_{n=1}^{\infty} C_n \bar{C}_n e^{-ik_n^2 t} - \mathcal{M}(-k_n, t) + \sum_{n=0}^{\infty} C_n^* \bar{C}_n^* \mathcal{M}(-k_n^*, t)$$

therefore:

$$A(t) = \sum_{n=1}^{\infty} C_n \bar{C}_n e^{-ik_n^2 t} - I(t) \tag{2.40}$$

where:

$$I(t) = \sum_{n=0}^{\infty} \big(C_n \bar{C}_n \mathcal{M}(-k_n, t) - C_n^* \bar{C}_n^* \mathcal{M}(-k_n^*, t) \big)$$

As it has been shown in the past paragraph, there are two possible contributions to the survival amplitude$A(t)$: One is exponential (given by the first term in the last relation), and the other is related with the analytic behavior of the Moshinsky function in the temporal domain $I(t)$.

What happened then when the first term, corresponding to the exponential contribution of the survival amplitude? Indeed, as the exponential contribution dominates over the nonexponential contribution, the value of the coefficients takes specific form, in accordance to the physical properties of the system. If there is only contributions of the poles near to the imaginary axis(the resonant poles in the “unphysical sheet”), and the nonexponential contribution is so small compared with the exponential one , the coefficients related with that poles are large compared with the coefficients of the other poles of the outgoing Green function. So, these coefficients are related to the importance of the low energies in the description of the cross section related with the scattering amplitude $f(\theta, \omega)$ through the optical theorem; as the scattering wavefunction can be spanned in terms of the spherical harmonics by the partial wave method , as it will be discussed in the next chapter, the s partial wave($l = 0$) takes a major contribution over the other partial waves, when the energy is low. So, the coefficients related with the poles near the real axis can be treated analogically as the contributions of the partial waves to the wavefunction that describes the state of the system. For low energies, the contribution of the $C_n \bar{C}_n$ related with the poles k_n near the real axis in the fourth quadrant are higher, compared with the poles far below them. So,$C_0 \bar{C}_0 = 1$, and the exponential contribution arises, giving the known exponential law of the survival probability, classically described (1.1).

As the survival probability $P(t)$ has been defined analytically, according to the relation (2.38), and the survival amplitude is given by the relation (2.40) it can be extended the behavior of the survival amplitude $A(t)$ at long times, in order to determine the survival probability that characterizes the metastable system, under the influence of the potential $V(r)$. As the survival probability depends on the behavior of the Moshinsky function, as t becomes large, this function can be spanned in a convergent series [24][6], from the asymptotic expansion of the Erfc(z) function:

$$\sqrt{\pi} z e^{z^2} \mathrm{Erfc}(z) \underset{z \to \infty}{\approx} 1 + \sum_{m=1}^{\infty} (-1)^m \frac{\prod_{n=1}^{m} (2n-1)}{(2z^2)^m} \tag{2.41}$$

The survival amplitude $A(t)$, given by (2.40), is related with the behavior of the Moshinsky function; so, as the time t gets larger, the behavior of the Moshinsky function

[6]page 298, equation 7.1.23

approaches to the asymptotic expansion given by (2.41), due to the relation between the definition of the Moshinsky function and the complementary error function Erfc(z); therefore, the survival amplitude $A(t)$ takes this form as long as t goes to infinity:

$$A(t) = \sum_{p=1}^{\infty} C_p \bar{C}_p e^{-ik_p^2 t} + \sum_{p=1}^{\infty} \left(\frac{C_p \bar{C}_p}{\pi \sqrt{itk_p}} + \frac{\bar{C}_p^* C_p^*}{\pi \sqrt{itk_p^*}} \right)$$
$$+ \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{(-1)^m C_p \bar{C}_p \prod_{n=1}^{m} (2n-1)}{2^m (\sqrt{itk_n})^(2m+1)} + \frac{(-1)^m C_p^* \bar{C}_p^* \prod_{n=1}^{m} (2n-1)}{2^m (\sqrt{itk_p^*})^{2m+1}} \right) \quad (2.42)$$

As there is a relationship between the coefficients, the second term of the survival probability goes to zero (2.34), the survival probability takes this form as t becomes large:

$$A(t) = \sum_{p=1}^{\infty} C_p \bar{C}_p e^{-ik_p^2 t} - \sum_{p=1}^{\infty} \left(\frac{C_p \bar{C}_p}{2(\sqrt{itk_n})^3} - \frac{C_p^* \bar{C}_p^*}{2(\sqrt{itk_n^*})^3} \right)$$
$$+ \sum_{m=2}^{\infty} \left(\frac{(-1)^m C_p \bar{C}_p \prod_{n=1}^{m} (2n-1)}{2^m (\sqrt{itk_n})^(2m+1)} + \frac{(-1)^m C_p^* \bar{C}_p^* \prod_{n=1}^{m} (2n-1)}{2^m (\sqrt{itk_p^*})^{2m+1}} \right) \quad (2.43)$$

If the analytic behavior of the survival probability $A(t)$ is analyzed, the first term of the nonexponential contribution to the survival probability has a dependence on time as $t^{\frac{3}{2}}$; so when t becomes large, this term becomes more important, and contributes to the decay, as long as the exponential contribution given by the first term of the survival probability vanishes slowly. Therefore, for large times, the survival probability, according to the Moshinshy model takes the form:

$$P(t) = \left| A(t) \right|^2 \underset{t \to \infty}{\approx} t^{-3} \quad (2.44)$$

In this model, the existence of resonances becomes relevant as the strength of the potential is large compared to the reach of it. As a matter of fact, as long as the strength of the potential becomes higher, the poles of the outgoing Green function $G^+(r, r', t)$ are isolated from each other, and, therefore, don't overlap, as the real part of the poles are so large compared with the imaginary part of them, that leads to the width of the resonance. In fact, the real importance that have the initial state in the development of this model becomes significant, as long as the wavenumber of the initial state, if it is considered a localized wavepacket as a description of it, determines the proximity to a resonance, and, therefore, according to the behavior of the propagator, its temporal evolution. The contributions to the survival probability corresponds to each pole of the Green function, as it was shown in the relation (2.43); as the exponential decaying part is given by the imaginary part of each pole, the pole that corresponds to the resonance with less width would contribute more than the others, even though the coefficients for that pole would be minimum. Also, as the initial state is near a resonance determine

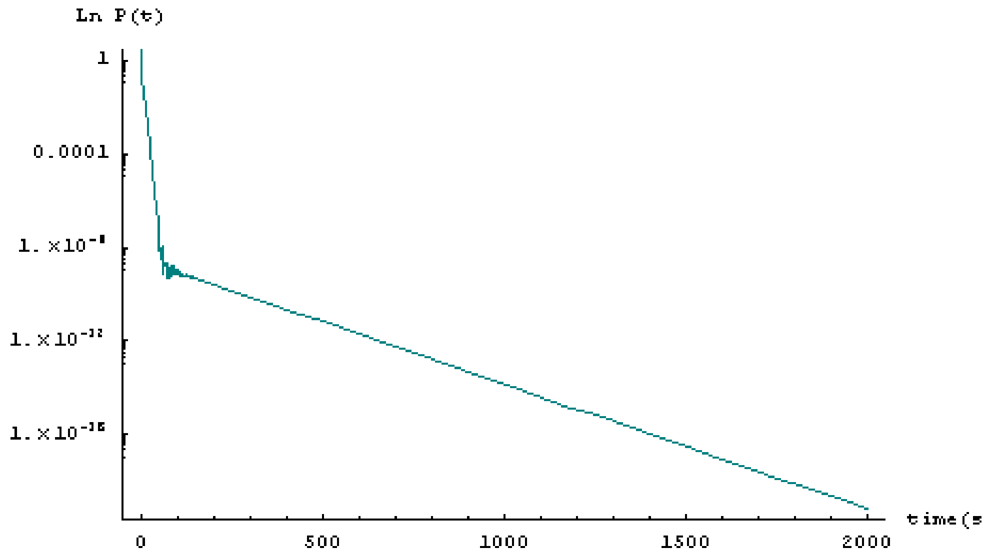

Figure 2.3: Logarithmic plot of the behavior of the survival probability $P(t)$ at all times, from the survival amplitude $A(t)$ given by the relation (2.43) , and (1.5)

by a pole of the Green function, the system would evolve as the contribution of that particular resonance to the decay would be greater than the others; but, in some graphics that develop the behavior of the survival probability against time, a fluctuation can be shown. As a system is initially between a couple of resonances, meaning that the wavenumber of the initial system would be different from an entire number of π, the system would go from a state to another, as it tries to stay in its initial state; so , the fluctuation is presented more over the initial wavefunction that corresponds to a system between resonances, that one near to a resonance. Nevertheless, the nonexponential contribution overlaps the exponential behavior of the survival probability as long as t goes to infinity. It is known that the behavior of the survival probability isn't exactly an exponential one at very short times, but during the so called intermediate times", the exponential behavior fits so well the analytic temporal expression of the survival probability. But, the main question that arises is the long time behavior; the transition period is led by fluctuations that becomes smaller in time, as the nonexponential contribution, given by the asymptotic expansion of the Moshinsky function, is greater. So, finally, the nonexponential behavior becomes more important; in the Fig. (2.3), this confrontation is evident, even though the scale doesn't help to find[7]; the oscillation last during the transition between the two domains: the intermediate times(exponential contribution) and the long times(nonexponential contribution). Van Dijk and Togami, [32], also worked in the subject of the decay of a quantum system, confined by a po-

[7]For operational calculations, only the 150 nearest poles to the real axis were taken, to calculate the survival probability $P(t)$ according to (2.40) and (1.5), in the Fig.(2.3)

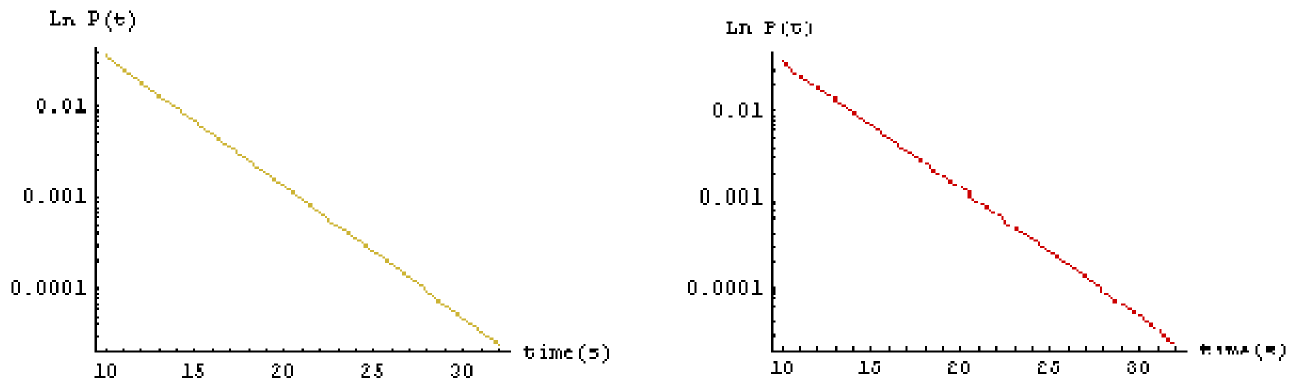

Figure 2.4: Right:Logarithmic plot of the behavior of the survival probability $P(t)$ during the period of intermediate times. The red plot corresponds to the survival probability defined from (1.5) and (2.40); left: Logarithmic plot of the exponential contribution to the survival probability $P(t)$, given by the relations (2.43) and (1.5)

tential, and found an expression of the wavefunction as an expansion of the Moshinsky functions,as the importance of the knowledge of the wavefunction becomes greater in some of the most important problems in physics, like the alpha decay, even though they didn't consider a boundary condition as Gamow stipulated for the so called "resonant states" [5]. They said that the Gamow wavefunctions corresponds to only outgoing waves outside the barrier, and the wavefunction they defined contains contributions inside the barrier and outside it; so the fluctuations that contain the survival probability are related with the behavior of the contributions of the incoming waves in the wavefunction during the transition time between the exponential and nonexponential behavior; they also found, as Moshinsky did [25], that the survival probability behaves as t^{-3} for long times; it is natural to think about the similarities between the two procedures, as the wavefunction that describes the physical state is spanned in terms of the Moshinsky function, calculated over all the possible poles of the Green function. Nevertheless, the supposition that led to a discrepancy between the two models is guided by the definition of the scattering solutions, related with the regular solutions given by (2.6); the most important thing to consider about these two models is the existence of a localized potential, and the initial state as a wavepacket confined by this potential, so that the width of it would be less than the distance that the decay products would follow, after the decay. van Dijk in his paper, extended his method to different kind of potentials, as a potential can be approximated to a potential with many steps, in order to express the generality of the method.

This model isn't the only one that describe the decay phenomena; as a matter of fact, the behavior of the survival probability depends on the analytic properties of the potential, as it was mentioned before, and the propagators, Green functions derived from it. Some models consider the decay phenomena as a particularization of a tunneling effect in which the particle is seeing as enclosed into a potential barrier. This picture will be discussed in the next chapter; for now, trying to understand the difference between the purposed models and confront their deductions is more relevant.

2.1.2 Formalism of Nicolaides

Nicolaides[33, 20, 34] took another perspective in his work on metastable systems. Throughout his works, Nicolaides tried to express the relation between the density of states $\omega(E)$, given by (1.17) and the behavior of the propagators, given by the outgoing Green function related with the temporal evolution of the system. The assumption he followed in order to find the analytic expression of the survival probability corresponds to localized initial states, as localized wavepackets $|\psi_0\rangle$; as a matter of fact, this initial state is confined, as the strength of the potential is less than the "length" of the wavepacket. In order to consider the temporal evolution of the initial wavepacket, it is necessary to apply the evolution operator $\hat{u}(t) = e^{-\frac{i\hat{H}t}{\hbar}}$, in order to find the final state $|\psi(t)\rangle$, that corresponds to the solution of the temporal Schrödinger equation. Nevertheless, Nicolaides and Douvropoulos claimed about a singularity of the solution of the Schrödinger equation in the initial time, so the temporal symmetry must be broken in that point [34]. This condition leads to a nonhermitian development, as the arrow of time must exclude the contribution of nonreversible states, since the temporal symmetry is broken. So, in order to specify the propagators of the system that causes the temporal evolution, it is necessary to build two functions of time, near the singularity given by the $t = 0$ condition: One corresponds to the positive temporal evolution, given by $G(E + i0)$, and the other, given by the negative temporal evolution, as $G(E - i0)$. According to the works of Nicolaides and Beck, [34], the survival amplitude $A(t)$ is related with a resolvent operator $R(z)$, that accomplish some properties, as its analyticity, except in the poles of the Green function, and the behavior according to the spectrum of energy. The survival amplitude can be calculated in terms of the resolvent operator,as it was shown in [33]:

$$A(t) = \frac{1}{2\pi} \oint \langle\psi_0|R(z)|\psi_0\rangle e^{-\frac{i}{\hbar}zt} dz \tag{2.45}$$

with $R(z) = \frac{1}{z-\hat{H}}$; as the resolvent operator $R(z)$ can be expressed in terms of a linear combination of the resolvent operator $R_0(z) = \frac{1}{z-\hat{H}_0}$, related with the unperturbed hamiltonian, H_0, which is the hamiltonian without the potential that leads to the decay. So, the resolvent operator is given by:

$$R(z) = R_0(z) + R_0(z)VR_0(z) + R_0(z)VR_0(z)VR(z) \tag{2.46}$$

As (2.45) involves the expectation value of the resolvent operator $R(z)$, it is necessary to calculate this factor, in order to determine the analytic form of the survival amplitude, according to (2.45); so, as the relation (2.46) is taken and a bracket product is taken on the relation, so that:

$$\langle\psi_0|R(z)|\psi_0\rangle = \frac{1}{z - E_0 - Ae(z)} \tag{2.47}$$

where:

$$Ae(z) = \langle\psi_0|VR_0(z)V|\psi_0\rangle$$

The operator $Ae(z)$ is called the "self energy" of the autoionizing state, as Nicolaides and Beck claimed in their work [34]. This self energy is the expectation value of an operator; in the many body theory, the self energy operator can be related with the weak interactions between the bodies that compose the system and it has some explicitly properties that define the behavior of the survival amplitude. As the survival amplitude depends on the expectation value of the resolvent operator, and this expectation value is given by the relation (2.47), so the survival amplitude can be affected by two principal conditions:

1. The analytic properties of the expectation value of the resolvent operator, given by (2.47)
2. The contour where the integral given by (2.45) is evaluated

Resolvent operator and the poles related to it

As the definition of the density of states $\omega(E)$ is a relevant problem in the many body theory, it is not an easy matter to find an analytic expression of the density of states; nevertheless, this approximation of the survival amplitude as it was stated above, depends on the behavior of the resolvent operator. If the expectation value has some poles in the domain enclosed by the contour where the integral (2.45) is evaluated, the residue theorem can be applied to find the analytic expression of $A(t)$; so, it is necessary to obtain the poles of the resolvent operator, i.e. the z points in the complex plane such as $z - E_0 - Ae(z) = 0$. This transcendental equation represents the points where the expectation value of the resolvent operator isn't analytic. The resolvent operator, as it could be related with the propagator, can have some of the poles of the Green function; Nicolaides and Beck pointed out, in his work [34], that if the $Im(Ae(z))$ would be zero, a pole would appear in the real axis, so that it could be considered a bound state, in the same way as a bound state means to the pole of the S matrix in the physical sheet. This condition can be only satisfy if: "The resolvent operator is unbounded, and defined on a set which is not dense in Hilbert Space"[34]. As Nicolaides and Beck worked on autoionization processes , they supposed that in the neighborhood of E_0, $Ae(z)$ must be analytic, except in $z = E_0$,according to the definition of $Ae(z)$; besides, as the energy width of the metastable state is energy independent, in the vicinity of E_0, $Ae(z)$ is energy independent; so, the value of the self energy must be related with the width and a small shift respect to the E_0. As the metastable state is characterized by a pole of the **S** matrix in the "unphysical" sheet $Im\ k < 0$, so the imaginary part of $Ae(z)$ must be less than zero. Therefore, the self energy of the autoionizing state can be defined as:

$$Ae(z) = Ae(E_0) = \Delta(E_0) - i\frac{\Gamma}{2} \tag{2.48}$$

As the survival amplitude must be a convergent integral, the existence of a negative term in the definition of the "self energy" term guarantees the convergence of the integral; so, as the $Ae(z)$ is taken in the vicinity of E_0, the pole of the expectation value of the resolvent operator is located at $G_d = E_0 + Ae(E_0)$, according to the transcendental equation described above. The most important thing to observe is that the Green function that describes the system is given by the expectation value of the resolvent operator as it was defined; that's the main reason why the poles of the expectation value and the poles of the Green function are related. so, the analytical properties of the expectation value are the same of those of the Green function: The Green function has a branch cut on the real axis, but the poles of them, in absence of bound states, are below the real axis, next to it("unphysical sheet"), in order to have physical significance. The Green function must be analytically continued in order to analyze the behavior of the poles in the second Riemannian sheet; so, the resolvent operator, related to it, must be continued analytically as well, as the pole is located in the third quadrant, and the chosen contour encloses it, in order to calculate the survival probability given by the relation (2.45); therefore, the expectation value of the resolvent operator is defined, on the "unphysical sheet", as:

$$\langle \psi_0 | R(E+i0) | \psi_0 \rangle = \frac{1}{z - z_0}\Big|_{Im\ z<0} \tag{2.49}$$

Where z_0 is defined as the pole of the expectation value of the resolvent operator: $z_0 = Ae(z) + E_0$. This fact is explained as the system is constrained by two conditions: t must be greater than zero, and E must be equal or greater than zero. As the evolution operator must be redefined as $u(t) = \theta(t)e^{-i\frac{\hat{H}t}{\hbar}}$, with $\theta(t)$ is the Heaviside function.

Influence of the path of integration in the determination of the survival amplitude

The other factor that arises in the analytic form of the survival probability is the path of integration. Nicolaides and Beck in their work, chose a contour similar to the contour Neelima Kelkar, Marek Nowakowski and M. Khemchandani [18] followed in order to find the survival probability from the density of states applying the Fock Krylov theorem, as it would be discussed later, in the next chapter.

As the initial state given by $|\phi_0\rangle$ can be spanned in terms of the eigenstates of the hamiltonian, and applying the relation (1.6), the survival amplitude can be found. As the exact behavior of the survival amplitude $A(t)$ depends on the density of states $\omega(E)$, the correct definition of density of states must be given; meanwhile, since the survival probability is given by (1.15)[8], the expansion of the survival amplitude arises from the expansion of the initial state in terms of the eigenstates of the hamiltonian that leads to the decay. The density of states is related with the threshold energy that makes the reaction possible as $\omega(E) = \theta(E - I_1)\left|\langle\psi_0|\psi(E)\rangle\right|^2$, where $\left|\psi(E)\rangle\right|$ are the eigenstates of the hamiltonian $\hat{H}$ that leads to the decay, with continuum spectrum of energy; therefore, the survival amplitude $A(t)$ is given by the expansion on the eigenstates of the hamiltonian, including the time reversal states.

$$A(t) = \sum_n e^{-i\frac{E_n t}{\hbar}}\left|\langle\psi_0|\psi_n\rangle\right|^2 + \int_{I_1}^{\infty} e^{-i\frac{Et}{\hbar}}\left|\langle\psi_0|\psi(E)\rangle\right|^2 dE \tag{2.50}$$

As Nicolaides and Beck pointed out in their work, in general,[34]:

$$\left|\langle\psi_0|\psi_n\rangle\right| \neq \left|\langle\psi_0|\psi(E)\rangle\right| \neq 0$$

so that, the operators commute with the hamiltonian but represent different observables.

If (2.50) is analyzed, the below limit is the first ionization threshold, as it is going to be taken in this work. Like it was mentioned before, Khalfin discovered that the survival probability can't be exponential over all the temporal dominium; as a matter of fact, the definition of the survival probability, that implies an asymmetry in the time(the reversed states and the no reversal states) defines the nonexponential behavior that can take the survival probability eventually. So, to determine the survival amplitude, it is necessary to calculate the analytic value of the integral (2.45) on the contour. This contour is divided in three parts: a straight line on the real axis from I_1 to $R_0(C_1)$; a straight line on the imaginary axis from R_0 to zero(C_2); and a quarter of arc with radius $R_0(C_3)$. This contour encloses the pole of the Green function ("the resonant pole"), located in the "unphysical sheet". As the integral is calculated, the R_0 radius is going to be taken to ∞; as the integral corresponding to arc decreases as an exponential function of R_0, as long as $R_0 \to \infty$, the integral over C_3 goes to zero.

If the line integral over the real axis is compared with the integral that defines the survival amplitude(2.50), the two integrals are the same, indeed. So, in order to determine the exact value of this integral, the residues theorem must be applied, as the pole is z_0 in the integral. The integral must be evaluated on the contour described above. As the integral over C_3 vanishes, the residue calculated over the pole z_0 of the function $f(z) = \frac{e^{-\frac{izt}{\hbar}}}{z-z_0}$ must be equal to the integral of the function over the C_1 and the

[8]The limits of this initial integral corresponds to $E_{threshold}$ to infinity, even though, initially it was discussed that the limits must be correspond to $-\infty$ to ∞, as the Fock Krylov theorem must be satisfied. Nevertheless, the energy spectrum is defined positive, as physically there's no meaning for negative energies; so, that's why the definition of the density of states (1.16) arises

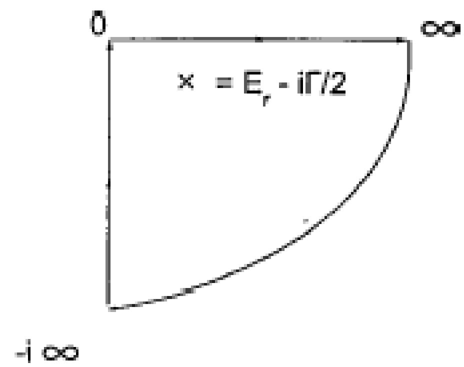

Figure 2.5: Integration contour used by Kelkar, Nowakowski and Khemchandani in their work[18], and Nicolaides and Beck[33], to obtain the analytic expression of the survival amplitude $A(t)$

C_2 contours, due to the residues theorem [28]:

$$2\pi i Res(f(z), z_0) = \int_{I_1}^{\infty} \frac{e^{-\frac{ixt}{\hbar}}}{x - z_0} dx + i \int_{-\infty}^{0} \frac{e^{\frac{yt}{\hbar}}}{iy - z_0} dy \tag{2.51}$$

The first integral of the expression (2.51) is the integral that appears in the relation (2.50); it is the integral to determine. The residue of the function $f(z)$ in z_0 is calculated from the definition of residue [28]:

$$Res(f(z), z_0) = \lim_{z \to \infty} \frac{e^{-\frac{izt}{\hbar}}}{z - z_0}(z - z_0)$$

Thereby, the residue is:

$$Res(f(z), z_0) = e^{-\frac{iz_0 t}{\hbar}} \tag{2.52}$$

The second integral of the relation (2.51) is the line integral over the complex semiaxis, from $-\infty$ to zero. This integral takes the form of the exponential integral function $Ei(z)$, given by[24][9]:

$$E_1(z) = \int_{z}^{\infty} \frac{e^{-t}}{t} dt \tag{2.53}$$

As the first integral is needed in order to determine the analytic expression of the survival amplitude $A(t)$, once the residue of $f(z)$ is calculated over the pole, the integral on the real axis C_1 can be determined, as the integral over the imaginary axis C_2 is

[9] page 228

expressed in terms of the $E_1(z)$ function given by the relation above (2.53); according to the definition of the survival amplitude given by (2.50), and replacing the value of the integral over the real axis C_1 by (2.51):

$$A(t) = 2\pi i Res(f(z), z_0) - i\int_{-\infty}^{0} \frac{e^{\frac{iyt}{\hbar}}}{iy - z_0} dy$$

if a couple of changes of variables is realized upon the second integral:

$$A(t) = 2\pi i Res(f(z), z_0) - e^{-\frac{iz_0 t}{\hbar}} \int_{\frac{iz_0 t}{\hbar}}^{\infty} \frac{e^{-x}}{x} dx$$

comparing with the definition of the $E_1(z)$ function, (2.53), and replacing (2.52) in the relation above

$$A(t) = e^{-\frac{iz_0 t}{\hbar}} \big(1 - \frac{E_1(-\frac{iz_0 t}{\hbar})}{2\pi i}\big) \tag{2.54}$$

This last expression (2.54) is, precisely, the analytic definition of the survival amplitude $A(t)$ at all times.

Long time behavior of $P(t)$

As it was mentioned before, the first contribution to the survival probability is given by the residue of the function $f(z)$ calculated on the pole; the second contribution, as a matter of fact, is given by the integral calculated on the imaginary axis, according to the contour. One relevant assumption must be taken into account: It was mentioned before that there are two constraints that must be taken as physical, according to the behavior of the physical systems: the energy spectrum defined as positive $E \geq 0$ and the positive temporal dominium $t \geq 0$. The latter of the condition implies that the total propagator(the Green function) must be separated in two, as there is a singularity of the Schrödinger equation in $t = 0$, as it was mentioned before in this chapter. As the pole of the expectation value of the resolvent operator z_0 is located in the "unphysical" sheet, it can be guaranteed that the first term of the survival amplitude $A(t)$ arises an exponential behavior, proportional to $e^{-\frac{\Gamma t}{2}}$; so, the survival probability takes from this contribution an exponential behavior as $e^{-\Gamma t}$, that corresponds to the classical description of the decay processes. Nevertheless, this isn't the only contribution to the survival probability; as (2.54) is analyzed, there's another term involved: The one related with the exponential integral function $E_1(-\frac{iz_0 t}{\hbar})$. If the long time domain is considered, the first term of the survival amplitude(the one that leads to the exponential behavior of the survival probability $P(t)$) becomes irrelevant. So, at large times, the behavior of the survival probability is related with the asymptotic expansion of the $E_1(-\frac{iz_0 t}{\hbar})$ function; the exponential function $E_1(z)$ is well defined in the literature, and its properties are known; explicitly, the definition of this function for large z [24][10]:

$$E_1(z) \underset{t\to\infty}{\approx} \frac{e^{-z}}{z}\left(1-\frac{1}{z}+\frac{2!}{z^2}-\frac{3!}{z^3}+...+\frac{(-1)^n n!}{z^n}\right) \qquad \text{If } |arg(z)| < \frac{3\pi}{2} \tag{2.55}$$

Therefore, as time becomes large, the survival amplitude takes an explicit form, if the integral exponential function $E_1(-\frac{iz_0t}{\hbar})$ is replaced by the asymptotic expansion given by (2.55). Meanwhile, the exponential contribution, given by the $e^{-\frac{iz_0t}{\hbar}}$ term, goes to zero, as t gets larger. This contribution is so small, so that the only important retribution is given by the asymptotic expansion of the exponential integral; thereby, the survival amplitude acts like[11]:

$$A(t) = -\frac{\hbar}{2\pi z_0 t} \tag{2.56}$$

According to the definition of survival probability $P(t)$ (1.5), it must be defined, at times becomes large, as:

$$P(t) = \frac{\hbar^2}{4\pi^2|z_0|^2t^2} \tag{2.57}$$

Where $|z_0|^2$ is the squared norm of the pole of the expectation value of the resolvent operator defined above; so, as the pole z_0 is given by $z_0 = E_0 - \Delta(E_0) + \frac{i\Gamma}{2}$, as $\Delta(E_0)$ goes to zero, the $|z_0|^2$ factor goes to $(E_0)^2 + \frac{\Gamma^2}{4}$. The most important issue to notice is precisely the temporal relation of the nonexponential contribution at large times, in contrast to what Moshinsky et all [25] found: a t^{-3} behavior on the survival probability $P(t)$ (2.44). This t^{-2} tail in the survival probability for long times has been predicted by Khalfin in his earlier works about the possible nonexponential behavior of the survival probability in different decay processes[7], given by (1.22). Nevertheless, there is a discrepancy between (2.57) and the one that Khalfin found years ago; this discrepancy in the coefficients is due to the inhomogeneity in the temporal domain, as Nicolaides explains in his work [33]. This singularity presented in the solution of the temporal Schrödinger equation affects the nonexponential contribution of the survival probability, whose domain is relevant at large times, as it has been demonstrated. So, the particularity of the physical systems also makes an influence on the possible coefficients in the decay law at large times, but the temporal relation would remain the same. What's the main difference between the two models? Well, as a matter of fact, the consideration of the propagators(The Outgoing Green function of the Moshinsky's treatment and the Green function that leads to the nonhermiticity of the evolution operator, related with the temporal singularity at $t = 0$) and the analytical properties that are related with the definition of the survival probability $P(t)$. But as it can be shown in the plot ((2.6)), the exponential contribution doesn't govern the entire temporal domain; as a matter of fact, after an interval when the fluctuations govern

[10] page 231

[11] As t becomes large, only the first term of the expansion (2.55) is relevant; the other terms are so small that doesn't contribute effectively to the survival amplitude, according to (2.54)

the survival probability, and the contributions given by the exponential part and the exponential integral function are similar, the survival probability $P(t)$ begins to behave different from what the exponential contribution would determine. This corresponds to the domain where the nonexponential behavior takes place, as it can be appreciated in the plot. ((2.7))[12] So, the distinction between the nonexponential behavior and

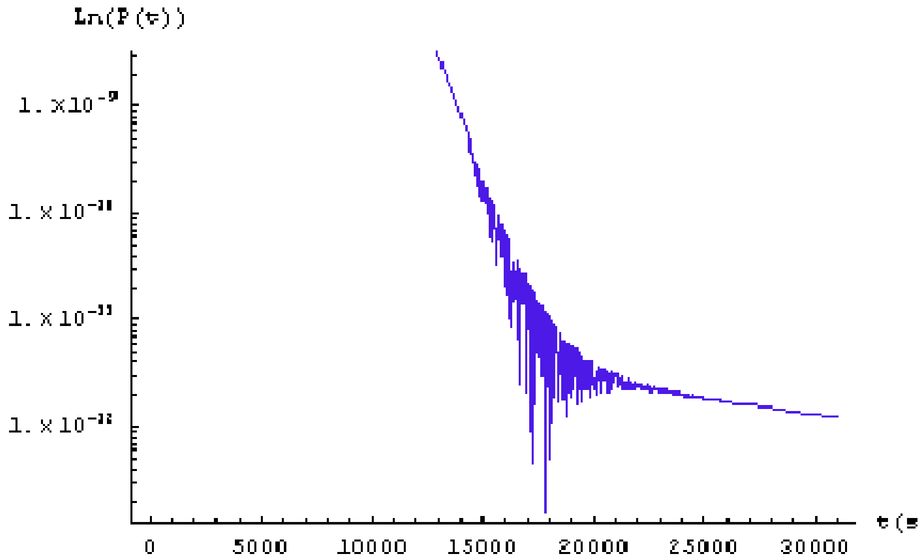

Figure 2.6: Logarithmic plot of the "Survival probability" $P(t)$ determined by Nicolaides and Beck [34]

the exponential one, as the time becomes large is essential. As the survival probability depends on the properties of the propagator(Green function), the long time effects appearing in the survival probability are given by the definition of the Green functions that determine the temporal evolution of the system, according to the hamiltonian that induces the decay. It is shown explicitly in the Fig.(2.7), the main distinction between the nonexponential contribution and the exponential behavior is evident in the long time dominium(in the Fig., the survival probability $P(t)$, given by (2.56) begins to separate from the exponential behavior at $t = 18000$ seconds[13]). This is the conclusion Khalfin found out in his earlier work.

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[12] For the both plots, z_0 is taken with the values of the mass and the energy width of ${}^5_3Li$

[13] This data is specific for the system that it was considered: ${}^5_3Li$

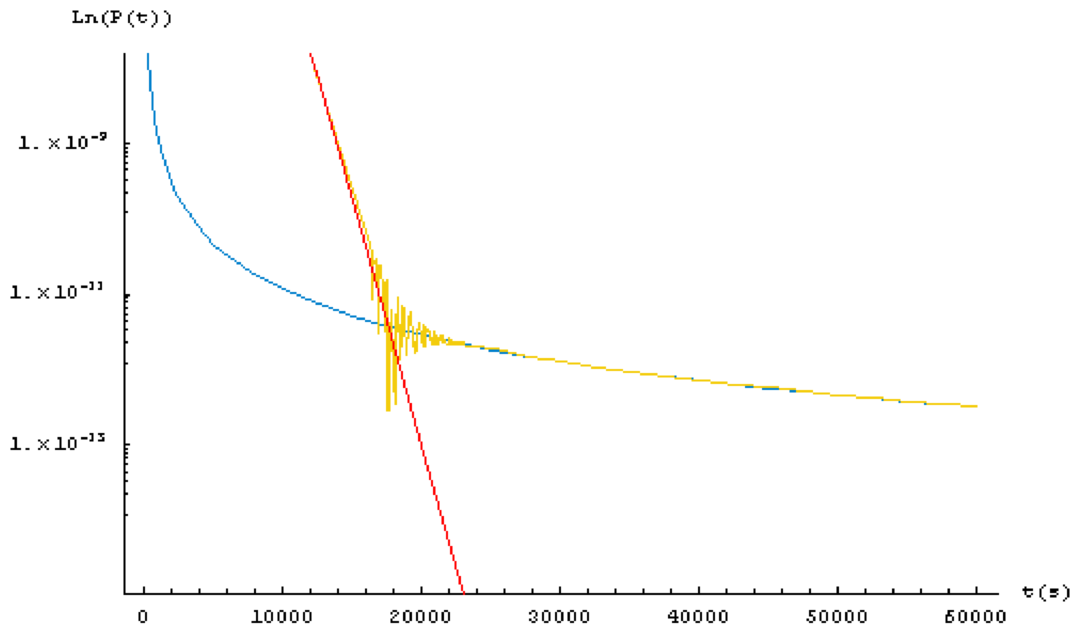

Figure 2.7: Logarithmic plot that illustrates the comparison between the "Survival probability" $P(t)$ determined by Nicolaides and Beck(yellow plot) [34], the exponential decay law(red plot), and the nonexponential power law(blue plot)

2.1.3 Analytic forms of the density of states proposed by Nicolaides

Nicolaides also supposed different kind of density of states, according to the analytical properties of the propagators and the physical constraints ligated with the evolution of the system, driven by the hamiltonian $\hat{H}$[33]; Dovropoulos and Nicolaides considered three kind of different spectral distributions, although the difficulties to obtain an analytic expression for the density of states, specially in the case of overlapping resonances, where the interactions make the problem a hard one to solve. The definition of the density of states given by (1.17) is related with the analytic properties of the propagator(the Green function), as it was pointed out by Nicolaides. But, he also considered an spectral distribution that Goldberger and Watson had already studied, and it is so relevant in the case of the s waves at low energy[35]: the Lorentzian modified distribution

$$G_L(E) = \begin{cases} \frac{N}{2\pi}\frac{\Gamma\sqrt{E}}{(E-E_r)^2+\frac{\Gamma^2}{4}} & \text{If } E \geq 0 \\ 0 & \text{If } E < 0 \end{cases} \tag{2.58}$$

Nicolaides and Dovropolous tried to look for a nonexponential propagator, as they applied the Fock Krylov theorem((1.1)) in order to find the survival amplitude $A(t)$

related with the density of states they supposed. So, the survival amplitude defined from the density of states (2.58), applying the Fock Krylov theorem(theorem (1.1)) is given by:

$$A(t) = \frac{K\Gamma}{2\pi}\int_0^{\infty}\frac{e^{\frac{-iEt}{\hbar}}\sqrt{E}}{(E-E_r)^2+\frac{\Gamma^2}{4}}dE \tag{2.59}$$

As the energy spectrum is taken as positive, and the density of states $G_L(E)$ has two poles: $z = E_r - \frac{i\Gamma}{2}$ and $z^* = E_r + \frac{i\Gamma}{2}$. As there are no bound states given by the potential, the chosen contour to find the integral must enclose the z pole of the density of states chosen analytically; so, Nicolaides and Dovropoulos chose the same contour described above, used by Kelkar, Nowakowski and Khemchandani [18]: The quarter of circle that enclose the resonant pole in the "unphysical sheet", on the fourth quadrant of the complex plane. As it was stated before, this contour is divided in three: An integration of $G_L(E)$ over the real axis, from zero to $R_0(C_1)$; an integration of $G_L(E)$ over the imaginary axis, from $-iR_0$ to zero(C_2), and at last, an integration of $G_L(E)$ over a quarter of circle with radius R_0, from $\theta = 0$ to $\theta = -\frac{\pi}{2}(C_3)$. As Nicolaides and Beck developed in their work[33], the integral (2.59) is the line integral over the contour C_1 of the density of states they supposed. So, in order to determine the integral (2.59), the residues theorem must be applied, as the contour ((2.5)) encloses the z pole of the density of states. Analogically to the work of Nicolaides and Beck[34], it is important to calculate the integral over the imaginary axis C_2, and the residue over the z pole, while the integral over the quarter of circle C_3 converges to zero as R_0 goes large, to infinity to extend to the complete "unphysical sheet". As the resolvent operator was analytically continued, to define its properties in the neighborhood of the pole z_0 of its expectation value, and construct the contour ((2.5)) to apply the residues theorem and find the survival probability, the properties of the density of states (2.58) demand that it must be analytic inside the contour, except in the pole itself; that's the main reason why the residues theorem can be applied in order to find out the analytic expression of the survival probability, according to the relation (2.59).

So, in first place, the residue of the function $G_L(E)e^{-\frac{iEt}{\hbar}}$ on the resonant pole z must be calculated. According to the definition of a residue [28]:

$$Res(e^{\frac{-iEt}{\hbar}}, z) = \lim_{E\to z}\Big(\frac{K\Gamma e^{\frac{-iEt}{\hbar}}\sqrt{E}}{2\pi\big((E-E_r)^2+\frac{\Gamma^2}{4}\big)}\Big)(E-z) \tag{2.60}$$

The denominator of the density of states can be taken as the product of $(E - E_r + i\frac{\Gamma}{2})(E - E_r - i\frac{\Gamma}{2})$, or $(E-z)(E-z^*)$, so that a factor can be canceled with the one in the numerator; applying the limit on the expression, the value of the residue on that pole is calculated:

$$Res(e^{\frac{-iEt}{\hbar}}, z) = -\frac{Ke^{-\frac{i(E_r-i\frac{\Gamma}{2})t}{\hbar}}\sqrt{E_r-i\frac{\Gamma}{2}}}{2\pi i} \tag{2.61}$$

Then, applying the residues theorem [28], the integral over the real axis of the function $G_L(E)e^{-\frac{iEt}{\hbar}}$ can be determined, and the survival probability is expressed as a relation between the residue calculated over the pole z that is enclosed in the contour ((2.5)), and the line integral over the imaginary axis(C_2), doing $E = iy$, for operational effects, from $-\infty$ to zero, initially; after a couple of changes of variables:

$$A(t) = 2\pi i Res(e^{\frac{-iEt}{\hbar}}, z) + \frac{iK\Gamma e^{\frac{i\pi}{4}}}{2\pi} \int_0^\infty \frac{e^{\frac{-yt}{\hbar}}\sqrt{y}}{\left((iy+E_r)^2 + \frac{\Gamma^2}{4}\right)} dy$$

thereby:

$$A(t) = -Ke^{-\frac{i(E_r - i\frac{\Gamma}{2})t}{\hbar}} \sqrt{E_r - i\frac{\Gamma}{2}} + \frac{iK\Gamma e^{\frac{i\pi}{4}}}{2\pi} \int_0^\infty \frac{e^{\frac{-yt}{\hbar}}\sqrt{y}}{\left((iy+E_r)^2 + \frac{\Gamma^2}{4}\right)} dy$$

As the relation above shows, there are two possible contributions to the survival amplitude, according to the properties of the density of states analytically imposed in (2.58): The first term is exponential, and it is the contribution of the pole z enclosed in the contour, as it obtained from the calculation of the residue on it (2.61); the other contribution is given by the behavior of the density of states in the imaginary axis; as the density of states is analytic in all the complex plane, except in its poles, the line integral over the imaginary axis converges and exists. This integral, in particular, is similar to the integral definition of the error complementary function Erfc(z) given earlier (2.30)[24]. If it is introduced a change of variables (x as $\sqrt{y}$), and, then, the integrand is separated in two partial fractions, such as $y - iE_r + \frac{\Gamma}{2}$ and $y - iE_r - \frac{\Gamma}{2}$, the integral over the imaginary axis(noted as $G_{\text{nonexponential}}(t)$, since it is the nonexponential contribution to the survival amplitude $A(t)$) is:

$$G_{\text{nonexponential}}(t) = -\frac{iKe^{\frac{i\pi}{4}}}{\pi}\Big(\int_0^\infty \frac{e^{-\frac{x^2t}{\hbar}}x^2}{\left(x^2 - iE_r - \frac{\Gamma}{2}\right)}dx - \int_0^\infty \frac{e^{\frac{x^2t}{\hbar}}x^2}{\left(x^2 - iE_r + \frac{\Gamma}{2}\right)}dx\Big) \quad (2.62)$$

This integral looks like a tricky one, since its complexity; nevertheless, it can be simplified by notation; for example, the term $iE_r + \frac{\Gamma}{2}$ that appears in the denominator of the first integral is precisely iz[14]; and the term $-iE_r + \frac{\Gamma}{2}$ is $-iz^*$; so, if $\frac{iz}{x^2 - iz}$ is added and subtracted to the integrand of the first integral in (2.62), and, at the same time $\frac{-iz^*}{x^2 - iz^*}$ is added and subtracted to the integrand of the second integral, the whole expression can be reduced, as there is a couple of integrals of exponentials that cancels, since they have different sign; so, the nonexponential propagator, as Nicolaides and Dovropoulos called the nonexponential contribution to the survival amplitude $G_{\text{nonexponential}}(t)$ is:

$$G_{\text{nonexponential}}(t) = \frac{Ke^{\frac{i\pi}{4}}}{\pi}\Big(z\int_0^\infty \frac{e^{-\frac{x^2t}{\hbar}}}{\left(x^2 - iz\right)}dx - z^*\int_0^\infty \frac{e^{-\frac{x^2t}{\hbar}}}{\left(x^2 - iz^*\right)}dx\Big) \quad (2.63)$$

[14] z defined as the pole of the density of states $G_L(E)$, i. e. $z = E_r - i\frac{\Gamma}{2}$

These two integrals look similar to the definition of the $w(z)$ function given by [24][15]. So, replacing this definition and the relation between the $w(z)$ function and Erfc(z) given by (2.30)[24][16] in (2.63), what it would be obtained is:

$$G_{\text{nonexponential}}(t) = \frac{-K}{2i}\Big(\sqrt{z}e^{-\frac{izt}{\hbar}}\,\text{Erfc}(-i\sqrt{\frac{izt}{\hbar}}) - \sqrt{z^*}e^{-\frac{iz^*t}{\hbar}}\,\text{Erfc}(-i\sqrt{\frac{iz^*t}{\hbar}})\Big) \quad (2.64)$$

The survival amplitude would be given by:

$$A(t) = -Ke^{-\frac{i(E_r - i\frac{\Gamma}{2})t}{\hbar}}\sqrt{E_r - i\frac{\Gamma}{2}} - \frac{K}{2i}\Big(\sqrt{z}e^{-\frac{izt}{\hbar}}\,\text{Erfc}(-i\sqrt{\frac{izt}{\hbar}}) - \sqrt{z^*}e^{-\frac{iz^*t}{\hbar}}\,\text{Erfc}(-i\sqrt{\frac{iz^*t}{\hbar}})\Big) \quad (2.65)$$

This relation expresses the analytic relation for the survival probability at any time. From this expression (2.65), it is deduced that $P(t)$ is given by two possible contributions, as it was noticed above: An exponential contribution, that domains in the intermediate times, and a nonexponential contribution, given by the analytic properties of the $Erfc(z)$ function, and its asymptotic behavior.

In the intermediate times, since the exponential contribution is greater than the nonexponential contribution to $P(t)$, the classical description of the decay processes adjust so well to the temporal development of the survival probability; nevertheless, there is a transition time between the intermediate times and the long time behavior, when, the fluctuations on the survival probability $P(t)$ are evident, due to the analytic properties of the hamiltonian that induces the decay, and the interference of the two contributions that describe the behavior of the survival probability $P(t)$. As the time approaches to the long time domain, these fluctuations become more relevant in the evolution of the system, even though are depreciable respect of the values of the energy that domain in the process. As the time continues growing, the discrepancy between the survival probability behavior and the exponential decay given by the classical description gets notorious(see Fig. (2.9)). The difference of the two possible behaviors of the decay gets larger as the system evolves at large times; that leads to a question: What happens in the large times domain? What's the behavior of the survival probability $P(t)$ on that domain? Since the survival amplitude is given by (2.65), the asymptotic expansion of the function $Erfc(z)$ gives the nonexponential contribution to the survival probability as t becomes larger. Meanwhile, the exponential contribution, given by the exponential term related with the residue calculated on the resonant pole, gets smaller, and doesn't contribute at all. Therefore, the survival amplitude at large times is related with the asymptotic expansion of the $Erfc(z)$ function (2.41):

$$A(t) \underset{t\to\infty}{\approx} -\frac{K}{2\sqrt{\frac{it\pi}{\hbar}}}\Big(\sum_{m=1}^{\infty}\frac{\Pi_{i=1}^{m}2i-1}{(\frac{2it}{\hbar})^m}\big(\frac{1}{z^m-(z^*)^m}\big)\Big) \quad (2.66)$$

[15] equation 7.1.4
[16] equation 7.1.3

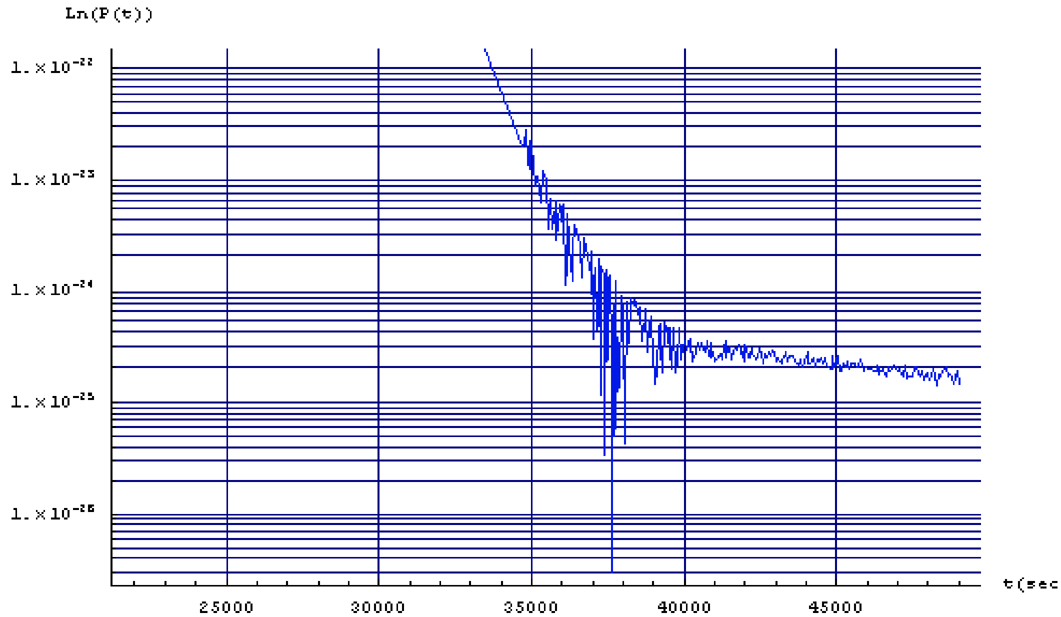

Figure 2.8: Logarithmic plot of the "Survival probability" $P(t)$ determined by Nicolaides and Dovropoulos(blue plot) [33]for a modified Lorentzian density of states given by (2.58)

The first term of the sum (2.66) is the biggest contribution as t goes to infinity; meanwhile, the rest of the terms go to zero. So, the first term of the sum is taken to describe the behavior of the survival amplitude $A(t)$. Once the survival amplitude has been determined, the survival probability $P(t)$ is defined in accordance to (1.5); the nonexponential behavior of the survival probability at large times is given by:

$$P(t) = \frac{K^2\,\hbar^3}{16\pi\Gamma^2 t^3} \qquad \text{Where K, due to normalization effects is: } K = \frac{2}{z - z^*} \qquad (2.67)$$

Therefore, at large times, the survival probability behaves differently from the classical description, and the power law(in this particular case $P(t) \propto t^{-3}$) determines the analytic properties of the decay. As the density of states is defined, the properties of the propagator and its poles are affected, so that, the survival amplitude $A(t)$ is affected also, in accordance to the Fock Krylov theorem (1.1), so that the survival probability (See (2.57),(2.44) and (2.67)). Since the most important relation to define the temporal evolution of the decaying systems is the survival probability $P(t)$, the nonexponential contribution of the survival amplitude as the time goes to infinity is relevant to describe the differences with the classical description, as it was mentioned in the first chapter of this work. Nevertheless, it is relevant to point out that the temporal transition of $P(t)$ from the intermediate times to the long times domain (where the nonexponential term contributes the most to $P(t)$, and the survival probability acts like a power law) changes as the propagator, that determines the temporal evolution of the quantum

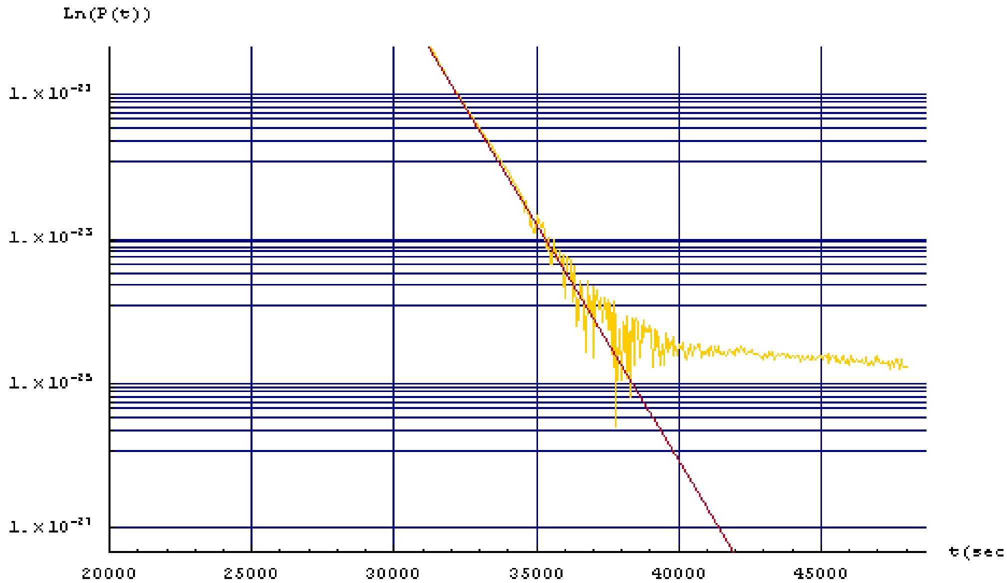

Figure 2.9: Logarithmic plot that lets analyze the comparison between the "Survival probability" $P(t)$ determined by Nicolaides and Dovropoulos(yellow plot)(2.59) and (1.5) [33]for a modified Lorentzian density of states given by (2.58)and the exponential decay law (red plot)

system, changes too. If the plots (2.9) and (2.7) are compared, the obvious issue to analyze is the discrepancy between the time interval where the transition from exponential decay law to nonexponential law occurs. Since the specific values of E_r and Γ were the same for both models[17], the difference between the temporal behavior given by the nonexponential decay law of the survival probability at long times, (2.67) and (2.57), for an isolated resonance must be explained by the analytic properties of the propagator that leads to the temporal evolution of the system, and, thereby, the definition of the density of states. The definition of the density of states, related with the survival probability through the Fock Krylov theorem gives the kinematical considerations that the evolution of the system must follow, and the physical constraints the system must take into account; therefore, as the analytic properties of the density of states changes, the behavior of the system and its evolution is affected, and the nonexponential decay law at large times gives a measure how the system evolves, in accordance to these considerations. Nevertheless, the works of Nicolaides, Beck and Dovropoulos helped to understand the nonexponential behavior of the survival probability, as Khalfin postulated its existence, due to lower bound of the energy spectrum as a physical constraint.

[17]These data are specific for the system that it was considered: ${}^{5}_{3}Li$

2.1.4 Fonda and the density of states $\omega(E)$ depending on l

Fonda, Ghirardi and Rimini defined a density of states $\omega(E)$ that expresses the existence of a possible resonant pole in the propagator[3]; in order to guarantee the analytic properties of the Green function as the propagator and the existence of a resonant pole in the "unphysical sheet", they considered a density of states related with the l angular momentum of the partial wave contribution to the wavefunction that determines the state of the system, that reproduces the behavior of the function as the energy takes closer values to the threshold[18]:

$$\omega(E) = \frac{\Gamma}{2\pi}\frac{(E - E_{\text{threshold}})^{l+\frac{1}{2}}g(E)}{(E - E_r)^2 + \frac{\Gamma^2}{4}} \tag{2.68}$$

They applied the Fock Krylov theorem (theorem (1.1)) in order to find the survival amplitude $A(t)$; besides, they decided to restrict the energy domain, as the energy is physically defined as a nonnegative variable; so, after a change of variables, the survival amplitude $A(t)$ is given by:

$$A(t) = \frac{\Gamma e^{-iE_{\text{threshold}}t}}{2\pi}\int_0^{\infty}\frac{e^{-ixt}(x)^{l+\frac{1}{2}}g(x + E_{\text{threshold}})}{(x + E_{\text{threshold}} - E_r)^2 + \frac{\Gamma^2}{4}}dx \tag{2.69}$$

In order to solve the integral, it is necessary to add and subtract the integration from $-\infty$ to 0, so that it can be taken as a contour integral calculated on the contour(Fig (2.5)); in accordance to the residues theorem[28], the integral described by (2.69) is the line integral over the real axis; it was mentioned that the integral over the quarter of circle shown in Fig. (2.5) vanishes, and the residue must be calculated on the pole $z_{\text{pole}} = E_r - i\frac{\Gamma}{2} - E_{\text{threshold}}$ of the integrand. If the residue is calculated on the pole, the exponential contribution of the survival probability is determined:

$$Res[f[z], E_r - i\frac{\Gamma}{2}] = \lim_{z\to z_{\text{pole}}}\frac{e^{-izt}z^{l+\frac{1}{2}}g(E_{\text{threshold}} + z)(z - z_{\text{pole}})}{(z + E_{\text{threshold}} - z_{\text{pole}})(z + E_{\text{threshold}} - z^*_{\text{pole}})} \tag{2.70}$$

Hence:

$$Res[f[z], E_r - i\frac{\Gamma}{2}] = \frac{e^{-i(E_r - E_{\text{threshold}})t}(E_r - E_{\text{threshold}} - i\frac{\Gamma}{2})^{l+\frac{1}{2}}g(E_r - i\frac{\Gamma}{2})}{2i\Gamma}e^{-\frac{\Gamma t}{2}} \tag{2.71}$$

Meanwhile, in order to obtain the nonexponential contribution to the survival amplitude $A(t)$, it is necessary to solve the integral (2.69) from $-i\infty$ to zero. A group of successive change of variables can be done in order to solve the integral[19]; after

[18]The description of the partial wave method will be treated in the next chapter

[19]The intermediate steps are going to be discussed in the next chapter, since the integral solution is similar to the model is going to be taken to find the survival probability $P(t)$, proposed by Kelkar, Nowakowski and Khemchandani [18]

these transformations, the final form is similar to the definition of the gamma function $\Gamma(z)$[24]

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2.72}$$

Fonda, Ghirardi and Rimini found that the nonexponential contribution is proportional to $t^{-(l+\frac{3}{2})}$[3]; so the survival amplitude is given by:

$$A(t) = \frac{e^{-i(E_r - E_{\text{threshold}})t}(E_r - i\frac{\Gamma}{2})^{l+\frac{1}{2}} g(E_r - i\frac{\Gamma}{2})}{2i\Gamma} e^{-\frac{\Gamma t}{2}} + D\frac{e^{-iE_{\text{threshold}}t}}{t^{l+\frac{3}{2}}} \tag{2.73}$$

where:

$$D = \frac{\Gamma}{2\pi} \frac{g(E_{\text{threshold}})(2l+1)!!\sqrt{\pi}(-i)^{l+\frac{3}{2}}}{2^{l+\frac{1}{2}}(E_r^2 + \frac{\Gamma^2}{4})}$$

Hence, the nonexponential contribution of the survival amplitude is proportional to $\frac{1}{t^{l+\frac{3}{2}}}$; therefore, in accordance to (1.5), the survival probability, at large times, is proportional to:

$$P(t) \sim \frac{1}{t^{2l+3}} \tag{2.74}$$

The exponential behavior, given by the residue calculated on the resonant pole, is characteristic of the intermediate times; but, as the density of states is defined by (2.68), the nonexponential contribution of the survival amplitude $A(t)$ is affected by the l dependence of $\omega(E)$; it is the nonexponential part which is related with the contribution of each partial wave, with angular momentum l; so, the power law, that describes the long time behavior, must be expressed in terms of the l angular momentum, as it can be shown in (2.74). In particular, for the first two partial waves P($l = 0$), and P($l = 1$), the power law describes the behavior of the survival probability at long times; meanwhile, for the S partial wave, the survival probability is proportional to t^{-3}; for the P partial waves, the survival probability is proportional to t^{-5}. If the figure (Fig. (2.10)) is analyzed, it can be deduced that for the two partial waves mentioned above, the nonexponential contribution of the survival probability overlaps the exponential contribution, after a large interval; so, eventually, the survival probability $P(t)$ will take the asymptotic form given by (2.74).

2.1.5 Nakazato and the time independent perturbation theory to find $P(t)$

These descriptions and models weren't the only ways to characterize the survival probability and figure the decay processes out; in fact, Nakazato, Namiki and Pascazio worked on the survival probability in accordance to the interaction picture, describing the evolution operator as the Fourier transform of the resolvent operator, defined above[16]. In the interaction picture, the state ket evolves determined by the behavior of the potential $V(r)$ that leads to the decay , and the observable evolves in time in accordance to the properties of the unperturbed hamiltonian; all possible operators, related

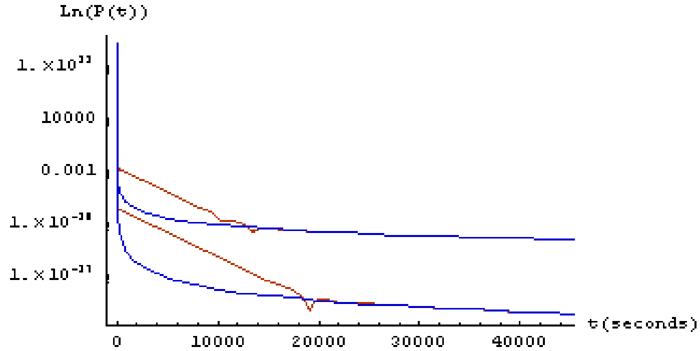

Figure 2.10: Logarithmic plot of the "Survival probability" $P(t)$ determined by Fonda, Ghirardi and Rimini for two different values of l: $l_0 = 0$ and $l_1 = 1$; and the comparison with the nonexponential contribution of the survival amplitude $A(t)$

with observables, suffer a transformation as they are considered as dynamical entities that evolve in time, in opposition to what the Schrödinger picture says. In particular, the potential that determines the evolution of the states in the interaction picture. As a matter of fact, the potential in the interaction picture is given by $V_I = e^{\frac{i\hat{H}_0 t}{\hbar}} V e^{-\frac{i\hat{H}_0 t}{\hbar}}$. In the interaction picture, the state of the system and the operator related with the observable evolve in time as:

$$i\hbar \frac{d}{dt} |\psi\rangle_I = V_I |\psi\rangle_I \qquad \text{with } |\psi\rangle_I = e^{\frac{i\hat{H}_0 t}{\hbar}} |\psi\rangle \tag{2.75}$$

Since the operators evolve in time, there exists a dynamical relation that describes the evolution of the operators; in particular, the temporal behavior of the evolution operator $U(t)$, unitary, is relevant, as the system must evolve in accordance to the analytic properties of the evolution operator at any time:

$$i\hbar \frac{dU_I(t)}{dt} = V_I U_I(t) \qquad \text{with boundary condition } U_I(0) = 1 \tag{2.76}$$

Indeed, since the evolution operator governs the temporal behavior of the system, the survival amplitude must be related to it(1.6); in the interaction picture, this relation doesn't change at all, even though the evolution operator must be transformed as any operator in the interaction picture; so, the main subject is to find the analytic expression of the evolution operator $U(t)$, solving the relation (2.76), or the next iterative relation to obtain $U_I(t)$:

$$U_I(t) = 1 - i \int_0^t V_I(t')dt' - \int_0^t dt_1 \int_0^{t_1} dt_2 V_I(t_1) V_I(t_2) U_I(t_2) \tag{2.77}$$

Nakazato et all considered the random phase approximation, so that the phases of the off diagonal matrix elements of the potential V that leads to the decay are randomized completely[16]; it means that $\langle n|V|n\rangle \neq 0$, as $\Big\{|n\rangle\Big\}$ is the basis of the undecayed space, and the eigenstates of the unperturbed hamiltonian $\hat{H}_0$. Since the evolution operator $U_I(t)$ gives the answer of the system in accordance to the properties of the potential that leads to the decay, the most important quantity to analyze is precisely the expectation value of the evolution operator $\langle\Psi_0|U_I(t)|\Psi_0\rangle$; in the interaction picture, the diagonal matrix element $\langle\Psi_0|U_I(t)|\Psi_0\rangle$ of the evolution operator $U_I(t)$ is defined as the survival amplitude $A(t)$; thereby,in order to obtain the survival amplitude, it is necessary to determine the diagonal matrix element, using the completeness properties of the basis of the unperturbed hamiltonian $\hat{H}_0$. Since the relation that lets determine the analytic behavior of the temporal evolution operator $U_I(t)$,is given by (2.76), in order to obtain the survival amplitude as it is defined in the interaction picture, a bracket product must be done on the relation (2.76), to obtain the diagonal matrix element; the form of the relation above leads to an integral differential equation, in accordance to the random phase approximation, so that:

$$\langle\Psi_0|V_I(t_1)V_I(t_2)U_I(t_2)|\Psi_0\rangle = \langle\Psi_0|V_I(t_1)V_I(t_2)|\Psi_0\rangle\langle\Psi_0|U_I(t_2)|\Psi_0\rangle \tag{2.78}$$

Applying the condition above in the relation (2.77), an integro differential equation can be obtained, in order to determine an analytic expression of the survival amplitude $A(t)$, related with the evolution operator $U_I(t)$ in the interaction picture:

$$\frac{dA(t)}{dt} = -\int_0^t dt_1 \Big(e^{iE_a(t-t_1)}\langle\Psi_0|Ve^{-i\hat{H}_0(t-t_1)}V|\Psi_0\rangle\langle\Psi_0|U_I(t_1)|\Psi_0\rangle\Big) \tag{2.79}$$

where E_a is the energy associated with the initial state, eigenstate of the unperturbed hamiltonian $\hat{H}_0$

Laplace transform to find $A(t)$

A Laplace transform is the natural method to solve an integro differential equation, given the complexity of the relation (2.79); the left part of (2.79) is, precisely, the convolution of the functions $\langle\Psi_0|U_I(t_1)|\Psi_0\rangle$ and $e^{iE_a(t)}\langle\Psi_0|Ve^{-i\hat{H}_0(t)}V|\Psi_0\rangle$; meanwhile, the Laplace transform of the evolution operator $U_I(t)$, in accordance to (2.79), is:

$$\mathcal{L}\big(A(t)\big) = \frac{1}{s+\mathcal{L}(e^{iE_at}f(t))} \tag{2.80}$$

where $f(t) = \langle\Psi_0|Ve^{-i\hat{H}_0(t)}V|\Psi_0\rangle$; Nakazato, Namiki and Pascazio [16] defined a function called $g(s,t)$ that is related with the Laplace transform of $e^{iE_at}f(t)$; so, in accordance to the definition of a Laplace transform, $g(s,t)$ is given by:

$$g(s,t) = s + t\int_0^\infty e^{-\frac{su}{t}}e^{iE_au}f(u)du \tag{2.81}$$

In order to find the behavior of the survival amplitude, an inverse Laplace transform must be done on the relation (2.80), in accordance to the Bromwich theorem (2.25) [28][20]; so, the survival amplitude $A(t)$ is given by:

$$A(t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{e^s}{g(s,t)} ds \tag{2.82}$$

Indeed, as the $g(s,t)$ function has been defined, some properties of the survival amplitude $A(t)$ can be deduced; for example, the convergence of the integral above can be guaranteed in order to define the analytic properties of the survival amplitude $A(t)$; in order to guarantee the analyticity of the survival amplitude along the complex plane, the function $g(s,t)$ must be defined for all the points in the complex plane. Nevertheless, if the $g(s,t)$ would have a zero in the left half complex plane, the exponential decay law would come naturally as a result; since the $g(s,t)$ function is not defined in the left half complex plane, it must be continued analytically, to do the inverse Laplace Transform. As a matter of fact, in accordance to the definition of the Laplace transform $g(s,t)$ can be written as:

$$g(s,t) = s + t\big\langle\Psi_0\big|V\frac{i}{E_a - \hat{H}_0 + i\frac{s}{t}}V\big|\Psi_0\big\rangle \tag{2.83}$$

In accordance to the time independent perturbation theory, the $g(s,t)$ function is related with the second term of the perturbation expansion[21]; so:

$$g(s,t) = s + it\int_{C_0} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2}{E_a - E_0 + i\frac{s}{t}} dE_0 \tag{2.84}$$

where the C_0 is the integration contour which extends along the real E_0 axis (Fig.(2.11)), and $\Big\{\big|E_0, r\big\rangle\Big\}$ is the orthonormal basis of the eigenstates of the unperturbed hamiltonian $\hat{H}_0$. Since the integrand in (2.84) has a simple pole at $E = E_0 + i\frac{s}{t}$, the $g(s,t)$ must be continued into the left half complex plane, so that, the contour C_0 must be deformed, as “the relative configuration with respect to the singularity is maintained”[16, 11].

Applying the residues theorem [28], the second term of the relation (2.84) becomes

[20]see footnote 4

[21]The second contribution given by the perturbation expansion, in accordance to the time independent perturbation theory is related with $\big\langle\Psi_0\big|V\frac{\phi_n}{E-\hat{H}_0}V\big|\Psi_0\big\rangle$, that can be expressed as $\big\langle\Psi_0\big|V\frac{\phi_n}{E-\hat{H}_0}V\big|\Psi_0\big\rangle \approx \sum_{k\neq n}\frac{\Big|\big\langle\Psi_k\big|V\big|\Psi_n\big\rangle\Big|^2}{E_n - E_k}$, where ϕ_n represents the so called complementary projection operator on the initial eigenstate of the unperturbed hamiltonian $\hat{H}_0$: $\phi_n = \mathbb{I} - \big|\Psi_0\big\rangle\big\langle\Psi_0\big|$ [39]

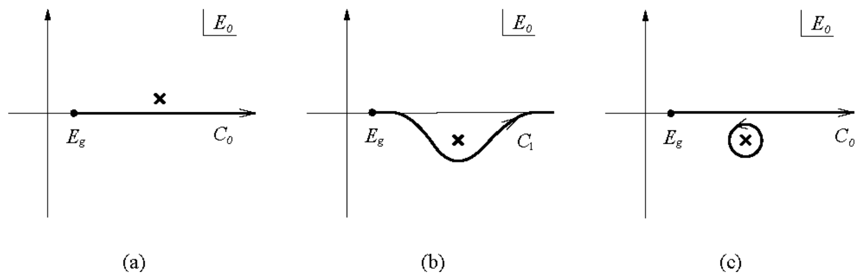

Figure 2.11: "Integration contours: a)C_0 for $Re(s) > 0$; b) C_1 for $Re(s) < 0$; c) The Contour C_1 can be further deformed and decomposed into the contour C_2 along the real E_0 axis and a circle surrounding the pole"[16])

in:

$$it\int_{C_0}\frac{\sum_r\left|\left\langle E_0,r\middle|V\frac{i}{E_a-\hat{H}_0+i\frac{s}{t}}V\middle|\Psi_0\right\rangle\right|^2}{E_a-E_0+i\frac{s}{t}}dE_0 = it\int_{C_2}\frac{\sum_r\left|\left\langle E_0,r\middle|V\middle|\Psi_0\right\rangle\right|^2}{E_a-E_0+i\frac{s}{t}}dE_0 + 2\pi t\sum_r\left|\left\langle E_0,r\middle|V\middle|\Psi_0\right\rangle\right|^2\Big|_{E_0=E_a+i\frac{s}{t}} \tag{2.85}$$

Therefore, the function $g(s,t)$ must have some properties, so that the relation (2.85) must be satisfy; as the total hamiltonian $\hat{H}$ acts on the system, the lower bound of the energy spectrum, $E_{threshold}$, lets define the existence of a branch cut on the s imaginary axis, from the branch point $s = it(E_a - E_{threshold})$ to $-\infty$. As the function $g(s,t)$ is continued on the second Riemannian sheet through the cut, the residue calculated on the pole is different from zero, so that, the second term of (2.85) appears as a contribution to the survival amplitude, in accordance to (2.82). Meanwhile, in the first Riemannian sheet, the function $g(s,t)$ must be analytic, since there is not a single singularity in that domain; that is the reason why the residue calculated on any point of the first Riemannian sheet is effectively zero.

In particular, a variable $s_0 = -\frac{\gamma t}{2} - i\delta Et$ is defined as the zero of $g(s,t)$ in the second Riemannian sheet. As $g(s,t)$ is given by (2.83), and the second term of this relation is calculated from the residue analysis as (2.85), the function $g(s,t)$ can be evaluated on s_0, so that $g(s_0,t) = 0$:

$$s_0 + it\int_{C_2}\frac{\sum_r\left|\left\langle E_0,r\middle|V\middle|\Psi_0\right\rangle\right|^2}{E_a-E_0+i\frac{s_0}{t}}dE_0 + 2\pi t\sum_r\left|\left\langle E_0,r\middle|V\middle|\Psi_0\right\rangle\right|^2\Big|_{E_0=E_a+i\frac{s_0}{t}} = 0 \tag{2.86}$$

Multiplying and dividing by the factor in the denominator, and integrating from

$E_{threshold}$ to ∞, in accordance to the energy spectrum:

$$\begin{aligned}-\frac{\gamma t}{2} - i\delta E t + it\int_{E_g}^{\infty} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2 (E_a - E_0 + \delta E - i\frac{\gamma}{2})}{(E_a - E_0 + \delta E - i\frac{\gamma}{2})^2} dE_0 \\ +2\pi t\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2\Big|_{E_0=E_a+\delta E - i\frac{\gamma}{2})} = 0\end{aligned} \tag{2.87}$$

This last expression can be reorganized in order to separate the complex part and the real part, so that:

$$\begin{aligned}&\Big\{ -\frac{\gamma t}{2} - \frac{\gamma t}{2}\int_{E_g}^{\infty} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2}{(E_a - E_0 + \delta E - i\frac{\gamma}{2})^2} dE_0 + 2\pi t\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2\Big|_{E_0=E_a+\delta E - i\frac{\gamma}{2}} \Big\} + \\ &\Big\{ - i\delta E t + it\delta E\int_{E_g}^{\infty} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2}{(E_a - E_0 + \delta E - i\frac{\gamma}{2})^2} dE_0 + \\ &\quad it\int_{E_g}^{\infty} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2 (E_a - E_0)}{(E_a - E_0)^2} dE_0 \Big\} = 0\end{aligned} \tag{2.88}$$

Therefore, the real part and the imaginary part of the expression above must be zero, in order to satisfy the equality, giving two relations, in terms of the two parameters of s_0:

$$-\frac{\gamma}{2}\Bigg[1 + \int_{E_g}^{\infty} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2}{(E_a - E_0 + \delta E - i\frac{\gamma}{2})^2} dE_0\Bigg] + 2\pi t\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2\Big|_{E_0=E_a+\delta E - i\frac{\gamma}{2}} = 0 \tag{2.89}$$

and:

$$-\delta E\Bigg[1 - \int_{E_g}^{\infty} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2}{(E_a - E_0 + \delta E - i\frac{\gamma}{2})^2} dE_0\Bigg] + \int_{E_g}^{\infty} \frac{\sum_r \Big|\big\langle E_0, r\big|V\big|\Psi_0\big\rangle\Big|^2 (E_a - E_0)}{(E_a - E_0)^2} dE_0 = 0 \tag{2.90}$$

with E_g defined as $E_{threshold}$, the bound limit of the energy spectrum of the hamiltonian $\hat{H}$. From expressions(2.89), some conclusions can be extracted; first of all, as Goto, and Namiki claimed in their work [11], the second term in the square brackets in both relations gives a finite contribution in accordance to the Payley Wiener theorem(Theorem (1.19)), and the convergence of the survival amplitude $A(t)$ at all times. The second term in the former equation, the one related with the parameter γ is related with the definition of the Fermi Golden rule[22]. This second term guarantees that the total expression vanishes, as the first one must converge; since s_0 is the pole of $\frac{1}{g(s,t)}$, the behavior of the residue calculated on s_0 depends on the chosen contour to do the inverse Laplace Transform (2.82); that is the main reason why the contour C_2 must be taken into account to determine the inverse Laplace transform. The contribution of the pole gives, as it was mentioned above, the exponential contribution to the survival amplitude $A(t)$ proportional to $e^{-is_0 t}$; thereby, in order to find the nonexponential contribution to the survival amplitude, this contour must be deformed(See Fig.(2.12)) to calculate the inverse Laplace transform given by (2.82). The integration to obtain the inverse Laplace Transform must be done on the s complex plane, but, in accordance to the properties of $g(s,t)$, the contribution of the branch point $s_{branch} = it(E_a - E_{threshold})$ vanishes.

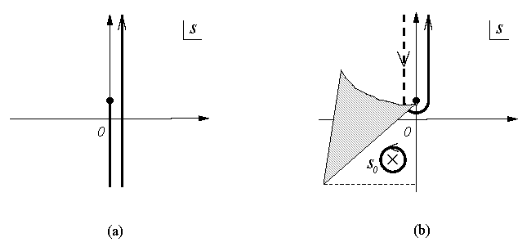

Figure 2.12: left: Original contour where the Laplace transform is evaluated to find the survival amplitude $A(t)$ (2.82); right: Deformed contour used to calculated the same inverse Laplace Transform

Long time behavior

The contribution described in the last subsection can be found analytically if the integral (2.82) is calculated along the deformed contour(Fig.((2.12)), excluding the vanishing contribution in the neighborhood of the branch point s_{branch}; so, since the $g(s,t)$ must be analytically continued in the second Riemannian sheet of the complex s plane, and the integral can be separated in two, as there are two contributions along the complex axis(from the branch point to $i\infty$ and from $i\infty$ to the branch point), the nonexponential contribution of $A(t)$, called X_C is given by:

$$X_C = \frac{1}{2\pi i}\int_{s_{branch}}^{\infty}\left\{\frac{e^s}{s + it\int_{E_g}^{\infty}\frac{\sum_r\left|\left\langle E_{0,r}\middle|V\middle|\Psi_0\right\rangle\right|^2}{E_a - E_0 + i\frac{s}{t}}dE_0} - \frac{e^s}{s + it\int_{E_g}^{\infty}\frac{\sum_r\left|\left\langle E_{0,r}\middle|V\middle|\Psi_0\right\rangle\right|^2}{(\frac{i}{t})(s - it(E_a - E_0)e^{-2\pi i})}dE_0}\right\}ds \tag{2.91}$$

[22]The Fermi Golden rule corresponds to a selection transition rule between the initial and final states, related with the behavior of the hamiltonian that expresses the interaction. The transition rate for the Fermi Golden Rule takes the form $w_{0\to[n]} = \frac{2\pi}{\hbar}\left|\left\langle\Psi_n\middle|V\middle|\Psi_0\right\rangle\right|^2\delta(E_n - E_0)$, if the energy spectrum is continuous.([39], page 332)

A change of variables can be done, to simplify the expression (2.91); if $s = i(y + t(E_a - E_g))$:

$$X_C = \frac{e^{\frac{i(E_a-E_g)t}{\hbar}}}{2\pi i} \int_0^\infty \Bigg\{ \frac{e^{iy}}{y + t(E_a - E_g) + t\int_{E_g}^\infty \frac{\sum_r \left|\left\langle E_0, r\middle|V\middle|\Psi_0\right\rangle\right|^2}{E_a - E_0 - \frac{y}{t}} dE_0} - \frac{e^{iye^{-2\pi i}}}{ye^{-2\pi i} + t(E_a - E_g) + t\int_{E_g}^\infty \frac{\sum_r \left|\left\langle E_0, r\middle|V\middle|\Psi_0\right\rangle\right|^2}{E_a - E_0 - \frac{ye^{-2\pi i}}{t}} dE_0} \Bigg\} dy \tag{2.92}$$

As the denominator of the integrand is a multivalued function in the complex s plane, an integration by parts can be done on the second term of the denominator(the one related to the second term of the time independent perturbation expansion), so that:

$$\int_{E_g}^\infty \frac{\sum_r \left|\left\langle E_0, r\middle|V\middle|\Psi_0\right\rangle\right|^2}{E_a - E_0 - \frac{y}{t}} dE_0 = \ln(E_0 - E_g + \frac{y}{t}) \sum_r \left|\left\langle E_0, r\middle|V\middle|\Psi_0\right\rangle\right|^2 \Bigg|_{E_g}^\infty + \int_{E_g}^\infty \ln(E_0 - E_g + \frac{y}{t}) \frac{d}{dE_0} \sum_r \left|\left\langle E_0, r\middle|V\middle|\Psi_0\right\rangle\right|^2 dE_0 \tag{2.93}$$

Thereby, since the logarithmic function $\ln(z)$ gains an extra imaginary part due to the argument, as its argument turns clockwise once around the origin, the main difference between the two terms in (2.92) would be given by the difference of the argument,as the second one has the same modulus that the former; nevertheless, its argument has rotated a complete round in the clockwise direction, so that the logarithmic function suffers a transformation $\ln(E_0 - E_g + \frac{y}{t}) \to \ln(E_0 - E_g + \frac{y}{t}) + 2\pi i\theta(\frac{y}{t} - |E_0 - E_g|)$, where $\theta(z)$ is the Heaviside function; thereby, the difference between the second terms of the denominators of (2.92) is proportional to $-2\pi i \sum_r \left|\left\langle E_g + \frac{y}{t}, r\middle|V\middle|\Psi_0\right\rangle\right|^2$. From this last statement, it can be inferred some properties that the term $\sum_r \left|\left\langle E_0, r\middle|V\middle|\Psi_0\right\rangle\right|^2$ must follow; it must assure the convergence of the integral (2.92), and it must vanish rapidly enough at the boundaries, in accordance to the localized behavior of the potential V; in particular, Nakazato, Namiki and Pascazio assumed an specific form for this expression, in order to guarantee the convergence of the survival amplitude $A(t)$:

$$\sum_r \left|\left\langle E_0, r\middle|V\middle|\Psi_0\right\rangle\right|^2 = \begin{cases} (E_0 - E_g)^\delta & \text{for } E_0 \approx E_g \\ E_0^{-\delta'} & \text{for } E_0 \to \infty \end{cases} \tag{2.94}$$

It is relevant to say that δ and δ' must be positive constants. Once the behavior of this term is determined, the integral (2.92) can be expressed in the limit when t becomes large. For that, it is necessary to make a new change of variables $u = \frac{y}{(E_a - E_g)t}$, so that the integration must be analyzed when u is really small, compared to zero.

$$X_C = -(E_a - E_g)e^{\frac{i(E_a-E_g)t}{\hbar}} \int_0^{\infty} \frac{B_u e^{\frac{i(E_a-E_g)tu}{\hbar}} du}{\left[(E_a - E_g)(1+u) + A_u\right]\left[(E_a - E_g)(1+u) + A_u - 2\pi i B_u\right]} \tag{2.95}$$

with:

$$A_u = \int_{E_g}^{\infty} \ln(E_0 - E_g + (E_a - E_g)u) \frac{d}{dE_0} \sum_r \left|\langle E_0, r \middle| V \middle| \Psi_0 \rangle\right|^2 dE_0 \tag{2.96}$$

and:

$$B_u = \sum_r \left|\langle E_g + (E_a - E_g)u, r \middle| V \middle| \Psi_0 \rangle\right|^2 \approx (E_a - E_g)^{\delta} u^{\delta} \tag{2.97}$$

Solving the integral, it can be shown [16] that the nonexponential contribution is:

$$X_C = -\frac{Ce^{\frac{i(E_a-E_g)t}{\hbar}}}{t^{1+\delta}} \tag{2.98}$$

So, the nonexponential contribution is given by a power function of t; nevertheless, the question about the nature of the δ exponent that defines the nonexponential behavior of $A(t)$ arises. In fact, there could be degeneracy in the energy spectrum, depending on the behavior of the hamiltonian, that affects the behavior of the survival probability; in their paper, Nakazato, Nimiki and Pascazio, talked about the possibility of degeneracies in the energy spectrum; if the long time behavior is considered, an expansion of $g(s,t)$ must be done,since this function carries the properties that the survival probability must satisfy. So, as the $g(s,t)$ gets spanned in terms of a power series of $\frac{1}{t}$, an exponential expression can be obtained [37, 16]. This expression contradicts whatever the quantum mechanics postulated about the behavior of the survival probability $P(t)$ at long times. Nevertheless, the exponential behavior at all times is ascribable to the finiteness of the considered system, as long as the exponential behavior cannot be expected if the number of degrees of freedom of the system is finite. In particular, if the system is restricted as physical constraints are essential to its evolution, in accordance to the properties of the hamiltonian, the continuous energy spectrum must be degenerated, as there would be quantum operators, representing observables that determine the state of the system, which can commute with the hamiltonian, as well. This latter condition guarantees the finite character of the number of the degrees of freedom. The survival amplitude, as it was stated above, must converge at any possible time; this statement means that, as a consequence of the Payley Wiener Theorem (Theorem (1.2)), the decay proceeds more slowly than exponentially at large times.

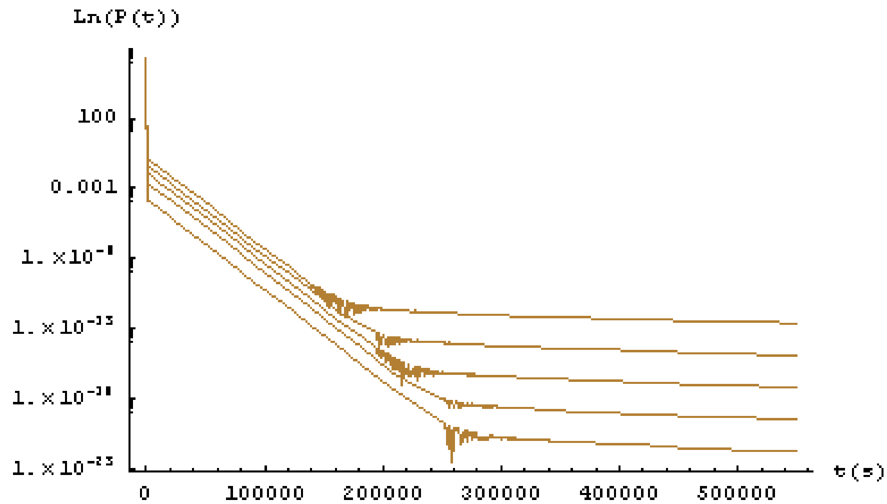

Figure 2.13: Logarithmic plot of the survival probability $P(t)$ determined in the works of Nakazato, Namiki and Pascazio [16], for different exponents($\delta_0 = 0.1$, $\delta_1 = 0.3$, $\delta_2 = 0.5$,$\delta_3 = 0.7$, $\delta_4 = 0.9$).

Fourier transform and $A(t)$

All the different physical constrains lead to redefine the behavior of the evolution operator in the interaction picture, as Nicolaides and Dovropoulos did it in their work [34]. Analogically to the Fock Krylov theorem ((1.1)), the evolution operator can be redefined as a Fourier Transform of an spectral density $G(x)$(different from the density of states (1.16)):

$$U(t) = \frac{i}{2\pi}\int_C G(x)e^{-\frac{ixt}{\hbar}}dx \tag{2.99}$$

The analytical properties of the evolution operator $U(t)$ must be satisfied by the physical constrains and the properties of the function $G(x)$; since it is not a density of states, as it was defined in the relation (1.16), the variable x runs from the entire real axis, in accordance to the integration contour C; the possible singularities on the spectral density appears as simple poles, that begin to get closer, until a branch cut on the real axis is formed; this spectral density takes the form of the resolvent operator defined above $R(z)$[33], at first sight. Nevertheless, a time independent perturbation theory can be applied, so that, the spectral function can be spanned in terms of the unperturbed hamiltonian operator and the potential that leads to the decay:

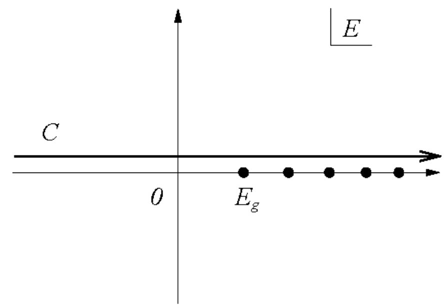

Figure 2.14: Contour where the integral (2.99) must be evaluated, to find an expression for the evolution operator [16]

$$\begin{aligned} G(x) = & \frac{1}{x-\hat{H}_0} + \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} \\ & + \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} \\ & + \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} \\ & + \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} V \frac{1}{x-\hat{H}_0} \\ & + \ldots \end{aligned} \tag{2.100}$$

Since the survival amplitude is related with the diagonal matrix elements of the evolution operator $\langle \Psi_0 | U(t) | \Psi_0 \rangle$, the survival amplitude is related with the expectation value of the spectral density $G(x)$, in accordance to (2.99):

$$A(t) = \frac{i}{2\pi} \int_C \langle \Psi_0 | G(x) | \Psi_0 \rangle e^{-\frac{ixt}{\hbar}} dx \tag{2.101}$$

The matrix element of the density operator must be calculated; as the initial state $|\Psi_0\rangle$ is an eigenstate of the unperturbed hamiltonian $\hat{H}_0$, the determination of this element is simplified; Nakazato, Namiki and Pascazio introduced a self energy part $\Sigma_a(x)$, so that, the $G_a(x) = \langle \Psi_0 | G(x) | \Psi_0 \rangle$ factor could be replaced by:

$$G_a(x) = \frac{1}{E - E_a - \Sigma_a(x)} \tag{2.102}$$

where $\Sigma_a(x) =$ is the self energy part, and it is built by the even contributions of the time independent perturbation expansion :

$$\Sigma_a(x) = \sum_{n \neq 0} \frac{\left| \langle \Psi_0 | V | \Psi_n \rangle \right|^2}{E - E_n} + \sum_{n \neq 0} \sum_{n' \neq 0, n} \frac{\left| \langle \Psi_0 | V | \Psi'_n \rangle \right|^2 \left| \langle \Psi_{n'} | V | \Psi_n \rangle \right|^2}{(E - E_n)(E - E_{n'})^2} + \ldots \tag{2.103}$$

The meaning of the self energy part $\Sigma_a(x)$ has been rebated; Nakazato, Namiki and Pascazio [16] used the random phase assumption in order to express the behavior of the survival amplitude $A(t)$, as the number of degrees of freedom is finite, but huge. The description of the terms of the expansion, in accordance to the time independent perturbation theory, are related with the so called "proper energy" terms in field theory; as a matter of fact, in field theoretical cases , the relation corresponds to particle systems that are coupled to fields. As the interaction between the systems and the fields takes a similar form of the expressions above, those terms are denoted as the "self energy" part, analogically; in the limit where the spectrum of the hamiltonian is continuum, the self energy part passes to the continuum, and the sum over the discrete eigenstates associated with the unperturbed hamiltonian $\hat{H}_0$ transforms into an integral, where the density of states defined above (1.16) expresses the contribution of each one of the eigenstates of the hamiltonian to describe the "self energy part" $\Sigma_a(x)$; since the energy spectrum has a physical constrain, $E \geq E_{threshold}$, the integral must be done based on the physical constrains the system must obey; so, the "self energy part" $\Sigma_a(x)$ must be given by:

$$\Sigma_a(E) = \int_{E_{threshold}}^{\infty} \omega(E') \frac{1}{E-E'} \left[\left| \left\langle \Psi_0 \middle| V \middle| \Psi_n \right\rangle \right|^2 + \sum_{n' \neq 0, n} \frac{\left| \left\langle \Psi_0 \middle| V \middle| \Psi'_n \right\rangle \right|^2 \left| \left\langle \Psi_{n'} \middle| V \middle| \Psi_n \right\rangle \right|^2}{(E-E_{n'})^2} + ... \right] dE' \quad (2.104)$$

As the expectation value of the spectrum density $G(E)$ is related with the behavior of the survival amplitude $A(t)$, it is necessary to review its properties, and, therefore, the properties of the "self energy part " $\Sigma_a(E)$; in the weak coupling limit(the system is taken as closed, only interacting weakly with its surroundings), the random phase approximation for all the elements of the matrix $\left\langle \Psi_n \middle| V \middle| \Psi_{n'} \right\rangle$ must be satisfied, so that $n \neq n'$; the exponential behavior, as it was stated before, is related with the residue of the function calculated on the pole below the real axis, where there is a branch cut, as all the simple poles of the spectral density function $G(x)$ approaches to each other. To determine the complete behavior of the survival amplitude $A(t)$, in accordance to (2.82), the contour C must be deformed into a new contour; the delimited portion of C, where the integral (2.82) is calculated can be expressed as the sum of a path running just below the real axis on the "unphysical" sheet from $E_{threshold}$ to ∞, called C', and a circle turning clockwise around the simple resonant pole in the second Riemannian sheet,as it is shown in the figure below Fig. (2.15):

The simple pole on the second Riemannian sheet is translated with respect to the pole defined by the singularity of the definition of the spectral density $G(x)$; this pole on the second is defined as $\bar{E} = E_a - \Sigma_a(\bar{E})$; so, applying the residues theorem [28] over the integral (2.101), it can be shown that the survival amplitude $A(t)$ has two different

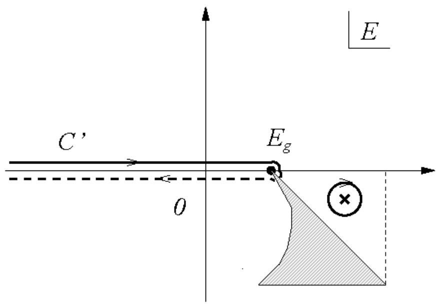

Figure 2.15: Deformed contour where the integral (2.99) must be evaluated, to find an expression for the evolution operator [16]

contributions, two relevant terms:

$$A(t) = \frac{1}{\frac{\partial\left(G_a(E)^{-1}\right)}{\partial E}\Big|_{E=\bar{E}}} e^{-\frac{i\bar{E}t}{\hbar}} + \frac{i}{2\pi}\int_{C'} e^{-\frac{iEt}{\hbar}} G_a(E)dE \tag{2.105}$$

Obviously, since the pole $\bar{E}$ is found below the real axis, in the "unphysical" sheet, the exponential contribution to the survival probability $P(t)$ would be given by the first term of (2.105); nevertheless, if all branch cut effects of the analytical properties of the spectral density $G(x)$, and the propagator related to it, are neglected, and only the effects from the single pole in the "unphysical" sheet are considered, the behavior of the survival probability $P(t)$ related with the temporal evolution of the unstable system would be exponential for all times; for that cases, the Weisskopf Wigner approximation[1] plays a huge role in the description of the analytical properties of the propagator and the temporal evolution of the unstable system; in accordance to that approximation, the self energy part Σ_a takes a constant value as a matter of fact, equal to the value of its pole. In accordance to the Weisskopf Wigner approximation, the survival probability $P(t)$, related with the properties of the spectral density $G_a(E)$, would take an exponential behavior for all times.

Nevertheless, if the branch cut effects are taken into account, the second term, given by the integration over the C' contour, is more relevant, and the exponential term vanishes at long times, at the decay process goes further. So, from the branch cut definition, this integral term takes the form:

$$\frac{i}{2\pi}\int_{C'} e^{-\frac{iEt}{\hbar}} G_a(E)dE = \frac{i}{2\pi}\int_{-\infty}^{E_{threshold}} e^{-\frac{iEt}{\hbar}} \left[G_a(E+i\epsilon) - G_a(E-i\epsilon)\right]dE \tag{2.106}$$

where the second term $G_a(E-i\epsilon)$ must be calculated on the second Riemannian sheet.

The distinction between the behavior of $G_a(E)$ above and below the real axis is necessary, as there is a branch cut on the real axis that restricts the analytic domain of the spectrum density; nevertheless, the difference between $G_a(E+i\epsilon)$ and $G_a(E-i\epsilon)$ is proportional to the strength of the interaction; so, as the $G_a(E)$ function is related with the potential, the only issue to determine is the behavior of the density of states, related with the "self energy part" $\Sigma_a(E)$ through (2.104), in order to find the analytic expression of the survival amplitude $A(t)$.

For weakly coupled systems, van Hove[38] found that it is possible to obtain a master equation(leading to an exponential behavior) for a quantum mechanical system endowed with many degrees of freedom, considering the limit when the strength of the potential that leads to the decay goes to zero, as long as λt remains finite. During the derivation of what it was called the "van Hove" limit, van Hove realized that the exponential behavior is inherent for all the systems that behave in accordance to the "van Hove" limit, meaning that all the terms in the diagonal of the matrix $\langle \Psi_n | V | \Psi_n \rangle$ have a predominance over the off diagonal terms of the representation of the potential in this basis. To guarantee the exponential behavior related with the "van Hove" limit, it is important to take into account the wavepacket form of the initial state, and the localized properties of the potential. The application of the weak coupling limit and the "van Hove" limit leads to a temporal scaling problem, which isn't the issue to discuss at this point. Nevertheless, the relevance of the "van Hove" limit related with the properties and the strength of the potential is given by the relation with the "self energy part",and, therefore, with the exponential behavior of the survival probability $P(t)$. In fact, it can be shown that the exponential behavior is a probabilistic law in the weak coupling limit.

Even though the time scaling problem is an important issue to solve, the most important thing to analyze is the asymptotic behavior that the survival amplitude $A(t)$ follows, as times becomes so large. In accordance to (2.105), the two contributions to the survival amplitude are related with analytical properties of the propagator that guides the evolution of the system; the exponential behavior is related with the pole, a single singularity, and the residue of the propagator calculated on it($\frac{1}{g(s,t)}$ in the case of the Laplace transform, or the spectral density, related with the resolvent operator $G(E)$ in the Fourier transform case). But, as the nonexponential contribution has a potential form in time (2.98), the main question to answer is the meaning of the exponent in that relation. When the two relations (2.98) and (2.105) are compared, it can be deduced that the behavior of the exponent in the power law is related with the properties of the potential V, and the analytic form of the density of states; in principle, these two factors affect the system throughout its evolution at large times, when the nonexponential contribution overlaps the exponential behavior, as it can be shown in the figure (Fig.(2.16)). The discrepancy with respect to the different exponents has been discussed along the literature, as different kinds of potentials have been supposed in order to study some unstable systems. Höhler, for example, showed explicitly a power behavior of $t^{-\frac{3}{2}}$ for the Lee model[40]; as the exponent changes, the temporal distinction between the two possible contributions to the survival amplitude and the survival probability changes as well, depending on the behavior of the potential V that

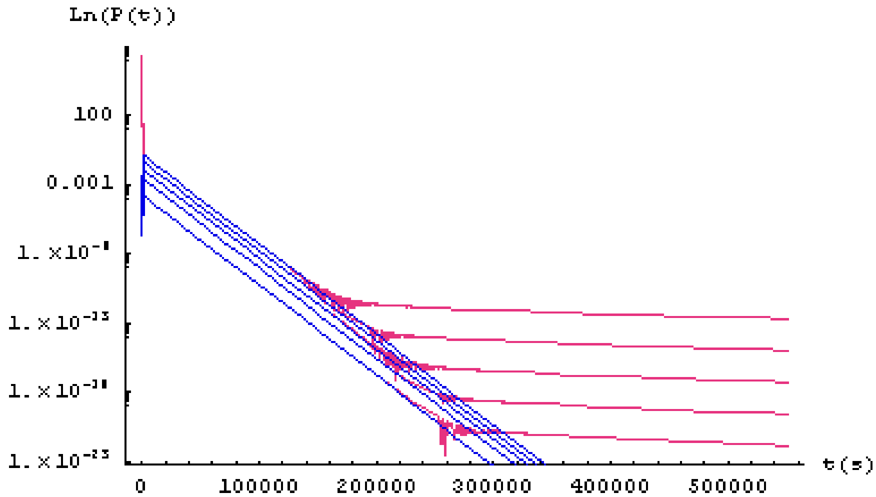

Figure 2.16: Logarithmic plot of the survival probability $P(t)$ determined in the works of Nakazato, Namiki and Pascazio [16], for different exponents($\delta_0 = 0.1$, $\delta_1 = 0.3$, $\delta_2 = 0.5$,$\delta_3 = 0.7$, $\delta_4 = 0.9$). The blue plots represents the exponential contribution to the survival probability, and the red plots are the survival probability $P(t)$ given by the nonexponential contribution (2.98) and the exponential one

leads the decay process. The evidence of the dominance of the long time behavior is so clear, in accordance to the physical constrains the system must accomplish in order to evolve in time(the behavior of the potential, the density of states defined analytically, the interaction with its surroundings)[23]. But the dynamical restrictions affect the behavior of the system at all times, and the temporal domain where the exponential contribution dominates over the power law. Nevertheless, the existence of poles in the propagator that arises the temporal evolution of the system makes this distinction clear, in accordance to the quantum mechanical postulates, as Khalfin predicted in his earlier works [7].

The main issue to conclude about the definition and the analytical properties of the survival probability $P(t)$ is, precisely, the relation between the system and all the physical constrains that affect its evolution; so that, the evolution operator must be redefined for this situation, and the singularities in the solutions of the Schrödinger equation appear naturally. Therefore, the confrontation between the two possible contributions to the survival probability, as the most interesting variable to analyze in the decay processes, is related with the possible interactions that delimitate the behavior of the system, in accordance to the explicit form of the hamiltonian; in these considerations, the density of states and its analytical properties plays a huge role into the determination of the possible deviations of the exponential behavior, as the quantum mechanical considerations affirm; all the possible physical constrains must be given so

[23]See Fig.(2.17)

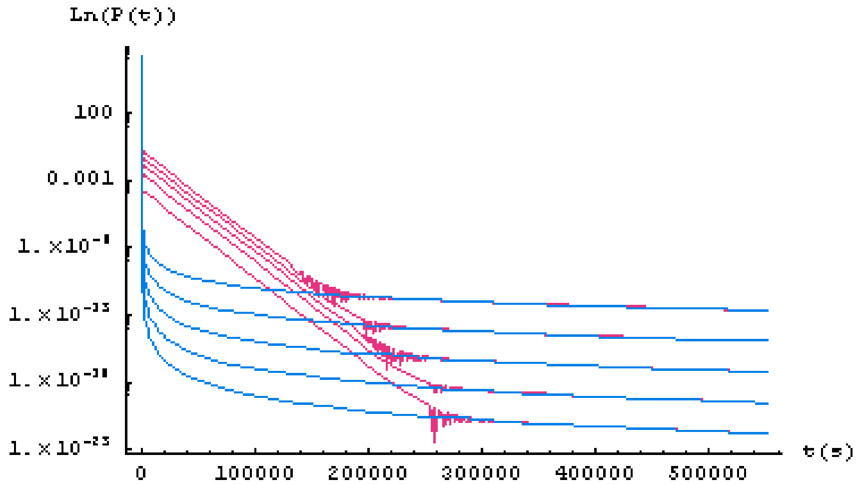

Figure 2.17: Logarithmic plot of the survival probability $P(t)$ determined in the works of Nakazato, Namiki and Pascazio [16], for different exponents($\delta_0 = 0.1$, $\delta_1 = 0.3$, $\delta_2 = 0.5$,$\delta_3 = 0.7$, $\delta_4 = 0.9$). The blue plots are the nonexponential contribution to the survival probability given by (2.98), and the red plots are the survival probability $P(t)$ given by the nonexponential contribution (2.98) and the exponential one

that the survival probability $P(t)$ would converge at all times, and it can be normalized. The other thing to consider is the mathematical description of the survival probability, since the choice of the integration contours, in accordance to the physical considerations that leads to the decay process. As it was mentioned earlier, from the works of Khalfin and others, the existence of the nonexponential contribution to the survival probability is related with the lower bound of the spectrum; as a matter of fact, Flambaum and Izrailev tried to find out the possible behavior of the survival probability, if the Fock Krylov theorem (theorem (1.1)) would be applied as the energy spectrum covers the entire real axis, contradicting the physical constrain $E > 0$[41].

2.1.6 Flambaum and Izrailev's exponential law

In their work, Flambaum and Izrailev [41] expressed the difficulties to find an analytic expression for the density of states $\omega(E)$ defined above (1.16), due to the interactions between the systems, when many body states are considered; indeed, the difficulties to obtain a potential that characterize the many body theory is reflected in that search to obtain the density of states that characterize the energy space, where the system follows its temporal evolution, in accordance to the properties of the evolution operator. Flambaum and Izrailev [41]introduced the strength function, which is the density of states defined above (1.16); indeed, they supposed different kind of density of states, as different kinds of interactions have been considered, acting on the system, and affecting the propagator that leads to the temporal evolution of the metastable state. In fact, there are a couple of corrections that can be done as the considered interaction changes to characterize the system; for example, the definition of the width could be independent of the energy, as weakly coupled interactions are considered, so that the width function would be broad. Since the relevance of the density of states is in the description of the dynamics of the wavepackets and physical states in the energy space, the change of the interaction that leads to the decay of the unstable system, must be involved in the physical properties of the system and their possible perturbations; therefore, if the interaction changes, the density of state must be redefined, and the propagator that leads the decay of the considered system, as well. Nevertheless, Khalfin in his work, said that the decay law that the physical systems follow would not be exponential at all times, as the energy spectrum has a lower bound[7]. He did not take a particular form of the interaction, so , this conclusion is quite general. Nevertheless, Flambaum and Izrailev [41] used different densities of states, for weak coupled and strong interactions[24], in order to analyze the survival amplitude $A(t)$, and the survival probability $P(t)$, in accordance to (1.5):

$$\omega(E) = \begin{cases} \frac{1}{2\Pi}\frac{\Gamma}{(E-E_r)^2+\frac{\Gamma^2}{4}} & \text{For weak coupled interactions, where the width } \Gamma \text{ is a constant} \\ \frac{N}{\sqrt{2\pi\sigma^2}}e^{-\frac{E^2}{2\sigma^2}} & \text{For strong interactions} \end{cases} \tag{2.107}$$

If the Fock Krylov theorem (theorem (1.1)) is applied on each one of the density of states defined above (2.107), as the physical constrain corresponding to the lower bound of the energy spectrum is not considered, the only contribution to the survival amplitude would be given by the exponential part; therefore, the decay law would be eminently exponential at all times:

$$P(t) = e^{-\sigma^2 t^2} \tag{2.108}$$

The relation above expresses the fast convergence of the survival probability, and its homogeneous behavior at all times , as it can be shown in the next figure (Fig. (2.18)) This fact affirms the hypothesis that relates the nonexponential behavior of the survival probability $P(t)$ at large times with the physical constrains that affect the temporal evolution of the system; even though the analytic expression of the propagators and therefore, the density of states are hard to obtain, since the interactions between the systems have not been determined explicitly, it can be shown that the nonexponential decay is a result of the inhomogeneities of the evolution operator, as the energy spectrum must be bounded from below, and the temporal domain must be taken as positive, as Nicolaides and Dovropoulos pointed out [33]; it is easy to demonstrate

[24] In accordance to what the work of Flambaum and Izraliev said[41], a weak coupled interaction must be so that the width should be so less than the width of the density of states δ_E, $\Gamma_0 \ll \delta_E$, given by the second moment $\delta_E = \sum_{f\neq i}\left|\left\langle\Psi_i\middle|V\middle|\Psi_f\right\rangle\right|^2$; meanwhile, the strong residual interaction must be so that the width Γ_0 would be greater than the width of the density of states $\Gamma_0 \geq \delta_E$

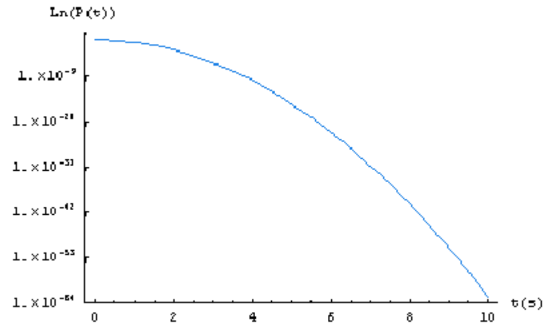

Figure 2.18: Logarithmic plot of the survival probability $P(t)$ given by (2.108) for strong interactions, [41]

that, since there is not a single singularity in the density of states, for strong interactions(the gaussian form), the exponential contribution must converge so quickly, and a metastable system would decay faster, as the Fig. (2.108) can show. So, in order to analyze the evolution of a physical system, under an interaction that leads it to a decay, the physical constrains must be taken into account, and the analytic properties of the propagators related with the dynamical evolution of the considered system. The overlapping of the exponential contribution by the nonexponential one is a physical fact, since the properties of the propagators that determine the evolution of the system, and the properties of the energy space(given by the analytic form of the density of states $\omega(E)$), where the physical state is projected in order to determine the representation in its basis, let the considered system behave in accordance to the postulates of the quantum mechanics, and the Schrödinger equation.

As it was developed along this chapter, there have been so different perspectives in order to analyze the properties of the decay processes. The behavior of the density of states and the physical constrains that limit the evolution of the system, in accordance to the Schrödinger equation and the evolution operator $U(t)$ affects the way how the decay is undergoing, and, therefore, the survival probability $P(t)$ as the most important variable that describes the physics of the process. In particular, the analytical properties of the potential $V(r)$ that leads to the decay, and the configuration of the initial state, as a localized wavepacket, let make a complete description of the process, and the temporal evolution of the considered system; the different definitions of densities of states affects the properties of the energy space, where the initial state is projected, in order to determine the survival probability $P(t)$; but as there are two possible contributions that domain in different temporal domains, the physical constrains that guide the evolution of the metastable system, in accordance to the properties of the hamiltonian $\hat{H}$, determine the analytical properties of the survival probability, and the relevance of each contribution with respect to the decay process. Since the main goal of this work is to study and characterize the decay process at large times, and energies near the threshold, one tool of the scattering theory would be so useful to find the properties of the evolution of the metastable states and resonances in the temporal domain that it is going to be analyzed: the partial wave method. What is the relevance of this method, and how can it help to understand the decay processes of resonances and metastable systems?

CHAPTER 3

Theoretical formalism to evaluate the survival probability $P(t)$

Since the evolution of all the quantum systems is governed by the Schrödinger equation and the temporal evolution operator $U(t)$, the determination of the systems is related with the expectation values of the operators that act on the systems, in particular, the way a system interacts with the other systems in its surroundings. As it was explained before, the analytical properties of the potential $V(r)$ that leads the physical processes development are so relevant in order to specify the characteristics of the system and its evolution in time; in particular, the metastable systems and resonances are susceptible and sensitive to the properties of the interaction that leads the decay process and the hamiltonian $\hat{H}$ that governs the state of the system at any particular time. The importance of the potential in the scattering theory arises in the dynamical and physical description of the wavefunctions, before and after an scattering process,as its properties let to understand the state of the system and the different variables whose behavior specifies the physical properties that define the system. Nevertheless, as the physical constrains take a huge importance over the evolution of the system, the properties of the potential must satisfy these restrictions; for example, one of the most important assumptions that can be done in scattering process is the analytic form of the wavefunction that describes the system after a scattering between two systems occurs: the asymptotic scattering wavefunction $\Psi_{\text{scattering}}$, that models the behavior of the system after a scattering process has happened. This assumption lets analyze the system at a distance so far with respect to the reach of the potential, since the wavefunction, as r goes to ∞ is taken as the superposition of an ingoing wave, representing the wavefunction of the incident particle before the scattering process takes place, and an spherical outgoing wave, representing the state of the system once the process has taken place, if the potential tends to zero faster than r^{-1}; in particular, the considered potential $V(r)$ depends on r, and the spherical symmetry is satisfied:

$$\Psi_{\text{scattering}} \underset{r\to\infty}{\approx} A\Big(e^{k_i\dot{r}} + \frac{f(\theta,\phi)e^{ikr}}{r}\Big) \tag{3.1}$$

with $f(\theta,\phi)$ is called the "scattering amplitude", related with the behavior of the outgoing waves at large distances from the origin; at comparable distances with respect to the reach of the potential $V(r)$, there are interference phenomena between the outgoing spherical waves, whose contribution is characterized by the scattering amplitude, and the incoming waves, that describes the state of the incident particle before the scattering process takes place, in the lab system. As a result of this fact, there can be found a relation between the scattering amplitude $f(\theta,\phi)$ and the cross section, in what it is called the "optical theorem"[30, 39]:

$$\sigma_{\text{total}} = \frac{4\pi}{k}\text{Im}f(\theta=0) \tag{3.2}$$

Nevertheless, in order to get closer to the physical processes given under that considerations, the description given by (3.1) could represent an idealization; so, as it was mentioned earlier in another chapter, the convenience to take wavepackets as the natural way to describe the states of the system, makes the right way to express the physical systems, and their localization, by definition of initial states, imposes some properties to the evolution of the system, even though the scattering processes, and the decay processes as it has been studied along the literature, are, in general, a markovian one.[1] As the systems are described by wavepackets, the phenomena related with the spread(width) and the evolution of the wavepackets arise naturally, and the interference effects that affect the description of the systems would play a huge role in the characterization of the physical systems, as well.

3.1 Method of partial waves

So, it was mentioned earlier in this chapter that the characteristics of the potential $V(r)$ would delimitate the evolution of the system. In particular, if the chosen potential is a central one(depending only on the radial coordinate), with some properties related with its convergence when the radius r goes larger, all the wavefunctions that evolve in accordance to the hamiltonian $\hat{H}$ can be spanned in the basis of the angular momentum space, since the $\hat{L}$ operator commutes with the hamiltonian, due to the central behavior of the potential, and the existence of the spherical symmetry of the hamiltonian, so that the Schrödinger equation can be separated in a radial part(the radial equation), depending on r:

$$-\frac{\hbar^2}{2m}\Big[\frac{1}{r^2}\Big(\frac{r^2 d}{dr}\Big) - \Big(\frac{l(l+1)}{r}\Big)\Big]R_l(k,r) + V(r)R_l(k,r) = ER_l(k,r) \tag{3.3}$$

[1] A markovian process doesn't have "memory" or susceptibility to the initial conditions

and an angular part, whose solution is the set of functions called the "spherical harmonics" $Y_{lm}(\theta, \phi)$, which are the basis of the angular momentum space; so the scattering wave function can be spanned in terms of the spherical harmonics; physically, this expansion can be understood as if the scattering wavefunction (3.1) could be a superposition of partial waves, so that each one of them contributes as $c_{lm}(k)$ to the definition of the physical state $\Psi_{\text{scattering}}$; this is the so called "partial wave method":

$$\Psi_{\text{scattering}} = \sum_{l=0}^{\infty}\sum_{l=0}^{\infty} c_{lm}(k) R_{lm}(k,r) Y_{lm}(\theta,\phi) \tag{3.4}$$

The determination of the $c_{lm}(k)$ coefficients depends on the behavior of the potential and its explicitly analytic form, given by the function $R_{lm}(k, r)$, defined as the radial solution of the Schrödinger equation. As a matter of fact, a description of the survival amplitude $f(\theta, \phi)$ can be done as the wavefunction $\Psi_{\text{scattering}}$ is spanned in the basis of the "spherical harmonics; in accordance to the physical behavior of the solution, it can be expected that outside the range of the potential, the radial function would behave as a superposition of the Bessel and Newmann functions:

$$R_{lm}(k,r) \underset{r\to\infty}{=} k\left[C_l^{(1)}(k) j_l(kr) + C_l^{(2)}(k) n_l(kr)\right] \tag{3.5}$$

Indeed, the radial solution $R_{lm}(k, r)$ can be expressed as superposition of the two Hankel functions:

$$R_{lm}(k,r) \underset{r\to\infty}{=} k\left[D_l^{(1)}(k) h_l^{(1)}(kr) + D_l^{(2)}(k) h_l^{(2)}(kr)\right] \tag{3.6}$$

When the coordinate r goes to ∞, the behavior of the radial solution is determined by the asymptotic form that the Bessel and Newmann function take in this limit[30]:

$$j_l(x) \underset{r\to\infty}{\approx} \frac{1}{x}\sin(x - \frac{\pi l}{2}) \tag{3.7}$$

and:

$$n_l(x) \underset{r\to\infty}{\approx} \frac{1}{x}\cos(x - \frac{\pi l}{2}) \tag{3.8}$$

for the Hankel functions, the asymptotic behavior takes the form:

$$h_l^{(1)}(x) \underset{r\to\infty}{\approx} -i\frac{e^{i(x-\frac{\pi l}{2})}}{x} \tag{3.9}$$

and:

$$h_l^{(2)}(x) \underset{r\to\infty}{\approx} i\frac{e^{-i(x-\frac{\pi l}{2})}}{x} \tag{3.10}$$

So, comparing (3.5) and (3.7), it can be shown that the radial function $u_l(kr) = rR_{lm}(k,r)$ can be expressed as:

$$u_(kr) \underset{r\to\infty}{\approx} A_l(k)\sin(kr - \frac{\pi l}{2} + \delta_l(k)) \tag{3.11}$$

or:

$$R_{lm}(k,r) \underset{r\to\infty}{\approx} A_l(k)\big[j_l(kr) - \tan(\delta_l(k))n_l(kr)\big] \tag{3.12}$$

Where the $\delta_l(k)$ parameter is called the "phase shift" associated with the partial wave with angular momentum l.

The wavefunction that describes the system must behave asymptotically in accordance to the scattering wavefunction defined above (3.1), as the wavefunction is spanned in terms of the basis of the spherical armonics[30], due to the expansion of the exponential function in terms of the basis, and the asymptotic behavior of the Bessel functions that expresses the radial solution (3.11) :

$$\Psi_{\text{scattering}}(k,r) = \sum_{l=0}^{\infty} c_{lm}(k)A_l(k)\frac{1}{2ir}\left(e^{i(kr-\frac{\pi l}{2}+\delta_l)} - e^{i(kr-\frac{\pi l}{2}+\delta_l)}\right)Y_{lm}(\theta,\phi) \tag{3.13}$$

From this last relation, the expansion coefficients can be found:

$$c_{lm}(k) = \frac{A(k)}{kA_l(k)}\sqrt{4\pi(2l+1)}i^l e^{i\delta_l}\delta_{m,0} \tag{3.14}$$

Replacing this coefficient in the relation (3.13), and comparing with the definition of the boundary scattering wavefunction $\Psi_{\text{scattering}}$, it can be found a definition of the scattering amplitude $f(\theta,\omega)$ in terms of the phase shifts and the contributions of each partial wave with angular momentum l:

$$f(k,\theta) = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos(\theta)) \tag{3.15}$$

Where $P_l(\cos(\theta))$ are the Legendre special functions and $a_l(k)$ are defined as the partial wave amplitudes, defined as:

$$a_l(k) = \frac{e^{2i\delta_l(k)} - 1}{2ki} \tag{3.16}$$

Since the scattering amplitude $f(\theta)$ can be expressed in terms of the set of phase shifts δ_l, related with each partial wave, a relation between the scattering cross section and the phase shifts can be found, due to the optical theorem (3.2)[30]:

$$\sigma_{\text{total}} = \sum_{l=0}^{\infty}\sigma_l(k) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l(k)) \tag{3.17}$$

Hence, each partial wave contributes to the total scattering cross section with a partial scattering cross section σ_l that depends on the behavior of the phase shift δ_l related to it.

3.1.1 The phase shifts $\delta_l(k)$ and their physical meaning

The relevance of the concept of the phase shifts arises from the properties of the potential that leads to the decay process; hence, there is a concrete relation between the **S** matrix and the phase shifts discussed[26]:

$$\mathbf{S_l} = e^{2i\delta_l} \tag{3.18}$$

The relation above gives some of the properties that characterize the phase shifts: As the **S** must be unitary, the phase shifts must be real in order to satisfy (3.18). The introduction of the phase shifts in the description of the scattering bound wavefunction $\Psi_{\text{scattering}}$ gives the importance of the properties of the interaction in the description of the scattering process: "The only material difference between the radial wave function $u_l(kr)$ in the presence of the scattered and in its absence is a shift in phase"[26, page 301]; so, the appropriate description of a system under a scattering process can be understood as a superposition of infinitely spherical waves each shifted in phase and added with a proper overall phase; it is important to notice that since the behavior of the potential is related with the properties of the defined phase shifts $\delta_l(k)$ for the scattering process, where the potential vanishes, the related phase shift goes to zero, so that, the radial function $u_l(kr)$ defined by (3.11), behaves as (3.5) where the localized potential $V(r)$ vanishes beyond its range R. This fact demonstrates how the analytical properties of the central potential $V(r)$ affect the definition of the phase shifts $\delta_l(k)$ related with each partial wave. In particular, if the potential $V(r)$ cannot support a bound state, and is very weak, the phase shift would quickly turn over and decrease, at low energies. Besides, if the potential is repulsive, the radial solution $u_l(k,r)$ is "pushed out" with respect to the free solution where the potential goes to zero; so that, the phase shifts are defined as negatives. Meanwhile, if the potential is attractive, the phase shifts are defined as positives, so that the radial solution is "pulled in" with respect to the free radial function in absence of a potential that leads to the decay process; as a consequence, if the potential is attractive, the phase shifts $\delta_l(k)$ tends to grow up easily. Since the contribution of the different partial waves is related with the range of potential, the contribution of the phase shifts associated with each one of them is also, related with the properties of the potential, as well; indeed, at low energies, the partial waves with lower l contribute the most to the description of the bound scattering wavefunction $\Psi_{\text{scattering}}$, since the centrifugal barrier term $\frac{l(l+1)}{r^2}$ becomes more and more important, as l increases; therefore, the contribution from the large l will tend to zero, or, as a matter of fact, in accordance to (3.18), they would tend to an integral multiple of π, in accordance to the number of bound states delimited by the properties of the potential $V(r)$[26] in absence of a bound state, the s wave phase shift(the phase shift related with the partial wave with $l = 0$) will tend to zero, effectively; nevertheless, the introduction of an additional bound state, given by the definition of the potential, raises the s wave phase shift by π, in what it has been called the "Levinson's theorem"[26] :

$$\delta_{l=0}(0) = m\pi \tag{3.19}$$

Meanwhile, at the so called “transitional” value, the value of the s partial wave phase shift is $\frac{\pi}{2}$, in absence of bound states ligated to the central potential $V(r)$. In general, the introduction of m bound states makes the s partial phase shift to arise in $(m+\frac{1}{2})\pi$. The importance to analyze the S partial wave phase shift relates with the low energy scattering, since the partial waves with lower l contribute the most to define the scattering amplitude and therefore, to the wavefunction that describes the state of the system, as it was mentioned above. The Levinson's theorem can be extended to any phase shift with $l > 0$, but the transitional value behavior belongs to the S partial wave phase shift in particular.

As a matter of fact, the description of a scattering process is so related with the definition of what it has been called the effective potential" $U_{\text{eff}} = V(r) + \frac{l(l+1)}{r^2}$; the second term corresponds to the centrifugal barrier, that restricts the behavior of the system, in accordance to the value of the angular momentum coefficient l, as it was states earlier in this subsection; the partial wave analysis could be a complement of the Born approximations, since it describes so well the behavior at low energies, or processes where the $\mathbf{S}$ matrix is diagonal; as it was mentioned before, the partial waves with the lower values contribute more than the ones with l higher values, to the description of the physical process that is taking place: The incident particle would need a higher energy to overcome the centrifugal barrier term and reach region where the central potential $V(r)$ acts, as the angular momentum l grows. Nevertheless, when the incident particle has a lower incident energy, only the partial waves with lower l would contribute, as the particle needs to overcome the repulsion of the centrifugal barrier term, and interacts with the potential, so that, the scattering process would be affected by the partial waves with lower l. In consequence, it takes less kinetic energy for an incident particle to overcome the centrifugal barrier given by a low angular momentum l, than the opposite case; that is the main reason why the scattering processes, at low energies, are described better by a partial wave analysis, with contributions of those partial waves with lower l, mostly.

How are the phase shifts defined? What is its concrete relation with the potential $V(r)$? Well, as the radial function $R_{lm}(k,r)$ (3.3), and its derivative, must be continuous in the limit inside and outside the range of the potential(where $r = R$), as a boundary condition, it can be delimited a condition to find the phase shifts[39][2]:

$$\tan(\delta_l(k)) = \frac{kRj_l'(kR) - \beta_l j_l(kR)}{kRn_l'(kR) - \beta_l n_l(kR)} \tag{3.20}$$

[2]In fact, if the derivative of the radial solution must be continued in the boundary corresponding to the reach of the potential, so does the logarithmic derivative of the radial equation $\frac{1}{u_l(kr)}\frac{du_l(kr)}{dr}$

with:

$$\beta_l = \left(\frac{r}{u_l(kr)} \frac{du_l(kr)}{dr} \right) \Bigg|_{r=R}$$

Once the phase shifts have been described in accordance to the analytical properties of the radial solution $u_l(kr)$, like it was shown in (3.20), the behavior of the phase shifts $\delta_l(kr)$ at low energies can be deduced[3]. Using the properties of the Bessel and Newmann functions , and considering the limit where the quantity kR, being R the strength of the potential, is so less with respect with the angular moment l of the considered partial wave[30](l), it can be shown that the behavior of the radial solution $u_l(kr)$ is determined by the centrifugal barrier term, compared with the contribution of the potential that leads to the decay process $V(r)$; the asymptotic form that takes the phase shift $\delta_l(k)$ with respect to the wavenumber k, is given by[30][4]:

$$\tan(\delta_l) \xrightarrow[k\to 0]{} \frac{(kR)^{2l+1}}{(2l+1)!!(2l-1)!!} \left(\frac{l-1-\hat{\beta}_l}{l+\hat{\beta}_l} \right) \tag{3.21}$$

with $\hat{\beta}_l = \lim_{k\to 0} \beta_l$, $(2l-1)!! = \prod_{i=1}^{l} 2i-1$ and $(2l+1)!! = \prod_{i=1}^{l} 2i+1$.

Hence the phase shifts $\delta_l(k)$ behaves, in the low energy limit as[39, 30, 26]:

$$\delta_l \sim k^{2l+1} \tag{3.22}$$

The relation above is called the "threshold behavior" of the phase shifts δ_l, for a threshold energy equal to zero. Since the wavenumber k depends on the energy as $k \sim \sqrt{2\,m\,E}$,the behavior of the phase shifts at low energies can be related with the energy itself:

$$\delta_l \sim E^{l+\frac{1}{2}} \tag{3.23}$$

The phase shifts δ_l must satisfy the continuity of its derivatives, as they characterize the properties of the **S** matrix in accordance to (3.18); so, the derivative of the phase shifts must exist, in particular, in the low energies domain; deriving the relation above (3.23), the threshold behavior of the derivative of the phase shifts δ_l with respect of the energy can be found:

$$\frac{d\delta_l}{dE} \sim \left(l+\frac{1}{2}\right)\left(E - E_{\text{threshold}}\right)^{l-\frac{1}{2}} \tag{3.24}$$

[3]The low energy scattering process is understood when the incident wavenumber k_i is comparable to or smaller than the range of the potential $V(r)$, so that, the contribution of the partial waves with higher l are unimportant, in accordance to what was explained in the last paragraph[39].

[4]page 82

3.2 Resonances or metastable states

The scattering theory is huge and lets describe different kinds of phenomena. One of the most interesting phenomena is the existence of resonances or "metastable states", as they have been called. As it was mentioned before, the existence of resonances is given by the analytical properties of the **S** matrix: in particular, the resonances are related to the simple poles of the **S** matrix in the so called "unphysical sheet", so close to the real axis; since the **S** matrix is defined in terms of the Jost Function and the propagator related to it, these single poles are the zeros of the Jost function in the upper half of the complex k plane. The most relevant characteristic that leads to the consideration of the existence of a resonance is a peak in the partial cross section; as the phase shifts are related with the **S** matrix, it can be found a relation between δ_l and the total cross section γ_l (3.17), related with each partial wave; in the presence of resonances, even though the phase shifts are considered as smooth varying functions of the energy, they suffer a quick variation for a determined potential in certain interval; hence, in that interval, the cross section related with the scattering amplitude for that partial wave changes dramatically; at low energies, since the partial waves with lower l contribute the most with the scattering amplitude, the total cross section, given by the sum of all the partial cross sections related with each partial wave with angular momentum l suffers a dramatic change, a peak, in the vicinity of the interval where the phase shifts δ_l change, due to the contribution of these partial waves, explicitly.

Particulary, the $\mathbf{S_l}$ matrix, related with the partial wave with angular momentum l, can take the next form:

$$\mathbf{S_l} = e^{2i\zeta_l}\frac{\beta_l - r_l + is_l}{\beta_l - r_l - is_l} \tag{3.25}$$

where the s_l and r_l factors are related with the Hankel functions $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$ in the boundary $r = R$:

$$r_l = \Re[\frac{kh_l'^{(2)}(kR)}{h_l^{(2)}(kR)}] \tag{3.26}$$

and:

$$r_l = \Im[\frac{kh_l'^{(1)}(kR)}{h_l^{(1)}(kR)}] \tag{3.27}$$

and the phase shift ζ_l is defined as:

$$e^{2i\zeta_l} = -\frac{h_l^{(2)}(kR)}{h_l^{(1)}(kR)} \tag{3.28}$$

Since the Hankel functions are related with the Bessel and Newmann functions, that describes the radial solution if the potential vanishes, the ζ_l phase shift can be expressed in terms of the Bessel and Newmann functions as:

$$\zeta_l = \tan^{-1}(\frac{j_l(kR)}{n_l(kR)}) \tag{3.29}$$

Hence, the ζ_l phase shift is not related with the behavior and properties of the potential $V(r)$; this is the reason why it is called the "background" scattering phase shift. This phase shift can be understood as a contribution given by the scattering of the all possible partial waves with different angular momentum from the one that is considered, or an expression for what it has been called the "hard sphere scattering phase shift", which expresses an analogy with the case of a hard sphere potential, since the effective potential $U_{\text{effective}}$ acts like a barrier for some values of l and incident energies. The contribution of the two effects are really small to be considered, in general; nevertheless, there are cases where the appearance of background effects introduces a damping in the total phase shift behavior, as it would be analyzed later.

Meanwhile, the second term of the **S** matrix gives the resonant pole in the "unphysical sheet"; as a matter of fact, since it depends on the logarithmic derivative of the radial solution $u_l(kr)$, on the boundary condition $r = R(\beta_l)$, it can be related with the properties of the considered potential $V(r)$. This second term can be expressed as a phase shift ρ_l, so that:

$$\rho_l = \tan^{-1}(\frac{s_l}{\beta_l - r_l}) \tag{3.30}$$

The phase shift ρ_l defined above is called the "resonant scattering phase shift", meaning the phase shift related with the resonant pole in the "unphysical sheet"; so, the total phase shift δ_l is given by two contributions: the one that originates the resonance as a pole of the **S** matrix, given by ρ_l, and the background scattering phase shift ζ_l; hence, the total phase shift δ_l related with the **S** matrix by (3.18) is given by:

$$\delta_l = \rho_l + \zeta_l \tag{3.31}$$

The resonant scattering phase shift ρ_l can be parameterized in terms of the energy and a function $\Gamma(E)$:

$$\rho_l = \tan^{-1}\frac{\Gamma(E)}{2(E_r - E)} \tag{3.32}$$

If the region near the value of the E_r parameter is taken, the function $\Gamma(E)$ can be considered as a constant. Outside the interval $(E_r - \frac{\Gamma}{2})$ the background scattering overlaps the resonance scattering, so that the incident wave cannot penetrate in the scattering region; within the interval, the resonant scattering domains, and the total phase shift δ_l increases rapidly through $\frac{\pi}{2}$ to π; in accordance to the behavior of the total cross section related with the description of the phase shifts (3.17), it can be shown that the partial cross section exhibits a peak when the value of the energy is so close to the resonance position E_r; in its neighborhood, the angular distribution only depends on the angular momentum, and nothing more; hence, the partial cross section, related with the partial wave with angular momentum l, takes he form:

$$\sigma_l = \Big(\frac{4\pi(2l+1)}{k^2}\Big)\Big(\frac{\Gamma^2}{4(E-E_r)^2\Gamma^2}\Big) \tag{3.33}$$

The function $\left(\frac{\Gamma^2}{4(E-E_r)^2\Gamma^2}\right)$ is known as the Breit Wigner distribution; this distribution is similar to the Normal Gaussian distribution; it exhibits a maximum at $E = E_r$; this quantity is known the energy or position of the resonance.

The Breit Wigner distribution(See Fig. (3.1)) is only an approximation, as the scattering resonant phase shift given by (3.32); in fact, Khalfin used a Breit Wigner distribution as a density of states in his work [7], and found that if the energy spectrum would run from $-\infty$ to ∞, the survival probability $P(t)$ would have an exponential behavior at all times: there would not be any distinction, as only the exponential contribution would remain and characterize the decay process.

$$\omega(E) = \begin{cases} \frac{\Gamma}{(E_r-E)^2+\frac{\Gamma^2}{4}} & \text{For } E \geq E_{\text{threshold}} \\ 0 & \text{For } E < E_{\text{threshold}} \end{cases} \tag{3.34}$$

Nevertheless, as it was stated before, Khalfin found a deviation from the exponential

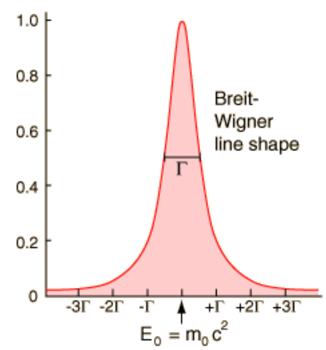

Figure 3.1: Plot of a Breit Wigner distribution [54]

behavior(1.22) in the survival probability $P(t)$ if the energy spectrum is taken from $E_{\text{threshold}}$ to infinity, for this particular density of states (3.34). In particular, what Khalfin found was that if the density of states would be given by (3.34), if the branch cut effects of the propagator would be neglected, and the Fock Krylov theorem(theorem (1.1)) would be applied on (3.34), for a continuum density spectrum from $-\infty$ to ∞, the distinction between short, intermediate and long times related with the behavior of the survival probability $P(t)$ would disappear: In fact, the survival probability $P(t)$ would be exponential at all times; the existence of a branch cut in the real axis, in the analytic domain of the propagator leads to the nonexponential behavior, characterized by the long time tail" in the survival probability behavior, as the residue calculated on the resonant pole of the propagator leads to the exponential contribution of the survival probability.[36]

Since the effects from the scattering and interference of the other partial waves, and the hard sphere scattering are not always regardless, the description of the total phase shift δ_l depends on the importance of these contributions, besides the analytic behavior of the resonance phase shift ρ_l. A Breit Wigner resonance phase shift, defined like (3.32), jumps from an integer value of π below the resonance position E_r to the next integer value of π above it, through the odd multiple of $\frac{\pi}{2}$ between the two of them; that is one of the most relevant characteristics that let define a resonance. The peak in the scattering partial cross section σ_l corresponds to a phase shift equal to an odd multiple of $\frac{\pi}{2}$; in absence of bound states ligated to the potential $V(r)$, the cross section has a peak in $\delta_l = \frac{\pi}{2}$, according to what the Levinson's theorem postulates. Nevertheless, when the background scattering phase shift ζ_l is close to $\frac{\pi}{2}$, the partial wave cross section would exhibit a dip as the resonant phase shift ρ_l goes to π near the position of the resonance E_r; this dip is due to the interference effect between the resonance amplitude and the nonresonance amplitude related with the background phase shift ζ_l; this interference effect between the two contributions of the total phase shift was discovered by Ramsauer and Towsend ; this fact implies that if there is a peak in the cross section, for each partial wave with angular momentum l, σ_l(and of course, a resonance), it must have a partial amplitude a_l equal to zero, so that, there would be no scattering from that partial wave specifically; this effect has been called the "Ramsauer Towsend effect"[44, 45].

The description of the resonant states is given by boundary conditions that the radial function must accomplish, the continuity of $u_l(kr)$ and its derivative inside and outside the potential $V(r)$; if the effective potential acted as an infinite barrier, for low incident energies, the formation of bound states as stationary states would be inherent, analogically to the case of a particle in a box, described quantum mechanically. Nevertheless, the effective potential $U_{\text{effective}}(r)$ gives a finite barrier, in accordance to the behavior of the centrifugal term $\frac{l(l+1)}{r}$. In this case, as a result of tunneling, there could be a possibility that incident particles with enough energy would enter into the region affected by the potential $V(r)$. Nevertheless, the penetration of the centrifugal barrier is small for low energies; if a "bound state" is formed, there is a possibility that this "bound state" decays, if the energy to form it is enough; besides, particles that get in can easily leak out once inside. So, the bound states energies are not sharp at all, since there can't be a formation of bound states at large energies. At low energies, there could be a formation of metastable states, which corresponds to the "bound states" mentioned above, that eventually decays, as there can be tunneling due to the finite centrifugal barrier of the effective potential $U_{\text{effective}}$. Since the appearing of metastable states is related with the behavior of the effective potential $U_{\text{effective}}$, there could not be formation of metastable states for s partial waves at low energies, since there is no centrifugal barrier as $l = 0$. This is the main reason why it is going to be analyzed the partial waves with the lowest l different from zero: the P partial waves($l = 1$), in order to find possible resonance behavior from particular scattering processes.

3.3 The Beth Uhlenbeck theorem and the density of states $\omega(E)$

In accordance to the Fock Krylov theorem, theorem (1.1), the survival amplitude $A(t)$ is defined as the Fourier transform of an entity called "density of states" that represents a continuum probability density to find the eigenstates of the unperturbed hamiltonian $\hat{H}_0$ in the initial state, according to the properties of the space where the set of eigenstates is its basis; the importance to specify the main analytical properties of the density of states, related with the behavior of the potential $V(r)$ that leads to the decay process, gets bigger, as the survival amplitude $A(t)$, as it was discussed in the first chapter, is the principal function to characterize the decay processes. In the second chapter, some of the analytical properties and models of the density of states, defined in (1.16), have been mentioned, in order to treat the decay processes problem, and discuss the physical implications that correspond to a suitable choice of a determine $\omega(E)$, in accordance to the physical constrains that the system must follow in its temporal evolution, given by the Schrödinger equation. It was shown that the difference of the choice of an analytical form of the density of states $\omega(E)$ lets obtain a sort of discrepancies in the time development of the wavefunction that represents the state of considered system, and hence, in the survival probability $P(t)$ related to it, so that, the power law behavior given by the nonexponential contribution of the survival amplitude determined by the analytical properties of the density of states and the physical constrains changes from one model to another.

In addition to that, it has been pointed out the relevance of the partial wave method in order to describe the scattering processes at low energies, where the behavior of the partial waves with lower l is so important in order to determine the physical description of the state of the considered system at any time; in particular, the definition of the phase shifts δ_l, related with the contribution of each partial wave to the $\mathbf{S}$ matrix and the influence of a localized spherically symmetric potential $V(r)$ in the description of the scattering process, and their behavior at low energies, near the threshold energy value, given by (3.23) and (3.24). It would be natural to think that, as there is a close relation between the definition of the phase shifts and their behavior with the analytical properties of the potential $V(r)$, it could be possible to find an expression that relates the phase shifts δ_l with the density of states, specially in the domain near the threshold energy, where the partial waves with lowest l describes the state of the system. Beth and Uhlenbeck [42] worked on the description of an equation of state in statistical mechanics, and when they began to search the second virial coefficient, they found a relation that it has been called the "Beth Uhlenbeck theorem"; this theorem says that there is a relation between the density of states $\omega(E)$ of an interacting system $\eta_l(k_{CM})$ and the rate of change of the phase shifts δ_l with respect to the center of mass energy[43, 42, 18, 46, 47]:

$$\eta_l(k_{CM}) - \eta_l^0(k_{CM}) = \frac{2l+1}{\pi}\frac{d\delta_l}{dE_{CM}} \tag{3.35}$$

where $\eta_l^0(k_{CM})$ is the density of states without interaction, k_{CM} is the momentum in the barycentric or center of mass frame, and E_{CM} is the energy of the center of mass.

As it was discussed in the earlier subsection, the appearance of a resonance depends on the characteristics and properties of the potential $V(r)$; if the potential vanishes, there is no way a resonance can be produced by the scattering process, since the contribution of each partial wave, given by the set of phase shifts $\delta_l(E_{CM})$ would be identically zero. So that, in accordance to the relation (3.35), when the potential $V(r)$ vanishes, the interacting density of states $\eta_l(k_{CM})$ tends to be the density of states of a noninteracting system $\eta_l^0(k_{CM})$; meanwhile, in accordance to the definition of the density of states $\omega(E)$ as a continuum probability density of the eigenstates of the unperturbed hamiltonian $\hat{H}_0$ in a resonance, as the potential $V(r)$ that leads the decay is switched down, the probability to find an eigenstate of $\hat{H}_0$ in the resonance state is zero, because there could not be a formation of resonance. Following these ideas, since the difference between the density of states $\eta_l(k_{CM})$ and $\eta_l^0(k_{CM})$ is positive for a resonance description($\eta_l(k_{CM}) - \eta_l^0(k_{CM}) \geq 0$), it can be assured that:

$$\omega(E_{CM}) \approx \frac{d\delta_l}{dE_{CM}} \tag{3.36}$$

This last relation (3.36) guarantees the connection between the experimental data that can be found from scattering processes, and the abstract definition of the density of states $\omega(E)$, which is necessary in order to obtain the description of the decay processes through the survival probability $A(t)$, in accordance to what the Fock Krylov theorem says (theorem (1.1)); since the partial wave analysis is more important when it describes the processes at low energies, near the threshold, the definition of the density of states $\omega(E)$ is quite general and doesn't depend explicitly on a determined form of the potential $V(r)$; nevertheless, the description of the density of states in accordance to the Beth Uhlenbeck relation (3.35) is better, if there is an isolated resonance in the domain which is going to be considered. If there are overlapping resonances, due to the definition of the $\mathbf{S}$ matrix (3.18), the distinction of the phase shift related with each partial wave would be difficult to express, so that, there could be interference effects between the partial waves that describes one particular resonant state, and the other; therefore, the Beth Uhlenbeck relation would not be accurate, since the $\frac{d\delta_l(E_{CM})}{dE_{CM}}$ would have several maxima and minima; these minima are related with the bound states appearing between resonances[2]. In order to avoid the effects of overlapping resonances, the energy interval that is taken to be analyzed extends from the threshold energy to the first isolated resonance appearing in the scattering cross section. So, that is the reason why it must be taken data at low energies, near the threshold term, so that, an isolated resonance would appear in the partial cross section, and a peak in $\frac{d\delta_l(E_{CM})}{dE_{CM}}$.

Since the partial wave analysis determines accurately the description of the scattering processes at low energies, the temporal behavior which is interesting to analyze by this description would be the long time behavior, due to the analytical properties of the phase shifts defined above (3.24); in fact, all it would be needed to extract from the experimentation is the behavior of the δ_l near the threshold and in the vicinity of the isolated resonance; hence, the long energy behavior of the survival probability $P(t)$ and the short time behavior related to it cannot be described accurately by the partial wave analysis and the properties of the phase shifts; so that, the behavior of the density of states at long energies is not relevant to determine the asymptotic behavior of the survival amplitude $A(t)$ at large times[18], even though it is known that the survival probability must fall off. Therefore, it would not be considered at all for the purposes of the present work.

3.4 The description of the survival probability $P(t)$ from the Beth Uhlenbeck formalism

The partial wave method is useful to analyze the scattering processes at low energies, and, in accordance to the Beth Uhlenbeck formalism(3.35), the phase shift behavior is related with the definition of the density of states $\omega(E)$, it is necessary to express the analytic properties of the density of states as a continuum probability density, as the survival amplitude $A(t)$ can be defined from the Fock Krylov theorem (theorem (1.1)), and the evolution of the system is related with the physical constrains the system must suffer; in particular, the energy spectrum doesn't run from $-\infty$ to ∞, but from the threshold energy, the minimum energy required to guarantee the existence of the scattering process, to ∞. It was explained in the last section that if the behavior of the phase shifts δ_l is considered, the density of states related to it lets to analyze the behavior of the survival amplitude $A(t)$ where the phase shifts are well behaved; hence, the partial wave analysis and the description of the density of states in terms of the phase shifts of each partial wave given by Beth and Uhlebeck is accurate at low energies; specially in systems where there is an isolated resonance near threshold, and the restriction over large times, that corresponds to small energies. The phase shifts can be extracted from different scattering experiments, even though the near threshold domain has not been taken into account, since some of the experimentalist have thought that it is not a spectrum region with physical importance.

The density of states must behave in accordance to the properties of the phase shifts, so that, the Beth Uhlebeck relation (3.35) and the asymptotic behavior of the phase shifts near threshold (3.24) must be satisfied; it is important to notice that, in order to simplify the analysis, the frame where the resonance is at rest must be taken; this frame corresponds to the barycentric frame or the so called "center of mass" frame. In order to guarantee the analytical properties of the density of states $\omega(E)$, it can be parameterized as :

$$\omega(E) = \mathcal{G}(E_{CM})(E_{CM} - E_{\text{threshold}})^{\gamma(l)} \tag{3.37}$$

The second term expresses the behavior of the phase shifts δ_l near the threshold energy, in accordance to (3.24); meanwhile, the first term must be related with the properties of the propagator that determines the temporal evolution of the physical system; so that, the function $\mathcal{G}(E_{CM})$ must behave so that:

- $\mathcal{G}(E_{\text{threshold}})$ must be different to zero, since the contribution of the threshold energy has been factorized in order to express the analytical properties of the density of states $\omega(E)$ near the threshold
- $\mathcal{G}(E_{CM})$ must tend to zero as E_{CM} goes to ∞, in order to guarantee the convergence of the survival probability $A(t)$, in accordance to the Fock Krylov theorem (theorem (1.1))
- Since the density of states is related with the properties of the propagator, the function $\mathcal{G}(E_{CM})$ can be allowed to have simple poles $z_{\text{pole}} = E_r - i\frac{\Gamma}{2}$ in the complex plane, given that the derivative of the phase shifts (3.24) carries the information about the poles [48, 49]. In particular, the resonant pole in the "unphysical sheet" gets importance in accordance to the chosen contour

In order to calculate the survival probability, the energy spectrum must be taken as positive, so that $E \geq E_{\text{threshold}}$. Hence, the survival amplitude can be defined as:

$$A(t) = \int_{E_{\text{threshold}}}^{\infty} \mathcal{G}(E_{CM})(E_{CM} - E_{\text{threshold}})^{\gamma(l)} e^{-iE_{CM}t} dE_{CM} \tag{3.38}$$

A change of variables can be introduced, so that , the limits of the integration corresponds from zero to infinity:

$$A(t) = e^{-iE_{\text{threshold}}t} \int_{0}^{\infty} \mathcal{G}(x + E_{\text{threshold}})(x)^{\gamma(l)} e^{-iExt} dx \tag{3.39}$$

Kelkar, Nowakowski and Khemchandani [18] chose the contour showed in the figure(Fig. (2.5)) in order to calculate the integral (3.39); this contour, as it was explained before, has three contributions: One line integral over the real axis, from 0 to infinity C_1; another line integral over the imaginary axis, from $-\infty$ to zero(C_2); and a third contribution from a quarter of circle in the fourth quadrant, enclosing the resonant pole $z_{\text{polemod}} = z_{\text{pole}} - E_{\text{threshold}}$, which is the pole of the function $\mathcal{G}(x + E_{\text{threshold}})(C_3)$; the integral over the quarter of circle vanishes if the radius of the domain goes to infinity, to cover the entire "unphysical sheet". Meanwhile, the C_1 line integral is precisely, the survival amplitude $A(t)$ given by (3.39); in accordance to the residues theorem [28], the survival amplitude can be defined as:

$$\begin{aligned} A(t) =& 2\pi i e^{-iE_{\text{threshold}}t} Res[\mathcal{G}(z+E_{\text{threshold}})(z)^{\gamma(l)}e^{-iEzt}, z_{\text{polemod}}] - \\ & e^{-iE_{\text{threshold}}t}\oint_{C_2}\mathcal{G}(z+E_{\text{threshold}})(z)^{\gamma(l)}e^{-izt}dz \end{aligned} \tag{3.40}$$

The residue of the function can be calculated, in accordance to the properties of $\mathcal{G}(z+E_{\text{threshold}})$:

$$Res[f[z], z_{\text{polemod}}] = \lim_{z\to z_{\text{polemod}}} \mathcal{G}(x+E_{\text{threshold}})(x)^{\gamma(l)}e^{-iExt}(z-z_{\text{polemod}}) \tag{3.41}$$

The L'hôpital rule can be applied to calculate the limit, and determine the residue; hence:

$$Res[f[z], z_{\text{polemod}}] = e^{iE_{\text{threshold}}t}\frac{(E_r - i\frac{\Gamma}{2} - E_{\text{threshold}})^{\gamma(l)}e^{-i(E_r-i\frac{\Gamma}{2})t}}{\left.\frac{d\mathcal{G}(z+E_{\text{threshold}})}{dz}\right|_{z=E_r-i\frac{\Gamma}{2}-E_{\text{threshold}}}} \tag{3.42}$$

It is important to notice that the nonexponential contribution of the survival amplitude $A(t)$, given by the residue calculated on the pole of the $\mathcal{G}(z+E_{\text{threshold}})$ is similar like the one Nakazato, Namiki and Pascazio found in their work[16](2.105); therefore, if the two relations are compared, it can be deduced that the $\mathcal{G}(z+E_{\text{threshold}})$ can be understood as a spectral density, so that its analytical properties are related with the temporal evolution of the system, and the form of the propagator itself.

Once the residue on the resonant pole has been determine, the next step is to calculate the line integral on the imaginary axis, from $-i\infty$ to zero, in order to find an analytical expression for the survival amplitude $A(t)$. So, since the line integral C_2 ought to be calculated, the complex variable z must be parameterized as $z = iy$, running from $-\infty$ to zero to cover the entire imaginary axis:

$$\begin{aligned} & e^{-iE_{\text{threshold}}t}\oint_{C_2}\mathcal{G}(z+E_{\text{threshold}})(z)^{\gamma(l)}e^{-iEzt}dz = \\ & \quad - ie^{-iE_{\text{threshold}}t}\int_{-\infty}^{0}\mathcal{G}(iy+E_{\text{threshold}})(iy)^{\gamma(l)}e^{yt}dy \end{aligned} \tag{3.43}$$

A change of variables must be done, in order to take the integral from zero to infinity:

$$\begin{aligned} e^{-iE_{\text{threshold}}t}\oint_{C_2}\mathcal{G}(z+E_{\text{threshold}})(z)^{\gamma(l)}e^{-iEzt}dz = -(-i)e^{-iE_{\text{threshold}}t} \\ \int_{0}^{\infty}\mathcal{G}(-iy+E_{\text{threshold}})(-iy)^{\gamma(l)}e^{-yt}dy \end{aligned} \tag{3.44}$$

The integral described above can be solved through the integration by parts method; since the Gamma function $\Gamma(z)$ is defined by (2.72)[24, 50], the integral defined above can be expressed like:

$$e^{-iE_{\text{threshold}}t}\oint_{C_2}\mathcal{G}(z+E_{\text{threshold}})(z)^{\gamma(l)}e^{-iEzt}dz = -(1+i)(-i)^{\gamma(l)+1}e^{-iE_{\text{threshold}}t} \left(\frac{\Gamma(\gamma(l)+1)}{t^{\gamma(l)+1}}\mathcal{G}(-iy+E_{\text{threshold}})\right)\Bigg|_0^{\infty} \tag{3.45}$$

In accordance to the properties of the function $\mathcal{G}(E_{CM})$, as the energy becomes larger, the function tends to zero in order to guarantee the convergence of the survival amplitude $A(t)$, in accordance to the Fock Krylov theorem (theorem (1.1)); so:

$$e^{-iE_{\text{threshold}}t}\oint_{C_2}\mathcal{G}(z+E_{\text{threshold}})(z)^{\gamma(l)}e^{-iEzt}dz = -(1+i)(-i)^{\gamma(l)+1}e^{-iE_{\text{threshold}}t} \left(\frac{\Gamma(\gamma(l)+1)}{t^{\gamma(l)+1}}\mathcal{G}(E_{\text{threshold}})\right) \tag{3.46}$$

Hence, the survival amplitude must be given by:

$$A(t) = 2\pi i\frac{(E_r - i\frac{\Gamma}{2} - E_{\text{threshold}})^{\gamma(l)}e^{-i(E_r)t}}{\frac{d\mathcal{G}(z+E_{\text{threshold}})}{dz}\Big|_{z=E_r-i\frac{\Gamma}{2}-E_{\text{threshold}}}}e^{-\frac{\Gamma}{2}t} + (1+i)(-i)^{\gamma(l)+1}e^{-iE_{\text{threshold}}t}\left(\frac{\Gamma(\gamma(l)+1)}{t^{\gamma(l)+1}}\mathcal{G}(E_{\text{threshold}})\right) \tag{3.47}$$

As the relation (3.49) shows, there are two different contributions that define the behavior of the survival amplitude $A(t)$; the exponential part, which leads to the classical law that describes the decay process, and the nonexponential part, which is given by the power law determined by the calculation of the integral over the imaginary axis. As long as the time becomes larger, the exponential contribution goes to zero, meanwhile the nonexponential contribution arises and defines the temporal behavior of the survival amplitude $A(t)$ and the survival probability $P(t)$ in accordance to (1.5); indeed, the tail of the survival probability at long times compared with the mean lifetime of the resonance is proportional to $t^{-(2\gamma+2)}$, from (3.49) and (1.5). At intermediate times, the exponential contribution dominates the nonexponential one, so the survival probability $P(t)$ shows naturally an exponential behavior. But, when the time begins to grow larger, the exponential contribution becomes smaller, since the exponential function converges faster than the temporal power law, implicit in the nonexponential contribution of the survival amplitude. So, a "tail" of a power law starts to be relevant in the temporal description of the decay process, and the behavior of the survival probability $P(t)$. It is important to realize that the $\gamma(l)$ exponent depends on the value of the angular momentum l ,and, hence, on the behavior of the partial wave with specific l; it was explained in the last section the possibility of a nonresonance or background phase shift presence, from the interference and scattering effects from the other phase shifts, and the hard sphere scattering from the partial wave that is currently analyzed in order to determine the existence of a resonance, and a damping effect on the values of the phase shifts, due to this background effect. So, since the nonexponential contribution is determined by the properties of the integral of the density of states defined by (3.37), it must show the behavior of the phase shifts near the threshold (3.24), in accordance to the Beth Uhlenbeck relation (3.35), (3.36). Therefore, if the relations (3.24), (3.36) and (3.37) are compared, it can be deduced a value of the γ coefficient in terms of the angular momentum related with each partial wave l. Hence, the parameter γ would be equal to $l - \frac{1}{2}$, so that, the threshold behavior of the phase shifts δ_l would be satisfied:

$$
\begin{aligned}
A(t) = 2\pi i \frac{(E_r - i\frac{\Gamma}{2} - E_{\text{threshold}})^{l-\frac{1}{2}} e^{-i(E_r)t}}{\left.\frac{d\mathcal{G}(z+E_{\text{threshold}})}{dz}\right|_{z=E_r - i\frac{\Gamma}{2} - E_{\text{threshold}}}} e^{-\frac{\Gamma}{2}t} + \\
(1+i)(-i)^{l+\frac{1}{2}} e^{-iE_{\text{threshold}}t} \left(\frac{\Gamma(l+\frac{1}{2})}{t^{l+\frac{1}{2}}} \mathcal{G}(E_{\text{threshold}}) \right)
\end{aligned}
\tag{3.48}
$$

What would happen if, there would be no threshold term in the definition of the density of states $\omega(E)$ (3.37), so that, the $\gamma(l)$ exponent would be zero? If that was the case, the survival amplitude would take this form:

$$
\begin{aligned}
A(t) = 2\pi i \frac{e^{-i(E_r)t}}{\left.\frac{d\mathcal{G}(z+E_{\text{threshold}})}{dz}\right|_{z=E_r - i\frac{\Gamma}{2} - E_{\text{threshold}}}} e^{-\frac{\Gamma}{2}t} + \\
(1+i)(-i) e^{-iE_{\text{threshold}}t} \left(\frac{1}{t} \mathcal{G}(E_{\text{threshold}}) \right)
\end{aligned}
\tag{3.49}
$$

So, the nonexponential contribution to the survival amplitude $A(t)$ would be proportional to $\frac{1}{t}$; as the nonexponential contribution dominates the exponential one at large times, the survival probability $P(t)$ will exhibit a power law in that domain proportional to $\frac{1}{t^2}$; this "tail" in the behavior of the survival probability $P(t)$ was the temporal power law that Khalfin (1.22) [7] and Nicolaides and Beck (2.57) [33] found in their work. Hence, a long time "tail" of the survival probability $P(t)$ proportional to $\frac{1}{t^2}$ represents a density of states $\omega(E)$ that does not have a threshold energy depending factor, so that, the asymptotic behavior that the phase shifts must follow at low energies (3.22) are not considered in order to determine the temporal evolution of the physical system.

It has been discussed the importance of the partial waves with lower l as the main contribution to the description of the scattering processes at low energies; so, in order to find resonances from experimental data, it is necessary to look for data corresponding to the lower l partial waves, in order to avoid the centrifugal barrier effect. Since each partial wave contributes to the survival amplitude in accordance to (3.48), the main contribution to the "tail" of the survival amplitude $A(t)$ would be given by the partial wave with lowest l, where a resonance behavior could be seen. In particular, the main partial waves to be observed would be the P partial waves ($l = 1$), since the S partial waves($l = 0$) don't have a centrifugal barrier $U_{\text{effective}}(r) = V(r)$ that let form a metastable state. For that particular phase shift, the survival probability tail at long times, calculated theoretically in accordance to (3.49), would be:

$$P(t) \underset{t\to\infty}{\sim} \frac{1}{t^3} \tag{3.50}$$

How accurate is this procedure? A particular form of a potential was not considered in order to obtain the survival amplitude $A(t)$ to characterize the decay of the metastable state. So, in order to check the validity of this model, it is necessary to take experimental data from different resonance formation processes, near the threshold as the partial wave analysis demands, in order to find out the long time behavior of the survival probability $P(t)$, and analyze the description of the phase shifts δ_l near the threshold energies.

CHAPTER 4

Survival probability $P(t)$ at large times from phase shift data

Although there are so many data from different kinds of scattering processes, the available phase shift data near the threshold behavior are only a few percent of the totality of the experimental data; the experimentalists understand that the threshold domain is not important in a scattering process for the characterization of some of the most important variables to be analyzed, like the scattering cross sections, and the polarization of the beams after the processes. Since the explained model in the last chapter analyzes the long temporal behavior of the survival probability $P(t)$ related with the properties of the phase shifts near the threshold of the scattering process, it is necessary to find scattering reactions which experimental data are restricted to the near threshold domain. In particular, two reactions would be analyzed in depth:

- The α particle - proton scattering process that creates the nuclear resonance ${}^{5}_{3}\mathrm{Li}$
- The α particle - neutron scattering process that creates the nuclear resonance ${}^{5}_{2}\mathrm{He}$

These scattering processes create the nuclear resonance that, after a time corresponding on the width of the resonance, emits the two initial particles. So, in order to analyze the proper development of the model based on the Beth Uhlenbeck theorem, it is necessary to find some experimental data in a region near the threshold, and then, in accordance to (3.24) and (3.36), find the survival amplitude from the Fock Krylov theorem (theorem (1.1)); in this work, a numerical integration would not be needed to determine the survival amplitude, once the fit to the phase shift data is made with a particular energy depending function, the exponent in the threshold factor will decide the power law behavior as can be seen from the analytical expressions of the survival amplitude $A(t)$ (3.48), and its accuracy would determine the behavior of the survival probability $P(t)$.

4.1 Experimental phase shifts obtained from scattering processes

Even though, as it was mentioned above, there are only a few available phase shifts data for scattering processes near the threshold energy domain, for the purposes of this work, the data published by Ardnt, Roper and Shotwell[52], in the case of the ${}^{5}_{3}$Li and Ardnt and Roper[51] in the case of ${}^{5}_{2}$He, are going to be used. Nevertheless, as it was also discussed before, the best reference frame to consider for the analysis of the evolution of the system in accordance to the laws of the quantum mechanics is precisely the center of mass frame, where the resonance is at rest. Therefore, since the incident energies, considered in the references to do the analysis, are in the lab system, where the target(proton in the case of ${}^{5}_{3}$Li, and neutron in the case of ${}^{5}_{2}$He) is at rest, it is necessary to make the conversion to the center of mass frame.

4.1.1 Conversion to the center of mass frame

Since the considered systems are relativistic, there can be found a relation that illustrates the pass from one reference frame to another, so that the laws of physics would take the same form in each one of the reference frames. One of the scalars defined under a Lorentz transformation is the center of mass energy E_{CM}; hence, it is convenient to express the relations that describe the temporal evolution of the system in terms of the center of mass energy. The center of mass energy is independent of the frame it is used to analyze the system; as a matter of fact, it can be defined as:

$$E_{CM}^2 = (E_t)^2 - (k_t^2) \tag{4.1}$$

In natural units, where $\hbar = c = 1$, and E_t is defined as the total energy measured in that particular frame, and k_t is the total momentum of the system measured in the frame.

In particular, if the analysis is made on the lab system, the total energy E_t would be the sum of the total energy of the incident particle, and the rest mass of the target; meanwhile, the total momentum k_t would be the momentum of the incident particle; hence, the center of mass energy would be:

$$E_{CM}^2 = (E_a + m_b)^2 - (k_a^2) \tag{4.2}$$

where E_a is the total energy of the incident particle, k_a is the momentum of the incident particle in the lab system and m_b is the rest mass of the target.

Since $E_a^2 - k_a^2 = m_a^2$, an expression can be found, that relates the center of mass energy and the total energy of the incident particle in the lab system:

$$E_{CM} = \sqrt{m_a^2 + 2(m_b)(E_a) + (m_b)^2} \tag{4.3}$$

In addition to the total energy of the incident particle in the lab system, E_a, it can be found a relation between the kinetic energy of the incident particle $T_a = E_a - m_a$ in terms of the center of mass energy:

$$E_{CM} = \sqrt{(m_a + m_b)^2 + 2(m_b)(T_a)} \tag{4.4}$$

This last relation (4.4) is the one it is going to be used in order to change the experimental data[51, 52], from the lab system to the frame where the nuclear resonance is at rest: the center of mass reference frame.

4.1.2 Phase shifts obtained from the $p + \alpha$ scattering(${}^5_3$ Li)

Once the transformation relation between the two frames of reference has been found (4.4), the data can be transformed in the center of mass system, where the metastable state is at rest.

The data contained in the next table (table(4.1)) are the phase shifts δ_l(measured in degrees) obtained from a elastic scattering between an α particle and a proton as a target. The energy values listed in the table corresponds to the kinetic energy T_a of the incident particle in the lab system [52]:

4.1.3 Phase shifts obtained from the $n + \alpha$ scattering(${}^5_2$ He)

In addition to the $p+\alpha$ scattering process considered in order to find out the possible behavior of the phase shifts near the threshold energy, the $n+\alpha$ elastic scattering process that creates the nuclear resonance ${}^5_2$ He is analyzed to determine the possible behavior near the threshold, and therefore, find some pertinent information about the analytical properties, in accordance to the Beth Uhlenbeck theorem (3.35); these data were taken from different scattering processes that different authors have done; nevertheless, Ardnt, Roper and Shotwell[51] collected the data and tried to analyze them, selecting some data in particular, according to the physical properties they considered interesting. For the purposes of the present work, the phase shift data were taken from this reference [51]: The table ((4.3)) shows the experimental phase shift data extracted from the α neutron for the P partial waves.

4.2 Analysis of the experimental data

Once the phase shifts from the scattering data are available, it is necessary to identify the partial waves where the metastable state can be determined, in accordance to the analytical properties of the phase shifts near the threshold energy. The best method to identify the formation of a resonance is through the plotting of the phase shift in order to analyze their behavior in the center of mass frame.

Energy(MeV)	$S_{\frac{1}{2}}$(degrees)	$P_{\frac{1}{2}}$(degrees)	$P_{\frac{3}{2}}$(degrees)	$D_{\frac{3}{2}}$(degrees)
0.94	-10.5 ± 0.39	0.508 ± 0.36	5.02 ± 0.32	
1.49	-18.3 ± 0.60	4.31 ± 0.65	20.05 ± 0.45	
1.70	-22.6 ± 0.85	7.56 ± 0.68	28.15 ± 0.72	
1.97	-22.9 ± 0.77	7.17 ± 0.36	47.46 ± 0.57	
2.18	-26.02 ± 1.48	9.217 ± 0.42	57.87 ± 1.50	
2.53	-28.33 ± 1.40	12.32 ± 0.66	78.06 ± 1.55	
3.006	-30.41 ± 0.40	15.01 ± 0.30	96.71 ± 0.518	
3.47	-33.14 ± 1.76	22.29 ± 2.34	106.7 ± 2.12	
4.006	-38.06 ± 0.38	24.57 ± 0.41	108.7 ± 0.49	
4.5	-40.14 ± 1.39	29.91 ± 0.66	113.2 ± 1.43	
6.02	-47.47 ± 0.32	42.94 ± 0.27	114.2 ± 0.37	
7.967	-55.57 ± 0.30	52.56 ± 0.35	111.92 ± 0.40	
8.5	-57.67 ± 0.57	54.61 ± 0.88	111.06 ± 0.98	
9.89	-64.37 ± 0.75	54.41 ± 0.74	106.44 ± 0.75	
9.89	-64.37 ± 0.75	54.41 ± 0.74	106.44 ± 0.75	
9.89	-62.69 ± 0.85	56.14 ± 0.38	107.91 ± 0.44	
10.00	62.57 ± 0.78	56.85 ± 1.2	108.43 ± 1.3	
11.16	-74.55 ± 1.6	53.50 ± 1.3	97.48 ± 1.72	-0.41 ± 0.77
12.04	-71.42 ± 0.2	53.35 ± 0.71	101.39 ± 0.47	-0.90 ± 0.26
14.23	-77.97 ± 0.51	53.54 ± 1.2	97.32 ± 0.96	-0.34 ± 0.58
17.45	-80.00 ± 1.00	60.20 ± 1.22	99.81 ± 1.1	3.29 ± 0.53
22.15	-87.56 ± 3.64	53.60 ± 3.27	91.73 ± 0.96	7.96 ± 1.28
22.41	-89.55 ± 3.33	50.58 ± 2.99	86.78 ± 3.11	6.96 ± 1.10
22.7	-94.03 ± 3.52	48.84 ± 3.16	86.23 ± 3.56	8.72 ± 1.12
22.93	-94.00 ± 2.22	51.63 ± 2.33	92.38 ± 2.72	13.42 ± 1.13
23.06	-93.09 ± 3.37	51.77 ± 2.79	89.60 ± 2.34	15.12 ± 1.41

Table 4.1: Experimental phase shifts δ_l extracted from the $p + \alpha$ scattering process [52]

4.2.1 Phase shifts and the formation of metastable states

The formation of a metastable state or resonance is characterized by the phase shifts behavior, as the phase shift suffers a rapid variation from zero to π through $\frac{\pi}{2}$(a "jump") (module π), if the nonresonant scattering contribution given by the background scattering phase shifts are so small to be considered. It was discussed earlier, in the last chapter, that the behavior of the phase shift is related with the centrifugal barrier that depends on the angular momentum l, which characterizes the partial wave. In the particular case of the S partial wave, there is no contribution to the effective potential by the centrifugal barrier, so there cannot be a metastable state that decays by tunneling through the barrier, since $l = 0$, for a localized potential with a finite barrier; even though, there can be a phenomena related with the S partial waves, called S partial

Energy(MeV)	$S_{\frac{1}{2}}$(degrees)
1.015	-24.00 ± 0.40
2.00	-34.64 ± 0.39
2.44	-38.24 ± 0.30
6.00	-60.37 ± 0.17
10.00	-71.8 ± 1.1
12.00	-66.6 ± 1.3
16.4	-105 ± 12

Table 4.2: Experimental phase shifts δ_l extracted from the $n + \alpha$ scattering process [51],for the $s_{\frac{1}{2}}$ partial wave

Energy(MeV)	$P_{\frac{1}{2}}$(degrees)	$P_{\frac{3}{2}}$(degrees)
0.202	0.5	2.8
0.303	0.9	4.6
0.402	1.2	7.2
0.501	1.4	12.2
0.599	2.6	18.6
0.704	2.8	26.4
0.799	4.0	34.9
0.899	4.5	47.5
1.008	5.8	62.5
1.106	6.5	74.9
1.207	6.0	85.4
1.306	13.5	98.6
1.700	12.7	111.0
1.961	14.6	118.4
2.200	17.7	119.4
2.454	21.0	122.4
2.980	28.8	123.9
5.028	46.1	118.1
5.505	49.6	117.4
5.988	51.3	115.7
6.523	52.1	113.7
7.013	53.0	112.2

Table 4.3: Experimental phase shifts δ_l extracted from the $n+\alpha$ scattering process [51] for the $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ partial waves

wave scattering; the phase shift associated with the S partial wave must tend to zero as k_{CM} goes to ∞, in order to guarantee the analytical properties of the **S** matrix, as an asymptotic behavior.

Meanwhile, in the presence of a bound state, the phase shifts associated with a partial wave behaves different, in accordance to the analytical properties of the effective potential $U_{\text{effective}}(r)$: Indeed, if the potential does not bind a bound state, the phase shift related with that partial wave in particular does not reach $\frac{\pi}{2}$; meanwhile, if the potential barely binds, the phase shift goes through $\frac{\pi}{2}$ in the low energy region. If a bound state is created in a particular partial wave, the phase shift value at zero energy must be equal to π and starts to tend to zero, as the energy goes larger[26]. As a matter of fact, the Levinson's theorem says that each newly introduced bound state raised the zero energy phase shift by π; so, if there is a bound state, the zero energy phase shift must be raised by π.

In particular, if the phase shifts related with the S partial wave of the proton α scattering and the neutron α scattering, are taken and plotted against the center of mass energy, (table (4.2) and table (4.1)), what it could be noticed is that the phase shifts don't increase their value as the energy grows larger. In fact, the S wave phase shift does not have a zero value at zero energy, but it doesn't have a π value either; that fact implies that the strength of the potential is not enough to create a bound state, for that particular phase shift; in absence of a bound state, the phase shift must be zero at zero energy, in accordance to the Levinson theorem; but, as the plots (Fig. (4.1)) are analyzed, there can't be a bound state, since the S wave phase shift at zero energy is not π, as it was discussed earlier. Besides, if the S wave phase shifts plots are analyzed for the two considered scattering processes (Fig. (4.1)), what it can be shown is that there is no jump in the value of the phase shifts, or a rising behavior; therefore, there can't be formation of metastable states or resonances in those phase shifts, that agrees with the theoretical behavior of the S wave phases described in the quantum scattering theory.

Since there is evidence that leads to the conclusion of the absence of resonance behavior in the S wave phase shifts (Fig. (4.1)), the next phase shifts corresponding to the P partial wave must be analyzed. There are two partial waves related with $l = 1$: The $P_{\frac{1}{2}}$ partial wave and the $P_{\frac{3}{2}}$ partial wave. In the case of the $P_{\frac{1}{2}}$ partial wave for both scattering processes(Table (4.1) and Table (4.3)), if the experimental phase shifts are plotted against the center of mass energy, as it is shown in the figures (Fig. (4.3) and Fig. (4.3)), what it can be seen is that the phase shifts arise from zero to an intermediate value(in the case of the $p + \alpha$ scattering, 60 degrees), and then begin to go down slowly. The phase shifts do not reach 90 degrees at their highest value, but a less one, before they start to fall. What would be the meaning of this behavior?

It was discussed earlier that, in accordance to the Levinson's theorem (3.19), the presence of a bound state arises the value of the phase shifts at zero energy by π[1]. Nevertheless, the experimental phase shifts, extracted from the scattering processes,

[1]in radians

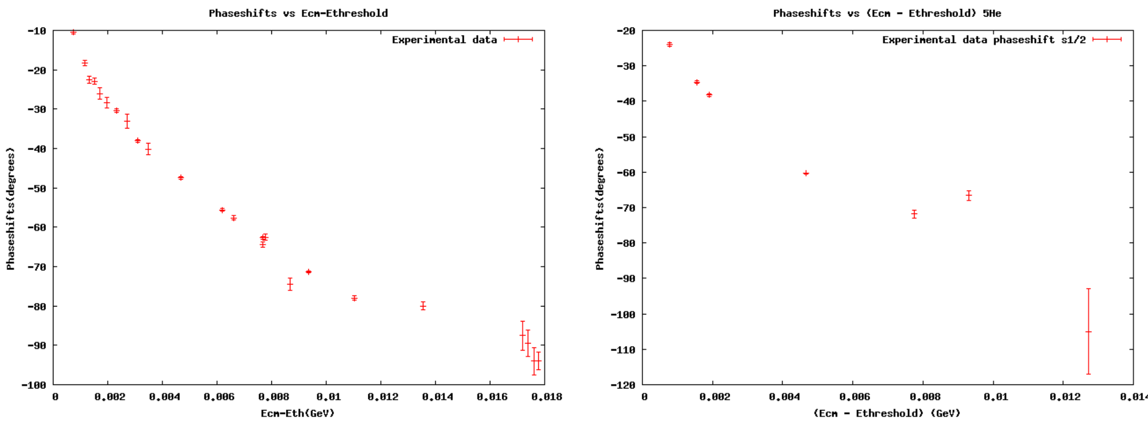

Figure 4.1: Plots of the S wave phase shifts given in the table (Table (4.1)), as a function of the center of mass energy, for proton α scattering and neutron α scattering

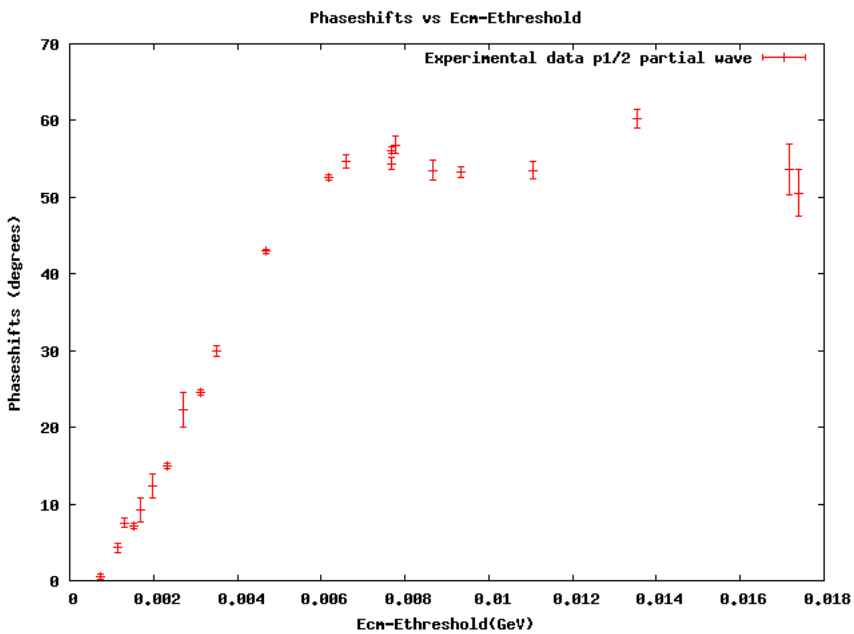

Figure 4.2: Plot of the $P_{\frac{1}{2}}$ wave phase shifts given in the table (Table (4.1)), as a function of the center of mass energy, for $p + \alpha$ scattering processes

are not greater than π or 180 degrees. Hence, there is no bound state associated with the considered partial wave ($P_{\frac{1}{2}}$). But, what it is important to analyze is the presence of a resonance, or its formation, according to the experimental phase shifts behavior. It seems so clear that the phase shifts rises their values as the center of mass energy increases; nevertheless, the phase shifts don't reach the $\frac{\pi}{2}$ limit[2], and start to fall when they reach intermediate values below. It was discussed earlier that one of the most

[2] in radians

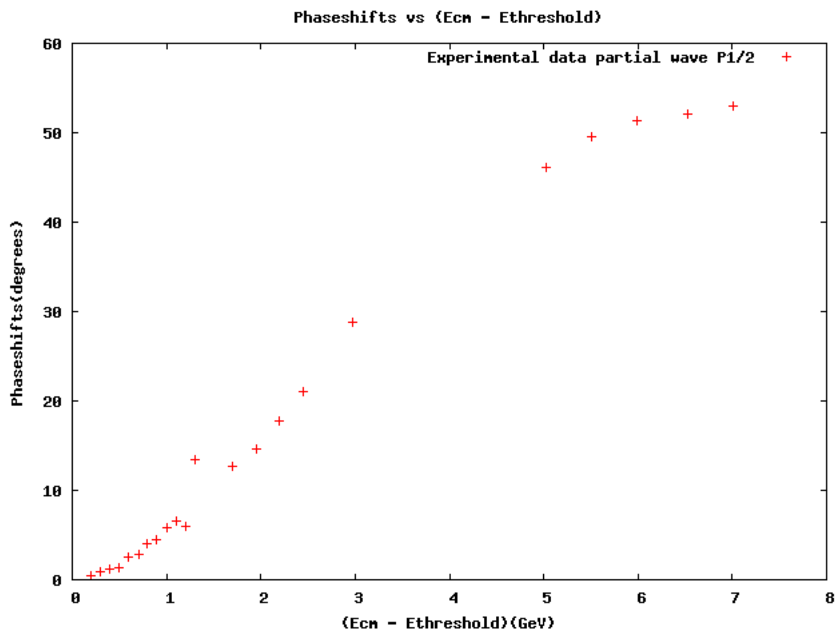

Figure 4.3: Plots of the $P_{\frac{1}{2}}$ wave phase shifts given in the table (Table (4.3)), as a function of the center of mass energy, for $n + \alpha$ scattering process

important characteristics that can showed the formation of a resonance is the phase shift behavior at low energies: The phase shifts related with an specific partial wave, so that $l \neq 0$, suffer a jump, in absence of a bound state formation. What the plots showed (Fig. (4.3) and Fig. (4.2)) is the presence of the rising of the value of the phase shift behavior. As it was mentioned before, the phase shifts start to increase their value until they take an intermediate limit below $\frac{\pi}{2}$, in this particular case 60 grades for both nuclear resonances; then, they start to fall again slowly, or stay constant, but their growth stops once they have reached their maximum. With that fact in mind, what it can be considered is the presence of a metastable state in this particular partial wave, because, even though the phase shift related with that partial wave does not reach the $\frac{\pi}{2}$ limit, its value rises as the center of mass energy increases; when it reaches a certain limit, the phase shift takes a constant value, and then starts to go down. But a questions emerges: If there is a metastable state formation associated with the phase shift behavior, related with the $P_{\frac{1}{2}}$ partial wave, why does the phase shift reach the intermediate value below $\frac{\pi}{2}$ as a superior limit, instead of the π value described in the scattering theory as a characterization of a resonance? Well, it is necessary to observe that in the experimental phase shift data extracted from the considered elastic scattering processes corresponds to the total phase shift δ_l defined by (3.31), the background scattering phase shifts are included, and their behavior affects the definition of the experimental phase shifts measured in the elastic scattering processes, as it was explained in the last chapter: The introduction of the background effects, like the "hard sphere" scattering phase shift related with the analyzed partial wave or the interaction between the other partial waves with the one that is analyzed can affect the description of the scattering processes. The presence of damping in the $P_{\frac{1}{2}}$ partial wave can be related with the nonresonant effects that affect the considered scattering processes, even though the formation of the metastable state agrees with the existence of the "jump" in the behavior of the scattering phase shifts related with the $P_{\frac{1}{2}}$ partial wave.

As the partial waves with lowest angular momentums are the biggest contributions to the description of the scattering processes at low energies, the $P_{\frac{3}{2}}$ partial wave phase shifts must be analyzed to find a resonance behavior, in the same way as it was found that there is a metastable state in the $P_{\frac{1}{2}}$ partial waves (but the damping in the values of the total phase shifts is a considerable effect from the background scattering as it was explained above), even though, there is no sign of formation of a metastable state in the case of the $S_{\frac{1}{2}}$ partial wave. Similarly to the analysis of the phase shifts in these two partial waves, a plot of the experimental phase shifts(Table (4.3) and Table (4.1)) against the center of mass energy is done, in order to find the possible resonance behavior on the phase shifts(Fig.(4.5) and Fig. (4.4)).

As the phase shifts are analyzed, a jump in the phase shifts behavior is found,

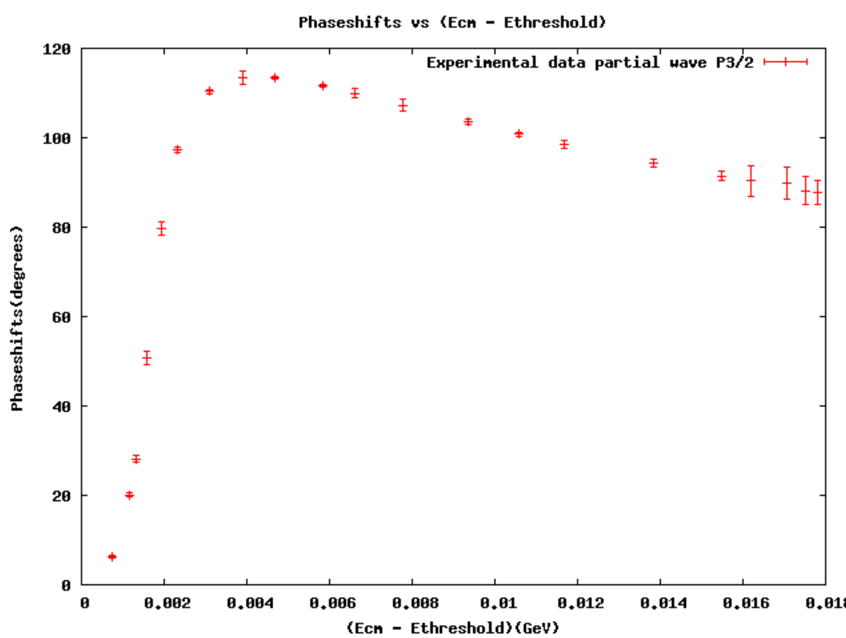

Figure 4.4: Plot of the $P_{\frac{3}{2}}$ wave phase shifts given in the table (Table (4.1)), as a function of the center of mass energy, for $p + \alpha$ scattering processes

through $\frac{pi}{2}$ or 90 degrees. In fact, compared with the other partial waves, the phase shifts related with the $P_{\frac{3}{2}}$ grows more than the values of the phase shifts related with the other analyzed partial waves, the $P_{\frac{1}{2}}$ partial wave phase shifts in particular. Nevertheless, the behavior of the phase shifts near the zero energy leads to think that there

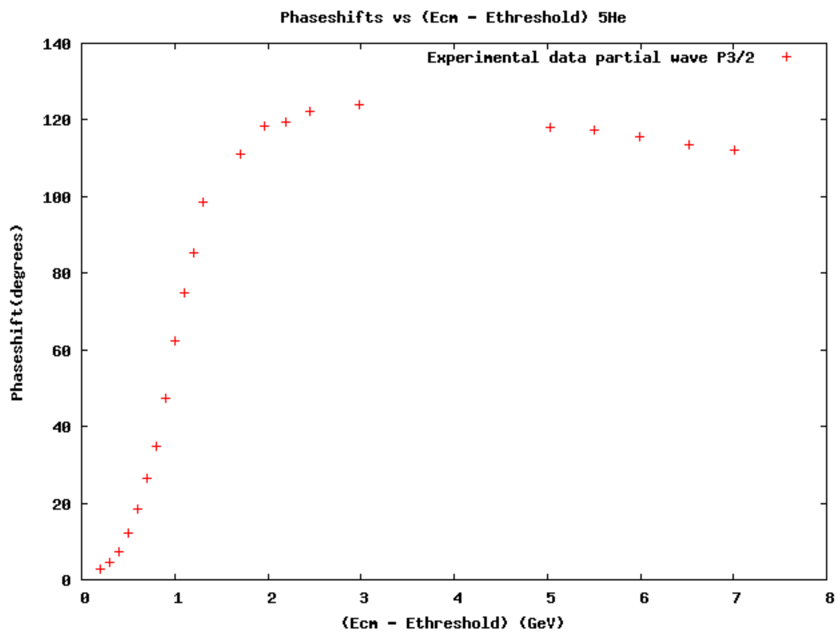

Figure 4.5: Plots of the $P_{\frac{3}{2}}$ wave phase shifts given in the table (Table (4.3)), as a function of the center of mass energy, for $n + \alpha$ scattering process

is not a bound state, since the phase shift value at zero energy goes to zero.

Since the phase shifts rises its value until they reach a higher limit, there can be a metastable state associated with this "jump"; nevertheless, the phase shifts does not reach the π value that stipulates the scattering theory, in absence of background effects; in order to explain the apparent discrepancy between the experimental data and the theoretical description of a resonance, the presence of interference effects of the other partial waves and the hard sphere scattering arising from the $P_{\frac{3}{2}}$ partial wave itself must be considered; like it was analyzed for the case of the experimental phase shifts related with the $P_{\frac{1}{2}}$ partial wave, the presence of the nonresonant or background scattering phase shifts affects the scattering process and the formation of the metastable state, as it was mentioned in the second section of the last chapter. Nevertheless, the increasing behavior of the phase shift[3] leads to suppose the existence of a metastable state for this particular partial wave. In fact, it is not the only partial wave with lower l, like the scattering theory at low energies demands, whose phase shifts behavior approximate to the implicit jump that the scattering theory predicts in the formation of a metastable state. Since the $P_{\frac{3}{2}}$ phase shifts take higher values than the $P_{\frac{1}{2}}$ partial wave phase shifts, so that the former experimental phase shifts goes through $\frac{\pi}{2}$, the damping phenomena in the description of the scattering processes is less relevant in the case of the former partial wave; so, the background scattering phase shifts affect

[3]It is important to notice that the experimental phase shifts extracted from the scattering processes (Table (4.2), Table (4.3) and Table (4.1)) are the total phase shifts defined by (3.31)

the scattering processes mostly in the case of the $P_{\frac{1}{2}}$ partial wave. In the case of the phase shifts related with the $P_{\frac{3}{2}}$ partial wave, the background scattering effects do not affect considerably the formation of the metastable state, and the limit the phase shifts reach before they start to go down is higher than the limit in the case of the scattering phase shifts related with the $P_{\frac{1}{2}}$ partial wave, where the damping effect is more important. Either of the two sets of experimental scattering phase shifts could be taken to analyze, since both of them show the formation of a metastable state in accordance to their behavior, as it was stated before; but, if the behavior of both phase shifts would be analyzed, the experimental phase shift data related with the $P_{\frac{3}{2}}$ partial wave look more uniform and less clumsy than the experimental data related with the $P_{\frac{1}{2}}$ partial wave. That is the main reason why the experimental $P\frac{3}{2}$ phase shifts of both ($n + \alpha$ scattering and $p + \alpha$ scattering processes) are going to be taken in order to do the decay process analysis: even the $P_{\frac{3}{2}}$ partial wave and the $P_{\frac{1}{2}}$ partial wave contribute the most to the formation of the metastable state at low energies, near the threshold, the behavior of the data related with the $P_{\frac{3}{2}}$ is more uniform and the damping effect arising from background scattering is less relevant. As the energy continues to grow, the contribution of the partial waves with lower l is even less relevant to the formation of the metastable state, and the possibility to find overlapping resonances increases.

Thus, in order to determine the possible density of states $\omega(E)$ that lets to express the survival amplitude $A(t)$ in accordance to the Fock Krylov theorem(theorem (1.1)), it is necessary to find an analytic expression that fits the behavior of the experimental phase shifts, related with the formation of the nuclear resonance, so that the density of states in accordance to the Beth Uhlenbeck formalism(3.35), and hence, find out the possible tail of the survival probability at long times given by (3.49).

4.2.2 Parametrization and nonlinear fit of the phase shifts of the $P_{\frac{3}{2}}$ partial wave

Once the phase shift behavior has been analyzed and compared with what the scattering theory must say related to the formation of the resonance, it is important to find a possible relation that expresses the phase shifts behavior at low energies, so that the density of states $\omega(E)$ can be determined. This density of states must behave in accordance to the analytical properties of the potential $V(r)$, with some requirements mentioned above, so as to guarantee the phase shift behavior at low energies, specially near the threshold (3.23).

In particular, one of the useful parameterizations used to characterize the resonance appearance in accordance to the phase shift behavior in function of the energy is the Breit Wigner phase shift given by (3.32). This analytical expression of a particular phase shift leads to a determined form of the scattering cross section in terms of the Breit Wigner distribution (3.33); the Breit Wigner distribution has a peak precisely at the resonance position E_r, since it has two single poles in the complex plane: The resonant pole $E_r - i\frac{\Gamma}{2}$, in the "unphysical sheet" and its complex conjugate $E_r + i\frac{\Gamma}{2}$. In addition to that fact, the Breit Wigner distribution has the described properties that the function $\mathcal{G}(E_{CM})$, which defines the behavior of the density of states $\omega(E)$ in accordance to the properties of the propagator that rules the temporal evolution of the system, should follow [4] Thus, the first option to fit the experimental phase shift would be, precisely, to take a similar analytic function of the Breit Wigner phase shift (3.32); but, as the phase shifts must tend to zero as long as the energy grows to infinity, there must be an exponential factor, so that, the chosen function that fits the experimental data would converge faster to zero at large energies. As it was discussed in the last subsection of this chapter, the experimental $\mathrm{P}_{\frac{3}{2}}$ partial wave scattering phase shifts don't show the jump to π, as the scattering theory demands, even though they rise their value as the energy increases through $\frac{\pi}{2}$, as well,due to the possible presence of nonresonance or background phase shifts that affect the scattering process. So, this possible appearance of background effects must affect the total phase shift behavior, and the analytical function that would be fitted on the data should express that fact. It is worth to mention that in the description of the Breit Wigner phase shift (3.32), the width $\Gamma(E)$ is a function of the energy in general, even though in the interval where the resonance phase shift domains over the background scattering phase shift, the width is taken as constant. In particular, the chosen function to parameterize the data can take a $\Gamma(E)$ function, depending on the center of mass energy, so that , it would adjust so well to the experimental phase shift data(Table (4.1) and Table (4.3)). In accordance to the behavior of the $\mathrm{P}_{\frac{3}{2}}$ phase shift data, showed in the figures(Fig. (4.5), Fig. (4.4)), the chosen function to parameterize the data and make the best nonlinear fit to adjust their behavior takes the form:

$$\delta(E_{CM}) = \tan^{-1}\left[\frac{\Gamma(E_{CM})}{E_0 - E_{CM}}\right] e^{-\beta(E_{CM} - E_{\text{threshold}})} \tag{4.5}$$

With:

$$\Gamma(E_{CM}) = \Gamma_0 \left(\frac{k_{CM}}{k_0}\right)^{\kappa}$$

where:

$$k_{CM} = \frac{1}{2E_{CM}}\left[E_{CM}^2 - (m_a + m_b)^2\right]^{\frac{1}{2}}\left[E_{CM}^2 - (m_a - m_b)^2\right]^{\frac{1}{2}}$$

and:

$$k_0 = \frac{1}{2E_0}\left[E_0^2 - (m_a + m_b)^2\right]^{\frac{1}{2}}\left[E_0^2 - (m_a - m_b)^2\right]^{\frac{1}{2}}$$

[4] These properties were described in the third chapter. $\mathcal{G}(E_{CM})$ must have a single pole on the resonant pole; $\mathcal{G}(E_{\text{threshold}})$ must be different from zero, and, besides, $\mathcal{G}(E_{CM})$ must tend to zero sufficiently fast as E_{CM} goes to infinity

m_a and m_b are defined as the masses of the incident particle and the target particle, respectively.

In fact, this parametrization is similar to the parametrization that Kelkar, Nowakowski and Khemchandani[18] followed in order to fit the D partial waves for the case of a $^{8}_{4}Be$; nonetheless, the analytic function is slightly different, since this fitting function only has a threshold term explicitly in the exponential term; meanwhile, Kelkar, Nowakowski and Khemchandani parameterized the $\Gamma(E_{CM})$ of the function in terms of the threshold energy. If the nonlinear fitting function $\delta_l(E_{CM})$ (4.5) is compared with the Breit Wigner phase shift (3.32), what it could be realized is that the width Γ, in the case of the Breit Wigner phase shift, is a constant; meanwhile, the function $\Gamma(E_{CM})$ that analogically could be related with the width, is depending on the wavenumber in the center of mass frame; therefore, the function $\Gamma(E_{CM})$ depends on the center of mass energy.

It is important to notice that the function k_{CM} in the parametrization function $\delta(E_{CM})$ is precisely the norm of the wavevector in the center of mass frame[5][30]. Meanwhile, the parameters $\kappa, \Gamma_0, \beta, E_0$ can change, in order to find the values that adjust the phase shifts(Fig. (4.5) and Fig. (4.4)) behavior with the greatest accuracy; since the nonlinear fit was done on the experimental P$_{\frac{3}{2}}$ phase shifts data, the program where the coefficients were recalculated in order to obtain the fitting curve for the experimental data, demanded some initial parameters. Once these parameters were given to the program, the calculation of the coefficients starts, in order to find the curve that adjusts the experimental phase shifts set.

The proximity of the final value of the fitting coefficients(Table (4.4) and Table

Parameters	**Initial value**	**Final value**
κ	3.0	3.00
E_0	4.6681	4.6680
β	0.0070	0.0078
Γ_0	0.006	0.0005

Table 4.4: Coefficients found after a nonlinear fit with a fitting function (4.5) is applied on the scattering phase shifts δ_l extracted from the $n+\alpha$ scattering process [51],for the $P_{\frac{3}{2}}$ partial wave

(4.5)) to some of the characteristics values that can take some properties of the nuclear resonances leads to compare some of the coefficients with properties of the compound nucleus formed after the scattering processes. It is known that the rest mass of the $^{5}_{2}He$ nucleus is 4.66784 $\frac{\text{Gev}}{c^2}$ and the rest mass of the $^{5}_{3}Li$ nucleus is 4.66762 $\frac{\text{Gev}}{c^2}$[53]; if the final value of the E_0 coefficient is compared with the theoretical masses of the

[5]page 28, equation (2.99)

Parameters	Initial value	Final value
κ	3.00	3.01719
E_0	4.669	4.66786
β	0.05815	0.0110032
Γ_0	0.0014	0.00101483

Table 4.5: Coefficients found after a nonlinear fit with a fitting function (4.5) is applied on the scattering phase shifts δ_l extracted from the $p+\alpha$ scattering process [51],for the $P_{\frac{3}{2}}$ partial wave

nuclear resonances, it can be deduced the proximity between the numerical value of this coefficient and the mass of the resonance nucleus created from the scattering process. This is not the only coincidence between the fitting coefficients and the values of the physical variables that define the resonance. If the final value of the Γ_0 parameter, found through the nonlinear fit program(Table (4.4) and Table (4.5)), is compared with the width Γ of each nucleus for the considered scattering processes[53][6], it is evident that the value of the coefficient Γ_0 is near to the theoretical value of the width, which characterizes the compound nucleus. Hence, it can be deduced that the Γ_0 parameter, expressed in the nonlinear function $\delta(E_{CM})$,(4.5) and (4.2.2), is related with the width of the nuclear resonance that has been created after the elastic scattering process.

As it was mentioned before, the β coefficient is related with the convergence of the nonlinear fitting function $\delta(E_{CM})$, (4.5); this parameter lets exhibit the asymptotic behavior that the phase shifts must follow in accordance to the scattering theory: As the energy tends to infinity, the phase shifts must tend to zero. This last condition is imposed also on the density of states definition, in accordance to the Beth Uhlenbeck theorem (3.35). As the $\delta(E_{CM})$ is the best fitting function to characterize the experimental phase shifts data, the β coefficient expresses the tendency of the phase shift data to tend to zero at large energies, as it was stated above.

The next question to solve is the meaning of the κ coefficient; since this coefficient appears in the parametrization function $\delta(E_{CM})$ as an exponent of the wavenumber $k(E_{CM})$, maybe its value is related with the angular momentum dependence of the exponent $\gamma(l)$ that leads to the nonexponential contribution of the survival amplitude $A(t)$ (3.49), since the phase shift behavior is related with the wavenumber in the center of mass frame k_{CM}, as a power law depending on the angular momentum coefficient l (3.22); so, maybe, the defined value of the κ parameter is related with the nonexponential tail presented in the survival probability at large times $P(t)$, and expresses the behavior of the system near the threshold energy.

[6]The widths of the two nuclear resonances are: $\Gamma_{{}^5_2He} = 0.6$ MeV and $\Gamma_{{}^5_3Li} = 0.1$ MeV

4.2.3 Survival probability at long times $P(t)$ and its behavior in accordance to the experimental phase shifts data

Once the coefficients of the nonlinear fitting function $\delta(E_{CM})$ has been determined, the parametrization function can be plotted in order to see how accurate is the nonlinear fit, and if the function $\delta(E_{CM})$ follows the behavior that the experimental phase shifts data shows. If the nonlinear fitting function adjusts so well the experimental phase shift data, it is necessary to do the derivative of the parametrization function, in order to find out the possible analytical properties of the density of states $\omega(E)$, according to what the Beth Uhlenbeck theorem says(3.35).

In particular, for the $p+\alpha$ and $n+\alpha$ scattering processes, the nonlinear fitting function $\delta_l(E_{CM})$ must adjust the experimental data; if this is the case, the derivative of the parametrization function could express the behavior that the derivative of the phase shifts follows, as a function of the center of mass energy E_{CM}, even though the fitting parameters Γ_0, β, E_0 and κ take particular values, depending on the scattering process that it is analyzed(Table (4.4) and Table (4.5)). Hence, if the derivative of the phase shifts is calculated and plotted, some properties of the metastable state, which formation is implicit in the $\mathrm{P}_{\frac{3}{2}}$ partial wave, in accordance to the phase shifts behavior, can be determined.

A proof of the accuracy of the nonlinear fit in order to adjust the experimental phase shift behavior is given by the confrontation between the nonlinear coefficients that define the parametrization function $\delta(E_{CM})$ and the theoretical values of some properties of the compound nucleus formed by the elastic scattering process. In the plots that show the behavior of the derivative of the nonlinear fitting function(Fig. (4.6)), the position of the resonance and the width have been determined, since the fitting coefficients are related to them, as it was mentioned earlier in this chapter(indeed, Γ_0 and E_0 are related with the width and the mass of the resonance respectively). The width values appearing in the Fig. (4.6) are determined through the nonlinear fit, and the final values of the coefficients. In the case of the $\,^5_2$He nucleus, the theoretical value of the width is near to 0.6 MeV, and the width value determined by the nonlinear fit of the phase shift data(related with the Γ_0 coefficient) is exactly 0.6MeV as the Fig.(4.6) shows. Meanwhile, for the $\,^5_3$Li, nucleus, the expected value of the resonance width is near to 1.5 MeV; in contraposition, the calculated width from the experimental phase shift data takes the value of 1.25 MeV. The proximity of the determined values by the nonlinear fit and the theoretical ones shows the accuracy of the nonlinear fit, and, hence, the description of the experimental data could be parameterized by the nonlinear fitting function $\delta(E_{CM})$.

In accordance to the Beth Uhlenbeck theorem (3.35), the density of states of an interacting system is related with the derivative of the phase shifts with respect of the center of mass energy (E_{CM}); so, as the parametrization function $\delta(E_{CM})$ expresses the behavior of the experimental phase shifts, specially at low energies near the threshold,

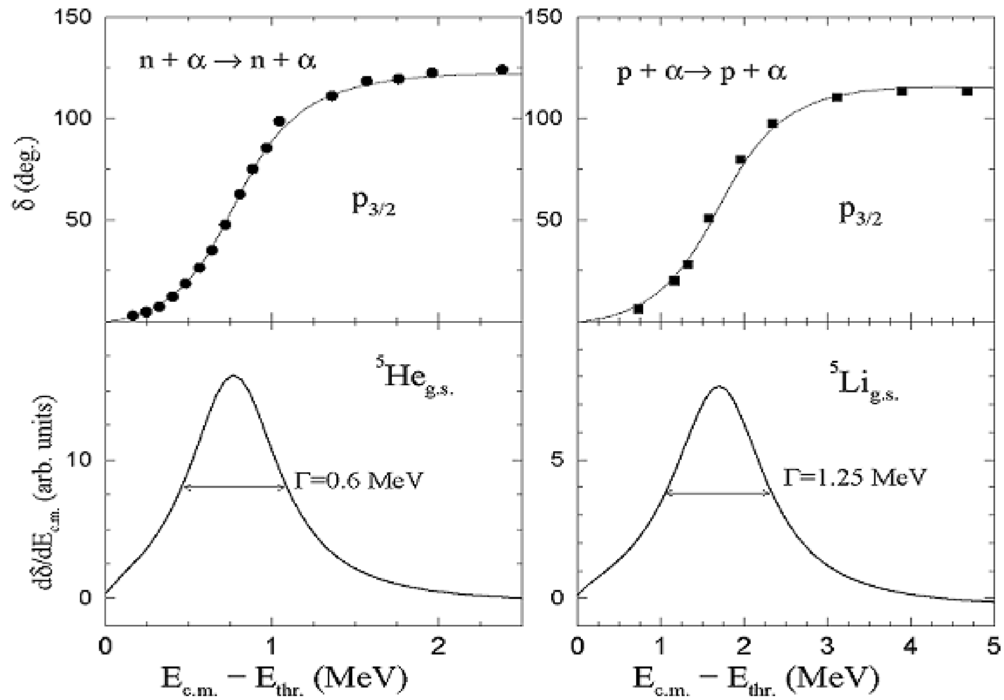

Figure 4.6: Left: Above, plot of the $P_{\frac{3}{2}}$ wave phase shifts given in the table (Table (4.3)), as a function of the center of mass energy, for $n+\alpha$ scattering process and the nonlinear fitting function $\delta(E_{CM})$ adjusting the data with the values of the coefficients given in the Table (4.4). *Below*, the plot of the derivative of the nonlinear fitting function $\delta_l(E_{CM})$, that adjusts the experimental data, with respect to the center of mass energy, for the same set of experimental data. Right: Above, plot of the $P_{\frac{3}{2}}$ wave phase shifts given in the table (Table (4.1)), as a function of the center of mass energy, for $p+\alpha$ scattering process and the nonlinear fitting function $\delta(E_{CM})$ adjusting the data with the values of the coefficients given in the Table (4.5). *Below*, the plot of the derivative of the parametrization function $\delta(E_{CM})$ with respect of the center of mass energy.

the density of states $\omega(E)$ should be related with the properties of the derivative of the parametrization function $\delta(E_{CM})$. Nevertheless, the density of states also must be defined in accordance to the behavior of the phase shifts, related with the partial waves with the biggest contribution to the description of the physical system; so that, the density of states must follow the threshold behavior, and, besides it must be defined in order to behave according to the properties of the propagator that express the temporal evolution of the system, as the decay process takes place. Kelkar, Nowakowski and Khemchandani parameterized the density of states $\omega(E)$ in order to separate the threshold behavior implicit in the definition of the density of states $\omega(E)$, in accordance to the phase shift behavior at low energies ,(3.36) and (3.22), and a function

$\mathcal{G}(E_{CM})$ that expresses the analytical properties of the propagator that leads to the decay process (3.37). This particular function must follow some properties that have been discussed earlier, in the third chapter. Hence, these properties should be reflected in the derivative of the nonlinear fitting function as a function of the center of mass energy (Fig.(4.6)); in particular, one of this properties is really interesting to recall, since it can be related with the physical description of the scattering process: The $\mathcal{G}(E_{CM})$ has a simple pole in the complex plane; indeed, this pole is the so called "resonant pole", and it is located in the "unphysical sheet"; this simple pole is defined as the pole of the **S** matrix in the case of a formation of a metastable state; so, the presence of this pole and its effects on the temporal evolution of the physical system should be observed in the derivative of $\delta(E_{CM})$(Fig. (4.6)), in principle. In the scattering process, the presence of a resonant pole in the "unphysical sheet" leads to an interesting behavior in the scattering cross section, as it was discussed earlier: The existence of a resonance is characterized by a peak in the scattering cross section; as a matter of fact, if the potential $V(r)$ is localized, the scattering cross section takes the form of a Breit Wigner distribution (3.33). In fact, the Breit Wigner distribution follows the requirements that $\mathcal{G}(E_{CM})$ has: It tends to zero as long as the center of mass energy goes to infinity, in order to guarantee the convergence of the survival amplitude $A(t)$ and the Breit Wigner function, evaluated in $E_{\text{threshold}}$ is different from zero, as the threshold energy does not make the Breit Wigner denominator vanish at all. Even though the derivative of the nonlinear fitting function $\delta(E_{CM})$, shown in the figure (Fig. (4.6)) is not exactly a Breit Wigner distribution, the similarity between the plots (Fig. (4.6)) and (Fig. (3.1)) is so evident, and hence, the derivative of the nonlinear fitting function follows the properties that the function $\mathcal{G}(E_{CM})$ and the density of states $\omega(E)$ must follow. In fact, this similar behavior between the Breit Wigner distribution and the derivative of $\delta(E_{CM})$ affirms the existence of a metastable state in the considered partial wave, in accordance to the behavior of the experimental phase shifts related to it.

It was explained above that the properties of the function $\mathcal{G}(E_{CM})$ are evidently shown in the behavior of the derivative of the parametrization function, as it behaves similarly to the Breit Wigner distribution, which is characteristic of resonances. But, in order to define an analytic form for the density of states $\omega(E)$ in accordance to (3.37), it is necessary to find the factor that expresses the dependence of the threshold energy $(E_{CM} - E_{\text{threshold}})^{\gamma(l)}$. This factor is the one is going to be relevant as the nonexponential contribution of the survival amplitude $A(t)$ is calculated, since the power behavior of t at large times will be related with the analytic expression of $\gamma(l)$. If the derivative of the nonlinear fitting function δE_{CM} expresses the properties that the function $\mathcal{G}(E_{CM})$ must follow, it also must behave in accordance to the threshold behavior of the phase shifts at low energy limit(3.24). It was mentioned above that, in particular, the κ coefficient in the parametrization function δE_{CM} could be related with the asymptotic behavior of the phase shifts near the threshold energy, since the parametrization (4.5) can be compared with the relation that expresses the phase shifts tangent of a particular partial wave at low energies (3.21). In particular, if the κ coefficient determined in the nonlinear fit of the experimental phase shifts data is taken as $\frac{\kappa}{2} = \gamma(l) + 1$, the density of states $\omega(E)$, defined by Kelkar, Nowakowski and Khemchandani in their work[18], takes the form:

$$\omega(E) = \mathcal{G}(E_{CM})(E_{CM} - E_{\text{threshold}})^{\frac{\kappa}{2}-1} \tag{4.6}$$

This particular definition of the density of states in terms of the κ coefficient, obtained through the nonlinear fit, leads to a new definition of the survival amplitude $A(t)$, according to the Fock Krylov theorem (theorem (1.1)); hence, the survival amplitude $A(t)$ is given by:

$$\begin{aligned} A(t) = 2\pi i \frac{(E_r - i\frac{\Gamma}{2} - E_{\text{threshold}})^{\frac{\kappa}{2}-1} e^{-i(E_r)t}}{\frac{d\mathcal{G}(z+E_{\text{threshold}})}{dz}\Big|_{z=E_r - i\frac{\Gamma}{2} - E_{\text{threshold}}}} e^{-\frac{\Gamma}{2}t} + \\ (1+i)(-i)^{\frac{\kappa}{2}} e^{-iE_{\text{threshold}}t} \left(\frac{\Gamma(\frac{\kappa}{2})}{t^{\frac{\kappa}{2}}} \mathcal{G}(E_{\text{threshold}}) \right) \end{aligned} \tag{4.7}$$

Hence, the nonexponential contribution at large times of $A(t)$ would be proportional to $\frac{1}{t^{\frac{\kappa}{2}}}$, and the survival probability $P(t)$ would be characterized,in accordance to the survival probability definition (1.5), by a temporal power law that dominates over the exponential contribution at large times:

$$P(t) \underset{t\to\infty}{\sim} \frac{1}{t^{\kappa}} \tag{4.8}$$

Thereby, the κ coefficient expresses the behavior of the density of states $\omega(E)$ near threshold and the temporal power form that takes the survival probability $P(t)$, once the exponential contribution vanishes at long times. It was stated in the third chapter that there could be a relation between the $\gamma(l)$ exponent and the angular momentum coefficient l,in order to follow the asymptotic behavior of the l partial wave phase shifts at low energies, near the threshold. In fact, it was deduced, by comparing the behavior of the phase shifts at low energies (3.24) with the definition of the density of states (3.37), that the $\gamma(l)$ can be expressed as $l - \frac{1}{2}$. So, there can be a relation between the κ coefficient determined in the nonlinear fit procedure, and the angular momentum l related with the partial wave:

$$\kappa = 2l + 1 \tag{4.9}$$

According to the last relation (4.9), for the particular case of the $P_{\frac{3}{2}}$ partial waves considered to do the phase shifts analysis, the theoretical value of the κ coefficient ought to be 3, since $l = 1$, for all P partial waves.

If the values of the κ parameter, obtained by the nonlinear fit of the scattering phase shifts for both scattering processes, are compared with the theoretical value of κ in accordance to the phase shift behavior at low energies, what it can be found is that there is no meaningful difference between the experimental results and the one obtained by the theoretical assumptions. As a matter of fact, the results are really closed, even though the scattering processes that has been taken to analyze the formation of

κ(**theoretical value**)	$\kappa(\,^{5}_{2}\mathbf{He})$	$\kappa(\,^{5}_{3}\mathbf{Li})$
3	3.01719	3.00

Table 4.6: Comparison between the values of the κ parameter: the theoretical one given by (4.9), the one calculated from the nonlinear fit of the phase shifts of the $n+\alpha$ scattering process, and the one calculated from the nonlinear fit of the phase shifts of the $p+\alpha$ scattering process

different nuclear resonances are different. Since the analyzed partial wave where the resonance formation has been found is the same for the both scattering processes($\mathrm{P}_{\frac{3}{2}}$), it is natural to think that the phase shifts related with that particular partial wave would follow the same asymptotic behavior as the energy goes near threshold (3.22). Thereby, the κ coefficient, which is related the phase shifts behavior at low energies, should be equal for the two considered scattering processes ($p+\alpha$ and $n+\alpha$ scattering processes). Since the κ coefficient lets to express the nonexponential contribution of the survival amplitude $A(t)$ in accordance to (4.7), the long time behavior of the survival probability can be characterized by the values of κ.

Theoretically, since $\kappa_{\text{theoretical}}$ is equal to 3, the power law that expresses the "tail" of the survival probability at long times is proportional to $\frac{1}{t^3}$(3.50); in the case of the $^{5}_{2}He$ nucleus, the κ coefficient takes the value of 3, exactly; so the power law calculated from (4.8), that expresses the nonexponential contribution to the survival probability $P(t)$ would be exactly as the theory affirms, proportional to $\frac{1}{t^3}$. Nevertheless, there is a discrepancy in the case of the $\,^{5}_{3}Li$ nucleus, since the κ coefficient is not exactly 3, but 3.01719. But this discrepancy is not relevant and does not look serious; due to the existence of this discrepancy, the tail of the survival probability $P(t)$ at large times, when the nonexponential contribution dominates the exponential one, which tends quicker to zero, is proportional to $\frac{1}{t^{3.10719}}$. The correspondence between the theoretical assumptions of the survival probability $P(t)$ at large times and the experimental results from the nonlinear fit of the scattering phase shift data indicates that the density of states $\omega(E)$ must follow the standard threshold behavior, in order to find the nonexponential contribution proportional to $\frac{1}{t^3}$; thereby, the density of states given by (3.37), with $\gamma(l) = l - \frac{1}{2}$ is the most natural way to define the continuum probability density of states in a resonance, and the survival amplitude $A(t)$ given by (3.48) characterizes the decay process, specially in the region near the threshold, where the partial wave analysis is accurate to describe the low energy scattering processes. This behavior contrasts with the description Fonda, Rimini, and Ghirardi[3] did on the analytical properties of the survival probability. Indeed, for a P partial wave, they found that the long time "tail" in the survival probability $P(t)$ would be proportional to $\frac{1}{t^5}$ in accordance to (2.74). Fonda, Rimini and Ghirardi considered a density of states which has a dependence on the threshold energy(2.68) like the one Kelkar, Nowakowski and Khemchandani supposed (3.37); nonetheless, they do not considered the behavior of the phase shifts related with the lowest partial waves near the threshold (3.23), so that, the exponent in the threshold factor is different compared with the one in (3.37); therefore, although the nonexponential contribution of the survival amplitude has a dependence on the angular momentum l, depending on the partial wave and its contribution to the description of the scattering processes, the density of states doesn't follow the threshold behavior, and the survival probability $P(t)$ won't follow the power law related with the phase shift behavior near the threshold. That is the main reason why there is a discrepancy between these two models. The nonlinear fit analysis leads to define a density of states that behaves in accordance to what the temporal evolution of the system requires and the phase shift behavior at low energies, where the long time description of the survival probability $P(t)$ becomes relevant. So, the description of the density of states $\omega(E)$ by (3.37) is the most appropriate one to determine the survival probability $P(t)$ in accordance to the physical constrains that the system must take into account and the description of its temporal evolution.

Since the density of states $\omega(E)$ has been determined in accordance to the analytical properties that the phase shifts, related with the partial wave which exhibits the resonant behavior, and the relation with the propagator that determines the temporal evolution of the system, from the analytical form that takes the hamiltonian $\hat{H}$, the survival probability $P(t)$ has been determined in accordance to the Fock Krylov theorem (theorem (1.1)) and the physical constrains imposed to the system($E \geq E_{\text{threshold}}$, and $t \geq 0$); the physical constrains restricts the physical properties that characterize the system, and its temporal evolution in accordance to the properties of the evolution operator $\bar{U}(t)$. Hence, under all these considerations, the survival probability $P(t)$ has two different contributions, as it was shown in (3.48): One exponential, that leads to the classical exponential decay law, that is determined from the residue of the spectral density $\mathcal{G}(E_{CM})$ calculated on the single resonant pole $E_r - i\frac{\Gamma}{2}$; and the nonexponential behavior, related, as it was mentioned before, on the phase shift behavior at low energies near the threshold. This nonexponential behavior, a power behavior on t, has an exponent related with l he angular momentum of the particular phase shift, related with the partial wave which contribution to the scattering process description is relevant. As it was mentioned earlier in this work, only the partial waves with lower l affects the scattering processes at low energies, due to the centrifugal barrier effect; so, the value that l takes in the survival amplitude description (3.48) is a low one, in order to characterize properly the behavior of the metastable state at energies near the threshold and long times: That implies that the characteristics of the density of states $\omega(E_{CM})$(3.37) are not important as the large energy domain is considered, since the main subject to analyze is the energies near the threshold.

It was stated before the similarity between the behavior of the derivative of the nonlinear fitting function $\frac{d\delta(E_{CM})}{E_{CM}}$, showed in figure(Fig. (4.6)) with the Breit Wigner distribution (Fig. (3.1)); as a matter of fact, it was explained earlier in the chapter that the Breit Wigner distribution follows all the properties that the spectral density

$\mathcal{G}(E_{CM})$ has. Therefore, a Breit Wigner distribution is a particular choice in order to characterize the survival probability behavior at long times. In particular, if a Breit Wigner distribution is taken as the spectral density $\mathcal{G}(E_{CM})$, the survival amplitude $A(t)$,(4.7), would be given by[7]:

$$\begin{aligned} A(t) = (-i)(E_r - i\frac{\Gamma}{2} - E_{\text{threshold}})^{\frac{\kappa}{2}-1} e^{-i(E_r)t} e^{-\frac{\Gamma}{2}t} + \\ \Gamma^2(1+i)(-i)^{\frac{\kappa}{2}} e^{-iE_{\text{threshold}}t} \left(\frac{\Gamma(\frac{\kappa}{2})}{((E_{\text{threshold}} - E_r)^2 + \frac{\Gamma^2}{4})t^{\frac{\kappa}{2}}} \right) \end{aligned} \tag{4.10}$$

The two possible contributions to the survival amplitude can be identified clearly in the relation (4.10); so, an analytic expression for the survival probability $P(t)$ can be found, in accordance to (1.5). If the κ coefficients, found from the nonlinear fitting procedure (Table (4.6)) are replaced in the relation (4.10) for each one of the considered elastic scattering process, the survival probability $P(t)$, that characterizes the decay process after the formation of the resonance, can be found.

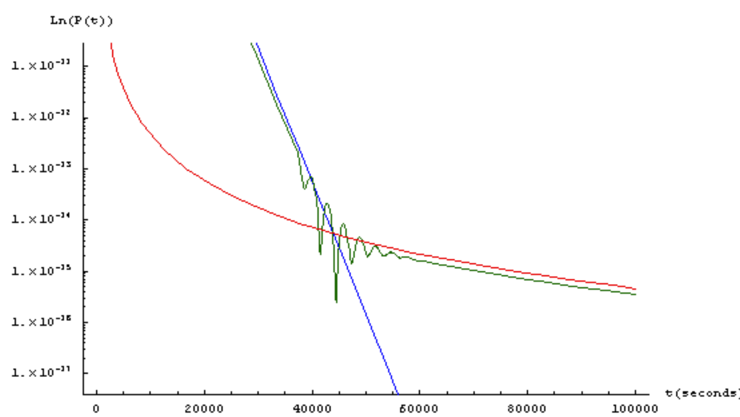

Figure 4.7: Logarithmic plot of the survival probability $P(t)$for a κ coefficient determined by the nonlinear fitting procedure for $n+\alpha$ scattering process(Table (4.6))(green plot); exponential contribution to the survival probability $P(t)$, that dominates at intermediate times(blue plot); nonexponential contribution to the survival probability $P(t)$ that leads to the power law at large times (red plot)

In fact, if the two graphics below are analyzed (Fig. (4.7) and Fig. (4.8)), the difference between the exponential and nonexponential behavior of the survival probability is evident. In the intermediate times, the exponential behavior dominates the description of the survival probability $P(t)$, overlapping the nonexponential contribution, which is small in that temporal domain, as the nonexponential contribution (blue

[7]the exponent in this relation is taken as the nonlinear fitting coefficient κ related with the angular momentum l as it was discussed earlier in this subsection(page101)

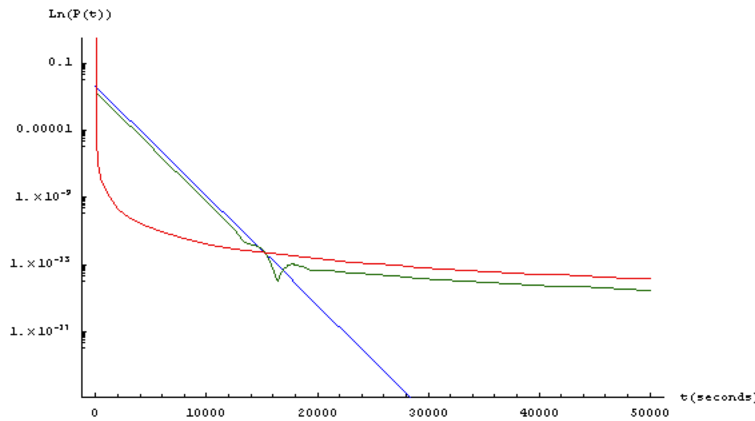

Figure 4.8: Logarithmic plot of the survival probability $P(t)$for a κ coefficient determined by the nonlinear fitting procedure for $p+\alpha$ scattering process(Table (4.6))(green plot); exponential contribution to the survival probability $P(t)$, that dominates at intermediate times(blue plot); nonexponential contribution to the survival probability $P(t)$ that leads to the power law at large times (red plot)

plot) is practically the same as the description of the survival probability $P(t)$ in the two figures above(Fig. (4.7) and Fig. (4.8)). Nevertheless, after a critical time t_c, the exponential contribution to the survival probability $P(t)$ begins to become less relevant than the nonexponential contribution, and the survival probability $P(t)$ starts to oscillate. Once the oscillating effect gets smaller and the survival probability $P(t)$ stabilizes, it moves away from the exponential behavior and starts to behave in accordance to the analytic form of the nonexponential contribution, leading to a power law, as it can be shown in the graphics[8](Fig. (4.7) and Fig. (4.8)). Thereby, the long time behavior is dominated by the nonexponential contribution to the survival probability $P(t)$, since the exponential contribution tends to zero faster after the critical time t_c, as Khalfin affirmed in his work[7], in accordance to the Payley Wiener theorem (theorem (1.2)). The definition of the energy spectrum as a physical constrain($E \geq E_{\text{threshold}}$) plays a huge role in the determination of the nonexponential contribution of the survival probability $P(t)$, since the nonexponential behavior arises from the calculation of the integral over the imaginary axis, in accordance to the contour(Fig. (2.5), and the definition of the density of states $\omega(E)$ (3.37), so that it could express the phase shift behavior at low energies near the threshold (3.23). It is important to realize that the exponential contribution to the survival probability $P(t)$ takes a different behavior for

[8]Before the critical time t_c the survival probability behaves in accordance to the exponential decay law, and the exponential contribution characterizes so well the behavior of $P(t)$; hence, the decay law and the survival probability are basically the same. After the critical time t_c the oscillation appears, and there is a discrepancy between the exponential behavior(blue plot) and the survival probability defined analytically (green plot); once the oscillation passes away, $P(t)$ seems to follow the nonexponential power law(red plot), and the exponential behavior goes to zero faster than $P(t)$

the two considered scattering processes($n + \alpha$ and $p + \alpha$ scattering processes), since the exponential decay law depends on the width of the resonance (1.1), so as the first term of the survival probability $P(t)$ (3.49). Nevertheless, the nonexponential contribution that leads to the long time "tail" of the survival probability $P(t)$ takes the same asymptotic form, since the resonance behavior can be appreciated in the same partial wave($P_{\frac{3}{2}}$) in the considered elastic scattering processes. As it was explained earlier in this chapter, the $P_{\frac{3}{2}}$ partial wave is not the only partial wave which presents in its phase shifts a particular behavior that leads to the existence of a metastable state at low energies, near the threshold (in particular, the phase shifts related with the $P_{\frac{1}{2}}$ partial wave show also the characteristics that a metastable state follows). So, the long time behavior of the survival probability $P(t)$, which is dominates by the nonexponential contribution, is basically the same for both scattering processes, even though there is a discrepancy between the experiment κ coefficient and the theoretical one, calculated from (4.9), specially in the case of the ${}^5_3He$. But this discrepancy is not relevant, since it is too small to consider, and the experimental phase shifts considered in order to find out the analytical form that takes the density of states $\omega(E)$ are the total phase shifts, so they include nonresonant and background effects that affect the scattering process(Fig.(4.9)). One of the manifestations of these background effects in the $P_{\frac{3}{2}}$ phase shifts behavior is the damping in the values of the phase shifts, as it has been discussed earlier. Nevertheless, the survival probability $P(t)$ behavior is really approximate to the one that the theory affirms, as the plot (Fig.(4.9)) shows.

A huge difference between the critical times t_c when the exponential contribution and the nonexponential one are comparable, and the survival probability $P(t)$ starts to oscillate, moving away from the exponential behavior that $P(t)$ exhibits at intermediate times, that can be appreciated in the considered scattering processes. In particular, in the case of the ${}^5_3Li$ nuclear resonance, the survival probability $P(t)$ looks to move away from the exponential behavior at $t_c \approx 15000$ seconds; meanwhile, the fluctuations in the survival probability starts after $t_c \approx 42000$ seconds for the ${}^5_2He$ nuclear resonance, in accordance to the plots that show the behavior of the survival probability $P(t)$ for these resonances(Fig.(4.7) and Fig.(4.8)). Hence, the value of this critical time t_c depends on the physical characteristics and the value of the variables that define the metastable state. But what kind of physical variables are involved in the determination of the critical time t_c, when the exponential contribution to the survival probability $P(t)$ becomes comparable with the nonexponential one, and the survival probability begins to fluctuate?

4.2.4 Critical time t_c and the transition from intermediate times to large times

It has been discussed that the description of the survival probability $P(t)$ is given by two analytic contributions: an exponential one that depends on the properties of the propagator, and an nonexponential one that depends on the behavior of the phase shifts

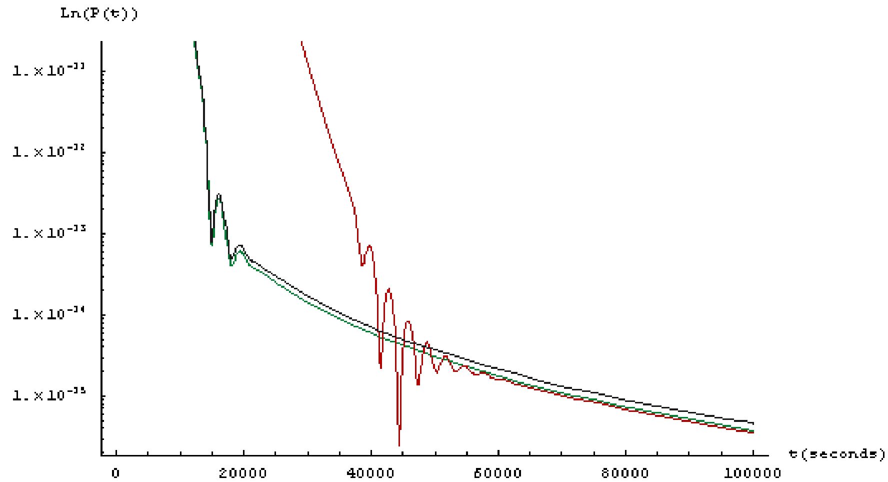

Figure 4.9: Logarithmic plot that shows a comparison between the survival probabilities $P(t)$ determined by different values of the κ coefficients(Table (4.6)): κ coefficient determined by the nonlinear fitting procedure for $p+\alpha$ scattering process(Table (4.6))(clear green plot); κ coefficient determined by the nonlinear fitting procedure for $n+\alpha$ scattering process(Table (4.6))(red plot); theoretical relation for the survival probability $P(t)$ given by (3.48), with $l=1$($\mathrm{P}_{\frac{3}{2}}$ partial wave)(dark green plot) for the ${}^{5}_{3}Li$ nuclear resonance

of the related partial wave near the threshold, in accordance to the relation (3.48); the exponential contribution characterizes the behavior of the survival probability $P(t)$ at intermediate times, where the nonexponential contribution is not meaningful, so that, the description of the decay process, and in particular, the survival probability $P(t)$ can be expressed as a decreasing exponential function. Nevertheless, at large times, the behavior of the survival probability changes considerably, since the exponential contribution tends to zero faster than the nonexponential contribution; hence, the survival probability is characterized by a power behavior, depending on the angular momentum coefficient l related with the partial wave, whose contribution to the description of the metastable state is more meaningful; indeed at long times, the characteristic behavior of the survival probability $P(t)$ leads to a long time "tail" that represents the power law that obeys $P(t)$ asymptotically. Meanwhile, there is a transition time between the two presented behaviors in the survival probability $P(t)$; as a matter of fact, in some temporal interval, the two contributions of the survival probability $P(t)$ become comparable, so that, in that interval, the behavior cannot be described as exponential or nonexponential, due to the fluctuations of the survival probability. Once the survival probability stabilizes, the transition between the exponential and nonexponential behavior is completed; therefore, the importance to determine the limit, when a system characterized by its survival probability $P(t)$, passes from an exponential behavior, at the intermediate times, to a nonexponential behavior, given by the long time "tail" in the survival probability $P(t)$ is relevant in order to characterize the decay process, in accordance to the analytical properties of the propagator and the evolution operator that expresses the temporal evolution of the system . This limit is what it has been called in this work the "critical time" t_c: The time when the survival probability abandons the exponential behavior, and the contribution of the nonexponential part is comparable with the exponential one, leading to some fluctuations in the description of $P(t)$.

As it was mentioned in the last subsection, the critical time t_c is a characteristic of each metastable state, since it depends on the physical properties of the resonance. So, the transition from exponential to nonexponential behavior is related with the nature of the resonance; it can be seen in the two figures that describes the survival probability $P(t)$ for the two considered scattering processes(Fig.(4.8) and Fig.(4.7)): The possible critical time for the ${}^5_3Li$ is 15000 seconds, approximately; meanwhile, for the ${}^5_2He$, the critical time is bigger: $t_c \approx 42000$ seconds in accordance to the plots.

It was mentioned in the first chapter that Bogdanowitz, Pindor and Raczka[17] found an expression that lets find out the value of this parameter, in order to characterize the transition between the exponential and nonexponential behavior of the survival probability $P(t)$, given by (1.18); this is a transcendental equation, so in order to solve it is necessary to see at what times does the function $\Gamma t^{-\frac{\Gamma t}{2}}$ intersects the line $\frac{1}{\pi}\left(\frac{\Gamma}{2(E_r - E_{\text{threshold}})}\right)$. Hence, if the resonance position E_r is so close to the threshold energy, the large solution to the transcendental equation (1.18) increases; in particular, if a resonance has a larger width Γ, the oscillations in the survival probability $P(t)$ will start earlier than a resonance with a smaller width: the critical time t_c for a metastable state with larger width is smaller.

In fact, for the two analyzed nuclear resonances, ${}^5_3Li$ and ${}^5_2He$, the critical time was determined through a numerical method in Mathematica, and a graphic method, plotting the $\Gamma t^{-\frac{\Gamma t}{2}}$ function, and determining the points where it intersects the constant line $\frac{1}{\pi}\left(\frac{\Gamma}{2(E_r - E_{\text{threshold}})}\right)^2$. Since the value of the widths Γ for both resonances are known[9], so as the resonance position(nucleus mass) and the threshold energy[10], the critical times t_c can be determined for the both nuclear resonances. If the logarithmic plots of the $\Gamma t^{-\Gamma t}$(Fig.(4.10)) are analyzed, the discrepancy between the intersections for both resonances results evident. For the ${}^5_2He$ nuclear resonance, the long time solution of the transcendental equation (1.18)(which is the more relevant one, since it is the critical time t_c) is near 12000 seconds; meanwhile, for the ${}^5_3Li$ nuclear resonance, the long time

[9] see footnote 7 of this chapter

[10] As it was stated before, the threshold energy is taken as the sum of the particles that are involved in the scattering process: the target and the incident particle(proton and α for the ${}^5_3Li$ resonance, and neutron and α for the ${}^5_2He$ nuclear resonance)

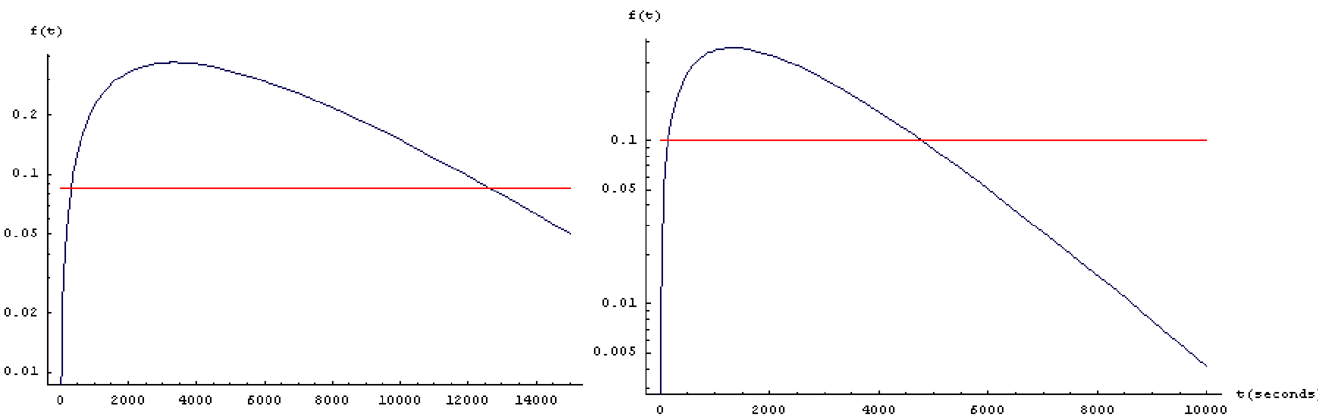

Figure 4.10: Logarithmic plots of the $\Gamma t^{-\frac{\Gamma t}{2}}$ function in order to determine the critical time, in accordance to (1.18); the constant line is the factor $\frac{1}{\pi}\left(\frac{\Gamma}{2(E_r - E_{\text{threshold}})}\right)^2$; the two intersection points are the solutions of the transcendental equation; but the physical solution is the large one, since it represents the critical time t_c; the left plot is related with the ^5_2He nucleus and the right plot is related with the ^5_3Li

solution of the transcendental equation is near 4000 seconds, as the plot (Fig.(4.10)) shows in each particular case. In fact, a Mathematica calculation of the solutions of the transcendental equation (1.18) gives a numerical values for t_c, related with each nuclear resonance: $t_c = 4700.7$ seconds for ^5_3Li and $t_c = 12660.1$ seconds in the case of ^5_2He; nevertheless, these results do not agree with the critical time values extracted from the plots that show the behavior of the survival probability $P(t)$ for the considered two scattering processes. In the last subsection, the critical time for the both processes were estimated as $t_c \approx 15000$ seconds for the ^5_3Li nucleus and $t_c \approx 40000$ in the case of the ^5_2He in accordance to the behavior of the survival probability in the plots(Fig.(4.7) and Fig.(4.8)). Naturally, a question arises: Why does this discrepancy between the two values of the critical time exist? How can this discrepancy be solved?

The description Bogdanowitz, Pindor and Raczka [17] did of the survival probability $P(t)$ implies a density of states $\omega(E)$ defined as a Breit Wigner distribution, similar to the one that Khalfin took in his work(3.34)[7]. Thereby, the power law behavior they found was explicitly the same Khalfin obtained in his work (1.22): The nonexponential contribution to the survival probability would be proportional to $\frac{1}{t^2}$. So, in order to determine the critical time, when the fluctuations in the survival probability $P(t)$ appear, they make the two contributions of the survival amplitude $A(t)$ equal; so, that is the method they used to determine the explicit value of the critical time parameter t_c. Nevertheless, as it was pointed out, the Breit Wigner distribution does not take into account the phase shift behavior at low energies near the threshold, where an isolated resonance can appear considerably, and the long time effects can be analyzed properly.

So, the description of the density of states $\omega(E)$ they used is incomplete; that is the main reason why the discrepancy between the critical time t_c values has been found. This fact is reflected in the behavior of the phase shifts at low energies; the density of states that Bogdanowitz, Pindor and Raczka [17] took in order to determine the survival probability $P(t)$ leads to a Breit Wigner phase shift, given by (3.32); the behavior of the Breit Wigner phase shift is an idealization, as the behavior of the phase shifts related with the partial wave analysis is not taken into account to describe the formation of the metastable state; so, the width is taken as a constant for this idealized phase shift. In the case of the density of states defined by (3.37), the description of the behavior of the phase shifts at low energies expresses the nonexponential contribution to the survival probability $P(t)$; since the phase shifts are characterized by the nonlinear fitting function $\delta_l(E_{CM})$, the function $\Gamma(E_{CM})$, analogically to the width Γ in the Breit Wigner phase shift, is not a constant, since it depends explicitly on the center of mass energy, as the parametrization function $\delta_l(E_CM)$ fits the experimental phase shifts extracted from the considered scattering processes, including the possible background effects that affect the description of the metastable state. Therefore, the Bogdanowitz, Pindor and Raczka model is an approximation, in the case where the background effects would be neglected, and the density of states would not depend on the behavior of the phase shifts related with the partial wave description, near the threshold energy. That is the main reason of the discrepancy between the calculated critical times from (1.18), given by Bogdanowitz, Pindor and Raczka [17], and the values of the critical times t_c behavior of the survival probability $P(t)$ for both scattering processes, showed in the plots (Fig.(4.7) and Fig.(4.8)).

Analogically to the procedure Bogdanowitz, Pindor and Raczka[17] followed in order to determine the critical time parameter t_c, the two contributions to the survival amplitude $A(t)$ can be equalized in order to define the time when the fluctuations begin to appear in the survival probability; so, it can be found a relation for this particular time, given by:

$$2\pi\left|\frac{(E_r - i\frac{\Gamma}{2} - E_{\text{threshold}})^{l-\frac{1}{2}}}{\left.\frac{d\mathcal{G}(z+E_{\text{threshold}})}{dz}\right|_{z=E_r-i\frac{\Gamma}{2}-E_{\text{threshold}}}}\right| e^{-\frac{\Gamma}{2}t_c} = \left(\frac{1}{(t_c)^{l+\frac{1}{2}}}\mathcal{G}(E_{\text{threshold}})\right) \tag{4.11}$$

In particular, if the Breit Wigner distribution is defined as the spectral density $\mathcal{G}(E_{CM})$, the transcendental equation (4.11) takes the form:

$$\left|(E_r - i\frac{\Gamma}{2} - E_{\text{threshold}})^{\frac{\kappa}{2}-1}\right| e^{-\frac{\Gamma}{2}t_c} = \Gamma^2\left|\left(\frac{\Gamma(\frac{\kappa}{2})}{((E_{\text{threshold}} - E_r)^2 + \frac{\Gamma^2}{4})(t_c)^{\frac{\kappa}{2}}}\right)\right| \tag{4.12}$$

with κ being the coefficient determined by the nonlinear fitting of the experimental scattering phase shifts of the $\mathrm{P}_{3\frac{3}{2}}$ partial wave, for the considered scattering processes.

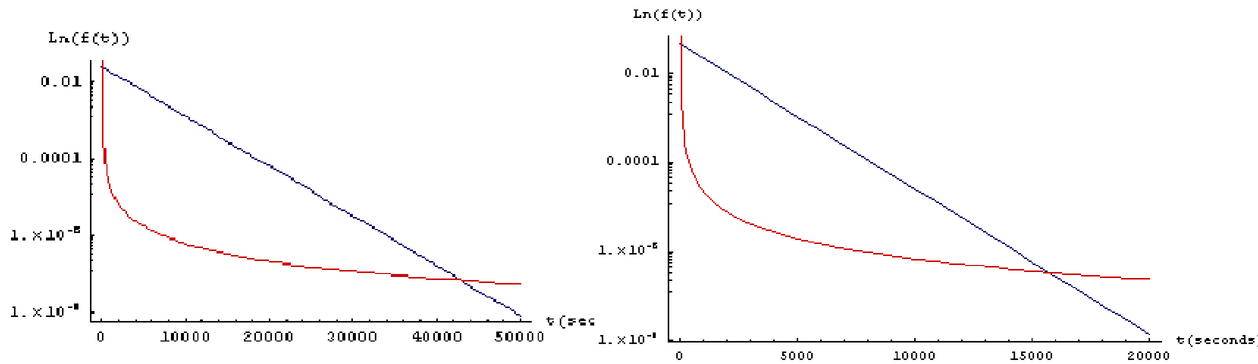

Figure 4.11: Logarithmic plots of the functions involved in the transcendental equation(left half side of (4.12)(blue plot) and right half side of (4.12)(red plot)) (4.12) in order to determine the critical time; the two intersection points are the solutions of the transcendental equation; but the physical solution is the large one, since it represents the critical time t_c; the left plot is related with the ${}^5_2He$ nucleus and the right plot is related with the ${}^5_3Li$

Therefore, in order to find the critical time t_c it is necessary to solve the transcendental equation (4.11); there are two procedures to solve it, analogically to what it was done with the transcendental equation determined by Bogdanowitz, Pindor, and Rackza[17]: Using a numerical method in order to find the intersection points between the two functions in the relation (4.11), or plotting the functions and find the possible intersection points between them. For the particular case of the considered nuclear resonances, since the width Γ, the resonance position E_r and the threshold energy of the scattering reaction $E_{\text{threshold}}$ are known, a critical time t_c value can be determined. So, in order to determine this numerical value, the transcendental equation (4.12) must be solved. Once the function on the left half side of the relation (4.12) is plotted and the intersection points of this function with the function in the right half side of the same relation are found, the critical time can be found. It is necessary to say that there are two solutions to the transcendental equation (4.11); only the large time solution has physical meaning, since it represents the critical time value t_c.

	$\mathbf{t_c}$(*Bogdanowitz[17]*)	$\mathbf{t_c}$((4.11))	$\mathbf{t_c}$*Plots(Fig.*(4.7) *and Fig.*(4.8)*)*
${}^5_2He$	12660.1s	42907.9s	42000s
${}^5_3Li$	4780.7s	15715.2s	15000s

Table 4.7: Critical time t_c values found from the transcendental equations (1.18)[17], (4.12), and determined from the behavior of the survival probability $P(t)$ showed in the plots (Fig.(4.7) and Fig.(4.8)) that characterize the transition from the exponential to the nonexponential behavior of the survival $P(t)$ for the considered scattering processes

If the plots (4.11) are analyzed, the long time solution to the transcendental equation would be $t_c \approx 42000$ seconds for the ${}^5_2He$ nucleus and $t_c \approx 15000$ seconds for the ${}^5_3Li$ nucleus; theses results agree with the transition time estimated in accordance behavior of the survival probability $P(t)$ for both processes (Fig.(4.7) and Fig(4.8)). Indeed, from a Mathematica Solve function, the critical time for the processes were estimated as the solutions of the transcendental equation (4.11). This procedure shows that the critical time, for the ${}^5_2He$ nucleus, is $t_c = 42908$ seconds; meanwhile, in the case of the ${}^5_3Li$, the estimated critical time is $t_c = 15715.2$ seconds; these results agree with the estimated time when the fluctuations show in the survival probability $P(t)$ behavior.

The different values that the critical time parameter t_c take are so long compared with the mean life time of the considered nuclear resonances; indeed, the consequent value that the survival probability $P(t)$ takes at that particular time t_c is really small; therefore, a direct measurement of this quantity looks not feasible. So, that fact indicates that the measurement of the survival probability $P(t)$ would be characterized by an exponential behavior, since the nonexponential contribution begins to appear at times so large compared with the mean life time, and at those considerable times, the deviation takes smallest values compared with the values that the survival probability $P(t)$ takes at the so called intermediate times. Nevertheless, near the threshold, the long time description of the survival probability is accurate, and the "tail" of the survival probability $P(t)$ can be identified, so that, as Kelkar, Nowakowski and Khemchandani quote as a final phrase of their work :

> We have shown that it is possible, as the information on the time evolutions is encoded in the scattering data[18]

Thus, as the long temporal evolution of the system is characterized by the phase shift behavior related with the partial waves at low energies, the nonexponential deviations implicit in the quantum mechanical description of the metastable systems, and the controverting power law behavior can be shown as a consequence of the dynamical and physical properties that a system must follow in its temporal evolution. The physical constrains restrict the evolution of the system and the physical variables that characterize the system, like the survival probability in the case of the metastable states. Besides, the evolution of the system must follow the postulates of the quantum mechanics; in the case of the metastable states, it was shown the importance of the partial wave method to characterize the scattering processes at low energies near the threshold; in particular, the behavior of the phase shifts at low energies (3.22) imposes an specific property on the density of states $\omega(E)$ that is related with the survival amplitude $A(t)$ in accordance to the Fock Krylov theorem (theorem (1.1)); as the density of states, by definition, is a continuum probability density, the restriction imposed by the phase shifts behavior affect the evolution of the metastable state in accordance to the analytical properties of the propagator or the hamiltonian $\hat{H}$ that leads to the decay process. The density of states is not only an spectral density, but it must reflect the physical constrains and the analytical properties that the system must follow in its temporal evolution. From this statement, the density of states $\omega(E)$ defined in (3.37) is the most appropriated to express the physical constrains, the analytical properties of the propagator and the phase shift behavior at low energies that restrict the temporal evolution of the system. This particular definition of $\omega(E)$ leads to a more accurate description of the temporal evolution of the system at long times, since it is in that domain when the low energy scattering theory is relevant to characterize the metastable state formation. So, the appearance of the deviations of the exponential behavior in the survival probability $P(t)$ at large times, as it can be shown in the plots (Fig.(4.7) and Fig.(4.8)) is a natural consequence of the physical constrains and the behavior of the phase shifts at low energies near threshold, as the experimental phase shifts data extracted from the considered elastic scattering reactions showed(Fig.(4.6)). Quoting what Nicolaides affirmed in one of his works [20]:

> "The search for the discovery of nonstationary states where the nonexponential decay is observable should focus on exceptional cases, i.e. on resonance states very close to the threshold."

The convenience of choosing a density of states with the analytical properties stated earlier in the third chapter arises from the description of the resonance state close to threshold, in order to find an accurate description of the survival probability at long times, and therefore, the deviations from the exponential behavior could be characterized as the formal theory predicted. The choice of the phase shifts with a particular behavior near the threshold should lead to a plausible "tail" of the survival probability at long times, as the logarithmic plots of $P(t)$ showed(Fig.(4.7) and Fig.(4.8)). In accordance to all these considerations, the best system to analyze in order to find the nonexponential behavior existing in the survival probability $P(t)$ is given by an isolated resonance, whose temporal evolution is characterized by the properties of the propagator and the hamiltonian that leads to the decay in one of its channel. Therefore, the experimental phase shifts must reflect the physical properties that must follow the density of states as it was discussed above; and the partial wave analysis is the useful tool to express the behavior of the system and its evolution, in particular at long times, since the partial wave method characterizes the low energy scattering processes better. So, what it could be a theoretical result(the power decay law and the deviations from the exponential behavior), it can be confirmed from the experimental data, in accordance to the physical constrains that must be imposed on the density of states in order to express the analytical behavior of the variable that illustrates the metastable state evolution: the survival probability $P(t)$.

CHAPTER 5

Concluding remarks

In this work, different kinds of aspects has been discussed in order to characterize the creation and evolution of the metastable states, under the influence of the analytical properties of a hamiltonian $\hat{H}$, and its importance under the assumptions of the quantum mechanics and the quantum scattering theory; from the general descriptions of the metastable state, related with the analytical properties of the hamiltonian that leads to the decay process, the description of the survival probability $P(t)$ as the most important variable that characterizes the decay, to an analysis of the experimental data, the possible meaning of the phase shifts and their relation with the dynamical properties that restrict the behavior of the physical system, and the influence of the physical constrains that lead to a nonexponential "tail" in the survival probability $P(t)$ at long times, the different aspects that determine the behavior of the unstable states has been discussed. The importance of the potential $V(r)$ and the analytic properties that it determines on the definition of the propagator and the evolution operator $\hat{U(t)}$ that shows the temporal evolution of the physical system defines the wavefunction that is related with the physical description of the state of the system, and the survival probability $P(t)$, in accordance to (1.5) and (1.6). As the survival probability is affected by the physical constrains that determines the evolution of the system, the behavior of the decay law that characterizes the decay processes is affected by those conditions, also. Some of the points that were discussed in this work were:

1. The survival probability $P(t)$, in accordance to the physical constrains that can affect the temporal evolution of the system, and the evolution operator, is not explicitly exponential for all times, as the classical picture shows; there are three different behaviors that the survival probability $P(t)$ follows: At small times, the survival probability is characterized by a gaussian behavior; at intermediate times, the survival probability follows an eminent exponential behavior that

corresponds to the classical description of the decay processes (1.1), and besides, there is a nonexponential contribution that dominates the behavior of the survival probability at long times, so that, the exponential contribution, that overlaps the nonexponential behavior at the intermediate times tends to zero faster than the nonexponential contribution, in accordance to what Khalfin postulates in his earlier works[7], from the Payley Wiener theorem(theorem (1.2)). In particular, the most important physical constrains that affect the evolution of the system are the definition of the energy spectrum $E \geq E_{\text{threshold}}$ and the inhomogeneity in the temporal domain $t \geq 0$, that leads to a new definition of the evolution operator and a nonhermiticity of the evolution operator.

2. The definition of the survival probability is determined by the analytical properties of the density of states $\omega(E)$(1.17), in accordance to what is has been called the Fock Krylov theorem(theorem (1.1))[6]:

   > The survival amplitude $A(t)$ is the Fourier transform of the density of states

   This density of states is defined as the continuum probability density to find eigenstates of a nonperturbed hamiltonian $\hat{H}_0$ in the initial physical wavefunction that defines the unstable state $\left|\Psi_0\right\rangle$. The behavior that the survival probability takes depends on the definition of the density of states, whose properties are defined by the propagator and the way the physical system evolves in time. Therefore, an analytic definition of the density of states $\omega(E)$, in accordance to the physical constrains that affect the evolution of the system, leads to different descriptions of the survival probability $P(t)$, specially in the nonexponential contributions and its behavior at long times(2.43), (2.108), (2.74), (2.67), (2.57). That is the main reason why there are some discrepancies with respect to the power law behavior that the survival probability $P(t)$ would follow at long times.

3. The appearance of a resonance is related with the analytical properties of the **S** matrix, as a single pole in the so called "unphysical sheet", which corresponds to a k complex semiplane, so that $Imk < 0$; in particular, this resonant pole is defined as $z_{\text{resonant}} = E_r - i\frac{\Gamma}{2}$; the appearance of the resonant pole arises from the zeros of the Jost Function $\mathcal{F}(-k)$, related with the irregular solutions of the Schrödinger equation, in accordance to the definition of the **S** matrix $\mathbf{S} = \frac{\mathcal{F}(k)}{\mathcal{F}(-k)}$ in the upper half of the complex k plane. Meanwhile, the appearance of poles in the so called "physical sheet" $Imk > 0$, specially in the complex positive semiaxis is related with the definition of bound states.

4. The partial waves method describes so well the behavior of a physical system at low energies, specially; if the scattering boundary wave function $\left|\Psi_{\text{scattering}}\right\rangle$ given by (3.1), is spanned in the spherical harmonics basis, as (3.13) shows, each partial wave contributes to the description of the scattering processes; nevertheless, at

low energies, only the partial waves with lower l makes a meaningful contribution, since the centrifugal barrier given by $\frac{l(l+1)}{2r^2}$ is more relevant as l takes higher values; so, for an incident particle, it would take a considerable amount of kinetic energy to overcome the centrifugal barrier and interacts with the target; meanwhile, if the incident particle has low energy, only the terms with lower l would affect the scattering process, since the centrifugal barrier with a high value of l would represent an impenetrable barrier for the incident particle.

5. The phase shifts δ_l describe the influence of the interaction or the potential in the description of a scattering processes, so that, the properties of the potential $V(r)$ affects the behavior of the phase shifts; in particular, the phase shifts have an special behavior in the presence of a metastable state: They suffer a quick variation, a "jump" in their values; the phase shifts associated with a resonance suffer a jump to π, through $\frac{\pi}{2}$. The presence of a bound state rises the zero energy value of the phase shifts by π in accordance to the Levinson's theorem. Nevertheless, there could be effects that are not related with the presence of a resonance, like the so called background effects, arising from a hard sphere scattering with the partial wave that it has been analyzed, or a scattering contributions of the other partial waves; in fact, these background scattering phase shifts could sometimes introduce a damping in the value of the total phase shift, which is the sum of the resonance phase shift and the nonresonance or background contributions (3.31). Since the partial waves method is accurate to analyze the low scattering processes, the main scattering process at low energies is determined by the behavior of the phase shifts related with the lower l, at low energies (3.22), specially near the threshold.

6. Khalfin[7] found in his work that the survival probability would behave at long time in accordance to a power law(1.22), if the density of states would be defined as (3.34), and the continuum energy spectrum would be defined as $E \geq E_{\text{threshold}}$. This analytic form that the density of states $\omega(E)$, for $E \geq E_{\text{threshold}}$ is known as a "Breit Wigner distribution" and it is the most natural result concerning to the behavior of the scattering cross section $\sigma(E)$ in presence of a resonance. Nevertheless, even though a density of states $\omega(E)$ like the one defined by Khalfin (3.34) is related with the analytical properties of the propagator that expresses the temporal evolution of the system, the description of the behavior of the physical system that the density of states gives would be incomplete, since the behavior of the phase shifts, related with the analytical properties of the potential and the **S** matrix, at low energies has not been taken into account, in accordance to the expression that Beth and Uhlenbeck[42, 43] found, which implies the relation between the density of states for an interacting system and the derivative of the phase shifts related with an specific partial wave, in function of the energy (3.35). The introduction of the phase shifts behavior at low energies would affect the behavior of the survival probability in accordance to the Fock Krylov theorem(theorem (1.1)), specially in the long time domain, where the effects of the

phase shifts at low energies, near the threshold would be more relevant. So, the density of states $\omega(E)$ would be defined as (3.37), so that, the effects of the phase shift behavior would be related with the exponent $\gamma(l)$, and the function $\mathcal{G}(E_{CM})$ would act like a spectral density, related with the properties of the propagator. So, it would have a complex pole in the "unphysical" sheet; it would tend to zero sufficiently fast as E_{CM} goes to infinity, and the function in $E_{\text{threshold}}$ would be different from zero[18]. With that in mind, the survival probability $P(t)$ would have two contributions(3.49): one exponential, determined by the residue of the function calculated on the resonant pole(3.42), and a nonexponential contribution, related with the behavior of the phase shifts at low energy near the threshold, in accordance to (3.24). In particular, it was found that the $\gamma(l) = l - \frac{1}{2}$. At long times, the nonexponential contribution overlaps the exponential one, giving the long time power law, whose exponent depends on the angular momentum$l$ as $t^{-(2l+1)}$.

7. The density of states (3.37) defined by Kelkar, Khemchandani and Nowakowski in their work[18] determines with accuracy the behavior of the survival probability $P(t)$ at long times; that is the main reason why it must be taken into account isolated resonances near the threshold, where the long time behavior of $P(t)$ would be feasible. The behavior of the density of states in the long energy domain does not matter for the purposes of this work, since its focus is precisely the long time domain, when the nonexponential contribution becomes relevant in order to describe the physical processes. Nevertheless, a Breit Wigner distribution could characterize so well the function $\mathcal{G}(E_{CM})$ implicit in the definition of the density of states $\omega(E_{CM})$, given by (3.37), since it follows all the properties that this particular function must behave; besides, the Breit Wigner distribution is related with the properties of the propagator and the evolution of the system as $\mathcal{G}(E_{CM})$ must be.

8. In particular, for the two considered elastic scattering reactions ($p + \alpha$ and $n + \alpha$ elastic scattering) that leads to the formation of the $^{5}_{3}Li$ and $^{5}_{2}He$ nuclear resonances, respectively, the S partial wave phase shifts do not show a resonance behavior (Fig.(4.1)),as it could be expected, since there is no centrifugal barrier that leads to the formation of metastable state, in accordance to the analytical properties of the effective potential $U_{\text{effective}} = \frac{l(l+1)}{2r^2} + V(r)$. Meanwhile, in the case of the P partial waves(l=1), the phase shifts show a "jump in their behavior, so that , it could be deduce the formation of a metastable state; nevertheless, the effect of the nonresonance and background effects are different for the two P partial waves, $P_{\frac{3}{2}}$ and $P_{\frac{1}{2}}$, so that , the presented damping on both phase shifts related with these partial waves is different. It can be observed that the damping in the case of the phase shifts related with the $P_{\frac{1}{2}}$ partial wave is higher, so the nonresonant and background effects are more meaningful(See Fig.(4.3) and Fig.(4.2)).

9. The behavior of the phase shifts is characterized by the nonlinear fitting function $\delta_l(E_{CM})$ given by (4.5); if this function is compared with the Breit Wigner phase shift, given by (3.32) that characterizes the resonance scattering phase shift in a localized potential, what it can be shown is the presence of background effects that affect the value of the total phase shifts value, measured in the scattering processes, and the difference between the definition of the function $\Gamma(E_{CM})$; in the case of the parametrization function, this factor $\Gamma(E_{CM})$ depends on the center of mass energy; meanwhile, in the case of the Breit Wigner phase shift, the width is a constant independent of the energy. So, the Breit Wigner phase shift is only an idealization.

10. The accuracy of the nonlinear fitting procedure is determined by the values of the coefficients E_0, Γ_0, β and κ; the parametrization function $\delta_l(E_{CM})$ adjusts so well the experimental phase shifts data, as it can be shown in the figure (Fig.(4.6)) and the values of the nonlinear fitting coefficients E_0 and Γ_0(see Table (4.5) and Table (4.4)) are near to the theoretical values of the mass and the width of the considered resonances, the nonlinear fit is accurate. So, the derivative of the nonlinear fitting function $\delta_l(E_{CM})$ can be related with the density of states $\omega(E)$ in accordance to the Beth Uhlenbeck theorem (3.35).

11. The κ coefficient is related with the behavior of the phase shifts at low energies, near the threshold. In particular, κ is related with the $\gamma(l)$ exponent in the density of stated described by (3.37), as $\frac{\kappa}{2} = \gamma(l) + 1$; so, theoretically, this coefficient would be related to l as (4.9). For the case of the P partial waves, where the resonance behavior was found from the experimental data, κ would take a theoretical value of 3, in accordance to (4.9), and the long time behavior of $P(t)$ would be proportional to t^{-3},(3.50). The values obtained by the nonlinear fit for both scattering processes are so closed with respect to the theoretical value of κ, in accordance to (4.9)(See table(4.6)); so, the experimental results agree with the theoretical assumptions that the model affirms. Besides, as the survival probability $P(t)$ can be defined from the value of the κ coefficient (4.7),and the power law characterizing the long time behavior of $P(t)$, as (4.8), it can be shown that the results for the both scattering processes agree with the theoretical expectations, as the power law for both elastic scattering processes takes the same analytical form, except for a discrepancy in the case of the ^5_3Li nuclear resonance, where $\kappa = 3.017$. Nevertheless, this is a nonessential discrepancy, and the power law behavior is so closed to the one predicted theoretically (Fig.(4.9)).

12. For the considered elastic processes, the dominance of the nonexponential behavior of the survival probability $P(t)$ was shown, as the tail that leads to the power law behavior begins to appear after the so called "critical time" t_c, when the exponential and nonexponential contributions of the survival probability $P(t)$ are comparable. Once the critical time has been reached, the survival probability starts to fluctuate, and then, the nonexponential contribution overlaps the

exponential behavior, and the tail in the survival probability appears, moving away from the exponential behavior, that dominates in the intermediate times (See Fig.(4.8) and Fig.(4.7)). This statement agrees with the statement Khalfin affirmed in his work, since the exponential contribution tends to zero faster than the nonexponential contribution at long times.

13. The critical time t_c depends on the properties of each resonance, and it is defined as the time when the exponential contribution is comparable to the nonexponential contribution of the survival probability $P(t)$. It is the long time solution of a transcendental equation given by (4.11), so depends on the behavior of the $\mathcal{G}(E_{CM})$ in the threshold,and its derivative calculated on the resonant pole $E_r - i\frac{\Gamma}{2}$. Even though Bogdanowitz, Raczka, and Pindor[17] estimated a possible value of the critical time for the survival probability, they considered a Breit Wigner distribution as a density of states, like Khalfin did; so the discrepancy between the theoretical results given by a numerical solution of the transcendental equation they found (1.18) with the experimental results given from the analysis of the plots (Fig.(4.7) and Fig.(4.8)) and the numerical solution of the transcendental equation that depends on the value of the κ coefficient, extracted from the non-linear fitting of the experimental data, (4.12) is explained by the consideration of the phase shifts behavior at low energies near threshold in the definition of the density of states given by (3.37). Even though the critical time results(see table (4.7)) are really large compared with the mean life time of these two nuclear resonances, the long time behavior of the survival probability is seen in the analytic description of the evolution of the resonances, and it has been shown in the analysis of the data(See Fig.(4.7) and Fig.(4.8)); therefore, it is not only a theoretical development in accordance to the postulates of the quantum mechanics; it's an experimental result from the behavior of the experimental phase shifts data that has been analyzed.

CHAPTER 6

Open questions and problems to solve

Once the most accurate description of the metastable states for long times has been discussed, there are a couple of open questions that can be analyzed, in order to find out a complete description of the evolution of the resonances and the decay processes:

- As the nonlinear fit was used in order to characterize the behavior of the phase shifts at low energies, and obtain the nonexponential contribution in accordance to the properties of the density of states, one open question would be to use a Fast Fourier Transform or another numerical method in order to extract from the phase shift behavior, the density of states in accordance to the Beth Uhlenbeck relation (3.35). It was not the focus of the present work, since the description of the properties of the density of states, given by (3.37) and the accuracy of the nonlinear fit were necessary to identify the nonexponential behavior at long times explicitly. But it would be interesting to find the possible behavior of the survival probability $P(t)$ from a numerical integration method, in accordance to the behavior of the phase shifts at low energies near the threshold, Fock Krylov theorem (theorem (1.1)) and the Beth Uhlenbeck relation (3.35).

- In the present work, only nuclear resonances with a low number of nucleons are analyzed, since the available data corresponds to these particular physical systems. It would interesting to analyze nuclei with a large number of nucleons, that experiment some sort of decay process, and characterize the survival probability for these processes as it could be done for the ${}^{5}_{2}He$, and ${}^{5}_{3}Li$ nuclei. Besides, it would be very interesting if the consequent critical time could be determined for the heavy nuclei, and its relation with the properties of the heavy nuclear resonances.

- One of the main questions to solve is an analytic expression of the nuclear potential. In fact, since the density of states is related with the analytic properties of the propagator, and hence the interacting potential, maybe in accordance to the definition of the phase shifts and their behavior, a nuclear potential could be characterized, extracted from the physical information that the phase shifts carry.

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