Introduction to the Black-Scholes Equations for Option Prices

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Abstract

This document offers an introduction for physicist to the financial world, particularly those aspects related to option pricing: First we introduce the first successful model in pricing an option by Fischer Black and Myron Scholes, they accomplished a reduction of the problem to a physical process, more specifically, a diffusion equation. Finally we discuss the main assumption made by Black and Scholes, i.e., price fluctuations follow a geometric Brownian motion, in order to describe the introduction of non-Gaussianities and their impact into the model.

Contents

Nomenclature

Roman Symbols

B	bond price
b	multiplicative factor for a bad event in the binomial model
C	option price
g	multiplicative factor for a good event in the binomial model
$\hat{r}$	$1 + r*$
K	strike price
$N(x)$	$\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-\frac{u^2}{2}} \, du$ standard Gaussian cumulative distribution function
$\bar{p}$	$\frac{p_* g}{\hat{r}}$
p_*	$\frac{\hat{r}-b}{g-b}$
r	risk free interest rate in continuous compounding
$r*$	constant rate of return over a period of length Δ_t
S	security price
N	number of intervals between 0 and maturity time T
T	maturity time

Greek Symbols

β in chapter 4: the number of bonds

ϕ wealth of a portfolio made to replicate an option price

Δ_C $\frac{\partial C}{\partial S}$, is one of the Greeks, referred as the Delta for the call option

α in chapter 4: number of the underlying security stocks

ξ event development, it takes values in the two set $\{g, b\}$

Acronyms

CRR Cox-Ross-Rubinstein model

IID independent, identically distributed

PDE partial differential equation

Chapter 1

Introduction

The field of finance, with increasing complexity, has seen a rapid development in recent years, mathematical sophistication is being implemented, given by the interest of mathematicians and physicist who find analogies to finance in nature. For instance, physics discover laws of nature and their ability to do so, acquired throughout centuries, may be applied to finance, considering that there may be laws in finance waiting to be discovered.

The first merge of the two fields, which really was the starting point for the introduction of advanced mathematics to finance, was the finding of the solution for the price of an option by Black and Scholes (Black & Scholes (1973)).

The determination of a fair option price depends on many attributes of the option, the underlying asset, its price history and some other external circumstances. Many of these factors are difficult, in some cases impossible, to determine, therefore stochastic calculus is introduced resembling random walks in physics.

The present work, intended for physicist readers in the first place, is a discussion of the calculation of specific formulas in finance. It contains a complete introduction to relevant concepts in finance and assumes some basic knowledge in statistical physics and mathematic methods for physicist. Throughout the course of this document the relation of finance and physics is alluded in order not to bored the reader and encourage the extrapolation of their knowledge.

Empirical data have successfully showed that the initial considered stochastic process for price fluctuations of assets may not be so accurate. Furthermore heavy tail process are needed to give a better representation. Despite this fact, Black-Scholes formula is still used, suggesting the urgently need of a better model. In this context, this project arose for the need of understanding how non-Gaussianities are introduced into finance as a first approach into the subject.

The long term goal of this project is to find further resemblance in physics that allow a reduction of the problem accounting for the inclusion non-Gaussianities, to a physic model.

This document is organised as follows: First a description of options and their role in finance is given alongside a description of the first model for the stochastic process behind the asset price fluctuations. This gives a glance of the roots that give rise to the relationship that finance has established with physics. Then a description of the simplest procedure to price an option leads us to a one-dimensional diffusion equation. Following this, we analyse the assumptions made in pricing the option, in particular we show that non-Gaussianities may exist. Finally, a more realistic stochastic processes is exposed, opening a discussion over their impact on option valuation.

Chapter 2

Financial Markets

This chapter is a review of all the basic concepts to understand what an option is in the context of financial markets. Following we introduce one key feature in option pricing, price time series. Their most basic types of modelling will give the first glace of why physics is important in finance.

2.1 Options

An option belongs to one family of financial products, the derivative contracts, also called contingent claims, for their value depend on the future dynamics of one or several underlying assets. Derivative types also include: forwards, futures, swaps, warrants and others more complex as barrier options or swaptions, etc.

An option is a written contract involving two parts, a holder and a writer. Basically it gives to the holder, in the one hand, the right, but not the obligation, to exercise a trade of an specific asset at a certain date and a fixed price. In the other hand, it gives to the writer, the obligation to exercise the trade. In this sense an option is a kind of insurance policy that protects the holder against unexpected potential price changes of the underlying asset.

The asymmetry between the holder who acquires a right and the writer who acquires an obligation is diminished by an intrinsic price of the option paid by the holder to the writer. The determination of a fair price depends on many attributes of the option, the underlying asset, its price history, and some other external circumstances, therefore its been topic of study.

There are three factors that gives an option its nature:

- the underlying asset can be of many types: stocks, bonds, commodities, currencies, futures, many indices measuring entire markets, etc.
- the type of trade distinguish an option for being a call option or a put option: in the case of call option the holder gets the right to buy, by the contrary in the case of a put option the holder gets the right to sell. Conversely the writer gets the obligation to sell in a call and to buy in a put.
- the date when trade can take place classifies the option into European option (also called plain vanilla), being the most commonly used, it gives a well defined maturity or expiry date as the only date for the option to be exercised and American option, whose maturity date states a limit, letting the trade to take place at any preceding time.

This work is centered on call options of the European type, therefore this is the type of option we will refer hereafter unless it is differently specified.

2.2 Price Dynamics

Price fluctuation of an underlying asset is certainly one of the most important features to look at when pricing an option, since price changes are the type of events an option insures against.

Unfortunately, a deterministic prediction of the price evolution of an underlying security is impossible to calculate. Correlations in price histories may exist in particular, correlations of the asset volatility are known to have the effect of oscillations in the volatility, however they do not lead to any profit opportunity. If there were the case a complex correlation leading to a profit opportunity existed, it must be very hard to find and once someone finds one, it is destined to disappear once it becomes known.

The tool used for modelling price time series is an stochastic process, which is a dynamic variable with unpredictable time evolution, its changes are drawn from probabilities distributions. The dynamics of an stochastic variable is usually given by a difference equation, in the case of discrete time, and a differential equation, in continuous time, such as:

$$\dot{x}(t) = ax(t) + b\epsilon(t) \tag{2.1}$$

$$\dot{x}(t) = ax(t) + bx(t)\epsilon(t) \tag{2.2}$$

where ϵ is a random variable drawn from a probability function such as gaussian distribution:

$$p(\epsilon) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\epsilon^2}{2\sigma^2}} \tag{2.3}$$

The second term in the right hand side of equation 2.1 represents an addition of noise whilst in equation 2.2 the second term in the right hand side adds a multiplicative noise. They both got the property of being independent, identically distributed (IID). This is because its autocorrelation vanishes for non-zero time windows, i.e.:

$$\langle \epsilon(t)\epsilon(t') \rangle = \sigma^2 \delta(t - t') \tag{2.4}$$

If we now define an stochastic process $z(t)$ whose changes are given by a gaussian distribution:

$$\frac{dz(t)}{dt} = \epsilon(t) \tag{2.5}$$

then the change of in an infinitesimal interval is given by:

$$dz(t) = \int_t^{t+dt} dt' \epsilon(t') \tag{2.6}$$

whose solution may seems trivial as $\epsilon(t)dt$, but actually it is not so for stochastic variables, instead expectations values can easily be calculated as:

$$\langle dz(t)\rangle = \int_t^{t+dt} dt' \langle \epsilon(t')\rangle = 0 \tag{2.7}$$

$$\langle dz(t)dz(t)\rangle = \int_t^{t+dt} dt_1 dt_2 \langle \epsilon(t_1)\epsilon(t_2)\rangle = \sigma^2 \int_t^{t+dt} dt_1 = \sigma^2 dt \tag{2.8}$$

In the latter equation we have used equation 2.4. These expectations values describes one of the main features for Brownian motion (also called Einstein-Wiener process). The second characteristic is that consecutive dz are statistically independent.

2.2.1 Bachelier's Model of Stock Prices

Bachelier's model was the first and the most basic model introduced to characterise price time series consisting as a random walk superimposed on a constant drift, i.e. an Einstein-Wiener process with zero mean value and a variance increasing linearly in time, if S stand for the price of an asset and $dz = \epsilon\sqrt{dt}$):

$$dS = \mu^* dt + \sigma^* dz \tag{2.9}$$

2.2.1.1 Problems in Bachelier's Model

Bachelier model had the following two problems:

1. It may happen for stock prices to become negative, which is not allowed in real world. This is because this model allows that accumulated price changes over a period of time T take a value such as $S(T) - S(0) < -S(0)$.

2. The profit of stock investments over a time interval T in Bachelier model is:

$$\langle S(T) - S(0)\rangle = \frac{dS}{dt} T = \mu^* T \tag{2.10}$$

which is independent of S. Real world requires the profit to depend on stock price.

2.2.2 Geometric Brownian Motion

The stochastic process that solve the troubles in Bachelier's model by replacing the drift μ^* by μS and the variance σ^{*2} by $\sigma^2 S^2$ is the so called geometric Brownian motion, which is a a random walk with multiplicative noise:

$$dS = \mu S dt + \sigma S dz \tag{2.11}$$

$$dS = \mu S dt + \sigma S \epsilon \sqrt{dt} \tag{2.12}$$

Geometric Brownian motion belong to a family of process called It$\hat{o}$ processes to be discussed in section 3.2.

Chapter 3

Black-Scholes Theory on Option Pricing

This chapter explains the first, most simple and still in use Back-Scholes model of option prices. This model reduce the option valuation problem to a heat diffusion equation. The solution is found by changing the border conditions and diffusion coefficient, compute the answer and then transform it back to the finance world.

3.1 The simplicity of the method

Here we make a list of necessary assumptions under which Black-Scholes model is based:

- efficient and complete market,
- no transaction costs,
- same taxation for all profits, making them irrelevant,
- same risk free interest rate r for every participant, which in addition compounds continuously,
- arbitrage is present,
- no payoffs such as dividends from the underlying security,

- the underlying security follows a geometric Brownian motion.

The most critic assumption is obviously the last one, which describes the price fluctuations as an stochastic process drawn from a Gaussian probability function. We will explore this assumption in more detail in chapter 5.

3.2 The Itô Lemma

Consider an stochastic variable to follow an Itô process:

$$dx = a(x,t)dt + b(x,t)dz \tag{3.1}$$

with again

$$dz = \epsilon\sqrt{dt} \tag{3.2}$$

A function $G(x,t)$ defines an also Itô process. If we Taylor expand the function $G(x+dx, t+dt)$ around (x,t), to second order, with the intention for the terms of dz (which is proportional to $\sqrt{dt}$) to be consistent with the terms of dt.

$$dG = \left(\frac{\partial G}{\partial t} + a\frac{\partial G}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial x^2}\right)dt + b\frac{\partial G}{\partial x}dz \tag{3.3}$$

3.3 The Risk Free Portfolio Method

The main idea behind Black-Scholes model is that, given the dependance of an option to the underlying security, it is possible to build a riskless portfolio, composed of a short position (to sell) of the option to be priced and the a long position (to buy) of the underlying security. This portfolio for being riskless must earn the risk free interest rate r. Consider the wealth of the portfolio ϕ:

$$\phi = -C + \frac{\partial C}{\partial S}S \tag{3.4}$$

where:

$$\frac{\partial C}{\partial S} \equiv \Delta_C \tag{3.5}$$

is the so called Delta for the call option and is one of the set of variables commonly referred in finance as the Greeks, for evident reasons. The Greeks stand for the derivatives of option prices with respect to the variables and parameters upon which the option price depends. Notice that Δ_C fluctuates with the underlying security price and therefore a continuous adjustment of this long position is required.

Now we will determine the differential for the wealth of the portfolio, but be careful because what we are about to do is not completely correct, mathematically speaking[1]:

$$d\phi = -dC + \frac{\partial C}{\partial S} dS \tag{3.6}$$

where C is a function of S, an stochastic variable following geometric Brownian motion,consequently they are both Itô processes, using $a(S,t) = \mu S$, and $b(S,t) = \sigma S$ in equation 3.1, we have:

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial G}{\partial x} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial x^2} \right) dt + b \frac{\partial G}{\partial x} dz \tag{3.7}$$

$$dS = \mu S dt + \sigma S dz \tag{3.8}$$

then, equation 3.12 becomes:

$$d\phi = \left(-\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial x^2} \right) dt \tag{3.9}$$

Notice that the terms involving the stochastic part dz, have disappear, leaving us with deterministic terms only. On the other hand $d\phi$ being riskless, must earn the risk free interest rate:

[1] An explanation of this is going to be given in section 3.3.1

$$d\phi = r\phi dt = r(-dC + \frac{\partial C}{\partial S} dS)dt \tag{3.10}$$

Computing equations and we get the acclaimed Black-Scholes differential equation:

$$-rC + \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0 \tag{3.11}$$

3.3.1 Accuracy of the Method

Although the risk free portfolio method is the most usual strategy to derive the Black-Scholes model, particularly for pedagogic purposes. This method is not correct. This is because in equation 3.12 there is a missing term, the correct equation would be:

$$d\phi = -dC + \frac{\partial C}{\partial S} dS + Sd(\frac{\partial C}{\partial S}) \tag{3.12}$$

The last term of above equation produces the effect of third orders derivatives into Black-Scholes PDE (equation 3.11), and even worse, it causes noise terms not to disappear completely. A correct procedure that leads us to Black-Scholes equation is described in chapter 4

The real equation we get to, by the risk free portfolio method is:

$$\left(-rC + \frac{\partial C}{\partial t} - S\frac{\partial^2 C}{\partial t \partial s} + rS\frac{\partial C}{\partial S} + \left(\frac{\sigma^2}{2} - \mu\right) S^2 \frac{\partial^2 C}{\partial S^2} - \frac{1}{2}\sigma^2 S^3 \frac{\partial^3 C}{\partial S^3}\right) dt - \sigma S^2 \frac{\partial^2 C}{\partial S^2} dz = 0 \tag{3.13}$$

where we have used again the Itô Lemma:

$$d(\frac{\partial C}{\partial S}) = \left(\frac{\partial C}{\partial t \partial S} + \mu S \frac{\partial^2 C}{\partial S^2} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^3 C}{\partial S^3}\right) dt + \sigma S \frac{\partial^2 C}{\partial S^2} dz \tag{3.14}$$

Very few reference allude this fact, for instance in Musiela & Rutkowski (1997) claim to show by using an again not completely correct procedure, that the additional terms in the PDE vanishes. They attempt this by differentiating Black-Scholes equation, then introducing it into equation 3.14 and finally showing that the remanning terms vanishes. The trouble in doing so is that by assuming the solution for Black Scholes to be also solution for equation 3.13, does not guarantee the other way round.

3.4 Diffusion Equation

We now want to solve the Black-Scholes PDE (equation 3.11), to do that, let us first rewrite it as:

$$-rC + \frac{\partial C}{\partial t} + \left(r - \frac{\sigma^2}{2}\right) S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 \left(S\frac{\partial}{\partial S}\right)^2 C = 0 \tag{3.15}$$

applying the change of coordinates to logarithmic: $S = e^Z$ and $\tau = T - t$, i.e. put C in terms of the remaining time to maturity instead of present time. This make the final condition of the option price becomes an initial condition for the diffusion equation we are about to find. The above equation becomes:

$$rC + \frac{\partial C}{\partial \tau} - \left(r - \frac{\sigma^2}{2}\right) \frac{\partial C}{\partial Z} - \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial Z^2} = 0 \tag{3.16}$$

to eliminate the first term of above equation, we use the change of variable: $C = De^{-r\tau}$, alluding the discount of a possible trade in the future. We get:

$$\frac{\partial D}{\partial \tau} - \left(r - \frac{\sigma^2}{2}\right) \frac{\partial D}{\partial Z} - \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial Z^2} = 0 \tag{3.17}$$

the final step is aimed to eliminate the first-order term. This is done through a shift of coordinates by:

$$y = Z + \left(r - \frac{\sigma^2}{2}\right) \tau \tag{3.18}$$

Leading us to put the Black-Scholes PDE in the form of a one-dimension diffusion equation:

$$\frac{\partial D}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 D}{\partial y^2} \tag{3.19}$$

Thus, the solution of Black-Scholes PDE is done by transforming the final condition of the option to an initial condition. Then solving the diffusion equation and transforming back we get the Black-Scholes formula for option valuation:

$$C = SN(d_1) - Ke^{r(T-t)}N(d_2) \tag{3.20}$$

with

$$N(d) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{u^2}{2}}\, du \tag{3.21}$$

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{3.22}$$

$$d_2 = \frac{ln(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{3.23}$$

Chapter 4

Binomial Model

In this chapter we show a correct derivation of the Black-Scholes model assuming in the first place a discrete binomial economy and taking it to the continuous limit. The assumptions of the market made here are the same used in the latter chapter, except for the stochastic process followed by the stock price, that, due to simplicity of the model, its Gaussianity is implied when taken to the continuous limit.

4.1 The Cox-Ross-Rubinstein Economy

Consider a discrete time model by an N-period economy over the time interval $[0, T]$ with N of the form $\frac{1}{2^k}$, $k \in \mathbb{N}$ and period length[1]:

$$\Delta_t = \frac{T}{N} \tag{4.1}$$

Parameterized as $\{0, \Delta_t, 2\Delta_t, ..., (N-1)\Delta_t, T\}$. During each of these periods the economy will develop independent of past events, i.e., statistically independent of subsequent price fluctuations, in such a way that every interval evolve in only one of two kinds: "good" or "bad". Additionally there is a defined probability p of the development being good, and consequently a probability $1 - p$ of

[1]Interested readers can find a more generalised look of multi-period market models in chapter 5 of Föllmer & Schied (2004)

the development being bad. In this context the relation between of The price for a security stock S_i in the ith interval and its price in the former interval is given by:

$$S_i = S_{i-1}\xi_i \tag{4.2}$$

where ξ can take one of the two values: u, when the period has had a good development; or d, when the contrary has happened.

If B_i represent an investment in a saving account (or a bond) in the ith period, there is a riskless constant rate of return $r*$, that allow us to relate write:

$$B_i = B_{i-1}(1 + r*) = B_0(1 + r*)^t \tag{4.3}$$

The numbers $r*$, g and b satisfy:

$$r* > 0 \tag{4.4}$$

$$g > 1 + r* > b > 0 \tag{4.5}$$

4.2 The CRR Pricing Formula

The intention of this section is to mimic the arbitrage price of an European call option with an appropriate mix of the underlying security stocks and bonds (or obligations) in a discrete economy. The idea behind this connection lie in the final condition (at maturity) of the option price:

$$C_N = max(0, S_N - K) \equiv (S_N - K)^+, \tag{4.6}$$

with K the strike (or delivery) price of the underlying. This final condition assumes for the buyer the wise decision to execute the option only when the underlying security price is higher than the strike price. This is because buying a stock at a price K and immediately sell it at its current value S_N when $S_N > K$ would leave a profit of $S_N - K$. In the other hand, when $K > S_N$ there would be no point in exercising the option because you can buy the stock cheaper and consequently the option becomes worthless.

Consider a portfolio whose wealth $\phi_i(\alpha_i, \beta_i)$ is given by:

$$\phi_i = \alpha_i S_i + \beta_i B_i \tag{4.7}$$

Where α and β stand for the number of the underlying security stocks and the number of obligations respectively. The Condition $\phi_N = C_N$ starts a backward strategy that proposes an adjustment of the parameters α and β in each period in such a way that no inputs or withdrawals of founds take place, thus a unique, dynamic, replicating, self-financing strategy is achieved guaranteeing that the wealth of the portfolio and the price of the option will equal at all times[1].

The evolution of ϕ considers two important events in each period: First we let the value of the underlying security and bonds evolve and second, once the new prices are known, we adjust the parameters α and β to keep the final condition valid.

The first iteration in the backward strategy is done just before the end of the last period, $(T - \Delta_t, T]$ when the value of the underlying security and bonds evolve from S_{N-1} and B_{N-1} to S_N and B_N, but the composition of the portfolio is still ruled by α_{N-1} and β_{N-1}. This is:

$$\alpha_{N-1} S_N + \beta_{N-1} B_N = (S_N - K)^+ \tag{4.8}$$

[1]In section 3.4 we saw that the equation we will derive by this method can be reduced to a diffusion equation with a negative diffusion coefficient, which explains the backward induction strategy; and a border condition in future, which explains that the condition of the option price is a final condition. Think of it as a group of particles that instead of diffuse, regroup to get to a final known point.

In either the good or bad scenarios for S_{N-1}, the condition in equation 4.6 must be satisfied, therefore the above equation becomes the set equations:

$$\alpha_{N-1} g S_{N-1} + \beta_{N-1} \hat{r} B_{N-1} = (g S_{N-1} - K)^+ \equiv C_N^g \tag{4.9}$$
$$\alpha_{N-1} b S_{N-1} + \beta_{N-1} \hat{r} B_{N-1} = (b S_{N-1} - K)^+ \equiv C_N^b \tag{4.10}$$

where $\hat{r} \equiv 1 + r^*$. The solution of above equations for α_{N-1} and β_{N-1} is:

$$\alpha_{N-1} = \frac{(g S_{N-1} - K)^+ - (b S_{N-1} - K)^+}{(g-b) S_{N-1}} \tag{4.11}$$
$$\beta_{N-1} = \frac{g (b S_{N-1} - K)^+ - b (g S_{N-1} - K)^+}{(g-b) \hat{r} B_{N-1}} \tag{4.12}$$

Here, it is convenient to define a new variable p_* as:

$$p_* \equiv \frac{\hat{r} - b}{g - b} \tag{4.13}$$

Notice that by conditions imposed in equations 4.4 and 4.5, p_* and consequently $(1 - p_*) \in [0, 1]$,i.e., they meet the requirements of probability number. The wealth of the portfolio during the period $(T - \Delta_t, T)$ takes the form:

$$\phi_{N-1} = C_{N-1} = \alpha_{N-1} S_{N-1} + \beta_{N-1} B_{N-1} \tag{4.14}$$
$$= \hat{r}^{-1} \left(p_* C_N^g + (1 - p_*) C_N^b \right) \tag{4.15}$$
$$= \hat{r}^{-1} \left(p_* \left(g S_{N-1} - K \right)^+ + (1 - p_*) \left(b S_{N-1} - K \right)^+ \right) \tag{4.16}$$

The right hand side of equation 4.15 can be seen as an average-like expression of the good and bad scenarios associated to an artificial probability p_*, (be careful not to think of it as the probability of the two happenings p), all of this discounted by the constant rate return $r*$.

The second iteration is obviously aimed to find α_{N-2} and β_{N-2}. Now with the new final condition given by C_{N-1}, we get again a set of two equations, depending on the development in the $(N-2)$th period:

$$\alpha_{N-2} g S_{N-2} + \beta_{N-2} \hat{r} B_{N-2} = C^g_{N-1} \tag{4.17}$$
$$\alpha_{N-2} b S_{N-2} + \beta_{N-2} \hat{r} B_{N-2} = C^b_{N-1}, \tag{4.18}$$

with

$$C^g_{N-1} = \hat{r}^{-1}\left(p_*\left(g^2 S_{N-2} - K\right)^+ + (1-p_*)\left(bgS_{N-2} - K\right)^+\right) \tag{4.19}$$
$$C^b_{N-1} = \hat{r}^{-1}\left(p_*\left(gbS_{N-2} - K\right)^+ + (1-p_*)\left(b^2 S_{N-2} - K\right)^+\right) \tag{4.20}$$

whose solution is given by:

$$\alpha_{N-1} = \frac{C^g_{N-1} - C^b_{N-1}}{(g-b)S_{N-2}} \tag{4.21}$$
$$\beta_{N-1} = \frac{gC^b_{N-1} - bC^g_{N-1}}{(g-b)\hat{r}B_{N-2}} \tag{4.22}$$

Consequently, the wealth of ϕ_{N-2} is:

$$\begin{aligned}
\phi_{N-2} = C_{N-2} &= \alpha_{N-2} S_{N-2} + \beta_{N-2} B_{N-2} && (4.23)\\
&= \hat{r}^{-1}\left(p_* C^g_{N-1} + (1-p_*) C^b_{N-1}\right) && (4.24)\\
&= \hat{r}^{-2}(p_*^2\left(g^2 S_{N-2} - K\right)^+ + 2p_*(1-p_*)\left(gbS_{N-2} - K\right)^+ \\
&\quad +(1-p_*)^2\left(b^2 S_{N-2} - K\right)^+) && (4.25)
\end{aligned}$$

Continuing this method will evidently find the option price C_{N-m} at any m periods before the maturity time T. Equations 4.16 and 4.25 let us see already the binomial behaviour of C_{T-m}, which can be written as:

$$C_{T-m} = \hat{r}^{-m} \sum_{j=0}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(g^j b^{m-j} S_{N-m} - K\right)^+ \tag{4.26}$$

If we let the function $a_m : \mathbb{R} \to \mathbb{N}$ be given by:

$$a_m = inf\{j \in \mathbb{N} | x g^j b^{m-j} > K\} \tag{4.27}$$

Equation 4.26 can be rewritten as:

$$C_{N-m} = \hat{r}^{-m} \sum_{j=a_m}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \left(g^j b^{m-j} S_{N-m} - K\right) \tag{4.28}$$

Defining $\bar{p} \equiv \frac{pg}{\hat{r}}$ we finally get to the standard form of the *Cox-Ross-Rubinstein valuation formula* for the option price at the time $(N-m)\Delta_t$:

$$C_{N-m} = S_{N-m} \sum_{j=a_m}^{m} \binom{m}{j} \bar{p}^j (1-\bar{p})^{m-j} - \hat{r}^{-m} K \sum_{j=a_m}^{m} \binom{m}{j} p_*^j (1-p_*)^{m-j} \tag{4.29}$$

Additionally from equations 4.15 and 4.24 we can also already be a little intuitive to see the recurrence equation, which is useful for section 4.3.1:

$$C_i = \hat{r}^{-1} \left(p_* C_{i+1}^g + (1-p_*) C_{i+1}^b\right) \tag{4.30}$$

4.3 The CRR Continuous Limit

The reminder of this chapter is dedicated to assess the asymptotic behaviour of CRR by studying the limit $N \to \infty$ ($\Delta_t \to 0$) in order to obtain Black-Scholes PDE and formula. First we must impose specific restrictions on g, b and $\hat{r}$:

$$g = e^{\sigma\sqrt{\Delta_t} + \mu\Delta_t}, \quad b = e^{-\sigma\sqrt{\Delta_t} + \mu\Delta_t}, \quad \hat{r} = e^{r\Delta_t} \tag{4.31}$$

4.3.1 From CRR to Black-Scholes PDE

The procedure to get the Black-Scholes PDE from CRR is basically to evaluate the limit $N \to \infty$ in the recurrence equation 4.30:

$$C(S,t) = \lim_{N\to+\infty} \hat{r}^{-1} \left(p_* C(gS, t+\Delta_t) + (1-p_*)C(bS, t+\Delta_t)\right) \tag{4.32}$$

To evaluate this limit we expand in Taylor series the following functions: g, b and r around $\Delta_t^* = 0$; $C(Sg, t+\Delta_t)$ and $C(Sb, t+\Delta_t)$ around (S,t):

$$\begin{aligned}
g &= e^{\sigma\sqrt{\Delta_t}+\mu\Delta_t} \cong 1 + \sigma\sqrt{\Delta_t} + (\frac{\sigma^2}{2} + \mu)\Delta_t & (4.33)\\
b &= e^{-\sigma\sqrt{\Delta_t}+\mu\Delta_t} \cong 1 - \sigma\sqrt{\Delta_t} + (\frac{\sigma^2}{2} + \mu)\Delta_t & (4.34)\\
\hat{r} &= e^{r\Delta_t} \cong 1 + r\Delta_t & (4.35)\\
C(gS, t+\Delta_t) &\cong C(S,t) + \frac{\partial C(S,t)}{\partial t}\Delta_t + \frac{\partial C(S,t)}{\partial S}(gS - S) \\
&+ \frac{1}{2}\frac{\partial^2 C(S,t)}{\partial S^2}(gS-S)^2 & (4.36)\\
C(bS, t+\Delta_t) &\cong C(S,t) + \frac{\partial C(S,t)}{\partial t}\Delta_t + \frac{\partial C(S,t)}{\partial S}(bS - S) \\
&+ \frac{1}{2}\frac{\partial^2 C(S,t)}{\partial S^2}(bS-S)^2 & (4.37)
\end{aligned}$$

Applying the expansions of equations 4.33-4.35 to p_* (equation 4.13), $C(gS, t+\Delta_t)$ (equation 4.36) and $C(bS, t+\Delta_t)$ (equation 4.37) gives:

$$p_* = \frac{\hat{r} - b}{g - b} \cong \frac{1}{2} + \sqrt{\Delta_t}\left(\frac{r-\mu}{2\sigma} - \frac{\sigma}{4}\right) \tag{4.38}$$

and

$$\begin{aligned}
C(gS, t+\Delta_t) &\cong C(S,t) + \frac{\partial C(S,t)}{\partial t}\Delta_t + \frac{\partial C(S,t)}{\partial S}(\sigma\sqrt{\Delta_t} + (\frac{\sigma^2}{2}+\mu)\Delta_t)S + \\
&\frac{1}{2}\frac{\partial^2 C(S,t)}{\partial S^2}(\sigma\sqrt{\Delta_t} + (\frac{\sigma^2}{2}+\mu)\Delta_t)^2 S^2 & (4.39)\\
C(bS, t+\Delta_t) &\cong C(S,t) + \frac{\partial C(S,t)}{\partial t}\Delta_t + \frac{\partial C(S,t)}{\partial S}(-\sigma\sqrt{\Delta_t} + (\frac{\sigma^2}{2}+\mu)\Delta_t)S + \\
&\frac{1}{2}\frac{\partial^2 C(S,t)}{\partial S^2}(-\sigma\sqrt{\Delta_t} + (\frac{\sigma^2}{2}+\mu)\Delta_t)^2 S^2 & (4.40)
\end{aligned}$$

Finally, by inserting equations 4.35 and 4.38-4.40, now in the desired form, into the recurrence equation (4.32) all drift (μ) terms disappear and survive only terms involving volatility (σ) and the riskless rate of return (r), then dividing by Δ_t gives the acclaimed Black-Scholes PDE using a correct procedure:

$$-rC + \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0 \tag{4.41}$$

4.3.2 From CRR to Black-Scholes Equation

We show in this section that it is also possible to go from the *Cox-Ross-Rubinstein valuation formula* (equation 4.29) straight to the Black-Scholes equation without going through the PDE. Let's bring the mentioned equation to this section:

$$C_{N-m} = S_{N-m}\sum_{j=a_m}^{m}\binom{m}{j}\bar{p}^j(1-\bar{p})^{m-j} - \hat{r}^{-m}K\sum_{j=a_m}^{m}\binom{m}{j}p_*^j(1-p_*)^{m-j}) \tag{4.42}$$

To solve the above equation in the limit $N \to +\infty$, let's start by applying Stirling approximation to the binomial term that altogether with the $\bar{p}$ terms take the form:

$$\begin{aligned}\binom{m}{j}\bar{p}^j(1-\bar{p})^{m-j} &= \frac{m!\bar{p}^j(1-\bar{p})^{m-j}}{j!(m-j)!} \cong \sqrt{\frac{m}{2\pi j(m-j)}}\frac{m^m\bar{p}^j(1-\bar{p})^{m-j}}{j^j(m-j)^{m-j}} \\ &= \sqrt{\frac{m}{2\pi}}exp\left[-m\left(\frac{j}{m}ln\left(\frac{j}{m}\frac{1}{\bar{p}}\right) + \left(1-\frac{j}{m}\right)ln\left(\frac{1-\frac{j}{m}}{1-\bar{p}}\right)\right)\right]\end{aligned} \tag{4.43}$$

If we now Taylor expand the argument of the exponential in the above equation around $\bar{p}$ where it has got its maximum, we get:

$$\binom{m}{j}\bar{p}^j(1-\bar{p})^{m-j} \cong \sqrt{\frac{m}{2\pi}}exp\left[-m\left(\frac{\left(\frac{j}{m}-\bar{p}\right)^2}{2\bar{p}(1-\bar{p})}\right)\right] \tag{4.44}$$

The above equation has been on purpose put in terms of $\frac{j}{m}$ so that a Riemann sum is constructed out of its sum. The limit $N \to +\infty$ will eventually form an integral, that, with the appropriate change of variables $x^2 = m\left(\frac{\left(\frac{j}{m}-\bar{p}\right)^2}{\bar{p}(1-\bar{p})}\right)$ gets the following look:

$$\lim_{N\to+\infty}\sum_{j=a_m}^{m}\binom{m}{j}\bar{p}^j(1-\bar{p})^{m-j}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1}e^{-\frac{x^2}{2}}\,dx \tag{4.45}$$

In the case we use p_* in the place of $\bar{p}$, the effect will be reflected in the limit by:

$$\lim_{N\to+\infty}\sum_{j=a_m}^{m}\binom{m}{j}p_*^j(1-p_*)^{m-j}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2}e^{-\frac{x^2}{2}}\,dx \tag{4.46}$$

with

$$d_1=\frac{ln(\frac{S}{K})+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{4.47}$$

$$d_2=\frac{ln(\frac{S}{K})+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{4.48}$$

The integral in equations 4.45 and 4.46, is well known as the standard Gaussian cumulative distribution function:

$$N(d)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}e^{-\frac{u^2}{2}}\,du \tag{4.49}$$

Finally, it is easy to see that equation 4.42 in the limit $N\to+\infty$ becomes the famous Black-Scholes formula:

$$C=SN(d_1)-Ke^{r(T-t)}N(d_2) \tag{4.50}$$

Chapter 5

Non-Gaussianities in Financial Markets

In this Chapter we examine further the stochastic processes followed by financial markets and realise they are not quite as assumed previously. Then at the light of a better representation we comment its consequences on option pricing.

5.1 Critical Examination of Geometric Brownian Motion

The stochastic process assumed in the Black-Scholes model is geometric Brownian motion. This process make two important hypothesis for price fluctuations:

1. Successive realizations of the stochastic variable are IID.
2. Relative changes of the stochastic variable are drawn form a Gaussian probability function.

In sections 5.1.1 and 5.1.2 we examine the fulfilment of these properties by analysing real data of price time series.

5.1.1 Statistical Independence of Price Fluctuations

To asses the IID property of stock prices, we must analyse correlations on single stock indices. A composed index such as the colombian IGBC may hide

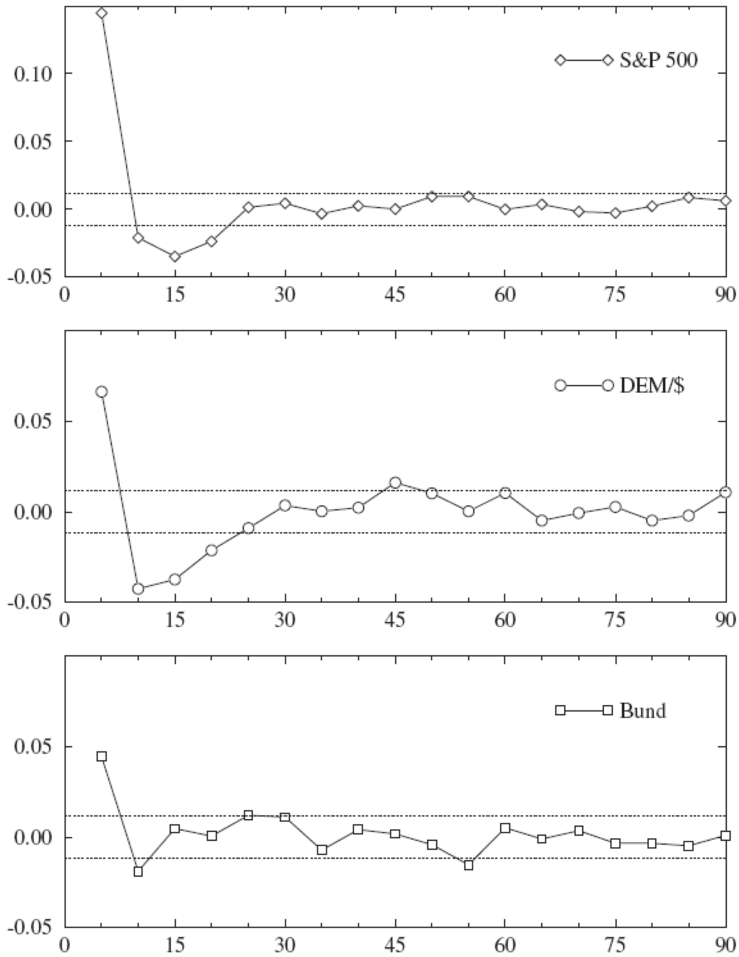

Figure 5.1: Autocorrelation functions of the $S\&P500$ index (top), the $DEM/US\$$ exchange rate (middle), and the BUND future (bottom), over a time scale of $\tau = 5$ minutes. The horizontal scale is the time window $t - t'$ in minutes. The horizontal dotted lines are the confidence levels of 3σ . From Bouchaud (2003)

correlations when superposing different processes. At a time window of τ the autocorrelation function is given by:

$$Q_\tau(t-t') = \frac{1}{var\left[\delta S_\tau(t)\right]} \left\langle \left[\delta S_\tau(t) - \langle \delta S_\tau(t) \rangle\right] \left[\delta S_\tau(t') - \langle \delta S_\tau(t') \rangle\right] \right\rangle \tag{5.1}$$

For IID data the autocorrelations should be $Q_\tau(t-t') = 0$ for $t \neq t'$ at least as allowed by the limit of large samples. Figure 5.1 shows the autocorrelation functions of the three assets with price changes evaluated on a time scale of $\tau = 5min.$. These correlations shows that for time lags below 30 minutes, there are weak correlations above the 3σ level. Above 30 minutes time lags, correlations are not significant. This result let us conclude that the hypothesis of IID price changes is correct according to empirical data.

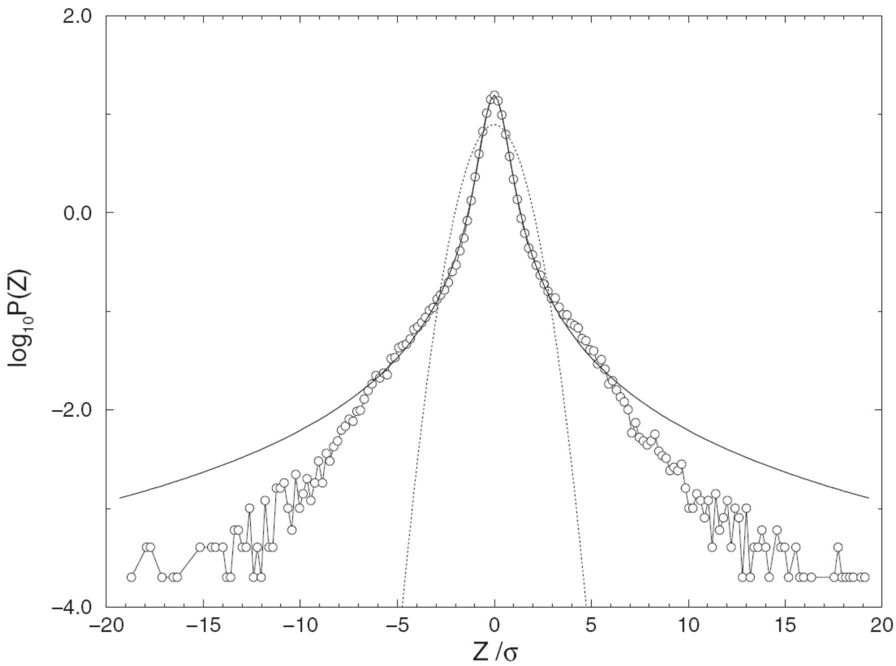

Figure 5.2: Probability distribution of changes of the $S\&P500$ index. Comparison of the $\tau = 1min.$ data (circles) with Gaussian (doted line) and stable Lévy distributions (continuous line). From Voit (2005)

5.1.2 Non-Gaussian Behaviour

Now we turn our attention to the second hypothesis in geometric Brownian motion. The deviation of price changes from Gaussian distribution was first found by Mandelbrot in 1963. He examined price changes for the cotton commodity and showed there is a disagreement between the collected data and predictions from geometric Brownian motion, furthermore he suggested a consistent behaviour to stable Lévy distribution. Figure 5.2 shows how Gaussian distribution (that in the log scale the the look of an inverted parabola) is a bad fit of real data, Lévy distribution is a better match but still overestimating the weight of the tails at long horizons.

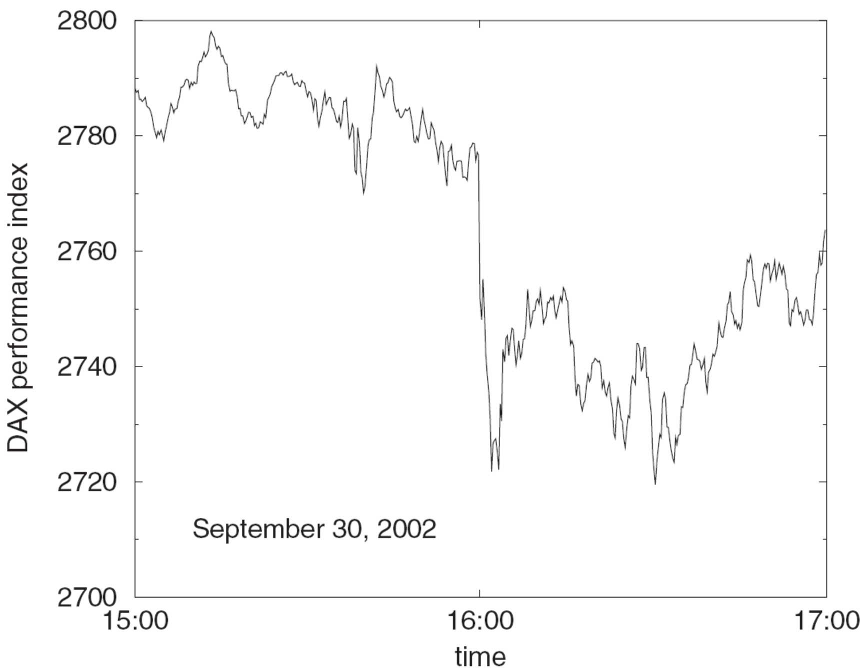

Figure 5.3: Variation of the DAX German blue chip index during two hours of September 30, 2002. No particular events were reported. Still, a 3% loss over a time scale of about one minute is reported around 16:00 h. From Voit (2005)

5.2 Heavy Tail Distributions

Figure 5.4 represents a computer simulation of data drawn from a heavy tail distribution. We can see that discontinuities in such time series are in better agreement to real data (figure 5.3). It is a didactic tool as well so see such process in a two-dimensions (figure 5.5).

5.2.1 Lévy Flights

An stable Lévy is defined by its characteristic function:

$$\hat{L}_{a,\beta,m,\mu}(z) = exp\left[-a|z|^{\mu}(1 + \imath\beta sign(t) tan(\frac{\pi\mu}{2})) + \imath mz\right] \tag{5.2}$$

Parameters β stand for the asymmetry characterization parameter (skewness), in such a way that $\beta = 0$ gives a symmetric distribution. μ is the index of the distribution which gives the exponent of the asymptotic power-law tail. a is a scale factor characterizing the width of the distribution, and m gives the peak position. For our purposes, we assume symmetric distributions ($\beta = 0$), a maximum at $x = 0$, leading to $m = 0$, and drop the scale factor a from the list of indices. The characteristic function then becomes:

$$\hat{L}_{\mu}(z) = e^{-a|z|^{\mu}} \tag{5.3}$$

In general, there is no analytic representation of the Lévy distributions. We have seen that $\mu = 2$ produces the Gaussian distribution and the special case $\mu = 1$ produces the Lorentz-Cauchy distribution:

$$L_{\mu=1}(x) = \frac{1}{\pi}\frac{a}{a^2 + x^2} \tag{5.4}$$

$$\tag{5.5}$$

for $\mu \neq 2$ behaves asymptotically as:

$$L_{\mu}(x) \sim \frac{\mu A^{\mu}}{|x|^{1+\mu}}, \quad |x| \to \infty \tag{5.6}$$

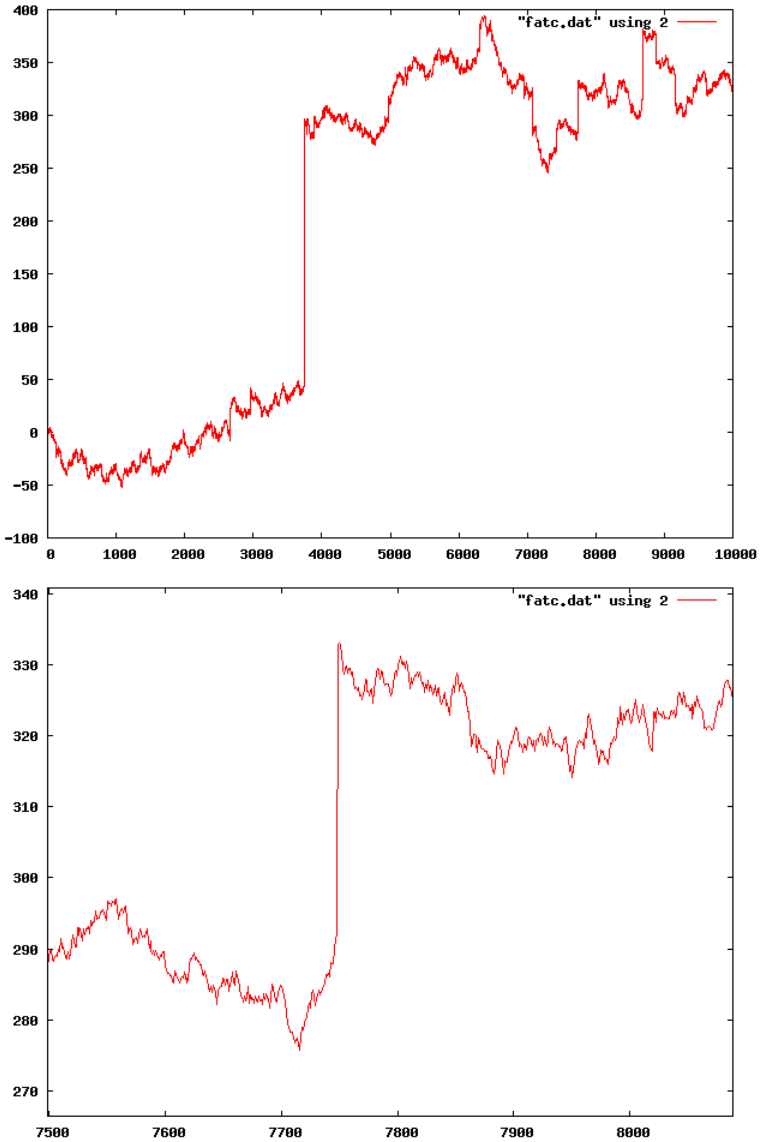

Figure 5.4: Heavy tailed time series numbers drawn from a Levy distribution conformed of 5% of Lorentz-Cauchy distribution ($\mu = 1$) and 95% of Gaussian distribution ($\mu = 2$). The lower panel is a zoom on the range (7500, 8100) and emphasizes the self-similarity of the flight. Notice the frequent discontinuities on all scales

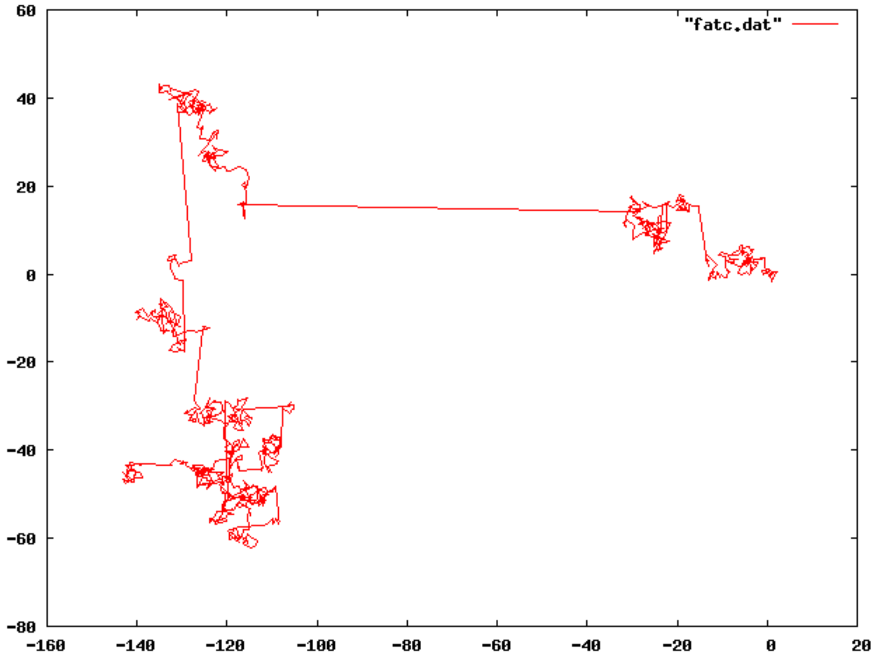

Figure 5.5: Computer simulation of a two-dimensional Lévy flight conformed of 5% of Lorentz-Cauchy distribution ($\mu = 1$) and 95% of Gaussian distribution ($\mu = 2$).

with $A^\mu \propto a$. For $1 < \ < 2$, the variance is infinite but the mean absolute value is finite. The particularity with infinite variance, and this is a very important characteristic, is that, no matter the number of realizations, the central limit theorem does not hold in the same way as it does for finite variances where any distribution probability will lead to a Gaussian distribution, In this case the infinite variance will always keep the distribution of Lévy form.

For $\mu > 2$ the function $L_\mu(x)$ is no longer positive semidefinite and therefore not suitable for a probability distribution.

5.2.1.1 Truncated Lévy Flights

Due to the overestimation of heavy tails in Lévy flights (see Figure 5.2), a truncation factor of $\frac{1}{\alpha}$ is introduced in order to achieve a better fit of the stochastic process and real data. Let the truncated distribution function be defined by its characteristic function:

$$\hat{T}_\mu(z) = \exp\left[-a\frac{(\alpha^2+z^2)^{\frac{\mu}{2}}\cos\left(\mu arctan\frac{|z|}{\alpha}\right)-\alpha^\mu}{\cos\frac{\pi\mu}{2}}\right] \tag{5.7}$$

For finite α, the variance and all the moments are also finite, letting the central limit theorem guarantee the convergence to a Gaussian distribution under addition of many random variables.

5.2.2 Heavy Tail Distributions in Physics

Although in physics most statistical processes follow Gaussian distribution. Some phenomena incur into Lévy distributions. Here we allude to the classical example taken from Voit (2005): "Second order phase transitions, from a paramagnet to a ferromagnet, as the temperature of, let's say, iron is lowered through the Curie temperature T_c. At a critical point, there are power-law singularities in almost all physical quantities, e.g., the specific heat, the susceptibility, etc. The reason behind these power-law singularities are critical fluctuations of the ordered phase (ferromagnetic in the above example) in the disordered (paramagnetic) phase above the transition temperature, resp. vice versa below the critical temperature, as a consequence of the interplay between entropy and interactions. In general, for $T = T_c$, there is a typical size for the correlation length of the ordered domains. At the critical point $T = T_c$, however, the correlation length diverges, and there is no longer a typical length scale. This means that ordered domains occur on all length scales, and are distributed according to a power-law distribution.", Although Lévy distributed we must clarify that, this example does not hold independence of their variables.

5.3 Non-Gaussian Option Pricing

Further option pricing strategies are left out this work, however it is important to mention that the basic idea of strategies beyond Gaussian world is to make no assumptions about the stochastic process followed by the underlying security. In this sense, the risk by hedging with options does not disappear automatically leaving a remanent risk minimised by the strategy[1]. We propose however to attempt through the binomial model, that has been subject of study here, to either make no assumption on the stochastic process, or to reformulate g and u, previously defined in chapter 4, in such a way that long tail distribution is accounted.

[1]Interested readers may refer to chapter 7 of Voit (2005)

Chapter 6

Conclusions

We have observed the importance of physics in finance as a tool for both modelling and solving interrogates, particularly those involving price time series.

We reviewed the traditional method to derive the Black-Scholes model for option valuation. We found that the risk-free portfolio strategy is not completely correct, therefore a discrete binomial model (neutral-risk) has been introduced in order to assure a correct derivation of Black-Scholes PDE, and consequently a correct reduction of the problem to a one dimensional heat diffusion equation.

We observed that the classic modelling of the underlying security price as geometric Brownian motion, where price fluctuations are drawn form Gaussian probabilities, does not represent reality in its whole extend. Different stochastic process such as truncated Lévy give a better representation. As a consequence Black Scholes theory should be modified to assume a different stochastic process or even better, not to make any assumption on the stochastic process.

Although in physics processes mostly follow Gaussian probabilities, heavy tails distribution can also be found in nature. Relationships of these process to finance also exist and offer help to solve the problem of option pricing.

Unfortunately creating risk-free portfolios is not possible at the light of a better representation. Nonetheless strategies exist to minimise the remanent risk.

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