Abstract

The subject of Neutron Star Cooling is discussed. First, some general properties of neutron stars are treated in order to put the problem of Neutron star cooling in context. Cooling curves of isolated Neutron stars undergoing neutrino emission are calculated, and are finally compared with actual observed temperatures.

Neutron Star Cooling

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Contents

Introduction

We have come a long way since Neutron Stars were first postulated as theoretical constructs in the 1930's. Ever since the first pulsar was observed in the 1960's, N.S's (Neutron stars) have been studied extensively and are still subject of research today. The very high densities present in N.S's ($\sim 10^{14}\, g/cm^3$) and extreme conditions present are by far unreachable here on earth. Consequently, N.S.s are almost ideal astrophysical laboratories that we can use to study matter in extreme conditions and to test our current knowledge of physics. As we shall see, a beautiful mixture of particle physics, statistical mechanics and general relativity, among other fundamental fields, are used to describe N.S.s.

The existence of Neutron stars was first proposed by Landau shortly after the Neutron was discovered (1931). The idea was then further refined by Baade and Zwicky in 1934, in their pioneering work on Supernovae. They made the accurate prediction that N.S.s were formed in Supernova explosions. Later, in 1938, the first N.S. model was made by Volkoff and Oppenheimer, a model being strongly based on General Relativity. For the next thirty years the idea of the neutron star was pretty much left aside on the basis that N.S. theory seemed to be a wild extrapolation of modern physics and also because it was thought that these objects would simply be too faint to be observed. It was not until 1967, when Jocelyn Bell and Anthony Hewish, discovered the first Pulsar (An object emmiting periodic radio signals). At first it was thought, that these periodic pulses came from extraterrestrial life. Soon a more convincing mechanism for the emission of these pulses was understood (Syncrotron radiation), and ever since, Neutron stars have been studied and observed extensively[1].

Natural questions concerning N.S's such as what are they made of (besides Neutrons, obviously), what are their dimensions, and how do they evolve are still being made today. Many plausible answers to these questions have been given, so all that can be done is try to discriminate dome of the answers given. How can this be done? Simple. We need to compare our theoretical models with observations. As we shall see, one of the most important observations that can be made on a N.S. is it's temperature.

[1]Historical facts were found in [2, 5, 4]

The temperature-Time curve depends strongly on the type of matter and it's structure, and the former determines the Equation of state. The E.O.S. (Equation of State), together with General Relativity give us the equilibrium conditions (maximum mass and radius). Consequently, by constructing cooling curves for various types of matter present [2] and comparing with observations one can not only constrain the E.O.S. and determine properties of the N.S., but also determine if exotic matter is actually present in the universe.

The "core" of this document is presented in chapter III, but the importance of the previous two chapters lies in the fact that they put the problem of Neutron star cooling in context. Also, some important results will be derived in the first two chapters, which besides from being important to the problem in question, are fundamental results in the field of astrophysics. This document tries to be as self contained as possible, so that some of the equations presented in the document are derived and treated a bit more deeply in the appendixes.

[2] Among the various types of matter considered are: Ordinary matter, quark matter, Pion condensates, Kaon condensates, etc.

Chapter 1

Neutron Stars

Main sequence stars produce energy by fussion, starting from hydrogen, and progressing towards heavier elements. When ^{56}Fe is reached, it is no longer energetically favorable to produce heavier elements, so gravity starts to take over until the counter-pressure produced by Pauli repulsion in the electrons may leave the star in equilibrium. This new equilibrium state is commonly called a *White Dwarf*. However, if the mass of the star is sufficiently large ($> 1.4\, M_\odot$, $M_\odot$ being the solar mass), the zero point pressure (produced by the Pauli exclusion principle) is no longer capable of supporting the star against gravitational collapse. A massive implossion and then an explosion, called a *Supernova*, takes place. The remnant of this spectacular event is what is now known as a *Neutron Star*.

This remnant Neutron star has a mass of $\sim 1.5 M_\odot$ and a radius of the order of $\sim 10km$ [1, 4, 2], thus corresponding to densities of the order of $10^{14} g/cm^3$. These densities correspond to energies much greater than the threshold energy for electron capture, so we expect that the neutron star is composed primarily of neutrons. The counter-pressure that keeps the N.S. from collapsing is the zero point pressure for a degenerate gas of neutrons. Just as in the case of White Dwarfs we expect there to be a limiting mass, and since a degenerate gas of neutrons might be the densest form of matter, the outcome of a possible collapse is a *Black Hole*.

Neutron Stars may emmit periodic radio signals and are thus called *Pulsars*(See chapter II). Currently, more than 1000 pulsars have been detected [5]. Neutron stars also emmit thermal radiation in the X-ray frequency range, and when in a binary accreting system, they also emmit radiation in the X-ray region. Neutrinos, that are virtually undetectable[1], are emmited in abundance. These neutrinos, as we shall see, are crucial in the cooling process. In the following subsections, some properties of neutron stars will be discussed in more detail, and the problem of neutron star cooling will be put in context.

[1]It is hard enough to detect neutrinos from the sun. Recall that neutrinos only interact via the weak interaction.

1.1 The Origin of Neutron Stars

We shall now discuss the case in which the mass of the initial main sequence star is greater than the limiting Chandrasekar[2] mass of $1.4M_{\odot}$. We shall briefly describe the Supernova event.

1.1.1 Type II Supernovae

Neutron Stars are formed in type II Supernovae[3]. This event only lasts a few seconds after nuclear fussion has completely halted. Since the electron pressure is not enough to keep the star from collapsing, an implosion occurs.

Soon, densities greater than $8 \times 10^6 g/cm^3$ begin to appear [4]. This is the minimum density for electron capture to take place ($p + e \rightarrow n + \nu$). Since electron capture is the key mechanism responsible for the Neutron Star formation, it might serve us to calculate the minimum density for which this process occurs.

The minimum energy required for ^{56}Fe to capture energetic electrons capture can be found by simply saying that

$$M(A{=}56, Z{=}26)c^2 + E_{min} = M(A{=}56, Z{=}25)c^2 \ , \tag{1.1}$$

giving
$E_{min} = 1.88 \times 10^{-13} J = 1.803 MeV$.

It is now plausible to say that this minimum energy is mostly supplied by the degenerate electrons with Fermi energy[4] $E_{min} = E_f(e)$

$$E_f(e) = \frac{\hbar}{2m_e}\left(3\pi^2 \frac{N_e}{V}\right)^{2/3} = 1.803 MeV \ . \tag{1.2}$$

Noting that $N_e = N_p$ because of charge conservation, and that the matter density can be expressed as

$$\rho_{min} = \frac{N_p}{V}\left(\frac{A}{Z}\right) m_B = \frac{N_e}{V}\left(\frac{A}{Z}\right) m_B \tag{1.3}$$

Were $\frac{A}{Z}$ is the number of barions per electron, and m_B is the mass of a barion. In the case of ^{56}Fe, $\frac{A}{Z} = 2.15$ [5]. We can now rewrite eq. (1.2) as

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[2]Chandrasekar predicted the maximum mass for a white dwarf by modeling it as a degenerate gas of electrons.

[3]In type I supernovae, there is initially a White Dwarf star that accretes mass from its binary companion. When the white Dwarf has accreted enough mass from its companion, a carbon-oxygen core is ignited and a shock wave is produced that destroys the star, there is no Neutron Star left [6].

[4]Eq. 1.2 is a known result for the energy of a gas of fermions at $T = 0$

$$E_f(e) = \frac{\hbar^2}{2m_e}\left(\frac{3\pi^2 Z}{m_B A}\right)^{2/3} \rho_{min}^{2/3} \,. \tag{1.4}$$

So

$$\rho_{min} = \left(\frac{E_f(e)2m_e}{\hbar^2}\right)^{3/2} \frac{m_p A}{3\pi^3 Z} = 8.64 \times 10^6 g/cm^3, \tag{1.5}$$

which is consistent with what is mentioned by Baym [4].

This density is still much less than the Nuclear density, this being $\rho_0 = 3 \times 10^{14} g/cm^3$ [4], consequently, there is a massive neutronization and neutrino production. The collapse halts when the density approximately reaches ρ_0, due to the high incompressibility of nuclear matter. The resistance to the collapse of this very dense core is in part responsible for a shock wave that turns the implosion into an explosion [6]. For an instant, the high opacities inside the star, make it impossible for the produced neutrinos to escape. Further neutrino production is momentarily suppressed because a degenerate gas of neutrinos is momentarily formed, i.e. for a neutrino to be produced, it would have to occupy a state that is probably already occupied.

It is thought that Neutrinos play a crucial role in the final ejection of the stellar mantle [1, 6]. Once the trapped neutrinos escape and thus, the stellar mantle is ejected, there is “room” (In phase space terms) for further neutronization to occur. Note that further neutronization causes the reduction of the number of electrons per unit volume, and thus a further decrease in the electron or zero point pressure[5]. Only when the concentration of neutrons becomes high enough, the star remnant can reach equilibrium because of the zero point pressure produced by the newly formed neutrons. What is now left is a star composed mainly of neutrons (hence the name of the star) with an initial temperature of $T_i \sim 10^{11}\ ^\circ K$ [3, 1].

It is now illuminating to calculate the Fermi energy of a gas of neutrons at nuclear density. Using eq.(1.2) we find $E_f(n) \approx 1.38 \times 10^{-11} J = 86.13 MeV$. This corresponds to a Fermi temperature of $10^{12}\ ^\circ K$. The temperature of the initial neutron star is roughly one order of magnitude less than the Fermi tempereture, *so the N.S. can be regarded as a degenerate neutron gas* (See figure 1.1.1). The fact that the initial N.S. is degenerate can be seen in figure . Note that as the temperature drops, the N.S. will become even more degenerate . This is the reason why a neutron star can be thought as a cold star. It is important to restate that the N.S. now supports itself against gravitational collapse by the neutron zero point pressure.

The problem of N.S. cooling will be treated with these initial conditions. As a final remark, it is worth it to say that the subject of Supernovae is by no means fully understood and is still subject to research today.

[5] $E_f(e) \propto \left(\frac{N_e}{V}\right)^{2/3}$ and $P_e \propto \left(\frac{N_e}{V}\right)^{4/3}$

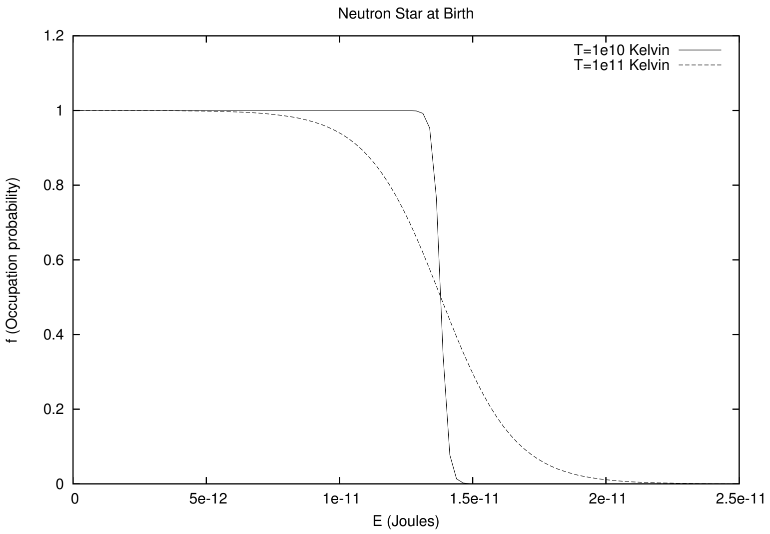

Figure 1.1: The Fermi distribution is plotted for an initial temperature of $10^{11}\,^{\circ}K$, and then for a temperature of $10^{10}\,^{\circ}K$, as we shall see, the former temperature is reached in only a few hours!. Note that almost all the energy levels beneath the Fermi energy are occupied. As the temperature keeps dropping, the N.S. will become even more degenerate.

1.2 Neutron Star Structure

The process of N.S. formation gives us a rather good idea about the global structure of a N.S. . Although a N.S. is composed mainly of neutrons, heavy nuclei also exist in the crust and some protons and electrons exist in the core along with the neutrons. First we shall discuss the global structure in more detail. Then, the equation of state, and the limiting masses of the N.S. will also be treated.

1.2.1 Composition

A cross section of a tipical N.S. is shown in figure (1.2). In the surface one expects there to be an abundance of ^{56}Fe, electrons are degenerate just as they are in a White dwarf star. As one goes further inside, the threshhold density for electron capure is reached[6] ($\rho_{min} = 8 \times 10^6$), so nuclei become more and more neutron rich, reaching nuclei such as ^{118}Kr at a density $\rho_d = 4.3 \times 10^{11} g/cm^3$ [4, 2]. These nuclei, although unstable in a laboratory, do not present β decay ($n \rightarrow e + p + \overline{\nu}$) because there is still a degenerate electron fluid. In other words, if a heavy nucleus such as ^{118}Kr, were to present β decay, the electron would have to go to an energy state that would probably be occupied. The outer region of the neutron star containing nuclei is called the *Crust* [4, 2].

[6]See subsection 1.1.1

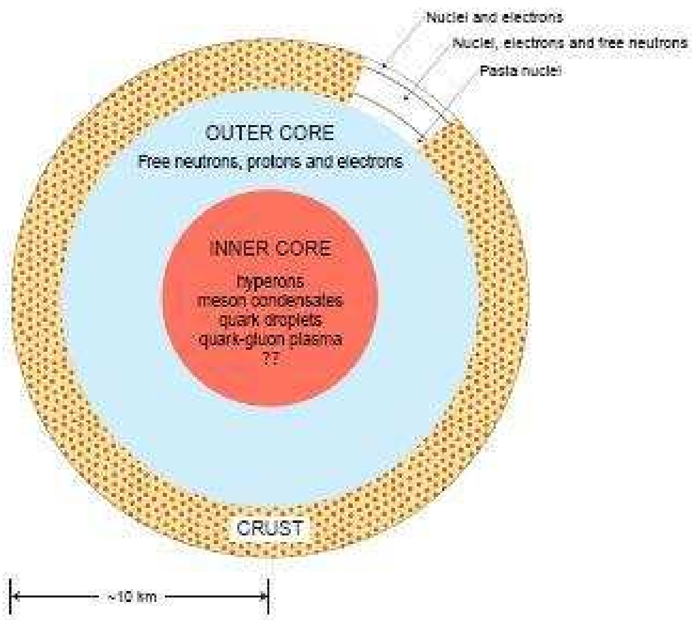

Figure 1.2: Schematic cross-section of a N.S.. Picture taken from the article of Baym [4]

The density $\rho_d = 4.3 \times 10^{11} g/cm^3$ is commonly called the neutron drip density. At densities greater than ρ_d, nuclear forces begin to saturate, as a consequence of them being short range. What happens now is that neutrons begin to "drip" out of the nuclei. As neutrons begin to drip out, unusual structures begin to appear. The shape of nuclei changes from spherical to rod like, to laminar. These unusual structures are called *Pasta Nuclei* [4]. Then, as densities keep increasing, and reach nuclear density ($\rho_0 = 3 \times 10^{14} g/cm^3$), matter dissolves into a uniform liquid that is primarily composed of degenerate neutrons with a few protons and electrons. The inner region, after the crust, is called the *Core*. As we shall see, the amount of protons inside the N.S. will slightly vary the equation of state and might have dramatic effects on the way the N.S. cools.

There are still uncertainties about the type of matter that exists at even higher densities. Pion or Kaon condensation may occur and form a superfluid Bose-Einstein condensate. Although superfluidity will not be treated in detail, it might also have strong effects on N.S. cooling. At ultrahigh densities, it is thought that free quarks might exist [4]. The core of the N.S. constitutes about 90% of the total volume. There is also an atmosphere composed of hydrogen that has no notable contribution on the total mass of the star, but plays an important role in shaping the emitted photon spectrum [1].

1.3 The Equation of State

In this section, we shall assume, as a first (and reasonable) approximation, that the N.S. is composed entirely of neutrons. This is a valid approximation because the core, which is composed primarily of neutrons constitutes 99% of the mass of the Neutron star [1]. We shall also assume, that the neutrons are in the ground state. This is valid to say because, the neutrons are highly degenerate[7]. One final and crucial assumption, is to take the neutrons as non interacting[8]. The main reference for this section is Reddy's article [5].

1.3.1 The pure Neutron Star

To obtain the energy of the ground state of a pure neutron gas, we can proceed as follows [5].

$$U = 2 \sum_{|\vec{p}|=0}^{|\vec{p}|=p_f} \sqrt{\vec{p}^2 c^2 + m^2 c^4} \tag{1.6}$$

Here the sum is made over the neutron momentum. The factor of 2 just takes into account the spin of the neutron and p_f is the Fermi momentum. We may now use the density of states (V/h^3) and write eq. (1.6) as an integral.

$$U = \frac{2V}{h^3} \int_{p=0}^{p=p_f} \sqrt{\vec{p}^2 c^2 + m^2 c^4}; d^3\vec{p}. \tag{1.7}$$

Now changing to spherical coordinates, the previous eq gives

$$U = \frac{8\pi V}{h^3} \int_{|p|=0}^{|p|=p_f} \sqrt{p^2 c^2 + m^2 c^4}\; p^2\; dp. \tag{1.8}$$

Solving this eq might not be as illuminating as looking at the relativistic and non relativistic limits of eq. (1.8).

First lets look at the non relativistic limit (making a series expansion for small momenta).

$$U = \frac{8\pi V p_f^5}{h^3 10 m_n} \tag{1.9}$$

We now need a relation between p_f and V given by

$$N = \frac{2V}{h^3} \int_0^{p_f} d^3\vec{p} = \frac{8\pi V p_f^3}{3h^3}, \tag{1.10}$$

[7]See subsection 1.1.1

[8]If interactions are taken into account, the bare mass of the neutron should be replaced by the effective mass.

So that

$$p_f = \left(\frac{3Nh^3}{8\pi V}\right)^{1/3}. \tag{1.11}$$

Eq. (1.9) becomes

$$U = \frac{8\pi}{10m_n h^3}\left(\frac{3Nh^3}{8\pi}\right)^{5/3} V^{-2/3}. \tag{1.12}$$

We can now calculate the pressure using $P = -\left(\frac{\partial U}{\partial V}\right)_{T=0}$, so that

$$P_{non-rel} = \frac{8\pi}{15m_n}\left(\frac{3Nh^3}{8\pi V}\right)^{5/3} \tag{1.13}$$

$$= \frac{8\pi}{15m_n}\left(\frac{3h^3}{8\pi m_n}\right)^{5/3} \rho^{5/3} \tag{1.14}$$

Note that $P \propto \rho^{5/3}$. This is commonly called a polytropic equation of state with $\gamma = 5/3$.

Next we calculate the equation of state for the very relativistic case.

$$U = \frac{8\pi V}{h^3}\int_0^{p_f} p^3 c\, dp = \frac{2\pi c}{h^3}\left(\frac{3Nh^3}{8\pi V}\right)^{4/3} \tag{1.15}$$

$$\Rightarrow P_{rel} = \frac{2\pi c}{3h^3}\left(\frac{3Nh^3}{8\pi V}\right)^{4/3} = \frac{2\pi c}{3h^3}\left(\frac{3h^3}{8\pi m_n}\right)^{4/3} \rho^{4/3}. \tag{1.16}$$

This is a polytrope with $\gamma = 4/3$. The non relativistic case corresponds to low pressures and low densities. In the relativistic case, neutrons are taken to be highly energetic and thus the approximation corresponds to high pressures and high densities within the star. More generally, one can express the equation of state as [5]

$$P = A_{n-r}\rho^{5/3} + A_r\rho^{4/3}. \tag{1.17}$$

These results are of course very approximate because we have not taken into account nucleon-nucleon interactions, a few more sophisticated models have been proposed that include interactions. The subject of the equation of state is still subject to research today[9].

[9] See Shapiro [2]

The methods used to calculate the equations of state, apply just as well for White dwarfs, but instead of there being a degenerate gas of neutrons, one takes there to be a degenerate gas of electrons. As we shall see in the next section, the equation of state is crucial in determining the maximum and minimum masses of a neutron star.

1.4 Neutron star stability

In section 1.1.1 we concluded that the equation of state could be expressed in two limits: A non relativistic limit (eq. 1.14) which corresponds physically to low pressures and densities, and a very relativistic limit (eq. 1.13) that corresponds to extremely high pressures and densities.

In previous sections (see 1.2.1) we have also stated that there is a density at which neutrons start to drip out of nuclei ρ_d. This density can be thought of as the limit of unstable white dwarfs because it corresponds to the transition from heavy nuclei to a neutron liquid. Similarly, we shall see that there is a maximum density allowed, which is thought to be $\rho_{max} = 5\times10^{15} g/cm^3$ (Pretty close to nuclear density.).

We shall now present the famous equations for stellar equilibrium, and along with the densities mentioned in the preceding paragraph and the equations of state, we can calculate the limiting masses and radii for Neutron stars.

1.4.1 Masses and Radii

In a normal main sequence star of $1M_\odot$, the Shwartzschild radius ($2GM/c^2$) is of the order of a few kilometers and is thus negligible to the radius of the star. This is why Newtonian gravity can be used to describe the main sequence star[10]. On the other hand, a typical N.S. will have a radius of a 10 or 12km, which is comparable to its Swartzchild radius, so a General Relativistic description of gravity must be used. The equations used to describe the hydrostatic equilibrium of a General Relativistic star are the following pair of coupled differential equations.

$$\frac{dp(r)}{dr} = \frac{-Gm(r)\,\rho(r)}{r^2}\left(1+\frac{p(r)}{\rho(r)c^2}\right)\left(1+\frac{1+4\pi r^3 p(r)}{m(r)c^2}\right)\left(1+\frac{2Gm(r)}{c^2 r}\right)^{-1} \quad (1.18)$$

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r) \quad (1.19)$$

These are commonly known as the Tolman-Oppenheimer-Volkov (O.V.) equations, and have a purely General relativistic origin[11]. Now that we know the equation of state $p(\rho)$, and values for central densities (ρ_{max}, ρ_{min}), we can solve this system

[10] If the star were to shrink beyond this radius, it would collapse into a Black Hole.

[11] See Appendix A for a derivation of these equations.

of differential equations (usually done numerically). What one obtains is a solution for $m(r)$ and $\rho(r)$. The radius 'r' at which the density drops to zero, is the radius of the star. Now with the O.V. equations and the equation of state we can construct a "Stellar model".

As an example of an application of the O.V. equations, we can calculate the minimum mass at which a Neutron star could possibly be stable. If one solves the O.V. equations taking the neutron drip density ρ_d as the minimum density and eq. (1.14) as the equation of state, one obtains what is shown in figure 1.3. One obtains the minimum stable mass of a N.S. to be around $0.03M\odot$, which is consistent with that obtained in the literature (see Shapiro [2]). This result is however a bit unrealistic in the sense that supernova remnants usually have masses larger than this. The minimum mass of a supernova remnant is usually of the order of $1M\odot$ [1], which is actually closer to the maximum allowed mass of a White Dwarf[12].

To find the radius and density corresponding to the alleged minimum mass of $1M\odot$, we can vary the central density and solve the O.V. equations. A mass of $1M\odot$ is found at a central density of $\rho_c = 1 \times 10^{14} g/cm^3$ using a relativistic equation of state (Eq. 1.13). This "Low density" star is found to have a radius of $\approx 91km$.

Now if one starts to increase the central density, and constructs a model of the form of figure 1.3, using a relativistic eq of state of the form 1.13, one obtains a maximum allowed mass of the order of $1.75M\odot$ as shown in figure 1.4. This maximum mass corresponds to a central density of the order of $5 \times 10^{15} g/cm^3$ and is consistent with what is mentioned by Lattimer [1]. It is natural to expect a maximum mass corresponding to a density a few times greater than nuclear density, since it is thought that no fundamental force can support densities greater than $5 \times 10^{15} g/cm^3$.

The results of this section will be used in later chapters and can be summarized as:

$$1M\odot < M < 1.75M\odot \tag{1.20}$$

$$12km < R < 91km \tag{1.21}$$

$$1 \times 10^{14} g/cm^3 < \rho_c < 5 \times 10^{15} g/cm^3 \tag{1.22}$$

These results of course vary (Drastically in some cases.) if one uses more sophisticated equations of state that take into account nucleon interactions, for this reason, there is still no agreement on the limiting masses (especially on the maximum mass). Nevertheless, these results are good enough for our present purposes.

[12] $M_{max} \sim 1.44M\odot$. This is the famous Chandrasekhar result.

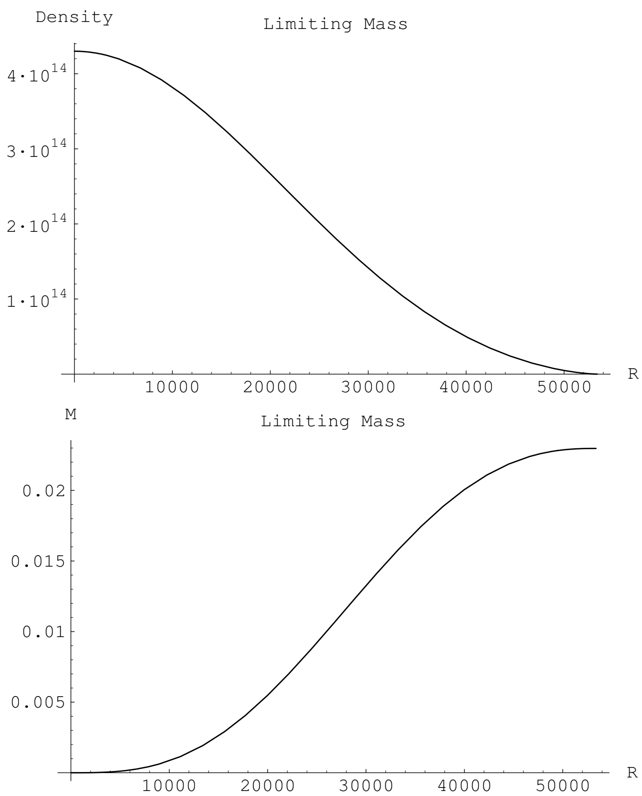

Figure 1.3: Stellar model for a low density neutron star (corresponding to an eq. of state of the form of eq. 1.14). The radius is measured in meters, the mass in solar masses and the density in $[g/cm^3]$. Note that the density goes to zero at the radius of the star. The minimum mass can be calculated by evaluating $m(r)$ were the density goes to zero. Note that the minimum mass is of the order of 0.03 $M\odot$.

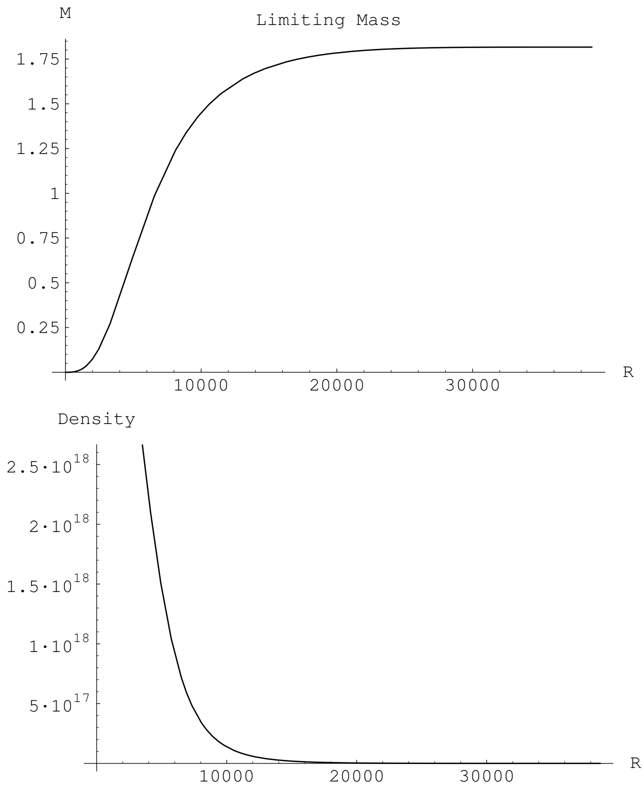

Figure 1.4: Model for a high density N.S., taking the central density to be $5 \times 10^{15} g/cm^3$. The density goes to zero at the radius of the star. The maximum mass can be calculated by evaluating $m(r)$ were the density goes to zero. $M_{max} \approx 1.75 M\odot$.

Chapter 2

Observations

Neutron Stars were first observed as *Pulsars*, which are objects emmiting periodic radio signals, today, more than 1000 pulsars have been detected. In this chapter, we shall briefly describe thermal and non thermal related radiation emission. Mass measurements will not be discussed; still, it is worth saying that these measurements are obtained with the help of Kepler's third law observing N.S. binaries. The inferred masses are consistent with the masses obtained in the previous chapter, and thus with our knowledge of late stellar evolution [2]. In this chapter we shall be discussing the observations made on isolated N.S.s, which can have masses and radii in the ranges calculated in the previous chapter.

2.1 Pulsars

The high spin rates[1] that N.S. possess give rise to extremely high magnetic fields of the order of $10^{12}G$, the magnetic axis usually makes an angle with the rotation axis as shown in figure (2.1). The rotating magnetic field thus produces an electric field (By Faraday's induction law.) that ejects electrons from the surface. These ejected electrons will follow helicoidal paths along the magnetic flux lines and will thus emmit radiation as shown in figure (2.1).

Now, if by any chance, an observer is momentarily colinear with the magnetic axis, he (or she) will observe periodic light signals, typically ranging from milliseconds to seconds. This radiation is usually in the broad-band radio range. What we just briefly described is called the pulsar mechanism, and is also known as the *Lighthouse model* [12]. The pulsar depicted in figure 2.1 has a relatively narrow radiation beam, of only a few degrees across [12]; as a consequence of this, not all N.S.s can be observed as pulsars. The radio pulses described can be detected by radio telescopes here on earth, and were first observed in 1967 by Jocelyn Bell, a graduate student;

[1] Since the radius of the star decreases during the Supernova event, conservation of angular momentum causes the newly born N.S. to have very high spin rates, of the order or 1 full revolution per second.

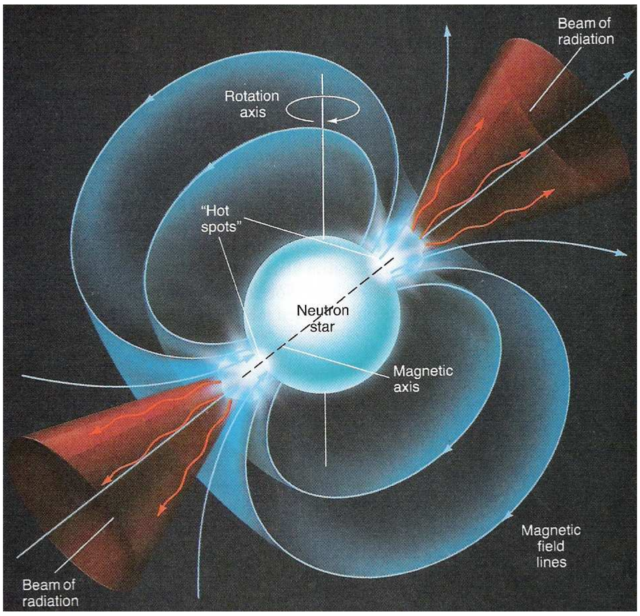

Figure 2.1: A schematic representation of a Pulsar. Taken from *Astronomy Today.* [12]

their interpretation however, was given by her thesis advisor, Anthony Hewish.

It is important to emphasize that this radiation is not thermally related, and is not emitted directly from the surface. Still, pulsars are worth being mentioned because once such an object is discovered, one is almost certain to have found a N.S., and thus, more observations can be made. In particular, once a pulsar is found, thermal observations can be performed.

2.1.1 Age determination

The age of N.S.s can only be known exactly if the pulsar is related to a historical supernova. For example, the Chinese documented in the year 1054, the first supernova. Today this supernova remnant is known as the *Crab Pulsar*. Kepler and T. Brahe also documented supernovae in 1604 and 1572 respectively, however, these are not documented pulsars. There are 7 historical supernovae that have occurred in this galaxy, but the only one were we can be sure that a N.S. exists is the Crab Pulsar because, as it's name suggests, it is a documented pulsar [2].

In case matter ejected from a supernova is detected, one could in principle measure its velocity. If the veocity can be measured (not always) one could calculate the time at which all matter was at the same place, and thus calculate the age of the supernova remnant, which is in many cases, a N.S.. This has actually been done for the Crab and Vela pulsars, and the age determined for the Crab pulsar coincides with what was documented by the Chineese [12].

For the rest of the observed pulsars, the age cannot be determined exactly. Still, a caracteristic time can be associated by knowing the period P and the period-time variation $\dot{P}$. A characteristic time τ can be defined by noticing that

$$\frac{dP}{P} \sim \frac{dt}{\tau} \tag{2.1}$$

so that

$$\tau \equiv \frac{P}{\dot{P}} \tag{2.2}$$

The estimation of a pulsar age was given by Gunn and Ostriker (1969), with the help of their magnetic dipole model for pulsars [2]. They found that to a good approximation, the age of a pulsar can be written as

$$t \approx \frac{\tau}{2} = \frac{P}{2\dot{P}} \tag{2.3}$$

The importace of the previous expression is that it only depends on measurable variables, in particular, the period of a pulsar can be measured with a precission of up to 14 significnt figures [2]. The previous equation gives a good estimation of the known age of the Crab Pulsar (A discrepancy of 325 years [2]).

2.2 Thermal emission

Thermal emission differs from non-thermal emission in the sense that the former originates at the surface of the N.S.. The wave length of the thermal light we perceive from N.S.s is in the X-Ray region. A rough estimation of the temperature using $E = k_B T = hc/\lambda$ yields a temperature of the order of $\sim MK$. Observations of thermal emission became possible after the launch of the *Einstein* satelite (1979) and *EXOSTAT* satellite (1983) [13], which are equipped with X-ray telescopes. The latest X-ray observations were made by the *Chandra* satellite.

There are several forms of surface emission. Perhaps the strongest signal is emitted by the magnetic poles of the N.S.. The accelerated particles create “hot spots” near the magnetic poles like the ones depicted in figure 2.1. These hot spots emmit radiation in the x-ray range, however, this radiation is considered non-thermal because it is related to accelerated charged particles. One can then observe x-ray pulsars. With the Chandra satelite, one can not only observe x-ray pulsars due to the magnetic poles, but also small pulses due to tiny variations in the surface temperature [13].

Taking the spectrum of the surface of a N.S. is not an easy task because non-thermal radiation is so bright that thermal radiation is hardly visible [13]. This is specially true with young N.S.s, with ages of the order of $10^3 yr$. Also, the ejected material from the supernova can be hot enough for x-rays to be emitted, these can shadow the ones being emitted from the N.S. itself.

Once the spectrum of the surface is taken[2], there are a few ways of finding the surface temperature. One is making a black-body fit of the spectrum, and associating a temperature to the fitted curve. If the curve cannot be fitted using a black-body curve, it can be fitted assuming that the atmosphere of the N.S. is composed mainly of hydrogen (Hydrogen atmosphere model.) [13]. The lack of spectral absorption lines corresponding to heavy elements or even Helium suggests that the Hydrogen atmosphere model is plausible; also, strongly magnetized hydrogen does not seem to have visible spectral lines in the x-ray region [13]. The obtained temperatures for N.S.s are shown in figure 2.2. For the purpose of this document, we would prefer cooling curves, but temperatures alone are hard enough to obtain. Not all objects in figure 2.2 are documented pulsars (The ideal case); in fact, most are "x-ray point sources" in the middle of supernova remnants. The former statement poses doubts as to wether some of the "x-ray point sources" are indeed N.S.s.

In the next section we shall discuss the relation between the observed temperature T_∞ and the actual surface temperature T_0.

2.3 The Observed Temperature

Recalling the discussion of section 1.4.1, which implies that General Relativistic effects become non-negligible, one can conclude something very important about the observed temperature. As we shall see, the observed temperature here on earth is actually less than the temperature one would observe at the surface of the N.S.. For the following discussion, some knowledge of General Relativity is required, the result final result of the "temperature redshift" is given in eq. (2.27) and (2.26).

2.3.1 The Temperature Redshift

If we neglect the effects of rotation, the geometry of space-time near the N.S. radius can be described by the spherically symmetric Schwarzschild metric[3]. This mertic ($g_{\mu\nu}$) can be written as [15]

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{2.4}$$

[2] A spectrum may be taken by graphing number of photons detected for a definite energy or wavelength.

[3] To neglect rotation is a very bold thing to say because of the high spin rates that N.S.s possess. Rotation could be taken into account using a more realistic *Kerr* metric, but the algebra is simply too long.

Source	t $[kyr]$	T_∞ $[MK]$
PSR J0205+5449	.82	< 1.1
PSR B0531+21 (Crab)	1	< 2.1
RX J0822-4330	2-5	1.6-1.9
1E 1207-52	≤ 7	1.1-1.5
PSR 0833-45 (Vela)	11-25	0.65-0.71
PSR B0656+14	~ 110	0.91 ± 0.05
PSR 0633+1748(Gemina)	~ 340	$5.6\,(+0.7, -0.9)$
RX J1856-3754	~ 500	0.52 ± 0.07
PSR 1055-52	~ 530	$0.82\,(+0.06, -0.08)$
Tycho	407	< 1.8
Kepler	375	< 2.1
SN 1006	973	< 0.8
RCW 86	1794	< 1.5
W28	3400	< 1.8

Figure 2.2: These temperatures were obtained by the Einstein and Chandra observatories, and were found in Shapiro's book [2], and in Yakovlev's article [14]. T_∞ refers to the temperature measured by an observer on earth (see next section). The ages that are known exactly correspond to historical supernovae. The uncertainties are basically due to interstellar absorption.

were dt,dr and dΩ are one-forms, and ds is the differential of the invariant space-time interval. Here we are using units were $c = \hbar = 1$

We are interested in null geodesics, which are the paths describe by free photons. Recall that a geodesic (Shortest-distance path) is a generalization of a straight line in a curved space-time.

To find the shortest-distance path $x^\mu(\lambda)$ in flat space time [4], it is natural to simply write

$$\frac{d}{d\lambda}\frac{dx^\mu}{d\lambda} = 0, \tag{2.5}$$

which is simply the equation of a straight line. A generalization of the previous equation to curved space-time can be written as

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0, \tag{2.6}$$

were $\frac{D}{d\lambda}$ is called the *directional covariant derivative*, and is defined as

[4]Here, λ is an appropriate parameter, and μ refers to the component of the four-vector. The $\mu = 0$ component refers to time and the rest refer to the spacial components.

$$\frac{D}{d\lambda} \equiv \frac{dx^{\mu}}{d\lambda}\nabla_{\mu} \tag{2.7}$$

Recall that ∇_{μ} is the covariant derivative. The covariant derivative of a vector x^{μ} can be expressed as

$$\nabla_{\mu} x^{mu} = \partial_{\mu} x^{\mu} + \Gamma^{\nu}_{\mu\lambda} x^{\lambda}. \tag{2.8}$$

In the previous equation we are using the Einstein summation convention. The covariant derivative may be thought as a partial derivative $\partial_{\mu} = \frac{\partial}{\partial x_{\mu}}$ plus a correction specified by a set of n matrices (n being the dimension of the manifold, in this case 4) $\Gamma^{\nu}_{\mu\lambda}$, one matrix for each direction μ. The Γs are known as the Christoffel connection and are derived from the metric, these vanish in flat space-time as expected [15].

The geodesic equation (2.6) can now be written as

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^{\rho}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0 \tag{2.9}$$

Now lets look at one of the geodesic equations, namely the $\mu = 0$ component. The only surviving connection coefficient is Γ^0_{01}, so that we can write

$$\frac{d^2 x^0}{d\lambda^2} + \Gamma^0_{01} \frac{dx^1}{d\lambda}\frac{dx^0}{d\lambda} = 0, \tag{2.10}$$

or equivalently

$$\frac{d^2 t}{d\lambda^2} + \Gamma^t_{tr} \frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0, \tag{2.11}$$

The Γ^t_{tr} can be calculated directly from the metric $g_{\mu\nu}$ using [15]

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu}). \tag{2.12}$$

So that

$$\begin{aligned} \Gamma^t_{tr} &= \frac{1}{2} g^{t\rho}(\partial_t g_{r\rho} + \partial_r g_{\rho t} - \partial_{\rho} g_{tr}) \\ &= \frac{1}{2} g^{tt} \partial_r g_{tt} \\ &= \frac{1}{2}\left(1 - \frac{2GM}{r}\right)^{-1} \frac{2GM}{r^2} \\ &= \frac{GM}{r(r-2GM)}. \end{aligned} \tag{2.13}$$

Now eq. (2.11) can be written as

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0. \tag{2.14}$$

We would like to have a differential equation for $\frac{dt}{d\lambda}$. We can use the metric to obtain an expression for $\frac{dr}{d\lambda}$ in terms of $\frac{dr}{d\lambda}$. By making $ds = 0$ in the Schwarzschild metric (2.4)[5], and also setting $d\Omega = 0$, which implies only radial paths, on obtains

$$\left(1 - \frac{2GM}{r}\right)\frac{dt^2}{d\lambda^2} = \left(1 - \frac{2GM}{r}\right)^{-1}\frac{d^2r}{d\lambda^2} \tag{2.15}$$

$$\frac{dt}{d\lambda} = \left(1 - \frac{2GM}{r}\right)^{-1}\frac{dr}{d\lambda}. \tag{2.16}$$

With the previous equation in mind we can write the geodesic equation 2.14 as

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r^2}\left(\frac{dt}{d\lambda}\right)^2 = 0. \tag{2.17}$$

As we shall see, we are interested in $\frac{dt}{d\lambda}$, rather than $t(\lambda)$. One can verify that

$$\frac{dt}{d\lambda} = \frac{\omega_0}{2\left(1 - \frac{2GM}{r}\right)^2} \tag{2.18}$$

is a solution to eq. (2.17), were ω_0 is a constant. The result we have just derived is important because it is the only non-zero component of the four-momentum $p_\mu = \frac{dx_\mu}{d\lambda}$ as measured by a comoving observer[6].

We are interested in the energy of the photon which is given by [15]

$$E = -p_\mu u^\mu, \tag{2.19}$$

were u^μ is the four-velocity of the comoving observer (At fixed spatial coordinates.). This observer would have a four-velocity such that $u^i = 0$ for the spatial components and u^0 is given by the normalization condition

$$u_\mu u^\nu = g_{\mu\nu}u^\mu u^\nu = g_{00}u^0u^0 = -1. \tag{2.20}$$

The previous equation implies that the time component gives

$$u^t = \sqrt{-g_{tt}} = \sqrt{1 - \frac{2GM}{r}}. \tag{2.21}$$

We can now calculate the photon's energy using equations (2.19) and (2.18), so that

[5]Recall that the proper time τ is related to the space-time interval as $d\tau = \sqrt{-ds^2}$, so making $ds = 0$ is equivalent to setting $d\tau = 0$, which is always true for a photon.

[6]Recall that one can always find an appropriate parameter λ for a null path such that $p_\mu = \frac{dx_\mu}{d\lambda}$ is the four-momentum [15]

$$\begin{aligned} E &= -g_{\mu\nu}p^{\nu}u^{\nu} = -g_{00}p^0u^0 \\ &\Rightarrow \quad E = \frac{\omega_0}{2\sqrt{1-\frac{2GM}{r}}}. \end{aligned} \tag{2.22}$$

Thus, if a photon is emmited from the N.S. surface at a radius R, it will have an energy

$$E_0 = \frac{\omega_0}{2\sqrt{1-\frac{2GM}{R}}}, \tag{2.23}$$

as measured by a comoving observer. When the photon has traveled a very long distance, it will have an energy given by

$$E_{\infty} = \lim_{r\to\infty} \frac{\omega_0}{2\sqrt{1-\frac{2GM}{R}}} = \frac{\omega_0}{2}. \tag{2.24}$$

The previous equation implies that

$$E_{\infty} = E_0\sqrt{1-\frac{2GM}{Rc^2}}, \tag{2.25}$$

in S.I units. The previous equation implies that *the photon has been gravitationally redshifted.*

We can now estimate the temperature as measured by a far away observer (T_{∞}), in terms of the surface temperature (T_0).

$$\begin{aligned} E &\approx k_BT \\ &\Rightarrow \quad T_o = \frac{\omega_0}{2\sqrt{1-\frac{2GM}{Rc^2}}k_B} \\ &\Rightarrow \quad T_{\infty} = T_0\sqrt{1-\frac{2GM}{Rc^2}} \end{aligned} \tag{2.26}$$

The previous result is known as the *Temperature Redshift*, in the sense that the temperatures we can measure here on earth are actually less than the ones at the surface of the N.S., we have just proved what is mentioned in Yakovlev's article [14]. The $\sqrt{1-\frac{2GM}{Rc^2}}$ factor is known as the **Redshift factor**, and can vary depending on the N.S. mass and radius. Using the limiting masses and radii that were calculated in Chapter I (Eqs. 1.20- 1.21) we can conclude that

$$0.75 < \sqrt{1-\frac{2GM}{Rc^2}} < 0.98. \tag{2.27}$$

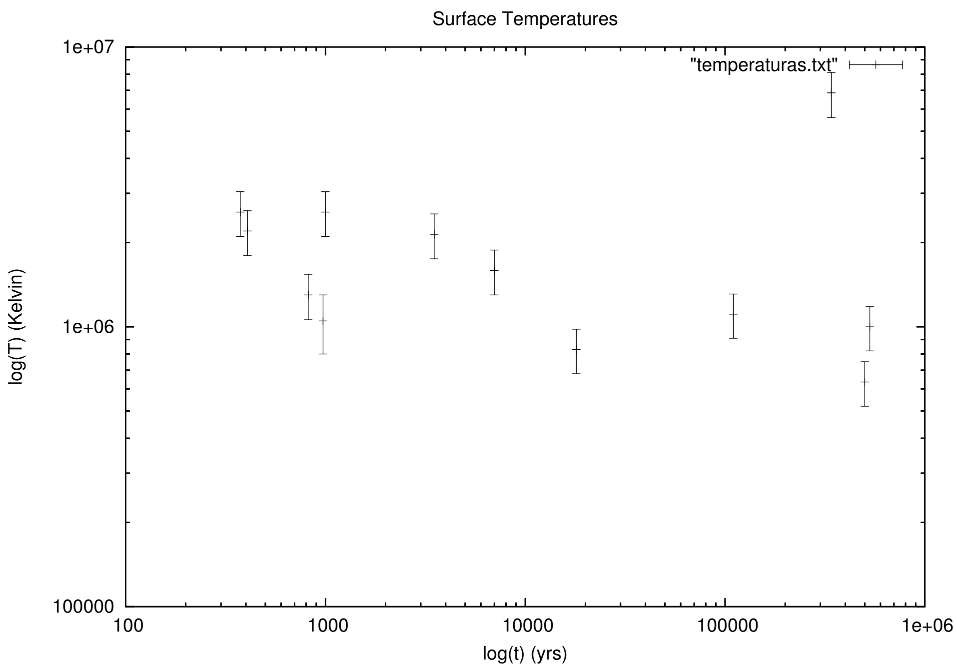

Figure 2.3: Here we are plotting actual surface temperatures (T_0), which differ from the observed temperatures (T_∞) by the redshift factor (eq. 2.27).

This factor will play an important role in the next chapter because it will give us an uncertainty in the surface temperature. This uncertainty is actually much greater than the one related to observational error or uncertainty in interstellar absorption. It is important to emphasize that this uncertainty in the temperature is due to the uncertainty in the masses of the observed N.S.s. We can now plot the actual surface temperatures (T_0) using what we have just learned (figure 2.3)

Chapter 3

Cooling

The origin of N.S.s as supernova remnants gives us a good idea of their global structure. Nontheless, there are still uncertainties on the matter composition of this astrophysical object. Different matter compositions will alter the equation of state and would thus alter some of the global properties treated in chapter I. The behavior of matter at extremely high densities is still not completely understood, and thus, questions such as what is the proton concentration in a N.S., and What kind of matter actually exists in the center of the N.S. could potentially be answered by studying the cooling mechanisms.

As we shall see, the cooling of a N.S. is very sensitive to the type of matter present. *The study of N.S. cooling provides us a way of looking directly inside the N.S.*, and to put to test much of our current knowledge of matter. In this chapter we shall discuss the two most favorite cooling mechanisms via neutrino emission, and the more inefficient cooling mode due to photon emission (Black body radiation). Finally, we will compare our theoretical cooling curves with observed temperatures.

3.1 Cooling modes

3.1.1 Photon Emission

The first cooling mechanism that comes to mind is black body radiation, one that we are more accustomed to; after all, many "earthly" objects cool this way. The energy lost via this mechanism can be expressed by using the Stephan-Boltzmann law:

$$\dot{E}_\gamma = 4\pi R^2 \sigma T^4, \tag{3.1}$$

were the Temperature T refers to the surface temperature of the outer crust mainly composed of heavy nuclei[1] and σ is the Stephan-Boltzmann constant. Note that the energy loss via photon emission is proportional to R^2. As we saw in section 1.4.1, the radius of a N.S. is of the order of $10km$, which is very small compared

[1]See section 1.2.1

for example to the radius of a White Dwarf star[2]. For this reason, the emission of photons is not an efficient cooling mechanism, and although a N.S. has very high temperatures, it may have a very low photon luminosity.

3.1.2 Neutrino Emission

The idea that N.S.'s might cool via neutrino emission was first stated by Gamow and Schoenberg (1941), at about the same time that the importance of neutronization in supernovae was considered [3]. The simplest neutrino emmiting processes are beta decay of the neutron and electron capture of electrons:

$$n \rightarrow p + e + \overline{\nu_e} \tag{3.2}$$

$$p + e \rightarrow n + \nu_e. \tag{3.3}$$

Both of these reactions would have to have the same rate if the system is in chemical equilibrium. That is, if it satisfies the condition

$$\mu_n = \mu_p + \mu_e, \tag{3.4}$$

and thus, neutrinos can be produced continously. Reactions of the type 3.2, 3.3 are possible via the weak interaction as shown in figure 3.1.2. Cooling mechanisms that emmit neutrinos are commonly called *Urca Processes*, reactions 3.2 and 3.3 are particularly known as the **Direct Urca process**.

The name Urca actually refers to a Casino in Rio de Janeiro, and was compared by Gamow to N.S. cooling in the sense that the casino was a perfect "sink" for money, just as neutrino emission is the perfect sink for the N.S.s thermal energy [3]. Neutrino emmision can be a good cooling mechanism because, since these do not interact "much", they can escape the N.S. quite easily.

The Direct Urca reactions are not the only possible neutrino emmiting processes, if for example a meson condensate were present in the N.S.[3], reactions such as

$$n + \pi^- \rightarrow n + e + \overline{\nu_e} \tag{3.5}$$

or

$$n + k^- \rightarrow n + e + \overline{\nu_e}, \tag{3.6}$$

and their accompanying electron capture reactions similar to eq. (3.3) would occur.

In the case that free quarks actually exist (Quark-Gluon Plasma), we may have a Quark Direct Urca process such as

[2] White dwarfs ($R \sim 100km$) actually cool via photon emission.

[3] See section 1.2.1

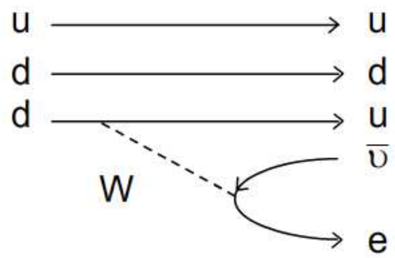

Figure 3.1: The figure shows a Feynmann diagram for reaction (3.2). During the beta decay of a neutron (udd), a "d" quark is turned unto a "u" quark via the emission of a "W", and thus a proton (uud) is formed. The other quarks are just spectators that go along for the ride.

$$u + e \rightarrow d + \nu_e \tag{3.7}$$

and

$$d \rightarrow u + e + \overline{\nu_e}, \tag{3.8}$$

may also be possible [3].

Each of these cooling mechanisms has its own characteristic cooling curve (T-time), so the N.S's cooling curve depends on the type of matter present.

3.2 The Direct Urca Threshhold

Let's look at the Direct Urca process (Eqs. 3.2 and 3.3) a bit more carefully. For this reaction to occur, all particles participating in it must have energies close to their Fermi surface, i.e. they must all have energies of the order of $K_B T_f$. For example, if the neutron were to decay, the states of the final proton and electron would have to be empty, these empty states are more likely to be at energies of the order of $K_B T_f$. If we impose this condition on the neutron decay we obtain the following restriction on the Fermi momenta [3]:

$$P(p)_f + P(e)_f \geq P(n)_f, \tag{3.9}$$

assuming that the momentum of the neutrino is negligible.

Recalling that the particle concentration and their Fermi momenta are related by eq. (1.10)

$$n(i) = \frac{P(i)_f^3}{3\pi^2\hbar^3}. \tag{3.10}$$

So we can say, because of charge neutrality that $P(p)_f = P(e)_f$, and thus rewrite eq. (3.9) as

$$2P(p)_f \geq P(n)_f \tag{3.11}$$

and using eq. (3.10) we can find a restriction on the proton concentration

$$n(p) \geq \frac{n(n)}{8}. \tag{3.12}$$

If we now define the proton fraction as

$$\chi = \frac{n(p)}{n(n) + n(p)}, \tag{3.13}$$

we find that [7]

$$\chi \geq \frac{1}{9} \simeq 11\%. \tag{3.14}$$

We have found a minimum proton fraction necessary for momentum to be conserved and for the Direct Urca process to occur. Because of this minimum proton fraction, the Direct Urca process was actually not accepted in the physics community for many years on the basis that the minimum proton fraction was considered to be too high [3]. The proton concentration is very sensitive to the eq. of state, but current nuclear matter models predict that it is possible to have $\chi \geq 1/9$ [3, 7].

3.3 The Modified Urca Process

If the proton fraction is less than 11%, the Direct Urca process is not possible, and a new particle must be introduced into inequality 3.9 so that momentum can be conserved. This new cooling mechanism can be written as

$$n + (n, p) \rightarrow (n, p)' + p + e + \overline{\nu_e} \tag{3.15}$$

and

$$p + e + (n, p) \rightarrow (n, p)' + n + \nu_e \tag{3.16}$$

This cooling mechanism is called the *Modified Urca Process*. Here, a bystander neutron or proton participates to allow momentum conservation (By absorbing momentum.). This is a second order process and is thus less likely to occur compared to the Direct Urca process. For this reason, a neutron star with a proton fraction less than 11% will cool much more slowly [8].

3.4 Cooling curves: An outline of the problem

The most important objective of this document is to calculate a temperature-time curve for a N.S. containing ordinary matter. For this, we must make the following assumptions:

- The N.S. is composed of ordinary matter (neutrons, protons and electrons), there is no exotic matter present. This could only be justified by comparing the final cooling curve with observations. Also, no superfluid phases will be considered; eventhough these might play an important role in the cooling of N.S.s, they are beyond the scope of this document.

- Since the core of the N.S. constitutes about 99% of the mass of the star, the Crust will not be considered (at least in the Neutrino Cooling modes).

- The N.S. may be considered as being highly degenerate. This can be justified because temperatures in N.S.s are much less than the Fermi temperature ($\sim 10^{12}K$)[4]. Fermi-Dirac statistics will be used to describe the N.S.

- The N.S. is roughly isothermal. This as a consequence of it being highly degenerate, i.e. the mean free path of a particle in a degenerate fluid is usually very long.

More assumptions will be made along the way, but these are probably the most important ones.

3.4.1 Procedure

In short, the procedure to calculate a cooling curve will be the following:

- First calculate the energy loss per unit time (Also called *Luminosity* or Emissivity.) due to neutrino and photon emission. These will be labeled $\dot{E}_{\overline{\nu}}$ and $\dot{E}_\gamma$ respectively. These luminosities as we shall see can be calculated using non relativistic quantum mechanics, in particular we will use the Fermi Golden rule of time dependent perturbation theory.

- Noting that the total luminosity (Photons plus neutrinos) is simply

$$\dot{E} = \dot{E}_{\overline{\nu}} + \dot{E}_\gamma = -\frac{dE}{dt}, \tag{3.17}$$

  and that

$$\frac{dE}{dT} = \frac{dE}{dt}\frac{dt}{dT} = C_v, \tag{3.18}$$

[4] See section 1.1.1

C_v being the Specific heat capacity at constant volume for the N.S.; we can finally write

$$-(\dot{E}_{\overline{\nu}} + \dot{E}_{\gamma}) = C_v \frac{dT}{dt}, \quad (3.19)$$

which can be called the **Cooling equation**. The next step consists in calculating the Specific heat capacity of the N.S., which can actually be found using statistical Mechanics, assuming that the star is made primarily of neutrons.

- Equation 3.19, simple as it may be, governs the cooling of the N.S., so now the final step is to solve this differential equation to obtain a temperature-time curve $T(t)$.

The same procedure will be done for both the Direct Urca and the Modified Urca processes, so that both cooling curves can be compared with each other and with actual observations.

3.5 Direct Urca Luminosity

The neutrino emisivity for the Direct Urca process shall now be calculated. ($n \rightarrow p + e + \overline{\nu}$).

The rate at which neutrons decay in a netron star can be calculated using the Fermi Golden rule. Qualitatively this can be written as

$$\Gamma_{\overline{\nu}} \propto \frac{2\pi}{\hbar} | < f|\hat{\mathcal{H}}|i> |^2 * (Phase - Space). \quad (3.20)$$

[5] Were the Bra-Ket is just the transition probability from an initial state to a final state and $\hat{\mathcal{H}}$ is the weak interaction Hamiltonian. The phase space is the one available for the transition. Note that since a neutron star is a degenerate fermi fluid, the phase space is severely restricted[6].The phase space is reduced because the probability that the final states with energies E_p and E_e are empty, is simply 1 minus the probability of them being occupied. This means that the terms $(1 - f_p)(1 - f_e)$ have to be included in equation (3.20)[7]. The f_is are simply the Fermi-Dirac distribution functions

$$f_i = \frac{1}{1 + e^{\frac{E_i - \mu_i}{k_B T}}}. \quad (3.21)$$

An extra term f_n has to be included because we have to take into account that the initial neutron with energy E_n actually exists. If we now want to calculate a luminosity, or energy loss, we can just use eq. 3.20 to write

[5] The Fermi golden rule is just an aplication Time-dependent perturbation theory to first order and is treated in Appendix A.

[6] This in contrast to the free neutron decay, were any final state is un-occupied

[7] Also called Pauli blocking factors.

$$\dot{E_{\overline{\nu}}} = 2\frac{2\pi}{\hbar}\sum|<f|\hat{\mathcal{H}}|i>|^2\, E_{\overline{\nu}}\, f_n(1-f_p)(1-f_e)\,\delta^4(\vec{P_n}-\vec{P_p}-\vec{P_e}-\vec{P_{\overline{\nu}}}), \qquad (3.22)$$

were the sum is made over all possible four-momenta. A factor of 2 has been added because the other Direct Urca reaction ($e+p\rightarrow n+\overline{\nu}$) has to be considered with equal probability (It's basically the same Feynmann diagram.). The delta function simply enforces consevarion of energy and momentum.

The Weak Interaction Matrix element

The matrix element strores the information of the strength of the interaction. When Fermi developed his Golden rule, he stored the information of the strength of the interaction in the coupling constant G_f and took the matrix element to be [10]

$$<f|\hat{\mathcal{H}}|i>=\frac{G_f}{V}. \qquad (3.23)$$

Here the volume V is the normalization volume of the free particle states, i.e. $|\Psi_i(\vec{r})\rangle = \frac{1}{\sqrt{V}}e^{i\vec{k}.\vec{r}}$. At the time this was done there was no knoledge of W bosons, so Fermi considered the inteaction potential as being a contact potential. A small correction to the matrix element is motivated by the more recent G.W.S. electroweak theory so that the matrix element becomes [3]

$$\mathcal{H}_{i\,f} = \frac{1}{V^2}G_F^2\cos^2\theta_c(1+g_a^2). \quad , \qquad (3.24)$$

This expression reminds us of the weak coupling vertex factor, were there is an axial and a vector coupling. In the previous equation, θ_c is the Cabibo angle and g_a is the Axial-Vector coupling constant.

The Phase Space Factor

Using the expression for the matrix element, eq. (3.22) becomes

$$\begin{aligned}\dot{E_{\overline{\nu}}} &= 2\frac{2\pi}{\hbar}G_F^2\cos^2\theta_c(1+g_a^2) \qquad (3.25)\\ &* \sum E_{\overline{\nu}}\, f_n(1-f_p)(1-f_e)\,\delta^4(\vec{P_n}-\vec{P_p}-\vec{P_e}-\vec{P_{\overline{\nu}}})\\ &= \frac{2\pi}{\hbar}G_F^2\cos^2\theta_c(1+g_a^2)\mathcal{P}\ ,\end{aligned}$$

were $\mathcal{P}$ is defined by

$$\mathcal{P} = \sum E_{\overline{\nu}}\, f_n(1-f_p)(1-f_e)\,\delta^4(\vec{P_n}-\vec{P_p}-\vec{P_e}-\vec{P_{\overline{\nu}}})\ , \qquad (3.26)$$

and is usually called The *Phase-Space factor.*

The problem of calculating the emisivity is basically reduced to calculating $\mathcal{P}$. If the reader is just interested in the final result, he (or she) may skip to eq. (3.60)

We can now change the sum to an integral over three momenta ($\vec{p_i}$) by introducing the density of states. For illustrative purposes we can first define a quantity $\mathcal{P}_0$ that corresponds to the Phase-space factor available for the Direct Urca process assuming that the initial neutron has a definite four-momentum.

$$\mathcal{P}_0 = \int d^3\vec{p}_p \frac{V}{h^3} \int d^3\vec{p}_e \frac{V}{h^3} \int d^3\vec{p_{\overline{\nu}}} \frac{V}{h^3} (1-f_p)(1-f_e)\,\delta^4(\vec{P_n} - \vec{P_p} - \vec{P_e} - \vec{P_{\overline{\nu}}}) \quad (3.27)$$

If we now integrate over all possible four-momenta of the neutron, we get

$$\mathcal{P} = \int d^3\vec{p}_n\,\mathcal{P}_0 \;\; = \frac{V^3}{h^9} \int d^3\vec{p}_n d^3\vec{p}_p d^3\vec{p}_e d^3\vec{p_{\overline{\nu}}}\, f_n(1-f_p)(1-f_e)\delta^4(\vec{P_n} - \vec{P_p} - \vec{P_e} - \vec{P_{\overline{\nu}}}) \quad (3.28)$$

If we now use spherical coordinates, we can separate $\mathcal{P}$ into a radial part and an angular part.

The angular part $\mathcal{A}$ can be written as:

$$\mathcal{A} = \int d\Omega_n d\Omega_p d\Omega_e d\Omega_{\overline{\nu}} \delta^3(\vec{p_n} - \vec{p_p} - \vec{p_e} - \vec{p_{\overline{\nu}}}). \quad (3.29)$$

Using the fact that $|\vec{p_{\overline{\nu}}}| \ll |\vec{p}_p + \vec{p}_e|$, we can rewrite the delta as $\delta^3(\vec{p}_n - \vec{p}_p - \vec{p}_e)$.

We can now use a property of the delta function:

$$\delta^3(\vec{p}_n - \vec{p}_p - \vec{p}_e) = \delta(|\vec{p}_n| - |\vec{p}_p + \vec{p}_e|)\delta(\Omega_n - \Omega_{e-p})\frac{1}{|\vec{p}_n|^2}. \quad (3.30)$$

Integrating over Ω_n we get,

$$\mathcal{A} = \int d\Omega_p d\Omega_e d\Omega_{\overline{\nu}} \delta(p_n - |\vec{p}_p + \vec{p}_e|)\frac{1}{p_n^2} \quad (3.31)$$

Noting that

$$\delta(p_n - |\vec{p}_p + \vec{p}_e|) = \delta(p_n - (p_e^2 + p_p^2 + 2p_e p_p \cos\theta)^{1/2}) = \delta(f(\cos\theta)) \quad (3.32)$$

and using

$$\delta(f(\cos\theta)) = \frac{1}{f'(\cos a)}\delta(\cos\theta - a) \quad (3.33)$$

were $f(a) = 0$.

The derivative f' can now be expressed as

$$f'(\cos\theta) = \frac{p_e p_p}{(p_e^2 + p_p^2 + 2p_e p_p \cos\theta)^{1/2}} = \frac{p_e p_p}{p_n} \quad (3.34)$$

So,

$$\delta(f(\cos\theta)) = \delta(\cos\theta - a)\,\frac{p_n}{p_e p_p}. \tag{3.35}$$

Now we can rewrite the angular part $\mathcal{A}$ as

$$\mathcal{A} = \int d\Omega_p d\Omega_e d\Omega_{\overline{\nu}} \frac{1}{p_e p_p p_n} \delta(\cos\theta - a). \tag{3.36}$$

Noticing that the delta function fixes θ, when we integrate over solid angle Ω_p or Ω_e we will obtain a factor of 2π instead of the usual 4π. So

$$\mathcal{A} = \frac{2\pi(4\pi)^2}{p_e p_p p_n} = \frac{32\pi^2}{p_e p_p p_n} \tag{3.37}$$

And

$$\mathcal{P} = \frac{32\pi^2 V^3}{h^9} \int p_e p_p p_n p_{\overline{\nu}}^2 E_{\overline{\nu}}\, dp_e dp_p dp_n dp_{\overline{\nu}}\, f_n(1 - f_p)(1 - f_e)\delta(E_n - E_p - E_e - E_{\overline{\nu}}). \tag{3.38}$$

We would like to integrate over energy instead of momentum. We can aproximate the protons and neutrons as being non relativistic particles [2].

$$dE_i = \frac{p_i dp_i}{m_i} \quad for \;\; i = n, p \tag{3.39}$$

This aproximation can be justified by racalling that at typical N.S. temperatures ($\sim 10^9\, K$) we have

$k_B(10^9)^\circ K \;\sim\; 10^{-14}\, J \;\sim\; m_n v_n^2$
$\Rightarrow\; v_n \;\sim\; 10^6\, m/s \;\sim\; 0.001c.$

On the other hand, we can take the electrons as being highly relativistic in tipical N.S. temperatures, so

$E_e = p_e c \;\Rightarrow\; dE_e = dp_e\, c$

For neutrinos it's always valid to write $dE_{\overline{\nu}} = dp_{\overline{\nu}}\, c$. So we can now write $\mathcal{P}$ in terms of energy. Thus

$$\begin{aligned} \mathcal{P} \;&=\; \frac{32\pi^3 V^3}{h^9} \int \left(\frac{E_e\, dE_e}{c^2}\right) (dE_p\, m_p)(dE_n\, m_n) \left(\frac{E_{\overline{\nu}}^3\, dE_{\overline{\nu}}}{c^3}\right) \\ &* \;\; f_n(1 - f_p)(1 - f_e)\delta(E_n - E_p - E_e - E_{\overline{\nu}}) \end{aligned} \tag{3.40}$$

Since the electrons are highly degenerate and only energies close to the fermi surface are considered, we may simply repalce E_e by the fermi energy μ_e and remove it from the integral[8].

[8] We cannot do the same with the neutrino since these are not degenerate, this is because they can escape the N.S. quite easily. Degenerate neutrinos are only thought to exist for a short period of time durig a supernova explosion when the opacity is extremely high.

$$\begin{aligned}\mathcal{P} &= \frac{32\pi^3 V^3 m_p m_n \mu_e}{h^9 c^5}\int dE_e dE_p dE_n\, E_{\overline{\nu}}^3\, dE_{\overline{\nu}} \\ &* \; f_n(1-f_p)(1-f_e)\delta(E_n - E_p - E_e - E_{\overline{\nu}}) \\ &= \mathcal{S}\int dE_e dE_p dE_n\, E_{\overline{\nu}}^3\, dE_{\overline{\nu}}\, f_n(1-f_p)(1-f_e)\delta(E_n - E_p - E_e - E_{\overline{\nu}}),\end{aligned} \tag{3.41}$$

Were $\mathcal{S}$ is defined by

$$\mathcal{S} = \frac{32\pi^3 V^3 m_p m_n \mu_e}{h^9 c^5}. \tag{3.42}$$

Now we make the following changes of variable:

$$\begin{aligned}\mathcal{E}_{\overline{\nu}} &= \frac{E_{\overline{\nu}}}{kT} \\ \mathcal{E}_n &= \frac{E_n - \mu_n}{kT} \\ \mathcal{E}_e &= \frac{-E_e + \mu_e}{kT} \\ \mathcal{E}_p &= \frac{-E_p + \mu_p}{kT}\end{aligned} \tag{3.43}$$

Recalling the chemical equilibrium condition ($\mu_n = \mu_p + \mu_e$), the delta function can be written as

$$\frac{1}{kT}\delta(\mathcal{E}_n + \mathcal{E}_p + \mathcal{E}_e - \mathcal{E}_{\overline{\nu}}) \tag{3.44}$$

Now $\mathcal{P}$ becomes

$$\mathcal{P} = \mathcal{S}(kT)^6 \int \mathcal{E}_{\overline{\nu}}^3\, \delta(\mathcal{E}_n + \mathcal{E}_p + \mathcal{E}_e - \mathcal{E}_{\overline{\nu}}) \frac{d\mathcal{E}_n}{1+e^{\mathcal{E}_n}} \frac{d\mathcal{E}_p}{1+e^{\mathcal{E}_p}} \frac{d\mathcal{E}_e}{1+e^{\mathcal{E}_e}} \tag{3.45}$$

.

Recalling that the delta function is just the Fourier transform of a constant, we can use the representation

$$\delta(\mathcal{E}_n + \mathcal{E}_p + \mathcal{E}_e - \mathcal{E}_{\overline{\nu}}) = \frac{1}{2\pi}\int e^{iz(\mathcal{E}_n + \mathcal{E}_p + \mathcal{E}_e - \mathcal{E}_{\overline{\nu}})}\, dz \tag{3.46}$$

So that

$$\mathcal{P} = \frac{\mathcal{S}}{2\pi}\int \mathcal{E}_{\overline{\nu}}^3\, d\mathcal{E}_{\overline{\nu}} \int e^{iz(\mathcal{E}_n + \mathcal{E}_p + \mathcal{E}_e - \mathcal{E}_{\overline{\nu}})}\, dz\, \frac{d\mathcal{E}_n}{1+e^{\mathcal{E}_n}} \frac{d\mathcal{E}_p}{1+e^{\mathcal{E}_p}} \frac{d\mathcal{E}_e}{1+e^{\mathcal{E}_e}} \tag{3.47}$$

$$= \frac{\mathcal{S}}{2\pi}\int \mathcal{E}_{\overline{\nu}}^3\, d\mathcal{E}_{\overline{\nu}} \int e^{iz(\mathcal{E}_n + \mathcal{E}_p + \mathcal{E}_e} e^{-\mathcal{E}_{\overline{\nu}})}\, dz\, \frac{d\mathcal{E}_n}{1+e^{\mathcal{E}_n}} \frac{d\mathcal{E}_p}{1+e^{\mathcal{E}_p}} \frac{d\mathcal{E}_e}{1+e^{\mathcal{E}_e}} \tag{3.48}$$

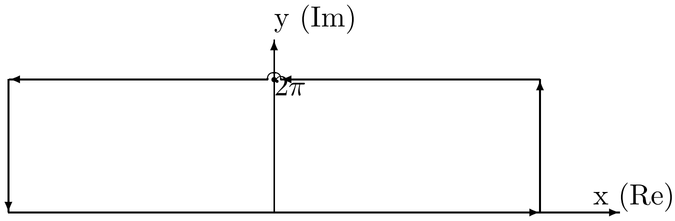

Figure 3.2: Path taken for integral 3.51. The vertical segments do not contribute to the integral when one integrates from $-\infty$ to ∞.

We can first calculate

$$\int_{-\infty}^{\infty} e^{iz\mathcal{E}_i} \frac{d\mathcal{E}_i}{1+e^{\mathcal{E}_i}} \tag{3.49}$$

by considering the closed contour integral in the complex plane

$$\oint e^{izw} \frac{dw}{1+e^{w}}. \tag{3.50}$$

Were w is complex ($w = x + iy$). Note that this contour integral has a pole at $w = i\pi$. Making a direct calculation for the residues, we find that the residue is $-e^{-\pi z}$. The most convenient contour for the calculation of eq. (3.49) is a closed contour that passes through 2π in the imagianry axis, and of course, it should pass through the real axis as shown in figure 3.2.

Now using the Residue Theorem.

$$\oint e^{izw} \frac{dw}{1+e^{w}}. = \int_{-\infty}^{\infty} \frac{e^{izx}\, dx}{1+e^{x}} - \int_{-\infty}^{\infty} \frac{e^{iz(2\pi i+x)}\, dx}{1+e^{2\pi i+x}} \tag{3.51}$$

$$= \int_{-\infty}^{\infty} \frac{e^{izx}\, dx}{1+e^{x}} - e^{2\pi z} \int_{-\infty}^{\infty} \frac{e^{izx}\, dx}{1+e^{x}} = -2\pi i e^{-\pi z} \tag{3.52}$$

And the definite integral (3.49) becomes

$$\int_{-\infty}^{\infty} e^{iz\mathcal{E}_i} \frac{d\mathcal{E}_i}{1+e^{\mathcal{E}_i}} = \frac{2\pi i e^{-\pi z}}{1-e^{-2\pi z}}. \tag{3.53}$$

We can now rewrite eq (3.48) as

$$\mathcal{P} = \frac{\mathcal{S}}{2\pi} \int_0^{\infty} \mathcal{E}_{\overline{\nu}}^3\, d\mathcal{E}_{\overline{\nu}} \int_{-\infty}^{\infty} e^{-iz\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi z}}{1-e^{-2\pi z}} \right)^3 dz. \tag{3.54}$$

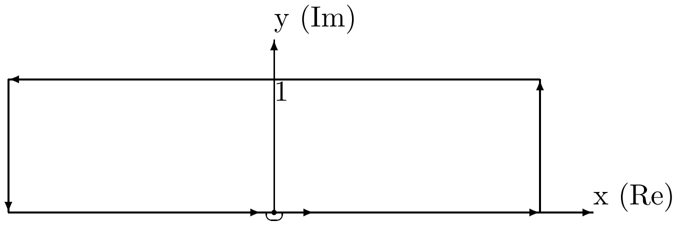

Figure 3.3: Path taken for integral 3.55

Once again we can make a contour integral to calculate the second integral in eq. (3.54). Note that this integral has a pole at $z = 0$. In this case a suitable contour is one that passes though 1 in the imaginary axis as shown in the figure.

$$\oint e^{-iz\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi z}}{1-e^{2\pi z}}\right)^3 dz = \tag{3.55}$$

$$\int_{-\infty}^{\infty} e^{-ix\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi x}}{1-e^{2\pi x}}\right)^3 dz - \int_{-\infty}^{\infty} e^{-i\mathcal{E}_{\overline{\nu}}(x+i)} \left(\frac{2\pi i e^{-\pi (x+i)}}{1-e^{-2\pi (x+i)}}\right)^3 dz$$

$$= \int_{-\infty}^{\infty} e^{-ix\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi x}}{1-e^{2\pi x}}\right)^3 \left(1+e^{\mathcal{E}_{\overline{\nu}}}\right) dz$$

$$= -2\pi i Res \left\{ e^{-ix\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi x}}{1-e^{2\pi x}}\right)^3 \right\}. \tag{3.56}$$

A calculation of the Residue can be made by making a Laurent expansion of the expression inside the residue brackets. This expansion gives

$$\frac{1}{z^3} + \frac{x}{z^2} - \frac{i(\pi^2+x^2)}{2z}. \tag{3.57}$$

This means that the residue is $\frac{1}{2}(\pi^2 + x^2)$. Finally eq. (3.54) can be exressed as

$$\mathcal{P} = \frac{i}{2}\mathcal{S} \int_0^{\infty} \mathcal{E}_{\overline{\nu}}^3 \left(1+e^{\mathcal{E}_{\overline{\nu}}}\right)^{-1} (\pi^2 + x^2) \tag{3.58}$$

$$\mathcal{P} = \mathcal{S}\frac{457\pi^6}{5040}. \tag{3.59}$$

Recalling eq. (3.42) for S, we can finally express the neutrino luminosity (eq. 3.26) for the Direct Urca process as

$$\dot{E_{\overline{\nu}}} = \frac{457\pi}{1080}\frac{Vm_pm_n\mu_e}{\hbar^{10}c^5}(kT)^6G_f^2\cos^2\theta_c(1+g_a^2)\ \ J/s \tag{3.60}$$

This is the energy loss via neutrino emision, it coincides with the result obtained by Pethick [3]. The $(kT)^6$ dependence is particular of the Direct Urca process as we shall see. The previous result, together with the specific heat, will allow us to calculate a cooling curve for neutrino cooling.

3.6 Modified Urca Luminosity

The Luminosity for the Modified Urca process shall now be calculated $(n_1 + n_1 \rightarrow n_2' + p + e + \overline{\nu}$. We shall follow a similar procedure to Shapiro [2]). The procedure followed is the same as in the Direct Urca process, only that now there is a bystander particle involved. Here we assume that the bystander particle is a neutron. We could also assume that the bystander is a proton, which would give us a different luminosity, but since there a more neutrons than protons, only the *neutron brach* shall be considered. Also, even if the proton concentration would be high enough for us to consider a *proton branch*, the Direct Urca process would occur and would be much more effective than the Modified Urca process as we shall see. The luminosity can be written as

$$\begin{aligned}\dot{E_{\overline{\nu}}} &= \frac{2\pi}{\hbar}\left|<f|\hat{\mathcal{H}}|i>\right|^2\sum_{P_i}E_{\overline{\nu}}\,\delta^4(\vec{P_{n_1}}+\vec{P_{n_2}}-\vec{P_{n_2'}}-\vec{P_p}-\vec{P_e}-\vec{P_{\overline{\nu}}})\\ &* \ \ f_{n_1}f_{n_2}(1-f_{n_2'})(1-f_p)(1-f_e).\end{aligned} \tag{3.61}$$

Here, the matrix element again refers to a transition probability, the P_i refer to the particles four-momenta, and the f_i refer to the Fermi-Dirac distributions. The matrix element however, is not as simple as the one used for beta decay (Direct Urca process), since there is now a bystander particle that interacts via the strong interaction.

The Weak Interaction Matrix element

The strong interaction can in this case, be modeled as a one pion (π) exchange between particles. This model is called the One Pion Exchange model (OPE)[9], a result for the matrix element was obtained by Yakovlev and Kaminker [11] and is written as

$$|<f|\hat{\mathcal{H}}|i>|^2 = \frac{16G_f^2\cos^2\theta_c g_a^2}{m_\pi^4E_{f_e}^2} \tag{3.62}$$

Let us first calculate the phase space factor for a definite energy for neutrons n_1 and n_2 (Just for illustrative purposes).

[9]The bystander particle interacts via the Strong force, which can be modeled by an effective theory that considers a Pion exchange. This was first done by Yukawa [10].

$$\mathcal{P}\prime = \int d^3\vec{P}_{n_2'} \int \frac{V}{h^3} d^3\vec{P}_p \frac{V}{h^3} d^3\vec{P}_e \int \frac{V}{h^3} d^3\vec{P}_{\overline{\nu}} E_{\overline{\nu}} \,\delta^4(\vec{P}_f - \vec{P}_i) * f_{n_1} f_{n_2}(1 - f_{n_2'})(1 - f_p)(1 - f_e). \tag{3.63}$$

The Phase Space Factor

If we now integrate over all possible four-momenta available for neutrons n_1 and n_2, we get

$$\begin{aligned} \mathcal{P} &= \int d^3\vec{P}_{n_1} \int d^3\vec{P}_{n_2} \mathcal{P}' \\ &= \frac{V^4}{h^{12}} \int d^3\vec{P}_{n_1} d^3\vec{P}_{n_2} d^3\vec{P}_p d^3\vec{P}_e d^3\vec{P}_{\overline{\nu}} E_{\overline{\nu}} \,\delta^4(\vec{P}_f - \vec{P}_i) \\ &* \ f_{n_1} f_{n_2}(1 - f_{n_2'})(1 - f_p)(1 - f_e). \end{aligned} \tag{3.64}$$

The problem is again reduced to calculating the previous phase space factor. If the reader is just interested in the final result, he (or she) may skip to eq. (3.96).

Using spherical coordinates, we can first integrate the angular part of $\mathcal{P}$.

$$\mathcal{A} = \int d\Omega_{n_1} d\Omega_{n_2} d\Omega_p d\Omega_e d\Omega_{\overline{\nu}} \delta^3(\vec{P}_f - \vec{P}_i). \tag{3.65}$$

If we now note that the four-momentum of the neutrino is small, i.e.

$$\left|\vec{P}_{\overline{\nu}}\right| \ll \left|\vec{P}_p + \vec{P}_e + \vec{P}_{n_2'}\right|, \tag{3.66}$$

so the delta function can now be written as

$$\delta^3(\vec{P}_f - \vec{P}_i) \simeq \delta(\vec{P}_{n_1} + \vec{P}_{n_2} - \vec{P}_{n_2'} - \vec{P}_p - \vec{P}_e). \tag{3.67}$$

Now we can exploit the fact that the delta function can be written as

$$\begin{gathered} \delta(\vec{P}_{n_1} + \vec{P}_{n_2} - \vec{P}_{n_2'} - \vec{P}_p - \vec{P}_e) = \\ \delta(|\vec{P}_{n_1}| + |\vec{P}_{n_2} - \vec{P}_{n_2'} - \vec{P}_p - \vec{P}_e|)\,\delta(\Omega_{n_1} - \Omega_{-n_2+n_2'+p+e})\tfrac{1}{P_{n_1}^2} \quad , \end{gathered} \tag{3.68}$$

were the P_i are the magnitudes of the four-momenta.

Now integrating over Ω_{n_1} we obtain

$$\mathcal{A} = \int d\Omega_{n_2} d\Omega_{n_2'} d\Omega_p d\Omega_e d\Omega_{\overline{\nu}} \frac{1}{P_{n_1}^2}\,\delta(|\vec{P}_{n_1}| + |\vec{P}_{n_2} - \vec{P}_{n_2'} - \vec{P}_p - \vec{P}_e|). \tag{3.69}$$

We can once again rewrite the delta function as

$$\delta(|\vec{P}_{n_1}| + |\vec{P}_{n_2} - \vec{P}_{n_2'} - \vec{P}_p - \vec{P}_e|) = \delta(|\vec{P}_{n_1}| - |\vec{P}_{n_2} + \vec{P}_R|), \tag{3.70}$$

were R stands for "the Rest of the particles ".

Now exploiting the delta identity the previous equation can be written as

$$\begin{aligned} &= \delta(P_{n_1} - (P_{n_2'}^2 + P_R^2 + 2P_{n_2'}P_R\cos\theta)^{1/2}) = \delta(f(\cos\theta)) \\ &= \frac{1}{f'(a)}\delta(\cos\theta - a), \end{aligned} \tag{3.71}$$

were $f'(\cos theta)$ just refers to the derivative and 'a' is the value at which $f(a) = 0$.

$$\begin{aligned} f'(\cos\theta) &= \frac{P_{n_2'}P_R}{(P_{n_2'}^2 + P_R^2 + 2P_{n_2'}P_R\cos\theta)^{1/2}} \\ &= \frac{P_{n_2'}P_R}{P_{n_1}} \end{aligned} \tag{3.72}$$

$$\Rightarrow \delta(f(\cos\theta)) = \frac{P_{n_1}}{P_{n_2'}P_R}\delta(\cos\theta - a) \tag{3.73}$$

$$\mathcal{A} = \int d\Omega_{n_2} d\Omega_{n_2'} d\Omega_p d\Omega_e d\Omega_{\overline{\nu}} \frac{1}{P_{n_2'}P_R P_{n_1}}\delta(\cos\theta - a) \tag{3.74}$$

Now recalling that θ is the angle between $\vec{P_{n_2'}}$ and $\vec{P_R} = \vec{P}_p + \vec{P}_e + \vec{P}_{n_2}$, we can note that there is a restriction over the solid angle. This means that when we integrate over $\Omega_{n_2'}$, the result will be 2π instead of the usual 4π. So integrating over all solid angles we obtain

$$\mathcal{A} = \frac{2\pi(4\pi)^4}{P_{n_2'}|\vec{P}_p + \vec{P}_e - \vec{P}_{n_2}|P_{n_2}}. \tag{3.75}$$

If we now remember that the particles involved in the reaction have energies close to the Fermi surface, we can write

$$|\vec{P}_p + \vec{P}_e - \vec{P}_{n_2}| \simeq P_{n_2}, \tag{3.76}$$

simply because the neutron concentration is much greater than the proton and electron concentration[10]. We now have for the angular part of the phase factor

$$\mathcal{A} = \frac{(4\pi)^5}{2P_{n_2'}P_{n_1}P_{n_2}}, \tag{3.77}$$

[10] Remember that the Fermi momentum is proportional to the concentration.

so the phase factor $\mathcal{P}$ (Eq. (3.64)) can now be written as (In spherical coordinates)

$$\begin{aligned} \mathcal{P} &= \frac{(4\pi)^5 V}{2h^{10}} \int P_{n_1} dP_{n_1} P_{n_2} dP_{n_2} P_{n_2'} dP_{n_2'} P_p^2 dP_p P_e^2 dP_e P_{\overline{\nu}}^2 dP_{\overline{\nu}} E_{\overline{\nu}} \\ &* \ \delta(E_{n_1} + E_{n_2} - E_{n_2'} - E_p - E_e - E_{\overline{\nu}}) \\ &* \ f_{n_1} f_{n_2} (1 - f_{n_2'})(1 - f_p)(1 - f_e). \end{aligned} \tag{3.78}$$

To integrate over energy instead of momentum, we can approximate the protons and neutrons as being non relativistic particles (Just as we did with the direct Urca process.).

$$dE_i = \frac{P_i dp_i}{m_i} \quad for \;\; i = n, p \tag{3.79}$$

We can also again take the electrons as being highly relativistic

$$E_e = P_e c \Rightarrow dE_e = dP_e c \tag{3.80}$$

For neutrinos we take as usual $dE_{\overline{\nu}} = dP_{\overline{\nu}}$. So now $\mathcal{P}$ becomes

$$\begin{aligned} \mathcal{P} &= \frac{(4\pi)^5 V m_{n_1} m_{n_2} m_{n_2'} m_p}{2h^{10}} \\ &* \int dE_{n_1} dE_{n_2} dE_{n_2'} dE_p \frac{E_e^2}{c^3} dE_e \frac{E_{\overline{\nu}}}{c^3} dE_{\overline{\nu}} f_{n_1} f_{n_2} \prod_{i=1}^{3} (1 - f_i)\, \delta \\ &= \mathcal{S} \int dE_{n_1} dE_{n_2} dE_{n_2'} dE_p E_e^2 c^3 dE_e E_{\overline{\nu}} c^3 dE_{\overline{\nu}} f_{n_1} f_{n_2} \prod_{i=1}^{3} (1 - f_i)\, \delta, \end{aligned} \tag{3.81}$$

were $\mathcal{S}$ si defined as[11]

$$\mathcal{S} = \frac{(4\pi)^5 V m_n^3 m_p P_{f_p} \mu_e^2}{2h^{10} c^6}. \tag{3.82}$$

Now we make the following changes of variable

$$\begin{aligned} \mathcal{E}_{n_1} &= \frac{E_{n_1} - \mu_{n_1}}{kT} \\ \mathcal{E}_{n_2} &= \frac{E_{n_2} - \mu_{n_2}}{kT} \\ \mathcal{E}_{n_2'} &= \frac{-E_{n_2} + \mu_{n_2'}}{kT} \end{aligned}$$

[11]Once again we have taken the electrons to be highly degenerate, so only energies close to the Fermi surface are considered. This means that we replace E_e by the Fermi energy μ_e which is constant.

$$\begin{aligned}\mathcal{E}_p &= \frac{-E_p+\mu_p}{kT}\\ \mathcal{E}_e &= \frac{-E_e+\mu_e}{kT}\\ \mathcal{E}_{\overline{\nu}} &= \frac{E_{\overline{\nu}}}{kT}\end{aligned} \tag{3.83}$$

(3.84)

Recalling the chemical equilibrium condition, the delta function can be written as

$$\frac{1}{kT}\delta(\mathcal{E}_{n_1}+\mathcal{E}_{n_2}+\mathcal{E}_{n_2'}+\mathcal{E}_p+\mathcal{E}_e-\mathcal{E}_{\overline{\nu}}), \tag{3.85}$$

so now $\mathcal{P}$ becomes

$$\mathcal{P}=\mathcal{S}(kT)^8\int \mathcal{E}_{\overline{\nu}}\delta(\mathcal{E}_{n_1}+\mathcal{E}_{n_2}+\mathcal{E}_{n_2'}+\mathcal{E}_p+\mathcal{E}_e-\mathcal{E}_{\overline{\nu}})\prod_{i=1}^{5}\frac{d\mathcal{E}_i}{1+e^{\mathcal{E}_i}}. \tag{3.86}$$

Again, recalling that the delta function is the Fourier transform of a constant, we can write

$$\delta\left(\sum_{i=1}^{5}\mathcal{E}_i-\mathcal{E}_{\overline{\nu}}\right)=\frac{1}{2\pi}\int\exp\left\{iz\left(\sum_{i=1}^{5}\mathcal{E}_i-\mathcal{E}_{\overline{\nu}}\right)\right\}\,dz, \tag{3.87}$$

so that

$$\mathcal{P}=\frac{\mathcal{S}}{2\pi}\int\mathcal{E}_{\overline{\nu}}^3\,d\mathcal{E}_{\overline{\nu}}\int\exp\left\{iz\left(\sum_{i=1}^{5}\mathcal{E}_i-\mathcal{E}_{\overline{\nu}}\right)\right\}\,dz\prod_{i=1}^{5}\frac{d\mathcal{E}_i}{1+e^{\mathcal{E}_i}} \tag{3.88}$$

We can first calculate

$$\int_{-\infty}^{\infty}e^{iz\mathcal{E}_i}\frac{d\mathcal{E}_i}{1+e^{\mathcal{E}_i}}, \tag{3.89}$$

which was already calculated in section (3.5). By using the residue theorem one obtains

$$\frac{2\pi i e^{-\pi z}}{1-e^{-2\pi z}}. \tag{3.90}$$

We can now write eq. (3.88) as

$$\mathcal{P}=\frac{\mathcal{S}}{2\pi}\int_0^{\infty}\mathcal{E}_{\overline{\nu}}^3\,d\mathcal{E}_{\overline{\nu}}\int_{-\infty}^{\infty}e^{-iz\mathcal{E}_{\overline{\nu}}}\left(\frac{2\pi i e^{-\pi z}}{1-e^{-2\pi z}}\right)^5 dz. \tag{3.91}$$

Again using the residue theorem, we can calculate the second integral in eq. (3.91). Making a contour integral around the path shown in figure (3.4) we get

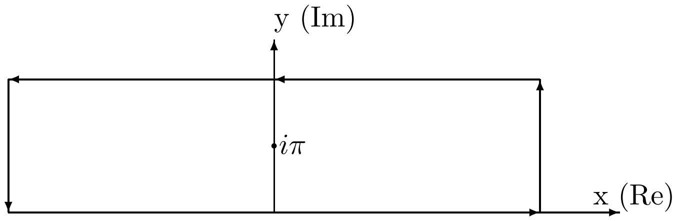

Figure 3.4: Path taken for integral (3.91). The vertical segments do not contribute to the integral when one integrates from $-\infty$ to ∞.

$$\oint e^{-iz\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi z}}{1 - e^{2\pi z}} \right)^5 dz =$$

$$\int_{-\infty}^{\infty} e^{-ix\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi x}}{1 - e^{2\pi x}} \right)^5 dz - \int_{-\infty}^{\infty} e^{-i\mathcal{E}_{\overline{\nu}}(x+i)} \left(\frac{2\pi i e^{-\pi(x+i)}}{1 - e^{-2\pi(x+i)}} \right)^5 dz$$

$$= \int_{-\infty}^{\infty} e^{-ix\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi x}}{1 - e^{2\pi x}} \right)^5 \left(1 + e^{\mathcal{E}_{\overline{\nu}}}\right) dz \tag{3.92}$$

$$= -2\pi i Res \left\{ e^{-iz\mathcal{E}_{\overline{\nu}}} \left(\frac{2\pi i e^{-\pi x}}{1 - e^{2\pi x}} \right)^5 \right\}. \tag{3.93}$$

Making a Laurent expansion of $\left(\frac{2\pi i e^{-\pi x}}{1 - e^{2\pi x}} \right)^5$ we find

$$\frac{-i}{z^5} - \frac{\mathcal{E}_{\overline{\nu}}^2}{z^4} + \frac{i(4\pi^2 + 3\mathcal{E}_{\overline{\nu}}^2)}{6z^3} + \frac{5\pi^2\mathcal{E}_{\overline{\nu}}^2 + \mathcal{E}_{\overline{\nu}}^3}{6z^2} - \frac{i(9\pi^4 + 10\pi^2\mathcal{E}_{\overline{\nu}}^2 + \mathcal{E}_{\overline{\nu}}^4)}{24z} + ... \tag{3.94}$$

so that the residue is $\frac{-i}{24}(9\pi^4 + 10\pi^2\mathcal{E}_{\overline{\nu}}^2 + \mathcal{E}_{\overline{\nu}}^4)$ and eq (3.91) becomes

$$\mathcal{P} = \frac{\mathcal{S}(kT)^8}{2\pi} \int_0^{\infty} \mathcal{E}_{\overline{\nu}}^3 \frac{-i}{24} (9\pi^4 + 10\pi^2\mathcal{E}_{\overline{\nu}}^2 + \mathcal{E}_{\overline{\nu}}^4) \left(1 + e^{\mathcal{E}_{\overline{\nu}}}\right) d\mathcal{E}_{\overline{\nu}} \tag{3.95}$$

$$= \frac{\mathcal{S}(kT)^8}{2\pi} \frac{11513\pi^8}{120960} = \mathcal{S}\frac{11513\pi^7}{241920}(kT)^8.$$

Recalling the definition of $\mathcal{S}$ (Eq. (3.82)), and the expression for the weak matrix element (Eq. (3.62)), we can finally we can express $\dot{E}_{\overline{\nu}}$ eq. (3.61) as

$$\dot{E}_{\overline{\nu}} = \frac{11513\, G_F^2 \cos^2\theta_c g_a^2 V m_n^3 m_p P_{f_p}}{60480\pi\hbar^{10}c^6 m_\pi^4}(kT)^8 \tag{3.96}$$

This is the Luminosity for neutrino emission for the Modified Urca process. Notice the $(kT)^8$ dependence, in contrast to the $(kT)^6$ dependence for the Direct Urca process. The Fermi momentum and energy terms contained in the constant depend on the concentration of protons and electrons respectively, so we expect there to be slight variations in Luminosity of N.S.s with different densities. As we shall see, this luminosity produces a much slower cooling than the Direct Urca process. We can now proceed to calculate a cooling curve, using the specific heat for a N.S..

3.7 Specific Heat Capacity

In this section we shall calculate the specific heat capacity for a N.S., assuming for simplicity that it is composed mainly of degenerate neutrons. This is a standard calculation for electrons and can be found in many references, here we shall rely mainly on Callen [16].

Recall that the specific heat capacity is given by

$$C_v = \left(\frac{\partial U}{\partial T}\right)_V, \tag{3.97}$$

were $U(T, V, \mu)$ is the internal energy, the quantity that we shall first calculate[12]. The internal energy can be calculated using the Fermi-Dirac distribution, and so we can write

$$U = \frac{2V}{h^3}\int d^3\vec{p}\frac{\mathcal{E}}{1+\exp\beta(\mathcal{E}-\mu)}. \tag{3.98}$$

Here the factor of 2 accounts for the spin of the neutron. Using spherical coordinates we can write

$$\begin{aligned} U &= \frac{8\pi V}{h^3}\int p^2 dp\frac{\mathcal{E}}{1+\exp\beta(\mathcal{E}-\mu)} \\ &= \frac{8\pi V}{h^3}\int\frac{\sqrt{2m\mathcal{E}}m\mathcal{E}d\mathcal{E}}{1+\exp\beta(\mathcal{E}-\mu)} \\ &= \frac{8\pi V m\sqrt{2m}}{h^3}\int\frac{\mathcal{E}^{3/2}d\mathcal{E}}{1+\exp\beta(\mathcal{E}-\mu)}. \end{aligned} \tag{3.99}$$

The previous integral can be calculated using the Sommerfeld expansion[13] [16]

[12] Recall that the internal energy was for the ground state ($T = 0$) of a N.S. was already calculated in section 1.3

[13]
$$\int_0^\infty \frac{\phi(\mathcal{E})d\mathcal{E}}{1+\exp\beta(\mathcal{E}-\mu)} = \int_0^\mu \phi(\mathcal{E})d\mathcal{E} + \frac{\pi^2}{6}(kT)^2\phi'(\mu) + ... \tag{3.100}$$

$$\Rightarrow U = \frac{8\pi V m^{3/2}\sqrt{2}}{h^3}\left\{\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}\mu^{1/2}(kT)^2 ...\right\} \tag{3.101}$$
$$= \frac{16\pi m^{3/2}\sqrt{2}V\mu^{5/2}}{5h^3}\left\{1 + \frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right\}$$

Note that the previous expression still depends on μ. To find C_v, we must also find the chemical potential which can be found with the help of the average particle number N.

$$N = \frac{2V}{h^3}\int_0^\infty \frac{d^3\vec{p}}{1+\exp\beta(\mathcal{E}-\mu)} \tag{3.102}$$
$$= \frac{8\pi V m^{3/2}\sqrt{2}}{h^3}\int \frac{\mathcal{E}^{1/2}d\mathcal{E}}{1+\exp\beta(\mathcal{E}-\mu)},$$

were we have again used spherical coordinates. Again, using the Sommerfeld expansion we obtain

$$N = \frac{8\pi V m^{3/2}\sqrt{2}}{h^3}\left\{\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}(kT)^2\mu^{-1/2}\right\} \tag{3.103}$$
$$= \frac{16\pi V m^{3/2}\sqrt{2}}{3h^3}\mu^{3/2}\left\{1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right\}. \tag{3.104}$$

To carry a solution for μ to second order in T, it is sufficient to replace μ by E_f (Chemical potential at $T = 0$, recall eq. (1.2)) in the second order term in eq. (3.104) [16], this gives

$$N = \frac{8\pi V m^{3/2}\sqrt{2}}{h^3}\left\{\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}(kT)^2 E_f^{-1/2}\right\} \tag{3.105}$$

$$\Rightarrow \quad \frac{Nh^3}{8\pi V m^{3/2}\sqrt{2}} = \left\{\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12}(kT)^2\mu_0^{-1/2}\right\} \tag{3.106}$$

$$\Rightarrow \quad \mu^{3/2} = \underbrace{\frac{3Nh^3}{16\pi V m^{3/2}\sqrt{2}}}_{E_f^{3/2}} - \frac{\pi^2}{12}(kT)^2 E_f^{-1/2} \tag{3.107}$$

$$\Rightarrow \quad \mu^{3/2} = E_f^{3/2}\left\{1 - \frac{\pi^2}{12}\left(\frac{kT}{E_f}\right)^2 + ...\right\} \tag{3.108}$$

So now μ can be approximated by

$$\mu = E_f \left\{ 1 - \frac{\pi^2}{12} \left(\frac{kT}{E_f} \right)^2 + ... \right\}. \tag{3.109}$$

The previous result conincides with what is obtained by Callen [16] and reduces to the knownh expression $\mu = E_f$ when $T = 0$.

To simplify some algebra, we can divide equation (3.102) by eq. (3.104) to write

$$\frac{3}{5} N\mu \left\{ 1 + \frac{5\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 \right\} \left\{ 1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 \right\}^{-1}. \tag{3.110}$$

The polynomial division can be carried out as follows. We need to find the coefficients of the polynomial $a + b\pi^2 (kT/\mu)^2$ such that

$$\begin{aligned} 1 + \frac{5\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 &= \left\{ 1 + \frac{\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 \right\} \left\{ a + b\pi^2 \left(\frac{kT}{\mu} \right)^2 \right\} \\ &= a + b\pi^2 \left(\frac{kT}{\mu} \right)^2 + \frac{a\pi^2}{8} \left(\frac{kT}{\mu} \right)^2 + \mathcal{O}(T^4). \end{aligned} \tag{3.111}$$

We find that $a = 1$ and $b = 1/2$ so that

$$\frac{3}{5} N\mu \left\{ 1 + \frac{\pi^2}{2} \left(\frac{kT}{\mu} \right)^2 + ... \right\}. \tag{3.112}$$

Now we can finally obtain $U(T, V, \mu)$ by replacing the expression for μ (eq. (3.109)) in the previous equation, and also replacing μ by E_f in the second order term in T to obtain.

$$\begin{aligned} U &= \frac{3}{5} N E_f \left\{ 1 - \frac{\pi^2}{12} \left(\frac{kT}{E_f} \right)^2 \right\} \left\{ 1 + \frac{\pi^2}{2} \left(\frac{kT}{E_f} \right)^2 \right\} \\ &= \frac{3}{5} N E_f \left\{ 1 - \frac{5\pi^2}{12} \left(\frac{kT}{E_f} \right)^2 \right\} \end{aligned} \tag{3.113}$$

Finally, the heat capacity can be obtained by taking the partial derivartive with respect to T.

$$C_v = \frac{3}{2} Nk \left(\frac{\pi^2 kT}{3E_f} \right). \tag{3.114}$$

The linear dependence on the temperature is something typical in a degenerate gas of fermions. This specific heat capacity, together with the neutrino emmisivities calculated in the previous sections, determine the N.S.'s cooling curve as we shall see in the next section.

3.8 Cooling Curves

In this section, a temperature-time curve will be derived for both the Direct and Modified Urca processes.

3.8.1 Direct Urca Cooling

We shall now calculate a cooling curve for a typical N.S..

Recalling the cooling equation (Eq. 3.19)

$-(\dot{E}_{\overline{\nu}} + \dot{E}_{\gamma}) = C_v \frac{dT}{dt}$,

and using eqs. (3.60) and (3.1) for neutrino and photon emission respectively, we may now write

$$CT = -\frac{dt}{dT}(UT^6 + GT^4) \tag{3.115}$$

were the constants C, U and G have been calculated for a typical N.S. with radius $R \approx 10km$ and density $\rho \approx 10^{15} g/cm^3$.

$C = 6.26 \times 10^9 V$ $[J/K]$
$U = 2.18 \times 10^{-25} V$ $[JK^{-6}s^{-1}]$
$G = 71.25$ $[JK^{-4}s^{-1}]$

So now,

$$\frac{-CT}{UT^6 + GT^4} dT = dt. \tag{3.116}$$

This equation can be directly integrated giving

$$t = -C\left(-\frac{1}{2GT^2} - \frac{U \ln T}{G^2} + \frac{U \ln G + T^2 U}{2G^2}\right) \tag{3.117}$$

The Neutrino Era

Before looking at eq. (3.117), lets look at some limiting cases. Prior to integrating differential equation (3.116) we should note that for initial temperatures ($T \sim 10^8$ K) we have for the luminosities

$\dot{E}_{\overline{\nu}} = 4.18 \times 10^{60}$ J/s

$\dot{E}_{\gamma} = 7.12 \times 10^{33}$ J/s, so $\dot{E}_{\overline{\nu}} \gg \dot{E}_{\gamma}$. This means that initially, cooling is dominated by neutrino emission (*The Neutrino Era*), so that the cooling equation (Eq. 3.116) becomes

$$CT\frac{dT}{dt} = -UT^6 \tag{3.118}$$

and so

$$T(t) = \left(\frac{C}{4Ut - C/T_0^4}\right)^{1/4} \tag{3.119}$$

were T_0 refers to the initial temperature of $T_0 \approx 10^{11}\,K$ (See section 1.1.1). Equation (3.119) can be called the *Neutrino era equation.* Note that the term C/T_0^4 is very small and can thus be discarded, so the cooling is actually independent of the initial temperature, and can thus be written

$$T(t) = \left(\frac{C}{4Ut}\right)^{1/4}. \tag{3.120}$$

The fact that the cooling is independent of the initial temperature is a big advantage because we don't have to rely on supernova theory. Replacing numerical values we obtain

$$T(t) \simeq 2.89 \times 10^8 t^{-1/4}\,K \tag{3.121}$$

for the neutrino era of the cooling. If a tipical N.S. were to cool via this mechanism, it would cool to $T \sim 10^7\,K$ in about a week [3]! As we shall see, this cooling rate is extremely high.

The Photon Era

We can now estimate the time at which Photon cooling becomes important by equating the neutrino and photon luminosities:

$$\dot{E}_{\gamma} = \dot{E}_{\overline{\nu}} \tag{3.122}$$

$$UT^6 = GT^4 \tag{3.123}$$

$$T = \sqrt{\frac{G}{U}} \sim 10^7\,K, \tag{3.124}$$

which corresponds to a time of $t \sim 2\,weeks$ according to eq. (3.117). So photon cooling becomes important after 2 weeks if the N.S. cools via the Direct Urca process [7].

Now lets look at the photon cooling era. In the photon cooling era, the neutrino luminosity becomes unimportant and eq. (3.116) can be written as

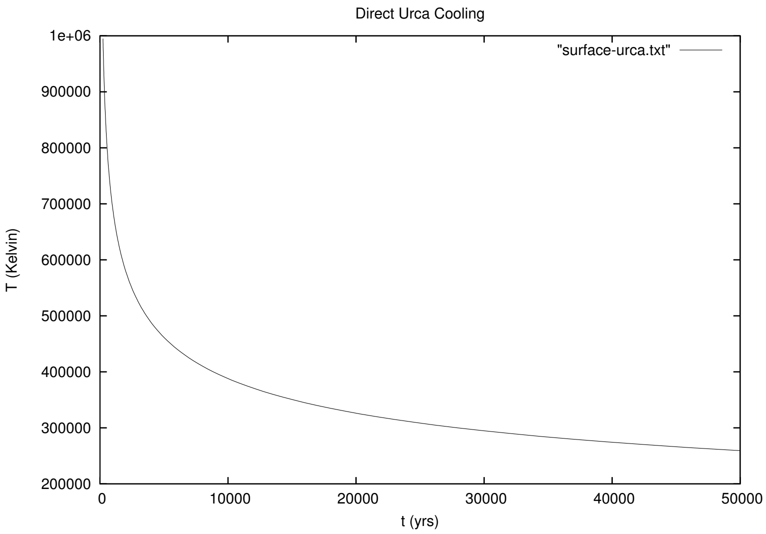

Figure 3.5: A cooling curve for the Direct Urca process is shown. Photon cooling becomes important at $t \sim 2\,weeks$. After the neutrino era is over, the cooling continues very slowly (Because R is small) due to photon emission. A "stable" temperature is reached at around $2 \times 10^5\,^\circ K$

$$CT\frac{dT}{dt} = -GT^4, \tag{3.125}$$

so

$$T(t) = \left(\frac{C}{2GT - C/T_0^2}\right)^{1/2} \simeq \left(\frac{C}{2G}\right) t^{-1/2} \tag{3.126}$$

for the photon cooling era.

Note that the neutrino cooling era goes as $\sim t^(- 1/4)$ and the photon cooling era goes as $\sim t^{-1/2}$, which proves the fact that the neutrino cooling is much faster than the photon cooling.

The Cooling Curve

Lets return to the full solution of the cooling equation (Eq. (3.117)).

$t = -C\left(-\frac{1}{2GT^2} - \frac{U\ln T}{G^2} + \frac{U\ln G + T^2 U}{2G^2}\right)$

This eq cannot be solved for T analytically, but with the help of "Mathematica", we can plot the temperature time curve $T(t)$ as shown in figure 3.5.

3.8.2 Modified Urca Cooling

Here we shall follow the same procedure as the one followed in section 3.8.1. Recalling the cooling equation 3.19, that includes photon and neutrino cooling, we obtain

$$CT = -\frac{dt}{dT}(UT^8 + GT^4). \tag{3.127}$$

The previous equation can be integrated but the algebraic result is not very illuminating, so it will be plotted in figure 3.6

The Neutrino Era

It is again instructive to look at the limiting case when the temperature is high and thus, neutrino cooling dominates. For this limiting case, eq. 3.127 becomes

$$CT\frac{dT}{dt} = -UT^8. \tag{3.128}$$

This equation can be integrated to give

$$T(t) = \left(\frac{C}{6Ut - \frac{C}{T_0^6}}\right)^{1/6} \approx \left(\frac{C}{6U}\right)^{1/6} t^{-1/6}, \tag{3.129}$$

were we have again neglected the contribution from the initial temperature. Notice that the Modified Urca cooling in the neutrino era goes as $\sim t^{-1/6}$, in contrast to the Direct Urca cooling, which goes as $\sim t^{-1/4}$, which provides a faster cooling. In fact, the neutrino era for the modified urca process is so long, compared to the one for the Direct Urca Process, that photon cooling only becomes important after $t \sim 10^5 \;\; yrs$ [14]. In other words, the neutrino era for the Direct Urca process looks much like a transient process (see figure 3.7.). A comparison of the two cooling mechanisms is plotted in figure (3.6)

3.9 Comparison to actual observations

Here we shall compare our theoretical cooling curves with observed temperatures. A good way of comparing the results with observations is to construct two curves for each process (Modified and Direct Urca), one curve corresponding to a N.S. with the maximum allowed mass and Minimum allowed radius, and the other corresponding to a N.S. with the minimum allowed mass and the maximum allowed radius[15]. One can think of the region in between these two curves as a *Cooling Band*.

[14] This can be found by making a similar analisis as the one done in the previous section, by setting $E_{\overline{\nu}} = E_\gamma$.

[15] See section 1.4

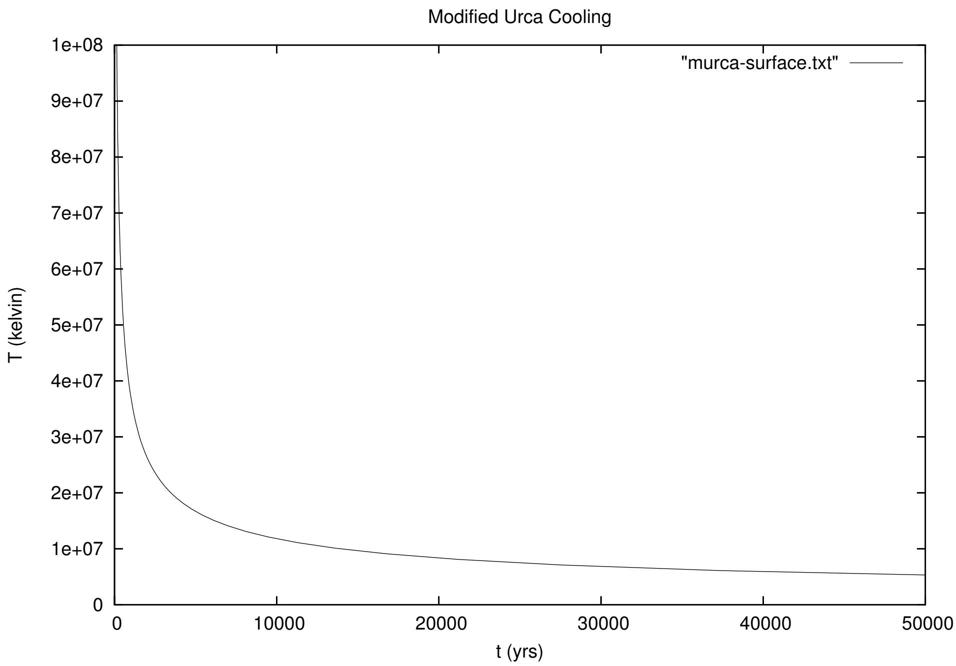

Figure 3.6: This is a temperature-time curve for a N.S. undergoing Modified Urca Cooling. The time is given in weeks. A "stable" temperature is reached at around $10^7\,{}^\circ K$

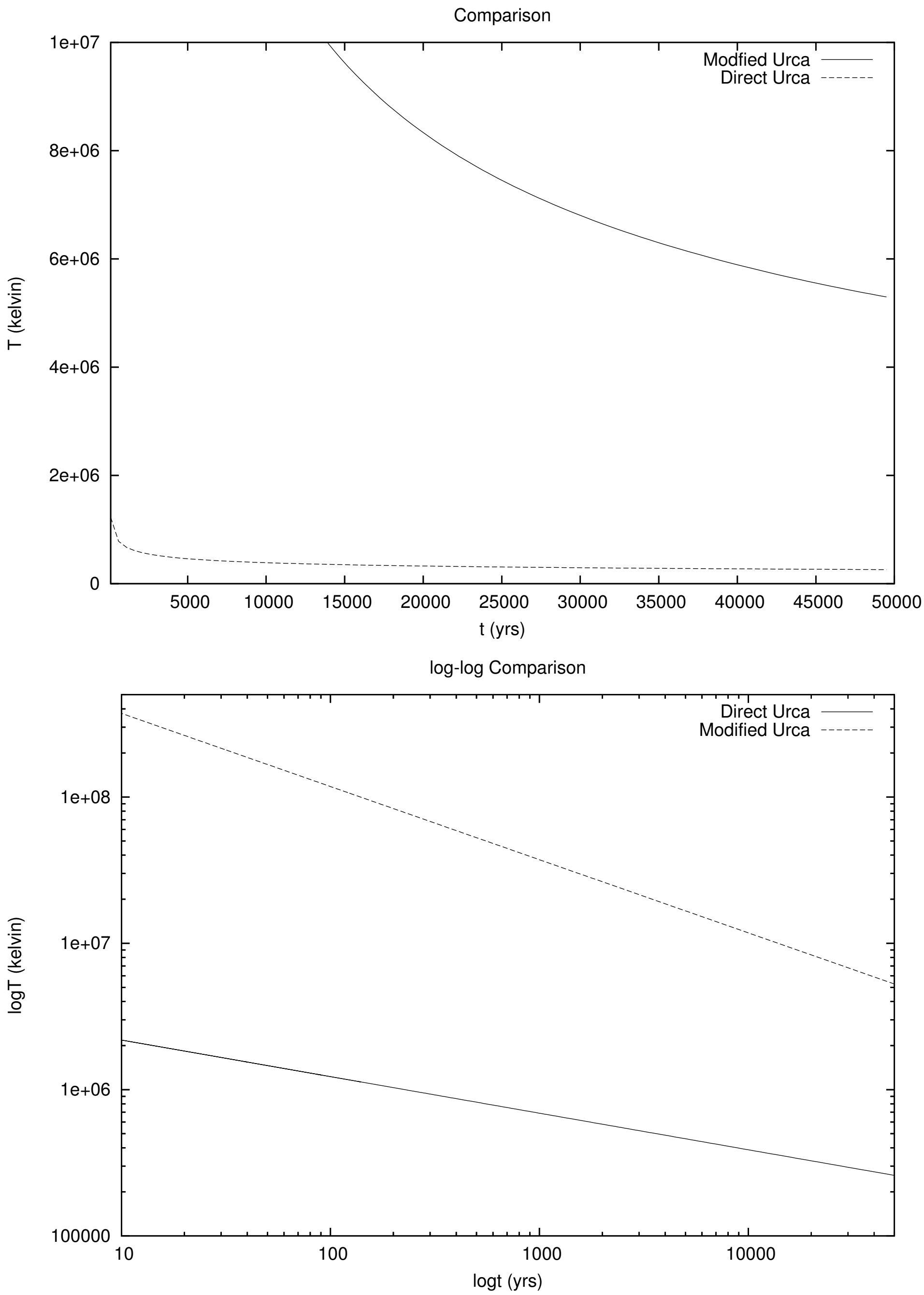

Figure 3.7: This curve compares the modified and direct Urca cooling, the upper curve is the Modified Urca curve. Notice how it takes much longer for the photon era to become important in Modified Urca cooling.

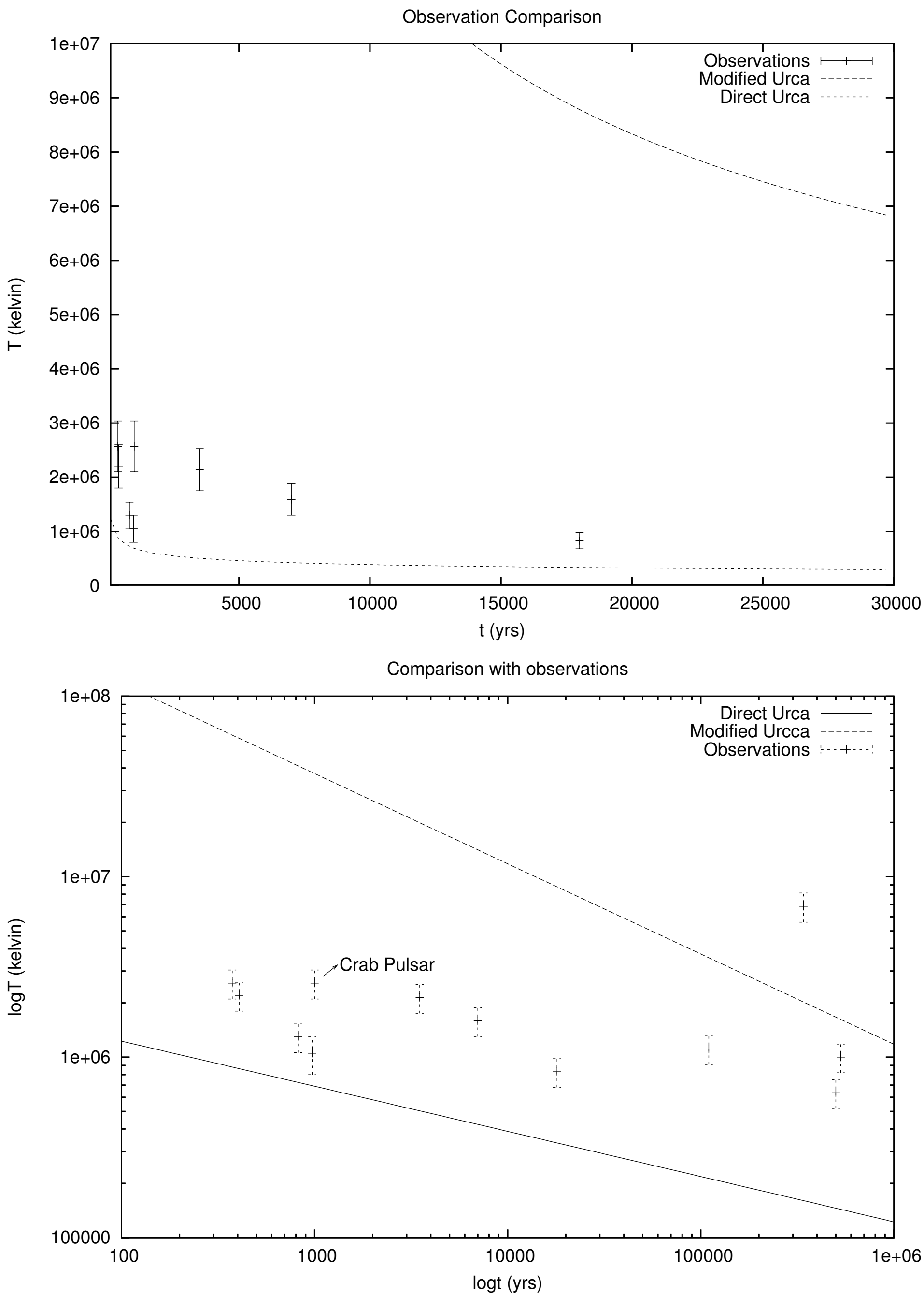

Figure 3.8: A comparison between the Modified Urca Process (Upper curve), the Direct Urca Process (Lower curve), and the observed temperatures. The error bars are a consequence of the redshift factor (See Chapter II). Notice how the observed temperatures tend to be higher than the Direct Urca temperatures, but are still much closer to the lower curve.

What we found is that the cooling band for both processes is extremely thin, in other words, the theoretical curves are insensitive to the N.S. mass, this confirms what is mentioned in Yakolev's article [14]. The cooling curves obtained in the previous sections suffice to compare N.S.s with any of the masses and radii found in section 1.4, and can thus be compared with the data in figure 2.2. The results plotted in figure (3.8) are by no means entirely conclusive, but one can dare say that N.S.s tend to prefer Direct Urca Cooling over Modified Urca Cooling[16], this may provide us information about the proton concentration inside the N.S. ($> 10\%$). We can also note that the observed temperatures are somewhat higher than the predicted temperatures ($\sim 10^6 K$ higher in some cases), these differences in temperature are thought (By many authors) to be consequences of superfluidity [17], which alters the specific heat capacity. Also, in the presence of superfluidity (or pairing between fermions inside the N.S.), one needs to break the cooper pair before the neutron decays.

Still, results are somewhat encouraging because the observed temperatures tend to have a rapidly decreasing pattern of the form $T \propto t^{-1/\gamma}$ (in the neutrino era), were $\gamma = 4$ for direct Urca Cooling and $\gamma = 6$ for Modified Urca Cooling. The observations tend to bo closer to $\gamma = 4$, that is, to Direct Urca Cooling.

3.10 Concluding Remarks

An enormous amount of physics is involved in the description of Neutron Stars. Up to now we have made use of General Relativity, Statistical Mechanics, and Elementary Particle Physics among others. This subject could potentially put a serious test to our current understanding of nature, but observations are still very scarce and in a few cases, unreliable.

In the absence of superfluidity, we found that we could construct a single cooling curve for each of the cooling modes. The insensitivity of the mass and initial temperature was responsible for there being only one cooling curve for each of the two mechanisms. This insensitivity was a great advantage when we compared with observations, because we didn't rely on complicated supernova models or different limiting masses which can vary with the equation of state. We found that these cooling curves have basically two stages: A transient stage corresponding to the neutrino era, and a stable state corresponding to the photon era. Since the observed temperatures are much lower than the Modified Urca curve, the slow cooling scenarios may be practically ruled out. It also seems that if only ordinary matter exists in the neutron star, the proton concentration is above the Direct Urca threshhold. Of course, we cannot be completely sure of these conclusions with only a "handfull" of observations. Thus, more observations are needed, and this is not an easy task for sure.

[16] One can dare say this, assuming only ordinary matter exists inside the N.S.

[17] See for example Page's article [19]

To account for the still large difference between the predicted and observed temperatures, one can come up with two non-excluding possibilities:

- The N.S. has an internal heating mechanism: One possible mechanism is due to external perturbations on the spherically symmetric geometry [18]. These perturbations cause the star to undergo radial pulsations. This heating mechanism may account for the difference in temperature, at least for the very studied case of the Vela pulsar [18].

- More recent models suggest theat the N.S. is in a superfluid state: The energy needed to break the cooper pairs will have to be supplied before the beta reaction can take place [19]. This can cause the neutrino emissivity to be severely suppressed. The results obtained by N.S. superfluid models are however not as encouraging as we would like, since the cooling band is very sensitive to the mass (Radically different from our case).

Other less plaussible possibilities include:

- Since observaitons have at least some resemblence to the predicted cooling curves, it may be possible that the proportionality factor of the neutrino emmisivity, i.e. the matrix elements (Eqs. 3.24, 3.62), are incorrect. This would imply that our knowledge of fundamental interactions is still "incomplete".

- Since many of the observations are not detected as pulsars, but just as localized x-ray sources in supernova sites, it is possible that some of the x-ray sources are not neutron stars.

The story is still not over, more observations and further research is needed, and since I am lost for words, it is perhaps best to end this document with a quote about what is to come:

> *"A lonely, uncompensated, perhaps even impossible Task,-yet some of us must ever be seeking, I suppose."* Thomas Pynchon

Appendix A

The Oppenheimer-Volkoff Equation

In this appendix we shall derive the Oppenheimer-Vokoff equations of hydrostatic equilibrium for a spherically symmetric object. These equations can be derived using the Einstein field equation in General Relativity. The main reference for this appendix is Carroll's text [15], but the formalism differs somewhat from Carroll's work.

A.1 The spherically symmetric metric

We cay start with the most general spherically symmetric metric $g_{\mu\nu}$,

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{A.1}$$

were the 'di' on the right of the equation are one-forms. $\mathrm{d}\Omega^2$ is the metric for a two-sphere[1] and ds^2 is the invariant space-time interval. In the particular case of vacuum, this metric reduces to the Schwarzschild metric.

A.1.1 The Field Equation

In the non-vacuum case, we turn to the full Einstein equation,

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}GT_{\mu\nu}, \tag{A.2}$$

were $G_{\mu\nu}$ is known as the Einstein tensor. $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar and $T_{\mu\nu}$ is the energy-momentum tensor[2]. The Einstein tensor can be completely calculated from the metric.

[1] $d\Omega = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$.

[2] Remember that the idea behind the Einstein equation is that $(Curvature) = (Energy)$.

A.2 Curvature

To proceed with the curvature calculations it is convenient to work with non-coordinate bases, also called orthonormal bases. We can choose the following basis one-forms

$$\hat{e}^t = e^{\alpha}\mathrm{d}t \quad (A.3)$$
$$\hat{e}^r = e^{\beta}\mathrm{d}r \quad (A.4)$$
$$\hat{e}^{\theta} = r\mathrm{d}\theta \quad (A.5)$$
$$\hat{e}^{\phi} = r\sin\theta\mathrm{d}\phi. \quad (A.6)$$

Note that with this choice of basis one-forms we have

$$\begin{aligned} ds^2 &= -(\hat{e}^t)^2 + (\hat{e}^r)^2 + (\hat{e}^{\theta})^2 + (\hat{e}^{\phi})^2 \\ &= \eta_{\mu\nu}\hat{e}^b \otimes \hat{e}^b \end{aligned} \quad (A.7)$$

were $\eta_{\mu\nu}$ is the Minkowski (flat space-time) metric. If for some reason we wish to go back to coordinate bases we can use the matrices ${e_\mu}^a$ such that

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab}. \quad (A.8)$$

These matrices are normally called "vielbeins" and are in this case

$${e_\mu}^a = \begin{pmatrix} e^{\alpha} & 0 & 0 & 0 \\ 0 & e^{\beta} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix} \quad (A.9)$$

A.2.1 The Spin Connection

The first step in a curvature calculation is to find the spin connection ${\omega^a}_b$ using the Cartan structure equation [15]

$$\omega^a_b \wedge \hat{e}^b = -\mathrm{d}\hat{e}^a, \quad (A.10)$$

were the '$\wedge$' is the wedge product[3], and the 'd' stands for the exterior derivative. Now solving for the spin connection is fairly straight-forward. The non zero two-forms give

$${\omega^t}_r = \frac{d\alpha}{dr}e^{-\beta}\hat{e}^t \quad (A.11)$$
$${\omega^r}_{\theta} = \frac{-e^{-\beta}}{r}\hat{e}^{\theta} \quad (A.12)$$

[3]In the case of two one-forms, $(A \wedge B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu$.

$$
\begin{aligned}
\omega^{r}{}_{\phi} &= \frac{-e^{-\beta}}{r}\hat{e}^{\phi} && \text{(A.13)} \\
\omega^{\theta}{}_{\phi} &= \frac{1}{r}\cot\theta. && \text{(A.14)}
\end{aligned}
$$

A.2.2 The Curvature Two-forms

The curvature two-forms and thus the Riemann tensor can now be calculated using [15]

$$\tilde{R}^{a}{}_{b} = \mathrm{d}\omega^{a}_{b} + \omega^{a}_{c} \wedge \omega^{c}_{b}. \tag{A.15}$$

For example, let's calculate $\tilde{R}^{t}{}_{r}$. Since $\omega^{t}_{c} \wedge \omega^{c}_{r} = 0$ we have

$$\tilde{R}^{t}{}_{r} = \mathrm{d}\omega^{t}_{r} = -\partial^{2}_{r}\alpha + \partial_{r}\alpha\partial\beta - (\partial\alpha)^{2}(\mathrm{d}r \wedge \mathrm{d}t). \tag{A.16}$$

The wedge products don't all give zero. For the rest of the curvature two-forms we have

$$
\begin{aligned}
\tilde{R}^{t}{}_{\theta} &= -\partial_{r}\alpha e^{\alpha-2\beta}\mathrm{d}t \wedge \mathrm{d}\theta && \text{(A.17)} \\
\tilde{R}^{t}{}_{\phi} &= -\partial_{r}\alpha e^{-2\beta+\alpha}\sin\theta\mathrm{d}t \wedge \mathrm{d}\phi && \text{(A.18)} \\
\tilde{R}^{r}{}_{\theta} &= e^{-2\beta} r\partial\beta\mathrm{d}r \wedge \mathrm{d}\theta && \text{(A.19)} \\
\tilde{R}^{r}{}_{\phi} &= r\partial_{r}\beta e^{-2\beta}\sin^{2}\theta\mathrm{d}r \wedge \mathrm{d}\phi && \text{(A.20)} \\
\tilde{R}^{\theta}{}_{\phi} &= \sin\theta\mathrm{d}\theta \wedge \mathrm{d}\phi && \text{(A.21)}
\end{aligned}
$$

Now the Riemmann tensor $R^{\sigma}_{\mu\nu\lambda}$ can be calculated using

$$\tilde{R}^{a}{}_{b} = R^{a}{}_{b\mu\nu}\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}. \tag{A.22}$$

Also, we can use the vielbein $e_{\mu}{}^{a}$ to transform back to coordinate bases,

$$R^{\mu}{}_{\nu} = e_{\nu}{}^{b} e^{\mu}{}_{a}\tilde{R}^{a}{}_{b}. \tag{A.23}$$

For example, $\tilde{R}^{t}{}_{r}$ can be transformed to a coordinate basis by

$$
\begin{aligned}
R^{t}{}_{r} &= e_{r}{}^{r} e^{t}{}_{t}\tilde{R}^{t}{}_{r} && \text{(A.24)} \\
&= \partial^{2}_{r}\alpha - \partial_{r}\alpha\partial_{r}\beta + (\partial_{r}\alpha)^{2}(\mathrm{d}r \wedge \mathrm{d}t), &&
\end{aligned}
$$

and so the curvature tensor component is

$$R^{t}{}_{rtr} = e_{r}{}^{r} e^{t}{}_{t}\tilde{R}^{t}{}_{r} = \partial^{2}_{r}\alpha - \partial_{r}\alpha\partial_{r}\beta + (\partial_{r}\alpha)^{2} \tag{A.25}$$

The rest of the non-vanishing components give

$$
\begin{aligned}
R^t{}_{\theta t\theta} &= -r\partial_r\alpha e^{-2\beta} && \text{(A.26)}\\
R^t{}_{\phi t\phi} &= -\partial_r\alpha e^{-2\beta}\sin^2\theta && \text{(A.27)}\\
R^r{}_{\theta r\theta} &= e^{-2\beta}r\partial_r\beta && \text{(A.28)}\\
R^r{}_{\phi r\phi} &= e^{-2\beta}r\partial_r\beta\sin^2\theta && \text{(A.29)}\\
R^\theta{}_{\phi\theta\phi} &= \sin^2\theta\left(1-e^{-2\beta}\right). && \text{(A.30)}
\end{aligned}
$$

The Ricci tensor $R_{\mu\nu}$ can now be calculated by making the contraction on the Riemmann Tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}. \tag{A.31}$$

For example, the R_{tt} component is just

$$R_{tt} = R^t{}_{trt} + R^\theta{}_{t\theta t} + R^\phi{}_{t\phi t}. \tag{A.32}$$

In terms of the curvature components we previously calculated, this becomes

$$
\begin{aligned}
R_{tt} &= g^{rr}g_{tt}R^t{}_{rtr} + g^{\theta\theta}g_{tt}R^t{}_{\theta t\theta t} + g^{\phi\phi}g_{tt}R^t_{\phi t\phi} && \text{(A.33)}\\
&= e^{2(\alpha-\beta)}\left(-\partial_r\alpha\partial\beta + \partial_r^2\alpha 9(\partial_r\alpha)^2 + \frac{2\partial_r\alpha}{r}\right).
\end{aligned}
$$

The rest of the components are calculated similarly and give

$$
\begin{aligned}
R_{rr} &= \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 + \frac{2\partial_r\beta}{r} && \text{(A.34)}\\
R_{\theta\theta} &= e^{-2\beta}(r(\partial_r\beta - \partial_r\alpha) - 1) + 1 && \text{(A.35)}\\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}. && \text{(A.36)}
\end{aligned}
$$

Now the Ricci scalar R can be calculated by contracting both indices in the Ricci tensor ($R = R^\mu{}_\mu$)

$$
\begin{aligned}
R &= R^t{}_t + R^r{}_r + R^\theta{}_\theta + R^\phi{}_\phi && \text{(A.37)}\\
&= g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}\\
&= \partial_r\alpha\partial_r\beta + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}\left(1 - e^{2\beta}\right)
\end{aligned}
$$

A.2.3 The Einstein tensor

Now using the field equation and the curvature results, we can write the Einstein tensor. For our present purposes we shall only need two components.

$$G_{tt} = \frac{1}{r^2}e^{2(\alpha-\beta)}(2r\partial_r\beta - 1 + e^{2\beta}) \tag{A.38}$$

$$G_{rr} = \frac{1}{r^2}(2r\partial_r\alpha + 1 - e^{2\beta}). \tag{A.39}$$

We can now couple this result to the Energy momentum tensor.

A.3 The Energy-Momentum Tensor

We can model the star as a perfect fluid, that has an energy momentum tensor of the form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{A.40}$$

were ρ is the energy density, 'p' is the pressure, and u_μ is the four-velocity. For an observer at rest, the spatial components are zero and the time-like component can be derived from the normalization condition

$$u^\mu u_\mu = g^{\mu\nu}u_\nu u_\mu = -1. \tag{A.41}$$

This implies that $u_0 = \sqrt{-1/g_{00}}$, and since our metric has become "Minkoskian" with our choice of basis, we have

$$u_\mu = (1, 0, 0, 0). \tag{A.42}$$

Now, the direct product $u_\mu u_\nu$ is straight-forward and has the only non-zero component $u_0 u_0 = 1$. The energy-momentum tensor now becomes

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{A.43}$$

We can now proceed to couple the previous result to the Einstein tensor.

A.4 Hydrostatic equilibrium

The tt component of the Einstein equation gives

$$\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\beta - 1 + e^{2\beta}\right) = 8\pi G\rho; \tag{A.44}$$

and the rr component gives

$$\frac{1}{r^2}e^{-2\beta}\left(2r\partial_r\alpha + 1 - e^{2\beta}\right) = 8\pi G p. \tag{A.45}$$

It is standard and also convenient to replace $e^{2\beta}$ by

$$e^{2\beta} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{A.46}$$

Notice the similarity with the Shwarzschild metric. Noting that

$$\frac{dm(r)}{dr} = \frac{1}{2G}e^{-2\beta}\left(2r\partial_r\beta - 1 + e^{2\beta}\right); \tag{A.47}$$

we can now rewrite the tt component of the Einstein equation (eq. A.44) as

$$\begin{aligned} \tfrac{2G}{r^2}\tfrac{dm}{dr} &= 8\pi G\rho \\ \Rightarrow \quad \tfrac{dm}{dr} &= 4\pi r^2\rho. \end{aligned} \tag{A.48}$$

The previous equation can be integrated[4] to obtain the mass of the star up to a radius r.

In terms if $m(r)$ the rr (eq. A.45) component can be written as

$$\frac{1}{r^2}\left(1 - \frac{2Gm(r)}{r}\right)\left(2r\frac{d\alpha}{dr} + 1 - \left(1 - \frac{2Gm(r)}{r}\right)^{-1}\right) = 8\pi G p \tag{A.49}$$

so that

$$\frac{d\alpha}{dr} = \frac{4\pi G p r^3 + Gm}{r(r - 2Gm)}. \tag{A.50}$$

We can now use the equation of energy-momentum conservation $\nabla_\mu T^{ab} = 0$ [5] which can be written as [15](using the spin connection found previously)

$$\nabla_\mu T^{ab} = \partial_\mu T^{ab} + \omega_{\mu\ c}^{\ a} + \omega_{\mu\ c}^{\ b}T^{ac}. \tag{A.51}$$

The $\omega_{\mu\ c}^{\ a}$s are related to the ω_c^as simply by

$$\omega_c^a = \omega_{\mu\ c}^{\ a}\mathrm{d}x^\mu. \tag{A.52}$$

Using the previous expression and eq. (A.12) we have that

$$\omega_{\mu\ t}^{\ r} = \partial_r\alpha e^{\alpha-\beta}. \tag{A.53}$$

Finally, the $\mu = 0$ and $\mu = 1$ components give

$$\frac{dp}{dr} = \frac{-(\rho + p)(Gm(r) + 4\pi G r^3 p)}{r(r - 2Gm(r))}. \tag{A.54}$$

---

[4] If one wants to be formal, the proper volume element should be $\sqrt{\gamma}\mathrm{d}^3x = e^\beta r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$

[5] ∇_μ is the Covariant derivative.

The previous equation and eq. (A.49) are known as the O.V. equations. These govern the hydrostatic equilibrium of a spherically symmetric star. It might be worth saying that these equations were deduced from a static, spherically- symmetric geometry. In the case of a N.S. the spin rates are extremely high (no spherical symmetry), and a better calculation could be made by taking this into account.

Appendix B

The Fermi Golden Rule

In this appendix we shall derive the Fermi Golden Rule, which is used as an important tool in chapter III. Most of this derivation can be found in J. Basdevant's book [17].

B.1 Time Dependent Perturbation Theory

The Golden Rule can be derived from time dependent perturbation theory in non-relativistic quantum mechanics. In time dependent perturbation theory one considers a Hamiltonian of the form.

$$\hat{H}_\lambda = \hat{H}_0 + \lambda \hat{H}_1. \tag{B.1}$$

Here we assume that the eigenvalues E_n and and eigenvectors $|n\rangle$ of $\hat{H}_0$ are known. Also, we assume that $\lambda \hat{H}_1$ is a small perturbation. In principle, what the perturbation will do, is that it will induce transitions between eigenstates of the non perturbed Hamiltonian.

At any time, the state of the system in consideration can be expanded in the basis of eigenstates $|n\rangle$ of the non perturbed Hamiltonian, so that one can write[1]

$$|\psi(t)\rangle = \sum_n \gamma_n(t) e^{-iE_n t/\hbar} |n\rangle \tag{B.2}$$

Our final objective is to calculate the transition probability

$$\mathcal{P}_{n\to m}(t) = |\langle m|\psi(t)\rangle|^2. \tag{B.3}$$

If we now plug eq. (B.2) in the Shrödinger equation, we get

$$i\hbar \textstyle\sum_n e^{-iE_n T/\hbar} \left(\dot{\gamma}(t) - \frac{iE_n}{\hbar}\gamma_n(t) \right) |n\rangle \tag{B.4}$$

[1]Recall that this follows from the fact that the Shrödinger equation can be written as $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$, and so one can obtain equation (B.2)

$$= \lambda \hat{H}_1 \sum_n \gamma_n(t) e^{-iE_n t/\hbar} |n\rangle + \sum_n \gamma_n(t) E_n e^{-iE_n T/\hbar}$$
$$\Rightarrow i\hbar \sum_n e^{-iE_n T/\hbar} = \lambda \hat{H}_1 \sum_n \gamma_n(t) e^{-iE_n T/\hbar} |n\rangle .$$

If now we multiply by the dual eigenstate $\langle k|$, we obtain by orthogonality ($\langle k|n\rangle = \delta_{k,n}$)

$$i\hbar \sum_n e^{-iE_k T/\hbar} \dot{\gamma}_k(t) = \sum_n \gamma_n(t) e^{-iE_n T/\hbar} \left\langle k|\lambda \hat{H}_1|n\right\rangle . \tag{B.5}$$

With an initial condition $|\psi(t_0)\rangle$, the problem is completely determined by this set of differential equations.

We can now assume that the $\gamma_k(t)$ are analytic functions of λ around the origin so that we can use a power series and write

$$\gamma_k(t) = \gamma_k^{(0)}(t) + \lambda \gamma_k^{(1)}(t) + ... + \lambda^p \gamma_k^{(p)}(t) + ... \tag{B.6}$$

If we now insert the previous expression in eq. (B.5) we get

$$i\hbar \sum_n e^{-iE_k T/\hbar} \left(\dot{\gamma}_k^{(0)}(t) + \lambda \dot{\gamma}_k^{(1)}(t) + ... \right) \tag{B.7}$$
$$= \sum_n e^{-iE_k T/\hbar} \left(\lambda \gamma_k^{(0)}(t) + \lambda^2 \gamma_k^{(1)}(t) + ... \right) \left\langle k|\hat{H}_1|n\right\rangle \quad .$$

Now comparing terms up to first order we can solve up to first order

$$\dot{\gamma}_k(t) = 0, \tag{B.8}$$
$$i\hbar \dot{\gamma}_k^{(1)}(t) = \sum_n \gamma_n(t) e^{-i(E_n - E_k)T/\hbar} \left\langle k|\hat{H}_1|n\right\rangle , \tag{B.9}$$

and so on. This system could ideally be solved up to any desired order by iteration, but for the present purposes if is enough to know the solution up to first order. If we choose the initial condition such that $|\psi(t_o)\rangle = |i\rangle$, we get

$$\gamma_k^{(0)}(t) = \delta_{k,i}. \tag{B.10}$$

Now, plugging the previous result in the first order solution (eq. (B.9))

$$i\hbar \dot{\gamma}_k^{(1)}(t) = \sum_n \delta_{k,i} e^{-i(E_n - E_k)T/\hbar} \left\langle k|\hat{H}_1|n\right\rangle \tag{B.11}$$
$$i\hbar \dot{\gamma}_f^{(1)}(t) e^{-i(E_f - E_i)T/\hbar} \left\langle i|\hat{H}_1|f\right\rangle$$
$$\Rightarrow \gamma_f^{(1)} = \frac{1}{i\hbar} \int_{t_0}^{t} e^{-i(E_f - E_i)T/\hbar} \left\langle i|\hat{H}_1|f\right\rangle dt.$$

Here we have assumed that $\gamma_f^{(1)}(t = 0) = 0$. The transition probability can now be written as

$$\mathcal{P}_{i\to f}(t) = \left|\gamma_f^{(1)}(t)\right|^2. \tag{B.12}$$

What we have just done, time dependent perturbation theory to first order, is known as **The Born approximation**.

B.2 Time independent perturbation

If the perturbation Hamiltonian does not depend on time we have

$$\gamma_f^{(1)}(t) = \frac{1}{i\hbar}\left\langle i|\hat{H}_1|f\right\rangle \frac{e^{i\omega_0 t}-1}{i\omega_0}, \tag{B.13}$$

were $\hbar\omega_0 = E_f - E_i$

$$\Rightarrow \mathcal{P}_{i\to f}(t) = \frac{1}{\hbar^2}\left|\left\langle i|\hat{H}_1|f\right\rangle\right|^2 \frac{\left|e^{i\omega_0 t}-1\right|^2}{\omega_0^2}. \tag{B.14}$$

Noting that

$$\begin{aligned} e^{i\omega_0 t}-1 &= \cos(\omega_0 t) - 1 + i\sin(\omega_0 t) && \text{(B.15)}\\ \Rightarrow \left|e^{i\omega_0 t}-1\right|^2 &= \cos^2(\omega_0 t) - 2\cos(\omega_0 t) + 1 + \sin^2(\omega_0 t) && \text{(B.16)}\\ &= 2(1-\cos(\omega_0 t))\\ &= 4\sin^2\left(\frac{\omega_0 t}{2}\right), \end{aligned}$$

so we can now write

$$\mathcal{P}_{i\to f}(t) = \frac{1}{\hbar^2}\left|\left\langle i|\hat{H}_1|f\right\rangle\right|^2 y(\omega_0, t), \tag{B.17}$$

were $y(\omega,t) \equiv \frac{\sin^2\left(\frac{\omega t}{2}\right)}{(\omega/2)^2}$.

B.3 Transition to a continuum of states

In some cases (as is the case in this document), we are interested in the transition probability to a domain of states D_f that are characterized by their direction within a solid angle $d\Omega$, in this case we can write

$$d^2\mathcal{P}_{i\to D_f}(t) = \frac{1}{\hbar^2}\sum_{D_f}\left|\left\langle i|\hat{H}_1|f\right\rangle\right|^2 y(\omega_{f,i}, t). \tag{B.18}$$

We can now use the density of states $\rho(E)$ to change the previous sum to an integral over the energy states, i.e.

$$\sum_{D_f} \rightarrow \int \rho(E_f) dE_f. \tag{B.19}$$

Now we have

$$d^2\mathcal{P}_{i\rightarrow D_f}(t) = \frac{1}{\hbar^2}\int_{D_f} \left|\left\langle i|\hat{H}_1|f\right\rangle\right|^2 y(\omega_{f,i},t)\rho(E_f)dE_f\frac{d\Omega}{4\pi}. \tag{B.20}$$

Now noting that

$$\begin{aligned} \lim_{t\rightarrow 0}\frac{1}{\pi t}\frac{\sin^2(\omega_{f,i}t/2)}{(\omega_{f,i}/2)^2} &= \delta(\omega_{f,i}/2) \\ &= 2\delta(\omega_{f,i}) \\ &= 2\hbar\delta(E_f - E_i). \end{aligned} \tag{B.21}$$

If we now plug the previous expression in the integral of eq. (B.20), we get

$$\frac{d^2\mathcal{P}_{i\rightarrow D_f}(t)}{t} = \frac{2\pi}{\hbar}\int \left|\left\langle f, E_f = E_i|\hat{H}_1|i\right\rangle\right|^2 \delta(E_f - E_i)\rho(E_i)dE_f d\Omega. \tag{B.22}$$

The delta function that appears in the integral can be interpreted as an energy conservation enforcer in the transition to the final state. We might as well enforce conservation of energy and momentum (four- momentum) and rewrite the previous expression like many authors do.

$$\Gamma = \frac{d^2\mathcal{P}_{i\rightarrow D_f}(t)}{t} = \frac{2\pi}{\hbar}\int \left|\left\langle f, E_f = E_i|\hat{H}_1|i\right\rangle\right|^2 \delta^4(\vec{P}_f - \vec{P}_i)\rho(E_i)d^3\vec{P} \tag{B.23}$$

Here Γ is just a transition rate. The previous expression is known as the **Fermi Golden Rule**, and can be interpreted as an integral over phase-space, i.e.

$$d\Gamma = \frac{2\pi}{\hbar}\left|\left\langle f, E_f = E_i|\hat{H}_1|i\right\rangle\right|^2 (Phase - Space). \tag{B.24}$$

Thus, the more phase-space is available for a reaction to take place, the higher the transition rate will be. In our problem of interest, phase-space is severely restricted because of the degenerate enviroment the fermions live in.

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