Ideal Solution to the King's Problem as an Optimal Average Solution

by

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To my parents

Acknowledgements

I consider myself very lucky to have found in my advisor, Alonso Botero, such a great physicist and person. No part of this paper would have been possible without his help and guidance. I wish to thank my family and friends for supporting me all these years, they are they reason why I am here.

John Jairo Molina

Chapter 1

Introduction

If you are not completely confused by quantum mechanics, you do not understand it.
John Wheeler

The *spooky* nature of quantum mechanics that has troubled so many generations of physicists has also proven to be a fertile breading ground for concepts that continue to push the frontiers of science and test the boundaries of human intellect. Two of the most famous examples of this are quantum teleportation and quantum cryptography. Consider the simple example of quantum teleportation given in every introductory quantum mechanics course (where Alice is able two send a two-state quantum system to Bob, using a previously shared *EPR* state and a classical communication channel). Now suppose that Alice wishes to send Bob a top secret message but the only resources at her disposal are the *EPR* states they had set aside for teleportation. Imagine that the number of *EPR* states that they have is just enough to transmit the message; in this case, when Alice and Bob tell each other the corresponding measurement basis they used for each particle they can no longer afford to discard the ones that don't coincide. Bob realizes that all the particles that Alice sends him are each half of an *EPR* pair (and that he has the other half) and proceeds to develop a strategy to guess the value of the "lost" spins. This example, using two of the most basic procedures in quantum information theory, presents the basic characteristics of the *King's Problem* to be introduced later.

The *King's Problem* is an inference problem, the challenge is to determine the best option (in this case the value of a spin measurement) based on incomplete information. The fact that probabilities will play a major part in this paper should surprise no one, especially since this is also a problem in quantum mechanics. But there is a distinction that must be made between the quantum mechanical probabilities, which reflect the basic laws of nature as we know them, and "ordinary" probabilities, which represent our state of knowledge [1]. We will use both with absolutely no distinction but failure to recognize the difference can prove to be very costly.

This paper presents a very brief summary of the *King's Problem*, it's variations, and some of the most important results that have been obtained. This is not done in more detail because most of the results are of no direct use to the

[1]This is an oversimplification

problem we will consider, and the results we do use admit a relatively simple derivation under the framework we will develop. We then formulate the retrodiction problem in terms of the relevant probabilities, which will lead us to a natural criteria for the optimality of a solution. The optimization problem, as such, is not solved but we prove that the ideal solution for the original *King's Problem* satisfies a particular extrema condition of this variational problem.

Chapter 2

The King's Problem

In 1987 Vaidman, Aharonov, and Albert (VAA) published a paper[18] entitled "How to Ascertain the Values of σ_x, σ_y, and σ_z of a Spin-$\frac{1}{2}$ Particle" which would give rise to the so called *King's Problem*. In it a hapless physicist gets captured by an evil king and is given one chance to win back his freedom. He must prepare a spin-$\frac{1}{2}$ particle which he will give to the king to perform one of three measurements, σ_x, σ_y, or σ_z. The king will then return the particle to the physicist and allow him the opportunity of carrying out one final measurement. The challenge ends with the king revealing the spin component that he measured and the physicist trying to determine which value was obtained. At first glance the physicist might appear to be doomed, since it is well known that the three spin components the king measures are mutually incompatible observables and the physicist doesn't know which one was chosen until after he performs his final measurement. Had he known this information beforehand the solution would be trivial.

The physicist's salvation lies in the fact that he is given complete freedom to do what he wants before and after the king's measurement, so he can decide which initial state to prepare and which control measurement to perform. Obviously, he wishes to find a combination of the two that will always allow him to retrodict the king's result with certainty (after being told of the spin measurement that was performed). Imagine the king was only permitted to measure one spin direction, $\sigma_{\hat{n}}$; the physicist could then prepare one of the two eigenstates $|n+\rangle$ or $|n-\rangle$ and without the need for any type of control measurements he could *predict* with absolute certainty the king's result. An equally valid alternative to *retrodict* the king's result would be to prepare an arbitrary initial state and just repeat the king's measurement at the end. Now consider the king is allowed to measure any two mutually perpendicular spin directions, for convenience choose σ_z and σ_x. Let the initial state of the system be an eigenstate of one of the the king's possible measurement directions, say $|n_z+\rangle$. If the king measures σ_z he will undoubtedly get the result $+1$, if he measures σ_x he can obtain ± 1 with equal probability. It is easy to see that the physicist's control measurement must then be σ_x; for if it coincides with the king's measurement the two obtained the same result and if not the king's result is determined uniquely by the initial state of the system (in this case $+1$). For three mutually perpendicular directions the solution isn't as straightforward and requires coupling the particle with another spin-$\frac{1}{2}$ particle in such a way that the composite system is in an EPR state (maximally entangled state). VAA proposed a procedure[18] (for one particular EPR state) that consists of measuring an observable, with non-degenerate eigenvalues, on the composite system such that each eigenstate is compatible with one and only one of the eigenstates of each of the three spin measurements. Throughout this work we will refer to this solution as the ideal solution (or equivalently the ideal basis) of the *King's Problem*. Additionally, they found that no solution of this type (for two spin-$\frac{1}{2}$ particles with projective measurements) exists if the three spin components of the king's measurement are not along mutually orthogonal directions or if more than three spin components are measured.

When one tries to generalize this problem to dimensions d higher than 2 it is necessary to determine the analogs of the Pauli matrices, which reduces to finding $d + 1$ mutually complementary observables (mutually unbiased bases) for this d-dimensional space[2]; whether or not this can be done is a definitive factor in finding a solution. Aharonov and Englert solved the Spin-1 ($d = 3$) problem[2] and later showed how to find a solution for any prime degree of freedom[6]. For $d = 6$ and $d = 10$ it is known that no solutions exist but for the other possible values of d (not a prime or a prime power) there is still no definite answer[8].

A further generalization for the Spin-$\frac{1}{2}$ problem (closer to the case we will consider in this paper) consists of working with observables that are not mutually complementary, by replacing σ_x, σ_y, and σ_z with $\hat{n}_k \cdot \vec{\sigma}$ (with $1 \leq k \leq m$ for some value m). The solutions involve either replacing the physicist's projective measurement in favor of a POVM measurement or working with an "external" particle of spin greater than $\frac{1}{2}$. Interestingly enough, both variations lead to exactly the same restriction on the vectors $\hat{n}_k$ in order to guarantee the existence of a solution[10][4].

Chapter 3

The King's Problem for a Uniformly Distributed Spin Measurement

We now consider a further variation of the King's Problem by allowing the initial measurement to be performed in any direction. It is known from the results previously stated that by removing all type of restrictions on these measurements we must forgo the possibility of finding an exact solution. Therefore we wish to provide a measure of the uncertainty involved in this retrodiction problem. This will allow us the possibility of comparing different measurement strategies against each other in order to choose the best one; which is the first step in trying to find an optimal solution. We will only consider solutions involving projective measurements and not the more general POVM measurements.

We adopt the following notation: the King's measurement direction is denoted by $\hat{n}$, with $\hat{n}$ a vector over the unit sphere; the physicist's measurement basis is given by $\{|\hat{\Phi}_j\rangle\}$, $j = 1 \ldots 4$; the results for the control and the initial measurements are represented by j and k respectively.

We begin by defining the following propositions:

X : *(Prior/Background Information) The physicist prepares a maximally entangled state of two Spin-$\frac{1}{2}$ particles (A and B).He gives particle A to the King, who measures it's spin along an arbitrary direction $\hat{n}$.The King then returns the particle to the physicist and allows him to carry out a control measurement on the composite system AB.*

K_k : *The King measures the spin of subsystem A and obtains the result k.*

J_j : *The physicist measures an observable $\hat{\Phi}$ on the composite system and obtains the result j.*

3.1 Conditional Probabilities

After the initial measurement has been performed and the physicist is again in possession of both particles he is free to chose which control measurement to use. Naturally, he wishes to do so in such a way that will allow him to extract the maximum amount of information about the initial measurement result as possible. In order to quantify this state of uncertainty we must first determine how knowledge of the value of j (for the composite system) would affect our knowledge of the prior value of k (for subsystem A). Thus, we must find the probabilities of the initial measurement results conditional on the results of the control measurement, $P(K_k|X\,\hat{n}\,J_j)$. A simple application of Bayes' Theorem gives

$$P(K_k|X\,\hat{n}\,J_j) = \frac{P(J_j|X\,\hat{n}\,K_k)\,P(K_k|X\,\hat{n})}{P(J_j|X\,\hat{n})}, \tag{3.1}$$

but on the information X provided, we know that $J_j = (K_- + K_+)J_j$ for all values of j (the results of the initial measurement are obviously ± 1, regardless of what control measurement is carried our or what results are obtained). Furthermore the $\{K_-, K_+\}$ are mutually exclusive propositions, so the sum and product rule can be used to obtain

$$P(K_k|X\,\hat{n}\,J_j) = \frac{P(J_j|X\,\hat{n}\,K_k)\,P(K_k|X\,\hat{n})}{\sum_{k'} P(J_j|X\,\hat{n}\,K_{k'})\,P(K_{k'}|X\,\hat{n})}. \tag{3.2}$$

All the probabilities on the left hand side are easily calculated using the basic measurement postulates of quantum mechanics. They are

$$P(J_j|X\,\hat{n}\,K_k) = \frac{\langle E|\hat{P}_k^A\hat{\Phi}_j^{AB}\hat{P}_k^A|E\rangle}{\langle E|\hat{P}_k^A|E\rangle} \tag{3.3}$$

$$P(K_k|X\,\hat{n}) = \langle E|\hat{P}_k^A|E\rangle \tag{3.4}$$

where $\hat{\Phi}_j$ and $\hat{P}_k$ are the projectors onto the eigenspaces corresponding to the eigenvalues j and k and $|E\rangle$ is any one of the *EPR* states

$$\hat{P}_k^A = \frac{1}{2}(\mathbb{I}^A + k\,\vec{\sigma}^A\cdot\hat{n}) \tag{3.5}$$

$$\hat{\Phi}_j^{AB} = |\Phi_j\rangle\langle\Phi_j|. \tag{3.6}$$

$$|E\rangle = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \\ |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \end{array}\right\} = \left\{\begin{array}{c} |E_1\rangle \\ |E_2\rangle \\ |E_3\rangle \\ |E_4\rangle \end{array}\right\}$$

The conditional probability for the initial measurement result (3.2), can then be written as

$$P(K_k|X\,\hat{n}\,J_j) = \frac{|\langle\Phi_j|\hat{P}_k^A|E\rangle|^2}{\sum_{k'}|\langle\Phi_j|\hat{P}_{k'}^A|E\rangle|^2}. \tag{3.7}$$

In the case where the $\langle\Phi_j|E\rangle \neq 0\,(\forall j)$ a simple calculation (see Appendix A) leads us to the following expression

$$P(K_k|X\,\hat{n}\,J_j) = \frac{1}{2}\left(1 + k\frac{2\,\vec{x}_j\cdot\hat{n}}{1+|\vec{z}_j\cdot\hat{n}|^2}\right), \tag{3.8}$$

where

$$\begin{aligned}\vec{z}_j &= \frac{\langle\Phi_j|\vec{\sigma}^A|E\rangle}{\langle\Phi_j|E\rangle} \\ &= \vec{x}_j + i\,\vec{y}_j\end{aligned} \tag{3.9}$$

with

$$\vec{x}_j = \Re\left(\frac{\langle\Phi_j|\vec{\sigma}^A|E\rangle}{\langle\Phi_j|E\rangle}\right), \qquad \vec{y}_j = \Im\left(\frac{\langle\Phi_j|\vec{\sigma}^A|E\rangle}{\langle\Phi_j|E\rangle}\right). \tag{3.10}$$

To get an idea of how these probabilities behave and what information they might be able to provide we will study three different measurement basis. The relevant calculations can be found in Appendix A.

3.1.1 Computational Basis

We must be careful to work with expression (3.7) and not (3.8) since $\langle\Phi_j|E\rangle = 0$ for some values of j (regardless of the EPR state we have chosen to use). The following conditional probabilities are obtained:

- For $|E\rangle \in \{|E_1\rangle, |E_2\rangle\}$

$$\begin{aligned}P(K_k|X\,\hat{n}\,J_1) &= \frac{1}{2} + k\frac{n_z}{n_z^2+1} \\ P(K_k|X\,\hat{n}\,J_2) &= \frac{1}{2} + k\frac{-n_z}{n_z^2+1} \\ P(K_k|X\,\hat{n}\,(J_3+J_4)) &= \frac{1}{2}\end{aligned} \tag{3.11}$$

- For $|E\rangle \in \{|E_3\rangle, |E_4\rangle\}$

$$\begin{aligned}P(K_k|X\,\hat{n}\,(J_1+J_2)) &= \frac{1}{2} \\ P(K_k|X\,\hat{n}\,J_3) &= \frac{1}{2} + k\frac{n_z}{n_z^2+1} \\ P(K_k|X\,\hat{n}\,J_3) &= \frac{1}{2} + k\frac{-n_z}{n_z^2+1}\end{aligned} \tag{3.12}$$

Thus we see that the outcome of our control measurement (for this choice of basis) yields useful information on the possible outcome of the king's measurement only when the initial EPR state has a non-zero projection over the final state of the system. The resulting probability function

$$f_{comp}^{\pm k} = \frac{1}{2} + k\frac{\pm n_z}{n_z^2+1} \tag{3.13}$$

depends solely on the z component of the initial measurement direction, which was to be expected given the polarization of the EPR states. A graph of these probability distributions is given in Figure 3.1.1. More important than the exact value of these distributions is whether or not this value is greater than or equal to $\frac{1}{2}$; since this is precisely the information one would need in order to make an educated guess on the result of the initial measurement. It is easy to see that this decision is based entirely on the sign of n_z, for a given value of k and j.

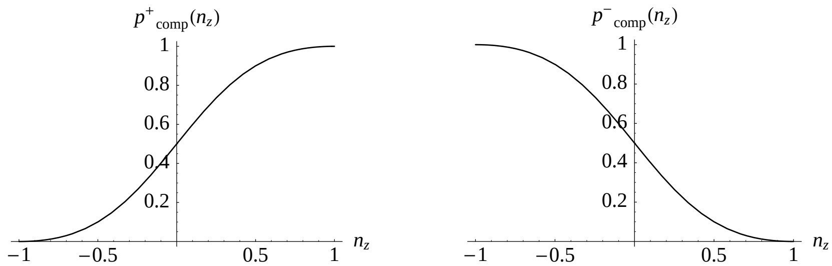

Figure 3.1: Non-trivial probability functions $P(K_k|X\,\hat{n}\,J_j)$.
(computational basis)

3.1.2 Ideal Basis

Recalling the solution to the original mean kings problem[18]

$$|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c} \sqrt{2}|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle e^{-\imath\pi/4} + |\downarrow\uparrow\rangle e^{\imath\pi/4} \\ \sqrt{2}|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle e^{-\imath\pi/4} - |\downarrow\uparrow\rangle e^{\imath\pi/4} \\ \sqrt{2}|\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle e^{\imath\pi/4} + |\downarrow\uparrow\rangle e^{-\imath\pi/4} \\ \sqrt{2}|\downarrow\downarrow\rangle - |\uparrow\downarrow\rangle e^{\imath\pi/4} - |\downarrow\uparrow\rangle e^{-\imath\pi/4} \end{array}\right\} = \left\{\begin{array}{c} |\Phi_1\rangle \\ |\Phi_2\rangle \\ |\Phi_3\rangle \\ |\Phi_4\rangle \end{array}\right\}$$

we now turn our attention to study the conditional probabilities $P(K_k|X\,\hat{n}\,J_j)$ obtained for this particular choice of measurement basis. The probability functions, for each possible initial state, are now

- $|E1\rangle$

$$\begin{aligned} P(K_k|X\,\hat{n}\,J_1) &= \frac{1}{2} + \frac{k}{2}\frac{n_x + n_y + n_z}{1 + n_x n_y + n_x n_z + n_y n_z} \\ P(K_k|X\,\hat{n}\,J_2) &= \frac{1}{2} + \frac{k}{2}\frac{-n_x - n_y + n_z}{1 + n_x n_y - n_x n_z - n_y n_z} \\ P(K_k|X\,\hat{n}\,J_3) &= \frac{1}{2} + \frac{k}{2}\frac{n_x - n_y - n_z}{1 - n_x n_y - n_x n_z + n_y n_z} \\ P(K_k|X\,\hat{n}\,J_4) &= \frac{1}{2} + \frac{k}{2}\frac{-n_x + n_y - n_z}{1 - n_x n_y + n_x n_z - n_y n_z} \end{aligned} \tag{3.14}$$

- $|E2\rangle$

$$\begin{aligned} P(K_k|X\,\hat{n}\,(J_1 + J_2)) &= \frac{1}{2} + \frac{k}{2}\frac{n_z}{1 - n_x n_y} \\ P(K_k|X\,\hat{n}\,(J_3 + J_4)) &= \frac{1}{2} + \frac{k}{2}\frac{-n_z}{1 + n_x n_y} \end{aligned} \tag{3.15}$$

- $|E3\rangle$

$$\begin{aligned} P(K_k|X\,\hat{n}\,(J_1 + J_3)) &= \frac{1}{2} + \frac{k}{2}\frac{n_x}{1 - n_y n_z} \\ P(K_k|X\,\hat{n}\,(J_2 + J_4)) &= \frac{1}{2} + \frac{k}{2}\frac{-n_x}{1 + n_y n_z} \end{aligned} \tag{3.16}$$

- $|E4\rangle$

$$\begin{aligned} P(K_k|X\,\hat{n}\,(J_1+J_4)) &= \frac{1}{2}+\frac{k}{2}\frac{n_y}{1-n_xn_z} \\ P(K_k|X\,\hat{n}\,(J_2+J_3)) &= \frac{1}{2}+\frac{k}{2}\frac{-n_y}{1+n_xn_z} \end{aligned} \tag{3.17}$$

Establishing a geometric interpretation for these probability distributions is relatively straightforward after some basic manipulations. We consider two cases, when the initial state is $|E1\rangle$ and when it is one of the other three EPR states.

The probability functions for $|E\rangle = |E_1\rangle$ are of the form

$$f_{id}^{s_1s_2s_3k} = \frac{1}{2}+\frac{k}{2}\frac{s_1n_x+s_2n_y+s_3n_z}{1+s_1s_2n_xn_y+s_1s_3n_xn_z+s_2s_3n_yn_z} \tag{3.18}$$

with $s_1, s_2, s_3 = \pm$. Again we are interested in determining the values of $\hat{n}$ for which this probability distribution is greater than or equal to $\frac{1}{2}$. Notice that

$$\begin{aligned} (s_1n_x+s_2n_y+s_3n_z)^2 &\geq 0 \\ n_x^2+n_y^2+n_z^2+2s_1s_2n_xn_y+2s_1s_3n_xn_z+2s_2s_3n_yn_z &\geq 0 \\ s_1s_2n_xn_y+s_1s_3n_xn_z+s_2s_3n_yn_z &\geq \frac{-1}{2} \\ 1+s_1s_2n_xn_y+s_1s_3n_xn_z+s_2s_3n_yn_z &\geq \frac{1}{2}, \end{aligned} \tag{3.19}$$

so the decision can be reduced to determining the the sign of

$$s_1n_x+s_2n_y+s_3n_z. \tag{3.20}$$

The borderline case $s_1n_x+s_2n_y+s_3n_z = 0$ defines a plane which divides the unit sphere in two halves; one corresponding to $s_1n_x+s_2n_y+s_3n_z > 0$, the other to $s_1n_x+s_2n_y+s_3n_z < 0$. This way of partitioning the unit sphere provides a simple way to "guess" the initial measurement result: knowledge of the value of j tells us which plane to use, then all that is needed is to know if $\hat{n}$ lies above or below this plane on the sphere (the appropriate guess would be $+1$ and -1 respectively). Figure 3.1.2 shows a graph of the four planes corresponding to the different values of j and their intersection with the unit sphere.

For $|E\rangle \in \{|E_2\rangle, |E_3\rangle, |E_4\rangle\}$ the probability functions take the form

$$f_{id}^{s_1s_2k} = \frac{1}{2}+\frac{k}{2}\frac{s_1n_a}{1+s_2n_bn_c} \tag{3.21}$$

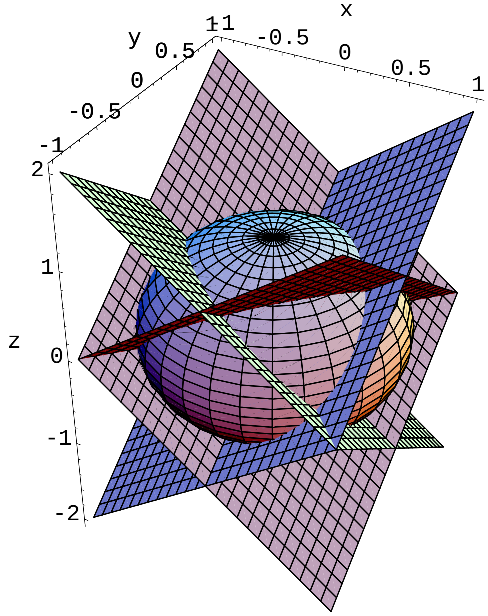

Figure 3.2: Graph of the unit sphere and the planes $z = \pm x \pm y$, a geometrical representation of the decision function.
(ideal basis, $|E_1\rangle$ initial state)

with (a, b, c) a permutation of (x, y, z). Proceeding in the same manner as before[1], we see that the term in the denominator is always greater than 0:

$$\begin{aligned} (n_a + s_2 n_b)^2 &\geq 0 \\ n_a^2 + n_b^2 + 2s_2 n_a n_b &\geq 0 \\ 1 + 2s_2 n_a n_b &\geq n_c^2 \\ 1 + s_2 n_a n_b &\geq \frac{n_c^2 + 1}{2} \\ 1 + s_2 n_a n_b &> 0. \end{aligned} \tag{3.22}$$

Which means that we only need to consider the sign of the term in the numerator, a trivial task since it is given by $s_1 n_a$. The decision function in this case is defined in the same manner as before, we take the $n_a = 0$ plane and partition the unit sphere to determine which value of k most likely corresponds to the value of j that has been measured. There is, however, a clear difference with the results obtained for the $|E1\rangle$ state, where each value of j defined a different plane. For $|E_2\rangle$, $|E_3\rangle$, and $|E_4\rangle$ we obtain two distinct probability functions for the four possible values of j and both of them define the same partitioning plane, regardless of the value of j. As such, these planes are determined uniquely by the EPR state that was initially prepared. This doesn't mean that the control

[1] These calculation are not completely necessary since (3.8) is valid for this choice of basis, and all we have to do is determine the sign of $\vec{x}_j \cdot \hat{n}$. We chose to do it in this manner in order to be consistent with the procedure followed for the other bases.

measurement isn't necessary, since the decision function depends on which half of the sphere is assigned to $k = +1$ and which to $k = -1$, and not just on which plane is used to partition the sphere. Intuitively we begin to expect that this ideal solution for the $|E_1\rangle$ initial state is probably not the best solution for $|E_2\rangle, |E_3\rangle$ or $|E_4\rangle$ since it is not taking advantage of the larger dimension of the entangled system with respect to the single spin-$\frac{1}{2}$ state. By assigning the same decision function to two distinct values of j it is in effect "throwing" away information that could have been used to narrow down the configuration space of the system and retrodict the value of k with greater certainty.

3.1.3 Bell Basis

For the Bell basis we obtain equal conditional probabilities, regardless of the initial state of the system or the result of the control measurement j. This basis clearly provides no advantage in retrodicting the value of k, which was to be expected since the initial state is itself a Bell state.

$$P(K_k|X\,\hat{n}\,J_j) = \frac{1}{2} \tag{3.23}$$

If the physicist was forced to use this basis he could simply try guessing the value of k and avoid any unnecessary hassle.

3.1.4 Discussion of the Results

The partitioning scheme just introduced could have been foreseen, since this is precisely what the physicist has to do in order to retrodict the value of k. For a given value of j he must assign every point of the unit sphere with one of two values $+1$ or -1, so that by learning of the king's choice of measurement direction he can immediately guess the appropriate value of k and do so consistently. Much care must be taken when interpreting these results because even though the partitioning planes all divide the unit sphere in halves this DOES NOT mean that half of the points are assigned to $+1$ and the other to -1. We must not forget that when we talk about $\hat{n}$ we are not really talking about a vector on the unit sphere but a spin measurement direction (which we represent by a vector on the unit sphere), and as such $\hat{n}_i$ and $-\hat{n}_i$ are really the same. If we interpreted the partitioning scheme blindly we would be led to contradictory results, with the same spin measurement defining two distinct values of k (for the same value of j)! The solution is very simple: we have to define a unique representation for the spin measurement directions (say $0 \leq \theta < \pi$ and $0 \leq \phi < \pi$) and be consistent with this choice whenever we refer to the vectors $\hat{n}$.

Looking forward towards finding an optimal solution to the problem, we begin to understand what needs to be done. We must determine the set of planes (consistent with the orthogonality relations of the measurement basis which defines them) that, on average, provide the best decision functions (i.e maximize the probability of success).

3.2 Probability of Success

Recalling the result we obtained for the respective probabilities of the king's measurement result (3.8), it can easily be seen that a suitable decision function to use is given by the sign of $(\vec{x}_j \cdot \hat{n})$. If $\delta_j^{\hat{n}}$ represents the value of the decision function for a given value of j and measurement direction $\hat{n}$ we have

$$\delta_j^{\hat{n}} = sgn(\vec{x} \cdot \hat{n}). \tag{3.24}$$

The probability of success for a given direction of the initial measurement $\hat{n}$ is by definition the probability that the King's initial measurement yield some value k and that the physicist correctly guess this value. Define the following propositions

Δ_k	:	The physicist guesses the value k for the King's measurement.
$\Delta_{k,j}^{\hat{n}}$	:	The physicist guesses the value $k = \delta_j^{\hat{n}}$ for the King's measurement with knowledge of both j and $\hat{n}$.

The probability of success is then defined as

$$P_s(\hat{n}) = \sum_k P(K_k \Delta_k | X),$$

but since $K_k = K_k(\sum_j J_j)$ we get

$$\begin{aligned} P_s &= \sum_k \sum_j P(K_k J_j \Delta_k | X) \\ &= \sum_k \sum_j P(\Delta_k | X K_k J_j) P(J_j | X K_k) P(K_k | X) \\ &= \sum_k \sum_j P(\Delta_{k,j}^{\hat{n}} | X K_k J_j) P(J_j | X K_k) P(K_k | X). \end{aligned}$$

Given the way we have defined the decision function, $P(\Delta_{k,j}^{\hat{n}} | X K_k J_j)$ is equal to either one or zero,

$$P(\Delta_{k,j}^{\hat{n}} | X K_k J_j) = \begin{cases} 0 & \text{if}\, k \neq \delta_j^{\hat{n}} \\ \\ 1 & \text{if}\, k = \delta_j^{\hat{n}}. \end{cases} \tag{3.25}$$

This allows us to simplify the expression considerably, since half of the terms vanish, leaving

$$\begin{aligned} P_s(\hat{n}) &= \sum_j P(J_j | X K_{\delta_j^{\hat{n}}}) P(K_{\delta_j^{\hat{n}}} | X) \\ &= \sum_j |\langle \Phi_j | \frac{1}{2} \left(\mathbb{I}^A + \delta_j^{\hat{n}} \vec{\sigma}^A \cdot \hat{n} \right) | E \rangle|^2 \qquad (3.26) \\ &= \frac{1}{4} \sum_j \Big\{ |\langle E | \Phi_j \rangle|^2 + |\langle E | \vec{\sigma}^A \cdot \hat{n} | \Phi_j \rangle|^2 \\ &\qquad + 2\delta_j^{\hat{n}} \Re \left(\langle E | \Phi_j \rangle \langle \Phi_j | \vec{\sigma}^A | E \rangle \cdot \hat{n} \right) \Big\} \end{aligned}$$

Using the completeness relation for the $|\Phi_j\rangle$, the properties of the Pauli matrices, the definition of the $\delta_j^{\hat{n}}$, and again assuming $\langle\Phi_j|E\rangle \neq 0$ we can reduce this expression down to

$$P_s(\hat{n}) = \frac{1}{2}\left(1 + \sum_j |\langle E|\Phi_j\rangle|^2 |\vec{x}_j \cdot \hat{n}|\right) \tag{3.27}$$

To obtain the average probability of success, assuming a uniform probability distribution for the King's choice of measurement direction, we simply integrate the previous expression over the unit sphere[2].

$$\begin{aligned} P_s &= \int P_s(\hat{n}) P(\hat{n})\, d\,\Omega \\ &= \frac{1}{2} + \frac{1}{8\pi}\sum_j |\langle E|\Phi_j\rangle|^2 \int |\vec{x}_j \cdot \hat{n}|\, d\,\Omega \\ &= \frac{1}{2} + \frac{1}{8\pi}\sum_j |\langle E|\Phi_j\rangle|^2\, |\vec{x}_j| \int |\cos\theta|\, d\,\Omega \\ &= \frac{1}{2} + \frac{1}{4}\sum_j |\langle E|\Phi_j\rangle|^2\, |\vec{x}_j| \end{aligned} \tag{3.28}$$

In essence, we are almost finished we our task, all that remains to be done is to find a measurement basis $\{|\Phi_j\rangle\}$ that maximizes (3.28). This turns out to be somewhat complex, so we will return to this subject further on when we develop a formalism better suited for this optimization problem.

In what follows we take a look at this probability of success for the two relevant measurement basis considered thus far.

3.2.1 Computational Basis

Using (3.26) we find that the probability of success for a given value of $\hat{n}$ is

$$P_s(\hat{n}) = \frac{1 + |n_z|}{2}, \tag{3.29}$$

independent of the initial state $|E\rangle$.[3]. To see how this probability is distributed we generated a uniform sampling of points over the unit sphere, calculated the probability at each point, and then generated a scaled histogram for this data (such that the area under the graph is equal to unity). The data obtained corresponds to what one would expect for a uniform distribution of the probability of success between $\frac{1}{2}$ and 1, corresponding to a mean of $\frac{3}{4}$ and a standard deviation of $\sqrt{\frac{7}{12} - \frac{9}{16}} \approx 0144$.

[2]The reader can verify that the problems discussed earlier about the spin measurement directions and their representations on the unit sphere are avoided since we have already taken into account the decision function and the probability of success depends only on the absolute value of $\vec{x}_j \cdot \hat{n}$.

[3]For this choice of basis the decision function must be changed to $sgn(sn_z)$, as defined by (3.13)

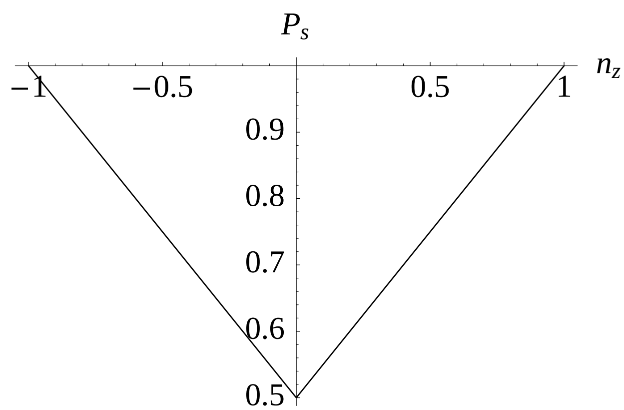

Figure 3.3: Graph of the probability of success $P_s(\hat{n})$ (computational basis).

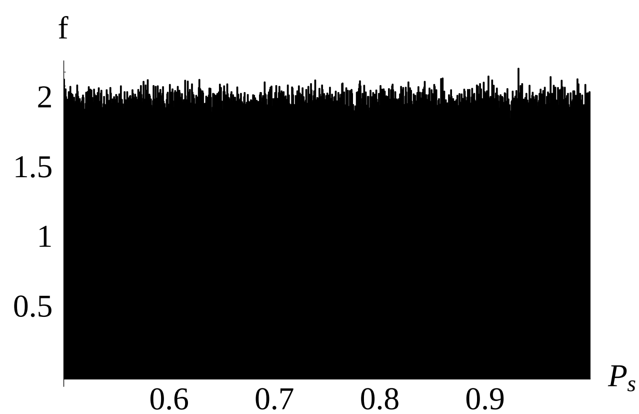

Figure 3.4: Scaled histogram of the probability of success for a uniform distribution of 10^6 points over the unit sphere: Mean = 0.749, Standard Deviation = 0.144.
(computational basis)

3.2.2 Ideal Basis

For this choice of basis the probability of success will depend on which initial state is chosen.

- $|E_1\rangle$

$$P_s(\hat{n}) = \frac{1}{2} + \frac{1}{8}\Big(|n_x + n_y + n_z| + |-n_x - n_y + n_z| + |n_x - n_y - n_z| + |-n_x + n_y - n_z|\Big) \quad (3.30)$$

- $|E_2\rangle$

$$P_s(\hat{n}) = \frac{1 + |n_z|}{2} \quad (3.31)$$

- $|E_3\rangle$

$$P_s(\hat{n}) = \frac{1 + |n_x|}{2} \quad (3.32)$$

- $|E_4\rangle$

$$P_s(\hat{n}) = \frac{1 + |n_y|}{2} \quad (3.33)$$

The only interesting case corresponds to $|E_1\rangle$, the corresponding histogram is given in Figure 3.2.2; the other states reproduce the results obtained for the computational basis (uniform distribution between 0.5 and 1) since they can be obtained from one another by interchanging the roles of n_x, n_y, and n_z and we our assuming the king's measurement direction is uniformly distributed. The histogram shows two very striking characteristics about the probability of success, it has a lower bound well above 0.5 and it presents a violent cutoff at a value close to the average. We can provide, *a posteriori*, a geometrical interpretation to try and explain why this distribution behaves the way it does. The terms in the absolute values of (3.30) each define a plane parallel to the partitioning plane and the intersections with each other and with the unit sphere will determine the points that may appear in the expression for the probability of success. For a given value p of the probability of success to be attainable, we would need to be able to arrange these four planes such that their intersections over the unit sphere corresponds to the points that make $P_s(\hat{n}) = p$. If no such configuration exists for a given value p than the probability of success can never be evaluated to this value. As we increase the value of p the admissible configurations seem to increase exponentially until they reach a maximum value, after which they fall drastically to settle on a constant value. Figure 3.2.2 shows a 3D spherical plot of the probability of success.

3.3 Projective Representation

We now consider an alternative formulation of the problem, in order to provide a greater geometrical intuition to the results we have already obtained. As has already been stated, the idea is to find an optimal solution to the modified King's problem and in order to do this we must find a measurement basis that

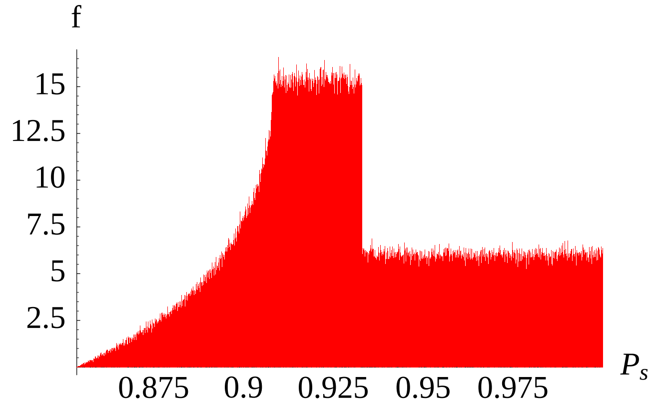

Figure 3.5: Scaled histogram of the probability of success for a uniform distribution of 10^6 points over the unit sphere: Mean = 0.933, Standard Deviation = 0.0326.
(ideal basis, $|E_1\rangle$ initial state)

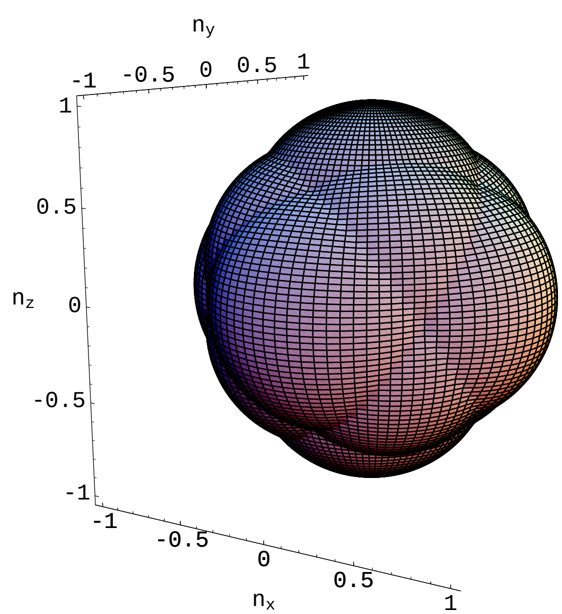

Figure 3.6: Spherical plot of the probability of success (3.30).

maximizes the average probability of success (3.28). In what follows we try to develop a framework better suited to achieve this goal. Let us consider a composite system AB consisting of two d-state systems, A and B. We can define the following privileged maximally entangled state (M.E.S)[4]

$$|\Phi_0\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha} |\alpha\rangle_A |\alpha\rangle_B. \tag{3.34}$$

A basis for the operators on this d-dimensional Hilbert space (in analogy with the Pauli matrices for the 2-dimensional case) are the identity matrix and the (d^2-1) traceless, hermitian matrices $\lambda^{(i)}$ obeying the additional property: $Tr(\hat{\lambda}^{(i)}\hat{\lambda}^{(j)}) = 2\delta_{i,j}$.[5]

For

$$\hat{e}^{(i)} = \sqrt{\frac{d}{2}}\hat{\lambda}^{(i)}, \tag{3.35}$$

the trace property now becomes

$$\frac{1}{d} Tr(\hat{e}^{(i)}\hat{e}^{(j)}) = \delta_{i,j}. \tag{3.36}$$

These matrices allow us to define the following (d^2-1) states

$$|\Phi^{(i)}\rangle = \hat{e}^{(i)}_A |\Phi_0\rangle, \tag{3.37}$$

which by construction are orthonormal to each other, $\langle \Phi^{(i)}|\Phi^{(j)}\rangle = \delta_{i,j}$. This means that the d^2 states

$$\{|\Phi_0\rangle, |\Phi^{(i)}\rangle\}, \qquad i = 1\ldots(d^2-1) \tag{3.38}$$

form a basis for this composite system. As such, any state vector $|\Psi\rangle$ can be represented uniquely as a linear combination of these base vectors; we adopt the following representation

$$|\Psi\rangle = \frac{|\Phi_0\rangle + \sum_{i=1}^{d^2-1} \mathfrak{z}_i |\Phi^{(i)}\rangle}{\sqrt{1+|\mathfrak{z}|^2}} \tag{3.39}$$

$$\equiv \frac{|\Phi_0\rangle + \vec{\mathfrak{z}} \circ |\vec{\Phi}\rangle}{1+|\vec{\mathfrak{z}}|^2} \tag{3.40}$$

with $|\vec{\mathfrak{z}}|^2 = \sum_i |\mathfrak{z}_i|^2$. By requiring that the leading coefficient be 1 we come across some difficulties when trying to represent any of the $|\Phi^{(i)}\rangle$ vectors in this manner, since they are being forced to adopt a component along $|\Phi_0\rangle$. We can work around this inconvenience by setting $\mathfrak{z}_j = 0$ for $j \neq i$ and letting $\mathfrak{z}_i \to \infty$, and in this way get as arbitrarily close to $|\Phi^{(i)}\rangle$ as we wish. This same limiting procedure can be used to represent any state vector that has a vanishing component along $|\Phi_0\rangle$. The following properties are immediately recognized

$$\langle \Phi_0|\Psi\rangle = \frac{1}{\sqrt{1+|\mathfrak{z}|^2}}, \tag{3.41}$$

$$\frac{\langle \Phi_0|\hat{e}^{(j)}|\Psi\rangle}{\langle \Phi_0|\Psi\rangle} = \mathfrak{z}_j. \tag{3.42}$$

[4]In the case where $d = 2$ this corresponds to the $|E_1\rangle$ state used previously.

[5]For $d = 2$ and $d = 3$ all we have done is to choose the Pauli matrices and the Gell-Mann matrices as the representations for $SU(2)$ and $SU(3)$.

The condition of orthogonality between two states $|\Psi\rangle$ and $|\Psi'\rangle$ takes the simple form

$$\begin{aligned} \langle\Psi|\Psi'\rangle &= 0 \\ \vec{\mathfrak{z}}^{\star} \cdot \vec{\mathfrak{z}}' &= -1. \end{aligned} \tag{3.43}$$

There is a clear correspondence between the vectors $\vec{\mathfrak{z}}$ and the $\vec{z}_j$ introduced earlier, which is the main reason for having introduced this formalism, but special care must be taken when passing between one and the other. Recall that the $\vec{z}_j$ were introduced as a mere notational shorthand for calculating probabilities and that they depend explicitly on the initial preparation of the system ($|E\rangle$) and the measurement basis ($\{|\Phi_j\rangle\}$). The $\vec{\mathfrak{z}}$ vectors, on the other hand, represent the expansion coefficients of an arbitrary vector $|\Psi\rangle$ in the $\{|\Phi_0\rangle, |\Phi^{(i)}\rangle\}$ basis as defined by 3.39. Taking this into account, we can establish the following correspondence

$$\begin{aligned} |\Phi_j\rangle &\equiv |\Psi_j\rangle \\ |E\rangle &\equiv |\Phi_0\rangle \\ \vec{z}_j &\equiv \vec{\mathfrak{z}}_j^{\star}. \end{aligned} \tag{3.44}$$

The results obtained in the previous sections for the conditional probabilities (3.8) and the probabilities of success (3.27-3.28) can be restated within this new formalism as

$$P(K_k|X\,\hat{n}\,J_j) = \frac{1}{2}\left(1 + k\frac{2\vec{x}_j\cdot\hat{n}}{1+|\vec{z}_j\cdot\hat{n}|^2}\right) \tag{3.45}$$

$$P_s(\hat{n}) = \frac{1}{2}\left(1 + \sum_j \frac{|\vec{x}_j\cdot\hat{n}|}{1+|\vec{z}_j|^2}\right) \tag{3.46}$$

$$P_s = \frac{1}{2} + \frac{1}{4}\sum_j \frac{|\vec{x}_j|}{1+|\vec{z}_j|^2}. \tag{3.47}$$

If we did not already know the ideal solution to the *King's Problem* we could obtain it from (3.45) by demanding that it equal either 1 or 0. We start looking for solutions of the form $\vec{z}_j = \vec{x}_j$ and since $\hat{n} \in \{\hat{n}_x, \hat{n}_y, \hat{n}_z\}$ we obtain the same equation for the three components of all four vectors

$$x_{jk} = \pm 1. \tag{3.48}$$

By requiring that the satisfy the orthogonalization condition we are led to (3.49).

This representation has a very interesting feature, all the probabilities depend exclusively on the vectors $\vec{z}_j$. If we interpret these not as the factors (3.9) but as the expansion coefficients of $|\Phi_j\rangle$ in the $\{|\Phi_0\rangle, |\Phi^{(i)}\rangle\}$ basis we acquire a powerful tool to interpret and generalize this problem. Consider the ideal solution to the original *King's Problem*, if we change basis from the computational basis to the Bell basis it takes the form

$$|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c} |E_1\rangle + |E_2\rangle + |E_3\rangle - \imath\,|E_4\rangle \\ |E_1\rangle + |E_2\rangle - |E_3\rangle + \imath\,|E_4\rangle \\ |E_1\rangle - |E_2\rangle + |E_3\rangle + \imath\,|E_4\rangle \\ |E_1\rangle - |E_2\rangle - |E_3\rangle - \imath\,|E_4\rangle \end{array}\right\} = \left\{\begin{array}{c} |\Phi_1\rangle \\ |\Phi_2\rangle \\ |\Phi_3\rangle \\ |\Phi_4\rangle \end{array}\right\}$$

or equivalently

$$|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c} |E_1\rangle + (\sigma_x^A + \sigma_y^A + \sigma_z^A)|E_1\rangle \\ |E_1\rangle + (-\sigma_x^A - \sigma_y^A + \sigma_z^A)|E_1\rangle \\ |E_1\rangle + (\sigma_x^A - \sigma_y^A - \sigma_z^A)|E_1\rangle \\ |E_1\rangle + (-\sigma_x^A + \sigma_y^A - \sigma_z^A)|E_1\rangle \end{array}\right\},$$

which is precisely of the form (3.39). By simple inspection we find the vectors z_j

$$\begin{aligned} z_1 &= (1,1,1) \\ z_2 &= (-1,-1,1) \\ z_3 &= (1,-1,-1) \\ z_4 &= (-1,1,-1). \end{aligned} \tag{3.49}$$

Recalling the role of the privileged state in this formalism we realize that the same development can be repeated for any choice of M.E.S, corresponding to the different possibilities for the initial preparation of the system in the Mean King's problem. The crucial factor here is the fact that all the probabilities depend exclusively on the the vectors $\vec{z}_j$, as such (3.49) not only represents the expansion coefficients of the solutions to the *King's Problem* for the $|E_1\rangle$ initial state but also for the other three maximally entangled states; where we must remember that every initial state (privileged state) defines it's own ordered basis. The general solution to the original Mean King's problem (for any initial M.E.S) can then be stated as

$$\Phi_{i1} = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{z}_1)\,|E_i\rangle \tag{3.50}$$

$$\Phi_{i2} = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{z}_2)\,|E_i\rangle \tag{3.51}$$

$$\Phi_{i3} = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{z}_3)\,|E_i\rangle \tag{3.52}$$

$$\Phi_{i4} = \frac{1}{2}(1+\vec{\sigma}\cdot\vec{z}_4)\,|E_i\rangle, \tag{3.53}$$

where the $\vec{z}_i$ refer to (3.49) and $i = 1\ldots 4$.

3.4 Optimization

As stated previously, we must maximize (3.47) subject to the orthogonality conditions (3.43). Applying the method of Lagrangian multipliers we get the following condition for the extrema of the average probability of success P_s

$$\nabla\left\{\left(\frac{1}{2}+\frac{1}{4}\sum_j \frac{|\vec{x}_j|}{1+|\vec{x}_j|^2+|\vec{y}_j|^2}\right)+\sum_{i<j}\left(\lambda^{ij}\,\vec{z}_i\cdot\vec{z}_j+1\right)\right\}=0. \tag{3.54}$$

Intuitively we expect that the optimal solution will be such that the $\vec{y}_j$ are ass small as possible consistent with (3.43), we will assume that they are identically zero ($\vec{z}_j = \vec{x}_j$). In this case we have the following four equations

$$\begin{aligned}
\frac{1}{4}\nabla_1\left(\frac{|\vec{x}_1|}{1+|\vec{x}_1|^2}\right) &= -\lambda^{12}\nabla_1(\vec{x}_1\cdot\vec{x}_2) - \lambda^{13}\nabla_1(\vec{x}_1\cdot\vec{x}_3) - \lambda^{14}\nabla_1(\vec{x}_1\cdot\vec{x}_4)\\
\frac{1}{4}\nabla_2\left(\frac{|\vec{x}_2|}{1+|\vec{x}_2|^2}\right) &= -\lambda^{12}\nabla_2(\vec{x}_1\cdot\vec{x}_2) - \lambda^{23}\nabla_2(\vec{x}_2\cdot\vec{x}_3) - \lambda^{24}\nabla_2(\vec{x}_2\cdot\vec{x}_4)\\
\frac{1}{4}\nabla_3\left(\frac{|\vec{x}_3|}{1+|\vec{x}_3|^2}\right) &= -\lambda^{13}\nabla_3(\vec{x}_1\cdot\vec{x}_3) - \lambda^{23}\nabla_3(\vec{x}_2\cdot\vec{x}_3) - \lambda^{34}\nabla_3(\vec{x}_3\cdot\vec{x}_4)\\
\frac{1}{4}\nabla_4\left(\frac{|\vec{x}_4|}{1+|\vec{x}_4|^2}\right) &= -\lambda^{14}\nabla_4(\vec{x}_1\cdot\vec{x}_4) - \lambda^{24}\nabla_4(\vec{x}_2\cdot\vec{x}_4) - \lambda^{34}\nabla_4(\vec{x}_3\cdot\vec{x}_4).
\end{aligned}$$

Evaluating the gradient in spherical coordinates, with $\vec{r}_i \equiv \vec{x}_i$ and $r_i = |\vec{x}_i|$, the terms on the left hand side can be simplified to

$$\begin{aligned}
\nabla_i\left(\frac{r_i}{1+r_i^2}\right) &= \frac{\partial}{\partial r_i}\left(\frac{r_i}{1+r_i^2}\right)\hat{r}_i\\
&= \left(\frac{1-r_i^2}{r_i(1+r_i^2)}\right)\vec{r}_i\\
&= \mathfrak{f}_i\,\vec{x}_i. \qquad (3.55)
\end{aligned}$$

The terms on the right hand side are easily calculated and we arrive at the following four vector equations

$$\begin{aligned}
\frac{1}{4}\mathfrak{f}_1\,\vec{x}_1 &= -\lambda^{12}\vec{x}_2 - \lambda^{13}\vec{x}_3 - \lambda^{14}\vec{x}_4\\
\frac{1}{4}\mathfrak{f}_2\,\vec{x}_2 &= -\lambda^{12}\vec{x}_1 - \lambda^{23}\vec{x}_3 - \lambda^{24}\vec{x}_4\\
\frac{1}{4}\mathfrak{f}_3\,\vec{x}_3 &= -\lambda^{13}\vec{x}_1 - \lambda^{23}\vec{x}_2 - \lambda^{34}\vec{x}_4\\
\frac{1}{4}\mathfrak{f}_4\,\vec{x}_4 &= -\lambda^{14}\vec{x}_1 - \lambda^{24}\vec{x}_2 - \lambda^{34}\vec{x}_3. \qquad (3.56)
\end{aligned}$$

The solution to this set of equations still presents some difficulties so we proceed to make one final simplification (based on our knowledge of the ideal solution to the original problem) by assuming that the norm of the four vectors is the same: $|\vec{x}_i| = r_i = r$, which in turn means that the $\mathfrak{f}_i$ are also equal. By setting $C = \mathfrak{f}/4$ we obtain

$$\begin{aligned}
C\vec{x}_1 &= -\lambda^{12}\vec{x}_2 - \lambda^{13}\vec{x}_3 - \lambda^{14}\vec{x}_4\\
C\vec{x}_2 &= -\lambda^{12}\vec{x}_1 - \lambda^{23}\vec{x}_3 - \lambda^{24}\vec{x}_4\\
C\vec{x}_3 &= -\lambda^{13}\vec{x}_1 - \lambda^{23}\vec{x}_2 - \lambda^{34}\vec{x}_4\\
C\vec{x}_4 &= -\lambda^{14}\vec{x}_1 - \lambda^{24}\vec{x}_2 - \lambda^{34}\vec{x}_3. \qquad (3.57)
\end{aligned}$$

Taking the inner product of each of these equations with the vector on the left hand side gives

$$\begin{aligned}
Cr^2 &= \lambda^{12} + \lambda^{13} + \lambda^{14}\\
Cr^2 &= \lambda^{12} + \lambda^{23} + \lambda^{24}\\
Cr^2 &= \lambda^{13} + \lambda^{23} + \lambda^{34}\\
Cr^2 &= \lambda^{14} + \lambda^{24} + \lambda^{34}, \qquad (3.58)
\end{aligned}$$

which gives the following restrictions on the Lagrange multipliers

$$\lambda^{13} = \lambda^{24},\ \lambda^{12} = \lambda^{34} \ \text{ and } \ \lambda^{23} = \lambda^{14}. \tag{3.59}$$

No we take the inner products of the first equation in (3.57) with the three vectors on the right hand side of this equation to obtain

$$\begin{aligned} -C &= -\lambda^{12} r^2 + \lambda^{13} + \lambda^{14} \\ -C &= \lambda^{12} - \lambda^{13} r^2 + \lambda^{14} \\ -C &= \lambda^{12} + \lambda^{13} - \lambda^{14} r^2, \end{aligned} \tag{3.60}$$

which leads us to

$$\lambda^{12} = \lambda^{13} = \lambda^{14} = \lambda. \tag{3.61}$$

From (3.58) we determine the value of λ, in terms of the constants r and C, to be

$$\lambda = \frac{Cr^2}{3}; \tag{3.62}$$

but from (3.60) we get

$$\lambda = \frac{C}{r^2 - 2}. \tag{3.63}$$

From these last two equations it follows that

$$r = \sqrt{3} \ \text{ and } \ \lambda = C = \frac{-1}{8\sqrt{3}}. \tag{3.64}$$

The condition for the extrema of the probability of success (3.57) can then be reduced to the following expression

$$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 + \vec{x}_4 = 0. \tag{3.65}$$

This takes us back to (3.49), and although it might not be *the* optimal solution for our version of the problem, it turns out that the solution to the original *King's Problem* is *an* optimal solution under the restrictions we imposed, which don't seem too severe when we consider how the probability of success was distributed.

Chapter 4

Conclusions

Given a measurement basis $\{|\Phi\rangle_j\}$ we have shown how a strategy for our version of the *King's Problem* can be constructed. All that needs to be done in order to choose one of the king's measurement results over the other, based on knowledge of the control measurement and the king's spin direction, is to determine the sign of $\vec{x_j} \cdot \hat{n}$. This decision function is just a way of expressing whether or not the conditional probability for a specific value of k (given $\hat{n}$ and j) is greater than (less than) $\frac{1}{2}$. We introduced a deceptively simple geometric representation for this decision function that can be used to visualize the steps involved in this retrodiction problem. Of the measurement bases we considered, only the ideal basis (and only for the EPR state for which it is actually an ideal solution) manages to take advantage of the higher dimensionality of the composite system. In all the other cases two or more values of j would end up assigned to the same value of k.

Knowing that no exact solution exists for this problem we focused our attention on trying to find or at least characterize the optimal solution. The natural way to do this was in terms of the average probability of success. We found that the ideal solution is actually a very good average solution, with a mean probability of success of 0.933 and a minimum of 0.85. The computational basis, chosen only as a reference point because of it's simplicity, doesn't turn out to be as bad as one would expect; it results in a uniform distribution for the probability of success with a mean value of 0.75. The ideal basis, when used with the "wrong" EPR state, reproduces the results of the computational basis; which is surprising since the conditional probabilities they define are different.

By working in the projective representation we were able to give the retrodiction problem, or more specifically the probabilities that define it, a geometrical interpretation that is easily generalized to higher dimensions. With this representation we were also able to express all the probabilities in terms four three-dimensional complex vectors, which allowed us to reformulate the optimization problem in a more *friendly* manner. We were able to prove that the ideal basis is not only a good average solution but that it is an optimal solution, provided some additional restrictions are imposed ($\vec{y}_j = 0$ and $|\hat{r}_i| = r$).

There are still some open questions that need to be answered in order to consider the problem solved. We were not able to find an analytical expression for the distribution of the probability of success for the ideal basis and a clear geometrical interpretation of its behavior (as evidenced by Figure 3.2.2) still eludes us. More importantly, we have still not found the optimal solution to this retrodiction problem. The first step towards this goal should be to consider vectors $\vec{z}_j$ with non-zero imaginary parts to study what happens with the optimal solution under these relaxed conditions.

Appendix A

Conditional Probability Distributions

First we give a derivation of (3.8) and then we calculate the conditional probability, as given by (3.7), for three different choices of the control measurement basis.

$$
\begin{aligned}
P(K_k|X\,\hat{n}\,J_j) &= \frac{|\langle\Phi_j|\hat{P}_k^A|E\rangle|^2}{\sum_{k'}|\langle\Phi_j|\hat{P}_{k'}^A|E\rangle|^2} \\
&= \frac{|\langle\Phi_j|E\rangle|^2 + 2k\Re(\langle E|\Phi_j\rangle\langle\Phi_j|\vec{\sigma}|E\rangle\cdot\hat{n}) + |\langle\Phi_j|\vec{\sigma}|E\rangle\cdot\hat{n}|^2}{2\left(|\langle\Phi_j|E\rangle|^2 + |\langle\Phi_j|\vec{\sigma}|E\rangle\cdot\hat{n}|^2\right)}
\end{aligned}
$$

provided $\langle\Phi_j|E\rangle \neq 0$, we can factor out this term and obtain

$$
\begin{aligned}
P(K_k|X\,\hat{n}\,J_j) &= \frac{1}{2}\,\frac{1 + 2k\Re\left(\frac{\langle E|\Phi_j\rangle\langle\Phi_j|\vec{\sigma}|E\rangle}{|\langle\Phi_j|E\rangle|^2}\cdot\hat{n}\right) + \left|\frac{\langle\Phi_j|\vec{\sigma}|E\rangle}{\langle\Phi_j|E\rangle}\cdot\hat{n}\right|^2}{1 + \left|\frac{\langle\Phi_j|\vec{\sigma}|E\rangle}{\langle\Phi_j|E\rangle}\cdot\hat{n}\right|^2} \\
&= \frac{1}{2}\left\{1 + k\frac{2\Re(\vec{z})\cdot\hat{n}}{1 + |\vec{z}\cdot\hat{n}|^2}\right\},
\end{aligned}
$$

where

$$
\vec{z} = \frac{\langle\Phi_j|\vec{\sigma}|E\rangle}{\langle\Phi_j|E\rangle}.
$$

Next we obtain explicit analytic expressions for these conditional probabilities, in terms of $\hat{n}, j,$ and k for the different possible initial states of the system (the four *EPR* states). Since expression (3.8) is not valid universally we will work instead with expression (3.7). It is necessary therefore, to evaluate expressions of the form

$$
|\langle\Phi_j|\hat{P}_k^A|E\rangle|^2.
$$

We start by recalling the expansion of the *EPR* states in the computational basis,

$$
|E\rangle = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \\ |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \end{array}\right\} = \left\{\begin{array}{c} |E_1\rangle \\ |E_2\rangle \\ |E_3\rangle \\ |E_4\rangle \end{array}\right\}
$$

and the effect of operating on them with the Pauli matrices

$$\sigma_x^A|E\rangle = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \\ |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \end{array}\right\} = \left\{\begin{array}{r} |E_3\rangle \\ -|E_4\rangle \\ |E_1\rangle \\ -|E_2\rangle \end{array}\right\}$$

$$\sigma_y^A|E\rangle = \frac{i}{\sqrt{2}}\left\{\begin{array}{c} |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \\ |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \end{array}\right\} = i\left\{\begin{array}{r} -|E_4\rangle \\ |E_3\rangle \\ -|E_2\rangle \\ |E_1\rangle \end{array}\right\}$$

$$\sigma_z^A|E\rangle = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \\ |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \end{array}\right\} = \left\{\begin{array}{c} |E_2\rangle \\ |E_1\rangle \\ |E_4\rangle \\ |E_3\rangle \end{array}\right\}.$$

The result of projecting the initial state over the spin k eigenstate in the $\hat{n}$ direction is

$$\begin{aligned} \hat{P}_k^A|E\rangle &= \frac{1}{2}\left(\mathbb{I}^A + k\left(n_x\sigma_x^A + n_y\sigma_y^A + n_z\sigma_z^A\right)\right) \\ &= \frac{1}{2}\left\{\begin{array}{c} |E_1\rangle \\ |E_2\rangle \\ |E_3\rangle \\ |E_4\rangle \end{array}\right\} + \frac{k}{2}\left\{\begin{array}{c} n_x|E_3\rangle - i\,n_y|E_4\rangle + n_z|E_2\rangle \\ -n_x|E_4\rangle + i\,n_y|E_3\rangle + n_z|E_1\rangle \\ n_x|E_1\rangle - i\,n_y|E_2\rangle + n_z|E_4\rangle \\ -n_x|E_2\rangle + i\,n_y|E_1\rangle + n_z|E_3\rangle \end{array}\right\}. \end{aligned}$$

A.1 Computational Basis

$$|\Phi_j\rangle = \left\{\begin{array}{c} |\uparrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \end{array}\right\} = \left\{\begin{array}{c} |\Phi_1\rangle \\ |\Phi_2\rangle \\ |\Phi_2\rangle \\ |\Phi_4\rangle \end{array}\right\}$$

- $|E\rangle = |E_1\rangle$

$$\langle\Phi_j|\hat{P}_k^A|E_1\rangle = \frac{1}{2\sqrt{2}}\left\{\begin{array}{c} 1 + kn_z \\ 1 - kn_z \\ k(n_x - i\,n_y) \\ k(n_x + i\,n_y) \end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_1\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c} (1 + kn_z)^2 \\ (1 - kn_z)^2 \\ n_x^2 + n_y^2 \\ n_x^2 + n_y^2 \end{array}\right\}$$

- $|E\rangle = |E_2\rangle$

$$\langle\Phi_j|\hat{P}_k^A|E_2\rangle = \frac{1}{2\sqrt{2}}\left\{\begin{array}{c} 1 + kn_z \\ -1 + kn_z \\ k(-n_x + i\,n_y) \\ k(n_x + i\,n_y) \end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_2\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c}(1+kn_z)^2\\(1-kn_z)^2\\n_x^2+n_y^2\\n_x^2+n_y^2\end{array}\right\}$$

- $|E\rangle = |E_3\rangle$

$$\langle\Phi_j|\hat{P}_k^A|E_3\rangle = \frac{1}{2\sqrt{2}}\left\{\begin{array}{c}k(n_x - i\,n_y)\\k(n_x + i\,n_y)\\1+in_z\\i-in_z\end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_3\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c}n_x^2+n_y^2\\n_x^2+n_y^2\\(1+kn_z)^2\\(1-kn_z)^2\end{array}\right\}$$

- $|E\rangle = |E_4\rangle$

$$\langle\Phi_j|\hat{P}_k^A|E_4\rangle = \frac{1}{2\sqrt{2}}\left\{\begin{array}{c}k(-n_x + i\,n_y)\\k(n_x + i\,n_y)\\1+kn_z\\-k+kn_z\end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_4\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c}n_x^2+n_y^2\\n_x^2+n_y^2\\(1+kn_z)^2\\(1-kn_z)^2\end{array}\right\}$$

A.2 Ideal Basis

$$|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c}\sqrt{2}|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle e^{-\imath\pi/4} + |\downarrow\uparrow\rangle e^{\imath\pi/4}\\\sqrt{2}|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle e^{-\imath\pi/4} - |\downarrow\uparrow\rangle e^{\imath\pi/4}\\\sqrt{2}|\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle e^{\imath\pi/4} + |\downarrow\uparrow\rangle e^{-\imath\pi/4}\\\sqrt{2}|\downarrow\downarrow\rangle - |\uparrow\downarrow\rangle e^{\imath\pi/4} - |\downarrow\uparrow\rangle e^{-\imath\pi/4}\end{array}\right\} = \left\{\begin{array}{c}|\Phi_1\rangle\\|\Phi_2\rangle\\|\Phi_3\rangle\\|\Phi_4\rangle\end{array}\right\}$$

- $|E\rangle = |E_1\rangle$

$$\langle E_1|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{4}\left\{\begin{array}{c}1+k(n_x+n_y+n_z)\\1+k(-n_x-n_y+n_z)\\1+k(n_x-n_y-n_z)\\1+k(-n_x+n_y-n_z)\end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_1\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c}1+k(n_x+n_y+n_z)+n_xn_y+n_xn_z+n_yn_z\\1+k(-n_x-n_y+n_z)+n_xn_y-n_xn_z-n_yn_z\\1+k(n_x-n_y-n_z)-n_xn_y-n_xn_z+n_yn_z\\1+k(-n_x+n_y-n_z)-n_xn_y+n_xn_z-n_yn_z\end{array}\right\}$$

- $|E\rangle = |E_2\rangle$

$$\langle E_2|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{4}\left\{\begin{array}{c} 1 + kn_z + i\ \{k(n_x - n_y)\} \\ 1 + kn_z + i\ \{k(-n_x + n_y)\} \\ -1 + kn_z + i\ \{k(-n_x - n_y)\} \\ -1 + kn_z + i\ \{k(n_x + n_y)\} \end{array}\right\}$$

$$|\langle \Phi_j|\hat{P}_k^A|E_2\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c} 1 + kn_z - n_x n_y \\ 1 + kn_z - n_x n_y \\ 1 - kn_z + n_x n_y \\ 1 - kn_z + n_x n_y \end{array}\right\}$$

- $|E\rangle = |E_3\rangle$

$$\langle E_3|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{4}\left\{\begin{array}{c} 1 + kn_x + i\ \{k(n_y - n_z)\} \\ -1 + kn_x + i\ \{k(n_y + n_z)\} \\ 1 + kn_x + i\ \{k(-n_y + n_z)\} \\ -1 + kn_x + i\ \{k(-n_y - n_z)\} \end{array}\right\}$$

$$|\langle \Phi_j|\hat{P}_k^A|E_3\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c} 1 + kn_x - n_y n_z \\ 1 - kn_x + n_y n_z \\ 1 + kn_x - n_y n_z \\ 1 - kn_x + n_y n_z \end{array}\right\}$$

- $|E\rangle = |E_4\rangle$

$$\langle E_4|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{4}\left\{\begin{array}{c} k(-n_x + n_z) + i\,(-1 - kn_y) \\ k(-n_x - n_z) + i\,(1 - kn_y) \\ k(n_x + n_z) + i\,(1 - kn_y) \\ k(n_x - n_z) + i\,(-1 - kn_y) \end{array}\right\}$$

$$|\langle \Phi_j|\hat{P}_k^A|E_4\rangle|^2 = \frac{1}{8}\left\{\begin{array}{c} 1 + kn_y - n_x n_z \\ 1 - kn_y + n_x n_z \\ 1 - kn_y + n_x n_z \\ 1 + kn_y - n_x n_z \end{array}\right\}$$

A.3 Bell Basis

$$|\Phi_j\rangle = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \\ |\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle \\ |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \end{array}\right\} = \left\{\begin{array}{c} |E_1\rangle \\ |E_2\rangle \\ |E_3\rangle \\ |E_4\rangle \end{array}\right\}$$

- $|E\rangle = |E_1\rangle$

$$\langle E_1|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c} 1 \\ kn_z \\ kn_x \\ -i(kn_y) \end{array}\right\}$$

$$|\langle \Phi_j|\hat{P}_k^A|E_1\rangle|^2 = \frac{1}{4}\left\{\begin{array}{c} 1 \\ n_z^2 \\ n_x^2 \\ n_y^2 \end{array}\right\}$$

- $|E\rangle = |E_2\rangle$

$$\langle E_2|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c} kn_z \\ 1 \\ i(kn_y) \\ -kn_x \end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_1\rangle|^2 = \frac{1}{4}\left\{\begin{array}{c} n_z^2 \\ 1 \\ n_y^2 \\ n_x^2 \end{array}\right\}$$

- $|E\rangle = |E_3\rangle$

$$\langle E_3|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c} kn_x \\ -i(kn_y) \\ 1 \\ kn_z \end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_3\rangle|^2 = \frac{1}{4}\left\{\begin{array}{c} n_x^2 \\ n_y^2 \\ 1 \\ n_z^2 \end{array}\right\}$$

- $|E\rangle = |E_4\rangle$

$$\langle E_4|\hat{P}_k^A|\Phi_j\rangle = \frac{1}{2}\left\{\begin{array}{c} i(kn_y) \\ -kn_x \\ kn_z \\ 1 \end{array}\right\}$$

$$|\langle\Phi_j|\hat{P}_k^A|E_1\rangle|^2 = \frac{1}{4}\left\{\begin{array}{c} n_y^2 \\ n_x^2 \\ n_z^2 \\ 1 \end{array}\right\}$$

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