ON THE THEORY OF THE LOGARITHMIC TRANSSERIES FIELD AS AN ORDERED VALUED LOGARITHMIC FIELD

PhD Thesis

Presented by

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To my daughter.

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1. Introduction

The study of the first order properties of ordered fields equipped with an exponential function has been an important subject of research since Tarski proved in [23] that the first order theory of the field of real numbers $Th(\mathbb{R})$ is decidable, and asked if his result can be extended to the first order theory of real numbers with exponentiation $Th(\mathbb{R}_{\exp})$. The main tool used by Tarski to prove his theorem consisted in showing that $Th(\mathbb{R})$ has quantifier elimination, but van den Dries proved in [24] that quantifier elimination fails for $Th(\mathbb{R}_{\exp})$. A better approximation to this question was given by Wilkie in [33] where he shows that this theory is model complete and is o-minimal, just like $Th(\mathbb{R})$ is.

In [33] Wilkie proved also that the theory of real numbers equipped with a restricted Pfaffian chain of functions is model complete. As examples he has the theories of the structures $(\mathbb{R}; \exp|_{[0,1]})$ and $(\mathbb{R}; \sin|_{[0,1]}, \exp|_{[0,1]}; r)_{r\in\mathbb{R}}$. The model completness of the last structure was proved originally by van den Dries in [26]. In the same way, van den Dries showed in [25] that the theory T_{an} of real numbers with all analytic functions $f(x_1, ..., x_n)$ restricted to $[-1,1]^n$ is model complete. Moreover, Denef and van den Dries showed in [9] that including a function for the multiplicative inverse, T_{an} has quantifier elimination and is o-minimal. In all these cases, the structures has an exponential map restricted to a compact set but the full exponential function is not definable.

Van den Dries, Macintyre and Marker studied in [27] the theory $T_{an}(\exp)$ of real numbers with restricted analytic functions and full exponentiation. Particularly, based on a re-examination of T_{an} from the point of view of valued fields and the embedding of models of T_{an} in Hahn series fields, they proved that adding a symbol for the logarithmic function, $T_{an}(\exp)$ has quantifier elimination and is o-minimal in the expanded language.

Although Hahn series fields with non-trivial ordered abelian groups as groups of monomials do not admit an exponential function (as shown in [18, 19]), chains of these fields have been used to construct non-archimedean models of exponential fields. Two non-isomorphic representative examples are the field of exponential-logarithmic series introduced by Kuhlmann and Kuhlmann in [16], and the field of logarithmic-exponential series, or field of transseries $\mathbb{T}$, defined by van den Dries, Macintyre and Marker in [30, 31], and independly in connection with the Tarski's problem, by Dahn and Göring in [8].

The exponential-logarithmic field was introduced in the study of non-archimedean exponential fields made in [18, 19, 17], which focuses on the search of conditions under which an exponential function is definable in a non-archimedean field in such way that the resultant structure behaves as the structure of the real numbers with ordinary exponentiation. Further, Kuhlmann and Kuhlmann showed in [18] that in such fields the exponential induces a contraction map on the value group of the field, which is surjective, contracts archimedean classes of the same sign, and is useful to understand how the growth of the exponential is related to the growth of polynomials, in particular to the Taylor polynomials associated to the natural exponentiation in $\mathbb{R}$. They showed in [14, 15] that the first order theory of a value group endowed with such contraction map is complete, decidable, admits quantifier elimination and is weakly o-minimal.

On the other hand, the transseries field $\mathbb{T}$ was introduced to obtain a non-standard model of the theory $T_{an}(\exp)$ and was utilized in first place to prove that some functions, like the Zeta-function, are not definable (see [30]). However, the main interest of the model theory of $\mathbb{T}$ is that this field carries a natural structure of ordered valued differential field. Van den Dries, Aschenbrenner and van der Hoeven have studied extensively the theory of this structure during the last two decades and the results of their work appears in [2, 3, 4, 5]. Among other things, they show that $\mathbb{T}$ is a model of one of the two completions of the theory of Newtonian Liouville ω-free H-fields. We notice that an important tool used by them to study differential fields is the structure induced by the differential map in the value group of the field, called the *asymptotic couple*. Particularly, they show that some properties of a differential field can be expressed in terms of its asymptotic couple and that the theory of the asymptotic couple of $\mathbb{T}$, that is the theory of closed H-asymptotic couples, admits quantifier elimination (see [1]).

As a special ordered valued differential subfield of $\mathbb{T}$, it was defined in [4] the field of logarithmic transseries $\mathbb{T}_{\log}$, informally speaking transseries with real coefficients and monomials not involving exponentiation. It was conjectured that it has good model theoretic properties and under this premise, Gehret focused in [10, 11, 12] on the theory of $\mathbb{T}_{\log}$ as an ordered valued differential field, showing that the theory of its asymptotic couple is model complete and has elimination of quantifiers in a natural language. He also outline a strategy for proving model completeness of $Th(\mathbb{T}_{\log})$.

The field of logarithmic transseries $\mathbb{T}_{\log}$ is not an exponential field, but it allows to define a logarithmic map, that is, an order preserving group morphism from the multiplicative group of positive elements of the field in its additive group, which has a nice behaviour with respect to the order and value structure of $\mathbb{T}_{\log}$. In that sense, the study of $\mathbb{T}_{\log}$ as an ordered valued logarithmic field becomes an interesting research subject and it is the main purpose of this thesis.

As a first approach to this study, we notice that $\mathbb{T}_{\log}$ can be expanded to a model of T_{an} and together with its convex valuation ring is a model of the theory T_{convex} introduced in [28]. Moreover, the logarithm of $\mathbb{T}_{\log}$ restricted to the multiplicative subgroup of positive units of the valuation ring has for image the valuation ring itself. Thus, we can define a partial exponential function on $\mathbb{T}_{\log}$ which extends the restricted analytic real exponential function defined in $[-1, 1]$. We focus on the class of valued models of T_{an} with a convex valuation ring and a partial exponential function satisfying some of the mentioned properties. Particularly, we show that the theory of such class is model complete, complete, admits quantifier elimination in a natural expansion of the language, is weakly o-minimal, and admits Skolem-functions.

In the same spirit, we study also a class of fields which are models of the universal part of T_{an}, and are equipped with a convex valuation ring and a partial exponential function that satisfies the same properties mentioned above. Working in a three sorted language for valued fields, we prove an Ax-Kochen-Ershov type equivalence theorem for this kind of valued fields.

As a next step and following the classical strategy used in model theory to study the theory of a valued field by studying first the theories of its value group and its residual field, and inspired by the ideas used in [14, 15] to study the theory of the contraction groups and those used in [10] to study the theory of the asymptotic couple of $\mathbb{T}_{\log}$, we study the theory of the associated precontraction group of $\mathbb{T}_{\log}$; that is, the structure given by its value group $\Gamma_{\log}$ endowed with a function χ induced by the logarithm map such that the image of $\Gamma_{\log}^{<0}$ by χ is a discrete set cofinal in $\Gamma_{\log}^{<0}$. Then we prove that the theory of the precontraction group is model complete and complete. We also expand the language of the theory to ensure that it has quantifier elimination, and characterize the definable subsets of the image of the group by the precontraction map.

Finally, we use the previous work to propose a theory of ordered valued logarithmic fields such that the field is a model of T_{an}, the valuation ring is definable, its value group is a model of the theory of the precontraction group of $\mathbb{T}_{\log}$, there is a partial exponential function defined in the field with the properties mentioned above, and such that the residual field becomes a model of $T_{an}(\exp)$. Particularly, we show that $\mathbb{T}_{\log}$ expands naturally to a model of such theory and in fact is the prime model. Finally, we sketch a path to obtain a possible proof of model completeness of this new theory.

The document is organized as follows. In the second chapter we present the preliminaries, including basic facts about valued fields, Hahn fields, and the construction of the field of logarithmic transseries $T_{\log}$ together with it's natural valuation and logarithm.

In chapter 3, we include a short description of the theory T_{convex}, and study the theory T_+ of structures of the form $(K, \mathcal{O}, \exp)$ where K is a model of T_{an}, $\mathcal{O}$ is a convex subring of K and exp is a partial exponential function defined in $\mathcal{O}$. Particularly, we show that T_+ is complete and model complete.

In chapter 4, we do a similar study for the models of the universal part of T_{an}. Particularly, we consider partial exponential functions defined in Hahn series fields whose value group is not necessarily divisible. For this purpose we work with valued fields equipped with a lifting of the residual field and a cross section for the value group. Our main result is an Ax-Kochen-Ershov equivalence theorem for the corresponding theory.

In Chapter 5, we focus on the precontraction map χ induced by the logarithm of $\mathbb{T}_{\log}$ in its value group $\Gamma_{\log}$ and study the theory T_{pdg} of the precontraction group $(\Gamma_{\log}, \chi)$. Particularly, we show that this theory is complete and model complete, and characterize all definable functions on the discrete set $\chi(\Gamma_{\log})$.

Finally, we study structures of the form $(K, \log, L)$, where K is an ordered field, the map

$\log : K^{>0} \to K$ is an ordered preserving embedding of groups, L is a definable subgroup of K such that $\log(K^{>0}) \subseteq L$, and it is possible to define a convex valuation ring of K in L. We call such structures L-fields. Next, we study the theory $T_+(\log)$ whose models are structures of the form $(K, \mathcal{O}, \exp, \log, L)$ where $(K, \mathcal{O}, \exp)$ is a model of T_+, $(K, \log, L)$ is an L-field with some extra properties, and the precontraction group associated to K is a model of T_{pdg}. We show that $(\mathbb{T}_{\log}, \mathcal{O}_{\log}, \exp, \log, \log(\mathbb{T}_{\log}^{>0}))$ is a model of $T_+(\log)$ and this theory has a prime model, and we outline a proof of model completeness for $\mathbb{T}_{\log}$, module some conjectures.

2. Preliminaries

This chapter contains the preliminaries for the rest of the document. In the first part we present a summary of some notions and facts about ordered groups, valued fields and Hahn fields. Next, we include a short description of the main results about the model theory of T_{an}, the theory of the reals with restricted analytic functions, and $T_{an}(\exp)$, the theory of reals with restricted analytic functions and exponentiation, and finally we introduce the field of logarithmic transseries $\mathbb{T}_{\log}$.

For general notions and facts about model theory, we refer the reader to [6, 13] or [4, Appendix B].

Throughout this document m and n range over $\mathbb{N} = \{0, 1, 2, 3, ...\}$. The letter Γ will be used to denote an additively written abelian group while $\mathfrak{M}$ will denote a multiplicative abelian group. If R is a ring, $R^\times$ will denote its multiplicative group of units.

If L is a first order language, and $\mathcal{M}, \mathcal{N}$ are L-structures, $\mathcal{M} \subseteq \mathcal{N}$ will denote that $\mathcal{M}$ is an L-substructure of $\mathcal{N}$.

2.1 Ordered sets and ordered groups

By an ordered set S we mean a set S equipped with a distinguished total order relation $\leq$. If B is a subset of S we define the set

$$S^{\geq B} = \{a \in S : a \geq b \text{ for all } b \in B\}.$$

In similar way we write $S^{>B}$, $S^{\leq B}$, $S^{<B}$ and $S^{\neq 0}$. Particularly, if $B = \{b\}$, then we write $S^{\geq b}$ for $S^{\geq B}$.

We say that a subset B of S is convex in S if for all $a, c \in B$ and $b \in S$ such that $a \leq b \leq c$ we have $b \in B$, and we define the *convex hull* of B in S as

$$\operatorname{conv}(B) = \{b \in S : a \leq b \leq c \text{ for some } a, b \in B\}.$$

Moreover, we say that $B \subseteq S$ is a *lower cut* in S if for all $b \in B$ and $a \in A$, $a < b$ implies $a \in B$.

Finally, we define intervals in S as usual and for $\infty \notin S$, we define the set $S_\infty = S \cup \{\infty\}$ and extend the order of S to S_∞ setting $a < \infty$ for all $a \in S$.

Ordered abelian groups

An *ordered abelian group* Γ, written additively, is an abelian group with an ordering such that for all $a, b, c \in \Gamma$ if $a < b$ then $a + c < b + c$. For $a \in \Gamma$ we set $|a| = \max\{a, -a\}$ and define the *archimedean class* of a in Γ as

$$[a] = \{b \in \Gamma : |a| \leq n|b| \text{ and } |b| \leq n|a| \text{ for some } n \geq 1\},$$

and we say that a is *archimedean equivalent* to b in Γ if $b \in [a]$. The set $[\Gamma]$ of all archimedean classes becomes an ordered set putting

$$[a] \leq [b] \Leftrightarrow |a| \leq n|b| \text{ for some } n \geq 1.$$

Moreover, we have

$$[a] < [b] \Leftrightarrow n|a| \leq |b| \text{ for all } n \geq 1.$$

Valued abelian groups

Let Γ be an abelian group. A *valuation* on Γ is a surjective map $v : \Gamma \to S_\infty$, where S is an ordered set, such that for all $a, b \in \Gamma$ the following conditions are satisfied:

(1) $v(a) = \infty \Leftrightarrow a = 0$.

(2) $v(-a) = v(a)$.

(3) $v(a + b) \geq \min\{v(a), v(b)\}$.

A *valued abelian group* is a structure conformed by an abelian group Γ, and ordered set S and a valuation v on Γ.

For example, for an ordered abelian group Γ if we put $S = [\Gamma]$ and equip S with the reversed ordering of $[\Gamma]$, then the map $v : \Gamma \to S$ defined as $v(a) = [a]$ is a valuation on Γ. We call this valuation the *natural valuation* of the group Γ.

2.2 Valued fields

In this section we include some basic facts about valued fields. A deeper study of the theory of valued fields and the proofs of the quoted results may be found in [4, Chapters 2,3].

Valuation rings of a field

A subring $\mathcal{O}$ of a field K is called a *valuation ring* of K if for each $x \in K^\times$ we have $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Thus, each valuation ring $\mathcal{O}$ of K is a local ring and if we denote by $\mathcal{o}$ the maximal ideal of $\mathcal{O}$, then we can associate to K the field $K_\mathcal{O} = \mathcal{O}/\mathcal{o}$ which is called the *residual field* of K relative to $\mathcal{O}$, $\text{res}(K)$ with a residual map $\text{res} : \mathcal{O} \to \text{res}(K)$, a surjective ring homomorphism.

Valuations on fields

Given a field K and an ordered abelian group Γ, a surjective map $v : K^\times \to \Gamma$ is called a *valuation* on K if for all $x, y \in K^\times$

(1) $v(xy) = v(x) + v(y)$, and

(2) $v(x + y) \geq \min\{v(x), v(y)\}$ for $x + y \neq 0$.

Moreover, if $\Gamma_\infty = \Gamma \cup \{\infty\}$ and we extend the addition and the order of Γ to Γ_∞ so that $\infty > a$ and $\infty + a = a + \infty = \infty + \infty = \infty$ for all $a \in \Gamma$, then we can extend v to $v : K \to \Gamma_\infty$ by putting $v(0) = \infty$. We call Γ_∞ the value group of K and Γ the value group of $K^\times$

The set $\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$ is a valuation ring of K and it has as maximal ideal the ring $\mathcal{o}_v = \{x \in K : v(x) > 0\}$.

Conversely, given a valuation ring $\mathcal{O}$ of K we can define a valuation $v_\mathcal{O}$ on K as follows: first, setting $\mathcal{O}^\times = \{x \in \mathcal{O} : x^{-1} \in \mathcal{O}\}$ we define $\Gamma_\mathcal{O}$ as the abelian group $K^{\neq 0}/\mathcal{O}^\times$ ordered by the relation

$$[x] \geq [y] \Leftrightarrow x/y \in \mathcal{O} \ ([x], [y] \in \Gamma_\mathcal{O}),$$

and then we define the valuation $v_\mathcal{O} : K \to \Gamma_\mathcal{O}$ as $v(x) = [x]$.

Dominance relations on a field

A *dominance relation* on a field K is a binary relation $\preceq$ that is reflexive, transitive and satisfies for all $x, y, z \in K$

(1) $\neg(1 \preceq 0)$,

(2) $x \preceq y$ or $y \preceq x$,

(3) $x \preceq y$ if and only if $xz \preceq yz$, for $z \neq 0$, and

(4) $x \preceq z$ and $y \preceq z$, imply $x + y \preceq z$.

The set $K^{\preceq 1} = \{x \in K : x \preceq 1\}$ is a valuation ring of K with maximal ideal

$$K^{\prec 1} = \{x \in K : x \prec 1\}.$$

Thus, we can associated to each dominance relation $\preceq$ in K, a valuation $v_{K^\preceq}$ on K. And, conversely, if $v : K \to \Gamma_\infty$ is a valuation, then we can define a dominance relation $\preceq_v$ on K as follows:

$$x \preceq_v y \Leftrightarrow v(x) \geq v(y) \ (x, y \in K).$$

From a dominance relation $\preceq$ on K we can define other useful relations on K:

$$x \prec y \Leftrightarrow x \preceq y \text{ and not } y \preceq x,$$

$$x \asymp y \Leftrightarrow x \preceq y \text{ and } y \preceq x,$$

and

$$x \sim y \Leftrightarrow x - y \prec x.$$

Valued fields

A *valued field* is a field K with a valuation ring O. Since there is a one-to-one correspondence between valuation rings, valuations and dominance relations, from now on for a valued field K we understand also the field K together with a valuation or a dominance relation.

When working with a fix valuation ring in a field K, we will denote by v_K its valuation, by $\mathcal{O}_K$ its valuation ring, by $\mathcal{o}_K$ the maximal ideal of $\mathcal{O}_K$, by $\mathrm{res}(K)$ the residual field of K, and by Γ_K the value group of $K^\times$. If the context is clear we will omit the subscripts.

Extension of valued fields

If E and K are valued fields with valuation rings $\mathcal{O}_E$ and $\mathcal{O}_K$, respectively, we say that K is a valued field extension of E, or E is a valued subfield of K (denote $E \subseteq K$), if K is a field extension of E such that $\mathcal{O}_E = E \cap \mathcal{O}_K$.

If $E \subseteq K$, we have a field embedding $\mathrm{res}(E) \to \mathrm{res}(K)$ and thus we can identify $\mathrm{res}(E)$ with a subfield of $\mathrm{res}(K)$. Similarly, there is an ordered group embedding $\Gamma_E \to \Gamma_K$ which allows us to identify Γ_E with an ordered subgroup of Γ_K.

If $E \subseteq K$, $\Gamma_E = \Gamma_K$ and $\mathrm{res}(E) = \mathrm{res}(K)$, we say that the extension is *immediate*. A valued field is said to be *maximal* if it has not proper immediate valued extensions.

pc-sequences

Let K be a valued field. A *pseudocauchy sequence* in K (from now on pc-sequence in K) is a well indexed sequence (a_α) in K such that for some index ρ and all $\gamma > \beta > \alpha > \rho$ we have

$$v(a_\gamma - a_\beta) > v(a_\beta - a_\alpha).$$

We say that the pc-sequence (a_α) *pseudoconverges* to a, denoted $a_\alpha \rightsquigarrow a$, if $(v(a - a_\alpha))$ is eventually strictly increasing, we mean that for some index ρ and all $\beta > \alpha > \rho$ we have $v(a - a_\beta) > v(a - a_\alpha)$. In this case we say that a is a pseudolimit of (a_α). Additionally, we say that the pc-sequence (a_α) is divergent in K if it has no pseudolimit in K.

An important fact is that each pc-sequence in K has a pseudolimit in an immediate valued field extension of K. Then we have that if K is maximal then every pc-sequence in K has a pseudolimit in K (see [4, Corollary 3.2.9]).

For each pc-sequence (a_α) in K and $P \in K[X]$ non-constant, $(P(a_\alpha))$ is a pc-sequence in K (see [4, Corollaries 2.2.8, 3.2.4]). Moreover, we say that a pc-sequence in K is of *algebraic type* over K if for some non-constant polynomial $P \in K[X]$, $(v(P(a_\alpha))$ is eventually strictly increasing, which is equivalent to saying that $P(a_\alpha) \rightsquigarrow 0$. On the other hand, we say that (a_α) is of *transcendental type* over K if for all non-constant polynomial $P \in K[X]$ it has $(v(P(a_\alpha))$ eventually constant, which is equivalent to saying that $P(a_\alpha) \not\rightsquigarrow 0$.

A valued field is *algebraically maximal* if it has no immediate proper algebraic valued field extensions, which is equivalent to saying that each pc-sequence in K of algebraic type over K has a pseudolimit in K (see [4, Corollary 3.2.12]).

Henselian fields

A valued field K is said to be *henselian* if for every polynomial $P \in \mathcal{O}[X]$ and $\alpha \in \mathrm{res}(K)$ with $\mathrm{res}(P)(\alpha) = 0$ and $\mathrm{res}(P')(\alpha) \neq 0$ there is $a \in \mathcal{O}$ with $P(a) = 0$ and $\mathrm{res}(a) = \alpha$. A *henselization* of K is a henselian valued field extension K^h of K such that any valued field embedding $K \to L$ into a henselian valued field L extends uniquely to an embedding $K^h \to L$. Thus, if K' is any henselian valued field extension of K, then by the henselization of K in K' we mean the henselization K^h of K such that $K \subseteq K^h \subseteq K'$ as valued fields.

The following are some important facts about henselian fields (for a proof see [4, Chapter 3]):

Lemma 2.1. *Let K be a valued field.*

(1) *K has a henselization.*

(2) *The henselization K^h of the valued field K in K' is an immediate extension of K which is algebraic over K.*

(3) *If K is of equicharacteristic 0 then K is henselian if and only if is algebraically maximal.*

(4) *If K is henselian of equicharacteristic 0, then every maximal subfield F of $\mathcal{O}$ is a lifting of the residual field $\operatorname{res}(K)$, this means that $\operatorname{res}(F) = \operatorname{res}(K)$*

(5) *If K is henselian and L is an algebraic valued field extension of K, then L is henselian.*

(6) *Let K have equi-characteristic 0. Then K is algebraically closed if and only if K is henselian, $\operatorname{res}(K)$ is algebraically closed, and Γ_K is divisible.*

Ordered valued fields and convex valuations

Let K be an ordered field. We say that a valuation v on K is *convex*, or compatible with the order, if for all $x, y \in K^{>0}$

$$x \leq y \to v(y) \leq v(x).$$

We call *ordered valued field* to an ordered field with a convex valuation.

The next lemma gives a useful characterization of this kind of valuations:

Lemma 2.2. *Let K an ordered valued field. The following conditions are equivalent:*

(1) *The valuation v is convex.*

(2) *The valuation ring $\mathcal{O}_K$ is a convex subset of K.*

(3) *The maximal ideal $\mathcal{o}_K$ of $\mathcal{O}_K$ is a convex subset of K, equivalently the set*

$$1 + \mathcal{o}_K = \{1 + \epsilon : \epsilon \in \mathcal{o}\},$$

*called the set of 1-**units**, is a convex subset of K.*

From [18] we have the following result:

Lemma 2.3 (Lexicographic decompositions of K). *Let K be an ordered valued field. Then the underlying additive group K decomposes as $K = A \oplus A' \oplus \mathcal{o}_K$, where A is a group complement*[1] *to $\mathcal{O}_K$ in K and A' is a group complement to $\mathcal{o}_K$ in $\mathcal{O}_K$.*

Additionally, if the multiplicative group $K^{>0}$ is divisible, then $K = B \cdot B' \cdot (1+\mathcal{o}_K)$, where B is a group complement to $(\mathcal{O}_K^\times)^{>0}$ in $K^{>0}$ and B' is a group complement to $1+\mathcal{o}_K$ in $(\mathcal{O}_K^\times)^{>0}$.

In the previous lemma A, A', B and B' are unique up to ordered isomorphism. Moreover, B is isomorphic to the ordered value group Γ, because the map $\phi : K^{>0} \to G$ given by $\phi(x) = -v(x) = v(x^{-1})$ is a surjective homomorphism of ordered groups with kernel $(\mathcal{O}_K^\times)$.

Natural valuation of an ordered field

If $[K]$ denotes the set of archimedean classes of an ordered field K and we equip $[K]$ with the reverse ordering, and make $[K^\times]$ an ordered abelian group defining $[x]+[y] = [x \cdot y]$ for $x, y \in K^\times$ and $0 = [1]$, then the function $w : K \to [K]$ is a convex valuation on K, that we call the *natural valuation* of K.

If $w : K^\times \to \Gamma_1$ and $v : K^\times \to \Gamma_2$ are valuations, we say that v is a *coarsening* of w, or w if *finer than* v, if for all $x, y \in K^\times$ we have

$$w(x) \leq w(y) \Rightarrow v(x) \leq v(y).$$

This implies that $\mathcal{O}_w \subseteq \mathcal{O}_v$, $\mathcal{o}_v \subseteq \mathcal{o}_w$, and if w is convex and v is a coarsening of w, then v is convex. Particularly, we have that the natural valuation is the finest convex valuation of an ordered field K.

From now on we will denote the natural valuation of an ordered field by w.

2.3 Well based series and Hahn Fields

Set E be a field and let $\mathfrak{M}$ be a multiplicative ordered abelian group with identity 1, then $E[[\mathfrak{M}]]$ is the set of all well based series, that is, the set of all functions $f : \mathfrak{M} \to E$, written as formal sums $f = \sum_{m \in \mathfrak{M}} f_m m$, with $f_m = f(m)$, whose support

$$\text{supp}(f) := \{m \in \mathfrak{M} : f_m \neq 0\}$$

has no strictly increasing sequences. Equipping $E[[\mathfrak{M}]]$ with the usual addition and multiplication of formal series:

$$f + g = \sum_{m \in \mathfrak{M}} (f_m + g_m) m$$

and

$$f \cdot g = \sum_{m \in \mathfrak{M}} \left(\sum_{n_1 \cdot n_2 = m} f_{n_1} \cdot g_{n_2} \right) m$$

[1] We say that M is a *group complement* of N in K if M is a submodule of the Z-module K such that $K = M \oplus N$

$E[[\mathfrak{M}]]$ becomes a field called the *Hahn field with monomials from* $\mathfrak{M}$ and *coefficients from* E (for details see [4, Lemma 3.1.3]).

Identifying E with the subfield $\{r \cdot 1 : r \in E\}$ of $E[[\mathfrak{M}]]$, the field $E[[\mathfrak{M}]]$ can be decomposed as an internal direct sum $E[[\mathfrak{M}^{>1}]] \oplus E \oplus E[[\mathfrak{M}^{<1}]]$, where

$$E[[\mathfrak{M}^{>1}]] = \{f \in E[[\mathfrak{M}]] : m > 1 \text{ for all } m \in \operatorname{supp}(f)\}$$

and

$$E[[\mathfrak{M}^{<1}]] = \{f \in E[[\mathfrak{M}]] : m < 1 \text{ for all } m \in \operatorname{supp}(f)\}.$$

Since for each non-zero $f \in E[[\mathfrak{M}]]$, $\operatorname{supp}(f)$ has a maximum element $\mathfrak{d}(f)$, called the *dominant monomial of* f, we can define a dominance relation $\preceq$ between non-zero elements of $E[[\mathfrak{M}]]$ as

$$f \preceq g \Leftrightarrow \mathfrak{d}(f) \leq \mathfrak{d}(g),$$

with corresponding valuation ring $\mathcal{O} = E \oplus E[[\mathfrak{M}^{<1}]]$ (the set of bounden elements), maximal ideal $\mathcal{o} = E[[\mathfrak{M}^{<1}]]$ (the set of infinitesimal elements) and residual field E.

Additionally, if E is an ordered field and $f \in E[[\mathfrak{M}]]$ we set $f > 0$ if and only if $f_{\mathfrak{d}(f)} > 0$. Hence, $E[[M]]$ becomes an ordered field, and identifying $\mathfrak{M}$ with $\{1 \cdot m : m \in \mathfrak{M}\}$ in $E[[\mathfrak{M}]]^{\neq 0}$, each $f \in E[[\mathfrak{M}]]^{>0}$ can be represented in an unique way as the product $\mathfrak{d}(f) \cdot f_{\mathfrak{d}(f)} \cdot (1 + \epsilon)$ where ϵ is an infinitesimal element. So we have

$$E[[\mathfrak{M}]]^{>0} = \mathfrak{M} \cdot E^{>0} \cdot (1 + E[[\mathfrak{M}^{<1}]]).$$

As important facts we have that each Hahn field $E[[\mathfrak{M}]]$ is maximal [4, Corollaries 2.2.7 and 3.2.9] and it is real closed if and only if E is real closed an $\mathfrak{M}$ is divisible [4, Corollary 3.2.10].

Some times we will work with Hahn fields with additive notation for the group. Let Γ be an additively written ordered abelian group and $\phi : \Gamma \to t^{\Gamma}$ and ordered-reversing group isomorphism, where t is just a symbol. Thus, $\mathfrak{M} := t^{\Gamma}$ is a multiplicative ordered group and for any field E we have the Hahn field $E[[t^{\Gamma}]]$, usually denoted $E((t^{\Gamma}))$. In this case we prefer to take $\operatorname{supp}(f)$ as a subset of Γ rather than $\mathfrak{M}$.

2.4 Ordered logarithmic fields

Let K be an ordered field. A *logarithm* of K is an ordered embedding log from the multiplicative group $K^{>0}$ into the additive group K. The structure $(K, \log)$ consisting of an ordered field K together a logarithm log is called *logarithmic field*. Particularly, if log is surjective then we say that log is an *exponential-logarithmic function* and the structure $(K, \log)$ an *exponential-logarithmic field*.

If $(K, \log)$ is an exponential-logarithmic field, then the inverse exp of log is called an *exponential function* of K and $(K, \exp)$ an *exponential field*.

Examples.

- *Let $E[[\mathfrak{M}]]$ be a Hahn field where E is a logarithmic field with logarithm $\log_E$. If s is an ordered embedding from the multiplicative group $\mathfrak{M}$ into the additive group $E[[\mathfrak{M}^{>1}]]$, then the map*
$$\log : E[[\mathfrak{M}]]^{>0} \to E[[\mathfrak{M}]]$$
given by
$$\log(f) = s(\mathfrak{d}(f)) + \log(f_{\mathfrak{d}(f)}) + \sum_{i\geq 1}(-1)^{i+1}\frac{\epsilon^i}{i},$$
for $f = \mathfrak{d}(f) \cdot f_{\mathfrak{d}(f)} \cdot (1+\epsilon)$ with $\mathfrak{d}(f)$ the dominant monomial of f and $\epsilon \in \mathcal{O}$, is the unique order preserving embedding of groups which extends $\log_E$, s and $\mathrm{L_u}$. The map s is called a logarithmic section and L is the logarithm associated to s.

  From [19] we know that for each field $E[[\mathfrak{M}]]$ with E logarithmic there is always a logarithmic section, but if $\mathfrak{M}$ is a non trivial ordered abelian group, then $E[[\mathfrak{M}]]$ does not admit an exponential-logarithmic function.

- *The structure of the real numbers $\mathbb{R}$ together the usual logarithm* log *is an ordered exponential-logarithmic field.*

Extensions of logarithmic fields

Let $(E, \log_E)$ and $(K, \log)$ be logarithmic fields. We say that $(K, \log)$ is an extension of $(E, \log_E)$, denoted by $(E, \log_E) \subseteq (K, \log)$, if K is an extension of E as ordered fields and $\log_E = \log|_E$. Moreover, if $(F, \log_F)$ and $(K, \log)$ are logarithmic field extensions of $(E, \log_E)$, $\varphi : F \to K$ is an ordered embedding over E of fields and $\varphi(\log_F(a)) = \log(\varphi(a))$ for all $a \in F$, then we say that $\varphi : F \to K$ is an embedding over E of ordered valued logarithmic fields.

By definition, the intersection of two logarithmic subfields of a logarithmic field is again a logarithmic field. Thus we have:

Lemma 2.4. *Let $(K, \log)$ be a logarithmic field and E a subfield of K. Then there is the smallest ordered logarithmic subfield $(F, \log_F)$ of $(K, \log)$ which contains E.*

2.5 The theories T_{an} and $T_{an}(\exp)$

Since most of the structures that we will study in the next chapter can be expanded to a models of the theories T_{an} or $T_{an}(\exp)$, we recall in this section the main results about such first order theories.

Let $\mathbb{R}\{X_1, ..., X_n\}$ be the ring of all real power series in $X_1, ..., X_n$ that converge in a neighbourhood of I^m, where $I = [-1, 1]$. For each $f \in \mathbb{R}\{X_1, ..., X_n\}$ we define the restricted analytic function $f_I : \mathbb{R}^n \to \mathbb{R}$ by
$$f_I(X) = \begin{cases} f(X), & \text{if } X \in I^n \\ 0, & \text{if } X \notin I^n. \end{cases}$$

Let L_{an} be the language of ordered rings $\{<, 0, 1, +, -, \cdot\}$ augmented by a new function symbol for each restricted analytic function $f_I : \mathbb{R}^n \to \mathbb{R}$, for each n. Then $T_{an} = Th(\mathbb{R}_{an})$, where $\mathbb{R}_{an}$ is the field of real numbers with its natural L_{an}-structure. By [25] we know that T_{an} is model complete and o-minimal.

Moreover, if we add to L_{an} the function symbol $^{-1}$ to denote $x \to \frac{1}{x}$, where $\frac{1}{0} = 0$, then it is shown in [9] that T_{an} has quantifier elimination. Adding also the radical symbols $\sqrt[n]{}$ for $n > 1$ and the axioms

$$(x > 0 \to ((\sqrt[n]{x})^n = x \wedge \sqrt[n]{x} > 0) \wedge (x \leq 0 \to \sqrt[n]{x} = 0)$$

then the extension T_{an^*} obtained in this way has a universal axiomatization (see [27]). Thus, in the language $L_{an^*} = L_{an} \cup \{^{-1}\} \cup \{\sqrt[n]{}\}_{n>1}$, each substructure of a model of T_{an^*} is in fact a model of T_{an^*}.

Example. *From [27] we know that for any ordered abelian group Γ, the Hahn field $\mathbb{R}((t^\Gamma))$ can be expanded to an L_{an}-structure and if Γ is divisible then in fact it can be expanded to a model of T_{an}.*

From now on, we single out the unary function symbol e of L_{an} for the restricted analytic function $e : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{e}(x) = e^x$ for $|x| \leq 1$ and $e(x) = 0$ for $|x| > 1$.

Let $L_{an}(\exp)$ be the language L_{an} augmented with a unary function symbol exp and $T_{an}(\exp)$ be the theory obtained by adding to T_{an} the universal closure of the following axioms:

(1) $\exp(x + y) = \exp(x)\exp(y)$.

(2) if $x < y$, then $\exp(x) < \exp(y)$.

(3) if $x > 0$, then there is y such that $\exp(y) = x$

(4) if $x > n^2$, then $\exp(x) > x^n$, for each $n > 0$.

(5) if $-1 \leq x \leq 1$, then $\exp(x) = \mathrm{e}(x)$; where e is the function symbol of L_{an} corresponding to $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

From [27], we know that the theory $T_{an}(\exp)$ is a complete axiomatization of $Th(\mathbb{R}_{an}, \exp)$. Moreover, if log is a new unary function symbol and $T_{an}(\exp, \log)$ is the $L_{an*}(\exp, \log)$-theory obtained by extending $T_{an}(\exp)$ by the following defining axiom

$$(x > 0 \to \exp(\log(x)) = x) \wedge (x \leq 0 \to \log(x) = 0),$$

then $T_{an}(\exp, \log)$ has quantifier elimination, has a universal axiomatization and is o-minimal[2]. The field of transseries $\mathbb{T}$, whose formal definition can be found in [4, 30, 31], can be expanded in a natural way to a model of $T_{an}(\exp)$.

[2]We recall that a theory in which an order is given or definable is called *o-minimal* if in every model of this theory, each definable subset is a finite union of points and intervals

2.6 The transseries field $\mathbb{T}$

The field of *Transseries* $\mathbb{T}$, or Logarithmic-Exponential series field as it was called in [30, 31], is an exponential ordered diferential field obtained as the direct union of Hahn fields with real numbers as coefficients and some divisible abelian groups as groups of monomials (for a formal construction see [4, 31]). Initially this field was defined in [30, 31] to exhibit a non-standard model of the theory $T_{an}(\exp)$, but in last two decades $\mathbb{T}$ has been studied extensively in [2, 3, 4] as valued ordered differential field. Particularly, it is shown in [4] that $\mathbb{T}$ is an Newtonian Liouville closed ω-free H-field, and it has the following main result:

Theorem 2.5. *Let $L = \{0, 1, +, -, ., ', \leq, \preceq\}$. The L-theory T^{nl} whose models are the Newtonian Liouville closed ω-free H-fields is model complete.*

Furthermore, $\mathbb{T}$ is a model of the completion of T^{nl} in which it has small derivation, this is $f \prec 1 \Rightarrow f' \prec 1$ for all f. Expanding the language L to L' to include a unary function symbol i and unary predicates Λ and Ω, and extending T^{nl} to the L' theory $T^{nl}_{i,\Lambda,\Omega}$ by adding as axioms the universal closure of

$$[a \neq 0 \rightarrow a \cdot i(a) = 1] \wedge [a = 0 \rightarrow i(a) = 0],$$

$$\Lambda(a) \leftrightarrow \exists y[y \succ 1 \wedge a = -y^{\dagger\dagger}],$$

and

$$\Omega(a) \leftrightarrow \exists y[y \neq 0 \wedge 4y'' + ay = 0],$$

where $y^\dagger = \dfrac{y'}{y}$ is the logarithmic derivative, then:

Theorem 2.6. *The L' theory $T^{nl}_{i,\Lambda,\Omega}$ admits elimination of quantifiers.*

2.7 The field $\mathbb{T}_{\log}$ of logarithmic transseries

The subfield $\mathbb{T}_{\log}$ of purely logarithmic transseries of $\mathbb{T}$ is a field whose elements are well based series with real coefficients and monomials which do not involve exponentiation.

We can construct $\mathbb{T}_{\log}$ independently as follows: Setting $\ell_0 = x$ and $\ell_{n+1} = \log(\ell_n)$ for all n, let $\mathcal{L}_n$ be the multiplicative ordered group of formal products given by

$$\mathcal{L}_n = \{\ell_0^{r_0}\ell_1^{r_1}\cdots\ell_n^{r_n} : r_0, r_1, ..., r_n \in \mathbb{R}\},$$

where $\ell_0^{r_0}\ell_1^{r_1}\cdots\ell_n^{r_n} > 1$ if and only if the exponents $r_0, r_1, ..., r_n$ are not all zero, and $r_i > 0$ for the least i with $r_i \neq 0$.

If $m \leq n$ then $\mathcal{L}_m$ is an ordered subgroup of $\mathcal{L}_n$, and the ordered group inclusions

$$\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq ... \subseteq \mathcal{L} = \bigcup_n \mathcal{L}_n,$$

induce field inclusions

$$\mathbb{R}[[x^{\mathbb{R}}]] = \mathbb{R}[[\mathcal{L}_0]] \subseteq \mathbb{R}[[\mathcal{L}_1]] \subseteq ...$$

where $\mathbb{R}[[\mathcal{L}_n]]$ is the Hahn field of well based series with real coefficients and monomials in $\mathcal{L}_n$. Finally we set

$$\mathbb{T}_{\log} = \bigcup_n \mathbb{R}[[\mathcal{L}_n]].$$

It follows that $\mathbb{T}_{\log}$ is a proper ordered subfield of $\mathbb{R}[[\mathcal{L}]]$ which contains $\mathbb{R}$. For example, the series

$$\frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \frac{1}{(\ell_0\ell_1\ell_2)^2} + \cdots$$

lies in $\mathbb{R}[[\mathcal{L}]]$, but not in $\mathbb{T}_{\log}$.

Further, since each group $\mathcal{L}_n$ is divisible, the fields $\mathbb{R}[[\mathcal{L}_n]]$ and $\mathbb{T}_{\log}$ are real closed. Moreover, as each Hahn field $\mathbb{R}[[\mathcal{L}_n]]$ can be expanded to a model of T_{an}, we have that $\mathbb{T}_{\log}$ can be expanded in a natural way to a model of T_{an}.

2.7.1 Logarithm and valuation in $\mathbb{T}_{\log}$

The field of logarithmic transseries $\mathbb{T}_{\log}$ carries on natural valuation and logarithmic maps:

The valuation

Let $\Gamma_{\log}$ be the $\mathbb{R}$-vector space $\bigoplus_{n>0} \mathbb{R}\ell_n$ ordered lexicographically, that is, $\alpha = \sum_{i=0}^{n} r_i l_{i+1} > 0$ if $r_k > 0$ for the least $k \in \{0, 1, ..., n\}$ such that $r_k \neq 0$. We can define a convex valuation v in $\mathbb{T}_{\log}$ as the unique map $v : \mathbb{T}_{\log} \to \Gamma_{\log} \cup \{\infty\}$ such that

(1) $v(r) = 0$ for all $r \in \mathbb{R}$, $v(\ell_0^{r_0}\ell_1^{r_1}\cdots\ell_n^{r_n}) = -r_0\ell_1 - r_1\ell_2 - \cdots - r_n\ell_{n+1}$,

(2) $v(f) = v(\mathfrak{d}(f))$ for all $f \in \mathbb{T}_{\log}^{\neq 0}$, where $\mathfrak{d}(f)$ is the dominant monomial of f.

(3) $v(0) = \infty$.

Here, the valuation ring associated to v is $\mathcal{O}_{\log} = \mathbb{R} \oplus \scriptstyle{\mathcal{O}}_{\log}$, with maximal ideal

$$\scriptstyle{\mathcal{O}}_{\log} = \{f \in \mathbb{T}_{\log} : v(f) > 0\},$$

and residue field $\mathbb{R}$.

The logarithm

Since each positive element $f \in \mathbb{T}_{\log}$ can be decomposed as $f = \mathfrak{d}(f) \cdot f_{\mathfrak{d}(f)} \cdot (1 + \epsilon)$ where $f_{\mathfrak{d}(f)} \in \mathbb{R}^{>0}$ and $\epsilon \in \scriptstyle{\mathcal{O}}_{\log}$, we define the logarithm of f as

$$\log(f) = \log_s(\mathfrak{d}(f)) + \log_{\mathbb{R}}(f_{\mathfrak{d}(f)}) + \log_{\mathcal{O}}(1 + \epsilon),$$

where $\log_{\mathbb{R}}$ is the natural logarithm in $\mathbb{R}$, $\log_{\mathcal{O}}$ is the logarithm on 1-units given by

$$\log_{\mathcal{O}}(1 + \epsilon) = \sum_{i>0} (-1)^{i+1} \frac{\epsilon^i}{i}$$

and $\log_s$ is the logarithmic section defined as

$$\log_s(\ell_0^{r_0}\ell_1^{r_1}\cdots\ell_n^{r_n}) = r_0\ell_1 + \cdots + r_n\ell_{n+1}.$$

Under this definition we see that the map log is a non surjective ordered embedding from the multiplicative group $\mathbb{T}_{\log}^{>0}$ into the additive group $\mathbb{T}_{\log}$. Particularly, we can see that $\log(\mathbb{T}_{\log}^{>0})$ is an $\mathbb{R}$-vector space. Moreover, since by [20] for each Hahn field $\mathbb{R}[[\mathcal{L}_n]]$ we have that $\log_{\mathcal{O}}(1+\mathcal{o}_{\mathbb{R}[[\mathcal{L}_n]]}) = \mathcal{o}_{\mathbb{R}[[\mathcal{L}_n]]}$, then

$$\log(\mathbb{T}_{\log}^{>0}) = \Gamma_{\log} \oplus \mathbb{R} \oplus \mathcal{o}_{\log}.$$

Additionally, the valuation and the logarithm are related by the following property known as the *Growth Axiom* (GA):

$$\text{for all } f \in \mathbb{T}_{\log}^{>0} \text{ with } v(f) < 0 \text{ we have that } v(\log(f)) > v(f),$$

which implies $f > \log(f^n) = n\log(f)$ for all $n \in \mathbb{N}$.

On the other hand, from the logarithmic map in $\mathbb{T}_{\log}$ we can define the valuation ring. Particularly, we have

$$\mathcal{O}_{\log} = \{f \in \mathbb{T}_{\log} : [0, |f|] \subseteq \log(\mathbb{T}_{\log}^{>0})\}.$$

Proof. First, since $\mathcal{O}_{\log} = \mathbb{R} \bigoplus \mathcal{o}_{\log} \subseteq \log(\mathbb{T}_{\log}^{>0})$ and $\mathcal{O}_{\log}$ is convex, we have the inclusion from left to right. Now, let $f \in \mathbb{T}_{\log}$ be such that $[0, |f|] \subseteq \log(\mathbb{T}_{\log}^{>0})$. If we suppose that $f \notin \mathcal{O}_{\log}$, then $v(f) = v(\mathfrak{d}(f)) < 0$. Defining $g = f \cdot (\mathfrak{d}(f))^{-1/2}$, we notice that $0 < |g| < |f|$ and

$$v(g) = v(f \cdot (\mathfrak{d}(f))^{-1/2}) = v(f) + v(\mathfrak{d}(f))^{-1/2}) = \frac{1}{2}v(f) < 0.$$

However, as $v(f) < 0$ and $f \in \log(\mathbb{T}_{\log}^{>0})$, we have $v(f) \in \chi(\Gamma_{\log}^{<0})$ and $v(g) = \frac{1}{2}v(f) \notin \chi(\Gamma_{\log}^{<0})$. This implies that $g \notin \log(\mathbb{T}_{\log}^{>0})$ which is a contradiction. Thus, $f \in \mathcal{O}_{\log}$. □

The precontraction map

The logarithm induce some extra structure on the value group $\Gamma_{\log}$ given by the map χ from $\Gamma_{\log}$ in $\Gamma_{\log}$ defined as

$$\chi(\alpha) = \begin{cases} v(\log(f), & \text{if } \alpha = v(f) < 0 \text{ with } f > 0, \\ 0, & \text{if } \alpha = 0, \\ -\chi'(-\alpha), & \text{if } \alpha > 0. \end{cases}$$

Remark. *Let $f \in \mathbb{T}_{\log}^{>0}$ with $v(f) < 0$. Then the dominant monomial of f is given by*

$$\mathfrak{d}(f) = x^{r_0}\ell_1^{r_1}\cdots\ell_k^{r_k}\cdots\ell_n^{r_n}$$

where for some $0 \leq k \leq n$ we have $r_k > 0$ and $r_m = 0$ for all $0 \leq m < k$. Thus we have:

(1) $\log(\mathfrak{d}(f)) = r_k\ell_{k+1} + \cdots + r_n\ell_{n+1}$,

(2) $v(f) = -\log(\mathfrak{d}(f)) = -(r_k\ell_{k+1} + \cdots + r_n\ell_{n+1})$,

(3) $\chi(v(f)) = v(\log(f)) = -\log(\mathfrak{d}(\log(f))) = -\log(\ell_{k+1}) = -\ell_{k+2}$,

From this observation, we see that if $f \in \mathbb{T}_{\log}^{>0}$ with $v(f) < 0$, then $v(f) < \chi(v(f))$ and

$$v(\log(\mathbb{T}_{\log}^{>0})) = \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0} \cup \{\infty\}.$$

Even more, if $f \in \mathbb{T}_{\log}^{\neq 0}$ then

$$f \in \log(\mathbb{T}_{\log}^{>0}) \leftrightarrow v(\mathrm{supp}(f)) \subseteq \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0}$$

Proof. If $f \in \log(\mathbb{T}_{\log}^{>0})$, then $f = r_{k_1}\ell_{k_1} + r_{k_2}\ell_{k_2} + ... + r_{k_n}\ell_{k_n} + r + \epsilon$ for some $k_1 > 1$, $r_{k_1}, ..., r_{k_n}, r \in \mathbb{R}$ and $v(\epsilon) > 0$. Thus $\mathrm{supp}(f) \subseteq \{\ell_{k_1}, \ell_{k_2}, ..., \ell_{k_n}, 1\} \cup \mathrm{supp}(\epsilon)$ and then

$$v(\mathrm{supp}(f)) \subseteq \{-\ell_{k_1+1}, -\ell_{k_2+1}, ..., -\ell_{k_n+1}, 0\} \cup \Gamma^{>0} \subseteq \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0}.$$

Conversely, if we suppose that $v(\mathrm{supp}(f)) \subseteq \Gamma^{>0}$, then $v(f) > 0$ and $f \in \log(\mathbb{T}_{\log}^{>0})$. If we suppose that $v(\mathrm{supp}(f)) \subseteq \chi(\Gamma^{<0})$, by construction of $\mathbb{T}_{\log}$, there are $r_{k_1}, ..., r_{k_n} \in \mathbb{R}$ such that $f = r_{k_1}\ell_{k_1} + r_{k_2}\ell_{k_2} + ... + r_{k_n}\ell_{k_n}$ with $k_1 > 1$. Thus $f \in \Gamma_{\log}$. Now, for any $f \in \mathbb{T}_{\log}^{\neq 0}$ such that $v(\mathrm{supp}(f)) \subseteq \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0}$ there are $g, h \in \mathbb{T}_{\log}$ and $r \in \mathbb{R}$ such that $f = g + r + h$ with $v(\mathrm{supp}(g)) \subseteq \chi(\Gamma^{<0})$, $v(\mathrm{supp}(r)) = v(1) = 0$ and $v(\mathrm{supp}(h)) \subseteq \Gamma^{>0}$. Thus, as $\log(\mathbb{T}_{\log}^{>0})$ is a vector space then we have that $f \in \log(\mathbb{T}_{\log}^{>0})$.

□

From this result we can give an explicit description of an element f in $\mathbb{T}_{\log} \setminus \log(\mathbb{T}_{\log}^{>0})$. First, we define the sub-series $f_{\log}$ as follows: If $f = \sum_{m \in \mathrm{supp}(f)} f_m m$ then

$$f_{\log} = \sum_{m \in \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0}} f_m m.$$

Thus, we have that $f_{\log} \in \log(\mathbb{T}_{\log}^{>0})$ and since $f - f_{\log} \neq 0$ we get

$$v(\mathrm{supp}(f - f_{\log})) \subseteq \Gamma^{<0} \setminus \chi(\Gamma^{<0}),$$

and

$$v(f - f_{\log}) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0}).$$

In conclusion, we have that

Lemma 2.7. *Let* $f \in \mathbb{T}_{\log}$.

$$f \notin \log(\mathbb{T}_{\log}^{>0}) \leftrightarrow \exists g \in \mathbb{T}_{\log}^{>0} \text{ such that } v(f - \log(g)) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0}).$$

3. $\mathbb{T}_{\log}$ as a restricted analytic field with partial exponentiation

We point out that $\mathbb{T}_{\log}$ can be expanded to a model of the theory of real numbers with restricted analytic functions T_{an} (see section 3.2), and we can define a map $\exp : \mathbb{T}_{\log} \to \mathbb{T}_{\log}^{\geq 0}$ as $\exp(f) = 0$ for $f \in K \setminus \mathcal{O}$ and

$$\exp(r + \epsilon) = e^r \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!},$$

for $r \in \mathbb{R}$ and $\epsilon \in \mathcal{O}_{\log}$, such that

(1) $\log(\exp(f)) = f$ for all $f \in \mathcal{O}_{\log}$, and

(2) exp restricted to $[-1, 1]$ agree with the interpretation in $\mathbb{T}_{\log}$ of the function symbol[1] e of T_{an}.

Under such observations, and as a first step to understand the theory of $\mathbb{T}_{\log}$ as an ordered valued logarithmic field, we study in this chapter structures of the form $(K, \mathcal{O}, \exp)$ where K is a model of T_{an}, $\mathcal{O}$ is a convex subring of K and $\exp : K \to K^{\geq 0}$ is a map such that its restriction to $\mathcal{O}$ is a surjective homomorphism from the additive group $\mathcal{O}$ onto the multiplicative group $((\mathcal{O}_{\log})^{\times})^{>0}$ and satisfies property (2).

In the fist part of this chapter we present some results of the theory T_{convex} that we will use to define the theory T_+ of the class of structures described above. Particularly, we show that $(\mathbb{T}_{\log}, \mathcal{O}_{\log}, \exp)$, with exp as was defined at the beginning of this chapter, is a model of T_+. Moreover, we show that this theory is complete, model complete and has quantifier elimination in an appropriate language.

3.1 The theory T_{convex}

Let L be a language extending the language of ordered rings, T a complete o-minimal L-theory extending the theory of real closed fields and K a model of T. A T-*convex* subring $\mathcal{O}$ of K is a convex subring of K such that $f(\mathcal{O}) \subseteq \mathcal{O}$ for each 0-definable continuous function

[1] We recall that through this document e is the unary function symbol of T_{an} which is interpreted in $\mathbb{R}_{an}$ as the restricted analytic function $e : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{e}(x) = e^x$ for $|x| \leq 1$ and $e(x) = 0$ for $|t| > 1$

$f : K \to K$. If $L_{\mathcal{O}}$ is the language L of T with an extra unary relation symbol to denote a T-convex ring, then T_{convex} denotes the theory of the pairs $(K, \mathcal{O})$ with K a model of T and $\mathcal{O}$ a proper T-convex subring of K.

From [28], if $(K, \mathcal{O})$ is a model of T_{convex} we have the following relevant facts:

($C1$) $\mathcal{O}$ is a valuation ring of K.

($C2$) The convex hull in K of any elementary substructure of K is a T-convex subring of K. Moreover, each T-convex subring of K contains the prime model of T.

($C3$) $\mathcal{O}$ is the convex hull of some $F \preceq K$, indeed $\mathcal{O} = F + o$ where

$$o = \{x \in K : |x| < y \text{ for all } y \in F^{>0}\}.$$

Moreover, if F and F_1 are maximal elementary substructures of K contained in $\mathcal{O}$, then there is a unique isomorphism $\phi : F \to F_1$ such that $\text{res}(\phi(x)) = \text{res}(x)$ for all $x \in F$.

($C4$) The residual field $\text{res}(K)$ corresponding to $\mathcal{O}$ can be expanded to a model of T. More precisely, taking any maximal elementary substructure F of K contained in $\mathcal{O}$, we can make $\text{res}(K)$ into a model of T such that the field isomorphism $F \to \text{res}(K)$ becomes an isomorphism of L-structures.

($C5$) If $F \preceq K$ with $F \subseteq \mathcal{O}$ and $a \notin F + m_F$, where $m_F = \{x \in K : |x| < y \text{ for all } x \in F^{>0}\}$, then $F\langle a\rangle \subseteq \mathcal{O}$.

($C6$) Let $K\langle a\rangle$ be a model of T which extends K with $a \notin K$. Let C be a cut in $\text{res}(K)$ and let $\mathcal{O}_1, \mathcal{O}_2$ be T-convex subrings of $K\langle a\rangle$ such that for $i = 1, 2$ we have $(K, \mathcal{O}) \subseteq (K\langle a\rangle, \mathcal{O}_i)$, $a \in \mathcal{O}_i$, and $\text{res}(a)$ realizes the cut C in $\mathcal{O}_i$. Then $\mathcal{O}_1 = \mathcal{O}_2$.

($C7$) Let $a \notin K$ such that $|\mathcal{O}| < a < |K \setminus \mathcal{O}|$. There is exactly one T-convex subring $\mathcal{O}'$ of $K\langle a\rangle$ such that $(K, \mathcal{O}) \subseteq (K\langle a\rangle, \mathcal{O}')$ with $a \notin \mathcal{O}'$.

($C8$) Each substructure $(F, \mathcal{O}_F)$ of $(K, \mathcal{O})$ has a T_{convex}-closure. That is, there is a model $(F', \mathcal{O}_{F'})$ of T_{convex} such that $(F, \mathcal{O}_F) \subseteq (F', \mathcal{O}_{F'})$ and for any other model $(G, \mathcal{O}_G)$ of T_{convex} which extends $(F, \mathcal{O}_F)$, there is an embedding of $(F', \mathcal{O}_{F'})$ over $(F, \mathcal{O}_F)$ into $(G, \mathcal{O}_G)$.

As main result about the model theory of T_{convex} we have from the above

Proposition 3.1. *T_{convex} is complete and:*

1. *If T has quantifier elimination and is universally axiomatizable, then T_{convex} has quantifier elimination.*
2. *If T is model complete, then T_{convex} is model complete.*
3. *T_{convex} is weakly o-minimal.*

In particular, if $L = L_{an^*}$ and $T = T_{an^*}$, then T_{convex} has quantifier elimination. Moreover, if $K \models T$, then each convex subring $\mathcal{O}$ of K is in fact a T-convex subring of K. Thus, for $K \neq \mathcal{O}$, $(K, \mathcal{O})$ is a model of T_{convex} where $\text{res}(K)$ can be made into a model of T_{an}.

3.2 The theory T_+

Throughout this section T denotes the theory T_{an^*} in the language L_{an*} and T_{convex} denotes its associated convex theory in the language $L_{an*,\mathcal{O}} = L_{an*} \cup \{\mathcal{O}\}$ as defined in the previous section.

Definition 3.2. *Let L_+ be the language $L_{an*,\mathcal{O}}$ expanded by a unary function symbol* exp*. We define the L_+-theory T_+ as the theory of the structures $(K, \mathcal{O}, \exp)$ where $(K, \mathcal{O})$ is a model of T_{convex} and $\exp : K \to K$ is a function such that $\exp(x) = 0$ for all $x \in K \setminus O$, $\exp(\mathcal{O}) \subseteq \mathcal{O}$ and for all $x, y \in \mathcal{O}$ we have:*

E1. $\exp(x + y) = \exp(x)\exp(y)$*;*

E2. $\exp(x) = \mathrm{e}(x)$ *if* $|x| \leq 1$*;*

E3. if $x > n^2$, then $\exp(x) > x^n$, for $n \geq 0$;

E4. if $y > 1$, there is $x \in \mathcal{O}$ such that $\exp(x) = y$.

Examples.

- *Let $\mathfrak{M}$ be a divisible ordered abelian group and $K = \mathbb{R}[[\mathfrak{M}]]$ be the Hahn field over $\mathbb{R}$ with monomials in $\mathfrak{M}$. We see K as a valued ordered field with the natural valuation and valuation ring $\mathcal{O} = \mathbb{R} + o$, where o is the maximal ideal of $\mathcal{O}$. Since K can be made a model of T, $(K, \mathcal{O})$ is a model of T_{convex}. Now, define $\exp : K \to K$ as $\exp(x) = 0$ for $x \in K \setminus \mathcal{O}$ and*

$$\exp(r + \epsilon) = e^r \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!},$$

  for $r \in \mathbb{R}$ and $\epsilon \in o$, then $(K, \mathcal{O}, \exp)$ is a model of T_+. From [20], for the Hahn field K we know that $\exp(o) = \mathrm{e}(o) = 1 + o$. Thus, since T_{convex} is complete, for any model $(F, \mathcal{O})$ of T_{convex} we have that $\mathrm{e}(o) = 1 + o$.

- *Since $\mathbb{T}_{\log} = \bigcup_{n=0}^{\infty} \mathbb{R}[[\mathcal{L}_n]]$, $\mathbb{T}_{\log}$ is a model of T and $(\mathbb{T}_{\log}, \mathcal{O}_{\log})$ is a model of T_{convex}. Thus, defining $\exp : \mathbb{T}_{\log} \to \mathbb{T}_{\log}$ as in the previous example, $(\mathbb{T}_{\log}, \mathcal{O}_{\log}, \exp)$ is a model of T_+.*

Now, for a model $(K, \mathcal{O}, \exp)$ of T_+, $\exp(0) = e(0) = 1$ and

$$\exp(x) = \exp(x/2 + x/2) = \exp(x/2)^2 > 0$$

for any $x \in \mathcal{O}$. Moreover, $\exp(x - x) = \exp(0) = 1$ and $\exp(-x) = (\exp(x))^{-1}$. By axiom E3, taking $n = 0$, we obtain that if $x > 0$ then $\exp(x) > 1$; thus, if $x, y \in \mathcal{O}$ with $x < y$, then $\exp(x) < \exp(y)$. Hence exp restricted to $\mathcal{O}$ is positive and strictly increasing.

By axiom E4, $\exp(\mathcal{O}) = (\mathcal{O}^\times)^{>0}$ and by the last observation in the first example, we obtain $\exp(o) = 1 + o$. Therefore, exp induces an exponential function on $\mathrm{res}(K)$:

Lemma 3.3. *Let $(K, \mathcal{O}, \exp)$ be a model of T_+. If we define $\exp' : \mathrm{res}(K) \to \mathrm{res}(K)$ as*

$$\exp'(\mathrm{res}(x)) = \mathrm{res}(\exp(x)),$$

then $(\mathrm{res}(K), \exp')$ is a model of $T_{an}(\exp)$.

Proof. First, we show that $\exp'$ is well defined. Let $x, y \in \mathcal{O}$ be such that $\mathrm{res}(x) = \mathrm{res}(y)$. Then $x = y + \epsilon$ for some $\epsilon \in \mathcal{o}$, and

$$\exp(x) = \exp(y)\exp(\epsilon) = \exp(y)(1 + \epsilon')$$

with $\exp(\epsilon) = 1 + \epsilon'$ and $\epsilon' \in \mathcal{o}$. Thus $\mathrm{res}(\exp(x)) = \mathrm{res}(\exp(y))$. Under this definition, for all $\mathrm{res}(x) \in \mathrm{res}(K)$, $\exp'(\mathrm{res}(x)) \in \mathrm{res}(K)^{>0}$, $\exp'$ is strictly increasing, and surjective since $\exp(\mathcal{O}) = (\mathcal{O}^\times)^{>0}$. Thus, by axiom $E1$, $\exp'$ is an order preserving embedding from the additive group $\mathrm{res}(K)$ onto the multiplicative group $K^{>0}$. Finally, since $\mathrm{res}(K)$ is a model of T, by $E2$ and $E3$, the structure $(\mathrm{res}(K), \exp')$ is a model of $T_{an}(\exp)$. □

Now, for a model $(K, \mathcal{O}, \exp)$ of T_+ define $\log : (\mathcal{O}^\times)^{>0} \to \mathcal{O}$ as the partial inverse of exp:

$$\log(y) = x \Leftrightarrow \exp(x) = y, \text{ for } y \in (\mathcal{O}^\times)^{>0} \text{ and } x \in \mathcal{O}.$$

Then, for $x, y \in (\mathcal{O}^\times)^{>0}$ with $\mathrm{res}(x) = \mathrm{res}(y)$, $x = y(1 + \epsilon)$ for some $\epsilon \in \mathcal{o}$ and we have $\log(x) - \log(y) \in \mathcal{o}$. If $\log' : \mathrm{res}(K)^{>0} \to \mathrm{res}(K)$ is the inverse of $\exp' : \mathrm{res}(K) \to \mathrm{res}(K)^{>0}$, then $\log'(\mathrm{res}(x)) = \mathrm{res}(\log(x))$.

3.2.1 Model completeness of T_+

Based on the proof of quantifier elimination of $T_{an}(\exp)$ given in [27] and of T_{convex} given in [28], we prove in this section that the theory T_+ is model complete and complete. The main step is to show that if $(E, \mathcal{O}_E, \exp_E) \subseteq (K, \mathcal{O}, \exp)$ are models of T_+ where $\mathrm{res}(E) \neq \mathrm{res}(K)$, then we can obtain, in a canonical way, a model $(H, \mathcal{O}_H, \exp_H) \subseteq (K, \mathcal{O}, \exp)$ of T_+ which properly extends $(E, \mathcal{O}_E, \exp_E)$ with $\mathrm{res}(H) = \mathrm{res}(K)$.

Since the language L_{an} has a name for each real number, $\mathbb{R}_{an}$ is an elementary substructure of K with $\mathbb{R} \subseteq \mathcal{O}$. Thus, $\exp(r) = e^r$ for $r \in \mathbb{R}$ and $\log(s)$ for $s \in \mathbb{R}^{>0}$ viewed as an element of $(\mathcal{O}^\times)^{>0}$ agrees with the usual real logarithm of s.

In a more general setting, if $F_0 \preceq K$ with $F_0 \subseteq \mathcal{O}$ and $\exp(F_0) = F_0^{>0}$, then $(F_0, \exp|_{F_0})$ is a model of $T_{an}(\exp)$, and the restriction of the residue map $\mathrm{res}(\bullet) : \mathcal{O} \to \mathrm{res}(K)$ to F_0 is an elementary embedding of $(F_0, \exp|_{F_0})$ into $(\mathrm{res}(K), \exp')$. Under these observations, we have the following:

Proposition 3.4. *For each $F_0 \preceq K$ with $F_0 \subseteq \mathcal{O}$ and F_0 log-closed (it means $\log(F_0^{>0}) \subseteq F_0$), there is an extension $F \preceq K$ of F_0 such that $F \subseteq \mathcal{O}$, $\exp(F) = F^{>0}$, and $\mathrm{res}(F) = \mathrm{res}(K)$.*

In order to prove this proposition we will use two valuations for K: the natural valuation, which we denote by w, and the valuation associated to $\mathcal{O}$, which we denote by v. Moreover, in this context, res will be the residual map corresponding to v. Since $\mathcal{O}$ is convex, v is a coarsening of w. Now, assuming $F_0 \preceq K$ with $F_0 \subseteq \mathcal{O}$, and F_0 log-closed, we have to deal with three steps: first we construct an extension F of F_0 in K such that $F \subseteq \mathcal{O}$, F is log-closed and $w(F^\times) = w(F_0^\times)$, then we extend F to be close under exponentiation and finally we extend this extension to include elements $x \in \mathcal{O}$ such that $w(x) \neq 0$.

Lemma 3.5. *Let $x \in \mathcal{O}$ be such that $\operatorname{res}(x) \notin \operatorname{res}(F_0)$ and $w(F_0(x)^\times) = w(F_0^\times)$. If $F := F_0\langle x\rangle$ (inside K), then $F \subseteq \mathcal{O}$ and F is log-closed.*

Proof. By fact (C5), $F \subseteq \mathcal{O}$. Moreover, by [27, Corollary 3.7], we have $w(F^\times) = w(F_0(x)^\times)$. If $a \in F^{>0}$, then by hypothesis there is $b \in F_0$ and $\epsilon \in F$ such that $w(\epsilon) > 0$ and $a = b(1+\epsilon)$. Thus, $\log(a) = \log(b) + \log(1+\epsilon)$. Since F_0 is log-closed and F is closed under restricted analytic functions we have $\log(a) \in F$. So, F is log-closed. □

Iterating Lemma 3.5 we can extend F_0 to a model F of T such that $F \subseteq \mathcal{O}$, $w(F^\times) = w(F_0^\times)$ and for all $x \in \mathcal{O}^\times$ with $\operatorname{res}(x) \notin \operatorname{res}(F_0)$, $w(F(x)^\times) \neq w(F^\times)$. Now, under the latter hypothesis, we extend F to be closed under exp:

Lemma 3.6. *Suppose that $w(F_0(x)^\times) \neq w(F_0^\times)$ for all $x \in \mathcal{O}$ with $\operatorname{res}(x) \notin \operatorname{res}(F_0)$. Let $x \in F_0$ be such that $\operatorname{res}(\exp(x)) \notin \operatorname{res}(F_0)$, and set $F := F_0\langle \exp(x)\rangle$ (inside K). Then $F \subseteq \mathcal{O}$ and F is log-closed.*

Proof. As in the previous lemma, the first affirmation follows from (C5). Now, we can see that $w(\exp(x)) \notin w(F_0^\times)$. Otherwise, $\exp(x) = a(1+\epsilon)$ for some $a \in F_0$ and $\epsilon \in F$ with $w(\epsilon) > 0$. Since F_0 is log-closed, if $b = \log(a) \in F_0$ then $\exp(x) = \exp(b)(1+\epsilon)$. Thus, $\exp(x-b) = 1+\epsilon$ and $x - b \in \mathcal{o}$. However, since F_0 is a model of T, $\exp(x-b) = \mathrm{e}(x-b) \in F_0$ and $\exp(x) \in F_0$, which is a contradiction.

On the other hand, if $w(\exp(x)) = g$, as $g \notin w(F_0^\times)$ and $\exp(x)$ is transcendental over F_0, by [27, Corollary 3.7] $w(F^\times) = w(F_0^\times) \oplus \mathbb{Q}g$. It follows that for each $a \in F^{>0}$ we have $a = b(1+\epsilon)\exp(qx)$ for some $b \in F_0$, $\epsilon \in F$ with $v(\epsilon) > 0$ and $q \in \mathbb{Q}$. Thus,

$$\log(a) = \log(b) + \log(1+\epsilon) + qx.$$

Because F_0 is log-closed and closed under restricted analytic functions, $\log(a) \in F$. So, F is log-closed. □

Now, using Lemmas 3.5 and 3.6, we can construct an extension F of F_0 such that $F \preceq K$, $F \subseteq \mathcal{O}$, F is closed under exponentiation, and for all $x \in \mathcal{O}$ with $\operatorname{res}(x) \notin \operatorname{res}(F)$, we have $w(F\langle x\rangle^\times) \neq w(F^\times)$. In this case, we can have an element $x \in \mathcal{O}$ such that $v(x) = 0$ but $w(x) \neq 0$.

Lemma 3.7. *Assume F_0 is closed under exponentiation and $w(F_0(x)^\times) \neq w(F_0^\times)$ for all $x \in \mathcal{O}$ with $\operatorname{res}(x) \notin \operatorname{res}(F_0)$. Let $x \in \mathcal{O}$ be such that $\operatorname{res}(x) \notin \operatorname{res}(F_0)$. Then there is $F \preceq K$ such that $F_0(x) \subseteq F \subseteq \mathcal{O}$ and F is log-closed.*

Proof. Since F_0 is maximal with value group $w(F_0^\times)$ and $x \in \mathcal{O}$, we can assume that $x > n$ for all n with $w(x) < 0$. Thus, $x > n\log(x)$ for all n and $w(x) < w(\log(x))$. We will built a sequence (x_n) in $\mathcal{O}^{>0}$ such that $w(x_n) < w(x_{n+1}) < 0$, $w(x_n) \notin w(F_0^\times)$ and a chain $(F_n)_{n\geq 0}$ of models of T with $F_{n+1} = F_n\langle x_n\rangle$, $F_n \preceq K$ and $F_n \subseteq \mathcal{O}$, and we set $F = \cup F_n$.

Let $x_0 = x$ and suppose x_n has been defined. Because F_0 is closed under exponentiation, we have $\log(x_n) \notin F_0$. Thus, by hypothesis $w(F_n(\log(x_n))^\times) \not\subseteq w(F_0^\times)$. Moreover, since $F_0 \subseteq F_0\langle\log(x_n)\rangle$ are real closed, by [27, Lemma 3.4] there is $a_n \in F_0$ such that

$$w(\log(x_n) - a_n) \notin w(F_0^\times).$$

Define $x_{n+1} = |\log(x_n) - a_n|$. Clearly $w(x_n) < w(\log(x_n) \leq w(x_{n+1})$. Moreover, $w(x_{n+1}) < 0$. Otherwise, $w(\log(x_n) - a_n) \geq 0$ and $w(\log(x_n)) = w(a_n) \in w(F_0^\times)$ since $w(\log(x_n)) < 0$.

On the other hand, $w(x_0), w(x_1), ...$ are $\mathbb{Q}$-linearly independent over $w(F_0^\times)$. Otherwise, there is a finite subset $x_{i_1}, x_{i_2}, ..., x_{i_n}$ with $i_1 < i_2 < \cdots < i_n$, rational numbers $q_2, ..., q_n$, and an element $b \in F_0$ such that

$$w(x_{i_1}) = \sum_{m=2}^{n} q_m w(x_{i_m}) + w(b).$$

Equivalently, for some $c \in \mathcal{O}^{>0}$ with $w(c) = 0$ we have

$$x_{i_1} = cb \prod_{m=2}^{n} x_{i_m}^{q_m}.$$

Since $x_{i_1+1} = |\log(x_{i_1}) - a_{i_1}|$,

$$w(x_{i_1+1}) = w(\log(cb \prod_{m=2}^{n} x_{i_m}^{q_m}) - a_{i_1}) = w(\log(c) + \log(b) + \sum_{m=2}^{n} q_m \log(x_{i_m}) - a_{i_1}).$$

Moreover, since $w(x_{i_1}) < w(\log(x_{i_m})) < 0$ for $m = 2, ..., n$ and $w(c) = 0$,

$$w(x_{i_1+1}) = w(\log(b) - a_{i_1}) \in w(F_0^\times),$$

which is a contradiction.

Finally, define $F_{n+1} = F_n\langle x_n\rangle$ and $F = \cup F_n$. By construction, $F_n \preceq K$ for all n, thus $F \preceq K$. Also, by fact (C5), we obtain $F \subseteq \mathcal{O}$. Now, since $x_1, x_2, ...$ are transcendental over F_0, and $w(x_1), w(x_2), ...$ are $\mathbb{Q}$-linear independent over $w(F_0^\times)$, for all n we have

$$w(F_{n+1}^\times) = w(F_0^\times) \oplus \mathbb{Q}w(x_0) \oplus \cdots \oplus \mathbb{Q}w(x_n).$$

Thus, if $a \in F^{>0}$, then $a \in F_{n+1}^{>0}$ for some $n \geq 0$, and $a = b(1+\epsilon)\prod_{i=0}^{n} x_i^{q_i}$ for some $a \in F_0$, $\epsilon \in F_{n+1}$ with $w(\epsilon) > 0$ and rational numbers $q_1, ..., q_n$. So,

$$\log(a) = \log(b) + \log(1+\epsilon) + \sum_{i=0}^{n} q_i \log(x_i) \in F_{n+2}.$$

In conclusion, F is log-closed. □

Iterating Lemmas 3.5, 3.6 and 3.7 we obtain the proof of Proposition 3.4.

Now, let $(E, \mathcal{O}_E, \exp_E) \subseteq (K, \mathcal{O}, \exp)$ be models of T_+ such that $\mathrm{res}(E) \neq \mathrm{res}(K)$. By fact (C2) we know that if F_0 and F_1 are maximal elementary substructures of E contained in $\mathcal{O}_E$, then there is a unique isomorphism $\phi : F_0 \to F_1$ such that $\text{res}(\phi(x)) = \text{res}(x)$ for all $x \in F_0$. Thus, taking any maximal elementary substructure F_0 of E such that $F_0 \subseteq \mathcal{O}_E$, $\exp(F_0) = F_0$ and $\text{res}(F_0) = \text{res}(E)$, and following the construction made in Lemmas 3.5, 3.6 and 3.7 we can prove the following:

Proposition 3.8. *Let* $(E, \mathcal{O}_E, \exp_E) \subseteq (K, \mathcal{O}, \exp)$ *be models of* T_+ *with* $\text{res}(E) \neq \text{res}(K)$. *There is a model* $(H, \mathcal{O}_H, \exp_H) \subseteq (K, \mathcal{O}, \exp)$ *of* T_+ *such that:*

(1) $(E, \mathcal{O}_E, \exp_E) \subseteq (H, \mathcal{O}_H, \exp_H) \subseteq (K, \mathcal{O}, \exp)$;

(2) $\text{res}(H) = \text{res}(K)$;

(3) *for any* κ^+*-saturated elementary extension* $(K^*, \mathcal{O}^*, \exp^*)$ *of* $(E, \mathcal{O}_E, \exp_E)$, *with* $\kappa = \text{card}(K)$, *there is an embedding* $(H, \mathcal{O}_H, \exp_H) \to (K^*, \mathcal{O}^*, \exp^*)$ *that extends the natural inclusion from* $(E, \mathcal{O}_E, \exp_E)$ *into* $(K^*, \mathcal{O}^*, \exp^*)$.

Proof. Take an $F_0 \preceq E$ maximal such that $F_0 \subseteq O_E$, $\exp_E(F_0) = F_0^{>0}$ and $\text{res}(F_0) = \text{res}(E)$.

(1) Suppose there is $x \in \mathcal{O}$ such that $\text{res}(x) \notin \text{res}(F_0)$, $w(F_0(x)^\times) = w(F_0^\times)$. Take such an x and set $E' := E\langle x\rangle$, $F := F_0\langle x\rangle$, $\mathcal{O}_{E'} = \mathcal{O} \cap E'$. By Lemma 3.5, we have $F \subseteq \mathcal{O}$ and F log-closed. Moreover, $E \preceq E' \preceq K$, $\mathcal{O}_{E'}$ is a T-convex subring of E' and $\mathcal{O}_E = \mathcal{O}_{E'} \cap E$.

By saturation, there is $y \in \mathcal{O}^*$ such that $z < x$ if and only if $z < y$ for all $z \in E$, and by o-minimality, there is an L_{an}-embedding $\phi : E' \to K^*$ such that $\phi(E) = E$ and $\phi(x) = y$. Thus we have that $\phi(F\langle x\rangle) = \phi(F)\langle y\rangle$ and by fact (C6) there is only one convex subring $\mathcal{O}_{\phi(E)}$ of $\phi(E')$ such that $\text{res}(y)$ realizes the image of the cut that $\text{res}(x)$ realizes over $\text{res}(F)$. Thus, we obtain that $(E', \mathcal{O}_{E'})$ embeds onto $(\phi(E'), \mathcal{O}_{\phi(E')})$.

On the other hand, for $a \in F^{>0}$, with $a = b(1+\epsilon)$, $b \in F_0$ and $\epsilon \in F$ with $w(\epsilon) > 0$, we have $\phi(a) = b(1+\phi(\epsilon))$. Now, by hypothesis $\log_E(b) \in E$, and since ϕ is an L_{an}-embedding, then we have $\log^*(1+\phi(\epsilon)) = \phi(\log_{E'}(1+\epsilon))$. Thus, $\log^*(\phi(a)) = \phi(\log_{E'}(a))$ for all $a \in F^{>0}$.

(2) Suppose that $w(F_0(x)^\times) \neq w(F_0^\times)$ for all $x \in \mathcal{O}$ with $\text{res}(x) \notin \text{res}(F)$ and let $x \in F_0$ be such that $\text{res}(\exp(x)) \notin \text{res}(F_0)$. Then set $E' := E\langle \exp(x)\rangle$, $F := F_0\langle \exp(x)\rangle$, $\mathcal{O}_{E'} = \mathcal{O} \cap E'$. By Lemma 3.6, we have $F \subseteq \mathcal{O}$ and F log-closed. Moreover, $E \preceq E' \preceq K$, $\mathcal{O}_{E'}$ is a T-convex subring of E' and $\mathcal{O}_E = \mathcal{O}_{E'} \cap E$.

Let $\exp^*(x)$ the exponential of x in K^*. Then for all $z \in \mathcal{O}_E$ we have $z < \exp(x)$ if and only if $z < \exp^*(x)$. So there is an L_{an}-embedding $\phi : E' \to K^*$ such that $\phi(\exp_{E'}(x)) = \exp^*(x)$. By the argument given in (1), we obtain an embedding of $(E', \mathcal{O}_{E'})$ onto $(\phi(E'), \mathcal{O}_{\phi(E')})$.

Moreover, from Lemma 3.6, for $a \in F^{>0}$, $a = b(1+\epsilon)\exp(qx)$ with $b \in F_0$, $\epsilon \in F$, $w(\epsilon) > 0$ and $q \in \mathbb{Q}$. So, $\log_E(b) \in F_0$, $\phi(\log_{E'}(1+\epsilon)) = \log_R(1+\phi'(\epsilon))$ and

$$\phi(\log_{E'}(\exp(qx))) = \phi(qx) = \log^*(\exp'(qx)),$$

thus $\log^*(\phi(a)) = \phi(\log_{E'}(a))$.

(3) Assume $\exp(F_0) \subseteq F_0$ and $w(F_0(x)^\times) \neq w(F_0^\times)$ for all $x \in \mathcal{O}$ with $\mathrm{res}(x) \notin \mathrm{res}(F_0)$. Take such an x and as in the proof of Lemma 3.7 define the sequence $(x_n)_n$ in $\mathcal{O}$ and the chain (F_n) of L_{an}-structures. Set $E_0 := E$, $E_{n+1} := E_n\langle x_n\rangle$ and $\mathcal{O}_n = \mathcal{O} \cap F_n$; $E' := \cup E_n$, $F := \cup F_n$ and $\mathcal{O}_{E'} = \cup \mathcal{O}_n$. Clearly, $F \preceq E'$ with $F \subseteq \mathcal{O}_{E'}$ and $\mathcal{O}_{E'}$ is a T-convex subring of E'.

Now, take $y \in K^*$ realizing the same cut that $x_0 = x$ realizes over F_0 and define the sequence (y_n) as $y_0 = y$ and $y_{n+1} = |\log^*(y_n) - \phi(a_n)|$, with (a_n) the sequence defined as in Lemma 3.7. By induction on n, we can easily see that y_n realizes the cut of x_n over F_0. Thus, for each n there is an L_{an}- embedding $\phi_n : E_n \to K^*$ such that $\phi(E) = E$ and $\phi(x_i) = y_i$ for all $i = 0, ..., n-1$. Finally, define $\phi : E' \to K^*$ as $\phi := \cup \phi_n$, then and as in case 1 we obtain that $(E', \mathcal{O}_{E'})$ embeds onto $(\phi(E'), \mathcal{O}_{\phi(E')})$. Moreover, for each element $a \in F^{>0}$ we have that $\log^*(\phi(a)) = \phi(\log_{E'}(a))$.

Iterating cases (1), (2) and (3), we obtain the model $(H, \mathcal{O}_H, \exp)$ of T_+ with the required properties.

□

We will prove now the that T_+ is model complete. To do this we will use the following test (see [4, Corollary B.10.4.]):

Lemma 3.9. *Let Σ be a consistent theory. The following are equivalent:*

(1) *Σ is model complete;*

(2) *for all models M, N of Σ with $M \subseteq N$ and every elementary extension M^* of M that is k-saturated for some $k > \mathrm{card}(N)$, there is an embedding $N \to M^*$ that extends the natural inclusion $M \to M^*$.*

Remark. *Let M, N, M^* be models of Σ where $M \subseteq N$, $M \preceq M^*$ and M^* is k-saturated for some $k > \mathrm{card}(N)$. If we want to show that Σ is model complete, by the last lemma and Zorn's lemma, it is enough to show that there is a substructure K of N that properly contains M, is model of Σ and embeds over M in M^*.*

Following the lines of the proof of quantifier elimination of T_{convex} given in [28, Theorem 3.10] we have:

Theorem 3.10. *The theory T_+ is model complete.*

Proof. Let $(E, \mathcal{O}_E, \exp_E) \subseteq (K, \mathcal{O}, \exp)$ and $(K^*, \mathcal{O}^*, \exp^*)$ be models of T_+ with $(K^*, \mathcal{O}^*, \exp^*)$ a κ^+-saturated elementary extension of $(E, \mathcal{O}_E, \exp_E)$, with $\kappa = \mathrm{card}(K)$. We have the following cases:

(1) If $\mathrm{res}(E) \neq \mathrm{res}(K)$, then by Proposition 3.8, there is $(H, \mathcal{O}_H, \exp_H) \subseteq (K, \mathcal{O}, \exp)$ a model of T_+ which properly extends $(E, \mathcal{O}_E, \exp_E)$, with $\mathrm{res}(H) = \mathrm{res}(E)$, and such that there is an L_+-embedding from $(H, \mathcal{O}_H, \exp_H)$ into $(K^*, \mathcal{O}^*, \exp^*)$ extending the natural inclusion of $(E, \mathcal{O}_E, \exp_E)$ into $(K^*, \mathcal{O}^*, \exp^*)$.

(2) Assume $\mathrm{res}(E) = \mathrm{res}(K)$ and there is $x \in K$ such that $|\mathcal{O}_E| < x < |E \setminus O_E|$. Clearly, $x \notin \mathcal{O}$. Then we define $H := E\langle x\rangle$ and $\mathcal{O}_H = \mathcal{O} \cap H$. By definition, $E \preceq H \preceq K$, $\mathcal{O}_H$ is a T-convex subring of H and $\mathcal{O}_E = \mathcal{O}_H \cap E$.

We claim that $(H, \mathcal{O}_H, \exp_H)$ is a model of T_+. Clearly, $(H, \mathcal{O}_H)$ is a model of T_{convex}. Moreover, since $\mathrm{res}(E) = \mathrm{res}(H) = \mathrm{res}(K)$, for $a \in \mathcal{O}_H$ there are $b \in \mathcal{O}_E$ and $\epsilon \in H$ such that $a = b + \epsilon$. Thus, $\exp_H(a) = \exp_E(b)\exp_H(\epsilon)$. Since $\exp_E(b) \in \mathcal{O}_E$ and H is closed under restricted analytic functions, then $\exp_H(\epsilon) \in H$. So $\exp_H(a) \in H$. On the other hand, if $a \in (\mathcal{O}_H^\times)^{>0}$ then $a = b(1+\epsilon)$ and $\log_H(a) = \log_E(b) + \log_H(1+\epsilon)$. By the same arguments we have $\log_H(a) \in H$.

Now, by saturation there is $y \in K^* \setminus \mathcal{O}^*$ that realizes the same cut that x realizes over E. By o-minimality of T, there is an isomorphism $\phi : H \to E\langle y\rangle$ with $E\langle y\rangle \subseteq K^*$. By fact (C7), we have that ϕ embeds $(H, \mathcal{O}_H)$ in $(K, \mathcal{O}^*)$ and thus ϕ embeds $(H, \mathcal{O}_H, \exp_H)$ into $(K^*, \mathcal{O}^*, exp^*)$.

(3) Assume $\mathrm{res}(E) = \mathrm{res}(K)$ and there is no $x \in K$ such that $|\mathcal{O}_E| < x < |E \setminus O_E|$. Thus, $\mathcal{O}$ is the convex hull of $\mathcal{O}_E$ in K. Otherwise, there would be $x \in \mathcal{O}$ with $x > \mathcal{O}_E$. However, the hypothesis implies that $y < x$ for some $y \in E$ and since $\mathcal{O}_E = \mathcal{O} \cap E$, we obtain $y \in \mathcal{O}_E$. Which is a contradiction. Since K^* is $|K|^+$ saturated and T_{convex} is model complete, there is an embedding $\phi : (K, \mathcal{O}) \to (K^*, \mathcal{O}^*)$ over $(E, \mathcal{O}_E)$, such that $\mathcal{O}^* \cap \phi(K)$ is the convex hull of $\mathcal{O}_E$ in $\phi(K)$, so $\phi(\mathcal{O}) = O^* \cap \phi(K)$ and thus ϕ embeds $(K, \mathcal{O}, \exp)$ into $(K^*, \mathcal{O}^*, \exp^*)$.

□

As we have seen, for each model $(K, \mathcal{O}, \exp)$ of T_+, $\mathbb{R}_{an} \subseteq \mathcal{O}$. Moreover, for all $x, y \in K^{>0} \setminus \mathcal{O}$ we have that $x > \mathbb{R}_{an}$ if and only if $y > \mathbb{R}_{an}$. By o-minimality of T, there is an L_{an}-isomorphism $\phi : \mathbb{R}\langle x\rangle \to \mathbb{R}\langle y\rangle$. Now, we define $\mathcal{O}_x$ and $\mathcal{O}_y$ as the convex hull of $\mathbb{R}_{an}$ in $\mathbb{R}\langle x\rangle$ and $\mathbb{R}\langle y\rangle$, respectively. Clearly, $\mathcal{O}_x$ is a proper T-convex subring of $\mathbb{R}\langle x\rangle$ and $\mathcal{O}_y$ is a proper T-convex subring of $\mathbb{R}\langle x\rangle$. But by fact (C7), we obtain that $\phi(\mathcal{O}_x) = \mathcal{O}_y$. Finally, since $\mathbb{R}_{an}$ is closed under exponentiation, logarithm, and restricted analytic functions, we obtain that $(\mathbb{R}\langle x\rangle, \mathcal{O}_x, \exp) \cong (\mathbb{R}\langle y\rangle, \mathcal{O}_y, \exp)$ as L_+-structures. In conclusion, setting

$$\wp := (\mathbb{R}\langle x\rangle, \mathcal{O}_x, \exp),$$

we have that $\wp$ is a prime model of T_+. Thus we have:

Corollary 3.11. *T_+ is complete.*

3.2.2 Quantifier elimination of T_+ and further results

Now, if we expand the language L_+ with the unary function symbol log and $T_{+,\log}$ is the extension of T_+ given by the following defining axiom

$$(x \in (\mathcal{O}^\times)^{>0} \to \exp(\log(x)) = x) \wedge (x \notin (\mathcal{O}^\times)^{>0} \to \log(x) = 0),$$

then each $L_{+,\log}$-substructure of a model of $T_{+,\log}$ has $T_{+,\log}$-closure in the following sense:

Lemma 3.12. *Let* $(K, \mathcal{O}, \exp, \log)$ *be a model of* $T_{+,\log}$ *and* $(E, \mathcal{O}_E, \exp_E, \log_E)$ *be a* $L_{+,\log}$*-substructure of* $(K, \mathcal{O}, \exp, \log)$. *There is a model* $(F, \mathcal{O}_F, \exp_F, \log_F)$ *of* $T_{+,\log}$ *such that*

(1) $(E, \mathcal{O}_E, \exp_E, \log_E) \subseteq (F, \mathcal{O}_F, \exp_F, \log_F)$, *and*

(2) $(F, \mathcal{O}_F, \exp_F, \log_F)$ *can be embedded over* $(E, \mathcal{O}_E, \exp_E, \log_E)$ *into every model of* $T_{+,\log}$ *which extends* $(E, \mathcal{O}_E, \exp_E, \log_E)$

Proof. If $E \neq \mathcal{O}_E$, then in fact $(E, \mathcal{O}_E, \exp_E, \log_E)$ is a model of $T_{+,\log}$ and we finish. Otherwise, there is $x \in K^{>0} \setminus \mathcal{O}$ such that $|E| < x$. So we define F as the L_+-substructure $E\langle x\rangle$ of K, $\mathcal{O}_F$ the unique convex subring of F such that $\mathcal{O}_E = \mathcal{O}_F \cap E$ with $x \notin \mathcal{O}_F$, given by fact $(C7)$, and $\exp_F = \exp|_F$. From this definition we have $w(F\langle x\rangle^\times) = w(E^\times) \oplus \mathbb{Q}w(x)$. Moreover, for each $a \in O_F^\times$, there are $b \in E$, $\epsilon \in F$ with $w(\epsilon) > 0$ and $q \in \mathbb{Q}$ such that $a = b(1+\epsilon)x^q$. Since $\log_E(b) \in E$, $\log_F(x) = 0$ and F is closed under restricted analytic functions we obtain that $\log_F(a) = \log_E(b) + \log_F(1+\epsilon) \in F$. Thus, $(F, \mathcal{O}_F, \exp_F, \log_F)$ is a model of $T_{+,\log}$, with $\log_F = \log|_F$.

Finally, given any model $(K^*, \mathcal{O}^*, \exp^*, \log^*)$ of $T_{+,\log}$ which extends $(E, \mathcal{O}_E, \exp_E, \log_E)$, there is $y \in (K^*)^{>0} \setminus \mathcal{O}^*$ such that $|\mathcal{O}_E| < y$. Clearly, x and y realize the same cut over E and defining the $L_{+,\log}$-substructure $(F', \mathcal{O}_{F'}, \exp_{F'}, \log_{F'})$ of $(K^*, \mathcal{O}^*, \exp^*, \log^*)$ using y as we use x to define $(F, \mathcal{O}_F, \exp_F, \log_F)$, then there is an embedding

$$\phi : (F, \mathcal{O}_F, \exp_F, \log_F) \to (K^*, \mathcal{O}^*, \exp^*, \log^*)$$

over $(E, \mathcal{O}_E, \exp_E, \log_E)$ such that $\phi(x) = y$. □

As a consequence of this lemma and mimicking the proof of Theorem 3.10 we have:

Proposition 3.13. *The theory* $T_{+,\log}$ *has quantifier elimination.*

If $L_{+,\log,c}$ is the language $L_{+,\log}$ expanded by a new constant symbol c and $T_{+,\log,c}$ is the theory $T_{+,\log} \cup \{c > 0 \wedge c \notin \mathcal{O}\}$, then $T_{+,\log,c}$ has quantifier elimination and has a universal axiomatization. Thus we obtain that:

Lemma 3.14. *Let* $(K, \mathcal{O}, \exp, \log, c)$ *be a model of* $T_{+,\log,c}$ *and let* $f : K^n \to K$ *be definable in* $L_{+,\log,c}(K)$. *Then there are finitely many functions* $f_1, .., f_m : K^n \to K$ *definable in* $L_{an*,c}(\exp, \log)(K)$ *such that for each* $a \in K^n$ *we have that* $f(a) = f_i(a)$ *for some* i *in* $\{1, ..., m\}$.

Proof. Since $T_{+,\log,c}$ has quantifier elimination and has a universal axiomatization, there are $L_{+,\log,c}(K)$-terms $t_1(\overline{x}), ..., t_m(\overline{x})$ such that for each $a \in K^n$ we have $f(a) = t_i(a)$ for some $i \in \{1, ..., m\}$. However, $t_1(\overline{x}), ..., t_m(\overline{x})$ are in fact $L_{an*,c}(\exp, \log)(K)$-terms. Thus, we define $f_i : K^n \to K$ as $f_i(a) = t_i(a)$ for each $i \in \{1, ..., m\}$. □

Weak o-minimality

We recall that a theory in which an order is given or defined is called *weakly o-minimal* if in every model of this theory, each definable subset is a finite union of convex subsets.

As a consequence of the o-minimality of T_{an}, the convexity of the subring $\mathcal{O}$ and the properties of exp and log we have the following result:

Proposition 3.15. *The theory $T_{+,\log}$ is weakly o-minimal.*

Proof. Let $(K, \mathcal{O}, \exp, \log)$ be a model of $T_{+,\log}$. Since $T_{+,\log}$ has quantifier elimination, it is enough to show that if $\phi(x)$ is an atomic $L_{+,\log}$-formula with parameters in K, then

$$\{x \in K : (K, \mathcal{O}, \exp, \log) \models \phi(x)\}$$

is a union of finitely many convex subsets of K. Now, $\phi(x)$ is of the form $f(x) = 0$, $f(x) > 0$ or $f(x) \in \mathcal{O}$ where $f : K \to K$ is a $L_{+,\log}$-definible function with parameters in K. The desired result is obtained by induction on the complexity of the term $f(x)$ and as a consequence of the following observations:

- If $A \subseteq K$ is convex, then $\exp(A)$ and $\log(A)$ are each one the union of finitely many convex subsets of $\mathcal{O}$.
- If $g, h : K \to K$ are L_{an*}-definable functions with parameters in K, then:
  - $\{x \in K : \exp(g(x)) = 0\}$, $\{x \in K : \exp(g(x)) > 0\}$ and $\{x \in K : \exp(g(x)) \in \mathcal{O}\}$ are union of finitely many convex subsets of K. This result is a consequence of the o-minimality of T_{an*} and the following equivalences:

    $$\exp(g(x)) = 0 \Leftrightarrow g(x) \in K \setminus \mathcal{O},$$

    $$\exp(g(x) > 0 \Leftrightarrow g(x) \in \mathcal{O},$$

    $$\exp(g(x)) \in \mathcal{O} \Leftrightarrow g(x) \in K.$$
  - $\{x \in K : g(\exp(x)) = 0\}$, $\{x \in K : g(\exp(x)) > 0\}$ and $\{x \in K : g(\exp(x)) \in \mathcal{O}\}$ are each one the union of finitely many convex subsets of K. This is a consequence of the fact that $\exp(K) = \{0\} \cup (\mathcal{O}^\times)^{>0}$ is an $L_{an*,\mathcal{O}}$-definable subset of K and the o-minimality of T_{an*}.
  - $\{x \in K : h(\exp(g(x))) = 0\}$, $\{x \in K : h(\exp(g(x))) > 0\}$ and

    $$\{x \in K : h(\exp(g(x))) \in \mathcal{O}\}$$

    are each one the union of finitely many convex subsets of K. As a consequence, the set $\{\exp(g(x) : x \in K\}$ is a union of finitely many convex subsets of K and h is an L_{an*}-function.
- We have similar results to those obtained in the previous item, taking log in place of exp and considering $L_{an*,\exp}$-definable functions $g, h : K \to K$.

□

Now, from [22, Appendix A.1.3] we have:

Corollary 3.16. *The theory $T_{+,\log}$ has NIP.*

Definable Skolem functions

Let L be a first order language with at least one constant symbol. An L-theory T admits *definable Skolem functions* if for each L-formula $\phi(x, y)$ there is a definable function $f(y)$ such that for any model M of T and $a \in M$ if $H := \{x \in M : M \models \phi(x, a)\}$ is non-empty, then $f(a) \in H$.

Examples.

(1) *The theory of real closed fields in the language of ordered rings admits definable Skolem functions.*

(2) *The theory $T_{convex,c}$ defined as T_{convex} plus the axiom $c > 0 \wedge c \notin \mathcal{O}$, in the language $L_{convex,c}$ with c a new constant symbol, has quantifier elimination, has universal axiomatization and admits definable Skolem functions (see [29, Remark 2.7]).*

From [21], we have the following theorem:

Theorem 3.17. *Let T be a model-complete L-theory and A a set of prenex $\prod_2$-sentences which axiomatizes T. If for every sentence $\forall\overline{x}\exists\overline{y}\phi(\overline{x}, \overline{y})$ in A there is a definable function f for which $T \vdash \forall\overline{x}\phi(\overline{x}, f(\overline{x}))$, then T admits definable Skolem functions.*

Let $L_{+,c}$ be the language L_+ expanded by a new constant symbol c and let $T_{+,c}$ be the theory $T_{convex,c}$ plus the axioms $E1$ to E_4. We observe that $T_{+,c}$ is model complete and is axiomatized by a set of prenex $\prod_2$-sentences. Since $T_{convex,c}$ admits definable Skolem functions, and by axiom $E4$, equivalent to

$$\forall y\exists x(y \in (\mathcal{O}^\times)^{>0} \to x \in \mathcal{O} \wedge \exp(x) = y),$$

we can define the function log such that

$$(x \in (\mathcal{O}^\times)^{>0} \to \exp(\log(x)) = x) \wedge (x \notin (\mathcal{O}^\times)^{>0} \to \log(x) = 0),$$

then by Theorem 3.17 we can conclude:

Lemma 3.18. *The theory $T_{+,c}$ admits definable Skolem functions.*

4. The theory of $\mathbb{R}((t^\Gamma))_{an}$ as valued field with partial exponentiation

In the previous chapter we observed that for each Hahn field $K = \mathbb{R}[[\mathfrak{M}]]$ over $\mathbb{R}$ with monomials in an ordered abelian group $\mathfrak{M}$, if $\mathcal{O}$ is the valuation ring of K under the natural valuation and $\scriptstyle\mathcal{O}$ its maximal ideal, we can define a partial exponential $\exp : \mathcal{O} \to (\mathcal{O})^{>0}$ given by

$$\exp(r + \epsilon) = e^r \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!},$$

for $r \in \mathbb{R}$ and $\epsilon \in \scriptstyle\mathcal{O}$. Thus, if $\mathfrak{M}$ is divisible, $(K, \mathcal{O}, \exp)$ can be made a model of T_+. In this chapter we study the theory of $(K, \mathcal{O}, \exp)$ without the assumption of the divisibility of the value group. Specifically, we work with valued fields as three sorted structures and prove an Ax-Kochen-Ershov theorem for this kind of fields along the lines of similar theorem for henselian fields (see for example [32]).

4.1 Technical results about valued fields

In the first part of this section we summarize some general lemmas about extensions of valued fields. After, we introduce some results about liftings and cross-sections. For a proof of results quoted here see, for example, [32] or [4].

Throughout this section K will be a valued field with valuation ring $\mathcal{O}$, residue field $\mathrm{res}(K)$, value group Γ and valuation v.

Lemma 4.1. *Let E be a valued subfield of K, $b_1, ...b_m \in \mathcal{O}$ be such that $\mathrm{res}(b_1), ..., \mathrm{res}(b_m)$ in $\mathrm{res}(K)$ are linearly independent over $\mathrm{res}(E)$, for $m > 0$, and $c_1, ..., c_n \in E^\times$ be such that $v(c_1), ...v(c_n) \in \Gamma$ lie in distinct cosets of Γ_E, for $n > 0$. Then*

$$v\left(\sum_{i,j} a_{ij} b_i c_j\right) = \min_{i,j} v(a_{ij} c_j)$$

with $a_{ij} \in E$. In particular, the family $(b_i c_j)$ is E-linearly independent.

Given a field extension $E \subseteq F$, $[F : E]$ denotes de dimension of F as vector space over E, with the convention that $[F : E] = \infty$ if this dimension is infinite, and $\mathrm{trdeg}(E|F)$ the transcendence degree of this extension, which we set ∞ if $\mathrm{trdeg}(F|E)$ is not finite. If $\Gamma \subseteq \Gamma'$ is an extension of abelian groups, $[\Gamma' : \Gamma]$ denotes its index, which by convention is ∞ if Γ'/Γ is infinite, and $\mathrm{rank}_{\mathbb{Q}}(\Gamma'/\Gamma)$ denotes the rational rank of Γ'/Γ, i.e $\mathrm{rank}_{\mathbb{Q}}(\Gamma'/\Gamma) = \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} (\Gamma'/\Gamma)$.

As a direct consequence of Proposition 4.1 we have:

Corollary 4.2. *Let E be valued subfield of K. then*

$$[K : E] \geq [\mathrm{res}(K) : \mathrm{res}(E)] \cdot [\Gamma : \Gamma_E]$$

and

$$\mathrm{trdeg}(K|E) \geq \mathrm{trdeg}(\mathrm{res}(K)|\,\mathrm{res}(E)) + \mathrm{rank}_{\mathbb{Q}}(\Gamma/\Gamma_E).$$

The following lemmas allow us to extend valued fields to include algebraic or transcendental elements which lie in an extension and show how to change the value group and residual field:

Lemma 4.3. *If β lies in an algebraic closure of $\mathrm{res}(K)$, $P(x) \in \mathcal{O}[x]$ is a monic polynomial such that $\mathrm{res}(P) \in \mathrm{res}(K)[x]$ is the minimum polynomial of β over $\mathrm{res}(K)$, and b is a zero of $P(x)$ in an algebraic closure of K, then the valuation v corresponding to $\mathcal{O}$ extends uniquely to a valuation $v' : K(b)^{\times} \to \Gamma$ on $K(b)$. Moreover, $\mathrm{res}(K(b)) \supseteq \mathrm{res}(K)$ is isomorphic over $\mathrm{res}(K)$ to $\mathrm{res}(K)(\beta)$, and $[K(b) : K] = [\mathrm{res}(K(b)) : \mathrm{res}(K)]$ (which implies that $\Gamma_{K(b)} = \Gamma$).*

In similar way we have the following two results on extensions by a transcendental element:

Lemma 4.4. *Let $L = K(x)$ be a field extension with x transcendental over K. There is a unique valuation ring $\mathcal{O}_L$ of L such that $\mathcal{O} = \mathcal{O}_L \cap K$, $x \in \mathcal{O}_L$, and $\mathrm{res}(x)$ is transcendental over $\mathrm{res}(K)$. Moreover, $\mathrm{res}(L) = \mathrm{res}(K)(\mathrm{res}(x))$ and $\Gamma_K = \Gamma_L$.*

Lemma 4.5. *Let $L = K(x)$ be a field extension with x transcendental over K. Let γ, in some ordered abelian group extending Γ satisfy $n\gamma \notin \Gamma$ for all $n > 0$. Then the valuation v corresponding to $\mathcal{O}$ extends uniquely to a valuation $w : L^{\times} \to \Gamma + \mathbb{Z}\gamma$ such that $w(x) = \gamma$. Moreover $\mathrm{res}(L) = \mathrm{res}(K)$.*

The next is straightforward but an important result that shows how the value group can change in an extension:

Lemma 4.6. *Let p be a prime number, and x be in a field extension of K such that $x^p = a \in K^{\times}$ but $v(a) \notin p\Gamma$. Then $X^p - a$ is the minimum polynomial of x over K, and v extends uniquely to a valuation $w : K(x)^{\times} \to \Delta$ with $\Delta \subseteq \mathbb{Q}\Gamma$ (as ordered groups). Moreover, $\mathrm{res}(K(x)) = \mathrm{res}(K)$ and $[\Delta : \Gamma] = p$, where*

$$\Delta = \bigcup_{i=0}^{p-1} (\Gamma + iw(x)) \text{ (disjoint union).}$$

Finally, the following lemma gives us information about the behaviour of extensions in the case that we have a henselian valued field of equicharacteristic 0:

Lemma 4.7. *Let E a valued subfield of K. If E is algebraically maximal and $x \in K \setminus E$, then one of the following holds:*

(1) *x is a pseudolimit of a pc-sequence in E of transcendental type over E;*

(2) *there is $a \in E$ such that $v(x - a) \notin \Gamma_E$;*

(3) *there exist $a, b \in E$, $b \neq 0$, such that $v(x - a) = v(b)$ and the residual class of $(x - a)/b$ does not lie in $\mathrm{res}(E)$.*

Valued fields as three-sorted structures

Throughout the rest of this chapter we work with valued fields as three-sorted structures of the form

$$\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v)$$

where K and $\mathbf{k}$ are fields, Γ is an ordered abelian group, $v : K^\times \to \Gamma$ is a valuation with corresponding valuation ring $\mathcal{O}$, and $\pi : \mathcal{O} \to \mathbf{k}$ is a surjective morphism of rings.

If $\mathcal{o}$ is the maximal ideal of $\mathcal{O}$, we observe that π has kernel $\mathcal{o}$ and there is an isomorphism $\phi : \mathbf{k} \to \mathrm{res}(K)$, with $\mathrm{res}(K)$ the residual field of K under v, such that $\phi(\pi(a)) = \mathrm{res}(a)$ for all $a \in \mathcal{O}$. Thus we may identify $\mathbf{k}$ with $\mathrm{res}(K)$.

By an ordered valued field we mean a three-sorted structure $\mathbf{K}$ as above such that K and $\mathbf{k}$ are ordered fields, v is a convex valuation and π is a surjective morphism of ordered rings.

Lifting of the residue field and cross-section

Let $\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v)$ be a valued field. A *lifting* of the residue field $\mathbf{k}$ of $\mathbf{K}$ is a field embedding $\mathrm{lif} : \mathbf{k} \to K$ such that $\pi(\mathrm{lif}(a)) = a$ for all $a \in \mathbf{k}$. A *cross-section* of the valuation v of $\mathbf{K}$ is a group morphism $\mathrm{s} : \Gamma \to K^\times$ such that $v(\mathrm{s}(\gamma)) = \gamma$ for all $\gamma \in \Gamma$.

Example. *Let Γ be an ordered abelian group. Interpreting the Hahn field $k((t^\Gamma))$ as the valued field*

$$(k((t^\Gamma)), k, \Gamma; \pi, v)$$

in the natural way, we can define the lifting $\mathrm{lif} : k \to K$ as the identity map and a cross section $s : \Gamma \to K^\times$ as $\mathrm{s}(\gamma) = t^\gamma$. We call these maps the natural lifting and the natural cross-section of the Hahn field $k((t^\Gamma))$.

From now on we say that $\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s})$ is an *lc-valued field* if $(K, \mathbf{k}, \Gamma; \pi, v)$ is a valued field, lif is a lifting of $\mathbf{k}$ and s is a cross section of v.

As is usual if $\mathbf{E} = (E, \mathbf{k}_E, \Gamma_E; \pi_E, v_E, \mathrm{lif}_E, \mathrm{s}_E)$ and $\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s})$ are lc-valued fields, we say that $\mathbf{K}$ is an extension of $\mathbf{E}$, denoted by $\mathbf{E} \subseteq \mathbf{K}$ if E is a valued subfield of K, $\mathbf{k}_E \subseteq \mathbf{k}$, $\Gamma_E \subseteq \Gamma$, $\mathrm{lif}_E = \mathrm{lif}\,|_E$ and $\mathrm{s}_E = \mathrm{s}\,|_E$.

An important observation about henselian fields is the following:

Lemma 4.8. *Let $\boldsymbol{K} = (K, \boldsymbol{k}, \Gamma; \pi, v)$ be a henselian valued field of equicharacteristic 0. Then*

(1) *There is a lifting of the residue field $\boldsymbol{k}$ of $\boldsymbol{K}$.*

(2) *$\boldsymbol{K}$ has an elementary extension with a cross section.*

In the particular case where both K and $\mathbf{k}$ are ordered fields, by a lifting of the residue field $\mathbf{k}$ into K we mean an ordered field embedding $\mathrm{lif} : \mathbf{k} \to K$.

4.2 $L_{an}(^{-1})$-structures

Let $T_{an,\forall}$ be the set of all universal consequences of the $L_{an}(^{-1})$-theory T_{an}. Since T_{an} has quantifier elimination in the language $L_{an}(^{-1})$, T_{an} is the model companion of $T_{an,\forall}$. By [27], we also know that if K is a model of $T_{an,\forall}$, then K is an ordered field and a henselian valued field under the natural valuation of K with residual field $\mathbb{R}$. Moreover, if $F \subseteq K$ is a $L_{an}(^{-1})$-substructure of K, then F is a model of $T_{an,\forall}$ and for each ordered subfield E of K its definable closure in K, which we denote in this chapter by $E_{an,\forall,K}$, is a model of $T_{an,\forall}$.

Example. *Let Γ be an ordered abelian group. By [27] we know that each Hahn field $\mathbb{R}((t^\Gamma))$ can be expanded to a model of $T_{an,\forall}$, usually denoted by $\mathbb{R}((t^\Gamma))_{an}$.*

Let $\alpha \in \mathbb{R}$ and $A_{n,\alpha} = \mathbb{R}\{X_1, X_2, ..., X_n\}$ be the ring of real power series in $X_1, ..., X_n$ with radius of convergence $\alpha \geq 1$ and $A = \{A_{n,\alpha} : n \in \mathbb{N}, \alpha \geq 1\}$. From [7, Section 3] we know that A is a real Weierstrass system that satisfies the following property:

Lemma 4.9 (Weierstrass division). *Let $f, g \in A_{n,\alpha}$. Suppose that f is regular in X_n of degree s at 0. Then there are uniquely determined elements $q \in A_{n,\alpha}$ and $r \in A_{n-1,\alpha}[X_n]$ of degree at most $s-1$ such that $g = qf + r$.*

Since L_{an} is the language of ordered rings together with a symbol function for each restricted analytic function in $\cup A$, any model K of $T_{an,\forall}$ has a real analytic A-structure in the sense of [7, Definition 3.2.3]. From this observation and the above lemma we have the following result:

Lemma 4.10. *Let K be a model of $T_{an,\forall}$. Then the real closure of K (K^{rc}) is a model of T_{an}.*

Proof. We just have to show that K^{rc} is a model of $T_{an,\forall}$. In this proof we consider K as a valued field with its natural valuation v. Thus the corresponding valuation ring $\mathcal{O}$ of K is the convex hull of $\mathbb{R}$ in K. Now, let $\alpha \in K^{rc} \setminus K$. Without loss of generality we can assume that $\alpha \in \mathcal{O}_{K^{rc}}$. Let $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ be the minimal polynomial of α over K. Since K is henselian, $a_1, ..., a_n \in \mathcal{O}$. Let $r_1, ..., r_n \in \mathbb{R}$ and $\epsilon_1, ..., \epsilon_n \in \mathcal{o}$ such that $a_i = r_i + \epsilon_i$ for $i : 1, ..., n$ and set

$$P_1(X_1, ..., X_n, X) = X^n + (r_1 + X_1)X^{n-1} + \cdots + (r_n + X_n) \in A_{n+1,\beta}$$

for some $\beta > 1$. Since P' is regular in X of degree n at 0, from Weierstrass division property, for any $g \in A_{n+1,\beta}$ there are uniquely determined elements $q \in A_{n+1,\beta}$ and $r \in A_{n,\beta}[X]$ of degree at most $n-1$ such that $g = qP_1 + r$. If g', q', f', P_1' and r' are the functions symbol in L_{an} corresponding to g, q, P_1 and r, respectively, then we have

$$g'(\epsilon_1, ..., \epsilon_n, \alpha) = r'(\epsilon_1, ..., \epsilon_n)$$

where $r'(\epsilon_1, ..., \epsilon_n)$ is a polynomial in α whose coefficients correspond to the interpretations in K of analytic functions applied to $\epsilon_1, ..., \epsilon_n$. Thus, $g'(\epsilon_1, ..., \epsilon_n, \alpha) \in K^{rc}$. Since any element of $A_{n,\beta}$ can be seen as an element of $A_{m,\beta}$ for $m > n$, we can conclude that K^{rc} is a model of T_{an}.

□

Remark 1. *Since T_{an} is o-minimal, if K, M_1, M_2 are models of $T_{an,\forall}$, $a_1 \in M_1 \setminus K$ and $a_2 \in M_2 \setminus K$, we have the following:*

(1) *If a_1 and a_2 realize the same cut over K and $P(a_1) = 0$, $P(a_2) = 0$ for $P(x) \in \mathcal{O}[x]$ (under the natural valuation), then there is an L_{an}-isomorphism from $K(a_1)_{an,\forall,M_1}$ onto $K(a_2)_{an,\forall,M_2}$ over K sending a_1 to a_2.*

(2) *If $a_1 \in \mathcal{O}_{M_1}$ and $a_2 \in \mathcal{O}_{M_2}$ are transcendental over K and for all polynomial $P(x) \in \mathcal{O}[X]$ we have $P(a_1) > 0$ if and only if $P(a_2) > 0$, then a_1 and a_2 realize the same cut over K^{rc}. Thus, there is an L_{an}-isomorphism from $K(a_1)_{an,\forall,M_1}$ onto $K(a_2)_{an,\forall,M_2}$ over K sending a_1 to a_2.*

4.2.1 The residue field and value group of the $L_{an}(^{-1})$-closure of some simple extensions

Let $\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v)$ be a valued field with K a model of $T_{an,\forall}$ and v a convex valuation on K. By definition, $\mathbf{k}$ can be made a model of $T_{an,\forall}$. Moreover, since v is convex, v is a coarsening of the natural valuation on K, which we denote by w.

If $w(K^\times) = \Gamma_w$, there is a convex subgroup Δ of Γ_w such that $\Gamma = \Gamma_w/\Delta$ and we have $v(a) = w(a) + \Delta$ for each $a \in K^\times$. From this we can define in a natural way a valuation $w' : \mathbf{k}^\times \to \Delta$ as $w'(r) = w(a)$ with $\pi(a) = r$. Thus, w' agrees with the natural valuation on $\mathbf{k}$ and its residual field is (isomorphic to) $\mathbb{R}$. Since (K, w) is henselian, $\mathbf{K}$ is henselian and $(\mathbf{k}, w')$ is henselian. Therefore, Δ is divisible if and only if $\mathbf{k}$ is real closed, and Δ is divisible if and only if $\mathbf{k}$ is a model of T_{an}.

Following the proof of Lemma 4.10 we observe that if $E \subseteq K$ is a model of $T_{an,\forall}$, then for each $a \in K \setminus E$ which is algebraic over E we have that $\mathrm{res}(E(a)_{an,\forall,K}) = \mathrm{res}(E(a))$, whenever $\mathrm{res}(E(a)) = \mathrm{res}(E)$. But, in fact we have a more general result:

Lemma 4.11. *For all $a \in K \setminus E$, if $\mathrm{res}(E(a)) = \mathrm{res}(E)$ then $\mathrm{res}(E(a)_{an,\forall,K}) = \mathrm{res}(E)$.*

Proof. Since $\mathrm{res}(E(a)) = \mathrm{res}(E)$, each element $x \in \mathcal{O}^n_{E(a)}$ is of the form $x = r + \epsilon$ for some $r \in (\mathcal{O}_E)^n$ and $\epsilon \in \mathcal{o}^n_{E(a)}$. Thus, to show that $\mathrm{res}(E(a)_{an,\forall,K}) = \mathrm{res}(E(a))$, it is enough to prove that if $x \in \mathcal{O}^n_{E(a)}$ with $x \in [-1,1]^n$, then for each analytic function $f \in A_{n,\alpha}$ we have $\mathrm{res}(f(x)) = \mathrm{res}(f(r))$ for some $r \in \mathcal{O}^\times_E$ with $\mathrm{res}(x) = \mathrm{res}(r)$.

Let L be a model of T_{an} such that $E(a) \subseteq L$, $f \in A_{n,\alpha}$ be an analytic function and $x \in \mathcal{O}^n_{E(a)}$. By o-minimality of L, we can partition $(-1,1)^n \cap L$ into finitely many cells such that the restriction of f to each cell of the partition is continuous. Thus, there is $y \in (\mathcal{O}_L)^n \cap (\mathcal{O}_E)^n$ such that f is continuous at y with $\mathrm{res}(x) = \mathrm{res}(y)$, and by Proposition 2.20 of [28] we obtain $f(x) - f(y) \in \mathcal{o}_L$. It follows that $f(x) - f(y) \in \mathcal{o}_{E(a)_{an,\forall,K}}$ and then $\mathrm{res}(f(x)) = \mathrm{res}(f(y)) \in \mathrm{res}(E)$. □

In relation with the value group of the $L_{an}(^{-1})$-closure of some simple extensions we have similar results for the residue field, but we need valued fields with cross section and lifting. Before stating the main result we notice the following:

Lemma 4.12. *Let $\boldsymbol{K} = (K, \boldsymbol{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s})$ be a lc-valued field with K a model of $T_{an,\forall}$, v a convex valuation on K and $\boldsymbol{k}$ a model of T_{an}. If $w(K^{\times}) = \Gamma_w$ with w the natural valuation on K, there is a cross section $\mathrm{s}' : \Gamma_w \to K^{\times}$.*

Proof. Let Δ be the divisible convex subgroup of Γ_w such that $v(a) = w(a) + \Delta$ for $a \in K^{\times}$. From the divisibility of Δ there is a $\mathbb{Z}$-submodule H of the $\mathbb{Z}$-module Γ_w such that $\Delta \oplus H \cong \Gamma_w$.

Since Δ is divisible and $\mathbf{k}$ is a model of T_{an}, by observation 3.2 of [27] there is a cross section $\mathrm{s}^* : \Delta \to \mathbf{k}^{\times}$. Thus, we define the cross section $\mathrm{s}' : \Gamma_w \to K^{\times}$ as $\mathrm{s}'(\alpha) = \mathrm{lif}(\mathrm{s}^*(\alpha))$ if $\alpha \in \Delta$ and $\mathrm{s}'(\alpha) = \mathrm{s}(\alpha + \Delta)$ if $\alpha \in H$. □

From this lemma and following the ideas of section 3 in [27], we will prove the next result:

Proposition 4.13. *Let $\boldsymbol{E} = (E, \boldsymbol{k}_E, \Gamma_E; \pi_E, v_E, \mathrm{lif}_E, s_E) \subseteq \boldsymbol{K} = (K, \boldsymbol{k}, \Gamma; \pi, v, \mathrm{lif}, s)$ be lc-valued fields with E and K models of $T_{an,\forall}$, $\boldsymbol{k}_E \subseteq \boldsymbol{k}$ models of T_{an} and v a convex valuations on K. If $a \in K \setminus E$, then for $v(E(a)^{\times}_{an,\forall,K})$ we have the following possibilities:*

(1) *If $v(E(a)^{\times}) = \Gamma_E$, then $v(E(a)^{\times}_{an,\forall,K}) = \Gamma_E$.*

(2) *If $a = \mathrm{s}(g)$ for some $g \in \Gamma \setminus \Gamma_E$, then*

$$v(E(a)^{\times}_{an,\forall,K}) = v(E(a)^{\times}).$$

The proof follows from the following observations and corollary 4.18 proved below.

Taking the natural valuation w on K, we can see that if $v(E(a)^{\times}_{an,\forall,K}) \neq v(E(a)^{\times})$ for some $a \in K \setminus E$, then $w(E(a)^{\times}_{an,\forall,K}) \neq w(E(a)^{\times})$. Thus, in order to prove Proposition 4.13 its enough to work with the fields $E \subseteq K$ as valued fields with the natural valuation w. Specifically we will analyse $w(E(a)^{\times}_{an,\forall,K})$ from $w(E(a)^{\times})$ for some $a \in K \setminus E$.

Now, if $w(E^{\times}) = \Gamma'_E$ and $w(K^{\times}) = \Gamma'$, by Lemma 4.12 there is a cross-section $\mathrm{s}' : \Gamma' \to K^{\times}$ such that $\mathrm{s}'|_{\Gamma'_E} : \Gamma'_E \to E^{\times}$ is also a cross section. Moreover the residual field of both fields under the natural valuation w is $\mathbb{R}$ because $E \subseteq K$ are models of $T_{an,\forall}$. Thus, the main idea is to examine embeddings of valued fields such as E into real Hahn series.

Lemma 4.14. *Suppose there is an L_{an}-embedding ϕ of E into $\mathbb{R}((t^{\Gamma'_E}))$ such that $\phi(\mathrm{s}'(g)) = t^g$ for all $g \in \Gamma'_E$. If $a \in K \setminus E$ with $w(E(a)^{\times}) = \Gamma'_E$, then we can extend ϕ to an L_{an}-embedding from $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma'_E}))$.*

Proof. Identifying E with its image in $\mathbb{R}((t^{\Gamma'_E}))$ under ϕ, we can see E as an L_{an}-substructure of $\mathbb{R}((t^{\Gamma'_E}))$. Moreover, since E is henselian, E is algebraically maximal, so by hypothesis $E(a)$ is a transcendental extension of E. Thus, there is a pc-sequence $(x_\alpha)_{\alpha<\lambda}$ in E of transcendental type in E whit pseudo-limit a in K. So by Remark 1 it is enough to find a pseudo limit b of $(x_\alpha)_{\alpha<\lambda}$ in $\mathbb{R}((t^{\Gamma'_E}))$ such that $c < a$ if and only if $c < b$ for all $c \in E^{rc}$, and take E^{rc} into $\mathbb{R}((t^{\Gamma^*_E}))$ with Γ^*_E the divisible hull of Γ'_E.

We build a such sequence $(x_\alpha)_{\alpha<\lambda}$ for some ordinal λ, as follows:

- Set $x_0 = 0$.

- Suppose we have defined $x_\alpha \in E$. Let $g_\alpha = w(a - x_\alpha) \in \Gamma'_E$ and set
$$a_\alpha = \mathrm{res}((a - x_\alpha)/t^{g_\alpha}),$$
then define $x_{\alpha+1} = x_\alpha + a_\alpha t^{g_\alpha}$.
- Let α be a limit ordinal and assume we have defined x_β for all $\beta < \alpha$. If there is $y \in E$ such that $w(a-y) > g_\beta$ for all $\beta < \alpha$, then define $x_\alpha = y$ and continue the construction. In other case we may define $\lambda = \alpha$ and finish the construction.

By definition, $w(x_{\alpha+1}-x_\alpha) = g_\alpha$ and $w(y-x_\alpha) = g_\alpha$ for all $\alpha < \lambda$. Even more, $w(x_\alpha - x_\beta) = g_\beta$ for $\beta < \alpha < \lambda$. Thus $(x_\alpha)_{\alpha<\lambda}$ is a pc-sequence in E of transcendental type over E and pseudo-limit a in K. On the other hand, x_α is of the form $\sum a_{\alpha,g} t^g$ for each $\alpha < \lambda$, thus since $w(x_{\alpha+1} - x_\alpha) = g_\alpha$, then $a_{\alpha,g} = a_{\alpha+1,g}$ for all $g < g_\alpha$. Therefore, if $g \in \Gamma'_E$ and $g < g_\alpha$ for some $\alpha < \lambda$, define $b_g = a_{\alpha,g}$, and $b_g = 0$ otherwise. Let $b = \sum b_g t^g$. Since $\mathrm{supp}(b)$ is well ordered then $b \in \mathbb{R}((t^{\Gamma'_E}))$. By definition $w(b - x_\alpha) = g_\alpha$, thus b is a pseudo-limit of $(x_\alpha)_{\alpha<\lambda}$ in $\mathbb{R}((t^{\Gamma'_e})) \setminus E$.

Now, since $(x_\alpha)_{\alpha<\lambda}$ is of transcendental type over E, then also is of transcendental type over E^{rc}. Finally, if we suppose, without loss of generality, that there is $c \in E^{rc}$ such that $a < c < b$, then $w(b - c) > g_\alpha$ for all $\alpha < \lambda$, which is equivalent to saying that
$$w(b - c) > w(x_{\alpha+1} - x_\alpha).$$
Thus, $c \in E^{rc}$ is pseudo-limit of $(x_\alpha)_{\alpha<\lambda}$. Which is a contradiction. Finally, by Remark 1 there is an L_{an}-isomorphism between $E(a)_{an,\forall,K}$ and $E(b)_{an,\forall,\mathbb{R}((t^{\Gamma'_E}))}$.

□

Now, if we assume that there is no $a \in K \setminus E$ such that $E(a)$ is an immediate extension of E, and there is $g \in \Gamma' \setminus \Gamma_E$ such that $ng \in \Gamma_E$ for some n, then we have the following:

Lemma 4.15. *Suppose $\phi : E \to \mathbb{R}((t^{\Gamma'_E}))$ is an L_{an}-embedding such that $\phi(\mathrm{s}'(g)) = t^g$ for all $g \in \Gamma'_E$. Let $g \in \Gamma' \setminus \Gamma'_E$ be such that $ng \in \Gamma_E$ for some n, $\mathrm{s}'(g) = a \in K \setminus E$ and $\Gamma'_{E(a)} = \Gamma'_E \oplus \mathbb{Z}g$. Then there is an L_{an}-embedding ϕ' of $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma'_{E(a)}}))$ which extends ϕ and such that $\phi'(a) = t^g$.*

Proof. Without loss of generality we can assume that $a > 0$ and, as in previous lemma, we can identify E with its image by ϕ in $\mathbb{R}((t^{\Gamma'_E}))$. By hypothesis there is $h \in \Gamma'_E$ such that $ng = h$. If $\mathrm{s}'(h) = b$, then we have that $a^n = b$ so a is algebraic over E and $t^g \in \mathbb{R}((t^{\Gamma'_{E(a)}}))$ is also algebraic over E. Moreover, $w(E(a)^\times) = \Gamma'_{E(a)}$ and a and t^g realize the same cut over E. Thus, by Remark 1, there is an L_{an}-embedding ϕ' of $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma'_{E(a)}}))$ which extends ϕ and such that $\phi'(a) = \phi(\mathrm{s}'(g)) = t^g$. □

Next, iterating Lemmas 4.14 and 4.15 we can assume that there is no $a \in K \setminus E$ such that $E(a)$ is an immediate extension of E, and $ng \notin \Gamma_E$ for all $g \in \Gamma' \setminus \Gamma'_E$ and $n > 0$. Thus we have the following extension lemma:

Lemma 4.16. *Suppose there is an L_{an}-embedding ϕ of E into $\mathbb{R}((t^{\Gamma'_E}))$ such that $\phi(\mathrm{s}'(g)) = t^g$ for all $g \in \Gamma'_E$. Let $g \in \Gamma' \setminus \Gamma'_E$ such that $ng \notin \Gamma_E$ for all $n > 0$, $\mathrm{s}'(g) = a \in K \setminus E$, and $\Gamma'_{E(a)} = \Gamma'_E \oplus \mathbb{Z}g \subseteq \Gamma'$. Then there is an L_{an}-embedding ϕ' from $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma'_{E(a)}}))$ which extends ϕ and such that $\phi'(a) = t^g$.*

Proof. Again, without loss of generality we can assume that $a > 0$ and we identify E with its image by ϕ in $\mathbb{R}((t^{\Gamma'_E}))$. By hypothesis a and $t^g \in \mathbb{R}((t^{\Gamma'_{E(a)}}))$ are transcendental over E and by Lemma 4.5 $w(E(a)^\times) = \Gamma'_{E(a)}$. Moreover, since for all $c \in E^{rc}$ we have $nw(c) = h$ for some $h \in \Gamma'_E$, if $c > 0$, we have $w(c) \neq g$ and then $c < a$ if and only if $c < t^g$. Thus, by Remark 1, there is an L_{an}-embedding ϕ' of $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma'_{E(a)}}))$ which extends ϕ and such that $\phi'(a) = \phi(\mathrm{s}'(g)) = t^g$. □

Finally, from Lemmas 4.14, 4.15 and 4.16 we have

Theorem 4.17. *Let K be a model of $T_{an,\forall}$ with w the natural valuation on K. If $w(K^\times) = \Gamma$ and s is a cross section of Γ, then there is an L_{an}-embedding ϕ of K into $\mathbb{R}((t^\Gamma))$ such that $\phi(\mathrm{s}(g)) = t^g$ for all $g \in \Gamma$.*

Proof. First, if $E = \mathbb{R}_{an}$, $\Gamma_E = \{0\}$ and $\mathrm{s}(0) = 1$, taking the identity map we show that there is an embedding of E into $\mathbb{R}((t^\Gamma))$. Next iterate Lemmas 4.14, 4.15 and 4.16. □

As a consequence of this construction we have:

Corollary 4.18. *Let $E \subseteq K$ be models of $T_{an,\forall}$, $\Gamma = w(K^\times)$, $\Gamma_E = w(E^\times)$, and s a cross section of Γ such that $\mathrm{s}|_{\Gamma_E}$ is a cross section of Γ_E. If $a \in K \setminus E$, then for $w(E(a)^\times_{an,\forall,K})$ we have the following possibilities:*

(1) *If $w(E(a)^\times) = \Gamma_E$, then $w(E(a)^\times_{an,\forall,K}) = \Gamma_E$.*

(2) *If $a = \mathrm{s}(g)$ for some $g \in \Gamma \setminus \Gamma_E$, then*

$$w(E(a)^\times_{an,\forall,K}) = w(E(a)^\times).$$

Proof. By Theorem 4.17 there is an L_{an}-embedding ϕ of E into $\mathbb{R}((t^{\Gamma_E}))$ such that $\phi(\mathrm{s}(g)) = t^g$ for all $g \in \Gamma_E$. If $w(E(a)^\times) = \Gamma_E$, then by Lemma 4.14 we can extend ϕ to an L_{an}-embedding of $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma_E}))$. So, $w(E(a)^\times_{an,\forall,K}) = \Gamma_E$.

Now, if $a = \mathrm{s}(g)$ for some $g \in \Gamma \setminus \Gamma_E$ such that $ng \in \Gamma_E$ for some $n > 0$, then by Lemma 4.15 we can extend ϕ to an L_{an}-embedding of $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma_1}))$. Thus

$$w(E(a)^\times_{an,\forall,K}) = \Gamma_1 = w(E(a)^\times).$$

Similarly, if $a = \mathrm{s}(g)$ for some $g \in \Gamma \setminus \Gamma_E$ such that $ng \notin \Gamma_E$ for all $n > 0$, then by Lemma 4.16 we can extend ϕ to an L_{an}-embedding of $E(a)_{an,\forall,K}$ into $\mathbb{R}((t^{\Gamma_1}))$. Thus

$$w(E(a)^\times_{an,\forall,K}) = \Gamma_1 = w(E(a)^\times).$$

□

The proof of Proposition 4.13 follows. Furthermore, from Lemma 4.11 and Proposition 4.13 we have the following result:

Corollary 4.19. *Let $E \subseteq K$ be model of $T_{an,\forall}$ and v a convex valuation of K. If $a \in K \setminus E$ and $E(a)$ is an immediate extension of E, then $E(a)_{an,\forall,K}$ is an immediate extension of E.*

The following result will be useful later:

Lemma 4.20. *Let* $(K, \boldsymbol{k}, \Gamma, \pi, v, \mathrm{lif}, \mathrm{s})$ *be an lc-valued field with* K *a model of* $T_{an,\forall}$, v *a convex valuation on* K *and* lif *a lifting of* $\boldsymbol{k}$ *into* K *such that* lif *is also an* L_{an}*-homomorphism of rings. Let* E *be an* $L_{an}(^{-1})$*-substructure of* K *with* $\boldsymbol{k}_E = \pi(\mathcal{O}_E)$, $\mathrm{lif}(\boldsymbol{k}_E) \subseteq \mathcal{O}_E$ *and* $\mathrm{s}|_{\Gamma_E}$ *a cross section of* $\Gamma_E = v(E^\times)$*. If* $a \in \boldsymbol{k} \setminus \boldsymbol{k}_E$*, and* $\boldsymbol{k}_E\langle a\rangle$ *denotes the* T_{an}*-closure of* $\boldsymbol{k}_E(a)$ *in* $\boldsymbol{k}$*, then there is the smallest* $L_{an}(^{-1})$*-substructure* F *of* K *such that* $\pi(\mathcal{O}_F) = \boldsymbol{k}_E\langle a\rangle$ *and* $\mathrm{lif}(\boldsymbol{k}_E\langle a\rangle) \subseteq \mathcal{O}_F$.

Proof. First we construct by transfinite induction a field $E' \subseteq K$, such that $\pi(\mathcal{O}_{E'}) = \mathbf{k}_E\langle a\rangle$ and $\mathrm{lif}(\mathbf{k}_E\langle a\rangle) \subseteq \mathcal{O}_{E'}$. Particularly, for some λ we define a chain of subfields $(E_\alpha)_{\alpha<\lambda}$ of K as follows:

- First, set $E_0 = E$.
- Suppose E_α has been defined and let $x \in \mathbf{k}_E\langle a\rangle \setminus \pi(\mathcal{O}_{E_\alpha})$. We have to deal with two cases:
  - x is algebraic over $\pi(\mathcal{O}_{E_\alpha})$. Set $E_{\alpha+1} = E_\alpha(\mathrm{lif}(x))$, then by Lemma 4.3, $E_{\alpha+1}$ is a valued subfield of K such that $\mathrm{res}(E_{\alpha+1})$ is isomorphic over $\pi(\mathcal{O}_{E_\alpha})$ to $\pi(\mathcal{O}_{E_\alpha})(x)$ and $v(E_{\alpha+1}^\times) = v(E_\alpha^\times)$.
  - x is transcendent over $\pi(\mathcal{O}_{E_\alpha})$. Set $E_{\alpha+1} = E_\alpha(\mathrm{lif}(x))$, then by Lemma 4.4, $E_{\alpha+1}$ is a valued subfield of K such that $\mathrm{res}(E_{\alpha+1})$ is isomorphic over $\pi(\mathcal{O}_{E_\alpha})$ to $\pi(\mathcal{O}_{E_\alpha})(x)$ and $v(E_{\alpha+1}^\times) = v(E_\alpha^\times)$.
- If $\alpha < \lambda$ is an ordinal limit, then define $E_\alpha = \cup_{\beta<\alpha} E_\beta$.

Now, define $E' = \cup_{\beta<\lambda} E_\beta$. By construction, $\pi(\mathcal{O}_{E'}) = \mathbf{k}_E\langle a\rangle$, $\mathrm{lif}(\mathbf{k}_E\langle a\rangle) \subseteq \mathcal{O}_{E'}$ and $v(E'^\times)$ is equal to $v(E^\times)$. Finally, define $F = E'_{an,\forall,K}$, thus $\pi(\mathcal{O}_F) = \mathbf{k}_E\langle a\rangle$ and after Proposition 4.13 we have $v(F^\times) = v(E^\times)$. □

4.3 The theory T_{pes}

As in the last chapter, we single out the unary function symbol e of $L_{an}(^{-1})$ to denote in $\mathbb{R}_{an}$ the restricted analytic function $e : \mathbb{R} \to \mathbb{R}$ such that $\mathrm{e}(x) = e^x$ for $|x| \leq 1$ and $e(x) = 0$ for $|x| > 1$.

Definition 4.21. *Let* $(K, \boldsymbol{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s})$ *be an ordered lc-valued field with* K *a model of* $T_{an,\forall}$ *and* v *a convex valuation of* K*. We say that the map* $\exp : K \to K$ *is a partial exponential function of* K *if* $\exp(x) = 0$ *for* $x \in K \setminus \mathcal{O}$, $\exp(\mathcal{O}) \subseteq \mathcal{O}$ *and for all* $x, y \in \mathcal{O}$ *we have:*

*E*1. $\exp(x + y) = \exp(x)\exp(y)$*;*

*E*2. $\exp(x) = \mathrm{e}(x)$ *if* $|x| \leq 1$*;*

*E*3. *if* $x > n^2$*, then* $\exp(x) > x^n$*, for* $n \geq 0$*;*

*E*4. *if* $y > 1$*, there is* $x \in \mathcal{O}$ *such that* $\exp(x) = y$*.*

Moreover, if $(\boldsymbol{k}, \boldsymbol{e})$ *is an exponential field, we call the structure* $(K, \boldsymbol{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s}, \exp)$ *a partial exponential* $L_{an}(^{-1})$*-structure if:*

(1) $(K, \boldsymbol{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s})$ *is an ordered lc-valued field with* K *a model of* $T_{an,\forall}$ *and* v *a convex valuation of* K*;*

(2) $(\boldsymbol{k}, \boldsymbol{e})$ *is a model of* $T_{an}(\exp)$*;*

(3) $\exp$ *is a partial exponential function of* K*;*

(4) $\pi(\exp(x)) = \boldsymbol{e}(\pi(x))$ *for all* $x \in \mathcal{O}$*;*

(5) lif *is an* L_{an^*}*-morphism and* $\mathrm{lif}(\boldsymbol{e}(x)) = \exp(\mathrm{lif}(x))$ *for all* $x \in \boldsymbol{k}$*;*

Example. *Let* Γ *be an ordered abelian group and*

$$(\mathbb{R}((t^{\Gamma})), \mathbb{R}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s})$$

the ordered lc-valued field with lif *the natural lifting and* s *the natural cross-section. Defining*

$$\exp : (\mathbb{R}((t^{\Gamma})) \to (\mathbb{R}((t^{\Gamma}))$$

$\exp(x) = 0$ *for* $x \in \mathbb{R}((t^{\Gamma}))$ *with* $v(x) < 0$ *and*

$$\exp(r + \epsilon) = e^r \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}$$

for $r \in \mathbb{R}$ *and* $\epsilon \in \mathcal{o}$*, and taking* e *as the usual exponential function of* $\mathbb{R}$*, then since* $\mathbb{R}((t^{\Gamma}))$ *expands naturally to a model of* $T_{an,\forall}$*,*

$$(\mathbb{R}((t^{\Gamma})), \mathbb{R}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s}, \exp)$$

is a partial exponential $L_{an}(^{-1})$*-structure.*

Let $\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s}, \exp)$ be a partial exponential $L_{an}(^{-1})$-structure. Since the language $L_{an}(^{-1})$ has a name for each real number, $\mathbb{R} \subseteq \mathcal{O}$. Thus, $\exp(r) = e^r$ for $r \in \mathbb{R} \subseteq \mathcal{O}$. Furthermore, as a direct consequence of the definition, the map exp restricted to $\mathcal{O}$ is positive and strictly increasing and we have:

Lemma 4.22. (1) $(K, \boldsymbol{k}, \Gamma; \pi, v)$ *is a henselian valued field.*

(2) $\exp(\mathcal{o}) = 1 + \mathcal{o}$*.*

Proof. Since K is a model of $T_{an,\forall}$, then K is a henselian valued field for the natural valuation. Thus, for each convex valuation v of K, we have that K is henselian too.

On the other hand if $\epsilon \in \mathcal{o}$, by axiom (4), $\pi(\exp(\epsilon)) = \mathrm{e}(\pi(\epsilon)) = \mathrm{e}(0) = 1$. Thus, $\exp(\epsilon) \in 1 + \mathcal{o}$. Moreover, if $1 + \epsilon > 1$ there is $x \in \mathcal{O}$ such that $\exp(x) = 1 + \epsilon$. Then, $\pi(\exp(x)) = \pi(1 + \epsilon) = 1$ and then $x \in \mathcal{o}$. Clearly, if $1 + \epsilon < 1$ then $(1 + \epsilon)^{-1} > 1$ and thus there is $x \in \mathcal{o}$ such that $\exp(x) = (1 + \epsilon)^{-1}$. Hence, $\exp(-x) = (1 + \epsilon)$. □

Definition 4.23. *Let L_{an-s} be a three-sorted language with sorts f (the field sort), r (the residue field sort), and v (the value group sort). This language consists of one copy of the language $L_{an}(^{-1})$ of sort f, one copy of the language $L_{an*}(\exp, \log)$ in the sort r, a copy of the language of ordered abelian groups in the sort v, and function symbols v of sort fv, π of sort fr, lif of sort rf and s of sort vf. We define the language $L_{an-s,\exp}$ as L_{an-s} augmented by a new function symbol exp of sort ff and T_{pes} as the theory whose models are the partial exponential $L_{an}(^{-1})$-structure seen as $L_{an-s,\exp}$-structures.*

Now, observe that if $K = \mathbf{k} = \mathbb{R}$, $\Gamma = \{0\}$, v is the trivial valuation, lif is de identity map and e is the usual exponential map of $\mathbb{R}$, then $\mathbf{R} := (\mathbb{R}, \mathbb{R}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{e})$ is a model of T_{pes}, actually a prime model of T_{pes}.

4.3.1 Good substructures and good maps

Let $\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s}, \exp)$ be a model of T_{pes}. A good substructure of $\mathbf{K}$ is a triple $\mathbf{E} = (E, \mathbf{k}_E, \Gamma_E)$ such that:

(1) E is an $L_{an}(^{-1})$-substructure of K such that $\exp(\mathcal{O}_E) \subseteq \mathcal{O}_E$ (here $\mathcal{O}_E := E \cap \mathcal{O}$),

(2) $\mathbf{k}_E$ is an $L_{an}(\exp, \log)$-substructure of $\mathbf{k}$ with $\pi(\mathcal{O}_E) \subseteq \mathbf{k}_E$ and $\mathrm{lif}(\mathbf{k}_E) \subseteq E$.

(3) Γ_E is an ordered abelian subgroup of Γ with $v(E^\times) \subseteq \Gamma_E$ and $\mathrm{s}(\Gamma_E) \subseteq E$

Moreover, if $\mathbf{E} = (E, \mathbf{k}_E, \Gamma_E)$ and $\mathbf{F} = (F, \mathbf{k}_F, \Gamma_F)$ are good substructures of $\mathbf{K}$, we say that $\mathbf{E} \subseteq \mathbf{F}$ if $E \subseteq F$, $\mathbf{k}_E \subseteq \mathbf{k}_F$ and $\Gamma_E \subseteq \Gamma_F$.

On the other hand, given

$$\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s}, \exp) \text{ and } \mathbf{K}^* = (K^*, \mathbf{k}^*, \Gamma^*; \pi^*, v^*, \mathrm{lif}^*, \mathrm{s}^*, \exp^*)$$

models of T_{pes}, and $\mathbf{E} = (E, \mathbf{k}_E, \Gamma_E)$ and $\mathbf{E}^* = (E^*, \mathbf{k}_{E^*}, \Gamma_{E^*})$ good substructures of $\mathbf{K}$ and $\mathbf{K}^*$, respectively, we say that a good map $\mathbf{f} : \mathbf{E} \to \mathbf{E}^*$ is a triple $\mathbf{f} = (f, f_r, f_v)$ consisting of an isomorphism of $L_{an}(^{-1})$-structures $f : E \to E^*$, an isomorphism of $L_{an}(\exp, \log)$-structures $f_r : \mathbf{k}_E \to \mathbf{k}_{E^*}$, and an ordered group isomorphism $f_v : \Gamma_E \to \Gamma_{E^*}$, such that

(1) $f_r(\pi(x)) = \pi^*(f(x))$ for all $x \in E$, and f_r is elementary as a partial map between the models $\mathbf{k}$ and $\mathbf{k}^*$ of $T_{an}(\exp)$;

(2) $f_v(v(x)) = v^*(f(x))$ for all $a \in E^\times$, and f_v is elementary as a partial map between the ordered abelian groups Γ and Γ^*.

(3) $f(\exp(x)) = \exp^*(f(x))$ for all $x \in E$.

(4) $f(\mathrm{lif}(x)) = \mathrm{lif}^*(f_r(x))$ for all $x \in \mathbf{k}_E$.

(5) $f(\mathrm{s}(\gamma)) = \mathrm{s}^*(f_v(\gamma))$

Finally, if $\mathbf{f} : \mathbf{E} \to \mathbf{E}^*$ is a good map we say that a good map $\mathbf{g} = (g, g_r, g_v) : \mathbf{F} \to \mathbf{F}^*$ extends $\mathbf{f}$ if $\mathbf{E} \subseteq \mathbf{F}$, $\mathbf{E}^* \subseteq \mathbf{F}^*$, and g, g_r, g_v extends f, f_r, f_v respectively.

If $\mathbf{E} = (E, \mathbf{k}_E, \Gamma_E)$ is a good substructure of $\mathbf{K}$, since $\pi(\mathcal{O}_E) \subseteq \mathbf{k}_E$ and $\mathrm{lif}(\mathbf{k}_E) \subseteq \mathcal{O}_E$, then $\mathrm{res}(E)$ is equal to $\mathbf{k}_E$. Similarly, we have that $v(E^{\times}) = \Gamma_E$. Moreover, since $\mathbf{k}_E$ is a model of $T_{an}(\exp)$ and $\exp(E) \subseteq E$, then for all $x \in (\mathcal{O}_E)^{\times}$ with $x > 0$, there is $y \in \mathcal{O}_E$ such that $\exp(y) = x$.

4.3.2 An equivalence theorem for T_{pes}

Let $\mathbf{K} = (K, \mathbf{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s}, \exp)$ and $\mathbf{K}^* = (K^*, \mathbf{k}^*, \Gamma^*; \pi^*, v^*, \mathrm{lif}^*, \mathrm{s}^*, \exp^*)$ be models of T_{pes}, then we have the following result:

Theorem 4.24. *$\boldsymbol{K} \equiv \boldsymbol{K}^*$ if and only if $\boldsymbol{k} \equiv \boldsymbol{k}^*$ as models of $T_{an}(\exp)$ and $\Gamma \equiv \Gamma^*$ as ordered abelian groups.*

Since $T_{an}(\exp)$ is a complete theory, this theorem is equivalent to saying that $\mathbf{K} \equiv \mathbf{K}^*$ if and only if $\Gamma \equiv \Gamma^*$ as ordered abelian groups.

Our strategy to prove this theorem consists of two steps. First we show that there is a good map between $\mathbf{K}$ and $\mathbf{K}^*$.

Lemma 4.25. *There are good substructures $\boldsymbol{E} = (E, \boldsymbol{k}_E, \Gamma_E)$ and $\boldsymbol{E}^* = (E^*, \boldsymbol{k}_{E^*}, \Gamma_{E^*})$ of $\boldsymbol{K}$ and $\boldsymbol{K}^*$, respectively, and a good map $\boldsymbol{E} \to \boldsymbol{E}^*$.*

Proof. Since $\mathbb{R}_{an}$ is an $L_{an}(^{-1})$-substructure of each model of $T_{an,\forall}$, then $(\mathbb{R}_{an}, \mathbb{R}_{an,\exp}, \{0\})$ is a good substructure of $\mathbf{K}$ and of $\mathbf{K}'$. Thus, we have a good map from $\mathbf{E} = (\mathbb{R}_{an}, \mathbb{R}_{an,\exp}, \{0\})$ onto $\mathbf{E}' = (\mathbb{R}_{an}, \mathbb{R}_{an,exp}, \{0\})$. $\square$

Next, we want to show that the good maps (with small domain) form a back-and-forth system from $\mathbf{K}$ to $\mathbf{K}^*$. Specifically, we have the following:

Theorem 4.26. *Let $\boldsymbol{E} = (E, \boldsymbol{k}_E, \Gamma_E)$ and $\boldsymbol{E}^* = (E^*, \boldsymbol{k}_{E^*}, \Gamma_{E^*})$ be good substructures of $\boldsymbol{K}$ and $\boldsymbol{K}^*$, respectively. Any good map $\boldsymbol{f}\colon \boldsymbol{E} \to \boldsymbol{E}^*$ is a partial elementary map between $\boldsymbol{K}$ and $\boldsymbol{K}^*$.*

Proof. Assume $\Gamma \neq 0$. Without of generality we can take $\mathbf{K}$ and $\mathbf{K}^*$ κ-saturated, for some uncountable cardinal κ such that $\mathrm{card}(\mathbf{k}_E), \mathrm{card}(\Gamma_E) < \kappa$. We say that a good substructure $\mathbf{F} = (F, \mathbf{k}_F, \Gamma_F)$ of $\mathbf{K}$ is small if $\mathrm{card}(\mathbf{k}_F), \mathrm{card}(\Gamma_F) < \kappa$. Our main purpose is to show that for each $a \in K$ there is a good map $\mathbf{g}$ extending $\mathbf{f}$ such that $\mathbf{g}$ has small domain $\mathbf{F} \supseteq \mathbf{E}$ with $a \in F$. To do this we have to deal with the following steps:

A. Extend the residue field. Let $a \in \mathbf{k} \setminus \mathbf{k}_E$. By saturation, there is $a' \in \mathbf{k}^* \setminus \mathbf{k}_{E^*}$ such that $x < a$ if and only if $f_r(x) < a'$ for all $x \in \mathbf{k}_E$. Let $\mathbf{k}\langle a\rangle_E$ and $\mathbf{k}\langle a'\rangle_{E^*}$ be the $T_{an}(\exp)$-closures of $\mathbf{k}_E(a)$ and $\mathbf{k}_{E^*}(a')$ in $\mathbf{k}$ and $\mathbf{k}^*$, respectively. By o-minimality of $T_{an}(\exp)$, there is an $L_{an,\exp,\log}$-isomorphism f_r' from $\mathbf{k}\langle a\rangle_E$ onto $\mathbf{k}\langle a'\rangle_{E^*}$ which extends f_r and such that $f_r'(a) = a'$.

By transfinite induction and for some ordinal λ we define chains $(E_\alpha)_{\alpha<\lambda}$, $(\mathbf{k}_\alpha)_{\alpha<}$, $(E_\alpha^*)_{\alpha<\lambda}$ and $(\mathbf{k}_\alpha^*)_{\alpha<\lambda}$ of $L_{an}(^{-1})$-substructures of K, $\mathbf{k}, K^*$ and $\mathbf{k}^*$, respectively; chains $(\Gamma_\alpha)_{\alpha<\lambda}$ and $(\Gamma_\alpha^*)_{\alpha<\lambda}$ of ordered subgroups of Γ and Γ^*; chains $(f_\alpha : E_\alpha \to E_\alpha^*)_{\alpha<\lambda}$, $(f_{r,\alpha} : \mathbf{k}_\alpha \to \mathbf{k}_\alpha^*)_{\alpha<\lambda}$ of L_{an}-isomorphisms, and $(f_{v,\alpha} : \Gamma_\alpha \to \Gamma_\alpha^*)_{\alpha<\lambda}$ of isomorphisms of ordered groups, such that for all $\alpha < \lambda$ we have:

1. $\pi(\mathcal{O}_{E_\alpha}) = \mathbf{k}_\alpha$ and $\pi^*(\mathcal{O}_{E^*_\alpha}) = \mathbf{k}^*_\alpha$,

2. $\mathbf{k}_\alpha$ and $\mathbf{k}_\alpha$ are models of T_{an},

3. $v(E_\alpha) \subseteq \Gamma_\alpha$ and $v(E^*_\alpha) \subseteq \Gamma^*_\alpha$,

4. $\pi(E_\alpha) \subseteq \mathbf{k}\langle a\rangle_E$ and $\pi(E^*_\alpha) \subseteq \mathbf{k}\langle a\rangle^*_E$.

Specifically,

- Set $(E_0, \mathbf{k}_0, \Gamma_0) = (E, \mathbf{k}_E, \Gamma_E)$, $(E^*_0, \mathbf{k}^*_0, \Gamma^*_0) = (E^*, \mathbf{k}^*_E, \Gamma^*_E)$, $f_0 = f$, $f_{r,0} = f_r$, and $f_{v,0} = f_v$.

- If $\alpha \le \lambda$ is limit, define $(E_\alpha, \mathbf{k}_\alpha, \Gamma_\alpha) = \cup_{\beta<\alpha}(E_\beta, \mathbf{k}_\beta, \Gamma_\beta)$, $(E^*_\alpha, \mathbf{k}^*_\alpha, \Gamma^*_\alpha) = \cup_{\beta<\alpha}(E^*_\beta, \mathbf{k}^*_\beta, \Gamma^*_\beta)$, and $(f_\alpha, f_{r,\alpha}, f_{v,\alpha}) = \cup_{\beta<\alpha}(f_\beta, f_{r,\beta}, f_{v,\beta})$.

- Suppose that we have defined $(E_\alpha, \mathbf{k}_\alpha, \Gamma_\alpha)$, $(E^*_\alpha, \mathbf{k}^*_\alpha, \Gamma^*_\alpha)$ and $(f_\alpha, f_{r,\alpha}, f_{v,\alpha})$, for some $\alpha < \lambda$. Let $x \in \mathbf{k}\langle a\rangle_E \setminus \mathbf{k}_\alpha$. Define $\mathbf{k}_{\alpha+1}$ as the T_{an*}-closure of $\mathbf{k}_\alpha(x)$ in $\mathbf{k}\langle a\rangle_E$; $\mathbf{k}^*_{\alpha+1}$ as the T_{an*}-closure of $\mathbf{k}^*_\alpha(f'(x))$ in $\mathbf{k}\langle a\rangle^*_E$; $E_{\alpha+1} = E_\alpha(\mathrm{lif}(x))_{an,\forall,K}$, and $E^*_{\alpha+1} = E_\alpha((\mathrm{lif}(f'_r(x)))_{an,\forall,K^*}$ as the fields given in Lemma 4.20. Since $v(E_\alpha(\mathrm{lif}(x))^\times) = v(E^\times_\alpha)$ and $v^*(E^*_\alpha(\mathrm{lif}(x))^\times) = v^*((E^*_\alpha)^\times)$, then by Proposition 4.13, $v(E^\times_{\alpha+1}) = v(E^\times_\alpha)$ and $v^*((E^*_{\alpha+1})^\times) = v^*((E^*_\alpha)^\times)$, thus we may set $\Gamma_{\alpha+1} = \Gamma_\alpha$ and $\Gamma^*_{\alpha+1} = \Gamma^*_\alpha$. Finally, by Remark 1 in section 4.2 there is a $L_{an}(^{-1})$-isomorphism $f_{\alpha+1}$ from $E_{\alpha+1}$ onto $E^*_{\alpha+1}$ which extends f_α and such that $f_{\alpha+1}(\mathrm{lif}(x)) = \mathrm{lif}(f'_r(x))$. Moreover, we may set $f_{v,\alpha+1} = f_{v,\alpha}$ and $f_{r,\alpha+1}$ as the restriction of f'_r to $\mathbf{k}_{\alpha+1}$.

Now, if we take $\mathbf{F} = (E_\lambda, \mathbf{k}_\lambda, \Gamma_\lambda)$, $\mathbf{F}^* = (E^*_\lambda, \mathbf{k}^*_\lambda, \Gamma^*_\lambda)$, and $g = (f_\lambda, f_{r,\lambda}, f_{v,\lambda})$, we have the following observations:

(1) $\Gamma_\lambda = \Gamma_E$, $\Gamma^*_\lambda = \Gamma_{E^*}$ and $f_{v,\lambda} = f_v$.

(2) $\mathbf{k}_\lambda = \mathbf{k}\langle a\rangle_E$, $\mathbf{k}^*_\lambda = \mathbf{k}\langle a\rangle_{E^*}$ and $f_{v,\lambda} = f'$.

(3) E_λ and E^*_λ are $L_{an}(^{-1})$-substructures of K and K^*, respectively, such that $E \subseteq E_\lambda$. Moreover, $v(E^\times_\lambda) \subseteq \Gamma_E$, $v((E^*_\lambda)^\times) \subseteq \Gamma_{E^*}$, $\pi(\mathcal{O}_{E_\lambda}) = \mathbf{k}_\lambda$ and $\pi(\mathcal{O}_{E^*_\lambda}) = \mathbf{k}^*_\lambda$.

(4) $\exp(E_\lambda) \subseteq E_\lambda$: if $x \in \mathcal{O}_{E_\lambda}$, then there are unique $a \in \mathrm{lif}(\mathbf{k}_\lambda)$ and $\epsilon \in \mathcal{O}_{E_\lambda}$ such that $x = a + \epsilon$. Now, since $\exp(a) \in \mathrm{lif}(\mathbf{k}_\lambda) \subseteq \mathcal{O}_{E_\lambda}$ and E_λ is closed under restricted analytic functions, then $\exp(x) = \exp(a)\exp(\epsilon) \in \mathcal{O}_{E_\lambda}$. In the same way we can show that $\exp(E^*_\lambda) \subseteq E^*_\lambda$ and by construction we have that for all $x \in \mathcal{O}_{E_\alpha}$,

$$f_\lambda(\exp(x)) = \exp^*(f_\lambda(x)).$$

If follows that $\mathbf{F}$ and $\mathbf{F}^*$ are small good substructures of $\mathbf{K}$ and $\mathbf{K}^*$, such that $\mathbf{E} \subseteq \mathbf{F}$ and $\mathbf{E}^* \subseteq \mathbf{F}^*$. Moreover, g is a good map from $\mathbf{F}$ onto $\mathbf{F}^*$ which extends f.

B. Extend the value group. Let $\gamma \in \Gamma \setminus \Gamma_E$, then we have two possibilities:

$(B.1)$ $p\gamma \in \Gamma_E$ for some prime number p. Let p a prime number such that $p\gamma \in \Gamma_E$ and take $x = \mathrm{s}(\gamma)$, thus $x^p = \mathrm{s}(p\gamma)$ and define, moreover, $x' \in \Gamma^* \setminus \Gamma_{E^*}$ as the unique element of Γ such that $x'^p = \mathrm{s}^*(f_v(p\gamma))$, and $\gamma' = v^*(x')$. Observe that x and x' are algebraic over E and E^*, respectively.

By Lemma 4.6, $\Gamma_{E(x)} = v(E(x)^\times) + \mathbb{Z}\gamma$, $\mathrm{res}(E(x)) = \mathrm{res}(E)$, $\Gamma_{E^*(x')} = v(E^*(x')^\times) + \mathbb{Z}\gamma'$, $\mathrm{res}(E^*(x'))$ es equal to $\mathrm{res}(E^*)$, and there is an isomorphism of ordered fields f' from $E(x)$ onto $E^*(x')$ which extends f with $f'(x) = x'$. Moreover, we can extend f_v to $f'_v : \Gamma_{E(x)} \to \Gamma_{E^*(x')}$ so that $f'_v(\gamma) = \gamma'$. On the other hand, if we define $F = E(x)_{an,\forall,K}$ and $F^* = E^*(x')_{an,\forall,K^*}$, then by Remark 1 there is an $L_{an}(^{-1})$-isomorphism g from F onto F^* which extends f'. By Proposition 4.13 we see that $v(F^\times) = v(E(x)^\times)$ and $v((F^*)^\times) = v(E^*(x')^\times)$, and by Lemma 4.11 we obtain $\mathrm{res}(F) = \mathrm{res}(E)$ and $\mathrm{res}(F^*) = \mathrm{res}(E^*)$. Moreover, $\exp(\mathcal{O}_F) \subseteq \mathcal{O}_F$ and $\exp(\mathcal{O}_{F^*}) \subseteq \mathcal{O}_{F^*}$.

For example, if $a \in \mathcal{O}_F$, then there are unique $b \in \mathrm{lif}(\mathbf{k}_E)$ and $\epsilon \in \mathcal{o}_F$ such that $a = b+\epsilon$. Since $\mathbf{k}_E$ is closed under exponentiation, $\mathrm{lif}(\mathbf{k}_E) \subseteq \mathcal{O}_F$ and F is closed under restricted analytic functions, then we have $\exp(a) = \exp(b)\exp(\epsilon) \in \mathcal{O}_F$.

Thus, $\mathbf{F} = (F, \mathbf{k}_E, \Gamma_{E(x)})$ and $\mathbf{F}^* = (F^*, \mathbf{k}_{E^*}, \Gamma_{E^*(x)})$ are good substructures of $\mathbf{K}$ and $\mathbf{K}^*$, respectively, and $\mathbf{g} = (g, f_r, f'_v)$ is a good map between $\mathbf{F}$ and $\mathbf{F}^*$.

$(B.2)$ $n\gamma \notin \Gamma_E$ for all $n > 0$. Without loss of generality we can assume that $\gamma > 0$. Take $x \in K \setminus E$ such that $x = \mathrm{s}(\gamma)$. Clearly, x is transcendental over E. By saturation, there is $x' \in K \setminus E^*$ such that for all $P(y) \in \mathcal{O}_E[y]$ we have $P(x) > 0$ if and only if $f(P)(x') > 0$ and $\mathrm{s}^*(v(x')) = x'$. Set $\gamma' = v^*(x')$. By Lemma 4.5, $\mathrm{res}(E(x)) = \mathrm{res}(E)$, $\mathrm{res}(E^*(x')) = \mathrm{res}(E^*)$, $\Gamma_{E(x)} = v(E(x)^\times) \oplus \mathbb{Z}\gamma$, $\Gamma_{E^*(x')} = v(E^*(x')^\times) \oplus \mathbb{Z}\gamma'$, and there is an isomorphism of ordered fields f' from $E(x)$ onto $E^*(x')$ which extends f with $f'(x) = x'$. Moreover, we can extend f_v to $f'_v : \Gamma_{E(x)} \to \Gamma_{E^*(x')}$ so that $f'_v(\gamma) = \gamma'$. As in previous case, we define $F = E(x)_{an,\forall,K}$ and $F^* = E^*(x')_{an,\forall,K^*}$ and by Remark 1 there is an $L_{an}(^{-1})$-isomorphism g from F onto F^* which extends f'. Again we have $v(F^\times) = v(E(x)^\times)$, $v((F^*)^\times) = v(E^*(x')^\times)$, $\mathrm{res}(F) = \mathrm{res}(E)$ and $\mathrm{res}(F^*) = \mathrm{res}(E^*)$. Moreover, $\exp(\mathcal{O}_F) \subseteq \mathcal{O}_F$ and $\exp(\mathcal{O}_{F^*}) \subseteq \mathcal{O}_{F^*}$.

Thus, $\mathbf{F} = (F, \mathbf{k}_E, \Gamma_{E(x)})$ and $\mathbf{F}^* = (F^*, \mathbf{k}_{E^*}, \Gamma_{E^*(x)})$ are good substructures of $\mathbf{K}$ and $\mathbf{K}^*$, respectively, and $\mathbf{g} = (g, f_r, f'_v)$ is a good map between them.

C. Extend the field E Let $x \in K \setminus E$. We want to extend $\mathbf{f}$ to a good map whose domain is small and contains x. Without loss of generality we assume that $x \in \mathcal{O}$. Here we also have two cases:

$(C.1)$ $E(x)$ is an immediate extension of E. In this case, since E is henselian then it is algebraically maximal and x is transcendent over E. Thus, there is a pseudo-cauchy sequence $(x_\alpha)_{\alpha<\lambda}$ in E of transcendental type over E with pesudo-limit x. Now, the sequence $(f(x_\alpha))_{\alpha<\lambda}$ in E^* is of transcendetal type over E^*. By saturation there is an element $x' \in K^*$ such that

- x' is pseudo-limit of $(f(x_\alpha))_{\alpha<\lambda}$, and
- for all $P(X) \in O_E[X]$ we have $P(x) > 0$ if and only if $f(P)(x') > 0$.

Fixing such element x' in K^* we have that $E^*(x')$ is an immediate extension of E and, by the first condition, there is an isomorphism of ordered valued fields f' from $E(x)$ onto $E^*(x')$ which extends f and such that $f'(x) = x'$. Moreover, by the second condition and Remark

1 we obtain an $L_{an}(^{-1})$-isomorphism g from $E(x)_{an,\forall,K}$ onto $E^*(x')_{an,\forall,K^*}$ which extends f'. By Lemma 4.19, we know that $E(x)_{an,\forall,K}$ and $E^*(x')_{an,\forall,K^*}$ are immediate extensions of E and E^*, respectively. Thus, since E and E^* are closed under exp and $E(x)_{an,\forall,K}$ and $E^*(x')_{an,\forall,K^*}$ are closed under restricted analytic functions, then they are closed under exp. Finally, we obtain that $\mathbf{F} = (E(x)_{an,\forall,K}, \mathbf{k}_E, \Gamma_E)$ and $\mathbf{F}^* = (E^*(x')_{an,\forall,K}, \mathbf{k}_{E^*}, \Gamma_{E^*})$ are good substructures of $\mathbf{K}$ and $\mathbf{K}^*$, respectively, and $\mathbf{g} = (g, f_r, f_v)$ is a good map between $\mathbf{F}$ and $\mathbf{F}^*$.

$(C.2)$ $E(x)$ is not an immediate extension of E. We will construct good substructures $\mathbf{F} = (F, \mathbf{k}_F, \Gamma_F)$ and $\mathbf{F}^* = (F^*, \mathbf{k}_{F^*}, \Gamma_{F^*})$ of $\mathbf{K}$ and $\mathbf{K}^*$, respectively, and a good map $g : \mathbf{F} \to \mathbf{F}^*$ which extends $\mathbf{f}$ such that $F(x)$ is an immediate extension of F. Particularly, we define by induction a chain $\mathbf{E}_n = (E_n, \mathbf{k}_{E_n}, \Gamma_{E_n})$ of good substructures of $\mathbf{K}$, a chain $\mathbf{E}_n^* = (E_n^*, \mathbf{k}_{E_n^*}, \Gamma_{E_n^*})$ of good substructures of $\mathbf{K}$, and a chain of good maps $\mathbf{f}_n : \mathbf{E}_n \to \mathbf{E}_n^*$, as follows:

- Set $\mathbf{E}_0 = \mathbf{E}$, $\mathbf{E}_0^* = \mathbf{E}^*$ and $\mathbf{f}_0 = \mathbf{f}$.
- Assume $\mathbf{E}_n$, $\mathbf{E}_n^*$ and $\mathbf{f}_n$ have been defined already. Let $\mathbf{k}_{E_{n+1}} = \mathbf{k}_{E_n}\langle x\rangle$ the $T_{an}(\exp, \log)$-closure of $\mathbf{k}_{E_n}(x)$ in $\mathbf{k}$ and $\Gamma_{E_{n+1}} = v(E_n(x)^\times)$. Using the steps A and B of the proof, we can extend $\mathbf{f}_n$ to a good map $\mathbf{f}_{n+1}$ with domain $\mathbf{E}_{n+1} = (E_{n+1}, \mathbf{k}_{E_{n+1}}, \Gamma_{E_{n+1}})$.

Finally, define $\mathbf{F} = \cup_n \mathbf{E}_n$, $\mathbf{F}^* = \cup_n \mathbf{E}_n^*$ and $\mathbf{g} = \cup_n \mathbf{f}_n$. Clearly, $F(x)$ is an immediate extension of F. Thus, we use the case C.1 to extend $\mathbf{g}$ to a good map such that its domain contains x. We have explained the Forth part of the Back-and-Fort, interchanging the role of $\mathbf{E}$ and $\mathbf{E}^*$ we obtain the Back part.

□

As a consequence of Lemma 4.25 and Theorem 4.26 we obtain

Proof of Theorem 4.24. Clearly, if $\mathbf{K} \equiv \mathbf{K}^*$ then $\Gamma \equiv \Gamma^*$, as ordered groups. Now, assume $\Gamma \equiv \Gamma^*$, as ordered groups. By lemma 4.25 $(\mathbb{R}_{an}, \mathbb{R}_{an,exp}, \{0\})$ is a good substructure of both $\mathbf{K}$ and $\mathbf{K}^*$. Now, apply Theorem 4.26.

□

From Theorem 4.24 we obtain that each model $\mathbf{K} = (K, \mathbf{k}, \Gamma; ...)$ of T_{pes} is elementarily equivalent to

$$(\mathbf{k}((t^\Gamma)), \mathbf{k}, \Gamma; \pi, v, \mathrm{lif}, \mathrm{s}, \exp),$$

defined above.

If $\mathbf{E} = (E, \mathbf{k}_E, \Gamma_E; ...) \subseteq \mathbf{K}$ is a model of T_{pes} then as a consequence of Theorem 4.26 we have the following:

Corollary 4.27. *If $\Gamma_E \preceq \Gamma$ as ordered abelian groups, then $\boldsymbol{E} \preceq \boldsymbol{K}$.*

Proof. It is enough to take an elementary extension $\mathbf{K}^*$ of E. Thus, $(E, \mathbf{k}_E, \Gamma_E)$ is a good substructure of $\mathbf{K}$ and $\mathbf{K}^*$, and the identity map on $(E, \mathbf{k}_E, \Gamma)$ is a good map from $\mathbf{K}$ into $\mathbf{K}^*$. Applying Theorem 4.26 we obtain that $\mathbf{K} \equiv \mathbf{K}^*$ over $\mathbf{E}$, and since $\mathbf{E} \preceq \mathbf{K}^*$ then $\mathbf{E} \preceq \mathbf{K}$. □

5. The theory of the precontraction group of $\mathbb{T}_{\log}$

Inspired by the ideas used in [14, 18] to study the theory of the contraction groups and those used in [10] to study the theory of the asymptotic couple of $\mathbb{T}_{\log}$, in this chapter we study the first order theory of the couple $(\Gamma_{\log}, \chi)$ as a precontraction group. Particularly, we notice that although the map χ is not surjective, since the logarithm is not surjective, the image of $\Gamma_{\log}^{<0}$ by χ is a discrete set cofinal in $\Gamma_{\log}^{<0}$ and using this fact we prove that the theory of the precontraction group $(\Gamma_{\log}, \chi)$ is model complete and complete and study the definable subsets of the image of $\Gamma_{\log}^{<0}$ by χ.

More specifically, in the first part of this chapter we include some definitions and results about precontraction groups. Next, we study the theory T_{pdg} of centripetal precontraction discrete groups. Particularly, we prove that this theory is model complete and complete. We also expand the language of the theory to ensure that it has quantifier elimination and characterize the definable subsets of the image of the group by the precontraction map. Finally we study the simple extensions of models of the theory and include other technical results about extensions of models of the theory which can be useful to work later with non-immediate extensions of valued fields whose value groups are centripetal precontraction discrete groups.

5.1 Precontraction groups

The notion of contraction map was introduced in [18] to study the structure of the value group of an exponential field and the theory of contraction groups was studied in [14, 15]. We list here some useful definitions and results of those papers. Specifically:

Definition 5.1. *Given a totally ordered abelian group Γ and a map $\chi : \Gamma \to \Gamma$, the pair (Γ, χ) is called a precontraction group and χ is called a precontraction map if it satisfies for all $a, b \in \Gamma$ the following axioms:*

(1) $\chi(a) = 0 \leftrightarrow a = 0$,

(2) $a \leq b \to \chi(a) \leq \chi(b)$,

(3) $\chi(-a) = -\chi(a)$,

(4) *if a is archimedean equivalent to b and* $\operatorname{sign}(a) = \operatorname{sign}(b)$, *then* $\chi(a) = \chi(b)$.

If in addition χ is surjective then χ is called a contraction map and (Γ, χ) is called a contraction group. Moreover, (Γ, χ) will be called centripetal if $|a| > |\chi(a)|$ for all $a \in \Gamma^{\neq 0}$.

Example. *The map χ defined in the value group $\Gamma_{\log}$ of $\mathbb{T}_{\log}$ is a precontraction map. Moreover, since the ordered valued logarithmic field $\mathbb{T}_{\log}$ satisfies the Growth Axiom, we have that in fact $(\Gamma_{\log}, \chi)$ is a centripetal precontraction group.*

Proof. We already see that χ is well defined. Now, let $v(f)$ be archimedean equivalent to $v(g)$ with $f, g \in \mathbb{T}_{\log}^{>0}$ and $v(f) \leq v(g) < 0$, then $nv(g) = v(g^n) \leq v(f)$ for some natural number n. By convexity of v, $g^n \geq f \geq g$, and then $\log(g^n) = n\log(g) \geq \log(f) \geq \log(g)$. Thus $v(\log(g)) = v(\log(f))$ and $\chi(v(f)) = \chi(v(g))$.

Finally, if $v(f) < 0$ with $f \in \mathbb{T}_{\log}^{>0}$, then by Growth Axiom $v(f) < v(\log(f)) = \chi(v(f))$. Thus, by definition of χ we conclude that $|v(a)| > |\chi(a)|$ for all $a \in \Gamma^{\neq 0}$, i.e. $(\Gamma_{\log}, \chi)$ is centripetal. □

We have some useful consequences of the axioms :

Lemma 5.2. *Let (Γ, χ) be a precontraction group and $a, b \in \Gamma$.*

(1) *Axiom (4) is equivalent to the single statement $\chi(2a) = \chi(a)$.*

(2) $\chi(\Gamma^{<0}) \subseteq \Gamma^{<0}$ *and* $\chi(\Gamma^{<0}) = -\chi(\Gamma^{>0})$.

(3) $\chi(a+b) \geq \min\{\chi(a), \chi(b)\}$.

(4) *If $\chi(a) < \chi(b) < 0$ then $\chi(a-b) = \chi(a)$.*

(5) *If $0 < \chi(a) < \chi(b)$ then $\chi(b-a) = \chi(b)$.*

(6) *Let $b > 0 > a$. If $\chi(|a|) > \chi(|b)|$ then $\chi(a-b) = \chi(a)$, and if $\chi(|b|) > \chi(|a|)$ then $\chi(b-a) = \chi(b)$.*

Proof.

(1) We just have to show that the statement $\forall a \in \Gamma\ \chi(2a) = \chi(a)$ implies axiom (4). First, by axiom (2) we can observe that if $\chi(2a) = \chi(a)$ then $\chi(na) = \chi(a)$ for all $n \in \mathbb{N}$. Now, If a is archimedean equivalent to b and $\mathrm{sign}(a) = \mathrm{sign}(b)$ then there is a natural number n such that $n|a| \geq |b|$ and $n|b| \geq |a|$, so $\chi(a) = \chi(na) \geq \chi(b) = \chi(nb) \geq \chi(a)$ and thus $\chi(a) = \chi(b)$.

(2) If $a < 0$, then by axioms 1 and 2 we obtain $\chi(a) < 0$ and by axiom (3) we have $\chi(-a) = -\chi(a) > 0$.

(3) Without loss of generality we can assume that $a < b$. Then

$$a + a < a + b < b + b$$

and

$$\chi(2a) \leq \chi(a+b) \leq \chi(2b).$$

Since $\chi(2x) = \chi(x)$ for all $x \in \Gamma$, we have

$$\chi(a) \leq \chi(a + b) \leq \chi(b),$$

so $\chi(a + b) \geq \min\{\chi(a), \chi(b)\}$.

(4) $a < b < 0$ and $a - b > a$ because $\chi(a) < \chi(b) < 0$. Thus $\chi(a - b) \geq \chi(a)$. On the other hand, as $\chi(b) > \chi(a)$ and by item (3)

$$\chi(a) = \chi((a - b) + b) \geq \min\{\chi(a - b), \chi(b)\},$$

then $\chi(a) \geq \chi(a - b)$. Thus, $\chi(a - b) = \chi(a)$.

Items (5) and (6) follow of item (4).

□

Working with the natural valuation of Γ, we obtain the following immediate properties:

Lemma 5.3. *Let (Γ, χ) be a precontraction group. Then*

(1) *For all $a, b \in \Gamma$, if $v(a) \leq v(b)$ then $|\chi(a)| \geq |\chi(b)|$.*

(2) *For all $a, b \in \Gamma$, if $v(a - b) > v(a)$ then $\chi(a) = \chi(b)$.*

(3) *For all $a_1, a_2, ..., a_n \in \Gamma$, if $v(a_k) < v(a_i)$ for all $i \neq k$ then $\chi(\sum_{i=1}^{n} a_i) = \chi(a_k)$.*

(4) *(Γ, χ) is a centripetal precontraction group if and only if $v(\chi(a)) > v(a)$ for all $a \in \Gamma^{\neq 0}$.*

The main result about contraction groups proved in [14, 15] is the following:

Theorem 5.4. *In the language of ordered groups expanded by a unary function symbol for the contraction map, the theory of nontrivial divisible centripetal contraction groups is complete, decidable, admits quantifier elimination and is weakly o-minimal, and it is the model completion of the theory of centripetal precontraction groups.*

5.2 The theory T_{pdg}

A key feature of the centripetal precontraction group $(\Gamma_{\log}, \chi)$ is that the image of $\Gamma_{\log}^{<0}$ by χ is a discrete set with first element and where the immediate successor of an element $a \in \chi(\Gamma_{\log}^{<0})$ is $\chi(a)$. Thus, to capture these properties we introduce the following definition:

Definition 5.5. *Let $L_{pdg} = \{+, -, 0, <, \chi, c\}$, be the language of ordered groups augmented by a unary function symbol χ and a constant symbol c. We say that a nontrivial centripetal precontraction group (Γ, χ) is a model of the L_{pdg}-theory T_{pdg} if:*

(1) *$\chi(\Gamma^{<0})$ has a least element c,*

(2) *$\chi : \chi(\Gamma^{<0}) \to \chi(\Gamma^{<0})^{>c}$ is a bijection,*

(3) $\forall a, b \in \chi(\Gamma^{<0})$ *if* $a < b$ *then* $a < \chi(a) \leq b$

(4) Γ *is a divisible ordered group.*

From the above definition, we can see that each substructure S of a model of T_{pdg} is a centripetal precontraction group where $\chi(S^{<0})$ has a least element and $\chi(a)$ is the immediate successor of a for each $a \in \chi(S^{<0})$.

Examples.

(1) *Clearly,* $(\Gamma_{\log}, \chi)$ *is a model of* T_{pdg}*. Moreover,*

$$\chi(\Gamma_{\log}^{<0}) = \{-\ell_2, -\ell_3, ...\},$$

where $\chi(-\ell_k) = -\ell_{k+1}$ *and* $-\ell_k < -\ell_{k+1}$*.*

(2) *Let* $\oplus_i \mathbb{Q} e_i$ *be a vector space over* $\mathbb{Q}$ *with enumerable ordered basis* (e_i)*. Under the usual lexicographic order, i.e.*

$$\sum a_i e_i > 0 \textit{ iff } a_k > 0 \textit{ for the least } k \textit{ such that } a_k \neq 0,$$

$\oplus_i \mathbb{Q} e_i$ *becomes an ordered abelian group and if we define the map* $\chi_{\mathbb{Q}} : \oplus_i \mathbb{Q} e_i \to \oplus_i \mathbb{Q} e_i$ *as*

$$\chi_{\mathbb{Q}}(\sum a_i e_i) = \operatorname{sign}(a_k) e_{k+1}$$

with k *the minimal index such that* $a_k \neq 0$*, then* $(\oplus_i \mathbb{Q} e_i, \chi_{\mathbb{Q}})$ *is a model of* T_{pdg}*.*

In addition to the properties listed in Lemmas 5.2 and 5.3, we can observe that if (Γ, χ) is a model of T_{pdg}, then the discrete set $\chi(\Gamma^{<0})$ is cofinal in $\Gamma^{<0}$ since for all $a \in \Gamma^{<0}$ we have $a < \chi(a)$. Now, although in the models of T_{pdg} the map χ is not surjective and we can not proceed as in [14] to prove the model completeness of T_{pdg}, here we will use the properties of the discrete set $\chi(\Gamma^{<0})$ to do that.

5.2.1 Some algebraic properties of models of T_{pdg}

First, we can observe the following

Lemma 5.6. *If* (Γ, χ) *is a model of* T_{pdg}*, then* Γ *is a vector space over* $\mathbb{Q}$ *and* $\chi(\Gamma^{<0})$ *is a linearly independent subset of* Γ*.*

Proof. Since Γ is a divisible ordered group, it follows that Γ is a vector space over $\mathbb{Q}$. Moreover, given $q_1, q_2, ..., q_n \in \mathbb{Q}$ with $q_1 \neq 0$ and $a_1, a_2, ..., a_n \in \chi(\Gamma^{<0})$ with $a_1 < a_2 < ... < a_n$ then

$$\chi(a_1) < \chi(a_2) < ... < \chi(a_n),$$

and if $\alpha = \sum_{i=1}^{n} q_i a_i$ then by Lemma 5.2 we have that $\chi(\alpha) = \chi(a_1)$ whenever $q_1 > 0$ and $\chi(\alpha) = -\chi(a_1)$ whenever $q_1 < 0$. Thus $\alpha \neq 0$.

□

Regarding the construction of new precontraction groups we have the following:

Lemma 5.7. *Let (Γ,χ) be a centripetal precontraction group and $\Delta \subseteq \Gamma$ be a nonempty convex subgroup such that if $\chi(x) \in \Delta$ then $x \in \Delta$ for $x \in \Gamma$. Then:*

(1) *There is a unique order $\leq'$ in Γ/Δ has a such that Γ/Δ is an ordered abelian group in which if $a \leq b$ then $\overline{a} \leq' \overline{b}$ for $a, b \in \Gamma$.*

(2) *The map $\chi' : \Gamma/\Delta \to \Gamma/\Delta$ given by $\chi'(\overline{a}) = \overline{\chi(a)}$ is well defined and makes $(\Gamma/\Delta, \chi')$ a centripetal precontraction group.*

Proof. Since the (1) is a general property of ordered abelian groups, it is enough to put $\overline{a} > 0$ if and only if $a > \Delta$. First we show that χ' is well defined. To do that we prove that if $a - b \in \Delta$ then $\chi(a) - \chi(b) \in \Delta$. If $\chi(a) = \chi(b)$ then clearly $\chi(a) - \chi(b) = 0 \in \Delta$. Now, if $\chi(a) \neq \chi(b)$ then we have the following cases:

- $\chi(a) > \chi(b) > 0$. Thus, $a-b > 0$ and by centripetal property we have $a-b > \chi(a-b) > 0$. Moreover, $\chi(a-b) = \chi(a)$. So, $a - b > \chi(a) > \chi(b) > 0$ which implies $\chi(a), \chi(b) \in \Delta$ and then $\chi(a) - \chi(b)$.

- $\chi(a) < \chi(b) < 0$. Similar to the previous case.

- $a < 0 < b$. Thus $a < \chi(a) < 0 < \chi(b) < b$, $a-b < 0$ and $a-b < \chi(a-b) < 0$. Moreover, $a - b < \chi(a) - \chi(b) < 0$ and then $\chi(a) - \chi(b) \in \Delta$.

Now, since $\chi(x) \in \Delta$ implies that $x \in \Delta$, we can prove that $(\Gamma/\Delta, \chi')$ is a centripetal precontraction group. □

5.2.2 Embedding lemmas

Let $(\Gamma,\chi), (\Gamma',\chi')$ be precontraction groups. We say that (Γ',χ') is an extension of (Γ,χ) if Γ' is an extension of Γ as valued groups $\chi(\Gamma) = \chi'(\Gamma') \cap \Gamma$ and $\chi'(a) = \chi(a)$ for $a \in \Gamma$. Moreover, we say that

$$\phi : (\Gamma,\chi) \to (\Gamma',\chi')$$

is an embedding of precontraction groups if $\phi : \Gamma \to \Gamma'$ is an embedding of ordered abelian groups such that

$$\phi(\chi(a)) = \chi'(\phi(a)) \text{ for all } a \in \Gamma.$$

From this definition it follows that if $(\Gamma,\chi) \subseteq (\Gamma',\chi')$ are models of T_{pdg}, then we have the following possibilities: First we can have $\chi(\Gamma^{<0}) = \chi'(\Gamma'^{<0})$, which is always true if $[\Gamma] = [\Gamma']$ and some times when $[\Gamma] \neq [\Gamma']$. Secondly, we can have $\chi(\Gamma^{<0}) \neq \chi'(\Gamma'^{<0})$, and here we have again two possibilities: either there is $b \in \chi'(\Gamma'^{<0})$ such that $b > \chi(\Gamma^{<0})$ or there is a nonempty lower cut G in $\chi(\Gamma^{<0})$ and $b \in \chi'(\Gamma'^{<0}) \setminus \chi(\Gamma^{<0})$ such that $\chi(G) \subseteq G$ and $G < b < \Gamma^{>G}$.

Definition 5.8. *From now on, we call $G \subseteq \chi(\Gamma^{<0})$ a special cut if G is a lower cut in $\chi(\Gamma^{<0})$ such that $\chi(G) \subseteq G$ and we denote by $scut(\chi(\Gamma^{<0}))$ the collection of all special cuts of $\chi(\Gamma^{<0})$.*

Based on Gehret's work about the theory of the asymptotic couple of $\mathbb{T}_{\log}$ in [10], in the following we present some embedding lemmas which deal with the above cases and that will be used to prove the model completeness of T_{pdg}.

Case 1. $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ **with** $[\Gamma] = [\Gamma']$**.**

From [14, Lemma 3.6] we have the following result:

Lemma 5.9. *Let* (Γ, χ) *be a centripetal precontraction group. Then for each extension* $(\Gamma', <)$ *of* $(\Gamma, <)$ *such that* $[\Gamma] = [\Gamma']$, χ *extends in a unique way to a centripetal precontraction* χ' *on* Γ' *and we have* $\chi(\Gamma') = \chi(\Gamma)$. *Particularly, if* $\mathbb{Q}\Gamma = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ *is the divisible hull of* Γ *then* $\chi(\mathbb{Q}\Gamma) = \chi(\Gamma)$ *since every element in* $\mathbb{Q}\Gamma$ *is archimedean equivalent to some element of* Γ.

Using the quantifier elimination of the theory of divisible ordered abelian groups we have:

Lemma 5.10. *Let* $(\Gamma, \chi), (\Gamma', \chi')$ *and* (Γ^*, χ^*) *be models of* T_{pdg}, *such that* $(\Gamma, \chi) \subseteq (\Gamma', \chi')$, $[\Gamma] = [\Gamma']$, (Γ^*, χ^*) *is* k*-saturated for some* $k > \operatorname{card}(\Gamma')$, *and* $\phi : (\Gamma, \chi) \to (\Gamma^*, \chi^*)$ *is an embedding, then there is an embedding* $\phi' : (\Gamma', \chi') \to (\Gamma^*, \chi^*)$ *which extends* ϕ.

Proof. Since Γ, Γ' and Γ^* are divisible ordered abelian groups and such theory has quantifier elimination, by saturation of (Γ^*, χ^*) there is an embedding $\phi' : \Gamma' \to \Gamma^*$ that extends the embedding $\phi : \Gamma \to \Gamma^*$. Moreover, if $b \in \Gamma'$, because $[\Gamma] = [\Gamma']$, there is $a \in \Gamma$ such that $[a] = [b]$ and $\operatorname{sign}(a) = \operatorname{sign}(b)$. Thus, $\chi'(b) = \chi'(a)$ and then

$$\phi'(\chi'(b)) = \phi'(\chi'(a)) = \phi(\chi(a)),$$

but as $[\phi'(b)] = [\phi(a)]$ in $[\Gamma']$, then $\chi^*(\phi'(b)) = \chi^*(\phi(a))$. Finally, since ϕ is an embedding of centripetal divisible precontraction groups, we obtain

$$\chi^*(\phi(a)) = \phi(\chi(a)) = \phi'(\chi'(b)).$$

□

Case 2. $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ **with** $[\Gamma] \neq [\Gamma']$ **and** $\chi(\Gamma^{<0}) = \chi'(\Gamma'^{<0})$**.**

From [14, Lemma 3.3] we know that:

Lemma 5.11. *Let* $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ *be precontraction groups. Let* $a \in \Gamma'$ *such that* $[a] \notin [\Gamma]$ *and* $\chi'(a) = b \in \Gamma$. *Then* $(\Gamma + \mathbb{Z}a, \chi_a)$ *is a precontraction group with* $\chi_a(\Gamma + \mathbb{Z}a) = \chi(\Gamma) \cup \{b, -b\}$ *in* Γ. *Moreover, the extension of* χ *from* (Γ, χ) *to* $\Gamma + \mathbb{Z}a$ *is uniquely determined by the assignment* $\chi'(a) = b$.

If Γ is divisible, $\Gamma + \mathbb{Q}a$ is the divisible hull of $\Gamma + \mathbb{Z}a$. Thus, by Lemmas 5.9 and 5.11 we have $[\Gamma + \mathbb{Q}a] = [\Gamma + \mathbb{Z}a]$ and the image under χ_a coincide. From this we have the following lemma:

Lemma 5.12. *Let* $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ *be models of* T_{pdg} *with* $\chi(\Gamma^{<0}) = \chi'(\Gamma'^{<0})$, $a \in \Gamma'^{<0}$ *such that* $[a] \notin [\Gamma]$ *and* C *is the lower cut in* $[\Gamma]$ *defined by* $[a]$. *Then there is a model* (Δ, χ_Δ) *of* T_{pdg} *such that:*

(1) $(\Gamma, \chi) \subset (\Delta, \chi_\Delta) \subseteq (\Gamma', \chi')$ *with* $[a] \in [\Gamma_\Delta]$, *and*

(2) *for any embedding* ϕ *of* (Γ, χ) *into a model* (Γ^*, χ^*) *of* T_{pdg} *and each* $a' \in \Gamma^{*<0}$ *with* $[a'] \notin [\phi(\Gamma)]$ *which realize the cut* $\{\phi(x) : x \in C\}$, *there is a unique embedding* ϕ' *from* (Δ, χ_Δ) *into* (Γ^*, χ^*) *that extends* ϕ *with* $\phi'(a) = a'$.

Proof. Let $a \in \Gamma'^{<0}$ with $[a] \notin [\Gamma]$ and $b = \chi'(a) \in \Gamma$. We define $(\Delta, \chi_\Delta) = (\Gamma + \mathbb{Q}a, \chi'_a)$ where χ'_a is the restriction of χ' to $\Gamma + \mathbb{Q}a$. As $\chi(\Gamma) = \chi'(\Gamma)$ then $\chi_\Delta(\Delta) = \chi(\Gamma)$, so $\chi_\Delta(\Delta)$ is a centripetal divisible precontraction group. □

Case 3. $(\Gamma,\chi) \subseteq (\Gamma',\chi')$ **with** $\chi(\Gamma^{<0}) \neq \chi'(\Gamma'^{<0})$**.**

As we saw above if $(\Gamma,\chi) \subseteq (\Gamma',\chi')$ are model of T_{pdg} and $\chi(\Gamma^{<0}) \neq \chi'(\Gamma'^{<0})$, we have two cases. First, we can have that there is $b \in \chi'(\Gamma'^{<0})$ such that $b > \chi(\Gamma^{<0})$. So we want to extend (Γ,χ) to a model (Δ,χ_Δ) of T_{pdg} in which $b \in \chi_\Delta(\Delta^{<0})$. To do that, we can observe that if $b > \chi(\Gamma^{<0})$, then $\chi'^k(b) > \chi(\Gamma^{<0})$ for any integer k, where $\chi^{n+1}(b) = \chi(\chi^n(b))$, $\chi^0(b) = b$ and $\chi^{-n}(b) = c$ means that $\chi^n(c) = b$. Thus, to define the model (Δ,χ_Δ) we need to add a copy of $\mathbb{Z}$ at the end of $\Gamma^{<0}$. Specifically, we have:

Lemma 5.13. *Let $(\Gamma,\chi) \subseteq (\Gamma',\chi')$ be divisible centripetal precontraction groups and $(b_n)_{n\geq 0}$ a family in $\chi(\Gamma'^{<0})$ such that $b_{n+1} = \chi'(b_n)$ and $b_n > \chi(\Gamma)$ for all $n \geq 0$, then there is a divisible centripetal precontraction group (Γ'',χ'') such that:*

1. $(\Gamma,\chi) \subset (\Gamma'',\chi'') \subseteq (\Gamma',\chi')$ *with $b_n \in \chi''(\Gamma''^{<0})$ for $n > 1$, and*
2. *for any embedding ϕ of (Γ,χ) into a divisible centripetal precontraction group (Γ^*,χ^*) and any family $(b'_n)_{n\geq 0}$ in $\chi^*(\Gamma^{*<0})$ such that $b'_{n+1} = \chi^*(b'_n)$ and $b_n > \phi(\chi(\Gamma^{<0}))$ for $n \geq 0$, there is a unique embedding $\phi' : (\Gamma'',\chi'') \to (\Gamma^*,\chi^*)$ which extends ϕ and such that $\phi'(b_n) = b'_n$ for all n.*

Proof. Let $((\Gamma_i,\chi_i)_{i\geq 0})$ be the family given by $\Gamma_0 = \Gamma$, $\Gamma_{n+1} = \Gamma_n + \mathbb{Q}b_n$ and χ_n the restriction of χ' to Γ_n. By Lemma 5.11 (Γ_n,χ_n) is a divisible precontraction group for each n and since

$$(\Gamma_n,\chi_n) \subseteq (\Gamma_{n+1},\chi_n)$$

and $\chi'(b_n) = b_{n+1}$, we have

$$(\Gamma'',\chi'') = \cup_{i\geq 0}(\Gamma_i,\chi_i) \subseteq (\Gamma',\chi')$$

is a divisible centripetal precontraction group which extends (Γ,χ). Now, by induction if we assume that $\phi_n : (\Gamma_n,\chi_n) \to (\Gamma^*,\chi^*)$ is an embedding such that $\phi_n(b_i) = b'_i$ for i in $\{0,1,...,b_{n-1}\}$, then by Lemma 5.11 there is a unique embedding

$$\phi_{n+1} : (\Gamma_{n+1},\chi_{n+1}) \to (\Gamma^*,\chi^*)$$

which extends ϕ_n and such that $\phi_{n+1}(b_n) = b'_n$. Thus, there is a unique embedding

$$\phi' = \cup_{i\geq 0}\phi_i : (\Gamma'',\chi'') \to (\Gamma^*,\chi^*)$$

which satisfies the required properties. □

Now, we use the above lemma to include the predecessors of the element b_0 of the family:

Lemma 5.14. *Let $(\Gamma,\chi) \subseteq (\Gamma',\chi')$ be divisible centripetal precontraction groups and $(b_k)_{k\in\mathbb{Z}}$ a family in $\chi(\Gamma'^{<0})$ such that $b_{k+1} = \chi'(b_k)$ and $b_k > \chi(\Gamma)$ for all $k \in \mathbb{Z}$, then there is a divisible centripetal precontraction group (Δ,χ_Δ) such that:*

1. $(\Gamma,\chi) \subset (\Delta,\chi_\Delta) \subseteq (\Gamma',\chi')$ *with $b_k \in \chi_\Delta(\Delta^{<0})$ for all $k \in \mathbb{Z}$, and*
2. *for any embedding ϕ of (Γ,χ) into a divisible centripetal precontraction group (Γ^*,χ^*) and any family $(b'_k)_{k\in\mathbb{Z}}$ in $\chi^*(\Gamma^{*<0})$ such that $b'_{k+1} = \chi^*(b'_k)$ and $b_k > \phi(\chi(\Gamma^{<0}))$ for $k \in \mathbb{Z}$, there is a unique embedding $\phi' : (\Delta,\chi_\Delta) \to (\Gamma^*,\chi^*)$ which extends ϕ and such that $\phi'(b_k) = b'_k$ for all k.*

Proof. First for each $n \in \mathbb{N}$ we define the family $(a_i^n)_{i\geq 0}$ where $a_i^n = b_{-n+i}$ for $i \geq 0$. Clearly, we have that $a_{i+1}^n = \chi'(a_i^n)$ and $a_{i+1}^{n+1} = a_i^n$. Now, using the Lemma 5.13 for each family $(a_i^n)_{i\geq 0}$ we obtain a divisible centripetal precontraction group (Γ''_n, χ''_n) such that $a_{i+1}^n \in \chi''_n(\Gamma''^{<0}_n)$ and $\chi''_n(a_i^n) = a_{i+1}^n$ and a unique embedding $\psi_n : (\Gamma''_n, \chi''_n) \to (\Gamma''_{n+1}, \chi''_{n+1})$ with $\psi_n(a_i^n) = a_{i+1}^{n+1}$. Thus we obtain the increasing chain

$$(\Gamma, \chi) \subset (\Gamma''_0, \chi''_0) \subset (\Gamma''_1, \chi''_1) \subset ...$$

and we define $(\Delta, \chi_\Delta) = \cup_{n\geq 0}(\Gamma''_n, \chi''_n)$.

Now, if $\phi : (\Gamma, \chi) \to \chi^*(\Gamma^{*<0})$ is an embedding with $(b'_k)_{k\in\mathbb{Z}}$ a family in $\chi^*(\Gamma^{*<0})$ such that $b'_{k+1} = \chi^*(b'_k)$ and $b_k > \phi(\chi(\Gamma^{<0}))$ for $k \in \mathbb{Z}$, then by Lemma 5.13 there is a unique embedding $\phi_n : (\Gamma''_n, \chi''_n) \to (\Gamma^*, \chi^*)$ that extends ϕ and such that $\phi_n(a_i^n) = \phi_n(b_{-n+i}) = b'_{-n+i}$. Moreover, $\phi_n \subseteq \phi_{n+1}$ because

$$\phi_{n+1}(a_i^n) = \phi_{n+1}(a_{i+1}^{n+1}) = b'_{-(n+1)+i+1} = b'_{-n+i} = \phi_n(a_i^n).$$

Thus we have that $\phi' = \cup\phi_n$ is the unique embedding from (Δ, χ_Δ) into (Γ^*, χ^*) that extends ϕ and such that $\phi'(b_k) = b'_k$ for all $k \in \mathbb{Z}$.

□

On the other hand, if $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ are models of T_{pdg}, $\chi(\Gamma^{<0}) \neq \chi'(\Gamma'^{<0})$ and there is a non-empty special cut G in $\chi(\Gamma^{<0})$ and $b \in \chi(\Gamma'^{<0}) \setminus \chi(\Gamma^{<0})$ such that $G < b < \Gamma^{>G}$, then there is a family $(b_k)_{k\in\mathbb{Z}}$ in $\chi'(\Gamma'^{<0})$ such that $G < b_k < \Gamma^{>G}$ and $b_{k+1} = \chi'(b_k)$. So, in order to extend (Γ, χ) to a model (Δ, χ_Δ) of T_{pdg} in which $b \in \chi_\Delta(\Delta^{<0})$ we have to add a copy of $\mathbb{Z}$ between some specific elements of $\chi(\Gamma)$.

Lemma 5.15. *Let $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ be divisible centripetal precontraction groups, G be a nonempty special cut in $\chi(\Gamma^{<0})$ and $(b_k)_{k\in\mathbb{Z}}$ be a family in $\chi'(\Gamma'^{<0})$ such that $G < b_k < \Gamma^{>G}$ with $b_{k+1} = \chi'(b_k)$, then there is a divisible centripetal precontraction group (Δ_G, χ_G) such that:*

(1) *$(\Gamma, \chi) \subset (\Delta_G, \chi_G) \subseteq (\Gamma', \chi')$ with $b_k \in \chi_G(\Delta_G^{<0})$ for all $k \in \mathbb{Z}$, and*

(2) *for any embedding ϕ of (Γ, χ) into a divisible centripetal precontraction group (Γ^*, χ^*) and any family $(b'_k)_{k\in\mathbb{Z}}$ in $\chi^*(\Gamma^{*<0})$ such that $b'_{k+1} = \chi^*(b'_k)$ and $\phi(G) < b'_k < \phi(\Gamma^{>G})$, there is a unique embedding $\phi' : (\Delta_G, \chi_G) \to (\Gamma^*, \chi^*)$ which extends ϕ and such that $\phi'(b_k) = b'_k$ for all k.*

Proof. It is enough to take $\Delta_G = \Gamma + \oplus_{k\in\mathbb{Z}}\mathbb{Q}b_k$ and χ_G the restriction of χ' to Δ_G. □

Under the hypothesis of the above lemma, for any element $a \in \Gamma^{<0} \setminus G$ we have $b_k < a < 0$ for all $k \in \mathbb{Z}$, and by item 2 of Lemma 5.2, we obtain

$$\chi'(b_k - a) = b_k.$$

Thus, taking $a_k = b_k - a$ we can define $\Delta_G = \Gamma + \oplus_{k\in\mathbb{Z}}\mathbb{Q}a_k$ and χ_G the restriction of χ' to Δ_G, with $b_k \in \chi_G(\Delta_G^{<0})$.

5.2.3 Model completeness of T_{pdg}

To prove the model completeness of T_{pdg} we use the the test 3.9 given in chapter 3. Thus, the desired result is a consequence of the following theorem:

Theorem 5.16. *Let $(\Gamma,\chi),(\Gamma',\chi')$ and (Γ^*,χ^*) be models of T_{pdg}, such that $(\Gamma,\chi)\subseteq(\Gamma',\chi')$ and (Γ^*,χ^*) is a k-saturated elementary extension of (Γ,χ), with $k>\mathrm{card}(\Gamma')$. Then there is a submodel (Δ,χ_Δ) of (Γ',χ') which properly extends (Γ,χ) such that (Δ,χ_Δ) embeds over (Γ,χ) in $(\Gamma^*,\chi*)$.*

Proof. We call ϕ the embedding of (Γ,χ) into $(\Gamma^*,\chi*)$ and just consider the following cases:

(1) $[\Gamma]=[\Gamma']$: By Lemma 5.10 it is enough to take $(\Delta,\chi_\Delta)=(\Gamma',\chi')$.

(2) $[\Gamma]\neq[\Gamma']$ and $\chi(\Gamma)=\chi'(\Gamma)$: By hypothesis there is an element $a\in\Gamma'^{<0}$ such that $[a]\notin[\Gamma]$, and by Lemma 5.12 there is a model $(\Delta,\chi_\Delta)\subseteq(\Gamma',\chi')$ of T_{pdg} that properly extends (Γ,χ). By saturation we can extend the embedding $\phi:(\Gamma,\chi)\to(\Gamma^*,\chi*)$ to an embedding ϕ' from (Δ,χ_Δ) in $(\Gamma^*,\chi*)$.

(3) $\chi(\Gamma^{<0})\neq\chi'(\Gamma^{<0})$ and there is $b\in\chi(\Gamma'^{<0})$ such that $b>\chi(\Gamma^{<0})$: If we define the family $(b_k)_{k\in\mathbb{Z}}$ of $\chi(\Gamma'^{<0})$ by $b_0=b$, $b_{k+1}=\chi'(b_k)$ for $k>0$ and b_{k-1} as the unique element of $\chi(\Gamma'^{<0})$ such that $\chi(b_{k-1})=b_k$ for $k<0$, then by Lemma 5.14 there is a model $(\Delta,\chi_\Delta)\subseteq(\Gamma',\chi')$ of T_{pdg} which properly extends (Γ,χ) and such that $b_k\in\chi_\Delta(\Delta^{<0})$.

   Using the saturation of $(\Gamma^*,\chi*)$ we can find a family $(b'_k)_{k\in\mathbb{Z}}$ in $(\Gamma^*,\chi*)$ such that $b'_k>\phi(\chi(\Gamma^{<0}))$ for all $k\in\mathbb{Z}$ and $b_{k+1}=\chi^*(b_k)$. Thus, again by Lemma 5.14 there is a unique embedding
$$\phi':(\Delta,\chi_\Delta)\to(\Gamma^*,\chi*)$$
   that extends ϕ and such that $\phi'(b_k)=b'_k$.

(4) There is $b\in\chi'(\Gamma'^{<0})\setminus\chi(\Gamma)$ such that b realize a special cut in $\chi(\Gamma^{<0})$: We define the set
$$G=\{a\in\chi(\Gamma^{<0}:a<b\}.$$
   Since the models of T_{pdg} are centripetal precontraction groups, we have $\chi(G)\subseteq G$ and by axioms (3) and (4) there is a family $(b_k)_{k\in\mathbb{Z}}$ in $\chi'(\Gamma'^{<0})$ such that $G<b_k<\Gamma^{>G}$, $b_0=b$, $b_{k+1}=\chi'(b_k)$ for $k>0$ and b_{k-1} is the unique element of $\chi'(\Gamma'^{<0})$ such that $\chi(b_{k-1})=b_k$ for $k<0$ then by Lemma 5.15 there is a model $(\Delta,\chi_\Delta)=(\Delta_G,\chi_G)\subseteq(\Gamma',\chi')$ of T_{pdg} which properly extends (Γ,χ) and such that $b_k\in\chi_\Delta(\Delta^{<0})$.

   By saturation there is a family $(b'_k)_{k\in\mathbb{Z}}$ in $(\Gamma^*,\chi*)$ such that $\phi(G)<b'_k<\phi(\Gamma^{>G})$, $b_{k+1}=\chi^*(b_k)$ for all $k\in\mathbb{Z}$, and again by Lemma 5.15 there is a unique embedding $\phi':(\Delta,\chi_\Delta)\to(\Gamma^*,\chi*)$ that extends ϕ and such that $\phi'(b_k)=b'_k$.

□

Corollary 5.17. *T_{pdg} is model complete.*

Now, we can observe that the model $(\oplus_i\mathbb{Q}e_i,\chi_\mathbb{Q})$ of T_{pdg} defined in the first example of section 5.2 embeds in any model (Γ,χ') of T_{pdg} since we can take any element $b\in\chi'(\Gamma^{<0})$, define the family $(b_n)_{n>0}$ such that $b_1=b$ and $b_{n+1}=\chi'(b_n)$, and identify the element $-e_n$ with the element b_n for all $n\geq1$. Thus we obtain that T_{pdg} has a prime model and:

Corollary 5.18. *T_{pdg} is complete.*

5.2.4 Quantifier elimination of T_{pdg*}

Expanding the language L_{pdg} to $L_{pdg*} = L_{pdg} \cup \{\infty, \chi^{-1}, \delta_1, \delta_2, \delta_3, ...\}$, where ∞ is a constant symbol, χ^{-1} and δ_n for $n > 0$ are unary function symbols, each model (Γ, χ) of T_{pdg} can be seen as a L_{pdg*}-structure with underlying set $\Gamma_\infty = \Gamma \cup \{\infty\}$ in which:

- ∞ is such that $\infty \notin \Gamma$, $\infty + \infty = \chi(\infty) = -\infty = \infty$ and for all $x \in \Gamma$ we have $x + \infty = \infty$, and
- we interpret δ_n as division by n and χ^{-1} as a function from Γ_∞ to Γ_∞ such that its restriction
$$\chi^{-1} : \chi(\Gamma^{<0})^{>c} \to \chi(\Gamma^{<0})$$
is the inverse of $\chi : \chi(\Gamma^{<0}) \to \chi(\Gamma^{<0})^{>c}$, $\chi^{-1}(0) = 0, \chi^{-1}(c) = \infty$ and $\chi^{-1}(a) = \infty$ for all a in $\Gamma_\infty^{\neq 0} \setminus \chi(\Gamma)$.

Thus, we define the theory T_{pdg*} as the L_{pdg*}-theory whose models are the expansion of models of T_{pdg}.

Now, we observe that each L_{pdg*}-substructure of a model of T_{pdg*} has a T_{pdg*}-closure in the following sense:

Lemma 5.19. *Let (Γ, χ) be a model of T_{pdg*} and (Γ_0, χ_0) be a L_{pdg*}-substructure of (Γ, χ). There is a model (Γ', χ') of T_{pdg*} such that*

(1) *$(\Gamma', \chi') \subseteq (\Gamma, \chi)$, and*

(2) *(Γ', χ') can be embedded over (Γ_0, χ_0) into every model of T_{pdg*} which extends (Γ_0, χ_0).*

Proof. If there is $a \in \Gamma_0$ such that $\chi(a) = c$, then in fact (Γ_0, χ_0) is a model of T_{pdg*} and we finish. Otherwise, there is $a \in \Gamma^{<0}$ such that $\chi(a) = c$, so we define Γ' as the divisible ordered abelian group generated by $\Gamma_0 \cup \{a\}$, and $\chi' = \chi|_{\Gamma'}$. Thus, (Γ', χ') is a model of T_{pdg*}.

Finally, given any model (Γ^*, χ^*) of T_{pdg*} which extends (Γ_0, χ_0), there is $b \in \Gamma^*$ such that $\chi(b) = c$. We see that a and b have the same type over Γ_0. Thus, we define the embedding $\phi : (\Gamma', \chi') \to (\Gamma^*, \chi^*)$ as $\phi(\Gamma_0) = \Gamma_0$ and $\phi(a) = b$.

□

As a consequence of this lemma and mimicking the proof of the Theorem 5.16, but considering L_{pdg*}-structures instead of L_{pdg}-structures, we can prove that the L_{pdg*}-theory T_{pdg*} has quantifier elimination.

5.2.5 Definable subsets of $\chi(\Gamma^{<0})$

In this section we mimic the study made by Gehret in [2] about some definable sets in the asymptotic couple of $\mathbb{T}_{\log}$ and show that given a model (Γ, χ) of T_{pdg*}, each definable subset of $\chi(\Gamma^{<0})$ is a finite union of intervals in $\chi(\Gamma^{<0})$ and singletons. To prove such result we will introduce a special kind of functions called χ-functions[1].

For any element $a \in \chi(\Gamma^{<0})$ and integer $k < 0$ we put $\chi^k(x) = (\chi^{-1})^{-k}(x)$ and $\chi^0(x) = x$.

[1] The notion of χ-function used here was inspired in the notion of χ-polynomial defined in [15]

Definition 5.20. *We say that a function $G : \chi(\Gamma^{<0}) \to \Gamma$ is a χ-function if it is constant or*

$$G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) - \alpha$$

for some $n > 0$, $k_1 < k_2 < ... < k_n$ in $\mathbb{Z}$, $q_1, ..., q_n \in \mathbb{Q}^{\neq 0}$ and $\alpha \in \Gamma$.

Since for each $k \in \mathbb{Z}^{<0}$, the χ-function $\chi^k(x)$ has image ∞ for $x < \chi^{-k}(c)$ with $x \in \chi(\Gamma^{<0})$, and it is injective and strictly increasing in $\chi(\Gamma^{<0})_k = \{x \in \chi(\Gamma^{<0}) : x \geq \chi^{-k}(c)\}$, then if for any χ-function $G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) + \alpha$ we define

$$\mathrm{Dom}_G = \begin{cases} \chi(\Gamma^{<0})_{k_1} \text{ if } k_1 < 0 \\ \chi(\Gamma^{<0}) \text{ if } k_1 \geq 0 \end{cases}$$

then we have:

Lemma 5.21. *Let $G : \chi(\Gamma^{<0}) \to \Gamma$ be the χ-function given by $G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) + \alpha$, then*

1. *$G(a) = \infty$ for any $a \in \chi(\Gamma^{<0}) \setminus \mathrm{Dom}_G$.*
2. *$G(x)$ is injective on Dom_G.*
3. *If $q_1 > 0$ then $G(x)$ is strictly increasing on Dom_G, and if $q_1 < 0$ then $G(x)$ is strictly decreasing on Dom_G.*

Proof. (1) If $k_1 < 0$ then $\chi^{k_1}(a) = \infty$ for all $a < \chi^{-k_n}(c)$. Now, if $k_n > 0$, then the proof is immediate.

(2) If $x \in \mathrm{Dom}_G$ then $\chi^{k_1}(x) < \chi^{k_2}(x) < ... < \chi^{k_n}(x)$. So, if $y, x \in \mathrm{Dom}_G \subseteq \chi(\Gamma^{<0})$ with $y \neq x$, then $\chi^{k_i}(y) \neq \chi^{k_i}(x)$ for all $1 \leq i \leq n$, and by Lemma 5.6 we have that $G(x) \neq G(y)$.

(3) If $a, b \in \mathrm{Dom}_G$ with $a < b$, then $[a] < [b]$, $\chi(a) < \chi(b)$ and by Lemma 5.2 $\chi(a-b) = \chi(a)$. Thus, for all $i, j \in \mathbb{Z}$ with $i < j$ we have that

$$[\chi^i(b) - \chi^i(a)] > [\chi^j(b) - \chi^j(a)]$$

and then

$$[\chi^{k_1}(b) - \chi^{k_1}(a)] > [\chi^{k_2}(b) - \chi^{k_2}(a)] > ... > [\chi^{k_n}(b) - \chi^{k_n}(a)].$$

So, since $\chi^{k_1}(b) > \chi^{k_1}(a)$, we have that $G(b) - G(a) = \sum_{i=1}^{n} q_i(\chi^{k_i}(b) - \chi^{k_i}(a)) > 0$ if and only if $q_1 > 0$.

□

Since by Lemma 5.6 we know that $\chi(\Gamma^{<0})$ is a linearly independent subset of Γ as $\mathbb{Q}$-vector space, depending on the constant value α we estimate how many images has the restriction of the χ-function

$$G(x) = \sum_{i=1}^{n} q_i \chi^{K_i}(x) - \alpha$$

to Dom_G in $\chi(\Gamma^{<0})$:

Lemma 5.22. *Given the χ-function $G(x) = \sum_{i=1}^{n} q_i \chi^{K_i}(x) + \alpha$ then we have one of the following possibilities:*

(1) $\alpha = 0$, $n = 1$, $q_1 = 1$ *and* $G(\chi(\Gamma^{<0})) \subseteq \chi(\Gamma^{<0})$, *or*

(2) $\mathrm{card}(G(\mathrm{Dom}_G) \cap \chi(\Gamma^{<0})) \leq 2$.

Proof. Considering the element α we have two main cases: α does not belongs to $\mathrm{span}_{\mathbb{Q}}\, \chi(\Gamma^{<0})$ or it does. In the first case, $G(x) \notin \chi(\Gamma^{<0})$ for all $x \in \mathrm{Dom}_G$. In the second case we can assume that for some natural $m > 0$ there are $r_1, r_2, ..., r_m \in \mathbb{Q}$ and $a_1, a_2, ..., a_m \in \chi(\Gamma^{<0})$ with $a_1 < a_2 < ... < a_n$ such that $\alpha = r_1 a_1 + r_2 a_2 + ... + r_m a_m$. Clearly, if $x \in \mathrm{Dom}_G$ then $G(x) \in \chi(\Gamma^{<0})$ if and only if $G(x) = \chi^{k_h}(x)$ for some $1 \leq h \leq n$ or $G(x) = a_s$ for some $1 \leq s \leq m$, which is possible only if all components except one of $G(x)$ are cancelled. We analyze the possible cases:

- If $m = 0$, i.e $\alpha = 0$ and $n = 1$, $q_1 = 1$ then $G(x) = \chi^{k_1}(x) \in \chi(\Gamma^{<0})$ for all $x \in \mathrm{Dom}_G$.
- If $|m - n| > 1$ then for each element x of Dom_G the value of $G(x)$ is a linear combination of at least two elements of $\chi(\Gamma^{<0})$. Thus, $G(\mathrm{Dom}_G) \cap \chi(\Gamma^{<0}) = \emptyset$.
- If $m = n + 1$, then $G(x)$ belongs $\chi(\Gamma^{<0})$ only if
$$\mathrm{card}(\{a_1, a_2, ..., a_m\} \cap \{\chi^{k_i}(x) : i \in \{1, 2, ..., n\}\}) = n.$$
Thus if $G(x) \in \chi(\Gamma^{<0})$ we have only two possibilities or $\chi^{k_1}(x) = a_1$ or $\chi^(k_n) = a_m$. So, since G is injective on Dom_G, we have $\mathrm{card}(G(\mathrm{Dom}_G) \cap \chi(\Gamma^{<0})) \leq 2$.
- If $m = n$, then $G(x)$ belongs $\chi(\Gamma^{<0})$ only if
$$\mathrm{card}(\{a_1, a_2, ..., a_m\} \cap \{\chi^{k_i}(x) : i \in \{1, 2, ..., n\}\}) = n$$
or equivalent $\chi^{k_i}(x) = a_i$ for all $1 \leq i \leq n$. Since G is injective on Dom_G, we have
$$\mathrm{card}(G(\mathrm{Dom}_G) \cap \chi(\Gamma^{<0})) = 1.$$
- If $n = m + 1$, then analysis is similar to the case $m = n + 1$.

□

Clearly if $G(x)$ and $H(x)$ are two χ-functions then $G(x) + H(x)$, $G(x) - H(x)$ and $\delta_n(G(x))$ for all $n > 0$ are again χ-functions. Thus

Lemma 5.23. *The set of χ-functions is closed under $+, -, \delta_n$.*

On the other hand, although the composition $\chi(G(x))$ of χ and a χ-function $G(x)$ is not necessarily a χ-function, we can prove that $\chi(G(x))$ is given piecewise by χ-functions (Lemma 5.24), which means that there are $a_1, a_2, ..., a_n \in \chi(\Gamma^0) \cup \{0\}$ with $c = a_1 < a_2 < ... < a_n = 0$ such that for any $i \in \{1, 2, ..., n-1\}$ the restriction of $\chi(G(x))$ to $[a_i, a_{i+1})_\chi$ is given by a χ-function.

To prove this, we first observe that by Lemma 5.6, for any element $\theta = \sum_{i=1}^{n} q_i a_i$ of Γ where $q_1, q_2, ..., q_n \in \mathbb{Q}^{\neq 0}$ and $a_1, a_2, ..., a_n \in \chi(\Gamma^{<0})$ with $a_1 < a_2 < ... < a_n$, we have that $\chi(\theta) = \chi(a_1)$ if $q_1 > 0$ and $\chi(\theta) = -\chi(a_1)$ if $q_1 < 0$. Thus we have:

Lemma 5.24. *Let $G(x)$ be a χ-function. Then $\chi(G(x))$ is given piecewise by χ-functions.*

Proof. If $G(x)$ is constant, then $\chi(G(x))$ is also a constant, which means that $\chi(G(x))$ is a χ-function. And if $G(x) = \sum_{i=1}^{n} q_i\chi^{k_i}(x) + \alpha$ then clearly, for all $x \in \chi(\Gamma^{<0}) \setminus \mathrm{Dom}_G$ we have $\chi(G(x)) = \chi(\infty) = \infty$ which is constant. So, from now on $G(x)$ will be a χ-function of the form $G(x) = \sum_{i=1}^{n} q_i\chi^{k_i}(x) + \alpha$ and we will focus on the values of G on Dom_G.

If $\alpha = 0$ by the above lemma $\chi(G(x)) = \mathrm{sign}(q_1)\chi(\chi^{k_1}(x))$ for all $x \in \mathrm{Dom}_G$. Putting now $\alpha \neq 0$ and $\theta(x) = \sum_{i=1}^{n} q_i\chi^{K_i}(x)$ we have $\chi(G(x)) = \chi(\theta(x) + \alpha)$.

Without loss of generality we can assume $q_1 > 0$. Thus $\chi(\theta(x)) = \chi(\chi^{k_1}(x))$ for all $x \in \mathrm{Dom}_G$, and there is a unique $x_0 \in \chi(\Gamma^{<0})$ such that $|\chi(\alpha)| = |x_0|$. Thus we have two possibilities:

(1) $|\chi(\theta(x))| \neq |x_0|$ for all $x \in \mathrm{Dom}_G$. If $\chi(\alpha) = x_0$ then either $x_0 < \chi(\theta(x))$ for all $x \in \mathrm{Dom}_G$ and $\chi(G(x)) = x_0$ for all $x \in \mathrm{Dom}_G$, or there is a unique $x_1 \in \mathrm{Dom}_G$ such that
$$\chi(\theta(x_1)) < x_0 < \chi(\theta(\chi(x_1))$$
and
$$\chi(G(x)) = \begin{cases} [\chi(\chi^{k_1}(x)) \text{ if } x < x_1 \\ x_0 \text{ if } x > x_1 \end{cases}$$
Now, if $\chi(\alpha) = -x_0$ then $\chi(G(x)) = \chi(\chi^{k_1}(x))$ for all $x \in \mathrm{Dom}_G$.

(2) There is a unique $x_1 \in \mathrm{Dom}_G$ such that $|\chi(\theta(x_1))| = |x_0|$. We can see that $\chi(G(x))$ has the same behavior for all $x \neq x_1$ that in the previous case . However, if $x = x_1$ then we have the following cases: If $\chi(\alpha) = x_0$ then $\chi(G(x)) = x_0$, but if $\chi(\alpha) = -x_0$ then: Let $\alpha_1 = \alpha + q_1\chi^{k_1}(x)$. If $\chi(\alpha_1) = \chi(\chi^{k_1}(x))$ then $\chi(G(x)) = \chi(\chi^{k_1}(x))$. In other case, we compare $|\chi(\chi^{k_2}(x))|$ with $|\chi(\alpha_1)|$. If $|\chi(\chi^{k_2}(x))| \neq |\chi(\alpha_1)|$ then the value of $G(x)$ is determined by the $\min\{\mathrm{sign}(q_2)\chi(\chi^{k_2}(x)), \chi(\alpha_1)\}$. If $|\chi(\chi^{k_2}(x))| = |\chi(\alpha_1)|$ then we have two cases, if $\mathrm{sign}(q_2)\chi(\chi^{k_2}(x)) = \chi(\alpha_1)$ then $\chi(G(x)) = \mathrm{sign}(q_2)\chi(\chi^{k_2}(x))$, but if not, then we define $\alpha_2 = \alpha_1 + q_2\chi^{k_2}(x)$ and repeat the analysis done for α_1. This process is finite because in the possible last step we analyze $\alpha_n = \alpha_{n-1} + q_n\chi^{k_n}(x)$.

In conclusion, for each χ-function $G(x) = \sum_{i=1}^{n} q_i\chi^{k_i}(x) + \alpha$, $\chi(G(x))$ is given piecewise by χ-functions.

□

From Lemmas 5.22, 5.23 and 5.24 we obtain:

Proposition 5.25. *Let $t(x) : \Gamma \to \Gamma$ be an L_{pdg*}-term and $G : \chi(\Gamma^{<0}) \to \Gamma$ the restriction of t to $\chi(\Gamma^{<0})$. Then G is given piecewise by χ-functions.*

Proof. The proof follows from Lemmas 5.22, 5.23 and 5.24 doing induction on the complexity of the L_{pdg*}-terms. □

As a consequence of this proposition and the quantifier elimination in T_{pdg*} we have:

Corollary 5.26. *Every definable $A \subseteq \chi(\Gamma^{<0})$ is a finite union of intervals in $\chi(\Gamma^{<0})$ and singletons.*

Remark. *For each model (Γ, χ) of T_{pdg*}, the definable set $\chi(\Gamma^{<0})$ is infinite and discrete, so (Γ, χ) is not weakly o-minimal.*

Now, if we expand the language L_{pdg*} by a new constant symbol d and define the theory T_{pdg**} as

$$T_{pdg*} \cup \{\chi(d) = c\}$$

then T_{pdg**} has quantifier elimination and a universal axiomatization. Thus, from Proposition 5.25 we have the following:

Theorem 5.27. *Let $G : \chi(\Gamma^{<0}) \to \Gamma$ be a definable function. Then G is given piecewise by χ-functions.*

Proof. Since T_{pdg**} has quantifier elimination and has a universal axiomatization, by Corollary B.11.15 of [4] there are L_{pdg**}-terms $t_1(x), t_2(x), ..., t_n(x)$ such that $G(x) = t_k(x)$ for $x \in \chi(\Gamma^0)$ and some $k \in \{1, 2, ..., n\}$. Thus, by Proposition 5.25 the restriction of $G(x)$ to

$$\mathrm{Dom}_k = \{x \in \chi(\Gamma^{<0}) : G(x) = t_k(x)\} \subseteq \chi(\Gamma^0),$$

is given piecewise by χ-functions.

□

5.2.6 Simple extensions

Let $\mathbb{M} = (M, \chi_M)$ be a monster model of T_{pdg*} and (Γ, χ) a small submodel of $\mathbb{M}$. In this section we show that each simple extension $\Gamma\langle a\rangle$ for $a \in M \setminus \Gamma$ of Γ is isomorphic to a specific extension of Γ utilizing the extensions given in Lemmas 5.14 and 5.15.

To do that, we first combine Lemmas 5.14 and 5.15 to define extensions of Γ which are built including many copies of $\mathbb{Z}$ in a specific and ordered way. Let $scut^{op}(\chi(\Gamma^{<0}))$ be the set of subsets $G \subseteq \chi(\Gamma^{<0})$ such that $\chi(\Gamma^{<0}) \setminus G$ is a special cut of $\chi(\Gamma^{<0})$, ordered by $G_1 \leq G_2$ if and only if $G_2 \subseteq G_1$ for $G_1, \mathbf{G}_2 \in scut^{op}(\chi(\Gamma^{<0}))$. Then given an ordinal δ and an increasing function $f : \delta \to scut^{op}(\chi(\Gamma^{<0})) \setminus \{\chi(\Gamma^{<0})\}$, for each $f(\alpha)$ with $\alpha < \delta$ we want to include a copy of $\mathbb{Z}$ between $\chi(\Gamma^{<0}) \setminus f(\alpha)$ and $f(\alpha)$. Moreover, if $\delta = \beta + 1$, it may happen that $f(\beta) = \emptyset$, which means we have to include a copy of $\mathbb{Z}$ at the end of $\chi(\Gamma^{<0})$.

Lemma 5.28. *Let δ be an ordinal. Given a increasing function*

$$f : \delta \to scut^{op}(\chi(\Gamma^{<0})) \setminus \{\chi(\Gamma^{<0})\},$$

there is a model (Γ_f, χ_f) of T_{pdg} and a family $(b_{k,\rho})_{k \in \mathbb{Z}, \rho < \delta}$ in $\chi(\Gamma_f^{<0})$ such that:

(1) $(\Gamma, \chi) \subset (\Gamma_f, \chi_f)$,

(2) $\Gamma^{<f(\rho)} < b_{k,\rho} < f(\rho)$ *and* $\chi_f(b_{k,\rho}) = b_{k+1,\rho}$ *for all* $k \in \mathbb{Z}$, *and* $\rho < \delta$,

(3) $b_{k_1,\rho_1} < b_{k_2,\rho_2}$ *for all* $k_1, k_2 \in \mathbb{Z}$ *and* $\rho_1 < \rho_2 < \delta$, *and*

(4) *for any embedding* ϕ *of* (Γ, χ) *into a model* (Γ^*, χ^*) *of* T_{pdg} *and any family* $(b^*_{n,k})_{n\in\mathbb{Z},k<\delta}$ *in* $\chi(\Gamma^{*<0})$*such that* $\phi(\Gamma^{<f(\rho)}) < b_{k,\rho} < \phi(f(\rho))$ *and* $\chi^*(b^*_{k,\rho}) = b^*_{k+1,\rho}$ *for all* $k \in \mathbb{Z}$, *and* $\rho < \delta$, *and* $b^*_{k_1,\rho_1} < b^*_{k_2,\rho_2}$ *for all* $k_1, k_2 \in \mathbb{Z}$ *and* $\rho_1 < \rho_2 < \delta$, *there is a unique embedding* ϕ' *from* (Γ_f, χ_f) *into* (Γ^*, χ^*) *which extends* ϕ *and such that* $\phi'(b_{k,\rho}) = b^*_{k,\rho}$ *for all* $k \in \mathbb{Z}$ *and* $\rho < \delta$.

Proof. The proof is by induction on δ and we only have to observe that for the successor step, if $\delta = \beta + 1$, then by inductive hypothesis there is an extension $(\Gamma_{f|\beta}, \chi_{f|\beta})$ of (Γ, χ) corresponding to

$$f|\beta : \beta \to scut^{op}(\chi(\Gamma^{<0})) \setminus \{\chi(\Gamma^{<0})\}$$

and $f(\beta) \in scut^{op}(\chi(\Gamma^{<0}_{f|\beta}))$. □

To study the simple extension $\Gamma\langle a\rangle$ of Γ with $a \in M \setminus \Gamma$, we consider first if $(\Gamma \oplus \mathbb{Q}a)^{<0}$ is closed under χ and to do that we use the set

$$\Delta_\Gamma = \chi((\Gamma + \mathbb{Q}^{\neq 0}a)^{<0}) = \{\chi(x + qa) : q \in \mathbb{Q}^{\neq 0},\ x \in \Gamma \text{ and } x + qa < 0\}.$$

Specifically, we have the following results:

Lemma 5.29.

(1) *For all* $x \in M^{<0}$ *and* $y \in \Delta_\Gamma$ *with* $x < y$, $x \in \Delta_\Gamma$ *if and only if* $x \in \chi(\Gamma^{<0})$.

(2) *For all* $x \in \chi(\Gamma^{<0})$ *and* $y \in \Delta_\Gamma \cap \chi(\Gamma^{<0})$, *if* $x < y$ *then* $x \in \Delta_\Gamma \cap \chi(\Gamma^{<0})$.

(3) $\mathrm{card}(\Delta_\Gamma \setminus \chi(\Gamma^{<0})) \leq 1$.

(4) *If* $\Delta_\Gamma \setminus \chi(\Gamma^{<0}) = \{b\}$ *with* $b \in \chi(M^{<0}) \setminus \chi(\Gamma^{<0})$, *then* b *realize the special cut*

$$(\Delta_\Gamma \cap \chi(\Gamma^{<0}), \chi(\Gamma^{<0}) \setminus \Delta_\Gamma)$$

in $\chi(\Gamma^{<0})$

Proof. (1) Let $y = \chi(b + qa)$ for some $b \in \Gamma$ and $q \in \mathbb{Q}^{\neq 0}$. If $x \in \Delta_\Gamma$, then $x = \chi(d + ra)$ for some $d \in \Gamma$ and $r \in \mathbb{Q}^{\neq 0}$. Without loss of generality we can assume that $q, r > 0$. Thus, $x = \chi(\frac{q}{r}d + qa)$ and as $x < y < 0$, then

$$x = \chi(\frac{q}{r}d + qa) = \chi((\frac{q}{r}d + qa) - (b + qa)) = \chi(\frac{q}{r}d - b) \in \chi(\Gamma^{<0}).$$

On the other hand, if $x \in \chi(\Gamma^{<0})$ then $x = \chi(d)$ for some $d \in \Gamma^{<0}$. Thus

$$x = \chi(d) = \chi(d + (b + qa)) = \chi((d + b) + qa) \in \Delta_\Gamma.$$

If $q < 0$ or $r < 0$, the demonstration is similar.

(2) It follows from (1).

(3) If we assume that there are $x, y \in \Delta_\Gamma \setminus \chi(\Gamma^{<0})$, with $x < y$, then as $x, y \in \Delta_\Gamma$, by item (1) we obtain that $x \in \chi(\Gamma^{<0})$, a contradiction.

(4) It follows by items (2) and (3).

□

As a consequence of the above, we have two possibilities $\chi(\Delta_\Gamma) \subseteq \Delta_\Gamma$ or $\chi(\Delta_\Gamma) \setminus \Delta_\Gamma \neq \emptyset$. Hence:

Corollary 5.30. *Exactly one of the following is true:*

(1) *There is a non-empty special cut B in $\chi(\Gamma^{<0})$ such that $\Delta_\Gamma = B$.*

(2) *There is $b \in \chi(\Gamma^{<0})$ such that $\Delta_\Gamma = (\chi(\Gamma^{<0}))^{\leq b} \subseteq \chi(\Gamma^{<0})$.*

(3) *There is a non-empty special cut B in $\chi(\Gamma^{<0})$ and $b \in \chi(M^{<0}) \setminus \chi(\Gamma^{<0})$ such that $B < b$, $b < (\chi(\Gamma^{<0}) \setminus B)$ and $\Delta_\Gamma = B \cup \{b\}$*

As a particular case, if $\Delta_\Gamma \subseteq \chi(\Gamma^{<0})$ then the ordered divisible abelian subgroup $\Gamma \oplus \mathbb{Q}a$ of M is closed under χ and $\Gamma\langle a\rangle = (\Gamma \oplus \mathbb{Q}a, \chi)$. In general we have the following:

Theorem 5.31. *If $a \in M \setminus \Gamma$, then $\Gamma\langle a\rangle$ is isomorphic over Γ to one of the following:*

(1) *Γ_f for some increasing function $f : n \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$ and some natural n.*

(2) *$\Gamma_f \oplus \mathbb{Q}a$ for some increasing function $f : n \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$ and some natural n*

(3) *$\Gamma_f \oplus \mathbb{Q}a$ for some increasing function $f : \omega \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$*

Proof. The main idea is to construct by induction a chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq ... \subseteq \Gamma\langle a\rangle$ of models of T_{pdg*} in $\mathbb{M}$, each one isomorphic to Γ_f for some increasing function

$$f : n \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}.$$

To do that, set $\Gamma_0 = \Gamma$. Clearly, Γ_0 is isomorphic to Γ_f for $f : 0 \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$. Assume we have built Γ_n with $\Gamma_n \cong \Gamma_f$ for some increasing $f : n \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$, then we have two possibilities:

(1) $\Gamma_n = \Gamma\langle a\rangle$, and then $\Gamma\langle a\rangle \cong \Gamma_f$.

(2) $a \notin \Gamma_n$. Thus we consider the set Δ_{Γ_n} for Γ_n, and we have other two cases:

- $\Delta_{\Gamma_n} \subseteq \chi(\Gamma_n)$. Thus, we put $\Gamma_{n+1} = \Gamma_n \oplus \mathbb{Q}a$. So, $\Gamma\langle a\rangle \cong \Gamma_f = \Gamma_{n+1}$ and $\Gamma_f \cong \Gamma_f \oplus \mathbb{Q}a$.
- $\chi(\Delta_{\Gamma_n}) \setminus \Delta_\Gamma$. Here, $\Delta_{\Gamma_n} = B \cup \{b\}$ for some special cut $B \subseteq \chi(\Gamma_n^{<0})$ and b in $\chi(M^{<0}) \setminus \chi(\Gamma_n^{<0})$ with $B < b < (\chi(\Gamma_n^{<0})) \setminus B$. Thus, we define Γ_{n+1} as the model of T_{pdg} given by Lemma 5.15 by including the copy of $\mathbb{Z}$ corresponding to b. Thus, there is $g : n+1 \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$ such that $\Gamma_{n+1} \cong \Gamma_g$.

Now, if $\Gamma\langle a\rangle = \Gamma_n$ for some n we have finished. Otherwise, we put $\Gamma\langle a\rangle = \cup_n F_n \oplus \mathbb{Q}a$. By construction, $\Gamma\langle a\rangle \cong \Gamma_f \oplus \mathbb{Q}a$ for some increasing $f : \omega \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$.

□

Examples.

(1) *Let* $(\oplus_i \mathbb{Q}e_i, \chi_{\mathbb{Q}}) \subseteq (\Gamma_{\log}, \chi)$ *be the model of* T_{pdg} *considered in the first example of section 5.2,* $r \in \mathbb{R}^{<0} \setminus \mathbb{Q}$ *and* $a = re_m \in \Gamma_{\log} \setminus \oplus_i \mathbb{Q}e_i$*, for some* m*. Since for each* $b + qa$ *in* $(\Gamma_{\log} + \mathbb{Q}^{\neq 0}a)^{<0}$ *the entry* m *never is* 0*, we have*

$$\Delta_{\Gamma_{\log}} = \chi(\Gamma_{\log} + \mathbb{Q}^{\neq 0}a)^{<0} = \{-e_i : 2 \leq i \leq m\} \subseteq \chi(\Gamma_{\log}^{<0}).$$

Hence,

$$(\oplus_i \mathbb{Q}e_i, \chi_{\mathbb{Q}})\langle a \rangle = (\oplus_i \mathbb{Q}e_i \oplus \mathbb{Q}a, \chi') \subseteq (\Gamma_{\log}, \chi)$$

where χ' *is the restriction of* χ *to* $\oplus_i \mathbb{Q}e_i \oplus \mathbb{Q}$*.*

(2) *Let* (Γ, χ) *be a model of* T_{pdg} *and* (Γ_f, χ_f) *be a fixed extension of* (Γ, χ) *for some increasing function*

$$f : n \to scut^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$$

with $n \geq 1$*. Let's take one element* $a_j \in (\mathrm{span}_{\mathbb{Q}}(b_{k,\mathrm{j}})_{k\in\mathbb{Z}})^{\neq 0}$ *for each* $j < n$*, where* $(b_{k,\mathrm{j}})_{k\in\mathbb{Z}}$ *are the elements of the* j*-th copy of* $\mathbb{Z}$ *added to* Γ *in* Γ_f*. Given* $c \in \Gamma$ *we define the element*

$$a = c + \sum_{j=0}^{n-1} a_j \in \Gamma_f.$$

Thus, $\Gamma\langle a \rangle = \Gamma_f$*.*

5.3 Other results about models of T_{pdg}

In this section we show some results about models of T_{pdg} that can be used to study the behaviour of non immediate extensions of valued fields whose value groups are models of T_{pdg}. The content of this section is based on sections 2.4 and 9.9 of [4]. Throughout this section (Γ, χ) will be a divisible centripetal precontraction group and v its natural valuation.

The next lemma contains some useful results:

Lemma 5.32. *For all* $a, b \in \Gamma$ *we have:*

(1) $v(\chi(a)) > v(a)$.

(2) *If* $v(a) < v(b)$ *then* $\chi(a + b) = \chi(a)$.

(3) $\chi(a - n\chi(a)) = \chi(a)$ *for all* $n > 0$

(4) *If* $a, b \neq 0$ *and* $a \neq b$, $v(\chi(a) - \chi(b)) > v(a - b)$.

Proof.

(1) Since χ is a centripetal precontraction map, $|\chi(a)| < |a|$ for all $a \in \Gamma$. If there is $n > 0$ such that $n|\chi(a)| > |a|$, then $|\chi(\chi(a))| > |\chi(a)|$ which is a contradiction. Thus, $n|\chi(a)| < |a|$ for all $n > 0$ and then $v(\chi(a)) > v(a)$.

(2) For all $a, b \in \Gamma$ if $v(a) < v(b)$ we have $\operatorname{sign}(a+b) = \operatorname{sign}(a)$ and $v(a+b) = v(a)$. Thus, $\chi(a+b) = \chi(a)$.

(3) By the previous items, for all $a \in \Gamma$ we have $v(a) < v(\chi(a))$ and $\chi(a - n\chi(a)) = \chi(a)$.

(4) If $\chi(a) = \chi(b)$ is immediate. Assume $\chi(a) \neq \chi(b)$ and without loss of generality $a > b$. If we suppose that $n(\chi(a) - \chi(b)) \geq a - b$, then

$$b - n\chi(b) \geq a - n\chi(a),$$

and

$$\chi(b - n\chi(b)) \geq \chi(a - n\chi(a)).$$

But by item (3), this implies $\chi(b) \geq \chi(a)$, which is a contradiction since we assume $\chi(a) \neq \chi(b)$ and $a > b$. Thus, we have $n(\chi(a) - \chi(b)) < a - b$ for all $n > 1$ and then $v(\chi(a) - \chi(b)) > v(a - b)$.

□

As a direct consequence of the item (3) of the previous lemma, the maps

$$\theta : \Gamma \to \Gamma \text{ and } \phi : \Gamma \to \Gamma$$

given by $\theta(a) = a + \chi(a)$ and $\phi(a) = a - \chi(a)$ are strictly increasing.

Definition 5.33. *Let U be a convex subset of Γ. We say that the map $f : U \to \Gamma$ has the intermediate value property if for all $a < b$ in U:*

- *$f(a) < f(b)$ implies that $(f(a), f(b)) \subseteq f(a, b)$*
- *$f(a) > f(b)$ implies that $(f(b), f(a)) \subseteq f(a, b)$.*

Moreover, we say that f is v-steady if f has the intermediate value property and

$$f(x) - f(y) = x - y + h$$

with $v(h) > v(x - y)$. On the other hand, we say that f is v-slow on the right if for all $x, y, z \in U$ we have:

- *$v(f(x) - f(y)) > v(x - y)$, and*
- *$f(y) = f(z)$ if $x < y < z$ and $v(z - y) > v(z - x)$.*

We say that f is v-slow on the left in the same way except that in the second clause we put $x > y > z$ instead of $x < y < z$.

We can prove that if $f : U \to \Gamma$ is v-steady and $g : U \to \Gamma$ is v-slow on the right or on the left, then $f + g : U \to \Gamma$ is v-steady (for a proof see [4, Lemma 2.4.9]).

Example. *The identity function $i : \Gamma \to \Gamma$ is v-steady. The map $\chi : \Gamma^{<0} \to \Gamma$ is v-slow on the left since for all $0 > y > z$ with $v(z - y) > v(z)$ we have that $\chi(y) = \chi(z - (z - y)) = \chi(z)$ and thus $i + \chi : \Gamma^{<0} \to \Gamma$ is v-steady. Similarly, $\chi : \Gamma^{>0} \to \Gamma$ is v-slow on the right, and $i + \chi : \Gamma^{>0} \to \Gamma$ is v-steady.*

Property $\star$

Now, let $a_1, ..., a_n \in \Gamma$ and $q_1, ..., q_n \in \mathbb{Q}$. We define the function $\chi_{a_1,...,a_n} : \Gamma \to \Gamma$ by recursion on n as follows:

$$\chi_{a_1}(a) = \chi(a - a_1) \text{ and } \chi_{a_1...,a_n}(a) = \chi(\chi_{a_1,...,a_{n-1}}(a) - a_n) \text{ for } n \geq 2.$$

We are interested in the following property:

Proposition 5.34. *Let* $(\Gamma, \chi) \subseteq (\Gamma^*, \chi^*)$ *be models of* T_{pdg}. *Suppose* $n > 0$, $a_1, ..., a_n \in \Gamma$, $q_1, ..., q_n \in \mathbb{Q}$ *and* $a \in \Gamma^*$ *such that* $a + q_1\chi^*_{a_1}(a) + \cdots q_n\chi^*_{a_1,...,a_n}(a) \in \Gamma$. *Then* $a \in \Gamma$.

In the rest of this section we work under the hypothesis of the previous proposition and show some results needed in its proof.

We notice first some properties about the functions $\chi_{a_1,...,a_n} : \Gamma \to \Gamma$. Since χ is increasing, it follows by induction that each function $\chi_{a_1,...,a_n}$ is increasing. For each $1 \leq i < n$ set

$C_{1,i} = \{a \in \chi_{a_1,...,a_i}(\Gamma) : v(a) < v(a_{i+1}) \text{ and } a < a_{i+1}\}$,
$C_{2,i} = \{a \in \chi_{a_1,...,a_i}(\Gamma) : v(a) = v(a_{i+1}) \text{ and } \operatorname{sign}(a) \neq \operatorname{sign}(a_{i+1})\}$,
$C_{3,i} = \{a \in \chi_{a_1,...,a_i}(\Gamma) : v(a) > v(a_{i+1})\}$,
$C_{4,i} = \{a \in \chi_{a_1,...,a_i}(\Gamma) : v(a) = v(a_{i+1}) \text{ and } \operatorname{sign}(a) = \operatorname{sign}(a_{i+1})\}$,
and
$C_{5,i} = \{a \in \chi_{a_1,...,a_i}(\Gamma) : v(a) < v(a_{i+1}) \text{ and } a > a_{i+1}\}$.

Clearly, $\chi_{a_1,...,a_{i+1}}(\Gamma) = \bigcup_{k=1}^{5} \chi(C_{k,i} - a_{i+1})$, $|C_{2,i}| = |C_{4,i}| \leq 1$, $C_{1,i} \subseteq \chi(\Gamma^{<0})$, $C_{5,i} \subseteq \chi(\Gamma^{>0})$ and $\chi(C_{3,i} - a_{i+1}) = -\chi(a_{i+1})$. Moreover, if $C_{1,i}$ and $C_{5,i}$ are nonempty we have:

- If $a_{i+1} > 0$, then $C_{1,i} < C_{2,i} < C_{3,i} < C_{4,i} < C_{5,i}$.
- If $a_{i+1} < 0$, then $C_{1,i} < C_{4,i} < C_{3,i} < C_{2,i} < C_{5,i}$.
- if $a_{i+1} = 0$ then $C_{3,i} = \emptyset$, $C_{2,i} = C_{4,i} = \{0\}$ and $C_{1,i} < 0 < C_{5,i}$.
- By definition of $\chi_{a_1,...,a_n}$, $C_{1,i}$ decomposes as $D_i^{<0} \cup E_i^{<0}$ where $D_i^{<0} < E_i^{<0}$, $D_i^{<0}$ is a convex subset of $\chi(\Gamma^{<0})$ and $E_i^{<0}$ is a finite subset of $\chi(\Gamma^{<0})$. Similarly, $C_{5,i}$ decomposes as $E_i^{>0} \cup D_i^{>0}$ where $E_i^{>0} < D_i^{>0}$, $D_i^{>0}$ is a convex subset of $\chi(\Gamma^{>0})$ and $E_i^{>0}$ is a finite subset of $\chi(\Gamma^{>0})$.
- For $i \geq 2$, $C_{3,i}$ if finite.
- If $D_i^{<0} \neq \emptyset$ then it has a minimum element and a maximum element. For $i = 1$, we see that $\chi_{a_1}(\Gamma) = \chi(\Gamma)$. If $v(a_2) \leq v(c)$ with c the first element of $\chi(\Gamma_{<0}$, then $C_{1,1} = \emptyset$. If $v(a_2) > v(c)$, the minimum of $C_{1,1}$ is c and there is $b \in \chi(\Gamma^{<0})$ such that $\chi^2(b) = \chi(-|a|)$. Clearly $v(b) < v(a)$, thus if $v(\chi(b)) < v(a)$ then $\chi(b)$ is the maximum of $C_{1,1}$ and if $v(\chi(b)) \geq v(a)$, then b is the maximum of $C_{1,1}$. Here, $D_1^{<0} = C_{1,1}$ and $E_1^{<0} = \emptyset$.

  Now, assuming that $D_{i-1}^{<0}$ is a convex subset of $\chi(\Gamma^{<0})$ with minimum p and maximum q, then for $C_{1,i}$ we have the following. If $v(a_{i+1}) \leq v(\chi(p))$ then $C_{1,i}$ is empty. If

$$v(\chi(p)) < v(a_{i+1}) < v(\chi(q)),$$

then $\chi(p)$ is the minimum element of D_i and by the same argument used for $C_{1,1}$ there is an element $b \in D_{i-1}$ such that $\chi(b)$ is the maximum of D_i. Finally, if $v(\chi(q)) \leq v(a_{i+1})$ then $D_i^{<0} = \chi(D_{i-1})$ and $\chi(D_{i-1}^{<0})$ is a convex subset of $\chi(\Gamma^{<0})$ with minimum element $\chi(p)$ and maximum $\chi(q)$. Thus $D_i^{<0}$ has a minimum and a maximum elements.

- If $D_i^{>0} \neq \emptyset$ then it has a minimum element and maximum element. The proof is similar to that for $D_i^{<0}$.

These observations and the model completeness of T_{pdg} allows us to obtain the following result:

Lemma 5.35. *Let $(\Gamma, \chi) \subseteq (\Gamma^*, \chi^*)$ be models of T_{pdg}. Suppose $n > 0$, $a_1, ..., a_n \in \Gamma$ and the functions $\chi_{a_1,...,a_n} : \Gamma \to \Gamma$ and $\chi^*_{a_1,...,a_n} : \Gamma^* \to \Gamma^*$ defined as above. Then for all $1 \leq i \leq n$ and $1 \leq j \leq 5$ we have $C_{j,i} = C^*_{i,j} \cap \Gamma$ and for $p, q \in \Gamma$:*

(1) *$D_i^{<0} = (D_i^{<0})^* \cap \Gamma$ and if $D_i^{<0}$ has minimum p and maximum q then $(D_i^{<0})^*$ has minimum p and maximum q.*

(2) *$D_i^{>0} = (D_i^{>0})^* \cap \Gamma$ and if $D_i^{>0}$ has minimum p and maximum q then $(D_i^{>0})^*$ has minimum p and maximum q.*

(3) *$E_i^{<0} = (E_i^{<0})^*$ and $E_i^{>0} = (E_i^{>0})^*$ are finite.*

(3) *$C_{2,i} = C^*_{2,i}$ and $C_{4,i} = C^*_{4,i}$.*

Now, we define the functions $f_n : \Gamma \to \Gamma$ by recursion on n as follows:

$$f_1(a) = q_1\chi_a(a) \text{ and } f_n(a) = f_{n-1}(a) + q_n\chi_{a_1,...,a_n}(a) \text{ for } n > 1$$

and the functions $g_n : \Gamma \to \Gamma$ as $g_n(a) = a + f_n(a)$.

Lemma 5.36. *The function g_n is strictly increasing and has the intermediate value property on $F_{n-1}^{<0} = \{a \in \Gamma : \chi_{a_1,...,a_{n-1}}(a) \in D_{n-1}^{<0}\}$ and $F_{n-1}^{>0} = \{a \in \Gamma : \chi_{a_1,...,a_{n-1}}(a) \in D_{n-1}^{>0}\}$.*

Proof. For all $a, b \in \Gamma$ with $a < b$, $v(\chi(a) - \chi(b)) > v(a - b)$. Thus, $v(\chi_{a_1}(a) - \chi_{a_1}(b)) > v(a - b)$ and if we assume that $v(\chi_{a_1,...,a_i}(a) - \chi_{a_1,...,a_i}(b)) > v(a - b)$ for $1 < i < n$, then

$$v(\chi_{a_1,...,a_{i+1}}(a) - \chi_{a_1,...,a_{i+1}}(b)) > v(\chi_{a_1,...,a_i}(a) - \chi_{a_1,...,a_i}(b)) > v(a - b).$$

Moreover,

$$\begin{aligned} v(f_n(a) - f_n(b)) &= v(q_1(\chi_{a_1}(a) - \chi_{a_1}(b)) + \cdots q_n(\chi_{a_1,...,a_n}(a) - \chi_{a_1,...,a_n}(b)) \\ &\geq \min\{v(\chi_{a_1}(a) - \chi_{a_1}(b)), ..., v(\chi_{a_1,...,a_n}(a) - \chi_{a_1,...,a_n}(b))\} > v(a - b). \end{aligned}$$

Thus, $g_n(a) - g_n(b) = a - b + f_n(a) - f_n(b)$ with $v(f_n(a) - f_n(b)) > v(a - b)$. So g_n is strictly increasing.

On the other hand, for all $n > 1$ the set $F_{n-1}^{<0}$ is convex on Γ. If $a < b < c$ in Γ with $a, c \in F_{n-1}^{<0}$, then we have $\chi_{a_1,...,a_{n-1}}(a) \leq \chi_{a_1,...,a_{n-1}}(b) \leq \chi_{a_1,...,a_{n-1}}(c) < a_n$ and

$$v(\chi_{a_1,...,a_{n-1}}(a)) \leq v(\chi_{a_1,...,a_{n-1}}(b)) \leq v(\chi_{a_1,...,a_{n-1}}(c)) < v(a_n),$$

which implies that $v(\chi_{a_1,...,a_{n-1}}(b)) \in F_{n-1}^{<0}$. Similarly, we can show that $F_{n-1}^{>0}$ is convex on Γ. Now, we will prove that the restriction of f to $F_{n-1}^{<0}$ is v-slow on the left. To do this we observe the following:

- Since for all $1 < i < n$, either $D_{i+1}^{<0} = \emptyset$ or $D_{i+1}^{<0} \neq \emptyset$ and $D_{i+1}^{<0} \subseteq \chi(D_i^{<0})$, it has $F_{i+1}^{<0} \subseteq F_i^{<0}$. Moreover, $F_i^{<0} \subseteq \Gamma^{<a_1}$.
- Since $\chi : \Gamma^{<0} \to \Gamma^{<0}$ is v-slow on the left, it has $f_1 : \Gamma^{<a_1} \to \Gamma^{<0}$ is v-slow on the left: let $a, b \in \Gamma^{<a_1}$ with $a < b$ and $v(a-b) > v(a-a_1)$, then $b - a_1 < a - b$ and

$$b - a_1 > a - a_1 = a - b + b - a_1 > 2(b - a_1),$$

thus, $\chi(b - a_1) = \chi(a - a_1)$ or equivalent $f_1(b) = f_1(a)$. As a consequence, since $F_i^{<0} \subseteq \Gamma^{<a_1}$ for all $i > 1$, for all $a, b \in F_i^{<0}$, with $a < b$ we have $a < b < a_1$. Thus if we assume $v(a-b) > v(a-a_1)$ then $f_{i+1}(a) = f_{i+1}(b)$. So, f_{i+1} restricted to $F_i^{<0}$ is v-slow on the left.

In conclusion, as the identity function is v-steady, g_n is strictly increasing and f_n restricted to $F_{n-1}^{<0}$ is v-slow on the left, then g_n restricted to $F_{n-1}^{<0}$ is v-steady and has the intermediate value property.

In similar way we can prove that f_n restricted to $F_{n-1}^{>0}$ is v-slow on the right and g_n restricted to $F_{n-1}^{>0}$ is v-steady and has the intermediate value property.

□

In the previous lemma we saw that $g_1 : \Gamma \to \Gamma$ given by $g_1(a) = a + q_1\chi_{a_1}(a)$ is strictly increasing. Moreover, g_1 is a bijection: if $b \in \Gamma$, setting $a = b - q_1\chi(b - a_1)$ we have

$$g_1(a) = b - q_1\chi(b - a_1) + q_1\chi(b - q_1\chi(b - a_1) - a_1).$$

However,

$$\chi(b - q_1\chi(b - a_1) - a_1) = \chi((b - a_1) - q_1\chi(b - a_1)) = \chi(b - a_1),$$

and thus $g_1(a) = b$. So g_1 is surjective.

Finally, we prove Proposition 5.34. Recall that $(\Gamma, \chi) \subseteq (\Gamma^*, \chi^*)$ are models of T_{pdg} and for $a_1, ..., a_n \in \Gamma$ and $q_1, ..., q_n \in \mathbb{Q}$ with $n > 0$ we defined the function $g_n : \Gamma \to \Gamma$ as

$$g_n(a) = a + q_1\chi_{a_1}(a) + \cdots + q_n\chi_{a_1,\ldots,a_n}(a),$$

and the corresponding function $g^* : \Gamma^* \to \Gamma^*$ as

$$g_n^*(a) = a + q_1\chi_{a_1}^*(a) + \cdots + q_n\chi_{a_1,\ldots,a_n}^*(a).$$

We want to show that for $a \in \Gamma^*$ if $g_n^*(a) \in \Gamma$ then $a \in \Gamma$.

Proof of Proposition 5.34. It is enough to prove that if $a \in \Gamma^* \setminus \Gamma$ then $g_n^*(a) \notin \Gamma$. We do this by induction over n. First, by the last observation, the map $g_1^* : \Gamma \to \Gamma$ defined as $g_1(a)^* = a + \chi_{a_1}^*(a)$ is a bijection. Thus, $a \in \Gamma^* \setminus \Gamma$ and $g_n^*(a) \notin \Gamma$.

Now, assume that the property is true for $1 < i < n$. That is, for all $a \in \Gamma^* \setminus \Gamma$ we have $g_i^*(a) = a + q_1\chi_{a_1}^*(a) + \cdots + q_i\chi_{a_1,\ldots,a_i}^*(a) \notin \Gamma$, and analyze the function

$$g_{i+1}^*(a) = g_i^*(a) + \chi_{a_1,\ldots,a_{i+1}}(a).$$

As for each $a \in \Gamma^*$ we have that $\chi_{a_1,\ldots,a_i}^*(a)$ is in $\bigcup_{k=1}^{5} C_{k,i}^*$ then if $a \in \Gamma^* \setminus \Gamma$ we have the following:

- If $\chi^*_{a_1,\ldots,a_i}(a) \in C^*_{2,i} \cup C^*_{4,i}$, as $C_{2,i} = C^*_{2,i}$ and $C_{4,i} = C^*_{4,i}$, then $\chi^*_{a_1,\ldots,a_{i+1}}(a) \in \Gamma$. Thus, since $g^*_i(a) \notin \Gamma$ and Γ is divisible, $g^*_{i+1}(a) \notin \Gamma$.
- if $\chi^*_{a_1,\ldots,a_i}(a) \in C^*_{3,i}$, then $\chi^*_{a_1,\ldots,a_{i+1}}(a) = -\chi(a_{i+1})$. Thus, since $g^*_i(a) \notin \Gamma$ and Γ is divisible, $g^*_{i+1}(a) \notin \Gamma$.
- if $\chi^*_{a_1,\ldots,a_i}(a) \in C^*_{1,i}$, then either $\chi^*_{a_1,\ldots,a_i}(a) \in (D_i^{<0})^*$ or $\chi^*_{a_1,\ldots,a_i}(a) \in (E_i^{<0})^*$. In the first case, $a \in (F_i^{<0})^*$ and by Lemma 5.36 we know that g^*_n is strictly increasing and restricted to $(F_i^{<0})^*$ has the intermediate value property, thus if a in the convex hull of $F_i^{<0}$ in $(F_i^{<0})^*$ then $g^*_{i+1}(a) \notin \Gamma$. Moreover, since $(D_i^{<0})^*$ and $D_i^{<0}$ has the same maximum and minimum, $(D_i^{<0})^*$ is the convex hull of $D_i^{<0}$ in Γ^*. In the other case, if $\chi^*_{a_1,\ldots,a_i}(a) \in (E_i^{<0})^*$, as $(E_i^{<0})^* = E_i^{<0}$ then, since $g^*_i(a) \notin \Gamma$ and Γ is divisible, $g^*_{i+1}(a) \notin \Gamma$.
- if $\chi^*_{a_1,\ldots,a_i}(a) \in C^*_{5,i}$ we proceed as in the case of $C^*_{1,i}$.

In conclusion, if $a \in \Gamma^* \setminus \Gamma$ then $g_{i+1}(a) \notin \Gamma$. □

6. $\mathbb{T}_{\log}$ as ordered valued logarithmic field

In section 2.7 we noticed that the valuation ring $\mathcal{O}_{\log}$ is definable in $\log(\mathbb{T}_{\log}^{>0})$ and for $f \in \mathbb{T}_{\log}$ we have $f \in \mathbb{T}_{\log} \setminus \log(\mathbb{T}_{\log}^{>0})$ if and only if there is $g \in \log(\mathbb{T}_{\log}^{>0})$ such that $v(f-g) \in \Gamma_{\log} \setminus \chi(\Gamma)$. Under this observations, in this chapter we study first structures of the form $(K, \log, L)$ where $(K, \log)$ is an ordered logarithmic field, L is a $\mathbb{Q}$-vector subspace of K with $\log(K^{>0}) \subseteq L$, there is a valuation ring $\mathcal{O}$ definable in L such that $\log((\mathcal{O}^{\times})^{>0}) = \mathcal{O} \cap \log(K^{>0})$ and $x \in K \setminus L$ if and only if there is $y \in L$ such that $v(x-y) \in \Gamma \setminus \chi(\Gamma)$. We call these structures *maximal log-fields.*

Next, we define the theory $T_{+}(\log)$ whose models are structures $(K, \mathcal{O}, \exp, \log, L)$ where $(K, \mathcal{O}, \exp)$ is a model of T_{+}, $(K, \log, L)$ is a maximal log-field and the value group is a model of T_{pdg}. Particularly, we show that $(\mathbb{T}_{\log}, \mathcal{O}_{\log}, \exp, \log, \log(\mathbb{T}_{\log}^{>0}))$ is a model of $T_{+}(\log)$ and this theory has a prime model. Moreover, we sketch a possible path to show that this theory is model complete.

6.1 L-fields

In Chapter 2 we saw that the valuation ring $\mathcal{O}_{log}$ of $\mathbb{T}_{\log}$ is definable as the convex component of 0 in $\log(\mathbb{T}_{\log}^{>0})$. Specifically, we have

$$\mathcal{O}_{\log} = \{f \in \mathbb{T}_{\log} : [0, |f|] \subseteq \log(\mathbb{T}_{\log}^{>0})\}.$$

Thus, we are interested in studying logarithmic fields in which the valuation is definable. Particularly, we introduce the following definition:

Definition 6.1. *Let $(K, \log)$ be a logarithmic field. We say that a $\mathbb{Q}$-vector subspace L of K is an L-set on K if:*

L1. $\log(K^{>0}) \subseteq L$,

L2. $[0,1] \subseteq L$ *and*

L3. *for all $a \in K$ if $[0, a] \subseteq L$ then $[0, a^2] \subseteq L$.*

Moreover, an L-field is a structure $(K, \log, L)$ consisting of an underlying logarithmic field $(K, \log)$ with a L-set L.

From this definition it follows that there is a convex valuation ring definable in each L-field:

Lemma 6.2. *Let $(K, \log, L)$ be an L-field. Then the set*

$$\mathcal{O}_L := \{a \in K : [0, |a|] \subseteq L\}$$

is a convex valuation ring of K.

Proof. For all $a, b \in \mathcal{O}_L$ and $c \in K$ we have:

(1) If $|c| < |a|$ then $[0, |c|] \subseteq L$.

(2) $-a \in \mathcal{O}_L$ and $a + b \in \mathcal{O}_L$ because L is a $\mathbb{Q}$-vector subspace of K.

(3) Assuming without loss of generality $|b| < |a|$ then we have that $|ab| < a^2$. Thus, by item (1) and L3 we have that $[0, |ab|] \subseteq L$ and then $ab \in \mathcal{O}_L$.

(4) Since either $|c| < 1$ or $|c^{-1}| < 1$, by item (1) and L2 either $[0, |c|] \subseteq L$ or $[0, |c^{-1}|] \subseteq L$. Thus, for all $c \in K$, either $c \in \mathcal{O}_L$ or $c^{-1} \in \mathcal{O}_L$.

□

From now on, we will understand that each L-field $(K, \log, L)$ is equipped with the valuation ring $\mathcal{O}_L$ induced by the L-set L on K. If the context is clear, we denote $\mathcal{O}_L$ just by $\mathcal{O}$, the valuation associated to $\mathcal{O}$ by v and the maximal ideal of $\mathcal{O}$ by $\mathcal{o}$.

Now, at the moment the only relation that we have between the logarithm of an L-field $(K, \log, L)$ and its induced valuation ring $\mathcal{O}$ is that if $-1 \leq \log(a) \leq 1$ for some $a \in K^{>0}$, then $\log(a) \in \mathcal{O}$; however, we don't know if $a \in \mathcal{O}$. Thus, based on what happens in $\mathbb{T}_{\log}$ we will study the L-fields $(K, \log, L)$ which satisfy the following condition:

L4. For all $a \in K^{>0}$ if $\log(a) \in \mathcal{O}$ then $a \in \mathcal{O}$.

For an L-field $(K, \log, L)$ satisfying L4, if $a \in K^{>0} \setminus \mathcal{O}$ then $\log(a) \in K^{>0} \setminus \mathcal{O}$. Moreover, if $a \in \mathcal{o}^{>0}$, then $a^{-1} \in K^{>0} \setminus \mathcal{O}$ and as $-\log(a) = \log(a^{-1}) \in K^{>0} \setminus \mathcal{O}$ then

$$\log(\mathcal{o}^{>0}) \subseteq K^{>0} \setminus \mathcal{O} \text{ and } \mathcal{O} \cap \log(K^{>0}) \subseteq \log((\mathcal{O}^{\times})^{>0}).$$

Additionally, if we want to ensure that $v(\log(a)) = v(\log(b))$ when $v(a) = v(b) \neq 0$, then we need an extra condition:

L5. $\log(a) \in \mathcal{O}$ for all $a \in (\mathcal{O}^{\times})^{>0}$.

If $(K, \log, L)$ is an L-field which satisfies L4 and L5, then

$$\mathcal{O} \cap \log(K^{>0}) = \log((\mathcal{O}^{\times})^{>0}).$$

Moreover, if $v(a) = v(b) < 0$, $a = bc$ for some $c \in (\mathcal{O}^{\times})^{>0}$, and $\log(a) = \log(c) + \log(b)$. Thus

$$v(\log(a)) \geq \min\{v(\log(c)), v(\log(b))\},$$

but by L5 we have that $v(\log(c)) \geq 0$. Thus, $v(\log(a)) = v(\log(b)) < 0$. On the other hand, if $v(a) = v(b) > 0$, then $v(a^{-1}) = v(b^{-1}) < 0$ and

$$v(\log(a^{-1})) = v(\log(a)) = v(\log(b)) = v(\log(b^{-1})).$$

Definition 6.3. *Let $(K, \log)$ be a logarithmic field and $\mathcal{O}$ be a convex valuation ring of K. We say that the logarithm and the valuation ring are compatible if*

$$\log((\mathcal{O}^\times)^{>0}) = \mathcal{O} \cap \log(K^{>0}).$$

Moreover, a valued logarithmic field is a structure of the form $(K, \log, \mathcal{O})$ where $(K, \log)$ is a logarithmic field and $\mathcal{O}$ is a convex valuation ring of K compatible with log. *Particularly, we say that an L-field $(K, \log, L)$ is a valued L-field if* log *is compatible with the valuation ring $\mathcal{O}$ induced by the L-set L on K.*

Examples.

(1) $(\mathbb{T}_{\log}, \log, \log(\mathbb{T}_{\log}^{>0}))$ *is a valued L-field.*

(2) *Let $(K, \log, L)$ be an L-field such that $L = \log(K^{>0})$. If for all $a \in K^{>0}$ it has*

A. if $[0, \log(a)] \subseteq \log(K^{>0})$ then $[0, a] \subseteq \log(K^{>0})$, and

B. if $a > 1$ then $a > \log(a)$,

then $(K, \log, L)$ is a valued L-field. Clearly, property A implies L4. Now, if $a \in \mathcal{O}^\times$ and $a \neq 1$ then $a > 1$ or $a^{-1} > 1$, thus, assuming $a > 1$, property B implies that $a > \log(a) > 0$ and then $\log(a) \in \mathcal{O}$.

(3) *If $(K, \log)$ is a logarithmic field in which the multiplicative group $K^{>0}$ is divisible, the logarithm satisfies properties A and B of the above item and $[0, 1] \subseteq \log(K^{>0})$, then $(K, \log, L)$ is a valued L-field with $L = \log(K^{>0})$. Further, if $a, b \in K^{>0}$ and $[0, a], [0, b] \subseteq \log(K^{>0})$ then*

$$[-a, 0], [0, a + b] \subseteq \log(K^{>0}).$$

Additionally, if $[0, a] \subseteq \log(K^{>0})$ for $a > 1$, as $\log(a^2) =< 2a$ and $[0, 2a] \subseteq \log(K^{>0})$, by property A it holds that $[0, a^2] \subseteq \log(K^{>0})$. Now, if $(K, \log, L)$ is a valued L-field with $L = \log(K^{>0})$ and log *satisfies property B then* log *satisfies property A.*

6.1.1 The natural precontraction map

Since in each ordered valued logarithmic field $(K, \log, \mathcal{O})$ it holds that $v(\log(a)) = v(\log(b))$ for all $a, b \in K^{>0}$ with $v(a) = v(b) \neq 0$, it follows that the logarithmic map induce a precontraction map in the value group:

Lemma 6.4. *Let $(K, \log, \mathcal{O})$ be an ordered valued logarithmic field and let $\chi : \Gamma \to \Gamma$ be the map defined as follows:*

- $\chi(0) = 0$,
- $\chi(v(a)) = v(\log(a))$ *for $a \in K^{>0}$ with $v(a) < 0$, and*
- $\chi(-\alpha) = -\chi(\alpha)$ *for $\alpha < 0$.*

Then (Γ, χ) is a precontraction group.

Proof. Since for all $a, b \in K^{>0}$ with $v(a) = v(b) \neq 0$ it has $v(\log(a)) = v(\log(b))$, χ is well defined. Now, let $v(a)$ be archimedean equivalent to $v(b)$ with $a, b \in K^{>0}$ and $v(a) \leq v(b) < 0$ then there is a natural number n such that $nv(b) = v(b^n) \leq v(a)$. By convexity of v we obtain that $b^n \geq a \geq b$, and then $\log(b^n) = n\log(b) \geq \log(a) \geq \log(b)$. Thus $v(\log(b)) = v(\log(a))$ and $\chi(v(a)) = \chi(v(b))$. □

We call the precontraction map defined in the previous lemma the *natural precontraction map* induced by log in Γ, and (Γ, χ) the natural precontraction group of $(K, \log, \mathcal{O})$. From now on we only work with the natural precontraction map associated to a valued logarithmic field, so we will omit the word natural.

For any logarithmic field $(K, \log, \mathcal{O})$ we say that the logarithmic map log satisfies the *Growth Axiom* (GA) if for all $a \in K^{>0} \setminus \mathcal{O}$ we have $v(a) < v(\log(a))$. From the definition of centripetal precontraction group, for any ordered valued logarithmic field $(K, \log, \mathcal{O})$ we have:

Lemma 6.5. *The natural precontraction group of K is centripetal if and only if* log *satisfies GA.*

Decomposition of log

In a valued L-field $(K, \log, L)$, for each $a \in \log(K^{>0})$ either $v(a) \geq 0$ or $v(a) \in \chi(\Gamma_K^{<0})$. In particular, if $L = \log(K^{>0})$ then $\mathcal{O} \subseteq L$ and each element of positive valuation is a logarithm of an element of $\mathcal{O}^{\times}$.

Now, in any ordered valued logarithmic field $(K, \log, \mathcal{O})$ the set

$$E(\scriptstyle\mathcal{O}) = \{a \in (\mathcal{O}^{\times})^{>0} : \log(a) \in \scriptstyle\mathcal{O}\}$$

is a convex subgroup of $(\mathcal{O}^{\times})^{>0}$. For example, for $\mathbb{T}_{\log}$ we have $E(\scriptstyle\mathcal{O}_{\log}) = 1 + \scriptstyle\mathcal{O}_{\log}$.

Lemma 6.6. *Let $(K, \log, L)$ be a valued L-field. Then the underlying additive group K decomposes as*

$$K = A \oplus A' \oplus \scriptstyle\mathcal{O},$$

where A is a group complement to $\mathcal{O}$ in K and A' is a group complement to $\scriptstyle\mathcal{O}$ in $\mathcal{O}$. Additionally, if the multiplicative group $K^{>0}$ is divisible, then

$$K = B \cdot B' \cdot E(\scriptstyle\mathcal{O}),$$

where B is a group complement to $(\mathcal{O}^{\times})^{>0}$ and B' is a group complement to $E(\scriptstyle\mathcal{O})$ in $\mathcal{O}^{\times}$, and the logarithm log *decomposes into three embedding of ordered groups:*

$\log_{>0} : E(\scriptstyle\mathcal{O}) \to \scriptstyle\mathcal{O}$
$\log_{=0} : B' \to A'$
$\log_{<0} : B \to A$

Remark.

(1) *In the previous lemma A, A', B and B' are unique up to ordered isomorphism, and the image of A' is isomorphic to the additive ordered group* $\mathrm{res}(K)$*.*

(2) *B is isomorphic to the ordered value group Γ_K, because the map $\phi : K^{>0} \to \Gamma_K$ given by $\phi(x) = -v(x) = v(x^{-1})$ is a surjective homomorphism of ordered groups with kernel $(\mathcal{O}^{\times})^{>0}$.*

In the context of the previous lemma, $\log_{=0}$ and $\log_{>0}$ are surjective because $\mathcal{O} \subseteq \log(K^{>0})$. Additionally, if $E(\mathcal{o}) = 1 + \mathcal{o}$ we have:

Lemma 6.7. *The map $\log^* : \mathrm{res}(K)^{>0} \to \mathrm{res}(K)$ given by $\log^*(\mathrm{res}(a)) = \mathrm{res}(\log(a))$ is an exponential-logarithmic function on $\mathrm{res}(K)$.*

Proof. First, we show that $\log^*$ is well defined. If $a, b \in (\mathcal{O}^{\times})^{>0}$ with $\mathrm{res}(a) = \mathrm{res}(b)$, there is $\epsilon \in \mathcal{o}$ such that $a = b + \epsilon$. Thus, $a = b(1 + \epsilon/b)$ and $\log(a) = \log(b) + \log(1 + \epsilon/b)$. As $1 + \mathcal{o} = E(\mathcal{o})$, $\log(1 + \epsilon/b) \in \mathcal{o}$ and then $\mathrm{res}(\log(a)) = \mathrm{res}(\log(b))$. Now, since log is an order preserving embedding of groups and $1 + \mathcal{o} = E(\mathcal{o})$, $\log^*$ is too. Moreover, as $\mathcal{O} \subseteq \log(K^{>0})$, $\log^*$ is surjective: if $\mathrm{res}(a) \in \mathrm{res}(K)$, $a \in \mathcal{O}$ and there is $b \in (\mathcal{O}^{\times})^{>0}$ such that $\log(b) = a$. Thus $\log^*(\mathrm{res}(b)) = \mathrm{res}(a)$. □

Extensions of L-fields

As is usual, given the L-fields $(E, \log_E, L_E)$ and $(K, \log, L)$, we say that $(K, \log, L)$ is an extension of $(E, \log_E, L_E)$ as L-fields, denoted by $(E, \log_E, L_E) \subseteq (E, \log_E, L_E)$, if $(K, \log)$ is an extension of $(E, \log_E)$ as logarithmic fields and $L_E = F \cap L$. Moreover, if the L-fields $(F, \log_F, L_F)$ and $(K, \log, L)$ are extensions of $(E, \log_E, L_E)$ and $\phi : F \to K$ is an embedding of logarithmic fields such that $\phi^{-1}(L) = (L_F)$, then we say that ϕ is an embedding over

$$(E, \log_E, L_E)$$

of L-fields.

Under this definition we observe that if $(E, \log_E, L_E) \subseteq (K, \log, L)$ then for all $a \in E$ such that $[0, |a|]_K \subseteq L$ we have $[0, |a|]_E = [0, |a|]_K \cap E$, thus $\mathcal{O}_L \cap E \subseteq \mathcal{O}_{L_E}$. Moreover, by definition $[0, 1]_E \subseteq L_E$ and $[0, 1]_K \subseteq L$. However, for $a \in E$ with $a > 1$ if $[0, a]_E \subseteq L_E$ then $a \in \mathcal{O}_{L_E}$ but not necessarily $[0, a]_K \subseteq K$.

In that sense, we say that $(K, \log, L)$ is a *good L-field extension* of $(E, \log_E, L_E)$ if $[0, a]_E \subseteq L_E$ implies that $[0, a]_K \subseteq L$. Thus, we obtain $\mathcal{O} \cap E = \mathcal{O}_E$, which is equivalent to say that the valued field K, with valuation ring $\mathcal{O}$, is an extension of the valued field E, with valuation ring $\mathcal{O}_E$.

6.2 Log-fields

A *log-field* $(K, \log, L)$ is a valued L-field in which $v(L) \subseteq \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0} \cup \{\infty\}$ and $\log(1 + \mathcal{o}) = \mathcal{o}$. In this case we call L a *log-set* on K.

As a first observation, we have that in a log-field $(K, \log, L)$, for each $a \in K$ with $v(a) \in \chi(\Gamma^{<0})$ there is $b \in K^{>0}$ such that $\chi(v(b)) = v(log(b)) = v(a)$. Thus, since $\log(K^{>0}) \subseteq L$ and $\mathcal{O} \subseteq L$, we have

$$v(L) = \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0} \cup \{\infty\}.$$

Other properties of log-fields are the following:

Lemma 6.8. *Let $(K, \log, L)$ be a log-field and $a, b \in K$, then:*

(1) *If $b \in L$ and $v(a - b) > 0$ then $a \in L$.*

(2) *If $b \in L$ and $v(a - b) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$ then $a \notin L$.*

Log-fields whose precontraction group is a model of T_{pdg}

If $(K, \log, L)$ is a log-field whose precontraction group (Γ, χ) is a model of T_{pdg}, then $(K, \log, L)$ satisfies GA. This implies that for all $a \in K^{>0}$ with $v(a) < 0$ we have $v(a) < v(\log(a))$. Moreover, since the valuation is convex, we have $a > \log(a)$ and by compatibility between the valuation and the logarithm we have $v(\log(a)) < 0$. In next lemma we list other properties:

Lemma 6.9. *Let $(K, \log, L)$ be a log-field such that (Γ, χ) is a model of T_{pdg}. For $a \in K^{>0}$ with $v(a) < 0$ and all $n > 0$ we have:*

(1) *$a > n \log(a)$.*

(2) *$a > (\log(a))^n$.*

(3) *If $a \in \log(K^{>0})$ then $v((log(a))^n) \in \Gamma^{<0} \setminus \chi(\Gamma)$ for $n > 1$*

Proof.

(1) If there is $n > 0$ such that $a \leq n \log(a)$, then $v(a) \geq v(n \log(a)) = v(\log(a))$ which contradicts GA. Thus, $a > n \log(a)$ for all $n > 0$.

(2) If there is $n > 0$ such that $a \leq (\log(a))^n$, then $\log(a) \leq n \log(\log(a))$, but this contradicts item (1). Thus $a > (\log(a))^n$ for all $n > 0$.

(3) If $a \in \log(K^{<0})$ and $v(a) < 0$ then $v(a) \in \chi(\Gamma^{<0})$, but $v(\log(a)^n) = nv(\log(a))$. Now, as (Γ, χ) is a model of T_{pdg}, then $v(\log(a)) \in \chi(\Gamma^{<0})$ but $nv(\log(a)) \notin \chi(\Gamma^{<0})$ for all $n > 1$. Thus, $(\log(a))^n$ is not a logarithm in K for $n > 1$ and $nv(\log(a)) \in \Gamma^{<0} \setminus \chi(\Gamma)$.

□

Using these properties we prove the next extension lemma:

Lemma 6.10. *Let $(K, \log, L)$ be a log-field such that (Γ, χ) is a model of T_{pdg} and $(F, \log_F)$ be a logarithmic subfield of $(K, \log)$. Then $(F, \log_F, L_F)$ is a log-field with $L_F = L \cap F$. Additionally, $(K, \log, L)$ is a good extension of $(F, \log_F, L_F)$.*

Proof. Clearly L_F is a $\mathbb{Q}$-vector subspace of F and $[0, 1]_F \subseteq L_F$. Moreover, since $(F, \log_F)$ is logarithmic subfield of K, we have $\log_F(F^{>0}) = \log(K^{>0}) \cap F$ and $\log(F^{>0}) \subseteq L_F$. Let

$$\mathcal{O}_F = \{a \in L_F : [0, |a|]_F \subseteq L_F\}.$$

We will show that $\mathcal{O}_F = \mathcal{O}_L \cap F$. Let $a \in L_F$ with $a > 0$. If $[0,a]_K \subseteq L$, then we obtain that $[0,a]_F = [0,a]_K \cap F$ and $[0,a]_F \subseteq L$. Thus, $\mathcal{O} \cap F \subseteq \mathcal{O}_F$. Now, if $[0,a]_F \subseteq L_F$ and $[0,a]_K \not\subseteq L$, since $a \in L_F^{>0} \subseteq L^{>0}$ and $v(a) < 0$, by the previous lemma we obtain $v((\log(a))^2) \in \Gamma^{<0} \setminus \chi(\Gamma)$. Thus, $(\log(a))^2 \notin L$ and $(\log(a))^2 \notin L_F$. However $0 < (\log(a))^2 < a$, so $[0,a]_F \not\subseteq L_F$, which is a contradiction. Thus, $\mathcal{O}_F \subseteq \mathcal{O} \cap F$.

Therefore, if $[0,a]_F \subseteq L_F$ then $[0,a^2]_F \subseteq L_F$. Thus, L is an L-set of F and $\mathcal{O}_F$ is a convex valuation ring of F. Additionally, since $(K, \log, L)$ is a log-field, we have

$$\log_F((\mathcal{O}_F^\times)^{>0}) = \mathcal{O}_F \cap \log_F(F^{>0}),$$

and

$$v_F(L_F) \subseteq \chi(\Gamma_F^{<0}) \cup \Gamma_F^{\geq 0} \cup \{\infty\}.$$

So $(F, \log_F, L_F)$ is a valued L-field and $(F, \log_F, L_F) \subseteq (K, \log, L)$ is a good extension of valued L-fields. □

6.2.1 Maximal log-fields

Let $(K, \log, L)$ be a log-field. We say that L is a *maximal log-set* on K if there is no log-set L' on K such that $L \subset L'$.

Let $(K, \log, L)$ be a log-field whose precontraction group (Γ, χ) is a model of T_{pdg}. If L is not a maximal log-set of K, there is $a \in K^{>0} \setminus L$ such that $v(a - L) \subseteq \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0}$. Since $\mathcal{O} \subseteq L$, we have $v(a) \in \chi(\Gamma^{<0})$. If we put $H = \mathbb{Q}a + L$, H is a $\mathbb{Q}$-vector subspace of K, $[0,1] \subseteq H$ and since for all $q > 0$ we have $(\log(qa))^2 \notin L$ and $0 < (\log(qa))^2) < qa$, $[0,qa] \not\subseteq H$. Thus, H is a log-set on K. Hence:

Lemma 6.11. *Let $(K, \log, L)$ be a log-field whose precontraction group is a model of T_{pdg}. The following conditions are equivalent:*

(1) *L is a maximal log-set on K.*

(2) *For each $a \in K \setminus L$ there is $b \in L$ such that $v(a - b) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$.*

Definition 6.12. *A maximal log-field is a log-field which satisfies the conditions of the Lemma 6.11.*

Example. $(\mathbb{T}_{\log}, \log, L)$ *is an maximal log-field with* $L = \log(\mathbb{T}_{\log}^{>0})$.

Now, the maximal log-sets have a nice behaviour with respect pc-sequences, namely:

Lemma 6.13. *Let $(K, \log, L)$ be a maximal log-field and (a_α) be a pc-sequence in K with pseudo limit $a \in K$. If $a_\alpha \in L$ for all α, then (a_α) has a pseudolimit $b \in L$.*

Proof. If $a \in L$ then we set $a = b$. Otherwise, there is $c \in L$ such that $v(a - c) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$. By definition of pseudo convergence we have $v(a - a_\beta) = v(a_\gamma - a_\beta)$ for some β big enough and $\gamma > \beta$ and since $a_\alpha \in L$ for all α, we obtain $a_\gamma - a_\beta \in L$ and

$$v(a - a_\beta) = v(a_\gamma - a_\beta) \in \chi(\Gamma^{<0}) \cup \Gamma^{>0} \cup \{\infty\}.$$

Thus, $v(a-a_\beta) \neq v(c-a)$ eventually. Now, if $v(a-a_\beta) > v(a-c)$ eventually, then

$$v(a_\beta - a) = v(c-a) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$$

which is a contradiction. So $v(a-a_\beta) < v(c-a)$ eventually and then $b = c$ is a pseudo limit of (a_α) in$L$. □

In consequence we have the following result:

Lemma 6.14. *Let $(K, \log, L)$ be a maximal log-field and $a \in K \setminus L$. Then $v(a-L)$ has a maximum in $\Gamma^{<0} \setminus \chi(\Gamma^{<0})$. Moreover, if $v(a-b) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$ for some $b \in L$ then $v(a-b) = \max v(a-L)$.*

Proof. First, if we assume that $v(a-L)$ does not have maximum then we can construct a sequence (a_α) in L such that $v(a-a_\alpha)$ is strictly increasing and cofinal in $v(a-L)$. Thus, (a_α) is a pc-sequence in L with pseudo limit a. By Lemma 6.13 there is $b \in L$ such that (a_α) is pseudo convergent to b. So $v(a-b) > v(a-L)$ which is a contradiction. Thus, $v(a-L)$ has a maximum. Now, assume $c \in L$ we have $v(a-c) = \max v(a-L) \in \chi(\Gamma^{<0}$. Since $a \in K \setminus L$, there is $b \in L$ such that $v(a-b) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$. Thus, we must have $v(a-b) < v(a-c)$, but

$$v(a-c) = v(a-b+b-c) \geq \min\{v(a-b), v(b-c)\},$$

and then $v(a-b) = v(b-c)$, which is a contradiction. Thus, $\max v(a-L) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$.

Finally, if for some $a, b \in L$ we have $v(a-b) = \max v(a-L)$ and $v(a-c) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$ with $v(a-c) < v(a-b)$, then

$$v(a-c-(a-b)) = v(b-c) = \min\{v(a-c), v(a-b)\} = v(a-c).$$

But this is a contradiction because $v(a-b) \in \chi(\Gamma^{<0}) \cup \Gamma^{\geq 0} \cup \{\infty\}$. Thus,

$$v(a-c) = \max v(a-L).$$

□

Remark. *Let $(K, \log, L)$ be a maximal log-field and $(F, \log_F, L_F) \subseteq (K, \log, L)$. If $a \in K \setminus L$, there is $\alpha = \max v(a-L)$. Let $\beta \in \chi(\Gamma^{<0})$ such that $\chi(\chi(\beta)) = \chi(\alpha)$. Clearly, $\beta < \alpha$ and*

$$\beta = \max v(a-L) \cap \chi(\Gamma^{<0}).$$

Thus, if $a \in F$, then $v(a-L) \cap \chi(\Gamma_F^{<0})$ has a maximum. Even more, if $(F, \log_F, L_F)$ is a maximal-field with F real closed $a \in K \setminus F$, $a \in K \setminus L$ and there is no $b \in L_F$ such that $v(a-b) \in \Gamma^{<0} \setminus \chi(\Gamma^{<0})$, then $v(a-F) \cap \chi(\Gamma_F^{<0})$ has a maximum.

6.3 The theory $T_+(\log)$

Let

$$L_+(\log) = L_+ \cup \{\log, L\}$$

be the language L_+ augmented by a unary function symbol log and a unary predicate symbol L. We define $T_+(\log)$ as the theory whose models are structures of the form

$$\mathbf{K} = (K, \mathcal{O}, \exp, \log, L)$$

such that

(1) $(K, \mathcal{O}, \exp)$ is a model of T_+;

(2) The precontraction group of K, (Γ, χ), is a model of T_{pdg}.

(3) $(K, \log, L)$ is a maximal log-field;

(4) $\mathcal{O}$ is the valuation ring defined by L;

(5) $\log(\exp(x)) = x$ for all $x \in \mathcal{O}$.

(6) For all $x \in O^\times$ there is $y \in \mathcal{O}^\times$ such that $\mathrm{res}(x) = \mathrm{res}(y)$ and $yL \subseteq L$.

(7) $L = \log(K^{>0})$

Example. $(\mathbb{T}_{\log}, \mathcal{O}_{\log}, \exp, \log, \log(\mathbb{T}_{\log}^{>0}))$ *is a model of* $T_+(\log)$.

For any model $\mathbf{K}$ of $T_+(\log)$ we notice that

$$A_K = \{x \in \mathcal{O} : xL \subseteq L\}$$

is a subring of $\mathcal{O}$, L is an A_K-module, $\exp(A_K) \subseteq A_K$, and by axiom (6), $\mathrm{res}(A) = \mathrm{res}(K)$. In fact, A_K is a local ring but not a valuation ring.

Furthermore, for any $a \in A_K$ we can define the function $f_a : K \to K$ as $f(x) = 0$ if $x \leq 0$ and $f_a(x) = y$ with $\log(y) = a\log(x)$ for $x > 0$. Thus, for $x > 0$ we can interpret $f_a(x)$ as the power x^a. Clearly, if $a, b \in A_K$ and $x > 0$, then $x^{a+b} = x^a x^b$ and $(x^a)^b = x^{ab}$. Moreover, if $\epsilon \in A_K$ with $v(\epsilon) > 0$, then $v(\epsilon) > v(\log(L^{>0}))$, $v(\epsilon \log(x)) \geq 0$ and $x^\epsilon \in \mathcal{O}^\times$ for all $x \in K^{>0}$.

Now, for $a, b \in A_K$ with $a = b + \epsilon$ for some $\epsilon \in o$, we have $x^a = x^{b+\epsilon} = x^b x^\epsilon$ and then $v(x^a) = v(x^b)$. Thus, we can say that Γ_K is a $\mathrm{res}(K)$-vector space in the following sense: for $r \in \mathrm{res}(K)$ and $x \in K^{>0}$, we define $rv(x) := v(x^a)$ for some $a \in A_K$ with $\mathrm{res}(a) = r$.

In preparation to prove some embedding lemmas using the above observations we recall the following:

Remark. *Let K be a model of T_{an}. Since T_{an} is o-minimal and has definable Skolem functions, we know that the operation of taking the definable closure of a subset A of K is a closure operation that satisfies the Steinitz exchange principle and then there is an associated notion of rank function. Particularly, the properties of this rank function imply that if $F \preceq K$ and $a, b \in K \setminus F$, then $F\langle a\rangle = F\langle b\rangle$ if $b \in F\langle a\rangle \setminus F$. Moreover, if $F \subseteq K$ are models of T_{an} and $y \in K \setminus F$ with $v(F(y)^\times) \neq v(F^\times)$, then there is $a \in F$ such that $v(y - a) \notin v(F^\times)$.*

Let $\mathbf{K} = (K, \mathcal{O}, \exp, \log, L)$ be a model of $T_+(\log)$. From the above remark we obtain the next extension lemma:

Lemma 6.15. *Let $\boldsymbol{E} = (E, \mathcal{O}_E, \exp_E, L_E)$ be a $L_+(\log)$-substructure of $\boldsymbol{K}$. If $\boldsymbol{E}$ satisfies axioms (1) to (6), and L_E is the A_E-module generated by $\log(E^{>0})$, then there is a model $\boldsymbol{F} = (F, \mathcal{O}_F, \exp_F, \boldsymbol{L}_F)$ of $T_+(\log)$ such that:*

(1) *$\boldsymbol{E} \subseteq \boldsymbol{F} \subseteq \boldsymbol{K}$, and*

(2) *for each $T_+(\log)$-model $\boldsymbol{K}^* = (K^*, \mathcal{O}^*, \exp^*, L^*)$ which extends $\boldsymbol{E}$, there is an $L_+(\log)$-embedding $\boldsymbol{F} \to \boldsymbol{K}^*$ over $\boldsymbol{E}$ that extends the natural inclusion $\boldsymbol{E} \to \boldsymbol{K}^*$.*

Proof. Trough this proof, if $B \subseteq K$ and $a \in K \setminus B$, then $B\langle a\rangle$ denotes the L_{an*}-substructure of K generated by B and a.

Existence. Define by transfinite induction an increasing chain $((F_\alpha, \mathcal{O}_\alpha, \exp_\alpha))_{\alpha<\lambda}$ of L_+-substructures of $(K, \mathcal{O}, \exp)$ and a chain $(L_\alpha)_{\alpha<\lambda}$ of subsets of L, with $\lambda < \operatorname{card}(E)^+$, as follows:

- Set $(F_0, \mathcal{O}_0, \exp_0) := (E, \mathcal{O}_E, \exp_E)$ and $L_0 = L_E$.
- Having defined $(F_\alpha, \mathcal{O}_\alpha, \exp_\alpha)$ and L_α, we have two cases:
  - For all $a \in L_\alpha$ there is $b \in F_\alpha^{>0}$ such that $a = \log(b)$. Then put $F_{\alpha+1} = F_\alpha$, $\mathcal{O}_{\alpha+1} = \mathcal{O}_\alpha$, $\exp_{\alpha+1} = \exp_\alpha$ and $L_{\alpha+1} = L_\alpha$.
  - There is $a \in L_\alpha$ such that $a \neq \log(b)$ for all $b \in F_\alpha$. Let $a \in L_\alpha \setminus \log(F_\alpha)$ such that $v(c) \notin \Gamma_\alpha$ for $c \in K \setminus F_\alpha$ with $\log(c) = a$. Then define $F_{\alpha+1} = F_\alpha\langle c\rangle$, $\mathcal{O}_{\alpha+1} = \mathcal{O} \cap F_{\alpha+1}$, $L_{\alpha+1} = L_\alpha + \mathcal{O}_{\alpha+1}$, and $\exp_{\alpha+1}$ the restriction of exp to $F_{\alpha+1}$.
- For ordinal limit $\alpha < \lambda$ we put $(F_\alpha, \mathcal{O}_\alpha, \exp_\alpha) = \cup_{\beta<\alpha}(F_\beta, \mathcal{O}_\beta, \exp_\beta)$ and $L_\alpha = L \cap F_\alpha$.

Setting $(F, \mathcal{O}_F, \exp_F) := \cup_{\beta<\lambda}(F_\beta, \mathcal{O}_\beta, \exp_\beta)$ and $L_F := \cup_{\beta<\lambda} L_\beta$, we have the following facts:

(1) Since $(E, \mathcal{O}_E, \exp_E)$ is a model of T_+, we can see that $(F_\alpha, \mathcal{O}_\alpha, \exp_\alpha)$ is a model of T_+ for all $\alpha < \lambda$. Thus, $(F, \mathcal{O}_F, \exp_F)$ is a model of T_+.

(2) Let Γ' be the $\operatorname{res}(E)$-vector space generated by Γ_E. For each $\alpha < \lambda$, if $F_{\alpha+1} \neq F_\alpha$, then $F_{\alpha+1} = F_\alpha\langle c\rangle$ for some $c \in K \setminus F_\alpha$ such that $\log(c) \in L_\alpha$ and $v(c) \notin \Gamma_\alpha$. By hypothesis and construction we have $v(c) \in \Gamma^*$, $v(c) < 0$, $\chi(v(c)) = v(\log(c)) \in \chi(\Gamma_\alpha^{<0})$,

$$\Gamma_{\alpha+1} := v(F_{\alpha+1}^\times) = \Gamma_\alpha \oplus \mathbb{Q}v(c),$$

and $\operatorname{res}(F_{\alpha+1}) = \operatorname{res}(F_\alpha)$. Thus, $\operatorname{res}(F) = \operatorname{res}(E)$ and $\Gamma_F = \Gamma'$. So, Γ_F is a model of T_{pdg}.

(3) If for each $\alpha < \lambda$ we put $\log_\alpha = \log|_{F_\alpha}$, then $\log_\alpha(F_\alpha^{>0})$. The proof is by induction. By hypothesis, $\log_{F_0}(F_0^{>0}) = \log_E(E^{>0}) \subseteq E$. Now, if we assume $\log_\alpha(F_\alpha^{>0}) \subseteq F_\alpha$ and $F_{\alpha+1} = F_\alpha\langle c\rangle$ with $c \in K \setminus F_\alpha$ such that $v(c) \notin \Gamma_\alpha$ and $\log(c) = a$ for some $a \in L_\alpha$, then for $d \in F_{\alpha+1}^{>0}$ there are $b \in F_\alpha$, $\epsilon \in \mathcal{O}_{\alpha+1}$ and $q \in \mathbb{Q}$ such that $a = b(1+\epsilon)c^q$. Thus

$$\log(a) = \log(b) + \log(1+\epsilon) + q\log(c) \in F_{\alpha+1}.$$

The case of limit ordinals is immediate.

As consequence of the above we have $\log_F(F^{>0}) \subseteq F$, with $\log_F = \log|_F$.

(4) By hypothesis L_E is the A_E-module generated by $\log(E^{>0})$. Thus, by construction for all $a \in L_F$ there is $b \in F^{>0}$ such that $\log_F(b) = a$. Thus, $L_F = \log(F^{>0})$.

(5) L_F is a maximal L-set on F. Clearly, L_F is an L-set on F. The proof of the maximality is again by induction. By hypothesis L_0 is a maximal L-set on F_0. Assume that L_α is maximal on F_α, $F_{\alpha+1} \neq F_\alpha$ and let $a \in F_{\alpha+1} \setminus F_\alpha$. If $a \notin L_{\alpha+1}$ and $v(a) \notin \chi(\Gamma_\alpha^{<0})$, then by observation 6.3 there are $b \in F_\alpha$ and $c \in F_{\alpha+1}$ such that $a = b + c$ with $v(c) \notin \Gamma_\alpha$. This implies that $v(c) \notin \chi(\Gamma^{<0})$. Since L_α is maximal on F_α, there is $d \in L_\alpha \subseteq L_{\alpha+1}$ such that $v(b-d) \in \Gamma_\alpha^{<0} \setminus \chi(\Gamma_\alpha^{<0})$. Thus, we obtain

$$v(a-d) = v(b-d+c) \in \Gamma_{\alpha+1}^{<0} \setminus \chi(\Gamma_{\alpha+1}^{<0}).$$

So, $L_{\alpha+1}$ is maximal on $F_{\alpha+1}$. Finally, the limit ordinal case is immediate.

From this observations we conclude that $\mathbf{F}$ is a model of $T_+(\log)$.

Embedding. Let $\mathbf{K}^* = (K^*, \mathcal{O}^*, \exp^*, L^*)$ be a model of $T_+(\log)$ such that $\mathbf{E} \subseteq \mathbf{K}$. We will prove by induction on $\alpha < \lambda$ that $\mathbf{F}$ embeds in $\mathbf{K}^*$.

- Let $\psi_0 : \mathbf{F}_0 = \mathbf{E} \to \mathbf{K}$ the natural embedding.
- Assume that $\psi_\alpha : \mathbf{H}_\alpha \to \mathbf{K}^*$ is an $L_+(\log)$-embedding which extends ψ_0 and define $\mathbf{F}'_\alpha = \psi_\alpha(\mathbf{F}_\alpha)$. If $F_{\alpha+1} = F_\alpha\langle c\rangle$ for $c \in K \setminus F_\alpha$ with $\log(c) = a$ for some $a \in L_\alpha$ and $v(c) \notin \Gamma_\alpha$, it follows that $a' := \psi_\alpha(a) \in L'_\alpha$ and $a' \neq \log(b)$ for any $b \in F'_\alpha$. Moreover, the element $c' \in K^* \setminus F'_\alpha$ with $\log^*(c') = a'$ realize the image under ψ_α of the cut that c made over F_α and $v^*(c') \notin \Gamma'_\alpha$. Thus we define $F'_{\alpha+1} = F'_\alpha\langle c'\rangle$, $\mathcal{O}_{\alpha+1} = \mathcal{O}^* \cap F'_{\alpha+1}$, $L'_{\alpha+1} = L'_\alpha + \mathcal{O}'_{\alpha+1}$, $\exp'_{\alpha+1}$ and $\log'_{\alpha+1}$ as the restrictions of $\exp^*$ and $\log^*$ to $F'_{\alpha+1}$, respectively. Now, since there is a unique convex ring $\mathcal{O}' \subseteq \mathcal{O}^*$ of $F'_{\alpha+1}$ such that $\mathcal{O}_\alpha \subseteq \mathcal{O}'$ and $c \notin \mathcal{O}'$, we have $O' = \mathcal{O}_{\alpha'}$ and $\psi_{\alpha+1} : \mathbf{F}_{\alpha+1} \to \mathbf{F}'_{\alpha+1}$ defined as the extension of ϕ_α with $\psi_{\alpha+1}(c) = c'$ is an $L_+(\log)$-embedding.
- For ordinal limit $\alpha < \lambda$ we define $\psi_\alpha := \cup_{\beta<\alpha}\psi_\beta : \mathbf{F}_\alpha \to \mathbf{K}^*$.

In conclusion there is an $L_+(\log)$-embedding $\psi : \mathbf{F} \to \mathbf{K}^*$. □

Let $\mathbf{K} = (K, \mathcal{O}, \exp, \log, L)$ be a model of $T_+(\log)$. We notice that for each $a \in K$ with $v(a) > \chi(\Gamma)$, $v(ax) = v(a) + v(x) > 0$ for all $x \in L$, that is $ax \in L$ and $a \in A_K$. Thus, under this observation and using the previous lemma we have the following result:

Proposition 6.16. *The theory $T_+(\log)$ has a prime model.*

Proof. Let $\mathbf{K} = (K, \mathcal{O}, \exp, \log, L)$ be a model of $T_+(\log)$. Trough this proof, if $B \subseteq K$ and $a \in K \setminus B$, then $B\langle a\rangle$ denotes the L_{an}-substructure of K generated by B and a in K.

Let $x \in K^{>0}$ be such that $v(x) < v(L)$ and (x_n) be the sequence in K defined by $x_0 = x$ and $x_{n+1} = \log(x_n)$. We define the chain $((F_n, \mathcal{O}_n, \exp_n))$ of L_+-substructures of $(K, \mathcal{O}, \exp)$ as $F_0 = \mathcal{O}_0 = \mathbb{R}_{an}$ with $\exp_0 = \mathrm{e}$, and $F_{n+1} = F_n\langle x_n\rangle$, $\mathcal{O}_{n+1} = \mathcal{O} \cap F_{n+1}$ with $\exp_{n+1} = \exp|_{F_{n+1}}$. Next, we set $(F, \mathcal{O}_F, \exp_F) := \cup(F_n, \mathcal{O}_n, \exp_n)$.

Since $v(x_n) < v(x_m)$ for all $n < m$ and $\{v(x_n) : n \geq 0\} \subseteq \Gamma$ is a $\mathbb{Q}$-linearly independent set, by construction of F we obtain $\mathrm{res}(F) = \mathbb{R}$ and

$$\Gamma_F := v(F^\times) = \mathbb{Q}v(x_0) \oplus \mathbb{Q}v(x_1) \oplus \cdots .$$

Moreover, taking $\log_F = \log|_F$, then $\log_F(F^{>0}) \subseteq F$. Indeed, if $a \in F^{>0}$, then $a \in F_n$ for some n, and there are $r \in \mathbb{R}$, $\epsilon \in \mathcal{O}$ and $q_0, ..., q_n \in \mathbb{Q}$ such that $a = r(1+\epsilon)\prod_{i=0}^{n-1} x_i^{q_i}$. Thus,

$$\log_F(a) = \log_F(r) + \log_F(1+\epsilon) + \sum q_i \log_F(x_i) \in F_{n+1} \subset F.$$

Setting $L_F := \mathbb{R}\log(F^{>0}) + \mathcal{O}_F$, we have that L_F is a maximal L-set of F. By definition, L_F is a L-set on F. Now, for $a \in F \setminus L_F$, with $v(a) \in \chi(\Gamma_F)^{<0}$, if $v(a-c) = v(x_m)$ for some $c \in L$ and m, then there is $r \in \mathbb{R}$ such that $v(a-c-rx_m) > v(x_m)$. However, since $a \in F_n$ for some n, $v(a - L_F) \cap \chi(\Gamma_F)$ has a maximum, so there is $d \in L$ such that $v(a-d) \in \Gamma_F^{<} \setminus \chi(\Gamma_F)$. Thus, L_F is a maximal L-set on F.

In sum, $\mathbf{F} := (F, \mathcal{O}_F, \exp_F, \log_F, L_F)$ is a $L_+(\log)$-substructure of $\mathbf{K}$ with $(F, \mathcal{O}, \exp_F)$ a model of T_+, $\Gamma_F \models T_{pdg}$, L_F a maximal L-set on F and L_F the $\mathbb{R}$-vector space generated by $\log(F^{>0})$ in F.

Claim. $\mathbf{F}$ embeds into any model $\mathbf{K}^* = (K^*, \mathcal{O}^*, \exp^*, \log^*, L^*)$ of $T_+(\log)$.

Proof of claim. Since $\mathbf{K}^*$ is a model of $T_+(\log)$, there is an element $y \in K^{>0}$ such that $v^*(y) < v^*(L^*)$. Let $y_0 \in K^{>0}$ with $v^*(y_0) < v^*(L^*)$ and define $y_{n+1} = \log^*(y_n)$. We use (y_n) to construct the chain $((F'_n, \mathcal{O}'_n, \exp'_n))$ in $\mathbf{K}^*$ in the same way we use (x_n) to build $((F_n, \mathcal{O}_n, \exp_n))$ in $\mathbf{K}$. Now, we will define an L_+-embedding ϕ_n from $(F_n, \mathcal{O}_n, \exp_n)$ onto $(F'_n, \mathcal{O}'_n, \exp'_n)$.

First, as $F_0 = \mathcal{O}_0 = F'_0 = \mathcal{O}'_0 = \mathbb{R}_{an}$, we define $\phi_0 : (F_0, \mathcal{O}_0, \exp_0) \to (F'_0, \mathcal{O}'_0, \exp'_0)$ as the natural embedding. Next, assume we have defined the L_+-embedding ϕ_n from $(F_n, \mathcal{O}_n, \exp_n)$ into $(F'_n, \mathcal{O}'_n, \exp'_n)$. Since for each n

- $x_n \notin \mathcal{O}$ and $|O_n| < x_{n+1} < |F_n \setminus \mathcal{O}_n|$, and
- $y_n \notin \mathcal{O}^*$ and $|O'_n| < y_{n+1} < |F'_n \setminus \mathcal{O}'_n|$,

by fact (C7) of chapter 3 there are unique subrings $\mathcal{O}_{n+1} \subseteq F_{n+1}$ and $\mathcal{O}'_{n+1} \subseteq F'_{n+1}$ such that $x_n \notin \mathcal{O}_{n+1}$ and $y_n \notin \mathcal{O}'_{n+1}$. Thus there is an L_{a+}-isomorphism

$$\phi_{n+1} : (F_{n+1}, \mathcal{O}_{n+1}, \exp_{n+1}) \to (F'_{n+1}, \mathcal{O}'_{n+1}, \exp'_{n+1})$$

which extends ϕ_n. Moreover, $\phi_{n+1}(\log(x)) = \log^*(\phi_{n+1}(x))$ for all $x \in F_{n+1}^{>0}$. Thus, we may define the L_+-isomorphism $\phi : (F, \mathcal{O}, \exp) \to (F', \mathcal{O}', \exp')$ as $\phi = \cup\phi_n$. Clearly, for all $x \in F^{>0}$ we have that $\phi(\log_F(x)) = \log^*(\phi(x))$, and defining $L_{F'} = \mathbb{R}\log((F')^{>0}) + \mathcal{O}_{F'}$ we have $\phi(L_F) = L_F$. Thus, there is an $L_+(\log)$-embedding of $\mathbf{F}$ into $\mathbf{K}^*$. Finally, by Lemma 6.15, there is a model $\mathbf{H}$ of $T_+(\log)$ such that $\mathbf{F} \subseteq \mathbf{H} \subseteq \mathbf{K}$ and for any other model $\mathbf{K}^*$ of $T_+(\log)$, $\mathbf{H}$ embeds into $\mathbf{K}^*$. So, $\mathbf{H}$ is a prime model of $T_+(\log)$. □

□

Following the proof of the last lemma, we can see $(\mathbb{T}_{\log}, \mathcal{O}_{\log}, \exp, \log, \log(\mathbb{T}_{\log}^{>0}))$ as the prime model of the theory $T_+(\log)$.

6.3.1 Towards the model completeness of $T_+(\log)$

Since for each model of the theory $T_+(\log)$ we can characterize the elements which are not logarithms using the set of logarithms and the valuation (under the notion of maximal log-fields), and the theories T_{pdg} and T_+ are model complete, we have the following conjecture:

Conjecture 1. *The theory* $T_+(\log)$ *is model complete and complete.*

Clearly, the completeness of $T_+(\log)$ follows of the model completeness of $T_+(\log)$ and Proposition 6.16.

We believe that a good strategy to prove that $T_+(\log)$ is model complete is to follow the lines of the model completeness proof of T_+. Specifically, if $\mathbf{E} = (E, \mathcal{O}_E, \exp_E, \log_E, L_E)$ and $\mathbf{K} = (K, \mathcal{O}, \exp, \log, L)$ are models of $T_+(\log)$ with $\mathbf{E} \subseteq \mathbf{K}$, we want to extend properly $\mathbf{E}$ to a submodel $\mathbf{F}$ of $\mathbf{K}$ such that $\mathbf{F}$ embeds into any κ^+-saturated model $\mathbf{K}^*$ of $T_+(\log)$ extending $\mathbf{E}$, with $\kappa = \operatorname{card}(K)$. To achieve this we need to study the following cases:

I. $\operatorname{res}(E) \neq \operatorname{res}(F)$.

II. $\operatorname{res}(E) = \operatorname{res}(F)$ and there is $y \in K \setminus E$ such that $|\mathcal{O}_E| < y < |E \setminus \mathcal{O}_E|$.

III. $\operatorname{res}(E) = \operatorname{res}(F)$ and there is no $y \in K \setminus E$ such that $|\mathcal{O}_E| < y < |E \setminus \mathcal{O}_E|$.

In the following we study extensions of models of $T_+(\log)$ related to each one of above cases and present some ideas and results that point to a possible way to prove conjecture 1.

I. Different residual fields

As a first step to prove that $T_+(\log)$ is model complete, we want show that if $\mathbf{E} \subseteq \mathbf{K}$ are models of $T_+(\log)$ with $\operatorname{res}(E) \neq \operatorname{res}(K)$, then there is a model $\mathbf{F}$ of $T_+(\log)$ such that $\mathbf{E} \subseteq \mathbf{F} \subseteq \mathbf{K}$ and $\operatorname{res}(F) = \operatorname{res}(K)$. To do this we need to show first the following conjecture:

Conjecture 2. *Let* $\boldsymbol{E} \subseteq \boldsymbol{K}$ *be models of* $T_+(\log)$ *with* $\operatorname{res}(E) \neq \operatorname{res}(K)$ *and* $a \in A_K$ *be such that* $\operatorname{res}(a) \notin \operatorname{res}(E)$*. If* $\boldsymbol{F}$ *is the* $L_+(\log)$*-substructure of* $\boldsymbol{K}$ *generated by* $E \cup \{a\}$*,* $b \in L_E$ *with* $v(b) < 0$ *and* $\epsilon \in \mathcal{O}_F$ *with* $b\epsilon \notin L_F$ *and* $|\chi(v(b))| = \chi(v(\epsilon))$*, then there is* $h \in L_F$ *such that* $v(b\epsilon - h) \in \Gamma_F^{<0} \setminus \chi(\Gamma_F)$*.*

Modulo this conjecture we have the following result:

Proposition 6.17. *Let* $\boldsymbol{E} \subseteq \boldsymbol{K}$ *be models of* $T_+(\log)$ *such that* $\operatorname{res}(E) \neq \operatorname{res}(K)$*. There is a model* $\boldsymbol{F}$ *of* $T_+(\log)$ *such that:*

(1) $\boldsymbol{E} \subseteq \boldsymbol{F} \subseteq \boldsymbol{K}$ *and* $\operatorname{res}(F) = \operatorname{res}(K)$*, and*

(2) *for any* κ^+*-saturated elementary extension* $\boldsymbol{K}^*$ *of* $\boldsymbol{E}$*, with* $\kappa = \operatorname{card}(K)$ *there is an embedding* $\boldsymbol{F} \to \boldsymbol{K}^*$ *that extends the natural inclusion* $\boldsymbol{E} \to \boldsymbol{K}^*$*.*

Proof. Since $\operatorname{res}(K) = \operatorname{res}(\mathcal{O}) = \operatorname{res}(A_K)$ and $\exp(A_K) \subseteq A_K$, we can modify the proof of Proposition 3.8, taking elements of $A_K^\times$ instead of $\mathcal{O}^\times$, to obtain a model $(E', \mathcal{O}_{E'}, \exp_{E'})$ of T_+ such that

- $(E, \mathcal{O}, \exp_E) \subseteq (E', \mathcal{O}_{E'}, \exp_{E'}) \subseteq (K, \mathcal{O}, \exp)$,
- $\mathrm{res}(E') = \mathrm{res}(A_{E'}) = \mathrm{res}(K)$, and
- for any κ^+-saturated elementary extension $\mathbf{K}^* = (K^*, \mathcal{O}^*, \exp^*, \log^*, L^*)$ of $\mathbf{E}$, with $\kappa = \mathrm{card}(K)$, there is an embedding $(E', \mathcal{O}_{E'}, \exp_{E'}) \to (K^*, \mathcal{O}^*, \exp^*)$ extending the natural inclusion of $(E, \mathcal{O}_E, \exp_E)$ into $(K^*, \mathcal{O}^*, \exp^*)$.

$v(E\langle a\rangle^\times) = v(E^\times)$ since $v(E(a)^\times) = v(E^\times)$ and $v(E\langle a\rangle^\times) = v(E(a)^\times)$ for all $a \in A_K$. So $\Gamma_{E'} := v(E'^\times) = v(E^\times)$. From this observation, $\log(E'^{>0}) \subseteq E'$, since for each $a \in E'^{>0}$, there are $b \in E$ and $c \in \mathcal{O}_{E'}^\times$ such that $a = bc$. Moreover, $\log(a) = \log(b) + \log(c) \in E'$ because $\log(E) \subseteq E \subseteq E'$ and $\log(O_{E'}^\times) \subseteq \mathcal{O}_{E'}$.

Claim. If we define $L_{E'} = A_{E'} \log((E')^{>0}) + \mathcal{O}_{E'}$, then $L_{E'}$ is an L-set on E' and we claim that L'_E is in fact a maximal L-set on E'.

Proof of claim. The first affirmation follows from the definition of $L_{E'}$. Now, let $a \in E' \setminus E$ be such that $a \notin L_{E'}$ and $v(a) \in \chi(\Gamma_{E'}^{<0})$. There are $b \in E$, $c \in A_{E'}^\times$ and $\epsilon \in E'$ such that $a = b(c+\epsilon)$. Thus, $v(a) = v(b)$. If $b \notin L_E$, then there is $h \in L_E$ such that $v(b-h) \in \Gamma_E^{<0} \setminus \chi(\Gamma_E)$ and then $v(a-ch) = v(b(c+\epsilon)-ch) \in \Gamma_E^{<0} \setminus \chi(\Gamma_E)$ with $ch \in L_{E'}$. Now, if we assume $b \in L_E$ then we have the following cases:

(1) If $|\chi(v(b))| \neq \chi(v(\epsilon))$, then $v(\epsilon) < |v(b)|$, $v(b\epsilon) \in \Gamma_E^{<0} \setminus \chi(\Gamma_E)$ and $v(a-bc) \in \Gamma_E^{<0} \setminus \chi(\Gamma_E)$ with $bc \in L_{E'}$.

(2) If $|\chi(v(b))| = \chi(v(\epsilon))$, by Conjecture 2 there is $d \in L_{E'}$ such that $v(a-d) \in \Gamma_E^{<0} \setminus \chi(\Gamma_E)$.

□

Now, there is an embedding $\phi : (E', \mathcal{O}_{E'}, \exp_{E'}) \to (K^*, \mathcal{O}^*, \exp^*)$ that extends the natural inclusion $(E, \mathcal{O}_E, \exp_E) \to (K^*, \mathcal{O}^*, \exp^*)$. Moreover, $\phi(\log(a)) = \log^*(\phi(a))$ for all $a \in E'^{>0}$ and $\phi^{-1}(L^*) = L'_E$. Thus, $\phi : \mathbf{E}' := (E', \mathcal{O}_{E'}, \exp_{E'}, \log_{E'}, L_{E'}) \to \mathbf{K}^*$ is in fact an $L_+(\log)$ embedding.

Finally, by Lemma 6.15, there is a model $\mathbf{F}$ of $T_+(\log)$ such that $\mathbf{E}' \subseteq \mathbf{F} \subseteq \mathbf{K}$ and $\mathbf{F}$ embeds over $\mathbf{E}$ into $\mathbf{K}$ as $L_+(\log)$-structures.

□

II. $\mathrm{res}(E) = \mathrm{res}(K)$ **and** $|O_E| < y < |E \setminus \mathcal{O}_E|$ **for some** $y \in K \setminus E$

Through this section $\mathbf{E} \subseteq \mathbf{K}$ will be models of $T_+(\log)$ with $\mathrm{res}(E) = \mathrm{res}(K)$ and $y \in K \setminus E$ be such that $y > 0$, $v(y) < 0$ and $|O_E| < y < |E \setminus \mathcal{O}_E|$. Thus, we see that $y \notin \mathcal{O}$. Moreover, $E(y)$ is not an immediate extension of E, on the contrary, there is $b \in E^{>0}$ such that $v(y) = v(b)$ and $\chi(v(b)) = v(\log(b)) > v(y)$ which is a contradiction since $v(y) \geq v(x)$ for all $x \in E \setminus \mathcal{O}_E$. Thus, $v(y) > v(x)$ for all $x \in E$ with $v(x) < 0$.

Now, let $y \in K \setminus E$ such that $|O_E| < y < |E \setminus \mathcal{O}_E|$ and $v(y) > v(x)$ for all $x \in E$ with $v(x) < 0$. Then we have:

Lemma 6.18. *There is an $L_+(\log)$-substructure $\boldsymbol{F}$ of $\boldsymbol{K}$ such that:*

(1) $\boldsymbol{E} \subseteq \boldsymbol{F}$ *and* $y \in F$*;*

(2) $L_F = L \cap F$.

Proof. Let (x_n) be the sequence in K given by $x_0 = \log(y)$ and $x_{n+1} = \log(x_n)$. We define a chain $((E_n, \mathcal{O}_n, \exp_n))$ of L_+-submodels of $(K, \mathcal{O}, \exp)$ and a chain (L_n) of subsets of L, such that $x_n \in F_{n+1}$ and $L_n \subseteq E_n$, as follows:

- Set $(E_0, \mathcal{O}_0, \exp_0) := (E, \mathcal{O}_E, \exp_E)$ and $L_0 = L_E$.
- Defined $(E_n, \mathcal{O}_n, \exp_n)$ and L_n, put $E_{n+1} = E_n\langle x_n \rangle$, $\mathcal{O}_n$ the unique convex subring of E_{n+1} such that $x_n \notin \mathcal{O}_{n+1}$, $\exp_n = \exp|_{E_n}$, and $L_{n+1} = B_{n+1} + \mathcal{O}_{n+1}$ where B_{n+1} is the A_E-module generated by L_n and x_n.

Now set $(E', \mathcal{O}_{E'}, \exp_{E'}) := \cup(E_n, \mathcal{O}_n, \exp_n)$ and $L_{E'} := \cup L_n$. We have the following observations:

- $x_0 > x_1 > \cdots$ and $\{v(x_0), v(x_1), ...\}$ is a res(E)-independent set over Γ_E.
- $|O_n| < x_n < |E_n \setminus \mathcal{O}_n|$, for each n. Thus, by fact $C7$ of Chapter 2 there is a unique convex subring $\mathcal{O}_n$ of E_{n+1} that does not contain y and $\mathcal{O}_n = \mathcal{O}_{n+1} \cap E_n$.
- $\Gamma_{n+1} := v(E_{n+1}^{\times}) = \Gamma_E \oplus \mathbb{Q}v(x_0) \oplus \mathbb{Q}v(x_1) \oplus \cdots \mathbb{Q}v(x_n)$.
- $\log(E'^{>0}) \subseteq E'$. If $a \in E' > 0$, then $a \in E_{n+1}^{>0}$ for some n, and $a = b(1+\epsilon)\prod_{i=0}^{n} x_i^{q_i}$ for some $b \in E, \epsilon \in \mathcal{O}_{E'}$ and $q_1, ..., q_n \in \mathbb{Q}$.Thus,

$$\log(a) = \log(b) + \log(1+\epsilon) + \sum_{i=0}^{n} q_i \log(x_i) \in E_{n+1}.$$

- $L_{E'}$ is a L-set on E', with $L_{E'} \subseteq L \cap E'$.

In sum $(E', \mathcal{O}_{E'}, \exp_{E'}, \log_{E'}, L_{E'})$ is a $L_+(\log)$-structure with $\log_{E'} = \log|_{E'}$ and $L_{E'}$ a L-set on E'. Next we close under exponentiation by transfinite induction. Thus we define a $L_+(\log)$-structure $\mathbf{F} := (F, \mathcal{O}_F, \exp_F, \log_F, L_F)$ such that $E(y) \subseteq F$ and $L_F = L \cap F$.

□

In the context of the above lemma, we have the following conjecture:

Conjecture 3. *The L-set* L_F *of* F *is a maximal L-set on* F*.*

If this conjecture is true and $(v(\mathbf{F}^{\times}), \chi)$ is a model of T_{pdg}, then $\mathbf{F} \subseteq \mathbf{K}$ is a model of $T_+(\log)$ which contains $\mathbf{E}$, and by construction of $\mathbf{F}$ and maximality of $L_F = \log(F^{>0})$ there is an embedding of $\mathbf{F}$ into any κ^+ saturated model $\mathbf{K}^*$ of $T_+(\log)$ which extends $\mathbf{E}$, with $\kappa = \mathrm{card}(K)$.

Now, if $(v(\mathbf{F}^{\times}), \chi)$ is not a model of T_{pgd}, then we have to extend $\mathbf{F}$ to complete the copy of $\mathbb{Z}$ corresponding to β in $\chi(\Gamma^{<0})$ at the end of Γ_E. In this case we have the following conjecture, based on Lemma 5.28:

Conjecture 4. *There is a model $\boldsymbol{F'} \subseteq \boldsymbol{K}$ of $T_+(\log)$ which extends $\boldsymbol{F}$ such that:*

(1) *The precontraction group $(\Gamma_{F'}, \chi)$ associated to $\boldsymbol{F'}$ is an extension of (Γ_E, χ) given by a family $(\beta)_{k\in\mathbb{Z}} \in \chi(\Gamma^{<0})$ such that $\beta_{k+1} = \chi(\beta_k)$, $\beta_k > v(x)$ for all $x \in E$ with $v(x) < 0$ and all $k \in \mathbb{Z}$, and $\chi(v(y)) = \beta_0$.*

(2) *For any model $\boldsymbol{K}^*$ of $T_+(\log)$ which extends $\boldsymbol{F}$ and family $(\beta^*_k)_{k\in\mathbb{Z}}$ in $\chi((\Gamma^*)^{<0})$ such that $\beta^*_{k+1} = \chi(\beta^*_k)$, $\beta^*_k > v(x)$ for all $x \in E$ with $v(x) < 0$ and all $k \in \mathbb{Z}$, and $\chi(v^*(y)) = \beta^*_0$, there is an embedding of $\boldsymbol{F'}$ into $\boldsymbol{K}^*$ over $\boldsymbol{F}$ which induces an embedding of $(\Gamma_{F'}, \chi)$ into (Γ^*, χ) over Γ_F such that β_k is sends to β^*_k.*

III. $\text{res}(E) = \text{res}(K)$ **and there is no** $y \in K \setminus E$ **with** $|O_E| < y < |E \setminus \mathcal{O}_E|$

Thought this section $\mathbf{E} \subseteq \mathbf{K}$ will be models of $T_+(\log)$ with $\text{res}(F) = \text{res}(K)$ and such that there is no $y \in K \setminus E$ with $|O_E| < y < |E \setminus O_E|$.

Let $y \in K \setminus E$. Without loss of generality we can assume that $y > 0$ and $v(y) < 0$. Here, we have two possibilities: either $E(y)$ is an immediate extension of E, or for all $\alpha, \beta \in \chi(\Gamma_E^{<0})$ with $\alpha < \chi(v(y)) < \beta$, we have $\alpha < \chi^k(\chi(v(y))) < \beta$ for all $k \in \mathbb{Z}$.

Let $y \in K \setminus E$ such that $|O_E| < y < |E \setminus \mathcal{O}_E|$ and $E(y)$ is an immediate extension of E. Then, in a routine way, we can construct the smallest $L_+(\log)$-substructure $\mathbf{F}$ of $\mathbf{K}$ such that $E(a) \subseteq F$, F is an immediate extension of E and $F \cap L = \log(F^{>0})$.

Conjecture 5. *$L_F = \log(F^{>0})$ is a maximal L-set on F.*

Under this conjecture we see that $\mathbf{F}$ is in fact a model of $T_+(\log)$ and by construction, $\mathbf{F}$ embeds into any κ^+ saturated model $\mathbf{K}^*$ of $T_+(\log)$ which extends $\mathbf{E}$, with $\kappa = \text{card}(K)$.

Now, let $y \in K \setminus E$ with $y > 0$, $v(y) < 0$ and such that $\alpha < \chi^k(v(y)) < \beta$ for all $k \in \mathbb{Z}$ and all $\alpha, \beta \in \chi(\Gamma_E^{<0})$ with $\alpha < \chi(v(y)) < \beta$. Here we can mimic the proof of Lemma 6.18 to obtain an $L_+(\log)$-substructure $\mathbf{F}$ of $\mathbf{K}$ such that $\mathbf{E} \subseteq \mathbf{F}$, $y \in F$ and $L_F = L \cap F$. Assuming conjecture 3, we also have two possibilities. Either $(v(F^\times), \chi)$ is a model of T_{pdg} and then $\mathbf{F}$ is a model of $T_+(\log)$ (and there is an embedding of $\mathbf{F}$ into any κ^+-saturated model $\mathbf{K}^*$ of $T_+(\log)$ which extends $\mathbf{E}$), or $(v(F^\times), \chi)$ is not a model of T_{pdg}.

In this last case, we need extend $\mathbf{F}$ to complete the copy of $\mathbb{Z}$ corresponding to $\chi(v(y))$ in $\chi(\Gamma^{<0})$ between all $\alpha, \beta \in \chi(\Gamma_E^{<0})$ such that $\alpha < \chi^k(\chi(v(y))) < \beta$ for all $k \in \mathbb{Z}$. Specifically, we have the following conjecture, based on Lemma 5.28:

Conjecture 6. *There is a model $\boldsymbol{F'} \subseteq \boldsymbol{K}$ of $T_+(\log)$ which extends $\boldsymbol{F}$ such that:*

(1) *The precontraction group $(\Gamma_{F'}, \chi)$ associated to $\boldsymbol{F'}$ is an extension of (Γ_E, χ) given by a family $(\beta)_{k\in\mathbb{Z}} \in \chi(\Gamma^{<0})$ with $\beta_{k+1} = \chi(\beta_k)$, $\chi(v(y)) = \beta_0$ and $\alpha < \beta_k < \beta$ for all $\alpha, \beta \in \Gamma_E$ such that $\alpha < \chi(v(y)) < \beta$.*

(2) *For each model $\boldsymbol{K}^*$ of $T_+(\log)$ that extends $\boldsymbol{F}$ and family $(\beta^*_k)_{k\in\mathbb{Z}}$ in $\chi((\Gamma^*)^{<0})$ such that $\beta^*_{k+1} = \chi(\beta^*_k)$, $\chi(v(y)) = \beta^*_0$ and $\alpha < \beta^*_k < \beta$ for all $\alpha, \beta \in \Gamma_E$ with $\alpha < \chi(v(y)) < \beta$, there is an embedding of $\boldsymbol{F'}$ into $\boldsymbol{K}^*$ over $\boldsymbol{F}$ which induces an embedding of $(\Gamma_{F'}, \chi)$ into (Γ^*, χ) over Γ_F such that β_k is sends to β^*_k.*

Modulo conjectures 2,5,3 and 6, we obtain easily the proof of model completeness of $T_+(\log)$.

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