A Noncommutative de Sitter Space

Juan Felipe Ariza Mejía

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Advisor: Prof. Andrés F. Reyes

Co-advisor: Prof. Mitsuru Wilson

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Abstract

The noncommutative nature of quantum mechanics and the geometrical nature of general relativity suggests noncommutative geometry as a possible meeting ground. In this work a version of a noncommutative de Sitter space is introduced. As a motivation for this some algebraic properties of differential geometry are discussed, and the classical de Sitter space and some of its properties are presented. The noncommutative catenoid of Arnlind and Holm is introduced as a guiding example for the noncommutative de Sitter space.

Resumen

La naturaleza no conmutativa de la mecánica cuántica y la naturaleza geométrica de la relatividad general sugieren adoptar la geometría no conmutativa como un posible punto de encuentro. En este trabajo se introduce una versión no conmutativa del espacio de de Sitter. Como motivación para esto, se discuten algunas propiedades algebráicas de la geometría diferencial y se presenta el espacio clásico de de Sitter junto con algunas de sus propiedades. Se introduce el catenoide no conmutativo de Arnlind y Holm como guía para la construcción del espacio de de Sitter no conmutativo.

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CHAPTER 1

Introduction

> A manifold corresponds naturally to our intuitive ideas of the continuity of space and time. So far this continuity has been established for distances down to about $10^{-15}cm$ by experiments on pion scattering (Foley *et al.* (1967)). It may be difficult to extend this to much smaller lengths as to do so would require a particle of such high energy that several other particles might be created and confuse the experiment. Thus it may be that a manifold model for space-time is inappropriate for distances less than $10^{-15}cm$ and that we should use theories in which space-time has some other structure on this scale. [**HE08**]
>
> Stephen W. Hawking
> George F. R. Ellis

The natural setting of classical and relativistic mechanics is a manifold, and the language of these theories is that of differential geometry. A manifold can be thought of as describing a configuration space. Then its tangent bundle represents the state space with coordinates of position and velocity, while its cotangent bundle represents the phase space of a system with coordinates which represent position and momentum. In this settings the usual Hamiltonian formalism and the Lagrangian formalism of classical mechanics is defined, as well as the gravitational theory of general relativity.

However, this does not fit the framework of quantum mechanics nor quantum field theory. In differential geometry, real (or complex) valued functions on the tangent or cotangent bundle are commutative. For example, when considering (classically) a particle moving in a straight line bound by some potential, the observables of position and momentum are functions on the cotangent bundle of $\mathbb{R}$. As expected, these observables commute. The case of quantum mechanics is quite the opposite since many observables do not commute.

This noncommutativity of observables is at the essence of quantum mechanics and quantum field theories as opposed to classical and relativistic mechanics. A possible approach to reconcile this differences is found in noncommutative geometry.

Many notions of classical geometry can be formulated in as properties of a commutative algebra. A concern of noncommutative geometry is how these notions can be generalized and interpreted when the requirement of commutativity is dropped.

The purpose of this work is to study a noncommutative version of an Einstein manifold. In particular, we will be interested in de Sitter space, since it exhibits many symmetries and interesting properties. As a long term goal, we are interested in studying quantum fields in a noncommutative space.

The structure of this work is as follows. In Chapter 2 a brief discussion of differential geometry is undertaken. The main reason to do this is to show the connections between algebra and geometry which are at the basis of noncommutative geometry. In Chapter 3 the classical (commutative) de Sitter space is studied. Some interesting properties of this space are discussed and a discussion of representations of its isometry group is presented. Chapter 4 presents a discussion of a recent work in noncommutative geometry. Namely, the noncommutative catenoid developed in [**AH17**] is discussed. The constructions presented in this Chapter serve as a guide in Chapter 5 where an approach to a noncommutative de Sitter space is presented.

CHAPTER 2

Differential geometry

In this Chapter various definitions and constructions of differential geometry shall be introduced in order to motivate the constructions of noncommutative geometry. The presentation follows the exposition of Wurzbacher [**Wur01**] and that of Tu [**Tu11**].

1. Differentiable Manifolds

DEFINITION 2.1. A *locally Euclidean space* is a topological space (M, τ) such that for every $p \in M$ there exists $m \in \mathbb{N}$, $U \subset M$ open neighbourhood of p, $V \subset \mathbb{R}^m$ open, and $\varphi : U \longrightarrow V$ homeomorphism. The maps φ are called *coordinate charts*. An *atlas of* M, $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ for some index set $\mathcal{I}$, is an open covering $\{U_\alpha : \alpha \in \mathcal{I}\}$ of M together with coordinate charts $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$.

REMARK 2.2. If $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is non-empty, then one defines $\varphi_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}\Big|_{\varphi_\beta(U_{\alpha\beta})}$. This map is a homeomorphism of open subsets of $\mathbb{R}^m$ and is called a *change of coordinates*.

In the spaces that will interest us, one shall impose further conditions (which are commonly required in the definition of a manifold). More precisely, we will require separability of points by open neighbourhoods, and the existence of a countable topological basis.

DEFINITION 2.3. A *topological manifold* M is a Hausdorff, second countable and locally euclidean space.

REMARK 2.4. The definitions 2.1 and 2.3 include the possibility of having different m's at different points. A topological manifold for which m is fixed is called *pure-dimensional* (of dimension m).

DEFINITION 2.5. Let M be a topological manifold and $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ an atlas of M. $\mathfrak{U}$ is called a *smooth atlas* if $\varphi_{\alpha\beta}$ is of class C^∞, whenever $\varphi_{\alpha\beta}$ is defined.

Remark 2.6. We can define an equivalence relation between smooth atlases as follows: let $\mathfrak{V} = \{(V_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}_1\}$ and $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}_2\}$ be smooth atlases on M, $\mathfrak{V}$ is said to be equivalent to $\mathfrak{U}$ if $\mathfrak{U} \cup \mathfrak{V}$ is a smooth atlas on M. This relation is clearly symmetric and reflexive. Transitivity can be proved as follows :

Let $\mathfrak{V} = \{(V_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}_1\}$, $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}_2\}$ and $\mathfrak{W} = \{(W_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}_3\}$ be smooth atlases on M such that $\mathfrak{V}$ is equivalent to $\mathfrak{U}$ and $\mathfrak{U}$ is equivalent to $\mathfrak{W}$. Pick $\alpha \in \mathcal{I}_1$ and $\beta \in \mathcal{I}_3$. For every $p \in U_{\alpha\beta} = V_\alpha \cap W_\beta$, exists $\gamma \in \mathcal{I}_2$ such that $p \in U_\gamma$. Define $U_{\alpha\beta\gamma} = V_\alpha \cap U_\gamma \cap W_\beta$ then $\varphi_{\alpha\beta}\Big|_{\varphi_\beta(U_{\alpha\beta\gamma})} = \varphi_\alpha \circ \varphi_\gamma^{-1} \circ \varphi_\gamma \circ \varphi_\beta^{-1}$ is C^∞.

Definition 2.7. A *differentiable* (also smooth or C^∞) *structure* on a topological manifold M is an equivalence class of smooth atlases on M. A *differentiable manifold* is a topological manifold M together with a differentiable structure.

Remark 2.8. The union of all atlases on a differentiable structure is a smooth atlas. It is called the *maximal smooth atlas* of the differentiable structure. Any chart contained in the maximal atlas is called an *admissible chart* (on the differentiable manifold).

Remark 2.9. Let M be a differentiable manifold and $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ a smooth atlas compatible with the differentiable structure. If $\Omega \subset M$ is open, then Ω (with the subspace topology) is a topological manifold and $\{(U_\alpha \cap \Omega, \varphi_\alpha|_\Omega) : \alpha \in \mathcal{I}\}$ is a smooth atlas on Ω. The differentiable structure on Ω is independent of the choice of atlas.

Remark 2.10. Let M, N be differentiable manifolds, and $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}_M\}, \mathfrak{V} = \{(V_\beta, \psi_\beta) : \beta \in \mathcal{I}_N\}$ be smooth atlases compatible with the differentiable structure of M and N respectively. Then $\mathfrak{W} = \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta) : \alpha \in \mathcal{I}_M, \beta \in \mathcal{I}_N\}$ is a smooth atlas on $M \times N$. The differentiable structure in $M \times N$ is independent of the choice of atlases.

Definition 2.11. Let M, N be differentiable manifolds and $F : M \longrightarrow N$ be a continuous map. F is called differentiable if for every $p \in M$ and for every admissible charts (U_p, φ) of M and $(U_{F(p)}, \psi)$ of N such that $p \in U_p$ and $F(p) \in U_{F(p)}$ the maps $\psi \circ F \circ \varphi^{-1}\big|_{\varphi(U_p \cap F^{-1}(U_{F(p)}))}$ is of class C^∞.

Remark 2.12. The differentiability of the change of coordinates maps allows to check differentiability with a single admissible chart of M and a single admissible chart of N

for each $p \in M$. Also, as differentiability is defined in terms of maps $\mathbb{R}^m \longrightarrow \mathbb{R}^n$, the differentiability of the composition of smooth functions follows.

DEFINITION 2.13. Let M, N be smooth manifolds and $F : M \longrightarrow N$ be a smooth function. F is called a *diffeomorphism* if it is bijective and its inverse is a smooth function.

An important tool in describing a manifold will be the set of smooth functions on the manifold.

DEFINITION 2.14. Let M be a smooth manifold. We will denote the set of continuous functions on M by $C^0(M)$. The set of smooth functions on M will be denoted by $\mathcal{E}(M)$.

PROPOSITION 2.15. *$C^0(M)$ and $\mathcal{E}(M)$, are commutative, associative, unital $\mathbb{R}-$ algebras, with the algebra operations being defined pointwise.*

PROOF. For real valued functions on $\mathbb{R}^m$ the sum, product and scalar multiple of functions preserves continuity and differentiability. The equivalent is true in M since it is locally euclidean. ■

2. Vector Bundles

DEFINITION 2.16. A *smooth real [respectively complex] vector bundle* of rank r over a manifold M is a smooth manifold E and a smooth projection $\pi : E \longrightarrow M$ such that each fibre of π, $E_p := \pi^{-1}(p)$, has the structure of a real [respectively complex] vector space of dimension r (for every $p \in M$) and such that E is locally trivial, that is:

For every $p \in M$, exists some $U \subset M$ open neighbourhood of p and a diffeomorphism $\Psi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{K}^r$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that $pr_1 \circ \Psi_U = \pi$ and such that for every $x \in U$ the restriction to the fibre, $\Psi_U|_{E_x}$, is a vector space isomorphism.

REMARK 2.17. A smooth vector bundle will be denoted as $\pi : E \longrightarrow M$. The space E is called *total space* and M the *base space.*

DEFINITION 2.18. Let $\pi : E \longrightarrow M$ be a smooth vector bundle. A *smooth section* of E is a smooth map $s : M \longrightarrow E$ such that $\pi \circ s = id_M$. The set of all smooth sections of E will be denoted by $\Gamma(M, E)$.

Remark 2.19. The $\mathbb{K}$ vector space structure of the fibres of E induces a $\mathbb{K}$ vector space structure on $\Gamma(M, E)$. Since sections can be added together and multiplied by smooth functions, $\Gamma(M, E)$ is an $\mathcal{E}(M)$ module.

Definition 2.20. Let $\pi_1 : E_1 \longrightarrow M_1$ and $\pi_2 : E_2 \longrightarrow M_2$ be smooth vector bundles and $f : M_1 \longrightarrow M_2$ be a smooth map. A smooth map $F : E_1 \longrightarrow E_2$ is called a *smooth vector bundle homomorphism* (over f) if for every $x \in M_1 \quad F((E_1)_x) \subset (E_2)_{f(x)}$ and $F|_{(E_1)_x} : (E_1)_x \longrightarrow (E_2)_{f(x)}$ is $\mathbb{K}$ linear. If f is a diffeomorphism and $F|_{(E_1)_x}$ is a vector space isomorphism for every $x \in M_1$, F is called a *vector bundle isomorphism.*

Remark 2.21. A vector bundle is called *trivial* if it is isomorphic to the *trivial bundle* $pr_1 : M \times \mathbb{K} \longrightarrow M$.

Definition 2.22. Let $\pi : E \longrightarrow M$ be a vector bundle of rank r and $\{U_\alpha : \alpha \in \mathcal{I}\}$ be an open covering of M together with local trivializations $\Psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{K}$ (diffeomorphisms). If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq 0$, the map $\Psi_\alpha \circ \Psi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{K}^r \longrightarrow U_{\alpha\beta} \times \mathbb{K}^r$ such that $(x, \vec{v}) \longmapsto (x, g_{\alpha\beta}(x) \cdot \vec{v})$ defines a map $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL(r, \mathbb{K})$ called the *transition functions* of E with respect to the local trivialization $\{U_\alpha : \alpha \in \mathcal{I}\}$.

Remark 2.23. The family of transition functions $\{g_{\alpha\beta} : \alpha, \beta \in \mathcal{I}\}$ fulfils the following identities (called cocycle identities):

i. $g_{\alpha\alpha}(x) = Id \quad$ for every $x \in U_\alpha$.
ii. $(g_{\alpha\beta}(x))^{-1} = g_{\beta\alpha}(x) \quad$ for every $x \in U_{\alpha\beta}$.
iii. $g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = Id \quad$ for every $x \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$.

Proposition 2.24. *Let M be a smooth manifold. If $\{U_\alpha : \alpha \in \mathcal{I}\}$ is an open covering of M and $\{g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL(r, \mathbb{K})\}$ is a family of maps fulfilling the cocycle conditions of remark 2.23, then the set $E := (\coprod_{\alpha \in \mathcal{I}} U_\alpha \times \mathbb{K}^r)/_\sim$ (where $(x, \vec{v}) \in U_\alpha \times \mathbb{K}^r$ and $(y, \vec{w}) \in U_\beta \times \mathbb{K}^r$ are said to be equivalent if $x = y \in U_{\alpha\beta}$ and $g_{\beta\alpha}(x) \cdot \vec{v} = \vec{w}$) together with the projection onto the first factor is a vector bundle of rank r over M.*

Proposition 2.25. *If $\pi : E \longrightarrow M$ is a vector bundle of rank r and $\{g_{\alpha\beta} : \alpha, \beta \in \mathcal{I}\}$ is a set of transition functions with respect to a trivializing cover $\{U_\alpha : \alpha \in \mathcal{I}\}$, then E is isomorphic as a vector bundle to $(\coprod_{\alpha \in \mathcal{I}} U_\alpha \times \mathbb{K}^r)/_\sim$.*

Proposition 2.26. *Under the same hypothesis of proposition 2.25, a smooth section s of E defines (and is defined by) a family of smooth maps $s_\alpha : U_\alpha \longrightarrow \mathbb{K}^r$ such that $s_\alpha(x) = g_{\alpha\beta}(x) \cdot s_\beta(x)$ for every $x \in U_{\alpha\beta}$ and every $\alpha, \beta \in \mathcal{I}$ such that $U_{\alpha\beta} \neq 0$.*

Remark 2.27. One can define common multilinear algebra constructions (as direct sum, tensor product, dual space, etc.) fibrewise and extend by the transition functions.

Remark 2.28. Let M, N be smooth manifolds, $\pi : E \longrightarrow N$ a smooth vector bundle and $f : M \longrightarrow N$ a smooth map. The pullback of the vector bundle is defined by $f^*E = \{(x, v) \in M \times E : f(x) = \pi(v)\}$ which is a closed submanifold of $M \times E$ and the projection $f^*\pi : f^*E \longrightarrow M$ is defined by $(f^*\pi)(x, v) = x$.

3. Germs of Differentiable Functions

Before introducing a particular (and very useful) example of a vector bundle (namely the tangent bundle) we shall introduce the concept of germs of functions. The locality of derivatives implies that two functions which agree in an open neighbourhood of a point will have the same differential at that point. This notion shall be made explicit as follows.

Definition 2.29. Let M be a smooth manifold and $p \in M$. Two smooth real-valued functions $f : U_f \longrightarrow \mathbb{R}$ and $g : U_g \longrightarrow \mathbb{R}$, with U_f ad U_g open neighbourhoods of p, are called equivalent, and denoted $f \sim_p g$, if there is an open neighbourhood V of p such that $V \subset U_f \cap U_g$ and $f|_V = g|_V$. An equivalence class $[f]_{\sim_p}$ of smooth real-valued functions at p on M is called a *germ* and will be denoted by f. The set of germs at p will be denoted by $\mathcal{E}_p(M)$.

Remark 2.30. The locality of germs implies $p \in U \subset M$ such that U open, then $\mathcal{E}_p(M) = \mathcal{E}_p(U)$. The pointwise addition and multiplication allows an algebraic structure to be defined on $\mathcal{E}_p(M)$, namely:

Proposition 2.31. *$\mathcal{E}_p(M)$ is an associative, commutative and unital $\mathbb{R}-$algebra.*

Proof. Algebraic operations on the functions are defined pointwise (same as for $\mathcal{E}(M)$ and the properties follow from the algebra structure of $\mathbb{R}$. The operations are well defined, since two representatives of an equivalence class always agree on a small enough open neighbourhood of p. ■

Definition 2.32. A linear map $D : \mathcal{E}_p(M) \longrightarrow \mathbb{R}$ is called a *scalar-valued derivation* of $\mathcal{E}_p(M)$ if it satisfies Leibniz rule (with respect to the evaluation map), that is: if $f, g \in \mathcal{E}_p(M)$, then $D(f \cdot g) = D(f) \cdot g(p) + f(p) \cdot D(g)$. The set of all scalar-valued derivations of $\mathcal{E}_p(M)$ is denoted by $Der(\mathcal{E}_p(M), \mathbb{R})$.

Remark 2.33. $Der(\mathcal{E}_p(M), \mathbb{R})$ is a vector space. Denoting the constant unit-valued function by $\mathbb{1} : M \longrightarrow \{1\}$, $\mathbb{1} \cdot \mathbb{1} = \mathbb{1}$ implies $2D(\mathbb{1}) = D(\mathbb{1}) = 0$. Therefore $D(f) = D(f - f(p) \cdot \mathbb{1})$.

Proposition 2.34. *Let M be a smooth manifold and $p \in M$. The kernel of the evaluation map at p, $m_p := \{f \in \mathcal{E}_p(M) : f(p) = 0\}$, and the set $m_p^2 := span\{fg : f, g \in m_p\}$ are both two sided ideals of the algebra $\mathcal{E}_p(M)$. There exists a vector space isomorphism $Der(\mathcal{E}_p(M), \mathbb{R}) \simeq Hom_{\mathbb{R}}({}^{m_p}\!/_{m_p^2}, \mathbb{R})$.*

Proof. Both m_p and m_p^2 are linear subspaces of $\mathcal{E}_p(M)$, and for every $z \in \mathcal{E}_p(M)$ and every $f, g \in m_p$ we have $(zf)(p) = 0 = (fz)(0)$, hence $(zfg), (fgz) \in m_p^2$, so they are ideals.

For every $f \in \mathcal{E}_p(M)$ there is a unique decomposition $f = (f(p)\mathbb{1}) + (f - f(p)\mathbb{1}) \in \mathbb{R} \cdot \mathbb{1} \oplus m_p$, and for every $D \in Der(\mathcal{E}_p(M), \mathbb{R})$ Leibniz rule implies $D(m_p^2) = 0$. Therefore the linear map $D \in Der(\mathcal{E}_p(M), \mathbb{R})$ factors to a linear map $\overline{D} : {}^{m_p}\!/_{m_p^2} \longrightarrow \mathbb{R}$. The map $D \mapsto \overline{D}$ is linear since $D|_{m_p}$ factors to the quotient. Injectivity follows from the decomposition $\mathcal{E}_p(M) = \mathbb{R} \cdot \mathbb{1} \oplus m_p$ and the fact that derivative of constant is 0.

In order to prove surjectivity let $\overline{D} \in Hom_{\mathbb{R}}({}^{m_p}\!/_{m_p^2}, \mathbb{R})$, there is a bijective correspondence to a $\phi \in Hom_{\mathbb{R}}(m_p, \mathbb{R})$ such that $\phi(m_p^2) = 0$. The unique decomposition $z = z(p)\mathbb{1} + f \in \mathcal{E}_p(M)$ such that $f \in m_p$ allows ϕ to be extended to $D : \mathcal{E}_p(M) \longrightarrow \mathbb{R}$ such that $D(z) = \phi(f)$ which is linear. Let $z = z(p) + f$ and $y = y(p) + g \in \mathcal{E}_p(M)$, then $D(yz) = D(y(p)z(p) + y(p)f + g \cdot z(p) + gf) = D(y(p)f) + D(g \cdot z(p)) = y(p)D(f) + D(g)z(p)$, therefore $D \in Der(\mathcal{E}_p(M), \mathbb{R})$. ■

4. Tangent Bundle

Definition 2.35. Let M be a smooth manifold and $p \in M$. A *smooth curve* at p is a smooth map $\gamma : I \longrightarrow M$, where I is an interval $0 \in I \subset \mathbb{R}$, such that $\gamma(0) = p$. Let (U, φ) be a chart of M such that $p \in U$, then two curves γ_1, γ_2 at p are called *tangent with respect to the chart* (U, φ) if $\frac{d}{dt}\big|_0 (\varphi \circ \gamma_1)(t) = \frac{d}{dt}\big|_0 (\varphi \circ \gamma_2)(t)$.

Lemma 2.36. *Let M be a smooth manifold and $(U_1, \varphi_1), (U_2, \varphi_2)$ two charts on M such that $p \in U_1 \cap U_2$, then two curves γ_1, γ_2 at p are tangent with respect to (U_1, φ_1) if and only if they are tangent with respect to (U_2, φ_2).*

Proof. Assume γ_1, γ_2 are tangent with respect to (U_1, φ_1), then $\frac{d}{dt}\big|_0 (\varphi_2 \circ \gamma_1)(t) = \frac{d}{dt}\big|_0 (\varphi_2 \circ \varphi_1^{-1} \circ \varphi_1 \circ \gamma_1)(t) = (D_{\varphi_1(p)}(\varphi_2 \circ \varphi_1^{-1})) \cdot \frac{d}{dt}\big|_0 (\varphi_1 \circ \gamma_1)(t) = (D_{\varphi_1(p)}(\varphi_2 \circ \varphi_1^{-1})) \cdot \frac{d}{dt}\big|_0 (\varphi_1 \circ \gamma_2)(t) = \frac{d}{dt}\big|_0 (\varphi_2 \circ \varphi_1^{-1} \circ \varphi_1 \circ \gamma_2)(t) = \frac{d}{dt}\big|_0 (\varphi_2 \circ \gamma_2)(t)$. ■

Definition 2.37. Let M be a smooth manifold. Two curves at p are called equivalent if they are tangent (with respect to any chart). The set of equivalence classes of curves at p is called the *tangent space to M at p*, and is denoted $T_pM := \{[\gamma]_p : \gamma \text{ curve at } p\}$. The disjoint union of all tangent spaces to M is called the tangent bundle $TM := \coprod_{p \in M} T_pM$.

Remark 2.38. Let $f : M \longrightarrow N$ be a smooth map of differentiable manifolds, $p \in M$, (U_p, φ_p) a chart of M, and $(U_{f(p)}, \varphi_{f(p)})$ a chart of N such that $p \in U_p$ and $f(p) \in U_{f(p)}$. If γ_1, γ_2 are two equivalent curves at p, then $\frac{d}{dt}\big|_0 (\varphi_{f(p)} \circ f \circ \gamma_1)(t) = \frac{d}{dt}\big|_0 (\varphi_{f(p)} \circ f \circ \varphi_p^{-1} \circ \varphi_p \circ \gamma_1)(t) = (D_{\varphi_p(p)}(\varphi_{f(p)} \circ f \circ \varphi_p^{-1})) \cdot \frac{d}{dt}\big|_0 (\varphi_p \circ \gamma_1)(t) = (D_{\varphi_p(p)}(\varphi_{f(p)} \circ f \circ \varphi_p^{-1})) \cdot \frac{d}{dt}\big|_0 (\varphi_p \circ \gamma_2)(t)$. Therefore, $f \circ \gamma_1$ and $f \circ \gamma_2$ are equivalent curves at $f(p)$.

Definition 2.39. Let $f : M \longrightarrow N$ be a smooth map of differentiable manifolds, the map $T_pf : T_pM \longrightarrow N$ sending $[\gamma]_p \mapsto [f \circ \gamma]_{f(p)}$ is called the *tangent of f at p*. The maps T_pf at each p induce a map $Tf : TM \longrightarrow TN$.

Lemma 2.40. *Let L, M, N be smooth manifolds and $f : M \longrightarrow N$, $g : L \longrightarrow M$ smooth maps, then $T(f \circ g) = (Tf) \circ (Tg)$.*

Proof. $\forall p \in L \quad T_p(f \circ g)[\gamma]_p = [f \circ g \circ \gamma]_{f(g(p))} = T_{g(p)}f[g \circ \gamma]_{g(p)} = (T_{g(p)}f)(T_pg[\gamma]_p)$. ■

Lemma 2.41. *Let $f : M \longrightarrow N$ be a diffeomorphism of smooth manifolds, then Tf is bijective and $(Tf)^{-1} = T(f^{-1})$. Moreover, $T\,Id_M = Id_{TM}$.*

Proof. $\forall p \in M \quad T_p(f^{-1}\circ f)[\gamma]_p = [f^{-1}\circ f\circ\gamma]_p = [\gamma]_p = (T_{f(p)}f^{-1})\circ(T_pf)[\gamma]_p$. Also $\forall q \in N \quad T_q(f\circ f^{-1})[\eta]_q = [f\circ f^{-1}\circ\eta]_q = [\eta]_q = (T_{f^{-1}(q)}f)\circ(T_qf^{-1})[\eta]_q$. From the two identities it follows $(Tf)^{-1} = T(f^{-1})$. ∎

Lemma 2.42. *Let $V \subset \mathbb{R}^m$ open and $p \in V$. For every $w \in \mathbb{R}^m$ define $\gamma_w(t) = p + tw$ (for a sufficiently small $\varepsilon_w > 0 \quad \gamma_w :]-\varepsilon_w, \varepsilon_w[\longrightarrow V$), then:*

i. $[\gamma_w]_p = [\gamma_v]_p$ *if and only if* $w = v$.

ii. *The map* $\chi_p : \mathbb{R}^m \longrightarrow T_pV$ *such that* $w \mapsto [\gamma_w]_p$ *is bijective.*

Proof. *i.* : $\forall w, v \in \mathbb{R}^m \quad [\gamma_w]_p = [\gamma_v]_p \iff \frac{d}{dt}\big|_0 \gamma_w(t) = w = v = \frac{d}{dt}\big|_0 \gamma_v(t)$.

ii. : *i.* implies injectivity of χ_p. Let $\gamma : I \longrightarrow V \subset \mathbb{R}^m$ be a curve at p, and define $w = \dot\gamma(0) = \frac{d}{dt}\big|_0 \gamma(t)$, then $\frac{d}{dt}\big|_0 \gamma_w(t) = w = \frac{d}{dt}\big|_0 \gamma(t)$. ∎

Proposition 2.43. *Let $V \subset \mathbb{R}^m$ open and $p \in V$, then T_pV has the structure of a real vector with a basis given by $[\gamma_{e_k}]_p$ for $k = 1, \dots, m$ (where $\{e_k : k = 1, \dots, m\} \subset \mathbb{R}^m$ is the canonical basis). The map $\chi_V : V \times \mathbb{R}^m \longrightarrow TV$, such that $(x, w) \mapsto [\gamma_w]_x$, is a trivialization giving TV the structure of a trivial vector bundle.*

Proof. By the preceding lemma any element of T_pV can be represented by a curve $\gamma_w = p + tw$. Let $[\gamma_w]_p, [\gamma_v]_p \in T_pV$ and $\alpha, \beta \in \mathbb{R}$, defining the addition and scalar multiplication as $\alpha[\gamma_w]_p + \beta[\gamma_v]_p := [\gamma_{\alpha w+\beta v}]_p$ induces the vector space structure on T_pV. By the preceding lemma χ_x is bijective for every $x \in V$, therefore χ_V is a bijection. TV is equipped with a projection $\pi : TV \longrightarrow V \quad [\gamma]_x \mapsto x$, satisfying $pr_1 = \pi\circ\chi_V$ (denoting $pr_1 : V \times \mathbb{R}^m \quad (x, w) \mapsto x$). Hence the bijection χ_V induces the vector bundle structure on TV. ∎

Proposition 2.44. *Let $U \subset \mathbb{R}^m$ open, $V \subset \mathbb{R}^n$ open and $f : U \longrightarrow V$ a smooth map. Then $Tf : TU \longrightarrow TV$ is a smooth vector bundle homomorphism, by the identifications $TU \simeq U \times \mathbb{R}^m$ and $TV \simeq V \times \mathbb{R}^n$ giving $(Tf)(x, w) = (f(x), (D_xf)(w))$. Moreover, if f is a diffeomorphism then Tf is a vector bundle isomorphism.*

PROOF. For every $x \in U$ and every $w \in \mathbb{R}^m$ $T_x f\,[\gamma_w]_x = [f \circ \gamma_w]_{f(x)}$, ordinary differentiation of f yields $\frac{d}{dt}\big|_0 (f \circ \gamma_w)(t) = (D_x f)(w)$ hence $(Tf)[\gamma_w]_x = [\gamma_{(D_x f)(w)}]_{f(x)}$. Define $\Psi_U = (\chi_U)^{-1} : TU \longrightarrow U \times \mathbb{R}^m$ and $\Psi_V = (\chi_V)^{-1} : TV \longrightarrow V \times \mathbb{R}^n$, then the map $\Psi_V \circ Tf \circ (\Psi_U)^{-1} : U \times \mathbb{R}^m \longrightarrow V \times \mathbb{R}^n$ sending $(x, w) \mapsto (f(x), (D_x f)(w))$ is a smooth vector bundle homomorphism over f. If f is a diffeomorphism, then $\forall x \in U$ $D_x f$ is a vector space isomorphism $\mathbb{R}^m \simeq \mathbb{R}^n$ $(m = n)$, hence Tf is a vector bundle isomorphism. ■

REMARK 2.45. The vector bundle structure given to the tangent bundle of an open subset of $\mathbb{R}^n$ in the preceding proposition can be extended similarly to other manifolds. The manifold structure on the tangent bundle is given by the charts on the base space.

DEFINITION 2.46. Let M be a smooth manifold and $\varphi : U \longrightarrow V \subset \mathbb{R}^m$. The map $T\varphi : TU \longrightarrow TV$ is called a *natural bundle chart.*

PROPOSITION 2.47. *Let M be a smooth manifold and $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathcal{I}\}$ an admissible atlas of M. The natural bundle atlas $T\mathfrak{U} := \{(TU_\alpha, T\varphi_\alpha) : \alpha \in \mathcal{I}\}$ is a smooth atlas on TM such that together with the canonical projection $\pi_{TM} : TM \longrightarrow M$ constitutes a smooth vector bundle.*

PROOF. $\mathfrak{U}$ covers M, therefore $TM \subset \bigcup_{\alpha \in \mathcal{I}} TU_\alpha$. Since φ_α is a diffeomorphism for every $\alpha \in \mathcal{I}$, then $T\varphi_\alpha : TU_\alpha \longrightarrow TV_\alpha$ is bijective for each α. The topology on TM is given by the bundle charts, then $\Omega \subset TM$ is open if and only if for every $\alpha \in \mathcal{I}$ $(T\varphi_\alpha)(\Omega \cap TU_\alpha) \subset TV_\alpha$ is open (making $T\varphi_\alpha$ a homeomorphism). The following diagram is commutative:

$$\begin{array}{ccc} TM \supset TU_\alpha & \xrightarrow{T\varphi_\alpha} & TV_\alpha \simeq V_\alpha \times \mathbb{R}^m \\ \big\downarrow{\scriptstyle \pi_{TM}} & & \big\downarrow{\scriptstyle \pi_{TV_\alpha}} \\ M \supset U_\alpha & \xrightarrow{\varphi_\alpha} & V_\alpha \subset \mathbb{R}^m \end{array}$$

hence the map π_{TM} is continuous. Moreover, the projections π_{TM} and π_{TV_α} are quotient maps. As a consequence TM is Hausdorff and second countable.

The coordinate changes in the atlas $T\mathfrak{U}$ are given by $(T\varphi_\beta) \circ (T\varphi_\alpha^{-1}) = T(\varphi_\beta \circ \varphi_\alpha^{-1})$ which is a diffeomorphism (since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism). Thus TM has a smooth vector bundle structure, with local trivializations given by $T\varphi_\alpha$. ■

PROPOSITION 2.48. *Let $f : M \longrightarrow N$ be a smooth map between smooth manifolds, then $Tf : TM \longrightarrow TN$ is a smooth vector bundle homomorphism.*

PROOF. Let $p \in M$, $\Psi_N : U_{f(p)} \longrightarrow V_{f(p)} \subset \mathbb{R}^n$ a chart of N such that $f(p) \in U_{f(p)}$, and $\Psi_M : U_p \longrightarrow V_p \subset \mathbb{R}^m$ a chart of M such that $p \in U_p \subset f^{-1}(U_{f(p)})$. The diagram

$$\begin{array}{ccccccc} V_p & \xleftarrow[\Psi_M]{} & U_p & \xleftarrow[\pi_{TM}]{} & TU_p & \xrightarrow{T\Psi_M} & TV_p \\ {\scriptstyle \Psi_N\circ f\circ\Psi_M^{-1}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Tf} & & \downarrow{\scriptstyle T\Psi_N\circ Tf\circ T\Psi_M^{-1}} \\ V_{f(p)} & \xleftarrow[\Psi_N]{} & U_{f(p)} & \xleftarrow[\pi_{TN}]{} & TU_{f(p)} & \xrightarrow{T\Psi_N} & TV_{f(p)} \end{array}$$

commutes. Since $\Psi_N \circ f \circ \Psi_M^{-1} : V_p \longrightarrow V_{f(p)}$ is smooth (between open subsets of vector spaces), then (by the lemma 2.40) $T(\Psi_N \circ f \circ \Psi_M^{-1}) = T\Psi_N \circ Tf \circ T\Psi_M^{-1}$ is a smooth vector bundle homomorphism, hence Tf is a smooth vector bundle homomorphism. ■

5. Vector Fields on Manifolds

LEMMA 2.49 (Hadamard's Lemma). *Let $V \subset \mathbb{R}^m$ open such that $p \in V$ and $f : V \longrightarrow \mathbb{R}$ smooth, then exists $\varepsilon > 0$ and smooth maps $f_j : B_\varepsilon(p) \longrightarrow \mathbb{R}$ for $j = a, \ldots, m$ such that $f_j(p) = \frac{\partial f}{\partial x_j}(p)$ and for every $\vec{x} = (x_1, \ldots, x_j) \in B_\varepsilon(p)$ $\quad f(\vec{x}) = f(p) + \sum_{j=1}^{m} f_j(\vec{x})(x_j - p_j)$.*

PROOF. Let $\varepsilon > 0$ such that $B_\varepsilon(p) \subset V$, then for every $\vec{x} \in B_\varepsilon(p)$ $\quad f(\vec{x}) - f(p) = \int_0^1 \frac{d}{dt} f(p + t(\vec{x} - p))dt = \int_0^1 \sum_{j=1}^{m} (\frac{\partial f}{\partial x_j}(p + t(\vec{x} - p)))(x_j - pj)dt$. Hence, we can define $f_j(\vec{x}) = \int_0^1 \frac{\partial f}{\partial x_j}(p + t(\vec{x} - p))dt$ which is smooth on $B_\varepsilon(\vec{0})$ and is such that $f_j(p) = \frac{\partial f}{\partial x_j}(p)$. ■

PROPOSITION 2.50. *Let $V \subset \mathbb{R}^m$ open and $p \in V$, then T_pV is $\mathbb{R}-$linearly isomorphic to $Der(\mathcal{E}_p(V), \mathbb{R})$.*

PROOF. Consider the map $\delta_p : T_pV \longrightarrow Der(\mathcal{E}_p(V), \mathbb{R})$ mapping $[\gamma_w] \mapsto \not{D}$ such that $\not{D}(f) = \frac{d}{dt}\big|_0 (f \circ \gamma_w)(t) = (D_pf) \cdot \frac{d}{dt}\big|_0 \gamma_w(t) = (D_pF) \cdot w$, the directional derivative in the direction w (is independent of the representative of $[\gamma_w]$, so is well defined), is an injective linear map.

Let $D_0 \in Der(\mathcal{E}_p(V), \mathbb{R})$, for $j = 1, \ldots, m$ let $a_j = D_0(x_j)$. Let $\vec{w} = (a_1, \ldots, a_m)$ and $\gamma_w(t) = p + t\vec{w}$, by Hadamard's lemma $f(\vec{x}) = f(p) + \sum_{j=1}^{m} f_j(\vec{x})(x_j - p_j)$, then $D_0(f) = \sum_{j=1}^{m} D_0(f_j(\vec{x})(x_j - p_j)) = \sum_{j=1}^{m} f_j(p)D_0(x_j) = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(p)D_0(x_j) = \sum_{j=1}^{m} a_j \frac{\partial f}{\partial x_j}(p)$ therefore

$(\delta_p([\gamma_w]_p))(f) = \frac{d}{dt}\big|_0 f(p + t\vec{w}) = \sum_{j=1}^{m} a_j \frac{\partial f}{\partial x_j}(p) = D_0(f)$. Thereby surjectivity is proven, so the map δ_p is a vector space isomorphism. ∎

PROPOSITION 2.51. *Let M be a smooth manifold and $p \in M$. Then $Der(\mathcal{E}_p(M), \mathbb{R})$ is $\mathbb{R}$-linearly isomorphic to T_pM.*

PROOF. Let $\varphi : U \longrightarrow V \subset \mathbb{R}^m$ be a chart such that $p \in U$, then $T_pU \simeq T_{\varphi(p)}V$ as vector spaces. The pullback of germs $\varphi^* : \mathcal{E}_{\varphi(p)}(V) \longrightarrow \mathcal{E}_p(U)$ sending $f \mapsto f \circ \varphi$ is an isomorphism of associative commutative unital $\mathbb{R}$ algebras, hence (by the previous proposition) $Der(\mathcal{E}_p(M), \mathbb{R}) \simeq Der(\mathcal{E}_{\varphi(p)}, \mathbb{R}) \simeq T_{\varphi(p)}V \simeq T_pM$ as vector spaces. ∎

DEFINITION 2.52. Let M be a manifold, a smooth section of the tangent bundle TM is called a *vector field on M*. We denote the set of all vector fields $\mathfrak{X}(M) = \Gamma(M, TM)$.

REMARK 2.53. The $\mathbb{R}-$vector space structure of T_pM for every $p \in M$ induces an $\mathbb{R}-$vector space structure on $\mathfrak{X}(M)$.

PROPOSITION 2.54. *Let M be a smooth manifold and $X : M \longrightarrow TM$ a section of TM, then X is a smooth section if and only if for every $f \in \mathcal{E}(M)$ the function Xf is smooth on M.*

PROOF. Assume X is a smooth vector field and let $f \in \mathcal{E}(M)$. Define $\delta_f : TM \longrightarrow \mathbb{R}$ $(p, [\gamma]_p) \mapsto \frac{d}{dt}\big|_0 (f \circ \gamma)(t)$. Let $p \in M$ and $\varphi : U \longrightarrow V$ be a chart about p, then $\delta_f \circ T\varphi^{-1} : V \times \mathbb{R}^m \longrightarrow \mathbb{R}$ $(\varphi(p), w_p) \mapsto \frac{d}{dt}\big|_0 ((f \circ \varphi^{-1}) \circ \gamma_{w_p})(t)$ where $\gamma_{w_p}(t) = \varphi(p) + tw$. Define $g : \mathbb{R} \times V \times \mathbb{R}^m \longrightarrow V$ by $g(t, x, w) = x + tw$, g is smooth and $\delta_f \circ T\varphi^{-1} = \frac{d}{dt}\big|_0 ((f \circ \varphi^{-1}) \circ g)(t, x, w)$ therefore $\delta_f \circ T\varphi^{-1}$ is smooth implying δ_f smooth. Hence $Xf = \delta_f \circ X$ is smooth.

Assume for every $f \in \mathcal{E}(M)$ Xf is smooth on M. Let $\varphi : U \longrightarrow V$ be a chart of M and define the functions $x^k : U \longrightarrow \mathbb{R}$ given by the projection onto the canonical basis of $\mathbb{R}^m$. Let $p \in U$, using bump functions, these x^k can be extended to $\tilde{x}^k : M \longrightarrow \mathbb{R}$ such that exists $W \subset U$ open neighbourhood of p and $x^k\big|_W = \tilde{x}^k\big|_W$. Hence in local coordinates $X\tilde{x}^k|_W = (\sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i})\tilde{x}^k = \sum_{i=1}^{m} a^i \frac{\partial \tilde{x}^k}{\partial x^i} = a^k$, then the local coefficients of the section are smooth. Therefore $T\varphi \circ X \circ \varphi^{-1}$ is C^∞ at every $p \in M$, then X is a smooth vector field. ∎

Before proving the isomorphism between smooth vector fields and derivations, we will prove a lemma on the locality of derivations.

LEMMA 2.55. *Let $f \in \mathcal{E}(M)$ such that exists $U \subset M$ open and $f|_U \equiv 0$ then for every $D \in Der(\mathcal{E}(M))$ $Df|_U \equiv 0$.*

PROOF. Let $D \in Der(\mathcal{E}(M))$ and $p \in M$. Let $w : M \longrightarrow \mathbb{R}$ be a bump function such that $w|_{U^c} \equiv 0$ and exists $V \subset U$ open $p \in V$ and $w|_V \equiv 1$. Then $fw \equiv 0$ hence $D(fw) = 0 = fD(w) + D(f)w$. Therefore $D(fw)|V = (D(f)w)|_V = D(f)|_V$, so $D(f)(p) = 0$. ∎

PROPOSITION 2.56. *Let M be a smooth manifold, then $\mathfrak{X}(M)$ and $Der(\mathcal{E}(M))$ are $\mathcal{E}(M)-$modules. Moreover, they are isomorphic as $\mathcal{E}(M)-$modules.*

PROOF. $\mathfrak{X}(M)$ has an additive abelian group structure induced (pointwise) by the $\mathbb{R}-$vector space structure on T_pM. Let $f \in \mathcal{E}(M)$ and $X \in \mathfrak{X}(M)$, then $(fX)_p = f(p)X_p$ is a section of TM. For every $g \in \mathcal{E}(M)$ $((fX)g)(p) = f(p)X_pg = f(Xg)(p)$, therefore $f(Xg)$ is smooth. Since the module operations are defined pointwise, the distributivity and associativity axioms are easily seen to be satisfied.

$Der(\mathcal{E}(M))$ has an additive abelian group structure given by its $\mathbb{R}-$vector space structure. Let $f, g \in \mathcal{E}(M)$ and $D \in Der(\mathcal{E}(M))$, then $(fD)g(p) = f(p)(Dg(p))$, hence $fD \in \mathcal{E}(M)$. Since the scalar multiple is defined pointwise, $fD \in Der(\mathcal{E}(M))$, the module axioms follow since the module operations are defined pointwise.

Let $\delta : \mathfrak{X}(M) \longrightarrow Der(\mathcal{E}(M))$ such that $(\delta(X))(f) = Xf$ for every $X \in \mathfrak{X}(M)$ and $f \in \mathcal{E}(M)$. Let $X, Y \in \mathfrak{X}(M)$ and $f \in \mathcal{E}(M)$, then $(\delta(X+Y))(f)(p) = (X+Y)_pf = X_pf + Y_pf = (\delta(X))(f)(p) + (\delta(Y))(f)(p)$, hence δ is additive. Let $g \in \mathcal{E}(M)$ $(\delta(gX))(f)(p) = (gX)_pf = g(p)X_pf = (g)(Xf)(p)$, hence $\delta(gX) = g\delta(X)$ so δ is $\mathcal{E}(M)-$linear.

Assume $\delta(X) = \delta(Y)$, then for every $f \in \mathcal{E}(M)$ and for every $p \in M$ $(\delta(X))(f)(p) = (\delta(Y))(f)(p) = X_pf = Y_pf$. Hence $X = Y$ and δ is injective.

Let $D \in Der(\mathcal{E}(M)$, define for each $p \in M$ $D_p : \mathcal{E}_p(M) \longrightarrow \mathcal{E}_p(M)$ such that if f is a representative of germ $[f]$ at p and $\tilde{f}$ is a global extension of f, then $D_p[f] = [D\tilde{f}]$. By the previous lemma D is a local operator, hence D_p is well defined and is $\mathbb{R}-$linear. Let $[f], [g] \in \mathcal{E}_p(M)$ and $\tilde{f}, \tilde{g} \in \mathcal{E}(M)$ be global extensions of some representative of the germs, then $D_p[fg] = [D(\tilde{f}\tilde{g})] = [D(\tilde{f})\tilde{g} + \tilde{f}D(\tilde{g})] = [D(\tilde{f})\tilde{g}] + [\tilde{f}D(\tilde{g})] = (D_p[f])[g] + [f][D_p(g)]$, hence $D_p \in Der(\mathcal{E}_p(M))$. The evaluation map at p produces $ev_p \circ D_p : \mathcal{E}_p(M) \longrightarrow \mathbb{R}$ which is a scalar valued derivation corresponding to a tangent vector $X_p \in T_pM$. The function $X : M \longrightarrow TM$ such that $p \mapsto X_p$ is a section of TM. Let $f \in \mathcal{E}(M)$, then $Xf(p) = X_pf = ev_p[Df] = (Df)(p)$. Since $Df \in \mathcal{E}(M)$, by the previous proposition X is a smooth vector field. Therefore δ is surjective, hence $\mathfrak{X} \simeq Der(\mathcal{E}(M))$ as $\mathcal{E}(M)-$modules. ∎

Remark 2.57. The isomorphism between vector fields and derivations allows us to think of vector fields as acting on functions, thus their composition is well defined. Nonetheless, if $X, Y \in Der(\mathcal{E}(M))$ the composition XY need not be a derivation. However, $[X, Y] = XY - YX$ is indeed a derivation.

Definition 2.58. Let $\mathbb{K}$ be a field and $\mathfrak{g}$ a $\mathbb{K}-$vector space equipped with a $\mathbb{K}-$bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is anti-symmetric and fulfils Jacobi's identity, namely $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$. Then $\mathfrak{g}$ is called a $\mathbb{K}-$*Lie algebra* and the product $[\ ,\ ]$ is called *Lie bracket*. If $\mathfrak{h} \subset \mathfrak{g}$ is a $\mathbb{K}-$subspace and is closed under the Lie bracket, then $\mathfrak{h}$ is called a *Lie subalgebra* of $\mathfrak{g}$.

Lemma 2.59. *Let $\mathcal{A}$ be a $\mathbb{K}-$algebra and $Der(\mathcal{A} := \{D \in End_{\mathbb{K}}(\mathcal{A} : D(ab) = D(a)b + aD(b)\}$, then $Der(\mathcal{A})$ is a Lie subalgebra of $End_{\mathbb{K}}(\mathcal{A})$ (Lie bracket given by the commutator).*

Corollary 2.60. *Let M be a smooth manifold, then $\mathfrak{X}(M)$ carries a Lie algebra structure given by the commutator of derivations.*

6. Differential forms and the Lie derivative

Definition 2.61. Let M be a manifold the dual of the tangent bundle $T^*M := (TM)^*$ (with projection π_{T^*M}) is called the *cotangent bundle of M*. A smooth section of T^*M is called a (differential)*1-form*, the space of all 1-forms is denoted by $\Omega^1(M)$. For $k \geq 1$ we construct the alternating product $\Lambda^k T^*M := \Lambda^k(TM)^*$, sections of $\Lambda^k T^*M$ are called (differential)*k-forms* and the space of its sections is denoted by $\Omega^k(M)$. We let $\Omega^0(M) = \mathcal{E}(M)$.

Remark 2.62. If $k > m = dim(M)$, then $\Lambda^k T_p^*M = 0$ for all $p \in M$.

LEMMA 2.63. *Let $V \subset \mathbb{R}^m$ open, then $\mathfrak{X}(V)$ and $\Omega^k(V)$ are free $\mathcal{E}(V)-$modules.*

DEFINITION 2.64. Let M be a manifold, $p \in M$, $\nu \in T_pM$ and $\eta \in \Lambda^k T_p^*M$, the *contraction* of ν and η is $i_\nu \eta \in \Lambda^{k-1}T^*M$ is given by $(i_\nu \eta)(v_1, \dots, v_{k-1}) := \eta(\nu, v_1, \dots, v_{k-1})$ for all $v_1, \dots, v_{k-1} \in T_pM$.

LEMMA 2.65. *Let $X \in \mathfrak{X}(M)$ and $\eta \in \Omega^k(M)$, then $i_X \eta \in \Omega^{k-1}(M)$.*

REMARK 2.66. On $\Lambda^*(T_p^*M) = \bigoplus_{k\geq 0} \Lambda^k(T_p^*M)$ one has exterior multiplication $\wedge$. This can be extended to $\Omega^*(M)$.

LEMMA 2.67. *Let $\eta \in \Omega^k(M)$ and $\mu \in \Omega^l(M)$, then $\eta \wedge \mu \in \Omega^{k+l}(M)$, which is given by $(\eta \wedge \mu)_p := \eta_p \wedge \mu_p$.*

Moreover, the space of sections of $\Lambda^*(T^*M)$ is canonically isomorphic to $\bigoplus_{k\geq 0} \Omega^k(M) = \Omega^*(M)$ which is a $\mathcal{E}(M)-$module. $\Omega^*(M)$ together with the wedge product is a super-commutative, associative, unital algebra over $\mathcal{E}(M)$.

DEFINITION 2.68. Let $f : M \longrightarrow N$ be a smooth map between manifolds and $\eta \in \Omega^k(M)$, then $f^*\eta \in \Omega^k(M)$, *the pullback of η by f*, is given by $(f^*\eta)_p(v_1, \dots, v_k) := \eta_{f(p)}((f_*)_p v_1, \dots, (f_*)_p v_k)$, for every $p \in M$ and every $v_1, \dots, v_k \in T_pM$.

LEMMA 2.69. *Let $f : M \longrightarrow N$ be a smooth map between manifolds, $\eta \in \Omega^k(N)$ and $\mu \in \Omega^l(N)$, then $f^*(\eta \wedge \mu) = (f^*\eta) \wedge (f^*\mu)$. Moreover, $f^* : \Omega^*(N) \longrightarrow \Omega^*(M)$ is an $\mathbb{R}-$linear even homomorphism of $\mathbb{R}-$super-algebras (in particular, for every $\psi \in \mathcal{E}(N)$, $f^*(\psi\eta) = f^*(\psi)f^*(\eta)$).*

DEFINITION 2.70. Let $X \in \mathfrak{X}(M)$, $\Phi_t^X : M \longrightarrow M$ a local flow for X and $\eta \in \Omega^k(M)$. The *Lie derivative of η with respect to X* is defined by $(\mathcal{L}_X \eta)_p := \frac{d}{dt}\big|_0 ((\Phi_t^X)^*\eta)_p$.

PROPOSITION 2.71. *Let $X \in \mathfrak{X}(M)$, $\eta, \eta' \in \Omega^k(M)$, $\mu \in \Omega^l(M)$ and $\lambda \in \mathbb{R}$, then:*

i. $\mathcal{L}_X \eta \in \Omega^k(M)$.

ii. $\mathcal{L}_X(\lambda\eta) = \lambda \mathcal{L}_X \eta$.

iii. $\mathcal{L}_X(\eta + \eta') = \mathcal{L}_X \eta + \mathcal{L}_X \eta'$.

iv. $\mathcal{L}_X(\eta \wedge \mu) = (\mathcal{L}_X \eta) \wedge \mu + \eta \wedge (\mathcal{L}_X \mu)$.

LEMMA 2.72. *Let $X \in \mathfrak{X}(M)$ and $f \in \mathcal{E}(M)$, then $\mathcal{L}_X f = Xf$.*

DEFINITION 2.73. Let $X, Y \in \mathfrak{X}(M)$ the *Lie derivative of Y with respect to X* is defined by $(\mathcal{L}_Y X)(p) := \frac{d}{dt}\big|_0 [((\Phi^X_{-t})_*)_{\Phi^X_t(p)}(\Phi^Y_t(p))]$.

7. Gelfand Naimark theorem

The theory of C^* algebras provides an important relation between the fields of algebra and geometry (more generally with topology). This Section follows [**Mur90**], the proofs of theorems in this Section appear in this reference.

DEFINITION 2.74. An *involutive algebra* $\mathcal{A}$ is an algebra over $\mathbb{C}$, together with an *involution.* That is, a map $x \mapsto x^*$ of $\mathcal{A}$ into itself such that for any $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:

i. $(x^*)^* = x$

ii. $(x + y)^* = x^* + y^*$

iii. $(\lambda x)^* = \overline{\lambda} x^*$

iv. $(xy)^* = y^* x^*$

The involution of $x \in \mathcal{A}$, x^*, is often called its adjoint.

DEFINITION 2.75. A normed *involutive algebra* $\mathcal{A}$ is a normed algebra together with an involution such that $\|x^*\| = \|x\|$ for every $x \in \mathcal{A}$. If $\mathcal{A}$ is complete, then $\mathcal{A}$ is called an *involutive Banach algebra.*

DEFINITION 2.76. A C^**−algebra* $\mathcal{A}$ is an involutive Banach algebra fulfilling the C^* identity, namely $\|x\|^2 = \|x^* x\|$ for every $x \in \mathcal{A}$.

DEFINITION 2.77. Let M be a locally compact space, the space of complex valued functions vanishing at infinity is defined as $C_0(M) := \{f : M \longrightarrow \mathbb{C} | \forall \varepsilon > 0 \; \exists K \subset M$ such that $|f(x)| < \varepsilon \;\; \forall x \notin K\}$

PROPOSITION 2.78. *Let M be a locally compact space, then $C_0(M)$ is an abelian C^*−algebra with the norm $\|f\|_\infty = \sup_{x \in M} |f(x)|$.*

In fact, it will turn out that all abelian C^*−algebras can be defined as the algebra of complex valued functions vanishing at infinity of some locally compact space. We shall sketch how this construction comes to be.

Definition 2.79. Let $\mathcal{A}$ be an abelian algebra. A *character* on $\mathcal{A}$ is a non-zero homomorphism $\tau : \mathcal{A} \longrightarrow \mathbb{C}$. The set of all characters on $\mathcal{A}$ is denoted by $\Omega(\mathcal{A})$.

The elements in the algebra are functionals on $\Omega(\mathcal{A}$ (this is so by defining $a(\tau) := \tau(a)$). Therefore, a topology on $\Omega(\mathcal{A}$ can be introduced by considering the coarsest topology such that the functionals defined by elements of $\mathcal{A}$ are continuous. This topology has some important properties.

Theorem 2.80. *If $\mathcal{A}$ is an abelian Banach algebra, then $\Omega(\mathcal{A})$ is a locally compact Hausdorff space. If $\mathcal{A}$ is unital, then $\Omega(\mathcal{A})$ is compact.*

Theorem 2.81. *Every abelian C^*-algebra is $*-$isomorphic to $C_0(\Omega)$ where Ω is a locally compact Hausdorff space. Moreover $\mathcal{A}$ is unital if and only if Ω is compact.*

Theorem 2.82 (Gelfand-Naimark). *If $\mathcal{A}$ is a C^*-algebra, then it has a faithful representation. Specifically, its universal representation is faithful*

CHAPTER 3

De Sitter Space

Before discussing de Sitter space, we shall briefly discuss some spaces of a more general kind. Namely, we will discuss spaces whose symmetries makes them possible candidates to model the universe.

1. Robertson-Walker spaces

Evidence from observation of the cosmic microwave background (CMB) serves as a solid ground to assert that the universe (at least from Earth's position) is spatially isotropic at large enough scales. The discovery of numerous planets, stars and galaxies serves to reassert the belief that Man occupies no special place in the universe, moreover, that there is no preferred spatial location in the universe. This last assertion is not so easily put to observational test, quoting from [**HE08**] "we are not able to make cosmological models without some admixture of ideology". Modern cosmological models often rely on the cosmological principle (see [**Wei15**]). This principle asserts that at a large enough scale the universe is spatially homogeneous and spatially isotropic.

A consequence of the cosmological principle (as stated in [**HE08**]) is that the universe, considered to be a $4-$dimensional Lorentzian manifold $\mathcal{M}$, admits a $6-$parameter group of isometries whose surfaces of transitivity are space-like $3-$surfaces of constant curvature. A Robertson-Walker space is a model of such a universe. $\mathcal{M}$ can be foliated by maximally symmetric euclidean $3-$manifolds Σ and there exist coordinates such that the metric takes the form

$$ds^2 = -dt^2 + R^2(t)d\sigma^2 \tag{1}$$

where $d\sigma^2$ is the metric in Σ (which is independent of t) and $R(t)$ is called the scale factor. Here we adopt the sign convention in [**HE08**] and [**Car14**], namely flat $(n+1)-$dimensional

Minkowski space will have the metric tensor given by the matrix $\mathrm{diag}(-1, 1, \dots, 1)$. Following also [**HE08**] and [**Car14**], we will use units such that $c = 1$.

It can be proven (see [**Wei72**]) that any maximally symmetric space is uniquely determined by the signature of the metric and the curvature (which is constant). Hence, the maximally symmetric hypersurface Σ is uniquely determined by its curvature. In particular there are three distinguished cases, namely positive curvature, zero curvature and negative curvature.

Homogeneity and isotropy of space imposes the condition that the Ricci tensor must be proportional to the metric tensor (see [**Car14**]), but also allows the choice of spherical polar coordinates so that the metric takes the form $d\sigma^2 = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$. Upon computing the associated Ricci tensor one obtains $\beta(r) = -\frac{1}{2}\ln(1 - Kr^2)$, hence the space-time metric is of the form:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right], \tag{2}$$

known as the Robertson-Walker metric.

Though the space curvature is constant (the curvature of Σ), the space-time curvature need not be constant. From (2) the non-zero Christoffel symbols of the Levi-Civita connection are computed.

$$\begin{aligned}
\Gamma^t_{rr} &= \frac{a(t)a'(t)}{(1-\kappa r^2)} & \Gamma^\theta_{\theta t} &= \frac{a'(t)}{a(t)} \\
\Gamma^t_{\theta\theta} &= r^2 a(t)a'(t) & \Gamma^\theta_{\theta r} &= \frac{1}{r} \\
\Gamma^t_{\phi\phi} &= r^2 a(t)\sin^2(\theta)a'(t) & \Gamma^\theta_{\phi\phi} &= -\cos(\theta)\sin(\theta) \\
\Gamma^r_{rt} &= \frac{a'(t)}{a(t)} & \Gamma^\phi_{\phi t} &= \frac{a'(t)}{a(t)} \\
\Gamma^r_{rr} &= \frac{\kappa r}{1-\kappa r^2} & \Gamma^\phi_{\phi r} &= \frac{1}{r} \\
\Gamma^r_{\theta\theta} &= r(-1+\kappa r^2) & \Gamma^\phi_{\phi\theta} &= \cot(\theta) \\
\Gamma^r_{\phi\phi} &= r(-1+\kappa r^2)\sin^2(\theta)
\end{aligned} \tag{3}$$

Further, the non-zero components of the Riemann tensor are computed from (3).

$$
\begin{aligned}
R^t_{rrt} &= -\frac{a(t)a''(t)}{1-\kappa r^2} & R^\theta_{t\theta t} &= -\frac{a''(t)}{a(t)} \\
R^t_{\theta\theta t} &= -r^2 a(t)a''(t) & R^\theta_{r\theta r} &= \frac{\kappa + a'(t)^2}{1-\kappa r^2} \\
R^t_{\phi\phi t} &= -r^2 a(t)\sin^2(\theta)a''(t) & R^\theta_{\phi\phi\theta} &= -r^2\sin^2(\theta)(\kappa + a'(t)^2) \\
R^r_{trt} &= -\frac{a''(t)}{a(t)} & R^\phi_{t\phi t} &= -\frac{a''(t)}{a(t)} \\
R^r_{\theta\theta r} &= -r^2(\kappa + a'(t)^2) & R^\phi_{r\phi r} &= \frac{\kappa + a'(t)^2}{1-\kappa r^2} \\
R^r_{\phi\phi r} &= -r^2\sin^2(\theta)(\kappa + a'(t)^2) & R^\phi_{\theta\phi\theta} &= r^2(\kappa + a'(t)^2)
\end{aligned}
\tag{4}
$$

From these the components of the Ricci tensor are computed.

$$
\begin{aligned}
R_{tt} &= -3\frac{a''(t)}{a(t)} & R_{\theta\theta} &= r^2(2\kappa + 2a'(t)^2 + a(t)a''(t)) \\
R_{rr} &= \frac{2\kappa + 2a'(t)^2 + a(t)a''(t)}{1-\kappa r^2} & R_{\phi\phi} &= r^2\sin^2(\theta)(2\kappa + 2a'(t)^2 + a(t)a''(t))
\end{aligned}
\tag{5}
$$

Finally, the scalar curvature of the Robertson-Walker space is obtained.

$$
R = 6\frac{\kappa + a'(t)^2 + a(t)a''(t)}{a^2(t)} = 6\left[\frac{\kappa}{a^2(t)} + \left(\frac{a'(t)}{a(t)}\right)^2 + \frac{a''(t)}{a(t)}\right] \tag{6}
$$

Taking into account the geodesics equation (7) and the values of the Christoffel symbols (3) it is straightforward to check that paths with constant coordinates (r, θ, ϕ) are geodesics. Hence, these coordinates (t, r, θ, ϕ) are called co-moving coordinates in the sense that freely falling objects at rest in these coordinates stay at rest.

$$
\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{du}\frac{dx^\rho}{du} = 0 \tag{7}
$$

One can notice that for any $\lambda > 0$, the substitutions $a(t) \mapsto \lambda^{-1}a(t)$, $r \mapsto \lambda r$ and $K \mapsto \lambda^{-2}K$ leave the metric (2) invariant. As such, units can be normalized so that $K \in \{-1, 0, 1\}$, r is dimensionless and the scale factor $a(t)$ has units of length. These are the three distinguished cases of positive curvature ($K = 1$), zero curvature ($K = 0$) and negative curvature ($K = -1$). They are also commonly called, respectively, closed, flat and open. Many spaces of such type exist which exhibit further particular properties. Of particular interest to us are spaces which are not only symmetric in space-like hypersurfaces, but are maximally symmetric space-times. The most well known of these is Minkowski space ($\mathbb{R}^{1,3}$), but we shall be specially interested in de Sitter space (maximally symmetric space-time of positive curvature).

2. Definition of de Sitter space

De Sitter universe can be defined as the maximally symmetric space-time of (constant) positive curvature (see [**HE08**]). The de Sitter universe [**HE08**] can be modelled as a hyperboloid in a five-dimensional Minkowski space $\mathbb{R}^{1,4}$ defined by the equation (for a non-zero real constant α):

$$v^2 + w^2 + x^2 + y^2 - z^2 = \alpha^2 \tag{8}$$

with metric inherited from the metric in $\mathbb{R}^{1,4}$ given by the line element:

$$ds^2 = dv^2 + dw^2 + dx^2 + dy^2 - dz^2. \tag{9}$$

This definition can be generalized to other dimensions by adding spatial dimensions to the ambient Minkowski space. The $n-$dimensional de Sitter space (henceforth denoted $d\boldsymbol{S}_n$) can be defined as the one sheet hyperboloid in Minkowski space $\mathbb{R}^{1,n}$ given by the equation

$$-(x_0)^2 + \sum_{i=1}^{n}(x_n)^2 = \alpha^2. \tag{10}$$

In view of equation (10) and the induced metric from $\mathbb{R}^{1,n}$, it is seen that an isometry of de Sitter space must correspond to an invertible linear transformation in $\mathbb{R}^{1,n}$ preserving the Lorentzian metric. Therefore, the isometry group of $d\boldsymbol{S}_n$ corresponds to $O(1,n)$, which is an $\frac{n(n+1)}{2}-$dimensional Lie group. The isometry group of de Sitter universe, namely $O(1,4)$, is often called the de Sitter group (see [**Wei72**]). Consider the point $\mathbf{x} = (0, \alpha, 0, \dots, 0) \in \mathbb{R}^{1,n}$. The isometries of $d\boldsymbol{S}_n$ fixing $\mathbf{x}$ correspond to linear transformations in $\mathbb{R}^{1,n-1}$ which leave $-(x_0)^2 + \sum_{i=2}^{n}(x_n)^2$ invariant. Hence, the stabilizer of $\mathbf{x}$ is isomorphic to $O(1, n-1)$, and $n-$dimensional de Sitter space can be defined as the quotient space $O(1,n)/O(1,n-1)$.

Since the isometry group of de Sitter space, $O(1,n)$ is an $\frac{n(n+1)}{2}-$dimensional Lie group, its Lie algebra, and hence the algebra of Killing vector fields, must have dimension $n(n+$

1)/2. Therefore, since $d\boldsymbol{S}_n$ is an $n-$dimensional manifold with $n(n+1)/2$ independent Killing vector fields, de Sitter space is a maximally symmetric manifold. The embedding of $d\boldsymbol{S}_n$ in $\mathbb{R}^{1,n}$ allows an easy description of a basis of Killing vector fields. Namely, letting X^μ (for $\mu = 0, 1, \ldots, n$) be the standard coordinate functions in $\mathbb{R}^{1,n}$ and $\partial_\mu := \frac{\partial}{\partial X^\mu}$, the vectors (11) and (12) for $i, j = 1, \ldots, n$ form a basis of the Lie algebra of $O(1, n)$.

$$J_{ij} = X^i\partial_j - X^j\partial_i \tag{11}$$

$$K_i = X^i\partial_0 + X^0\partial_i \tag{12}$$

From which the commutation relations of the Lie algebra of $O(1, n)$ can be directly computed.

$$[K_i, K_j] = J_{ij} \tag{13}$$

$$[K_i, J_{jk}] = \delta_{i,j}K_k - \delta_{i,k}K_j \tag{14}$$

$$[J_{ij}, J_{kl}] = \delta_{i,l}J_{jk} - \delta_{i,k}J_{jl} + \delta_{j,k}J_{il} - \delta_{j,l}J_{ik} \tag{15}$$

According to the equation (10), for each $z \in \mathbb{R}$ the restriction $x_0 = z$ defines an $(n-1)-$sphere with radius $\alpha^2 + z^2$. Hence, de Sitter space is homeomorphic to $\mathbb{R} \times S_{n-1}$ (in fact this homeomorphism is a diffeomorphism). This homeomorphism, is made explicit by the parametrization of $d\boldsymbol{S}_n$ by:

$$x_0 = \alpha \sinh(t/\alpha) \tag{16}$$

$$x_i = \alpha \cosh(t/\alpha)\zeta_i \tag{17}$$

where $\zeta_1, \ldots, \zeta_n$ is a parametrization of the unit $(n-1)-$sphere embedded in $\mathbb{R}^n$. From this parametrization $d\boldsymbol{S}_n$ is seen to be an embedded submanifold of $\mathbb{R}^{1,n}$. Therefore, the metric in de Sitter space can be computed from its embedding in $\mathbb{R}^{1,n}$.

With coordinates as in (16) - (17) the metric in $d\boldsymbol{S}_n$ is

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha) d\Omega^2_{n-1} \tag{18}$$

where $d\Omega_{n-1}$ is the metric in S_{n-1}. As mentioned in the previous Section, de Sitter universe $d\boldsymbol{S}_4$ is a Robertson-Walker space.

3. Properties

Let us consider the parametrization of $d\boldsymbol{S}_4$ as given by (16) - (17), together with the standard parametrization of the $3-$sphere given by $\psi,\theta\in[0,\pi[$ and $\phi\in[0,2\pi[$ as:

$$\zeta_1=\cos\psi \tag{19}$$

$$\zeta_2=\sin\psi\cos\theta \tag{20}$$

$$\zeta_3=\sin\psi\sin\theta\cos\phi \tag{21}$$

$$\zeta_4=\sin\psi\sin\theta\sin\phi \tag{22}$$

From this parametrization $d\Omega_3^2=d\psi^2+\sin^2\psi(d\theta^2+\sin^2\theta d\phi^2)$ is obtained, from which the non-zero Christoffel symbols are computed.

$$\begin{aligned}
\Gamma^t_{\psi\psi}&=\frac{1}{2}\alpha\sinh\left(\frac{2t}{\alpha}\right) & \Gamma^\theta_{\theta t}&=\cot\psi\\
\Gamma^t_{\theta\theta}&=\frac{1}{2}\alpha\sinh\left(\frac{2t}{\alpha}\right)\sin^2\psi & \Gamma^\theta_{\phi\phi}&=-\cos\theta\cos\theta\\
\Gamma^t_{\phi\phi}&=\frac{1}{2}\alpha\sinh\left(\frac{2t}{\alpha}\right)\sin^2\psi\sin^2\theta & \Gamma^\theta_{t\theta}&=\frac{1}{\alpha}\tanh\left(\frac{t}{\alpha}\right)\\
\Gamma^\psi_{\theta\theta}&=-\cos\psi\sin\psi & \Gamma^\phi_{\phi\psi}&=\cot\psi\\
\Gamma^\psi_{\phi\phi}&=-\cos\psi\sin\psi\sin^2\theta & \Gamma^\phi_{\phi\theta}&=\cot\theta\\
\Gamma^\psi_{t\psi}&=\frac{1}{\alpha}\tanh\left(\frac{t}{\alpha}\right) & \Gamma^\phi_{t\phi}&=\frac{1}{\alpha}\tanh\left(\frac{t}{\alpha}\right)
\end{aligned} \tag{23}$$

Next we compute the non-zero components of the Riemann tensor.

$$\begin{aligned}
R^\psi_{\theta\theta\psi}&=-\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi & R^\phi_{\psi\phi\psi}&=\cosh^2\left(\frac{t}{\alpha}\right)\\
R^\psi_{\phi\phi t}&=-\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi\sin^2\theta & R^\phi_{\theta\phi\theta}&=\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi\\
R^\psi_{tt\psi}&=\frac{1}{\alpha^2} & R^\phi_{tt\phi}&=\frac{1}{\alpha}\\
R^\theta_{\psi\theta\psi}&=\cosh^2\frac{t}{\alpha} & R^t_{\psi t\psi}&=\cosh^2\left(\frac{t}{\alpha}\right)\\
R^\theta_{\phi\phi\theta}&=-\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi\sin^2\theta & R^t_{\theta t\theta}&=\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi\\
R^\theta_{tt\theta}&=\frac{1}{\alpha^2} & R^t_{\phi t\phi}&=\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi\sin^2\theta
\end{aligned} \tag{24}$$

Finally the non-zero components of the Ricci tensor are obtained.

$$
(25) \qquad
\begin{aligned}
R_{\psi\psi} &= 3\cosh^2\left(\frac{t}{\alpha}\right) & R_{\phi\phi} &= 3\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi\sin^2\theta \\
R_{\theta\theta} &= 3\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi & R_{tt} &= -\frac{3}{\alpha}
\end{aligned}
$$

And the scalar curvature of $d\boldsymbol{S}_4$ is obtained.

$$
R = \frac{12}{\alpha^2} \qquad (26)
$$

From (26) and (25) the non-zero components of Einstein's tensor are obtained.

$$
(27) \qquad
\begin{aligned}
G_{\psi\psi} &= -3\cosh^2\left(\frac{t}{\alpha}\right) & G_{\phi\phi} &= -3\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi\sin^2\theta \\
G_{\theta\theta} &= -3\cosh^2\left(\frac{t}{\alpha}\right)\sin^2\psi & G_{tt} &= \frac{3}{\alpha^2}
\end{aligned}
$$

Taking into account Einstein's field equations in the form (28) together with the Einstein tensor (27), it is seen that de Sitter universe corresponds to an "empty" universe, everywhere vanishing energy-momentum tensor, with a positive cosmological constant $\Lambda = \frac{3}{\alpha^2}$.

$$
G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (28)
$$

For a sufficiently large t, the scale factor $\alpha^2\cosh^2(t/\alpha)$ grows exponentially in time as is expected of a vacuum-dominated universe (see [**Wei15**]). In fact, an expanding universe with non-zero vacuum energy is asymptotically a de Sitter space. This is due to the reduction of matter and radiation energy density due to the expansion, while the vacuum energy density is kept constant.

An important feature of de Sitter space is the overall shape of its geodesics. Let us consider $d\boldsymbol{S}_n$ as given by equation (10). A linear change of coordinates $x_0 \mapsto ix_0$ yields the equation

$$
-(ix_0)^2 + \sum_{i=1}^{n}(x_n)^2 = \sum_{i=0}^{n}(x_i)^2 = \alpha^2 \qquad (29)
$$

which is simply the equation of a $n-$sphere. The geodesics for the $n-$sphere correspond to great circles, that is, geodesics are the curves of intersection of the $n-$sphere with planes in $\mathbb{R}^{(n+1)}$ which intersect the origin. Therefore, since the transformation $x_0 \mapsto ix_0$ is linear and homogeneous, the geodesics in $d\boldsymbol{S}_n$ correspond to the intersections of $d\boldsymbol{S}_n$ with planes passing through the origin. Such geodesics can be characterized by the angle between the plane of intersection and the x_0 axis (in $\mathbb{R}^{1,n}$). Let us consider three cases separately.

Consider a plane with zero angle with respect to x_0 (that is, a plain containing the x_0 axis) given by $x_2 = 0, \ldots, x_n = 0$. Its intersection with $d\boldsymbol{S}_n$ is given by the equations $-(x_0)^2 + (x_1)^2 = \alpha^2$ and $x_2 = 0, \ldots, x_n = 0$. Hence this plane contains two non-intersecting geodesics, namely two hyperbola branches, which in the parametrization (16) - (17) corresponds to constant ζ_i. Clearly these two geodesics are time-like geodesics.

Now consider a plane with angle $\pi/4$ given by $x_0 = x_1$ and $x_3 = 0, \ldots, x_n = 0$. Its intersection with $d\boldsymbol{S}_n$ is given by the equations $(x_2)^2 = \alpha^2$ and $x_3 = 0, \ldots, x_n = 0$. Therefore, the intersection contains two non-intersecting geodesics which are actually straight lines. According to the embedding in $\mathbb{R}^{1,n}$ these two straight lines are light-like geodesics.

Finally, let us consider a plane with angle $\pi/2$ given by $x_0 = 0, x_3 = 0, \ldots, x_n = 0$. The intersection of this plane with $d\boldsymbol{S}_n$ is given by the equations $(x_1)^2 + (x_2)^2 = \alpha^2$ and $x_0 = 0, x_3 = 0, \ldots, x_n = 0$. Therefore, it defines one connected geodesic corresponding to a space-like circumference.

The "spatial isotropy" of $d\boldsymbol{S}_n$ allows us to extend the previous examples to arbitrary planes with angles $0, \pi/4, \pi/2$. By continuity the examples can be extended to arbitrary angles, hence describing every possible geodesic in $d\boldsymbol{S}_n$. The previous discussion prompts us with a particular feature of de Sitter space, namely the causal independence of antipodal events.

For any point $\mathbf{x} = (x_0, \ldots, x_n)$ in $d\boldsymbol{S}_n$ embedded in $\mathbb{R}^{1,n}$, its antipode is defined as $\bar{\mathbf{x}} = -\mathbf{x}$. Let us consider the point $\mathbf{x} = (0, \alpha, 0, \ldots, 0)$. Planes containing $\bar{\mathbf{x}}$ and $\mathbf{x}$ are either at an angle less than or equal to $\pi/4$ in which case the geodesics containing $\bar{\mathbf{x}}$ and $\mathbf{x}$ do not intersect, or the angle is greater than $\pi/4$ in which case there is an space-like geodesic connecting the two antipodal points. This results in the fact that the interior light cones of antipodes have no point in common. Moreover, consider an observer moving in a time-like world line from infinite past to infinite future. The events accessible to such observer are all the events which lie in the interior of the light cone originating in any position of the observer. As such, if an event $\mathbf{x}$ is accessible to the observer, then its antipode $\bar{\mathbf{x}}$ is not accessible. Hence the accessible region covers only half of the entire de Sitter space.

A parametrization of de Sitter space, covering only one half of the space-time, such that no pair of antipodal points are contained is possible. We shall describe this parametrization together with the parametrization yielding a metric of the form (2) in the next Section.

4. Lemaître-Robertson frame

In Section 2 of the present Chapter, the global parametrization of de Sitter space was described, namely equations (16) and (17). The adjective "global" is owed to the fact that the metric in this parametrization is non-singular and allows a parametrization of the entire de Sitter space. We shall now introduce a non-global parametrization with some interesting features.

Let us start from the equation (10) defining $d\boldsymbol{S}_n$. A "time" coordinate $\widehat{t}$ will be introduced such that in $d\boldsymbol{S}_n$ hypersurfaces of constant $\widehat{t}$ correspond to the intersection of $d\boldsymbol{S}_n$ with hyperplanes in $\mathbb{R}^{1,n}$ of constant $x_0 + x_n$. In the case $x_0 + x_n = 0$, such hypersurface would contain light-like geodesics (which is contrary to the intention that surfaces of constant $\widehat{t}$ be space-like). But in the case $x_0 + x_n \neq 0$, such hypersurfaces would be space-like. Hence, the Lemaître-Robertson frame shall only describe the region $x_0 + x_n > 0$ in $d\boldsymbol{S}_n$.

The other imposition which shall define the Lemaître-Robertson frame is that after introducing "space" coordinates $\widehat{\mathbf{r}} = (\widehat{x_1}, \dots, \widehat{x_{n-1}})$ the line $\widehat{\mathbf{r}} = \text{constant}$ be perpendicular (in the sense of the Lorentzian metric) to the hypersurfaces $\widehat{t} = \text{constant}$ (a further description of this frame is found in [**Sch56**]).

These restrictions can be guaranteed by defining (for $i = 1, \dots, n-1$):

(30) $$\widehat{t} = \alpha \ln\left(\frac{x_0 + x_n}{\alpha}\right)$$

(31) $$\widehat{x_i} = \alpha \frac{x_i}{x_0 + x_n}$$

This frame is a parametrization of one half of $d\boldsymbol{S}_n$, which gives an embedding into $\mathbb{R}^{1,n}$ by:

$$x_0 = \frac{1}{2}\left(\alpha e^{\widehat{t}/\alpha} - \alpha e^{-\widehat{t}/\alpha} + \frac{1}{\alpha} e^{\widehat{t}/\alpha} \sum_{i=1}^{n-1} \widehat{x}_i^{\,2}\right) \tag{32}$$

$$x_i = e^{\widehat{t}/\alpha} \widehat{x}_i \tag{33}$$

$$x_n = \frac{1}{2}\left(\alpha e^{\widehat{t}/\alpha} + \alpha e^{-\widehat{t}/\alpha} - \frac{1}{\alpha} e^{\widehat{t}/\alpha} \sum_{i=1}^{n-1} \widehat{x}_i^{\,2}\right) \tag{34}$$

In this frame the metric takes the form:

$$ds^2 = -d\widehat{t}^2 + e^{2\widehat{t}/\alpha} \sum_{i=1}^{n-1} d\widehat{x}_i^{\,2} \tag{35}$$

As such "contemporary spaces" (spaces of constant $\widehat{t}$) are infinite and flat, which is to be contrasted with the "contemporary spaces" in the global frame appear as spheres (compact and curved). In fact, the patch of $d\boldsymbol{S}_n$ described by this frame is homeomorphic to $\mathbb{R}^n$. The fact that the scale factor in (35) grows exponentially in time ($\widehat{t}$) is of interest in the field of cosmology since it describes inflationary expansion of space-time.

5. The reduced model

De Sitter universe $d\boldsymbol{S}_4$ is certainly a more appropriate, though not completely satisfying, cosmological model than Minkowski space-time. Hence, $d\boldsymbol{S}_4$ should be our main object of study. However, for practical reasons, we shall be mainly focused on the (so-called in [**Sch56**]) reduced model $d\boldsymbol{S}_2$. In itself, the reduced model exhibits certain interesting features. Thus, we shall briefly discuss this particular space.

Let us take the global coordinates on $d\boldsymbol{S}_2$ to be given by the embedding:

$$x_0 = \alpha \sinh(t/\alpha) \tag{36}$$

$$x_1 = \alpha \cosh(t/\alpha) \cos\theta \tag{37}$$

$$x_2 = \alpha \cosh(t/\alpha) \sin\theta \tag{38}$$

In such parametrization one can study the Killing vectors (11) and (12) which become

$$J = J_{12} = \partial_\theta \tag{39}$$

$$K_1 = \alpha \cos\theta \partial_t - \tanh(t/\alpha) \sin\theta \partial_\theta \tag{40}$$

$$K_2 = \alpha \sin\theta \partial_t + \tanh(t/\alpha) \cos\theta \partial_\theta \tag{41}$$

Which obey the commutation relations:

$$[K_1, K_2] = J \tag{42}$$

$$[K_2, J] = -K_1 \tag{43}$$

$$[J, K_1] = -K_2 \tag{44}$$

In principle, the vector fields J, K_1, K_2 can be integrated in order to obtain three different one-parameter groups which act on $d\boldsymbol{S}_2$. However, integrating the vector fields $K - 1$ and K_2 is not as direct as the case for J. Instead of following this approach, one could study the isometries of $d\boldsymbol{S}_2$ by studying groups which locally look like $O(1,2)$. Let $SO^+(1,n)$ be the identity component of the group $O(1,n)$. We shall discuss some groups which share the same Lie algebra as $SO^+(1,2)$.

Let $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^{1,2}$, this vector can be arranged as a hermitian matrix in $\mathbf{M}_2(\mathbb{C})$ such that the determinant of said matrix corresponds to the Lorentzian magnitude of the vector (actually to the opposite sign convention, but it shall make no difference), namely:

$$\mathbf{x} \sim \mathbf{X} = \begin{pmatrix} x_0 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 \end{pmatrix} \tag{45}$$

Hence, a transformation of $\mathbf{M}_2(\mathbb{C})$ which preserves determinant, self-adjointness and such that the resulting transformation of $\mathbf{X}$ is constant in the diagonal will correspond to an element of $O(1,2)$.

Consider the group $SU(1,1)$, that is the group of complex, invertible matrices of determinant 1 which preserve the indefinite bilinear form, namely:

$$SU(1,1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2(\mathbb{C}) : ad - bc = 1, \right.$$
$$\left. \text{and } \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{46}$$

Hence, $SU(1,1)$ can also be defined by

$$SU(1,1) = \left\{ \begin{pmatrix} \beta & \gamma \\ \bar{\gamma} & \bar{\beta} \end{pmatrix} \in \mathbf{M}_2(\mathbb{C}) : |\beta|^2 - |\gamma|^2 = 1 \right\} \tag{47}$$

If $A \in SU(1,1)$, then:

$$A\mathbf{X}A^\dagger = \begin{pmatrix} (|\beta|^2+|\gamma|^2)x_0+(\bar{\beta}\gamma+\beta\bar{\gamma})x_1+i(\bar{\beta}\gamma-\beta\bar{\gamma})x_2 & 2\beta\gamma x_0+(\beta^2+\gamma^2)x_1-i(\beta^2-\gamma^2)x_2 \\ 2\bar{\beta}\bar{\gamma}x_0+(\bar{\beta}^2+\bar{\gamma}^2)x_1+i(\bar{\beta}^2-\bar{\gamma}^2)x_2 & (|\beta|^2+|\gamma|^2)x_0+(\bar{\beta}\gamma+\beta\bar{\gamma})x_1+i(\bar{\beta}\gamma-\beta\bar{\gamma})x_2 \end{pmatrix} \tag{48}$$

This transformation of $\mathbb{R}^{1,2}$ corresponds to the matrix $B \in O(1,2)$.

$$B = \begin{pmatrix} |\beta|^2 + |\gamma|^2 & \bar{\beta}\gamma + \beta\bar{\gamma} & i(\bar{\beta}\gamma - \beta\bar{\gamma}) \\ \bar{\beta}\bar{\gamma} + \beta\gamma & \frac{1}{2}(\beta^2 + \bar{\beta}^2 + \gamma^2 + \bar{\gamma}^2) & \frac{i}{2}(\bar{\beta}^2 - \beta^2 + \gamma^2 - \bar{\gamma}^2) \\ i(\beta\gamma - \bar{\beta}\bar{\gamma}) & \frac{i}{2}(\beta^2 - \bar{\beta}^2 + \gamma^2 - \bar{\gamma}^2) & \frac{1}{2}(\beta^2 + \bar{\beta}^2 - \gamma^2 - \bar{\gamma}^2) \end{pmatrix} \tag{49}$$

Notice that A and $-A$ correspond to the same element $B \in O(1,2)$. In fact, the map $SU(1,1) \longrightarrow SO^+(1,2)$ such that $A \mapsto B$ is a double covering (surjective 2 to 1 map). One could also define a double covering of $SO^+(1,2)$ by $SL(2,\mathbb{R})$, since there is an isomorphism $SL(2,\mathbb{R}) \longrightarrow SU(1,1)$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a+d+i(b-c) & b+c+i(a-d) \\ b+c-i(a-d) & a+d+i(c-b) \end{pmatrix} \tag{50}$$

Therefore, an element of $SL(2,\mathbb{R})$ corresponds to an element of $O(1,2)$ as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}(a^2 + b^2 + c^2 + d^2) & ab + cd & \frac{1}{2}(b^2 - a^2 + d^2 - c^2) \\ ac + bd & ad + bc & bd - ac \\ \frac{1}{2}(d^2 + c^2 - a^2 - b^2) & cd - ab & \frac{1}{2}(a^2 + d^2 - b^2 - c^2) \end{pmatrix} \tag{51}$$

One parameter subgroups of $SL(2,\mathbb{R})$ are easily obtained by the Iwasawa decomposition $SL(2,\mathbb{R}) = \boldsymbol{KAN}$ (this is in fact a homeomorphism, though not a homomorphism), and one also has the decomposition $SL(2,\mathbb{R}) = \boldsymbol{KAH}$. Denoting the one parameter subgroups $\boldsymbol{K}, \boldsymbol{A}, \boldsymbol{N}, \boldsymbol{H}$ as follows.

$$\boldsymbol{N} = \left\{ u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\} \tag{52}$$

$$\boldsymbol{A} = \left\{ a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\} \tag{53}$$

$$\boldsymbol{H} = \left\{ h(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\} \tag{54}$$

$$\boldsymbol{K} = \left\{ r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} \tag{55}$$

This one parameter groups allow us to define various isometric action of $\mathbb{R}$ on $d\boldsymbol{S}_2$.

6. Group contractions

An interesting feature of de Sitter space comes from considering its structure when the parameter α tends to infinity. One can study different limiting procedures on the Lie algebra of $SO^+(1,n)$. In particular we shall show how $SO^+(1,n)$ contracts to the Galilei group $E(n)$ and how it contracts to the Poincaré group $E(1,n-1)$. We shall borrow from [**Gil74**] in order to discuss the meaning of this limiting procedures.

Let us consider a $3-$sphere of radius R. Let $d > 0$ and consider an observer at the north pole (described as $(0,0,R)$) of the sphere. We can let this observer move in a geodesic of length d in the direction $\vec{j_1}$ and then move in a geodesic of length d in a direction $\vec{j_2}$ perpendicular to his first displacement. The curvature of the sphere would cause that this combined displacement be different from the one he would have achieved had he started moving in a direction $\vec{j_2}$ (parallelly transported to the north pole) and then in a direction $\vec{j_1}$ (parallelly transported from the north pole).

If the radius of the sphere were to be fixed, the difference between the two paths could be made arbitrarily small taking an arbitrarily small distance d. It is so, since the difference of the two paths is a continuous function of the distance travelled, which goes to zero, trivially, when the distance travelled is zero. Equivalently, the distance travelled could be a fixed value, then the difference of the two paths could be made arbitrarily big letting the radius assume an arbitrarily large value.

This argument suggests a limiting procedure by which the sphere comes to resemble the plane. The sphere can be interpreted as the quotient $S_2 \simeq SO(3)/SO(2)$, while the plane can be interpreted as the quotient $\mathbb{R}^2 \simeq E^+(2)/SO(2)$ (where $E(2)$ represents the euclidean group in two dimensions, the semidirect product $\mathbb{R}^2 \rtimes O(2)$). Then, the limiting process can be thought of as a limit $SO(3) \longrightarrow E^+(2)$. This leads to the concept of a group contraction.

A group contraction is defined in terms of the corresponding Lie algebras. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The Inönü-Wigner contraction (which is the group contraction that shall interest us) corresponds to a sequence $(\mathfrak{g}_n)_n$ of Lie algebras isomorphic to $\mathfrak{g}$ obtained by a non-singular change of basis, which becomes singular in the limit. A main requirement of such contraction is that the Lie bracket be well defined in the non-singular limit, hence a different Lie algebra arises in the limit.

Let us regress to the sphere example in order to illustrate the idea.

The rotation group $SO(3)$ in three dimensions has three infinitesimal generators corresponding to rotations about each axis in $\mathbb{R}^3$. Namely, the generators L_x, L_y, L_z fulfil the following commutation relations.

$$[L_x, L_y] = L_z \qquad [L_y, L_z] = L_x \qquad [L_z, L_x] = L_y \tag{56}$$

The euclidean group in two dimensions $E(2)$ has three infinitesimal generators, one corresponding to the rotation of the plane, and the other two corresponding to translations. Namely, the generators L, P_1, P_2 fulfil the following commutation relations.

$$[P_1, P_2] = 0 \qquad [L, P_1] = P_2 \qquad [L, P_2] = -P_1 \tag{57}$$

Now one can consider the change of basis in $\mathfrak{so}(3)$ by defining $P_x = L_x/R$ and $P_y = L_y/R$. The corresponding variation of the commutation relations is as follows.

$$[P_x, P_y] = \frac{L_z}{R^2} \qquad [L_z, P_x] = P_y \qquad [L_z, P_y] = -P_x \tag{58}$$

This relations define the same algebra $\mathfrak{so}(3)$. But in the limit when $R \longrightarrow \infty$ the relations for $\mathfrak{e}(2)$ are obtained.

This limiting procedure can be thought of in geometric terms referring back to the description of an observer in S_2 standing in the north pole. Exponentiation of the infinitesimal generators L_x, L_y, L_z produces the rotations of S_2. To the observer at the pole the rotations generated by L_z are indeed rotations (the observer stays static), while the rotations generated by the exponentiation of L_x and L_y move the the observer an angle of 1 radian, which corresponds to a distance of R. While the exponentiation of P_x and P_y produce a rotation in an angle of $1/R$ radians, hence a unit displacement of the observer. Thus, the change of basis can be thought of as introducing an scale factor such that the displacement induced upon the observer are constant, regardless of the sphere's radius.

The previous procedure can be carried over to the de Sitter group. There, however, the geometric interpretation is not so straightforward. The infinitesimal generators of the group $SO(1, n)$ are K_i, J_{jk} (for $i, j, k \in \{1, \ldots, n\}$ and $j \neq k$) with the following commutation relations.

$$[K_i, K_j] = J_{ij} \tag{59}$$

$$[K_i, J_{jk}] = \delta_{i,j} K_k - \delta_{i,k} K_j \tag{60}$$

$$[J_{ij}, J_{kl}] = \delta_{i,l} J_{jk} - \delta_{i,k} J_{jl} + \delta_{j,k} J_{il} - \delta_{j,l} J_{ik} \tag{61}$$

We shall study two different contractions.

(i.) $\mathfrak{so}(1,n) \longrightarrow \mathfrak{e}(n)$

Let us define $P_i = K_i/\alpha$. Then the commutation relations are as follows.

$$[P_i, P_j] = \frac{J_{ij}}{\alpha^2} \longrightarrow 0 \tag{62}$$

$$[P_i, J_{jk}] = \delta_{i,j} P_k - \delta_{i,k} P_j \tag{63}$$

$$[J_{ij}, J_{kl}] = \delta_{i,l} J_{jk} - \delta_{i,k} J_{jl} + \delta_{j,k} J_{il} - \delta_{j,l} J_{ik} \tag{64}$$

In the limit $\alpha \longrightarrow \infty$ these are the commutation relations for the euclidean group in n dimensions.

(ii.) $\mathfrak{so}(1,n) \longrightarrow \mathfrak{e}(1,n-1)$

We shall consider the cases $n = 2, 4$. The others can be generalized by extending the matrices η and M which will be introduced for the two examples.

1) $\mathfrak{so}(1,2) \longrightarrow \mathfrak{e}(1,1)$

Let $P^0 = \frac{K_1}{\alpha}$, $P^1 = \frac{J_{12}}{\alpha}$, $\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M^{\mu\nu} = \begin{pmatrix} 0 & K_2 \\ -K_2 & 0 \end{pmatrix}$. Then the commutation relations are as follows.

$$[P^0, P^1] = \frac{K_2}{\alpha^2} \longrightarrow 0 \tag{65}$$

$$[M^{\mu\nu}, P^\sigma] = \eta^{\mu\sigma} P^\nu - \eta^{\nu\sigma} P^\mu \tag{66}$$

$$[M^{\mu\nu}, M^{\sigma\tau}] = \eta^{\mu\sigma} M^{\nu\tau} - \eta^{\mu\tau} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\tau} + \eta^{\nu\tau} M^{\mu\sigma} \tag{67}$$

The matrices $M^{\mu\nu}$ and $\eta^{\mu\nu}$ were introduced to explain and motivate the generalization to higher dimensions. This case ($n = 2$) is easier to describe since the $E(1,1)$ is generated by a boost K_2, time translations P^0 and space translations P^1. The translations commute, and the remaining commutation relations correspond to $[K_2, P^0] = -P^1$ and $[K_2, P^1] = -P^0$.

2) $\mathfrak{so}(1,4) \longrightarrow \mathfrak{e}(1,3)$

Let us define $P^0 = \frac{K_1}{\alpha}$ and $P^i = \frac{J_{1,i+1}}{\alpha}$. Consider the matrices η, M given by $\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $M^{\mu\nu} = \begin{pmatrix} 0 & K_2 & K_3 & K_4 \\ -K_2 & 0 & -J_{23} & -J_{24} \\ -K_3 & J_{23} & 0 & -J_{34} \\ -K_4 & J_{24} & J_{34} & 0 \end{pmatrix}$. The obtained commutation relations are the following.

$$[P^\mu, P^\nu] \longrightarrow 0 \tag{68}$$

$$[M^{\mu\nu}, P^\sigma] = \eta^{\mu\sigma} P^\nu - \eta^{\nu\sigma} P^\mu \tag{69}$$

$$[M^{\mu\nu}, M^{\sigma\tau}] = \eta^{\mu\sigma} M^{\nu\tau} - \eta^{\mu\tau} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\tau} + \eta^{\nu\tau} M^{\mu\sigma} \tag{70}$$

These correspond to the infinitesimal generators of the Poincaré group, as in [**Sch13**], Section 6.3.1. (notice the sign convention is opposite in the reference listed and the physicist convention[1] is in use in the reference).

Our interest in the study of group contractions comes from the fact that a particular contraction of the de Sitter group yields the Poincaré group. Since a further work shall involve an investigation on field theories defined in a noncommutative de Sitter space.

7. Representations of the reduced de Sitter group

In this Section we shall briefly discuss representations of the group $SO(1,2)$. This will be of particular interest to us when, in future work, the approach of [**Thi67**] to define a scalar field in $d\boldsymbol{S}_2$ is studied in order to develop a generalization to the noncommutative setting.

Keeping with the notation of [**Thi67**] we shall redefine J, K_1, K_2 the Killing fields in $d\boldsymbol{S}_2$. In future work, we shall be interested in Quantum Field Theories in de Sitter space (in particular in the reduced model), and the approach we shall adopt is the study of representations of the de Sitter group. Hence we will be interested in the complexification of the Lie algebra $\mathfrak{so}(1,2)$. Therefore, a complex unit factor can be introduced as follows (this is the physicist convention mentioned in the footnote[1]).

$$\boldsymbol{K_1} \longrightarrow -iK_1 \qquad \boldsymbol{K_2} \longrightarrow -iK_2 \qquad \boldsymbol{K_3} \longrightarrow -iJ \tag{71}$$

[1]As commented in [**BD85**] (2.21) it is customary in physics to define the exponential of a Lie algebra as $\exp : B \mapsto e^{iB}$, hence the Lie algebra elements include a factor of $-i$. Up to Section 6 all arguments have been mainly geometrical, though with important physical implications. Thus the physicists convention is not in use in Section 6, but we shall employ it in Section 7.

With (71) the redefined commutation relations take the following form.

$$[\boldsymbol{K_1}, \boldsymbol{K_2}] \longrightarrow [-iK_1, -iK_2] = -J \longleftarrow -i\boldsymbol{K_3} \tag{72}$$

$$[\boldsymbol{K_2}, \boldsymbol{K_3}] \longrightarrow [-iK_2, -iJ] = K_1 \longleftarrow i\boldsymbol{K_1} \tag{73}$$

$$[\boldsymbol{K_3}, \boldsymbol{K_1}] \longrightarrow [-iJ, -iK_1] = K_2 \longleftarrow i\boldsymbol{K_2} \tag{74}$$

Then, a Casimir element can be defined as follows.

$$\boldsymbol{I} = \boldsymbol{K_3^2} - \boldsymbol{K_1^2} - \boldsymbol{K_2^2} \tag{75}$$

It can be noticed that $\boldsymbol{K_3}$ represents a rotation. Thus we can consider unitary representations of $\mathfrak{so}(1,2)$ where the $\boldsymbol{K_i}$ are self-adjoint operators and $\boldsymbol{K_3}$ is diagonal with integer eigenvalues. As such the following relation must hold.

$$\langle m | \boldsymbol{K_3} | m' \rangle = m\delta_{mm'} \tag{76}$$

In direct analogy to the harmonic oscillator, the following ladder operators can be defined.

$$\boldsymbol{K_\pm} = \boldsymbol{K_1} \pm i\boldsymbol{K_2} \tag{77}$$

Which have the following properties.

$$\begin{aligned} \boldsymbol{K_3}\boldsymbol{K_\pm} &= \boldsymbol{K_3}(\boldsymbol{K_1} \pm \boldsymbol{K_2}) = [\boldsymbol{K_3}, \boldsymbol{K_1}] + \boldsymbol{K_1}\boldsymbol{K_3} \pm i([\boldsymbol{K_3}, \boldsymbol{K_2}] + \boldsymbol{K_2}\boldsymbol{K_3}) \\ &= i\boldsymbol{K_2} + \boldsymbol{K_\pm}\boldsymbol{K_3} \pm \boldsymbol{K_1} = \boldsymbol{K_\pm}(\boldsymbol{K_3} \pm 1) \end{aligned} \tag{78}$$

Therefore, $\boldsymbol{K_\pm}|m\rangle$ is an eigenvector of $\boldsymbol{K_3}$ with eigenvalue $m \pm 1$.

The product of the ladder operators gives the following equation.

$$\boldsymbol{K_\pm}\boldsymbol{K_\mp} = \boldsymbol{K_1^2} + i(\mp\boldsymbol{K_1}\boldsymbol{K_2} \pm \boldsymbol{K_2}\boldsymbol{K_1}) + K_2^2 = -\boldsymbol{I} + \boldsymbol{K_3^2} \mp \boldsymbol{K_3} \tag{79}$$

Which, together with $\boldsymbol{K_\pm^\dagger} = \boldsymbol{K_\mp}$, implies the following.

$$\| \boldsymbol{K_\mp} | m \rangle \|^2 = \langle m | \boldsymbol{K_\pm}\boldsymbol{K_\mp} | m \rangle = (-I + m(m \mp 1))\|m\|^2 \tag{80}$$

Thus, the coefficient is non negative, that is $-I + m(m \mp 1) \geq 0$. It can be noticed that for $m \in \mathbb{Z}$ one has $m(m \mp 1) \geq 0$. One can study two cases, either $I \leq 0$ and no restriction on m arises, or $I > 0$ and $m(m \pm 1) \geq I > 0$.

Let us consider $I > 0$. Notice that equations (79) and (80) imply the following result.

$$\boldsymbol{K_\pm}|m\rangle = \sqrt{-I + m(m \pm 1)}\; |m \pm 1\rangle \tag{81}$$

Hence, for some integers $m_u < 0 < m_l$ one must have $\boldsymbol{K}_-|m_l\rangle = 0$ and $\boldsymbol{K}_+|m_u\rangle = 0$. Hence, the next equation holds.

$$m_u(m_u+1) = I = m_l(m_l-1) \tag{82}$$

At this point we have a description of the unitary representations where the operators $\boldsymbol{K_i}$ are self-adjoint. In a future work we shall return to this point in order to study quantum field theories in de Sitter space. However, before proceeding to the study of noncommutative geometry, we shall discuss the effect on the representations of the group contraction $\mathfrak{so}(1,2) \longrightarrow \mathfrak{e}(1,1)$.

From the previous Section we know that this group contraction can be achieved by introducing a change of basis $\boldsymbol{P_1} = \frac{\boldsymbol{K_1}}{\alpha}$ and $\boldsymbol{P_3} = \frac{\boldsymbol{K_3}}{\alpha}$. Let us define $\boldsymbol{\mu}^2 = \boldsymbol{P_3}^2 - \boldsymbol{P_1}^2$. The change of basis has an effect on the Casimir element as shown in the following equation.

$$\boldsymbol{I} = \alpha^2\left(\frac{\boldsymbol{K_3}^2}{\alpha^2} - \frac{\boldsymbol{K_1}^2}{\alpha^2} - \frac{\boldsymbol{K_2}^2}{\alpha^2}\right) = \alpha^2\left(\boldsymbol{P_3}^2 - \boldsymbol{P_1}^2 - \frac{\boldsymbol{K_2}^2}{\alpha^2}\right) \longrightarrow \alpha^2(\boldsymbol{P_3}^2 - \boldsymbol{P_1}^2) = \alpha^2\boldsymbol{\mu}^2 \tag{83}$$

Let us denote the eigenvalues of $\boldsymbol{P_3}$ by p_3, then $p_3 = m/\alpha$. Thus the spectrum of $\boldsymbol{P_3}$ becomes continuous in the limit. While $\boldsymbol{P_1}^2 = \boldsymbol{P_3}^2 - \boldsymbol{\mu}^2$, thus $\boldsymbol{P_1}$ has eigenvalues $p_1 = \pm\sqrt{-\mu^2 + (m/\alpha)^2}$. Once again, we shall consider two separate cases depending on I, namely $I > 0$ and $I < 0$.

If $I > 0$, then $\mu^2 > 0$. This restricts the values of p_1 and p_3 to $\mu < p_3 < \infty$ and $-\infty < p_1 < \infty$. Thus, p_3 can be thought of as the energy and p_1 as the momentum of a particle with mass μ. This interpretation would imply that the Killing vector K_3 (corresponding to rotations of $d\boldsymbol{S}_2$) be associated to time. Hence this would represent a universe infinite in space and periodic in time.

If $I < 0$, then $\mu^2 < 0$. Hence the restrictions obtained are $-\infty < p_3 < \infty$ and $-\infty < p_1 < -|\mu|$ or $|\mu| < p_1 < \infty$. This can be thought of as representing energy with p_1 and momentum with p_3. Thus it refers to the more usual interpretation of de Sitter space as describing compact space-like surfaces and a universe infinite in time.

CHAPTER 4

The noncommutative catenoid of Arnlind and Holm

Minimal surfaces arose as solutions to Plateau's problem. This is the problem of finding the surfaces minimizing area with the constraint of a fixed boundary. In an analogue fashion as curves of minimal length are geodesics, and a local characterization of such minimality property can be given, so can a local characterization of minimal surfaces be given. It turns out that surfaces of minimal area are characterized by having mean curvature zero everywhere.

One of the first minimal surfaces to be discovered and studied was the catenoid. This is actually a surface of revolution whose profile curve is a catenary. A useful parametrization of the catenoid is given by:

$$x_1 = \cosh u \sin v \tag{84}$$

$$x_2 = \cosh u \cos v \tag{85}$$

$$x_3 = u. \tag{86}$$

1. The deformed algebra

The algebra of complex functions generated by the functions $x_1(u,v),\ x_2(u,v),\ x_3(u,v)$ can also be generated by the functions $u, e^{\pm u}, e^{\pm iv}$. And so an action can be defined, $\alpha : \mathbb{R}^2 \times \mathscr{A} \longrightarrow \mathscr{A}$, where $\mathscr{A}$ is the algebra of complex-valued functions generated by

$$R = e^u \quad W = e^{iv} \quad U = u.$$

The action is given by:

$$\alpha(t,s)(u) = u + t \tag{87}$$

$$\alpha(t,s)(e^u) = e^(u + t) \tag{88}$$

$$\alpha(t,s)(e^{iv}) = e^{i(v+s)}. \tag{89}$$

Following Rieffel's deformation quantization (as in [**Rie93**]), a deformed product can be introduced in $\mathscr{A}$ (therefore obtaining a deformed algebra $\mathscr{A}_\theta$) by:

$$f \times_\theta g = \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} \alpha(\theta(t,s))(f)\alpha(t',s')(g)e^{2\pi i(tt'+ss')}dtdt'dsds' \tag{90}$$

where $\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an antisymmetric linear transformation given by the matrix

$$\theta = \begin{pmatrix} 0 & \pi\hbar \\ -\pi\hbar & 0 \end{pmatrix}.$$

We shall follow Arnlind and Holm's ([**AH17**]) notation and introduce the following functions (which we will prove correspond, respectively, to the inverses of R and W)

$$\tilde{R} = e^{-u} \quad \tilde{W} = e^{-iv}$$

The following are the deformed product of the functions $R, W, U, \tilde{R}, \tilde{W}$.

$$\begin{aligned} e^{iv} \times_\theta e^{-iv} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{i(v-\pi\hbar t)}e^{-i(v+s')}e^{2\pi i(tt'+ss')}dtdt'dsds' \\ &= \int\limits_{\mathbb{R}^2} e^{2\pi it(t'-\hbar/2)}dtdt' \int\limits_{\mathbb{R}^2} e^{2\pi is'(s-1/2\pi)}ds'ds = 1 \end{aligned} \tag{91}$$

$$\begin{aligned} e^{-iv} \times_\theta e^{iv} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{-i(v-\pi\hbar t)}e^{i(v+s')}e^{2\pi i(tt'+ss')}dtdt'dsds' \\ &= \int\limits_{\mathbb{R}^2} e^{2\pi it(t'+\hbar/2)}dtdt' \int\limits_{\mathbb{R}^2} e^{2\pi is'(s+1/2\pi)}dsds' = 1 \end{aligned} \tag{92}$$

$$\begin{aligned} e^{u} \times_\theta e^{-u} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{u+\pi\hbar s}e^{-(u+t')}e^{2\pi i(tt'+ss')}dtdt'dsds' \\ &= \int\limits_{\mathbb{R}^2} e^{s\pi\hbar}e^{2\pi iss'}ds'ds \int\limits_{\mathbb{R}^2} e^{-t'}e^{2\pi itt'}dtdt' = 1 \end{aligned} \tag{93}$$

$$
\begin{aligned}
e^{-u} \times_\theta e^u &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{-(u+\pi\hbar s)} e^{u+t'} e^{2\pi i(tt'+ss')} dtdt'dsds' \\
&= \int\limits_{\mathbb{R}^2} e^{-\pi\hbar s+t'} \int\limits_{\mathbb{R}} e^{2\pi i tt'} \int\limits_{\mathbb{R}} e^{2\pi i ss'} ds'dtdsdt' = 1
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
e^u \times_\theta u &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{u+\pi\hbar s}(u+t') e^{2\pi i(tt'+ss')} dtdt'dsds' \\
&= e^u \int\limits_{\mathbb{R}^2} e^{\pi\hbar s} e^{2\pi i ss'} ds'ds \int\limits_{\mathbb{R}^2} (u+t') e^{2\pi i tt'} dtdt' = e^u u
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
u \times_\theta e^u &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} (u+\pi\hbar s) e^{u+t'} e^{2\pi i(tt'+ss')} dtdt'dsds' \\
&= e^u \int\limits_{\mathbb{R}^2} (u+\pi\hbar s) e^{2\pi i ss'} ds'ds \int\limits_{\mathbb{R}^2} e^{t'} e^{2\pi i tt'} dtdt' = e^u u
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
u \times_\theta e^{-u} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} (u+\pi\hbar s) e^{-(u+t')} e^{2\pi i(tt'+ss')} dtdt'dsds' \\
&= e^{-u} \int\limits_{\mathbb{R}^2} (u+\pi\hbar s) e^{2\pi i ss'} ds'ds \int\limits_{\mathbb{R}^2} e^{-t'} e^{2\pi i tt'} dtdt' = e^{-u} u
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
e^{-u} \times_\theta u &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{-(u+\pi\hbar s)} (u+t') e^{2\pi i(tt'+ss')} dtdt'dsds' \\
&= e^{-u} \int\limits_{\mathbb{R}^2} e^{-\pi\hbar s} e^{2\pi i ss'} ds'ds \int\limits_{\mathbb{R}^2} (u+t') e^{2\pi i tt'} dtdt' = e^{-u} u
\end{aligned} \tag{98}
$$

$$
\begin{aligned}
e^u \times_\theta e^{iv} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{u+\pi\hbar s} e^{i(v+s')} e^{2\pi i(tt'+ss')} dtdt'dsds' \\
&= e^u e^{iv} \int\limits_{\mathbb{R}^2} e^{\pi\hbar s} e^{2\pi i s'(s+1/2\pi)} ds'ds \int\limits_{\mathbb{R}^2} e^{2\pi i tt'} dtdt' = e^u e^{iv} e^{-\hbar/2}
\end{aligned} \tag{99}
$$

$$
\begin{aligned}
e^{iv} \times_\theta e^u &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{i(v-\pi\hbar t)} e^{u+t'} e^{2\pi i(tt'+ss')} dtdt'dsds' \\
&= e^u e^{iv} \int\limits_{\mathbb{R}^2} e^{t'} e^{2\pi i t(t'-\hbar/2)} dtdt' = e^u e^{iv} e^{\hbar/2}
\end{aligned} \tag{100}
$$

$$\begin{aligned}e^{-u} \times_\theta e^{iv} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-(u+\pi\hbar s)} e^{i(v+s')} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\ &= e^{-u}e^{iv}\int_{\mathbb{R}^2} e^{-\pi\hbar s} e^{2\pi i s'(s+1/2\pi)} ds' ds \int_{\mathbb{R}^2} e^{2\pi i t t'} dt dt' = e^{-u}e^{iv}e^{\hbar/2}\end{aligned} \tag{101}$$

$$\begin{aligned}e^{iv} \times_\theta e^{-u} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i(v-\pi\hbar t)} e^{-(u+t')} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\ &= e^{-u}e^{iv}\int_{\mathbb{R}} e^{-t'} \int_{\mathbb{R}} e^{2\pi i t(t'-\hbar/2)} = e^{-u}e^{iv}e^{-\hbar/2}\end{aligned} \tag{102}$$

$$\begin{aligned}e^{-iv} \times_\theta e^{u} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(v-\pi\hbar t)} e^{u+t'} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\ &= e^{u}e^{-iv}\int_{\mathbb{R}} e^{t'} \int_{\mathbb{R}} e^{2\pi i t(t'+\hbar/2)} dt' dt = e^{u}e^{-iv}e^{-\hbar/2}\end{aligned} \tag{103}$$

$$\begin{aligned}e^{u} \times_\theta e^{-iv} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{u+\pi\hbar s} e^{-i(v+s')} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\ &= e^{u}e^{-iv}\int_{\mathbb{R}} e^{\pi\hbar s} \int_{\mathbb{R}} e^{2\pi i s'(s-1/2\pi)} ds' ds = e^{u}e^{-iv}e^{\hbar/2}\end{aligned} \tag{104}$$

$$\begin{aligned}e^{-u} \times_\theta e^{-iv} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-(u+\pi\hbar s)} e^{-i(v+s')} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\ &= e^{-u}e^{-iv}\int_{\mathbb{R}} e^{-\pi\hbar s} \int_{\mathbb{R}} e^{2\pi i s'(s-1/2\pi)} ds' ds = e^{-u}e^{-iv}e^{-\hbar/2}\end{aligned} \tag{105}$$

$$\begin{aligned}e^{-iv} \times_\theta e^{-u} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(v-\pi\hbar t)} e^{-(u+t')} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\ &= e^{-u}e^{-iv}\int_{\mathbb{R}} e^{-t'} \int_{\mathbb{R}} e^{2\pi i t(t'+\hbar/2)} dt dt' = e^{-u}e^{-iv}e^{\hbar/2}\end{aligned} \tag{106}$$

$$\begin{aligned}e^{iv} \times_\theta u &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i(v-\pi\hbar t)} (u+t') e^{2\pi i(tt'+ss')} dt dt' ds ds' \\ &= e^{iv}\int_{\mathbb{R}} (u+t') \int_{\mathbb{R}} e^{2\pi i t(t'-\hbar/2)} dt' dt = e^{iv}(u+\hbar/2)\end{aligned} \tag{107}$$

$$
\begin{aligned}
u \times_\theta e^{iv} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (u + \pi\hbar s) e^{i(v+s')} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\
&= e^{iv} \int_{\mathbb{R}} (u + \pi\hbar s) \int_{\mathbb{R}} e^{2\pi i s'(s+1/2\pi)} ds' ds = e^{iv}(u - \hbar/2)
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
e^{-iv} \times_\theta u &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(v-\pi\hbar t)} (u + t') e^{2\pi i(tt'+ss')} dt dt' ds ds' \\
&= e^{-iv} \int_{\mathbb{R}} (u + t') \int_{\mathbb{R}} e^{2\pi i t(t'+\hbar/2)} dt dt' = e^{-iv}(u - \hbar/2)
\end{aligned} \tag{109}
$$

$$
\begin{aligned}
u \times_\theta e^{-iv} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (u + \pi\hbar s) e^{-i(v+s')} e^{2\pi i(tt'+ss')} dt dt' ds ds' \\
&= e^{-iv} \int_{\mathbb{R}} (u + \pi\hbar s) \int_{\mathbb{R}} e^{2\pi i s'(s-1/2\pi)} ds' ds = e^{-iv}(u + \hbar/2)
\end{aligned} \tag{110}
$$

The former products allow us to state the commutation relations:

$$
\begin{aligned}
\tilde{W}W &= \mathbb{1} & W\tilde{W} &= \mathbb{1} \\
R\tilde{R} &= \mathbb{1} & \tilde{R}R &= \mathbb{1} \\
RU &= UR & \tilde{R}U &= U\tilde{R} \\
WR &= e^{\hbar}RW & W\tilde{R} &= e^{-\hbar}\tilde{R}W \\
\tilde{W}R &= e^{-\hbar}R\tilde{W} & \tilde{W}\tilde{R} &= e^{\hbar}\tilde{R}\tilde{W} \\
WU &= UW + \hbar W & \tilde{W}U &= U\tilde{W} - \hbar\tilde{W}
\end{aligned} \tag{111}
$$

At this point it should be noted that the approach in [**AH17**] is different. Namely, they start from a Weyl algebra consisting of two hermitian generators U, V satisfying $[U, V] = i\hbar\mathbb{1}$, and obtain the relations (111) by exponentiation guided by a Baker-Campbell-Hausdorff type formula. From these relations, an algebra $\mathcal{C}_\hbar$ is defined as a quotient of $\mathbb{C}\langle U, R, \tilde{R}, W, \tilde{W}\rangle$, the free associative unital algebra on the letters $U, R, \tilde{R}, W, \tilde{W}$, by the two-sided ideal generated by the relations. The involution introduced on $\mathcal{C}_\hbar$ agrees with the involution of $\mathscr{A}$. Thus, $\mathcal{C}_\hbar$ is a subalgebra (closed under the operations of addition and product) of $\mathscr{A}_\theta$.

Since many functions in the original (commutative) algebra of the catenoid have no analogue in $\mathcal{C}_\hbar$, the approach in [**AH17**] is to describe an extension of $\mathcal{C}_\hbar$ denoted by $\hat{\mathcal{C}}_\hbar$. Nonetheless, most of the results proved in [**AH17**] can be reduced to the properties of $\mathcal{C}_\hbar$, hence we shall not elaborate on the construction of $\hat{\mathcal{C}}_\hbar$.

2. Calculus in the deformed algebra

A first geometric notion that can be introduced in the algebra $\mathcal{C}_\hbar$ is that of vector fields. Two different approaches can be given. As described in Chapter 2, the smooth vector fields are derivations of the algebra and form a finitely generated projective module over the algebra.

Let us first think in terms of the algebra derivations, particularly we shall describe the tangent space at each point). The formal correspondence $U \sim u$, $W \sim e^{iv}$ and $R \sim e^u$ suggests the introduction of derivations ∂_u, ∂_v such that the following holds.

$$\begin{array}{lll} \partial_u U = \mathbb{1} & \partial_u R = R & \partial_u W = 0 \\ \partial_v U = 0 & \partial_v R = 0 & \partial_v W = iW \end{array} \tag{112}$$

From ∂_u and ∂_v an abelian complex Lie algebra $\mathfrak{g}$ is constructed, consisting of a particular set of derivations in $\hat{\mathcal{C}}_\hbar$.

Let us consider the second approach. As an embedded surface in $\mathbb{R}^3$, tangent vectors to the catenoid at a given point are spanned by two linearly independent vectors in $\mathbb{R}^3$. Namely, given the parametrization

$$\vec{z}(u,v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u) \tag{113}$$

the derivations $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ of $\vec{z}$ give the tangent vectors

$$\frac{\partial \vec{z}}{\partial u} = (\sinh(u)\cos(v), \sinh(u)\sin(v), 1) \tag{114}$$

$$\frac{\partial \vec{z}}{\partial v} = (-\cosh(u)\sin(v), \cosh(u)\cos(v), 0) \tag{115}$$

which span the tangent space at every point $\vec{z}(u,v)$. Since the interest is put on the complex vector fields, one can define the following vector field.

$$\phi = (\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})\vec{z} = (\sinh(u+iv), -i\cosh(u+iv), 1) \tag{116}$$

Thus, as a module over the algebra $\mathscr{A}$, the vector fields on the catenoid are spanned by ϕ and $\bar{\phi}$. Elements Φ and $\bar{\Phi}$ of $(\hat{\mathcal{C}}_\hbar)^3$, the free right module generated by a set of generators

$\{e_1, e_2, e_3\}$, are defined in analogy to ϕ and $\bar{\phi}$. Namely

$$\Phi = e_1 \frac{1}{2} e^{\hbar/2}(RW - R^{-1}W^{-1}) + e_2 \frac{-i}{2} e^{\hbar/2}(RW + R^{-1}W^{-1}) + e_3 \mathbb{1} \tag{117}$$

$$\bar{\Phi} = e_1 \frac{1}{2} e^{\hbar/2}(W^{-1}R - WR^{-1}) + e_2 \frac{i}{2} e^{\hbar/2}(W^{-1}R + WR^{-1}) + e_3 \mathbb{1} \tag{118}$$

The two elements $\Phi = e_1 \Phi^1 + e_2 \Phi^2 + e_3 \Phi^3$ and $\bar{\Phi} = e_1 (\Phi^1)^* + e_2 (\Phi^2)^* + e_3 (\Phi^3)^*$ are linearly independent. It is so, since

$$\begin{aligned} (\Phi^1)^2 + (\Phi^2)^2 + (\Phi^3)^2 = \tfrac{e^\hbar}{4}\big(&RWRW - RWR^{-1}W^{-1} - R^{-1}W^{-1}RW \\ &+R^{-1}W^{-1}R^{-1}W^{-1}\big) - \tfrac{e^\hbar}{4}\big(RWRW + RWR^{-1}W^{-1} \\ &+R^{-1}W^{-1}RW + R^{-1}W^{1-}R^{-1}W^{-1}\big) + \mathbb{1} \\ =& -\frac{e^\hbar}{2}\big(RWR^{-1}W^{-1} + R^{-1}W^{-1}RW\big) + \mathbb{1} \\ =& -\frac{e^\hbar}{2}\big(Re^{-\hbar}R^{-1}WW^{-1} + R^{-1}e^{-\hbar}RW^{-1}W\big) + \mathbb{1} = 0 \end{aligned} \tag{119}$$

and

$$\begin{aligned} \Phi^1(\Phi^1)^* + \Phi^2(\Phi^2)^* + \Phi^3(\Phi^3)^* = \mathbb{1} &+ \tfrac{e^\hbar}{4}\big(R^2 - RW^2R^{-1} - R^{-1}W^{-2}R + R^{-2}\big) \\ &+\tfrac{e^\hbar}{4}\big(R^2 + RW^2R^{-1} + R^{-1}W^{-2}R + R^{-2}\big) \\ = \mathbb{1} &+ \tfrac{e^\hbar}{2}\big(R^2 + R^{-2}\big) \neq 0 \end{aligned} \tag{120}$$

Therefore, if $\Phi a + \bar{\Phi} b = 0$ (for some $a, b \in \hat{\mathcal{C}}_\hbar$), then

$$\begin{aligned} 0 &= (\Phi^1)^2 a + \Phi^1(\Phi^1)^* b + (\Phi^2)^2 a + \Phi^2(\Phi^2)^* b + (\Phi^3)^2 a + \Phi^3(\Phi^3)^2 b \\ &= \big(\Phi^1(\Phi^1)^* + \Phi^2(\Phi^2)^* + \Phi^3(\Phi^3)^*\big)\, b \end{aligned} \tag{121}$$

so necessarily $b = 0$ (in [**AH17**] it is proven that $\hat{\mathcal{C}}_\hbar$ has no zero divisors). Hence, the submodule of $(\hat{\mathcal{C}}_\hbar)^3$ spanned by $\{\Phi, \bar{\Phi}\}$, denoted by $\mathfrak{X}(\hat{\mathcal{C}}_\hbar)$, is a free right $\hat{\mathcal{C}}_\hbar$ module of rank 2.

Several other constructions are defined in [**AH17**] such as a connection, curvature and integration (which shall interest us in future work). Before considering how the previous framework can be applied to de Sitter space, we will mention how the concept of a metric is introduced in the noncommutative catenoid.

In the commutative setting, a metric is a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{E}(M)$ which is $\mathcal{E}(M)-$bilinear. Therefore, a hermitian form (corresponding to a metric) on the noncommutative catenoid is a sesquilinear map $h : \mathfrak{X}(\hat{\mathcal{C}}_\hbar) \times \mathfrak{X}(\hat{\mathcal{C}}_\hbar) \longrightarrow \hat{\mathcal{C}}_\hbar$. A way to define h is taking into account $\mathfrak{X}(\hat{\mathcal{C}}_\hbar) \subset (\hat{\mathcal{C}}_\hbar)^3$ just as the catenoid is embedded in $\mathbb{R}^3$. Letting

$X = e_1X^1 + e_2X^2 + e_3X^3 \in (\hat{\mathcal{C}}_\hbar)^3$ and $Y = e_1Y^1 + e_2Y^2 + e_3Y^3 \in (\hat{\mathcal{C}}_\hbar)^3$ one can define

$$h(X,Y) = \sum_{i=1}^{3}(X^i)^*Y^i \tag{122}$$

and obtain a hermitian form on the noncommutative catenoid by restricting h to the subset $\mathfrak{X}(\hat{\mathcal{C}}_\hbar) \times \mathfrak{X}(\hat{\mathcal{C}}_\hbar)$. In the generators $\{\Phi, \bar{\Phi}\}$ this takes the form.

$$h(\Phi,\Phi) = \mathbb{1} + \frac{1}{2}e^{-\hbar}(R^2 + R^{-2}) \tag{123}$$

$$h(\bar{\Phi},\bar{\Phi}) = \mathbb{1} + \frac{1}{2}e^{\hbar}(R^2 + R^{-2}) \tag{124}$$

$$h(\bar{\Phi},\Phi) = 0 \tag{125}$$

In the next Chapter, we shall follow a similar procedure to define vector fields in the noncommutative de Sitter space.

CHAPTER 5

Noncommutative de Sitter model

In this Chapter we will study a noncommutative version of de Sitter space. Our main focus will be in $d\boldsymbol{S}_2$ for computational reasons. The procedure we will undertake consists of deforming the algebra $\mathcal{E}(d\boldsymbol{S}_2)$ by means of Rieffel's deformation quantization (as presented in [**Rie93**]), then we shall follow Arnlind's approach (as presented in Chapter 4) to introduce a quantized calculus in the deformed algebra

1. Algebra deformation

Let us consider the global coordinates described in Section 2 of Chapter 3.

$$x_0 = \alpha \sinh(t/\alpha) \tag{126}$$

$$x_1 = \alpha \cosh(t/\alpha) \cos\psi \tag{127}$$

$$x_2 = \alpha \cosh(t/\alpha) \sin\psi \tag{128}$$

From this parametrization one can define the following functions.

$$\begin{aligned} R &= e^{t/\alpha} & \qquad W &= e^{i\psi} \\ \tilde{R} &= e^{-t/\alpha} & \qquad \tilde{W} &= e^{-i\psi} \end{aligned} \tag{129}$$

These generate the algebra $\mathcal{E}(d\boldsymbol{S}_2)$. An action $\beta : \mathbb{R}^2 \times \mathcal{E}(d\boldsymbol{S}_2) \longrightarrow \mathcal{E}(d\boldsymbol{S}_2)$ can be obtained by the action on the generators given by:

$$\beta(r,s)(e^{t/\alpha}) = e^{(t+r)/\alpha} \tag{130}$$

$$\beta(r,s)(e^{i\psi}) = e^{i(\psi+s)} \tag{131}$$

Then, picking an antisymmetric matrix $\theta = \begin{pmatrix} 0 & \pi\hbar\alpha \\ -\pi\hbar\alpha & 0 \end{pmatrix}$ one can compute the deformed product.

$$\begin{aligned} e^{i\psi} \times_\theta e^{-i\psi} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{i(\psi - r\pi\hbar\alpha)} e^{-i(\psi+s')} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\ &= \int\limits_{\mathbb{R}^2} e^{2\pi i r(r'-\hbar\alpha/2)} dr dr' \int\limits_{\mathbb{R}^2} e^{2\pi i s'(s-1/2\pi)} ds' ds = 1 \end{aligned} \tag{132}$$

$$\begin{aligned} e^{-i\psi} \times_\theta e^{i\psi} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{-i(\psi - r\pi\hbar\alpha)} e^{i(\psi+s')} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\ &= \int\limits_{\mathbb{R}^2} e^{2\pi i r(r'+\hbar\alpha/2)} dr dr' \int\limits_{\mathbb{R}^2} e^{2\pi i s'(s+1/2\pi)} ds' ds = 1 \end{aligned} \tag{133}$$

$$\begin{aligned} e^{t/\alpha} \times_\theta e^{-t/\alpha} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{(t+s\pi\hbar\alpha)/\alpha} e^{-(t+r')/\alpha} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\ &= \int\limits_{\mathbb{R}^2} e^{s\pi\hbar} e^{-r'/\alpha} \int\limits_{\mathbb{R}} e^{2\pi i r r'} \int\limits_{\mathbb{R}} e^{2\pi i s s'} ds' dr ds dr' \\ &= \int\limits_{\mathbb{R}} e^{s\pi\hbar} \delta(s) ds \int\limits_{\mathbb{R}} e^{-r'/\alpha} \delta(r') dr' = 1 \end{aligned} \tag{134}$$

$$\begin{aligned} e^{-t/\alpha} \times_\theta e^{t/\alpha} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{-(t+s\pi\hbar\alpha)/\alpha} e^{(t+r')/\alpha} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\ &= \int\limits_{\mathbb{R}^2} e^{-s\pi\hbar} e^{r'/\alpha} \int\limits_{\mathbb{R}} e^{2\pi i r r'} \int\limits_{\mathbb{R}} e^{2\pi i s s'} ds' dr ds dr' = 1 \end{aligned} \tag{135}$$

$$\begin{aligned} e^{t/\alpha} \times_\theta e^{i\psi} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{(t+s\pi\hbar\alpha)/\alpha} e^{i(\psi+s')} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\ &= e^{t/\alpha} e^{i\psi} \int\limits_{\mathbb{R}} e^{\pi\hbar s} \int\limits_{\mathbb{R}} e^{2\pi i s'(s+1/2\pi)} \int\limits_{\mathbb{R}^2} e^{2\pi i r r'} dr dr' ds' ds \\ &= e^{t/\alpha} e^{i\psi} \int\limits_{\mathbb{R}} e^{\pi\hbar s} \delta(s+1/2\pi) ds = e^{u} e^{i\psi} e^{-\hbar/2} \end{aligned} \tag{136}$$

$$\begin{aligned} e^{i\psi} \times_\theta e^{t/\alpha} &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} e^{i(\psi - r\pi\hbar\alpha)} e^{(t+r')/\alpha} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\ &= e^{t/\alpha} e^{i\psi} \int\limits_{\mathbb{R}} e^{r'/\alpha} \int\limits_{\mathbb{R}} e^{2\pi i r(r'-\hbar\alpha/2)} dr dr' = e^{u} e^{i\psi} e^{\hbar/2} \end{aligned} \tag{137}$$

$$
\begin{aligned}
e^{-t/\alpha} \times_\theta e^{i\psi} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-(t+s\pi\hbar\alpha)/\alpha} e^{i(\psi+s')} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\
&= e^{-t/\alpha} e^{i\psi} \int_{\mathbb{R}} e^{-\pi\hbar s} \int_{\mathbb{R}} e^{2\pi i s'(s+1/2\pi)} ds' ds = e^{-t/\alpha} e^{i\psi} e^{\hbar/2}
\end{aligned}
\tag{138}
$$

$$
\begin{aligned}
e^{i\psi} \times_\theta e^{-t/\alpha} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{i(\psi-r\pi\hbar\alpha)} e^{-(t+r')/\alpha} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\
&= e^{-t/\alpha} e^{i\psi} \int_{\mathbb{R}} e^{-r'/\alpha} \int_{\mathbb{R}} e^{2\pi i r(r'-\hbar\alpha/2)} = e^{-t/\alpha} e^{i\psi} e^{-\hbar/2}
\end{aligned}
\tag{139}
$$

$$
\begin{aligned}
e^{-i\psi} \times_\theta e^{t/\alpha} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(\psi-r\pi\hbar\alpha)} e^{(t+r')/\alpha} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\
&= e^{t/\alpha} e^{-i\psi} \int_{\mathbb{R}} e^{r'/\alpha} \int_{\mathbb{R}} e^{2\pi i r(r'+\hbar\alpha/2)} dr dr' = e^{t/\alpha} e^{-i\psi} e^{-\hbar/2}
\end{aligned}
\tag{140}
$$

$$
\begin{aligned}
e^{t/\alpha} \times_\theta e^{-i\psi} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{(t+s\pi\hbar\alpha)/\alpha} e^{-i(\psi+s')} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\
&= e^{t/\alpha} e^{-i\psi} \int_{\mathbb{R}} e^{\pi\hbar s} \int_{\mathbb{R}} e^{2\pi i s'(s-1/2\pi)} ds' ds = e^{t/\alpha} e^{-i\psi} e^{\hbar/2}
\end{aligned}
\tag{141}
$$

$$
\begin{aligned}
e^{-t/\alpha} \times_\theta e^{-i\psi} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-(t+s\pi\hbar\alpha)/\alpha} e^{-i(\psi+s')} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\
&= e^{-t/\alpha} e^{-i\psi} \int_{\mathbb{R}} e^{-\pi\hbar s} \int_{\mathbb{R}} e^{2\pi i s'(s-1/2\pi)} ds' ds = e^{-t/\alpha} e^{-i\psi} e^{-\hbar/2}
\end{aligned}
\tag{142}
$$

$$
\begin{aligned}
e^{-i\psi} \times_\theta e^{-t/\alpha} &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-i(\psi-r\pi\hbar\alpha)} e^{-(t+r')/\alpha} e^{2\pi i(rr'+ss')} dr dr' ds ds' \\
&= e^{-t/\alpha} e^{-i\psi} \int_{\mathbb{R}} e^{-r'/\alpha} \int_{\mathbb{R}} e^{2\pi i r(r'+\hbar\alpha/2)} dt dt' = e^{-t/\alpha} e^{-i\psi} e^{\hbar/2}
\end{aligned}
\tag{143}
$$

The former products allow us to state the commutation relations, which gives the same algebraic structure as in the case of Arnlind's catenoid.

$$\begin{aligned} \tilde{W}W &= \mathbb{1} & W\tilde{W} &= \mathbb{1} \\ R\tilde{R} &= \mathbb{1} & \tilde{R}R &= \mathbb{1} \\ WR &= e^{\hbar}RW & W\tilde{R} &= e^{-\hbar}\tilde{R}W \\ \tilde{W}R &= e^{-\hbar}R\tilde{W} & \tilde{W}\tilde{R} &= e^{\hbar}\tilde{R}\tilde{W} \end{aligned} \tag{144}$$

Let us denote the deformed algebra (with the same underlying set as $\mathcal{E}(d\boldsymbol{S}_2)$, but with deformed product) by $\mathcal{E}(d\boldsymbol{S}_2)_\theta$.

2. Quantized calculus in de Sitter

We shall follow [**AH17**] in introducing derivations and a right $\mathcal{E}(d\boldsymbol{S}_2)_\theta-$module corresponding to vector fields. Let us define ∂_t and ∂_ψ by their action on the generators.

$$\begin{aligned} &\partial_t R = \frac{1}{\alpha}R && \partial_t R^{-1} = -\frac{1}{\alpha}R^{-1} && \partial_t W = 0 && \partial_t W^{-1} = 0 \\ &\partial_\psi R = 0 && \partial_\psi R^{-1} = 0 && \partial_\psi W = iW && \partial_\psi W^{-1} = -iW^{-1} \end{aligned} \tag{145}$$

This derivations form a complex abelian Lie algebra just as in the case of the catenoid.

The right module $\mathfrak{X}(\mathcal{E}(d\boldsymbol{S}_2)_\theta)$ can be obtained following the same procedure as in the previous Chapter. The basis for $\mathfrak{X}(\mathcal{E}(d\boldsymbol{S}_2)_\theta)$ shall have the same elements as in the catenoid, since the catenoid and $d\boldsymbol{S}_2$ are diffeomorphic to $\mathbb{R}\times S_{n-1}$. Therefore, their tangent bundles are isomorphic.

We are interested, however, in introducing a Lorentzian metric in de Sitter space. Let us denote the basis of the free right $\mathcal{E}(d\boldsymbol{S}_2)_\theta-$module by $\{e_0, e_1, e_2\}$, and define

$$\Psi = e_0\,\mathbb{1} + e_1\,\frac{1}{2}e^{\hbar/2}(RW - R^{-1}W^{-1}) + e_2\,\frac{-i}{2}e^{\hbar/2}(RW + R^{-1}W^{-1}) \tag{146}$$

$$\bar{\Psi} = e_0\,\mathbb{1} + e_1\,\frac{1}{2}e^{\hbar/2}(W^{-1}R - WR^{-1}) + e_2\,\frac{i}{2}e^{\hbar/2}(W^{-1}R + WR^{-1}) \tag{147}$$

By the same argument as in the previous Chapter, this vectors span a free right $\mathcal{E}(d\boldsymbol{S}_2)_\theta-$module. Let us define a metric in $(\mathcal{E}(d\boldsymbol{S}_2)_\theta)^3$ by

$$g(X,Y) = -(X^0)^*Y^0 + (X^1)^*Y^1 + (X^2)^*Y^2 \tag{148}$$

This defines a different structure on the module as compared to the one introduced in the previous Chapter.

CHAPTER 6

Concluding remarks

During the course of this work several questions have been raised which, for the time being, are left unanswered. Nonetheless, it is expected that the points missed in this exposition can be addressed in future work, and that the unanswered questions shall guide further investigation.

As to the classical de Sitter space, this exposition does not include a study of quantum fields in de Sitter space. A Section on representation theory of the de Sitter group (in 2 dimensions) has been included, guided by the Thirring's article ([**Thi67**]) on quantum field theory in de Sitter space. This article deals mainly with $2-$dimensional de Sitter space. A further work on quantum field theory in de Sitter space should take into account Thirring's results [**Thi67**] and a consideration of the $4-$dimensional de Sitter space (as in [**CT68**]).

As for the results of [**AH17**] on the noncommutative catenoid and their implementation on the noncommutative de Sitter space, the concepts of integration (and hence total curvature) were not part of this exposition. Our interest in this respect is twofold. First, it is expected that a form of integration in the noncommutative de Sitter space can be used to compute the total curvature of the space, in the same way as it is used in [**AH17**]. This should serve to established how properties of the classical space are preserved in the noncommutative case. On the other hand, it is expected that the notion of integration in a noncommutative manifold be of use when defining quantum fields.

In Minkowski space, the quantized real scalar field takes the form

$$\phi(t,\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \left(e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-E_p t)}\hat{a}(p) + e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}-E_p t)}\hat{a}^\dagger(p) \right) \frac{d^3p}{2E_p} \tag{149}$$

with $E_p = \sqrt{\boldsymbol{p}^2 + m^2}$. The factors $e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-E_p t)}$ are introduced to take the Fourier transform from momentum space to position space. These factors take into account the Lorentzian nature of the metric in Minkowski space. Since our interest is set on the study of quantum fields in a noncommutative de Sitter space, some questions have been raised regarding the generalization of this result to the noncommutative setting. One of those has to do with the comments of the preceding paragraph. Namely, how can a notion of integration be introduced in the noncommutative space such that these expansions (Fourier) are possible and meaningful. Another question question that has been raised is how the hermitian form defined for the noncommutative algebra can be used to implement the nature of the metric in the classical de Sitter space.

An alternative proposal on quantum field theories in noncommutative spaces has already been laid down and studied for versions of flat Minkowski space. In particular it is worth mentioning the work of Doplicher, Fredenhagen and Roberts [**DFR95**]. In their theory, the noncommutative space-time algebra is an operator algebra acting on some Hilbert space. In this sense, the factors $e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-E_p t)}$ are again operators acting on the same Hilbert space, while the operators $\hat{a}(p), \hat{a}^\dagger(p)$ act on Fock space and their product is actually a tensor product. This approach is also discussed in [**SM15**], where a method to deal withdivergences and renormalization is proposed. It would be interesting to compare the DFR approach with the one set out in this work, and expected to be further developed in future work.

Hopefully, some of this questions can be addressed in future work.

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