THE MODULI STACK OF ELLIPTIC CURVES

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By moduli space of Riemann surfaces of genus g, where g is a non-negative integer, we mean the set of isomorphism classes of complex analytic structures on a closed oriented surface of genus g, fixed once and for all. It is not clear *a priori* why this definition makes sense, nor whether this set has an extra structure, turning it into some kind of *space*. In this memoir, we focus on these questions in the genus 1 case. As a matter of fact, we shall also fix some extra data: a base point on our genus 1 curve. The moduli space we obtain by doing this is called the moduli space of elliptic curves and the goal of the thesis is to show that this space is a complex analytic space, in a sense that we will make precise along the way. The main result we prove is the following classical theorem:

Theorem. *The moduli stack of complex elliptic curves is a complex analytic stack, isomorphic to the orbifold* $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$*, where* $\mathfrak{h} := \{z \in \mathbb{C} \mid \mathrm{Im}\,(z) > 0\}$ *is the hyperbolic plane and*

$$\mathbf{SL}(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is the so-called modular group, acting on $\mathfrak{h}$ *via homographic transformations*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The material presented here is well-known but we set ourselves the task of writing an expository account of the proof, providing precise references whenever we cannot be self-contained. This memoir also provides an introduction to the notion of stack, which is ubiquitous in moduli theory, by showing how it can be used in the concrete case of elliptic curves.

The manuscript is divided into three parts. In the first one, we recall the definition of an elliptic curve and we review the classical material that will be useful to construct the moduli stack. Then in the second one, we show how the intuitive picture of a *family* of elliptic curves gives rise to the notion of stack. Finally, in the third section, we show that the moduli stack of elliptic curves is a complex analytic stack.

1. Elliptic curves

In this chapter we will discuss what the elliptic curve structure over a compact orientable surface of genus 1 is.

The main goal in this section is to understand, that an elliptic curve can be seen in different ways and each of these points of view will help us to characterize the moduli space of Riemann surfaces structures over a fixed compact orientable surface of genus 1. More precisely, the goal is to understand various possible definitions of an elliptic curve, starting with the following one.

Definition 1.1. Let X be a compact Riemann surface. The sheaf of holomorphic $1-$forms on X is the sheaf of sections of the complex cotangent bundle $T_X \to X$. This sheaf will be denoted by Ω^1_X.

Definition 1.2. An elliptic curve is a pair (X, e) where X is a compact Riemann surface such that the dimension of globally defined holomorphic $1-$forms, denoted by $H^0(X, \Omega^1_X)$, is 1 [1] and e is a point of X.

Implicitly within this definition our discussion is set in the category of complex manifolds together with holomorphic mappings, which will be denoted by $\mathscr{A}n$. We also note that the name "elliptic curve" is derived from the algebro-geometric point of view which will be explained later on.

Remark 1.3. Additionally we say that a morphism of elliptic curves is a morphism of the underlying compact Riemann surface such that the marked point is preserved, and we note such morphisms as $(X, e) \to (X', e')$.

Now we want to characterize isomorphism classes of these objects. In order to do this, let us give a couple of definitions.

Definition 1.4. A lattice Λ is a discrete subgroup of a real vector space V such that V/Λ is a compact space. This definition is used in [Hai08].

Definition 1.5. A complex one-dimensional torus is the quotient space of $\mathbb{C}$ (seen as a real vector space) by a lattice Λ, which will be denoted by $\mathbb{C}/\Lambda$. More precisely Λ acts on $\mathbb{C}$ by translations along elements of the lattice, (i.e. the action map is $(\tau, z) \in \Lambda \times \mathbb{C} \mapsto \tau + z \in \mathbb{C}$) and the holomorphic structure on the quotient is the one given by the projection map $\pi : (\mathbb{C}, 0) \to (\mathbb{C}/\Lambda, p)$.

Proposition 1.6. *The complex dimension of the space of holomorphic* $1-$*forms over a complex one-dimensional torus* $\mathbb{C}/\Lambda$ *is 1.*

Proof. Since we defined the complex structure on the torus by requiring the cover π to be holomorphic, then we have that a holomorphic differential defined on a sufficiently small chart V of the torus is identified with a holomorphic differential defined on $\pi^{-1}(V) \subseteq \mathbb{C}$. In order to get a globally defined holomorphic form on the torus it is necessary that the holomorphic forms defined on the preimages of charts coincide. Since the holomorphic $1-$forms on $\mathbb{C}$ are $f(z)dz$ with f holomorphic, then $f(z)$ has to be a doubly periodic function (i.e. $f(z + \Lambda) = f(z)$) in order to define a $1-$form of the torus, luckily the only such elements are constant multiples of dz this is a consequence of the fact that the only doubly periodic holomorphic functions over $\mathbb{C}$ are the constant ones which is a consequence of Liouville's theorem [Ahl66]. $\square$

This implies that if we choose a point on a torus then we have that $(\mathbb{C}/\Lambda, p)$ is an elliptic curve.

Remark 1.7. Since we are only going to talk about one-dimensional complex torus in this document, we will call them complex torus without risk of confusion.

Now we define what a divisor over a Riemann surface is in order to formulate Abel's theorem.

[1] in this case one says that the genus of X is 1

Definition 1.8. Let X be a Riemann surface then a divisor of X is a mapping

$$D : X \to \mathbb{Z} \tag{1.1}$$

such that for any compact subset $K \subseteq X$ there are only finitely many points $x \in K$ such that $D(x) \neq 0$. In particular if X is a compact Riemann surface then the set of points where a divisor is different from 0 (this set will be called the support of the divisor) is finite. Thus every divisor in a compact Riemann surface is a finite collection of points of the surface with some integer multiplicity (this is denoted by $\sum n_i p_i$). The set of all divisors $Div(X)$, has a sum induced by the sum on the additive group $\mathbb{Z}$.

To every non-zero meromorphic function $f : X \to \mathbb{C}$ and for every point $x \in X$ we associate the order of f at that point by:

$$ord_x(f) := \begin{cases} 0 & \text{if f is holomorphic and non-zero at } x, \\ k & \text{if f has a zero of order } k \text{ at } x \\ -k & \text{if f has a pole of order } k \text{ at } x \end{cases} \tag{1.2}$$

Then the mapping $x \to ord_x(f)$ is a divisor for any non-zero meromorphic function. The divisors that come from meromorphic functions will be called *principal divisors.*

Remark 1.9. Note that this definition is extended to meromorphic 1-forms of X in the following manner. Let ω be a 1-form then for any point $x \in X$ there is a coordinate neighborhood (U, z) of x and in this neighborhood we have $\omega = f dz$, where f is a meromorphic function then define $ord_x(\omega) = ord_x(f)$.

This definition is independent of coordinates because composition with the transition function (which is holomorphic) doesn't change the order at x. Thus define a divisor for every meromorphic $1-$form in the exact same way as before.

Proposition 1.10. *Let f be a meromorphic function on a complex manifold X and let ω be a $1-$form, then the divisors associated to them satisfy the following properties:*

$$(fg) = (f) + (g), \quad (1/f) = -(f), \quad (f\omega) = (f) + (\omega) \tag{1.3}$$

The proposition is an immediate consequence of the Laurent expansion of a meromorphic function at a pole or at a zero.

Definition 1.11. (Degree of a divisor) Let D be a divisor such that $D = \sum n_i p_i$ with $p_i \in X$ then $\sum n_i$ is a finite sum which will be called the *degree* of the divisor D and will be noted by $deg(D)$.

Proposition 1.12. *The degree of any principal divisor on a compact Riemann surface is 0.*

Proof. We will prove this fact in two statements, firstly we will show that for any meromorphic 1-form on X a compact Riemann surface, the sum of all its residues vanish. Secondly we will prove the proposition using the above statement.

Let ω be a meromorphic 1-form on X and let U_α be charts around each pole of ω (they will be finite since ω can only have a discrete set of poles). Let γ_α be a small loop around the pole $P_\alpha \in U_\alpha$ that avoids the other poles and D_α the disk that bounds the corresponding loop. Then we have

$$2\pi i Res(\omega) = 2\pi i \sum Res_{P_\alpha}(\omega) = \sum_\alpha \int_{\gamma_\alpha} \omega = \int_{\sum \gamma_\alpha} \omega = \int_{X \setminus \cup \{D_\alpha\}} d\omega = 0 \tag{1.4}$$

Since $X \setminus \{D_\alpha\}$ is an analytic submanifold of X then Stokes Theorem [GH11], page 33, guarantees that the last integral equals 0.

Now if we let f be a meromorphic function on X, then let us consider the integral $\int_\gamma df/f$ where γ is a sum of loops that avoid the zeroes and poles of f. Then by the argument principle we have that this integral counts the number of poles and zeroes with multiplicities which is exactly $deg(f)$ and by the above result we have that this number is 0. $\square$

A consequence of the above proposition 1.12 is that any meromorphic function has as many zeros as poles, counting multiplicity. In particular a holomorphic $1-$form cannot have zeros, unless it is identically 0.

Now we give some necessary definitions in order to state Abel's theorem.

Definition 1.13. (1-chains) A 1-chain on a Riemann surface X is a formal finite linear combination with integer coefficients,

$$c = \sum n_j c_j \text{ where } n_j \in \mathbb{Z} \tag{1.5}$$

and where $c_j : [0,1] \to X$ is a curve (i.e. continuous piece-wise differentiable function) on X.

We will denote the group of 1-chains on X by $C_1(X)$, where the group operation is the sum of formal linear combinations, on this group we define the following boundary operator:

$$\begin{aligned} \partial : C_1(X) &\to Div(X) \\ c = \sum n_j c_j &\to \partial c = \sum n_j \big(c_j(1) - c_j(0)\big). \end{aligned} \tag{1.6}$$

The following convention is established, we set the divisor $c(1) - c(0)$ to be 0 if $c(1) = c(0)$ is the same point on X. We denote the kernel of the map ∂ by $Z_1(X, \mathbb{Z})$.

We integrate $1-$forms along chains in the natural way, that is:

$$\int_{\sum n_j c_j} \omega = \sum n_j \int_{c_j} \omega.$$

Theorem 1.14. *(Abel's theorem) Suppose D is a divisor on a compact Riemann surface X with $deg(D) = 0$. Then D is a principal divisor if and only if there exists a 1-chain $c \in C_1(X)$ with $\partial c = D$ such that*

$$\int_c \omega = 0 \textit{ for every } \omega \in \Omega^1_X. \tag{1.7}$$

A proof of this theorem can be found in [GF12] theorem 20.7 page 163.

Definition 1.15. (Period map) Let (X, e) be an elliptic curve and let $\omega \in H^0(X, \Omega^1(X))$ be a non-zero holomorphic $1-$form on X then we define the following map:

$$\begin{aligned} \Phi : H_1(X, \mathbb{Z}) &\to \mathbb{C} \\ \gamma &\mapsto \int_\gamma \omega \end{aligned} \tag{1.8}$$

Now that the *period map* has been defined, we associate a group to every elliptic curve (X, e) in the following way:

Fix a non-zero holomorphic $1-$form $\omega \in H^0(X, \Omega^1(X))$ and let:

$$\Lambda(\omega) := \{\int_\gamma \omega \in \mathbb{C} \mid \gamma \in H_1(X, \mathbb{Z})\} = Im(\Phi : H_1(X, \mathbb{Z}) \to \mathbb{C}) \tag{1.9}$$

The elements of $\Lambda(\omega)$ are called periods, which inherit the group structure from the codomain of the period map.

Remark 1.16. We note that choosing another non-zero holomorphic 1-form will yield an isomorphic group and this will give us an isomorphic torus (a proof of this statement will be given later on).

Theorem 1.17. *The group of periods $\Lambda(\omega)$ is a lattice over $\mathbb{C}$.*

Proof. The proof of this statement is carried out in two parts. First the discreteness, which will heavily rely on Abel's theorem:

Let (U, z) be a simply connected coordinate neighborhood of p with $z(p) = 0$. With respect to these coordinates let

$$\omega = z_0 dz \text{ where } z_0 \in \mathbb{C} \tag{1.10}$$

Now define the map $F : U \to \mathbb{C}$ as $F(x) = \int_p^x \omega = \int_p^x z_0 dz$ where this integral is taken along any path from p to x. It is well defined since the integral only depends on the homotopy class of the path and, since U is contractible, there is only one class for each path.

This map is complex differentiable and its differential is just $z_0 \neq 0$, since $F(y) = F(x) + \int_x^y \omega$ and this integral is bounded by the distance between x and y. As a consequence of the open mapping theorem [Ahl66] we have that $F(U)$ is a neighborhood of $0 \in \mathbb{C}$. Now we have to show that it doesn't intersect $\Lambda(\omega)$ at any other point.

If we suppose there is a point t in the intersection this implies that there is a point $y \in U$ and a path γ from p to y also in U such that $F(y) = t = \int_\gamma \omega$ where γ is in a different homology class. Thus we can consider a divisor such that its degree is 0 and whose support is y and p. Since there is a contractible path from p to y then the hypothesis of Abel's theorem are satisfied, in conclusion there must be a meromorphic function f on X which has a pole of order 1 at y and a zero of order 1 at p and is holomorphic otherwise. Let cz^{-1} be its principal part at y then

$$0 = Res(f\omega) = z_0 c. \tag{1.11}$$

The first equality comes from 1.12, and the second equality from the fact that y is the only pole of f, but this is a contradiction since none of these constants are zero.

The second part is the compactness of $\mathbb{C}/\Lambda(\omega)$. Note that if $\Lambda(\omega)$ is contained in a proper real vector subspace of $\mathbb{C}$ then the quotient would either be diffeomorphic to a cylinder or to a plane, since none of those are compact. Then we need to prove that Λ is generated by two $\mathbb{R}$-linearly independent vectors of $\mathbb{C}$.

Suppose it isn't the case, then there would exist a non trivial real linear form $g : \mathbb{R}^2 \to \mathbb{R}$ that vanishes precisely on the real vector space generated by elements of Λ. Since any real linear form is the real part of a unique complex one, then we get a complex non trivial linear form $G : \mathbb{C} \to \mathbb{C}$ and since $\mathbb{C}^* \cong \mathbb{C}$ then we also get an element $c \in \mathbb{C} \setminus \{0\}$ with $G(z) = cz$ and such that

$$g(\int_\gamma \omega) = Re(c \int_\gamma \omega) = 0 \text{ for every } \gamma \in H_1(X, \mathbb{Z}) \tag{1.12}$$

but we know that $Re(c\omega)$ is exact because $F(x) = \int_p^x Re(c\omega)$ is well defined over X (since it is independent of the chosen path) and $F(x)$ is holomorphic since its differential is $Re(c\omega)$, in particular F is a primitive of $Re(c\omega)$. Now we know that $Re(c\omega)$ is a harmonic $1-$form since it is the real part of a holomorphic $1-$form. This implies that it must vanish because the space of exact forms is orthogonal to the space of harmonic ones [GF12] pages 155, 156. Which is a contradiction because the real linear form is not trivial. $\square$

Theorem 1.18. *Let (X, e) be an elliptic curve, and let $\Lambda(\omega)$ be the lattice defined in* 1.9. *Then (X, e) is isomorphic to the marked complex torus $(\mathbb{C}/\Lambda(\omega), 0)$.*

Proof. Let (X, e) be an elliptic curve and let $\omega \in H^1(X, \mathbb{C})$ be a non-zero holomorphic differential. We define the following biholomorphism:

$$\begin{aligned} F : X &\to \mathbb{C}/\Lambda(\omega) \\ x &\mapsto F(x) = \int_e^x \omega \mod \Lambda(\omega) \end{aligned} \tag{1.13}$$

where the integral is meant as a path integral from any path in X from e to x, and clearly $F(e) = 0$. This map is well defined precisely because those paths differ by elements of $H_1(X, \mathbb{Z})$ which are naturally identified with elements of $\Lambda(\omega)$ (to every cycle $[\gamma]$ we assign its period $\int_\gamma \omega$).

Let us see that F is holomorphic, for that let $x \in X$ be a point and let $U \subseteq X$ be a neighborhood of x. If $y \in U$ then there is a path γ in U from x to y, which implies that:

$$F(y) = \int_P^y \omega \mod \Lambda(\omega) = F(x) + \int_\gamma \omega \mod \Lambda(\omega) \tag{1.14}$$

but we can take U to be a contractible neighborhood so that $\omega = f(z)dz$ inside of it and $\int_\gamma \omega$ doesn't depend on γ this implies that in U we have $F'(y) = \lim \frac{|F(y)-F(x)|}{|y-x|} = \lim \frac{|\int_x^y \omega|}{|y-x|} = f(y)$ which is holomorphic thus F is holomorphic inside of U, since we can choose a cover of X by this sort of open sets and apply the same procedure in each of them we conclude that F is holomorphic in all X.

Since F is a holomorphic mapping between compact Riemann surfaces then F is an open map, thus $F(X)$ is open and additionally we have that $F(X)$ is compact thus being inside a Hausdorff space it is closed, since $\mathbb{C}/\Lambda(\omega)$ is connected this means that F is surjective.

The map F is also injective, for otherwise we could replicate the proof of the discreteness of $\Lambda(\omega)$ 1.17 and Abel's theorem would again imply that there exists a meromorphic function f on X having a single pole of order one, which is impossible since the $Res(f\omega) \neq 0$ for ω a non-zero 1-form as we have already seen 1.12. □

As corollaries of this theorem we have:

Corollary 1.19. *Since the underlying Riemann surface X of an elliptic curve (X, e) is in particular homeomorphic to the torus $\mathbb{T} = S^1 \times S^1$, which has topological genus equal to 1. Then the topological genus of an elliptic curve is 1.*

Corollary 1.20. *Let (X, e) be an elliptic curve, then (X, e) has a group structure with identity e determined by an isomorphism $(X, e) \cong (\mathbb{C}/\Lambda(\omega), 0)$. Where the latter is known to be a group (with the addition of the complex numbers modulo Λ).*

Before studying some properties of complex tori that will be necessary for what follows, we will present an example that will explain why these objects are called elliptic curves.

Example 1.21. (Plane algebraic curves) Consider the following family of plane algebraic curves: Let R_λ be the the set of points $(x, y) \in \mathbb{C}^2$ that solves a polynomial equation of the following form:

$$y^2 = x(x-1)(x-\lambda) \text{ where } \lambda \in \mathbb{C} \setminus \{0, 1\}. \tag{1.15}$$

Additionally to the set of points that solve this equation we add the point $p_\infty = (\infty, \infty)$, this new set will be noted by R. Define a complex manifold structure over R by considering the projection $\pi : R_\lambda \to \mathbb{C}, \pi(x, y) = x$.

Since $\mathbb{C}$ is a Riemann surface then we can define over it the unique complex manifold structure such that π is a holomorphic map (i.e. the charts over R are the preimages of charts of $\mathbb{C} \cup \{\infty\}$ through π), or equivalently it can be checked that the algebraic (complex analytic) function $f(x, y) = y^2 - x(x-1)(x-\lambda)$ has 0 as a regular value and thus by the inverse function theorem for holomorphic functions we would get a complex manifold structure. Note that building an explicit biholomorphism between this construction and a complex torus will be difficult.

However we note that π is a two sheeted branched covering since over any $x \in \mathbb{C} \cup \{\infty\}$ that isn't $0, 1, \lambda, \infty$ we know that $\pi^{-1}(x)$ is the set with elements:

$$\{(x, \sqrt{x(x-1)(x-\lambda)}), (x, -\sqrt{x(x-1)(x-\lambda)})\}.$$

Now if we consider the function $f : R \to \mathbb{C} \cup \{\infty\}, f(x, y) = y$ then we note that R is the Riemann surface where the function f is defined and is single valued. A deeper analysis of elliptic integrals (which are the ones that lend the name to these curves), namely integrals of the form:

$$\int_{\infty}^{x} \frac{dt}{\sqrt{t(t-1)(t-\lambda)}} \tag{1.16}$$

shows that these integrals are path dependent and in order to end this problem one can make branch cuts between $0, \infty$ and $1, \lambda$ in each copy of $\mathbb{C} \cup \{\infty\}$ and the glue them together along the corresponding cuts. This will not only produce a torus, it will produce the complex torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.[Sil09] Chapter 6.

Remark 1.22. Note that using the theory of Weierstrass elliptic functions, it can be proved that all complex tori are algebraic curves. This is an important fact but its proof is beyond the scope of this document and it isn't required for what follows so we refer to [Cle80] chapter 3, [GH11] chapter 6 or [Ser73] chapter 7, for a complete treatment of this subject.

Theorem 1.23. *Suppose that Λ_1, Λ_2 are two lattices over $\mathbb{C}$. If there is a holomorphic map $f : (\mathbb{C}/\Lambda_1, 0) \to (\mathbb{C}/\Lambda_2, 0)$, then there exists $c \in \mathbb{C}$ such that $c\Lambda_1 \subseteq \Lambda_2$ and*

$$f(z + \Lambda_1) = cz + \Lambda_2 \tag{1.17}$$

Proof. One knows that $(\mathbb{C}, 0) \to (\mathbb{C}/\Lambda, 0)$ is a pointed universal covering, thus the map f induces a map $F : \mathbb{C} \to \mathbb{C}$ between universal coverings such that it commutes with the projections which means that

$$f(z + \Lambda_1) = F(z) + \Lambda_2 \tag{1.18}$$

So one only need to prove that F is linear. Let $\omega_j = dz \in H^0(\mathbb{C}/\Lambda_j, \Omega^1_{\mathbb{C}/\Lambda_j})$ for $j = 1, 2$. As it is known that the space of differential one forms is one-dimensional for both tori, this implies that there is a constant $c \in \mathbb{C}$ such that $f^*\omega_2 = c\omega_1$. Consequently on each open ball the projections of the universal covering are just identities thus $dF = df = cdz$ and since $F(0) = 0$ this implies that $F(z) = cz$ □

Corollary 1.24. *Two complex tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic if and only if there exists $c \in \mathbb{C}^*$ such that $\Lambda_2 = c\Lambda_1$.*

Then we can deduce that $Aut(\mathbb{C}/\Lambda, 0) = \{u \in \mathbb{C}^* \mid u\Lambda = \Lambda\}$

Remark 1.25. Since the complex dimension of $H^0(X, \Omega^1_X)$ is 1, then for every $\omega, \omega' \in \Omega^1_X$ there is a $c \in \mathbb{C}$ such that $\omega = c\omega'$. This implies that $\Lambda(\omega) = c\Lambda(\omega')$ and consequently $\mathbb{C}/\Lambda(\omega) \cong \mathbb{C}/\Lambda(\omega')$, this proves 1.16.

More can be said about the automorphism group and its structure if we forget that the elliptic curve has a marked point.

Proposition 1.26. *The group of automorphisms of $(\mathbb{C}/\Lambda)$ is $\mathbb{C}/\Lambda \rtimes Aut(\mathbb{C}/\Lambda, 0)$ where $\mathbb{C}/\Lambda$ acts on itself by translations.*

Proof. Note that we can reproduce the proof of theorem 1.23 on two unmarked tori $\mathbb{C}/\Lambda, \mathbb{C}/\Lambda'$ to conclude that any holomorphic map f in between them induces a map $F : \mathbb{C} \to \mathbb{C}$ of covering spaces such that $F(z) = az + b$ and such that $a\Lambda \subseteq \Lambda'$. Consequently an automorphism f of $\mathbb{C}/\Lambda$ induces a map $F(z) = az + b$, obviously any translation satisfies this condition but notice that if $a = 1$ then only translations can satisfy this condition since $f(z) = F(z) + \Lambda = z + b + \Lambda$ for any $z \in \mathbb{C}/\Lambda$.

On the other hand if $a \neq 1$ we see that $\mathbb{C}/\Lambda$ acts on the automorphisms by translations (in the above case translating by b), thus we may consider $b = 0$ then we obtain the automorphisms of the form $\{u \in \mathbb{C}^* \mid u\Lambda = \Lambda\}$. Thus if we let $f \in Aut(\mathbb{C}/\Lambda)$ and let $F(z) = az + b$ be the lift of f to the universal covering of $\mathbb{C}/\Lambda$ then we can define the following map:

$$\begin{aligned} m : Aut(\mathbb{C}/\Lambda) &\to \mathbb{C}/\Lambda \rtimes Aut(\mathbb{C}/\Lambda, 0) \\ f &\mapsto (b, a) \end{aligned} \tag{1.19}$$

This map is injective since its kernel is just the identity, and is surjective by the considerations above. □

As an additional comment we note that if we use the isomorphism

$$Aut(\mathbb{C}/\Lambda, 0) \cong \{u \in \mathbb{C}^* \mid u\Lambda = \Lambda\} \tag{1.20}$$

to identify the elements of both groups, then the only elements that can be a part of this group are roots of unity. To see this firstly let us see that any element in $\{u \in \mathbb{C}^* \mid u\Lambda = \Lambda\}$ has modulus equal to 1. This is because the set of distances $\{d(u, v) = |u - v| \mid u, v \in \Lambda\}$ is a discrete set (if it wasn't it would contradict the discreteness of Λ). Then for any $a \in \{u \in \mathbb{C}^* \mid u\Lambda = \Lambda\}$ we have that

$$d(au, av) = |a|d(u, v) \text{ for all } u, v \in \Lambda \tag{1.21}$$

thus if u, v are elements which attain the minimum non-zero distance, then every automorphism has to send the set of this elements to itself otherwise we could not generate them with elements which have bigger modulus, thus $d(u, v) = d(au, av) = |a|d(u, v)$ and therefore $|a| = 1$.

Now set $\sigma = min\{|z| : z \in \Lambda \setminus \{0\}\}$, which is reached because Λ is a discrete set, and Λ_σ denote the finite set of all $z \in \Lambda$ such that $|z| = \sigma$ then for any $z \in \Lambda_\sigma$ and any automorphism $a \in \{u \in \mathbb{C}^* \mid u\Lambda = \Lambda\}$ we have that $|az| = |a||z| = \sigma$ thus a permutes the elements of Λ_σ, which is finite, thus there is a $k \in \mathbb{N}$ such that $a^k = 1$.

A consequence of this fact is that $Aut(\mathbb{C}/\Lambda, 0)$ is a discrete group and as an abstract group it should be a subgroup of $Gl(2, \mathbb{Z})$ since an element $u \in \mathbb{C}^*$ such that $u\Lambda = \Lambda$ induces an element of $Gl(2, \mathbb{Z})$ by choosing a basis of Λ, and the complex dimension of $Aut(\mathbb{C}/\Lambda)$ is 1.

Definition 1.27. A framing of a lattice Λ in $\mathbb{C}$ is an ordered basis (λ_1, λ_2) such that λ_2/λ_1 has positive imaginary part.

Remark 1.28. We say that two elements $a, b \in H_1(X, \mathbb{Z})$ have intersection number (noted $a.b$) equal to 1 if for any non-zero holomorphic $1-$form of X, we have that $\frac{\int_b \omega}{\int_a \omega}$ has positive imaginary part. We also acknowledge that the intersection number can also be defined in terms of the fundamental form of X but we consider that this definition will fit better for our purposes.

We can also define a similar apparatus on an elliptic curve.

Definition 1.29. A framed elliptic curve is an elliptic curve (X, e) together with an ordered basis (a, b) of $H_1(X, \mathbb{Z})$ with the property that the intersection number $a.b = 1$.

Note that for a complex torus $\mathbb{C}/\Lambda$ its first homology group $H_1(\mathbb{C}/\Lambda, \mathbb{Z})$ is isomorphic to Λ, thus a framing on the lattice induces a framing on the torus. With these new objects at hand, we say that a morphism between the framed elliptic curves $(X, e; a, b)$ and $(X', e'; a', b')$ is a morphism of elliptic curves $f : (X, e) \to (X', e')$ such that the induced map on homology preserves the frame (i.e. $f_*(a) = a'$ and $f_*(b) = b'$).

Similarly we say that two framed lattices $(\Lambda, \lambda_1, \lambda_2)$ and $(\Lambda', \lambda_1', \lambda_2')$ are isomorphic if there is a non-zero complex number u such that $\lambda_j' = u\lambda_j$ for $j = 1, 2$. A consequence of this fact is that every framed lattice is isomorphic to a unique framed lattice of the form

$$\Lambda_\tau = (\mathbb{Z} \oplus \mathbb{Z}\tau, 1, \tau) \tag{1.22}$$

where τ lies in the upper half plane $\mathfrak{h}$.

Theorem 1.30. *For every $\tau \in \mathfrak{h}$ let $a(\tau), b(\tau) \in H_1(\mathbb{C}/\Lambda_\tau, \mathbb{Z})$ be the framing for which $a(\tau)$ and $b(\tau)$ correspond to the cycles $0 \to 1$ given by the loop $t + 0i \mod \Lambda$ for $t \in [0,1]$ and $0 \to \tau$ given by the loop $t\tau \mod \Lambda$ for $t \in [0,1]$ respectively. Then $(\mathbb{C}/\Lambda_\tau, 0; a(\tau), b(\tau))$ and $(\mathbb{C}/\Lambda_{\tau'}, 0; a(\tau'), b(\tau'))$ are isomorphic if and only if $\tau = \tau'$.*

Proof. Assume first that they are isomorphic, then we have a biholomorphism

$$f : (\mathbb{C}/\Lambda_\tau, 0) \to (\mathbb{C}/\Lambda_{\tau'}, 0) \tag{1.23}$$

such that $f_*(a(\tau)) = a(\tau')$ and $f_*(b(\tau)) = b(\tau')$. Then by theorem 1.23 we know that a lift $\overline{f}$ of f to the covering space is of the form $\overline{f}(z) = az$ for some $a \in \mathbb{C}$. Hence we conclude that $\overline{f}(1) = a = 1$ and $\overline{f}(\tau) = a\tau = \tau'$ therefore $\tau = \tau'$. The converse is clear, the isomorphism is the identity. □

Corollary 1.31. *For every framed elliptic curve $(X, e; a, b)$ we have that $Aut(X, e; a, b) = Id$.*

2. Families of elliptic curves

In this chapter we will discuss families of elliptic curves. These objects allow us to study what happens when we allow a fixed topological torus to have an elliptic curve structure that varies holomorphically, this variation will be parametrized by a base space which will be an object of $\mathscr{A}n$.

2.1. **Elliptic curves over an arbitrary base.**

Definition 2.1. A family of elliptic curves is a triple (X, π, e) such that X is in $\mathscr{A}n$ and π, s are morphisms such that:

$$\begin{tikzcd} X \arrow[d, bend left, "\pi"] \\ U \arrow[u, bend left, "e"] \end{tikzcd} \tag{2.1}$$

where $s : U \to X$ is a section of π and π is a surjective holomorphic map of maximal rank such that $(\pi^{-1}(u), e(u))$ is an elliptic curve .

Definition 2.2. (Pullback) Let $f : V \to U$ be a morphism of complex manifolds and $\pi : X \to U$ be a family of elliptic curves. We say that $\pi^* : X^* \to V$ is the pullback of the family over U through f if for any other family $X_2 \to V$ with $X_2 \to X$ such that the following diagram commutes

$$\begin{tikzcd} X_2 \arrow[dr, dotted, "\psi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X^* \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & V \arrow[r, "f"] & U, \end{tikzcd} \tag{2.2}$$

there exists a unique morphism $\psi : X_2 \to X^*$ that makes everything commute.

Fortunately, we can characterize very precisely the pullback family in this category.

Theorem 2.3. *The following family is the pullback for the above diagram*

$$f^*X := \{(x, v) \in X \times V | \pi(x) = f(v)\} \tag{2.3}$$

it inherits the projections from the product $pr_1 : f^*X \to X, pr_2 : f^*X \to V$

Proof. Let us see firstly show that $f^*X \to V$ together with the section $f \circ s$ is in fact a family of elliptic curves.

Note that f^*X is a complex manifold, consider the map $pr_2 \circ \pi : X \times V \to f(V)$ which is holomorphic and a submersion since π and pr_2 are, thus the complex implicit function theorem implies that $f^*X = (pr_2 \circ \pi)^{-1}(f(V))$ is a complex submanifold of $X \times V$. Since $pr_2 : f^*X \to V$ is a projection, since $pr_2^{-1}(a) = \{(x, a) | \pi(x) = f(a)\} \cong \pi^{-1}(f(a))$ and since $(\pi^{-1}(f(a)), s \circ f(a))$ is an elliptic curve, then we are done.

Now let us have another family of elliptic curves such that the following diagram commutes

$$\begin{tikzcd}
X_2 \arrow[drr, bend left, "h"] \arrow[dr, dotted, "\psi"] \arrow[ddr, bend right, "\tau"'] & & \\
& f^*X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& V \arrow[r, "f"] & U
\end{tikzcd} \tag{2.4}$$

then we define $\psi : X_2 \to f^*X$ by $\psi(t) = (h(t), \tau(t))$. This map is well defined since $\pi(h(t)) = f(\tau(t))$ and $pr_1 \circ \psi = h, pr_2 \circ \psi = \tau$ and is clearly the only map that makes everything commute. $\square$

Now we can introduce a category where we can work with these objects.

Definition 2.4. (The category of families of elliptic curves) We define the category $\mathscr{M}_{ell}$, where objects are pairs $(\pi : X \to U, e : U \to X)$ where π is a surjective holomorphic map of maximal rank such that $(\pi^{-1}(u), e(u))$ is an elliptic curve, and morphisms are cartesian squares

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & Q \arrow[d] \\
U \arrow[r, "\varphi"] & V.
\end{tikzcd}$$

This means that P is the pullback of Q through φ and the section over the pullback is $e \circ \varphi$.

We remark that this category is equipped with a functor $\pi : \mathscr{M}_{ell} \to \mathscr{A}n$ which is the forgetful functor which preserves the base space. This means:

$$\pi\left(\begin{tikzcd} P \arrow[d] \\ U \end{tikzcd}\right) = U \text{ and } \pi\left(\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ U \arrow[r, "\varphi"] & V \end{tikzcd}\right) = \begin{tikzcd} U \arrow[r, "\varphi"] & V \end{tikzcd} \tag{2.5}$$

In this case we say that $\mathscr{M}_{ell}$ is a category *over* $\mathscr{A}n$.

We will usually omit the section from the discussion to make the diagrams more readable, but never forgetting that it exists since it is the one that allows the family to have elliptic curves in each fibre.

From this category we can construct a functor which will be called $\mathscr{M}_{1,1}$. The codomain of this functor is the category of groupoids, which are defined as follows.

Definition 2.5. (Groupoid) A groupoid is a category in which every morphism is invertible. Thus the category of all groupoids *Grpds* has groupoids as objects and functors between them as morphisms.

Note that for every $U \in \mathscr{A}n$ we can consider the collection of objects $\pi^{-1}(U)$. They have a natural collection of morphisms associated to them, that is if $X_1, X_2 \in \pi^{-1}(U)$ then a morphism between them $f : X_1 \to X_2$ is such that $\pi(f) = id : U \to U$, this collection is a category and more importantly a groupoid.

Lemma 2.6. *The category $\pi^{-1}(U)$ defined above, is a groupoid.*

Proof. Let $\bar{f} : X_1 \to X_2$ be a morphism of $\pi^{-1}(U)$, we immediately have that this morphism comes from the following cartesian square

$$
\begin{tikzcd}
X_1 \arrow[r, "\bar{f}"] \arrow[d] & X_2 \arrow[d] \\
U \arrow[r, "id"] & U
\end{tikzcd}
\tag{2.6}
$$

Since this is a cartesian square then there exists a unique $\bar{g}$ that makes this diagram commute

$$
\begin{tikzcd}
X_2 \arrow[dr, dotted, "\bar{g}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "\pi"'] & & \\
& X_1 \arrow[r, "\bar{f}"'] \arrow[d] & X_2 \arrow[d, "\pi"] \\
& U \arrow[r, "id"] & U.
\end{tikzcd}
\tag{2.7}
$$

This implies that $\bar{g}$ is a left inverse for $\bar{f}$ and in much the same way we can interchange $\bar{f}$ by $\bar{g}$ in the preceding diagram, and by the uniqueness of the morphism we can conclude that they are the inverse of each other. $\square$

2.2. **Prestacks.** Thus we define the following functor:

Definition 2.7. Let us define the functor

$$
\begin{aligned}
\mathscr{M}_{1,1} : \mathscr{A}n &\longrightarrow Grpds \\
U &\longmapsto \mathscr{M}_{1,1}(U) := \pi^{-1}(U)
\end{aligned}
\tag{2.8}
$$

additionally, note that if we have $f : V \to U$ a morphism in $\mathscr{A}n$ then we can define the following functor:

$$
f^* : \mathscr{M}_{1,1}(U) \longrightarrow \mathscr{M}_{1,1}(V)
$$

$$
\begin{tikzcd}
X_1 \arrow[r, "g"] \arrow[d] & X_2 \arrow[dl] & \longmapsto & f^*X_1 \arrow[r, "f^*(g)"] \arrow[d] & f^*X_2 \arrow[dl] \\
U & & & V &
\end{tikzcd}
\tag{2.9}
$$

where f^*X is the pullback family along f, this is the same as saying that the diagram

$$
\begin{tikzcd}
f^*X_i \arrow[r, "\bar{f}_i"] \arrow[d] & X_i \arrow[d] \\
V \arrow[r, "f"] & U
\end{tikzcd}
\tag{2.10}
$$

is cartesian for $i = 1, 2$.

In order to check that $\mathscr{M}_{1,1}$ is in fact a functor, we have to check that f^* is a functor between $\mathscr{M}_{1,1}(U)$ and $\mathscr{M}_{1,1}(V)$ thus we have to define the morphism $f^*(g)$ and check that this definition is compatible with composition on $\mathscr{M}_{1,1}(U)$. Additionally we will get a natural transformation every time we have two composable morphisms in $\mathscr{A}n$ and an associativity condition induced by the composition of three morphisms in $\mathscr{A}n$. The two previous properties tell us that $\mathscr{M}_{1,1}$ is not just a functor but *a prestack* whose formal definition will be stated after this example.

So let $f : V \to U$ be a map in $\mathscr{A}n$ and let $g : X_1 \to X_2$ be a map in $\mathscr{M}_{1,1}(U)$ then we have the following maps:

(2.11)
$$
\begin{tikzcd}
f^*X_i \arrow[r, "\bar{f}_i"] \arrow[d] & X_i \arrow[d] \\
V \arrow[r, "f"] & U
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*X_1 \arrow[dr, dotted, "G"] \arrow[drr, bend left, "g\circ\bar{f}_1"] \arrow[ddr, bend right] & & \\
& f^*X_2 \arrow[r, "\bar{f}_2"'] \arrow[d] & X_2 \arrow[d] \\
& V \arrow[r, "f"] & U
\end{tikzcd}
$$

As the right diagram commutes we get a *unique* morphism G, thus we can define $f^*(g)$ as the morphism G.

It is clear that f^* respects the composition law of $\mathscr{M}_{1,1}(V)$ since if we allow:

$$X_1 \xrightarrow{g} X_2 \xrightarrow{h} X_3 \ \in \mathscr{M}_{1,1}(U),$$

then the map $f^*(g \circ h)$ has to be equal to $f^*(g) \circ f^*(h)$ by uniqueness of the morphism that makes following diagram commute

(2.12)
$$
\begin{tikzcd}
f^*X_1 \arrow[dr, dotted, "f^*(h)\circ f^*(g)"] \arrow[drr, bend left, "h\circ g\circ\bar{f}_1"] \arrow[ddr, bend right] & & \\
& f^*X_3 \arrow[r, "\bar{f}_3"'] \arrow[d] & X_3 \arrow[d] \\
& V \arrow[r, "f"] & U.
\end{tikzcd}
$$

Since $\bar{f}_3 \circ f^*(h) \circ f^*(g) = h \circ \bar{f}_2 \circ f^*(g) = h \circ g \circ \bar{f}_1$. Thus the identification f^* identifies the maps $f^*(g \circ h)$ and $f^*(g) \circ f^*(h)$, hence it respects the associativity of morphisms in $\mathscr{M}_{1,1}(U)$ and $\mathscr{M}_{1,1}(V)$. In conclusion, f^* is indeed a functor.

Note that additionally if we allow $W \xrightarrow{g} V \xrightarrow{f} U$ two composable morphisms in $\mathscr{A}n$ then we have two functors $\mathscr{M}_{1,1}(U) \underset{(f\circ g)^*}{\overset{g^*\circ f^*}{\rightrightarrows}} \mathscr{M}_{1,1}(W)$ and we can define a natural transformation between them which is:

$$\Phi_{f,g} : \ \mathscr{M}_{1,1}(U) \underset{(f\circ g)^*}{\overset{g^*\circ f^*}{\rightrightarrows}} \mathscr{M}_{1,1}(W)$$

(2.13)
$$
\begin{tikzcd}
X_1 \arrow[d, "t"] \\
X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
g^* \circ f^*(X_1) \arrow[r, "\Phi_{f,g}(X_1)"] \arrow[d, "f^*\circ g^*(t)"] & (f \circ g)^*(X_2) \arrow[d, "(f\circ g)^*(t)"] \\
g^* \circ f^*(X_2) \arrow[r, "\Phi_{f,g}(X_2)"'] & (f \circ g)^*(X_2)
\end{tikzcd}
$$

Where the morphism $\Phi_{f,g}(X_i)$ is defined as the unique morphism given by the universal property of the pullback $(g \circ f)^*(X_i)$, for example:

$$\begin{tikzcd} g^* \circ f^*(X_1) \arrow[dr, dotted, "\Phi_{f,g}(X_1)"] \arrow[drr, bend left, "t\circ\bar{f}_1\circ\bar{g}_1"] \arrow[ddr, bend right] & & \\ & (g\circ f)^* X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ & W \arrow[r, "f\circ g"] & U. \end{tikzcd} \tag{2.14}$$

This morphism makes the above diagram commute as a consequence of all the previous pullback diagrams commuting. Additionally if we have 3 composable morphisms h, g, f in $\mathscr{A}n$, we have associativity on the natural transformation (i.e $\Phi_{f\circ g,h} = \Phi_{f,h\circ g}$).

This kind of functor is known as a 2-functor between the category $\mathscr{A}n$ and Groupoids or equivalently as a *prestack* and the category $\mathscr{M}_{ell}$ is known as a *fibred category* over $\mathscr{A}n$.

Definition 2.8. (Prestack) A prestack $\mathscr{N}$ is a (2)-functor

$$\mathscr{N} : \mathscr{A}n \to Gpds \tag{2.15}$$

this means:

(1) For any manifold $U \in \mathscr{A}n$ we get a groupoid $\mathscr{N}(U)$.
(2) For any morphism $f : V \to U$ we get a functor

$$f^* : \mathscr{N}(U) \to \mathscr{N}(V)$$

and id^* has to be the identity functor.

(3) For any $W \xrightarrow{g} V \xrightarrow{f} U$ two composable morphisms in $\mathscr{A}n$ we get a natural transformation $\Phi_{f,g} : g^* \circ f^* \to (g \circ f)^*$ that identifies the two functors (since the maps of the natural transformation are all invertible because they are morphisms of the groupoid $\mathscr{N}(W)$). Additionally the natural transformation is associative whenever we have 3 composable morphisms.

We present more examples of this sort of object:

Example 2.9. (Manifolds as prestacks) If $U \in \mathscr{A}n$ we can define the category $\mathscr{A}n_U$ where the objects are $f : V \to U \in \mathscr{A}n$ and morphisms are commutative triangles

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & U \\ W \arrow[ur, "g"'] & \end{tikzcd}$$

(that we will call φ by an abuse of notation).

We define $\pi : \mathscr{A}n_U \to \mathscr{A}n$ as the forgetful functor which sends $V \xrightarrow{f} U$ to V and which on morphisms acts as follows:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & U & \longmapsto & V \arrow[d, "\varphi"] \\ W \arrow[ur, "g"'] & & & W. \end{tikzcd}$$

We notice that in this case the groupoid associated to a manifold V is the category $\mathscr{A}n_U(V)$ where objects are maps $V \to U$ and morphisms are commutative triangles:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & U \\ V \arrow[ur, "g"'] & \end{tikzcd}$$

such that $\pi(\varphi) = id_V$. Consequently there is only one morphism for each object in $\mathscr{A}n_U(V)$ which is the identity of V, this is precisely the groupoid generated by the set $Hom_{\mathscr{A}n}(V, U)$. Thus we define the prestack

$$\underline{U} : \mathscr{A}n \longrightarrow Gpds \tag{2.16}$$
$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi"'] & U & \longmapsto & \underline{U}(V) := Hom_{\mathscr{A}n}(V, U) \\ W \arrow[ur, "g"'] & & & \underline{U}(W) := Hom_{\mathscr{A}n}(W, U) \arrow[u, "\underline{U}(\varphi) := \varphi\circ \_"'] \end{tikzcd}$$

Since the only morphisms in the groupoid $Hom_{\mathscr{A}n}(V, U)$ are the identity of V over every object then it is easy to check all the requirements for being a prestack. Eventually we will drop the underline on this notation and we will consider all complex manifolds as prestacks. The next example is of particular interest for the moduli problem of elliptic curves.

Example 2.10. Let G be a complex Lie group acting by biholomorphisms on a complex manifold M. Let $[M/G]$ be the category of pairs $(P \to U,\ P \xrightarrow{f} M)$ where the first element of the pair is a G-principal bundle and the second element is a G-equivariant analytic map. The definition of a morphism between two pairs in $[M/G]$, $(P \to U,\ P \xrightarrow{f} M\ )$ and $(Q \to V,\ Q \xrightarrow{g} M\ )$ will be a cartesian square:

$$\begin{tikzcd} P \arrow[r, "\beta"] \arrow[d] & Q \arrow[d] \\ U \arrow[r, "\varphi"] & V. \end{tikzcd}$$

This means that P is isomorphic to the pullback of Q through φ, such that β is equivariant and $g \circ \beta = f$, and the functor $\pi : [M/G] \to \mathscr{A}n$ is the forgetful functor which assigns $(P \to U,\ P \xrightarrow{f} M)$ to U and the previous cartesian square to $\varphi : U \to V$, we see that this example is in fact a prestack because for any map $U \xrightarrow{\varphi} V$ and any principal G-bundle $Q \longrightarrow V$, we can form the pullback.

Note that $[M/G](U)$ is the category of principal G-bundles over U such that there is an equivariant map from the total space to M but since morphisms in this category are cartesian squares where the forgetful functor sends the square to the identity of U (i.e. the bottom map of the square is the identity), this implies that the only morphisms that can complete such a diagram are precisely the action by elements of G since the following diagram

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d] & P \arrow[d] \\ U \arrow[r, "Id"] & U \end{tikzcd} \tag{2.17}$$

where $g : P \to P$ is the action by the element $g \in G$, is cartesian since it commutes and P is clearly the pullback along the identity. Any other morphism $P \to P$ that makes this diagram commute has to send every orbit of the G-action to itself thus it can only permute the elements of each orbit.

Since P is a principal G-bundle then every orbit of an element of P is isomorphic to G and the multiplication by an element of G on a fibre commutes with the action of G on P. Consequently the restriction of the map to the orbits has to be a group homomorphism (i.e. multiplication by some element of the group) and the map $P \to P$ is the action of an element on P.

We conclude that the groupoid $[M/G](U)$ has principal G-bundles over U as objects. The only non trivial morphisms are in between isomorphic bundles and the set $Hom_{[M/G](U)}(P, P)$ is in bijection with G.

Let $\mathfrak{X}$ be a prestack over $\mathscr{A}n$. We will introduce some notation that will be needed in order to make some maps more readable. If $f : U \to V \in \mathscr{A}n$ and $\phi : A \to B \in \mathfrak{X}(V)$ we will denote $A_{|U}, B_{|U}$ the pullbacks $f^*(A), f^*(B) \in \mathfrak{X}(U)$ and $\phi_{|U} := f^*(\phi)$ the morphism between them induced by ϕ, this notation will come in handy specially when we talk about open covers of objects.

Note that the prestack $\mathscr{M}_{1,1}$ has the following interesting properties:

Theorem 2.11. *Any maximal rank surjective holomorphic map $\pi : X \to U$ such that each fibre $\pi^{-1}(x)$ is a compact Riemann surface, defines a C^∞ locally trivial bundle.*

Proof. Any submersion is an open map because we can always find charts around any point of the domain and its image such that the submersion restricted to this chart is a projection, which is an open map. On the other hand we know that any open map with compact fibres is proper because if we take any $K \subset U$ compact and $\mathfrak{U}$ a cover of $\pi^{-1}(K)$ then for any $k \in K$ we have that $\mathfrak{U}$ in particular covers the fibre above it, hence the compacity of the fibre implies that there are $V_1^k, ...V_{n_k}^k \in \mathfrak{U}$ that cover it. Then $k \in f(\bigcup_i V_{i_k}^k)$, as k was arbitrary and f is open then the collection of open sets $\{\bigcup_{k\in K} \bigcup_i V_{i_k}^k\}$ are such that $f(\bigcup V_{n_k}^k)$ covers K, thus there is a finite subcollection that covers K. After establishing this fact we can now use Ehresmann's fibration lemma [Ehr52] which states that a surjective submersion that is also a proper map is a locally trivial C^∞ fibration. ☐

2.3. **Stacks.** Let us consider a cover $(V_i)_{i\in I}$ of $U \in \mathscr{A}n$. We have that if for two families $\pi : X \to U$, $\pi' : X' \to U$ in $\mathscr{M}_{1,1}(U)$ and any collection $(\phi_i : X_{|V_i} \to X'_{|V_i})_{i\in I}$ such that $\phi_i|_{V_i\cap V_j} = \phi_j|_{V_i\cap V_j}$ of isomorphisms. Then we have that $(\pi^{-1}(V_i))$ and $(\pi'^{-1}(V_i))$ are covers of X and X' respectively. Since the collection of isomorphisms ϕ_i is defined over these covers, we can consider the map defined by parts

$$\begin{aligned} \phi : X &\longrightarrow X' \\ x &\mapsto \phi(x) = \phi_i(x) \quad \text{if } \pi(x) \in V_i \end{aligned} \tag{2.18}$$

which is a well-defined map of complex manifolds because the family ϕ_i coincides on the intersections of the cover $(\pi'^{-1}(V_i))$. Additionally this map is the unique one such that $\phi|_{V_i} = \phi_i$ since if any other satisfy these conditions, then it is bound to have the exact same values.

On the other hand we also notice that if we have a cover $(V_i)_{i\in I}$ of $U \in \mathscr{A}n$ and families $\pi_i : X_i \to V_i$ together with isomorphisms $\varphi_{ij} : X_i|_{V_i\cap V_j} \to X_j|_{V_i\cap v_j}$ which satisfies the cocycle condition on threefold intersections i.e. $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}|_{V_i\cap V_j\cap V_k}$. Then we can glue together this local data in order to construct the complex manifold $X = \sqcup X_i/\sim$ where we identify $x, y \in \sqcup X_i$ if $x, y \in X_i \cap X_j$ and $\varphi_{ij}(x) = y$ (this is an equivalence relation as a consequence of the cocycle condition).

Additionally we can define the following family:

$$\begin{aligned} \pi : X = \sqcup X_i/\sim &\longrightarrow U \\ [x] &\mapsto \pi_i(x) \text{ if } x \in X_i. \end{aligned} \tag{2.19}$$

Which is well defined, since if $y \in [x]$ then $\pi_j(\varphi_{ij}(x)) = \pi_i(x)$, as isomorphisms of families have to commute with the projections. Also note that $\pi : X \to U$ is clearly a holomorphic map of max rank since it is so for every open set in the cover $\{X_i\}$ of X. Additionally for every $m \in U$ we have that $m \in V_i$ thus $\pi^{-1}(m) = \pi_i^{-1}(m)$ which is in fact an elliptic curve (and the marked point is given by the section of the family $X_i \to V_i$). In conclusion $\pi : X \to U$ is a family of elliptic curves.

It turns out that the difference between prestacks and stacks is given by exactly the two conditions mentioned above, that we will now state in a broader sense, this conditions are called descent data.

Definition 2.12. (Stack over $\mathscr{A}n$) A prestack $\mathfrak{X}$ over $\mathscr{A}n$ is a *stack* if, for any $U \in \mathscr{A}n$ and any open cover $(V_i)_{i\in I}$ of U, the following two conditions are satisfied:

(1) Gluing morphisms: Given two objects $A, B \in \mathfrak{X}(U)$ and any family $(\phi_i : A_{|V_i} \to B_{|V_i})_{i\in I}$ such that $\phi_i|_{V_i\cap V_j} = \phi_j|_{V_i\cap V_j}$ then there is a unique morphism $\phi : A \to B$ such that $\phi_{|V_i} = \phi_i$. This condition implies that the presheaf $V \to Hom_{\mathfrak{X}(V)}(A_{|V}, B_{|V})$ is a sheaf.

(2) Gluing objects: Let $A^i \in \mathfrak{X}(V_i)$ be a collection of objects, together with isomorphisms $\phi_{ij} : A^j_{V_i\cap V_j} \to A^i_{|V_i\cap V_j}$ in $\mathfrak{X}(V_{ij})$ such that they satisfy the cocycle condition $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $\mathfrak{X}(V_{ijk})$ for any triple indices. Then there is an $A \in \mathfrak{X}(U)$, together with isomorphisms $\varphi_i : A_{|V_i} \to A_i$ such that

$$\begin{tikzcd} A_{|V_i} \arrow[r, "\phi_j"] \arrow[d, "\varphi_i"'] & A_j \\ A_i \arrow[ur, "\phi_{ij}"'] & \end{tikzcd}$$

commutes.

Note that this object is unique up to isomorphism because any two such objects are isomorphic locally and we can glue together these local isomorphisms to produce a global one.

Proposition 2.13. *The above examples 2.9, 2.10 satisfy the condition for being a stack over $\mathscr{A}n$. Additionally theorem 2.11 implies that $\mathscr{M}_{1,1}$ is a stack.*

Proof. Since a map on a manifold is determined by its values on a cover and we can construct bundles by gluing local data, then we can repeat the proof done for $\mathscr{M}_{1,1}$, in 2.18, on both examples. □

Thus we can form the category of prestacks. Where we say that a morphism between prestacks $\mathfrak{X}, \mathfrak{X}'$ is a collection of functors $F_U : \mathfrak{X}(U) \to \mathfrak{X}'(U)$ for every $U \in \mathscr{A}n$ such that for any $f : V \to U$ morphism in $\mathscr{A}n$ we have a natural transformation $F_f : f^*F_U \to F_V f^*$ we will refer to this condition as *pullback compatibility.*

Since all morphisms in each groupoid are invertible we can easily find an inverse for the natural transformation ϕ, which will consist of the inverse of every map of the original natural transformation. Thus $Hom_{\mathfrak{pst}}(F; G)$ is a groupoid.

The category of prestacks ($\mathfrak{pst}$) satisfies a version of Yoneda's lemma.

Lemma 2.14. *(Yoneda's lemma for prestacks) Let U be a manifold, $\underline{U}(V) = Hom(V, U)$ the prestack associated to it and $\mathfrak{X} : \mathscr{A}n \to Gpds$ a prestack. The natural functor $Hom_{\mathfrak{pst}}(\underline{U}, \mathfrak{X}) \to \mathfrak{X}(U)$ is an equivalence of groupoids.*

Proof. The natural functor is defined by:

$$\begin{aligned} ev : Hom_{\mathfrak{pst}}(\underline{U}, \mathfrak{X}) &\longrightarrow \mathfrak{X}(U) \\ f : \underline{U} \to \mathfrak{X} &\longmapsto f(id_U) \end{aligned} \tag{2.20}$$

where clearly $id_U \in \underline{U}(u)$ and $f(id_U) \in \mathfrak{X}(U)$, since $f : \underline{U}(U) \to \mathfrak{X}(U)$ is a functor. We will construct an inverse of the natural functor. Let $A \in \mathfrak{X}(U)$, now consider the following morphism

$F_A : \underline{U} \to \mathfrak{X}$ which to every $g : V \to U$ in $\underline{U}(V)$ assigns the object $A_{|V} = g^*A \in \mathfrak{X}(V)$. This assignment defines a morphism from $\underline{U}$ to $\mathfrak{X}$ that we will call p. Now we compute the composition on an element

$$\begin{aligned} p \circ ev(f : \underline{U} \to \mathfrak{X}) &= p(f(id_U)) \\ &= F_{f(id_U)} : \underline{U} \to \mathfrak{X} \end{aligned} \tag{2.21}$$

If we evaluate this morphism in $V \in \mathscr{A}n$ then we get

$$\begin{aligned} F_{f(id_U)} &: \underline{U}(V) \to \mathfrak{X}(V) \\ g : V \to U &\mapsto g^*(f(id_U)) \cong f(g^*(id_U)) \\ &= f(g) \end{aligned} \tag{2.22}$$

and let $A \in \mathfrak{X}(U)$ then we have that:

$$\begin{aligned} ev \circ p(A) &= ev(F_A : \underline{U} \to \mathfrak{X}) \\ &= F_A(id_U) = id_U^* A = A \end{aligned} \tag{2.23}$$

□

Remark 2.15. Since the category of stacks has the same morphisms as the category of prestacks then Yoneda's lemma is also true for the category of stacks.

Note that a consequence of this lemma is that there is an equivalence of categories between $Hom_{\mathfrak{pst}}(\underline{U}, \underline{V})$ and $\underline{V}(U) = Hom(U, V)$. If we restrict this equivalence to objects or if we remember that the only morphisms in the groupoid generated by the elements of $Hom(U, V)$ is the identity of V over each object, then there is a bijection between the sets of morphisms $Hom_{\mathfrak{pst}}(\underline{U}, \underline{V})$ and $Hom(U, V)$.

This will be important in the moduli problem because we want the stack $\mathscr{M}_{1,1}$ to be isomorphic to a stack that comes from a manifold. Ideally the moduli problem would be solved if there is a universal elliptic curve over it.

This is, if $\mathscr{M}_{1,1} \cong \underline{V}$ where $V \in \mathscr{A}n$ then, for any $U \in \mathscr{A}n$ and its realization as a stack (that we will write as U from now on), that any map of stacks $U \to V$ induces a family of elliptic curves over U. If that is the case, then isomorphism classes of families of elliptic curves are in one to one correspondence with maps in $Hom_{\mathfrak{pst}}(U, V)$, which we know are identified with maps in $Hom(U, V)$.

3. Moduli

The moduli problem at hand is the classification of families of elliptic curves, the standard stack associated to this moduli problem was defined in the previous chapter, definition 2.7. In this section, we will study the stack $\mathscr{M}_{1,1}$ in more detail.

Definition 3.1. (Pullback of stacks) Let $f : \mathfrak{Y} \to \mathfrak{X}$ and $g : \mathfrak{Z} \to \mathfrak{X}$ be two morphisms in the category of stacks then we say $(\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z};\ f' : \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Z};\ g' : \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \to \mathfrak{Y})$ is the pullback of the two morphisms above and it is defined as follows:

For every $U \in \mathscr{A}n$ we have the groupoid whose objects are

$$\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}(U) := \langle (F, F', \phi) | F \colon U \to \mathfrak{Y},\ F' \colon U \to \mathfrak{Z},\ \phi \colon f \circ F \Rightarrow g \circ F' \rangle \tag{3.1}$$

and where morphisms between two objects $(F, F', \phi) \to (G, G', \varphi)$ are pairs of morphisms $(a_{F,G} : F \to G, a_{F',G'} F' \to G')$ such that $\varphi \circ f(a_{F,G}) = g(a_{F',G'}) \circ \phi$. Moreover we define $f' := pr_1$, $g' := pr_2$. Note that the above definition makes the following diagram commute

$$\begin{tikzcd} \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z} \arrow[r, "g'"'] \arrow[d, "f'"] & \mathfrak{Y} \arrow[d, "f"] \\ \mathfrak{Z} \arrow[r, "g"] & \mathfrak{X}. \end{tikzcd} \tag{3.2}$$

The stack $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}$ is also known as the fibre product of $\mathfrak{Y}$ and $\mathfrak{Z}$. If $\mathfrak{Y}, \mathfrak{Z}$ and $\mathfrak{X}$ are complex manifolds then the pullback of stacks coincides with the pullback of complex manifolds, defined in 2.2.

Definition 3.2. A representable (stack) morphism $\mathscr{M} \to \mathscr{N}$ is a morphism such that if we have $V \in \mathscr{A}n$ and a morphism $V \to \mathscr{N}$ then there exists $U \in \mathscr{A}n$ such that the stack $\mathscr{M} \times_{\mathscr{N}} V$ is isomorphic to the stack $Hom_{\mathscr{A}n}(-, U)$ defined in example 2.9. Equivalently we say that $\mathscr{M} \times_{\mathscr{N}} V$ is a complex manifold.

Definition 3.3. (Stack epimorphism) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a stack morphism, we say that f is an epimorphism if for every $U \in \mathscr{A}n$ and every $Y \in \mathfrak{Y}(U)$ there is an open cover of U, $i : V \to U$ and $X \in \mathfrak{X}(V)$ such that $f_V(X) = i^*Y$ see [LMB99] definition 3,6.

Definition 3.4. (Atlas over a stack) Let $\mathfrak{X}$ be a stack we say that an atlas over $\mathfrak{X}$ is a complex manifold $U \in \mathscr{A}n$ and a representable morphism $p : U \to \mathfrak{X}$, which is also an epimorphism.

Luckily the condition of being an epimorphism for a representable morphism is characterized by:

If $V \in \mathscr{A}n$ and there is a morphism $f : V \to \mathfrak{X}$ then the fibre product $U \times_{\mathfrak{X}} V$ (which is isomorphic to a complex manifold by the representability of the morphism p) is equipped with a map between complex manifolds $U \times_{\mathfrak{X}} V \to V$ that is a surjective submersion in $\mathscr{A}n$.

Now we say that, to solve the moduli problem $\mathscr{M}_{1,1}$ is to find a space M in $\mathscr{A}n$ and a representable epimorphism of stacks such that $M \to \mathscr{M}_{1,1}$, this is the same as giving the moduli stack a structure of a analytic stack which will allow us to study the moduli stack in a geometric way.

We could also say that the moduli problem is solved if the moduli stack is represented by a complex manifold, meaning that there is an isomorphism of stacks between the moduli stack and the complex manifold, but as we will see in the next section, there are moduli problems that can be represented in this way but in general this condition is much too restrictive. We also could define what a coarse moduli space is, and say that the moduli problem is solved by the stack associated to it, but we restrict ourselves to this way of solving the moduli problem for the sake of conciseness.

Furthermore, the goal of the next sections is to show that the moduli stack $\mathscr{M}_{1,1}$ is isomorphic to a quotient stack, defined in 2.10, of the proper action of a discrete group over a complex manifold.

3.1. **The moduli problem of framed elliptic curves.** Let us consider a supplementary moduli problem, that is the moduli problem of families of framed elliptic curves. We will call the stack associated to this problem $\hat{\mathscr{M}}_{1,1}$ and see that it is isomorphic to a complex analytic manifold.

As this stack has a natural map to our original moduli stack namely the map which forgets the frame over an elliptic curve, it will suffice to show how this map translates to the complex analytic manifold that represents it.

We start by remembering that any elliptic curve (X, p) is isomorphic to a complex torus of the form $(\mathbb{C}/\Lambda, 0)$ proved in theorem 1.18, we also recall that the map:

$$\begin{aligned} F : \Lambda = \gamma\mathbb{Z} \oplus \tau\mathbb{Z} &\longrightarrow H_1(\mathbb{C}/\Lambda, \mathbb{Z}) \\ a\gamma + b\tau &\longmapsto a\gamma' + b\tau' \end{aligned} \tag{3.3}$$

where γ' and τ' are the cycles given by the straight line paths $(0, \gamma)$ and $(0, \tau)$. Is an isomorphism of groups consequently we can identify Λ and $H_1(\mathbb{C}/\Lambda, \mathbb{Z})$ using this isomorphism. Thus a positive basis of the lattice Λ corresponds to a framing of $(\mathbb{C}/\Lambda, 0)$ and since every elliptic curve is isomorphic to a torus then this also defines what a frame over an elliptic curve is, see definition 1.29.

Definition 3.5. We say that two framed curves $(X, e; a, b)$ and $(X', e'; a', b')$ are isomorphic if there is $f : (X, e) \to (X', e')$ an isomorphism of elliptic curves such that $f_*(a) = a'$ and $f_*(b) = b'$.

A consequence of this definition is that the elliptic curve $(X, e; a, b)$ is isomorphic to $(\mathbb{C}/\Lambda, 0; \int_a \omega, \int_b \omega)$ where $\Lambda = \{\int_\gamma \omega \mid \gamma \in H_1(C, \mathbb{Z})\}$ is the period lattice and ω is a non-zero holomorphic 1-form of

X. Similarly the marked elliptic curve $(\mathbb{C}/\Lambda, 0; \int_a \omega, \int_b \omega)$ is isomorphic to $(\mathbb{C}/\Lambda', 0; 1, \frac{\int_b \omega}{\int_a \omega})$ where this time $\Lambda' = (\int_a \omega)^{-1}\Lambda$, or equivalently we choose $\omega' = \omega / \int_a \omega$, the isomorphism in between the complex tori is clearly $f([z]) = [z/(\int_a \omega)]$, thus we have the following proposition:

Proposition 3.6. *We can say that for any framed elliptic $(X, e; a, b)$ there is $\tau \in \mathfrak{h}$ such that $(X, e; a, b) \cong (\mathbb{C}/\Lambda, 0; 1, \tau)$.*

Remark 3.7. Note that this implies that the automorphisms of a framed elliptic curve are trivial. Since an automorphisms f of $(\mathbb{C}/\Lambda, 0)$ induces an element $c \in \mathbb{C}^*$ such that $c\Lambda = \Lambda$, this was proved in corollary 1.24. Thus this automorphisms induces an isomorphisms between the marked elliptic curves $(\mathbb{C}/\Lambda, 0; 1, \tau)$ and $(\mathbb{C}/\Lambda, 0; c, c\tau)$. Note that this is an automorphism of $(\mathbb{C}/\Lambda, 0; 1, \tau)$ only if $c = 1$ but this implies by 1.23, that $f = Id$.

Now the aim is to construct a framing for a family of elliptic curves, that is a continuously varying choice of homology element of every fibre of the family. To do this, we construct various fibre bundles.

Let $R_p := (\texttt{Homeo}((S,a) \to (X_p, e(p)))/\sim)$ be the topological space of all equivalence classes of homeomorphisms of marked topological spaces, where (S,a) is a fixed marked torus (meaning a topological compact oriented manifold of topological genus 1). The equivalence relation identifies two of this homeomorphisms f_1, f_2 if $f_2^{-1} \circ f_1 \cong id_{(S,a)}$ meaning that this composition is homotopic to the identity of the torus.

Note that

$$G := \texttt{Homeo}((S,a),(S,a))/\sim, \tag{3.4}$$

is a group and acts on R_p by right composition. Moreover as we know that the family above X is locally trivial as a topological bundle, then by fixing an open cover $\{V_\alpha\}$ that trivializes the family we get transition functions $\phi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times (S,a)$ and trivializations $\psi_{\alpha,\beta} : V_\alpha \cap V_\beta \to G$. We use the trivializations in order to glue $R := \sqcup_\alpha(\cup_{x\in V_\alpha} R_x)/\diamond \ \to U$ (where $(x,y) \diamond (x',y')$ if $x = x' \in V_\alpha \cap V_\beta$ and $y' = \psi_{\alpha\beta}(x)(y)$), which is a G-principal bundle.

Note that the group G acts on the homology, cohomology and even homotopy groups of (S,a) by composition, for example $\Delta : G \to Aut(H_1(S,\mathbb{Z})) \cong Gl(2,\mathbb{Z})$ $[f] \to f_*([\gamma]) = [f \circ \gamma]$. Also note that given an element γ of $Gl(2,\mathbb{Z})$ we can produce a linear homeomorphism of $\mathbb{R}^2$ that is equivariant with the action by translation of $\mathbb{Z}^2$ through $\gamma(1,0), \gamma(0,1)$ thus it descends to a map $f : S \to S$ and $f_* = \gamma$.

Since G acts both on the first homology group of S and the the bundle $R \to U$, then we can construct an associated fibre bundle with fibre the group $H_1(S,\mathbb{Z})$. We will call the latter bundle:

$$R \times_G H_1(S,\mathbb{Z}) = P' \to U. \tag{3.5}$$

If we evaluate a local section of this bundle at a point $p \in U$ we will get a G-orbit: $[f : S \to X_p, \gamma]$ where $\gamma \in H_1(S,\mathbb{Z})$. Since we can produce from γ the map $h : S \to S$ then this orbit is actually a map $F_p = f \circ h : S \to X_p$ (because the action of any element of G on (f,γ) will result in the same map F_p). Note that this map induces a map $F_* : H_1(S,\mathbb{Z}) \to H_1(X_p,\mathbb{Z})$, but $H_1(S,\mathbb{Z})$ is identified with $\mathbb{Z}^2$ by choosing a basis.

We have the canonical basis of $H_1(S,\mathbb{Z})$ which is the homology classes of the longitudinal $S^1 \times \{0\}$ and meridional loops $\{0\} \times S^1$. Then by evaluating this two homology classes in F_*, we get two generators of $H_1(X_p,\mathbb{Z})$, since F is a homeomorphism. The collection of these generators, on each fibre of the family, will be called local system of homology.

It is also known that G acts on the group of $H^2(S,\mathbb{Z}) \cong \mathbb{Z}$. Thus we have a subgroup of index 2 in G formed by the the classes of homeomorphisms that preserve orientation, we will call this subgroup G_+. This information induces the following exact sequence:

$$1 \to G_+ \to G \to \mathbb{Z}/2\mathbb{Z} \to 1. \tag{3.6}$$

Thus if we have a continuously varying choice of orientation in each fibre of $X \to U$ (i.e. a section of the bundle $X \times_G \mathbb{Z}/2\mathbb{Z} \to U$), then the transition functions of the bundle $X \to U$ (seen as a locally trivial bundle in the topological way) will lie in G_+, this is a reduction of the structure group.

Since the bundle $R \to U$ was built with the transition functions of $X \to U$, then the knowledge of a continuously varying choice of orientation in each fibre is equivalent to a reduction of structure group from G to G_+ of the bundle $R \to U$. Thus we can call $R' \to U$ the bundle with reduced structure group G_+. We decide to give this bundle its own name for clarity.

Note that this reduction of structural group also carries over to the homology bundle $P' \to U$, since we have the exact sequences and commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & G_+ \arrow[r] \arrow[d, "\Delta"] & G \arrow[r] \arrow[d, "\Delta"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & \mathbf{SL}(2,\mathbb{Z}) \arrow[r] & Gl(2,\mathbb{Z}) \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 1, \end{tikzcd} \tag{3.7}$$

and the group $\mathbf{SL}(2,\mathbb{Z})$ is precisely the group that preserves a given orientation of (S,a). Thus the structural group G_+ has a map to $\mathbf{SL}(2,\mathbb{Z})$ defined by the action of G_+ over $H_1(S,\mathbb{Z})$ by homeomorphisms that preserve a given orientation (recall that this maps also preserve the intersection number of homology elements).

Note that as $X \to U$ is in particular a complex fibration. This means that X is a complex manifold, hence it is oriented (its orientation bundle is trivial thus there is a section). This orientation carries over to each fibre since they are complex submanifolds (thus oriented) that are included $X_p \hookrightarrow X$. If we choose the orientation on X_p that agrees with the induced one by X, then as was stated in 3.6, this implies that the structural group of $P' \to U$ reduces to $\mathbf{SL}(2,\mathbb{Z})$.

Now since $\mathbf{SL}(2,\mathbb{Z})$ is discrete then we can give the total space of this fibration a complex manifold structure, given by asking its projection to X, recall 3.5, to be a local biholomorphism and this structure endows each fibre with an orientation, which is inherited by the orientation of the total space.

Definition 3.8. (Framed family of elliptic curves) We say that a family of elliptic curves $\pi : X \to U, e : U \to X$ is a framed family or admits a framing if the fibration P' has a section.

We say that a framing is the homology elements extracted from a section (if it exists) of the fibration $P' \to X$ as done above, we will thus sometimes refer to a framing as just the section since they are identified bijectively.

Thus a framing is of the form:

$$\{(a(p), b(p)) \in H_1(X_p, \mathbb{Z})|\ a(p).b(p) = 1, p \in M\} \tag{3.8}$$

and $a(p), b(p)$ are locally constant functions. This means that over any open contractible set $V \subseteq U$, which trivializes $X \to U$ as a topological bundle, we have that for any two points m, t in it we have that the maps:

$$H_1(X_m, \mathbb{Z}) \longrightarrow H_1(X_V, \mathbb{Z}) \longleftarrow H_1(X_t, \mathbb{Z}) \tag{3.9}$$

induced by the inclusions are isomorphisms and it is required that $i_*a(m) = i_*a(t)$ under this isomorphisms, and analogously for $b(m)$ and $b(t)$.

Proposition 3.9. *The fibration constructed above is functorial meaning that if we consider the stack $[*/G_+]$ of G_+-principal bundles over $\mathscr{A}n$ where $G_+ \subseteq Homeo(S,S)$ such that they preserve a given orientation and $U \in \mathscr{A}n$ we have:*

$$R : \mathscr{M}_{1,1}(U) \to [*/G_+](U)$$

$$\begin{tikzcd} X^1 \arrow[r, "f"] \arrow[d] & X^2 \arrow[d] & R(X^1) \arrow[r, "R(f)"] \arrow[d, "\pi"] & R(X^2) \arrow[d, "\pi'"] \\ U \arrow[r, "id"] & U & U \arrow[r] & U \end{tikzcd} \tag{3.10}$$

where $R(f) : R(X^1) \to R(X^2)$ is defined in a fibrewise fashion by:

$$\begin{aligned} R(f) : (Homeo[(S,a) \to (X^1_p, e(p))]/\sim) &\to Homeo[(S,a) \to (X^2_p, e(p))]/\sim' \\ [\psi] &\longmapsto R(f)(\psi) = [f \circ \psi] \end{aligned} \tag{3.11}$$

The functor R is known as a rigidifying functor.

Proof. It is clear that R is a morphism of stacks since if we have a map $f : M \to N$ in $\mathscr{A}n$ then R is compatible with pullbacks since the fibres of f^*X are identified with the fibres of X through f then the fibres of $R(f^*X)$ are also identified with the fibres of $R(X)$ through f which is the same as saying that $R(f^*X) \cong f^*R(X)$. □

Now we will characterize the moduli space of framed elliptic curves (also known as Teichmüller space of elliptic curves [Gro61a] and [Gro61b]).

Let us construct a framed elliptic curve over $\mathfrak{h}$. It is easy to see that $\mathbb{Z}^2$ acts properly and freely over $(\mathbb{C} \times \mathfrak{h})$ by

$$(m,n) : (z,\tau) \to (z + m\tau + n, \tau) \tag{3.12}$$

because it is an action by translation.

Thus the map $\epsilon := (\mathbb{C} \times \mathfrak{h})/\mathbb{Z}^2 \longrightarrow \mathfrak{h}$ is an analytic map and the fibre over every point $\tau \in \mathfrak{h}$ is precisely the elliptic curve $(\mathbb{C}/\Lambda_\tau, 0)$. Additionally we know that this elliptic curve is naturally framed by $s(\tau) : (S, a) \cong \mathbb{R}^{\not{2}}/\mathbb{Z}^2 \to (\mathbb{C}/\Lambda_\tau, 0)$, which takes $(x, y) \mod \mathbb{Z}^2$ and sends it to $(x + iy)$ mod Λ_τ. This gives us the pair of homology elements $(1, \tau)$ using the identification $\Lambda_\tau \cong \mathbb{Z} \oplus \tau\mathbb{Z}$, thus this is a family of framed elliptic curves over $\mathfrak{h}$. We will call this family of elliptic curves $\epsilon \to \mathfrak{h}$. This family is the universal framed elliptic curve and we will prove this statement in the following theorem.

Theorem 3.10. *The stacks $\hat{\mathscr{M}}_{1,1}$ and $\mathfrak{h}$ are isomorphic.*

Proof. To prove this we need to show that given a family of framed elliptic curves over U given by $(\pi : X \to U, e : U \to X)$ and (a(p),b(p)) the framing of $(\pi^{-1}(p), e(p))$. Then there is a unique map $U \to \mathfrak{h}$, such that the pullback along this map of the family $\epsilon \to \mathfrak{h}$, defined previously 3.12, is isomorphic to the initial family. Thankfully we have the period map

$$\begin{aligned} \Phi : U &\longrightarrow \mathfrak{h} \\ p &\mapsto \frac{\int_{b(p)} \omega_p}{\int_{a(p)} \omega_p} \end{aligned} \tag{3.13}$$

where ω_p is a non-zero holomorphic one form of the elliptic curve over p. We need to show that this map is holomorphic, thus we need to choose "holomorphically" ω_p. In order to obtain this object, let us consider the following sheaf:

$$\pi_* \Omega^1_{X/U} := \pi_*(\Omega^1_X / \pi^* \Omega^1_U) \tag{3.14}$$

(this means that $\Omega^1_{X/U}$ is the associated sheaf to the quotient of the two sheaves).

We know that this sheaf, called the relative dualizing sheaf, is locally free of rank 1, this result is proven in [BLR90] Proposition 5 page 37.

Moreover note that for any $x \in X$ there is a neighborhood O_x, biholomorphic to the product of two neighborhood $W_x \times V_{\pi(x)}$ where W_x is a neighborhood of $\pi^{-1}(\pi(x))$ and $V_{\pi(x)}$ is a neighborhood of $\pi(x)$. This is a consequence of π being a submersion (as we had already discussed in theorem 2.11).

This implies that $\Omega^1_X(W_x) \cong \Omega^1_X(U_x) \times \Omega^1_X(V_{\pi(x)})$. On the other hand we know that since π is a submersion then $\pi^*\Omega^1_U(W_x) \cong \Omega^1_X(V_{\pi(x)})$ by definition of the inverse image of a sheaf. Thus $\Omega^1_{X/U}(W_x) \cong \Omega^1_X(U_x)$.

Now for the direct image of this sheaf we can cover each fibre by this sort of neighborhood, and since the fibre is compact we can choose $\{x_j\}_{0<j<n}$ where W_{x_j} cover the fibre. Then $\bigcap \pi(W_{x_j}) \cong \bigcap V_{\pi(x_j)}$ is an open neighborhood of $\pi(x)$ that we will call V, consequently the preimage $\pi^{-1}(V) \cong \pi^{-1}(\pi(x)) \times V$ and for this sort of open sets we know that $\pi_*\Omega^1_{X/U}(V) \cong \Omega^1_{X/U}(\pi^{-1}(x) \times V) \cong \Omega^1_X(\pi^{-1}(x))$. Thus if we take the direct limit we note that the stalk of this sheaf is $(\pi_*\Omega^1_{X/U})_p \cong \Omega^1_X(X_p, \mathbb{C})$, which means that a non-zero section of this sheaf assigns to every point of U, a non-zero differential form on the elliptic curve above that point.

So a local non-zero section of this sheaf is exactly the 1-form needed to define the map 3.13. Note also that a local section over every open set of a cover will be enough to define the period mapping, since on the intersections of the covers the choosing of the 1-form isn't important (the fact that the space of 1-forms of an elliptic curve is one dimensional, implies that the quotient $\frac{\int_{b(p)} \omega_p}{\int_{a(p)} \omega_p}$is well defined for any two section of the sheaf at the intersection of two open sets).

This map is unique because if there was a different one then its image would have to differ for some point but this cannot be, because if the image is different it would mean that the elliptic curve above this point is another framed elliptic curve (since any framed elliptic curve is completely determined by its period).

Thus we can pullback a family of elliptic curves $\Phi^*(\epsilon) \to U$, we now have to show that there is an isomorphism f between the two families:

(3.15)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & \Phi^*\epsilon \arrow[dl, "pr"] \\ U. & \end{tikzcd}$$

Thus we have to show that f is a biholomorphism that commutes with both projections, we will build f locally. Firstly note that $(X_p, e(p); a(p), b(p))$ is isomorphic to $(\mathbb{C}/\Lambda_{\Phi(p)}, 0; 1, \Phi(p))$ and we will call the isomorphism in between them

(3.16)
$$\begin{aligned} f_p : X_p &\longrightarrow \Phi^*\epsilon \\ x &\mapsto \frac{\int_c \omega_p}{\int_{a(p)} \omega_p} \mod \Lambda_{\Phi(p)} \end{aligned}$$

where c is a path in X_p from $e(p)$ to x.

Note that since π is a submersion then for each $x \in X$ there are open neighborhoods W, V of X_p and X respectively such that for a neighborhood O of x we have that $O \cong W \times V$ (they are biholomorphic). Thus we can define f_p locally by composing it with this biholomorphism we will call this extended map f.

This will be the required isomorphism, because from its definition it is clear that f is fibrewise, meaning that sends fibres to fibres, this is equivalent to saying that f commutes with the projections.

On the other hand, we know that f is holomorphic because it is a composition of two holomorphic maps (we already showed that the period map is biholomorphic) and it is invertible because for each $z \in (\mathbb{C}/\Lambda_{\Phi(p)}, 0; 1, \Phi(p))$ we can define its image in $\pi^{-1}(\Phi(p))$ through the isomorphism between $(X_p, e(p); a(p), b(p))$ and $(\mathbb{C}/\Lambda_{\Phi(p)}, 0; 1, \Phi(p))$ much in the same way as we defined f.

Now we prove the universality of the family

(3.17)
$$\begin{tikzcd} X' \arrow[dr, dotted, "L"] \arrow[drr, bend left] \arrow[ddr, bend right, "\pi'"'] & & \\ & X \arrow[r, "f"'] \arrow[d, "\pi"] & \epsilon \arrow[d] \\ & U \arrow[r, "\Phi"] & \mathfrak{h} \end{tikzcd}$$

Let $X' \to U$ be another family such that everything commutes, since it has the same period map as $X \to U$ we deduce that the fibres of X' are isomorphic to the ones of $\Phi^*(\epsilon)$ in the exact same way as the ones from X thus by composing $X' \longrightarrow \Phi^*(\epsilon) \longleftarrow X$ we get the unique morphism L that makes everything commute. $\square$

Remark 3.11. In this case as $\mathfrak{h} \to \hat{\mathscr{M}}_{1,1}$ is an isomorphism. In particular, it is a representable epimorphism, thus this solves the moduli problem. Moreover $\mathfrak{h}$ is a fine moduli space because any framed family of elliptic curves over a complex manifold $X \to U$ is obtained by pulling back the universal curve over $\mathfrak{h}$ through the period map.

Theorem 3.12. *Any family of elliptic curves $\pi : X \to U$ can be locally framed.*

Proof. It suffices to take a local trivialization by contractible open sets $\mathfrak{U} = (V_i)$ of π seen as a C^∞ locally trivial bundle over each of this open sets we know that $\pi^{-1}(V_i) \cong V_i \times X_p$ for some $p \in V_i$ then a framing of this family of curves is just a framing of the curve X_p and then transported to a different fibre X_y through the isomorphisms

$$H_1(X_p, \mathbb{Z}) \longrightarrow H_1(X_{V_i}, \mathbb{Z}) \longleftarrow H_1(X_y, \mathbb{Z}) \tag{3.18}$$

Induced by the retraction $r : V_i \to p$ and $r' : V_i \to y$ □

In particular this tells us that any family over a contractible base is trivial and can be framed. But in particular we also have:

Theorem 3.13. *A family of elliptic curves over a simply connected base has a framing.*

Proof. Let $\pi : X \to U$ be a family of elliptic curves over a simply connected U. Since a framing of the family is induced by a section of the fibration $P' \to U$ and since its fibres are discrete, in particular they are $H_1(X_p, \mathbb{Z}) \cong \mathbb{Z}^2$. Thus by the long exact sequence of homotopy groups of a fibration we have that the total space P' is simply connected (since the fibre $\mathbb{Z}^2$ and the base space U are simply connected), thus it is the universal covering of U but since U is simply connected we know that $id : U \to U$ is also the universal covering of U, thus there is a homeomorphism $f : U \to P'$ such that $\pi \circ f = id_U$, additionally this map is a section of the bundle $P' \to U$. Then it has a section, thus the family $X \to U$ is framed. □

3.2. **The moduli stack $\mathscr{M}_{1,1}$.** Now we need to see what does the morphism "forget the frame" does to $\mathfrak{h}$, in order to do this, we note that the group $\mathbf{SL}(2, \mathbb{Z})$ acts on framings by left multiplication in the following way

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} a\beta + b\alpha \\ c\beta + d\alpha \end{bmatrix} \tag{3.19}$$

and we can extend this action to families of elliptic curves. Note that the elliptic curve $(\mathbb{C}/\Lambda_\tau, 0)$ can be framed by $(1, \tau)$, but another possible frame for it can be given by $(c\tau + d, a\tau + b)$ if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, \mathbb{Z})$. At the same time, note that the elliptic curve $(\mathbb{C}/\Lambda_\tau, 0)$ is isomorphic to $(\mathbb{C}/(c\tau + d)^{-1}\Lambda_\tau, 0)$ as previously discussed in corollary 1.24, and the isomorphism f between these two acts as follows on the frame:

$$f_*(c\tau + d) = c\tau + d/c\tau + d = 1$$

$$f^*(a\tau + b) = \frac{a\tau + b}{c\tau + d}.$$

Thus choosing a frame in the class of isomorphism of elliptic curves is equivalent to choosing an $\mathbf{SL}(2, \mathbb{Z})$ orbit over $\mathfrak{h}$. This allows us to conclude that

Theorem 3.14. *The points of the topological space $\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})$ are in bijection with all isomorphism classes of elliptic curves.*

Proof. Clearly every point of $\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})$ represents an isomorphism class of elliptic curves, as a consequence of 3.6. If we have two elliptic curves $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{\tau'}$ such that τ and τ' are in the same $\mathbf{SL}(2, \mathbb{Z})$-orbit, then there exists $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2, \mathbb{Z})$ such that $(a\tau + b, c\tau + d) = (\tau', 1)$.

As we saw 3.19, the isomorphism $z \mapsto (c\tau + d)^{-1}z$ induces the required isomorphism between the elliptic curves. $\square$

Proposition 3.15. *The stack* $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ *is analytic.*

Proof. Let $f \times - : V \times \mathbf{SL}(2,\mathbb{Z}) \to \mathfrak{h}$ be the map $(v,\gamma) \mapsto \gamma f(v)$. Note that this map is equivariant, thus it induces the following projection map

$$\pi : \mathfrak{h}(V) \longrightarrow [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](V)$$

$$f : V \to \mathfrak{h} \mapsto \begin{tikzcd} V \times \mathbf{SL}(2,\mathbb{Z}) \arrow[r, "f\times -"'] \arrow[d] & \mathfrak{h} \\ V. & \end{tikzcd} \tag{3.20}$$

This projection is a representable epimorphism of stacks, because:

Let U be a manifold and let $U \to [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ be a stack morphism, we can identify this morphism with an $\mathbf{SL}(2,\mathbb{Z})$-principal bundle $P \to U$ together with an equivariant map from $f : P \to \mathfrak{h}$. Then the fibred product $\mathfrak{h} \times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U(V)$ is the category whose objects are triples, where the first element is a morphism $\phi : V \to U$ in the stack U, the second is a morphism $f : V \to \mathfrak{h}$ in the stack $\mathfrak{h}$ and the third is a cartesian square of the form:

$$\begin{tikzcd} V \times \mathbf{SL}(2,\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "pr"] & \phi^*(P) \arrow[d] \\ V \arrow[r, "Id"'] & V \end{tikzcd}$$

where the top arrow is equivariant.

Note that if the bundle $P \to U$ were a trivial bundle, then the fibred product $\mathfrak{h} \times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U(V)$ would be just the collection of all maps $V \to U \times \mathbf{SL}(2,\mathbb{Z})$, since any morphism $g : V \to U \times \mathbf{SL}(2,\mathbb{Z})$ is obtained as a map $V \to U$ and a section $s : V \to V \times \mathbf{SL}(2,\mathbb{Z})$, which can be taken as a map $V \to \mathbf{SL}(2,\mathbb{Z})$. This would imply that $\mathfrak{h} \times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U \cong U \times \mathbf{SL}(2,\mathbb{Z})$.

In particular we can cover U with open subsets $\{O_i\}_{i\in I}$ that trivialize the bundle over it. Thus we obtain a collection of isomorphisms of stacks over every O_i given by $\mathfrak{h} \times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} O_i \cong O_i \times \mathbf{SL}(2,\mathbb{Z})$.

Using the descent condition we conclude that we can glue together all the objects $\{O_i \times \mathbf{SL}(2,\mathbb{Z})\}_{i\in I}$ to obtain the complex manifold P. Similarly we can glue the morphisms to get the isomorphism $\mathfrak{h} \times_{\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})} U \cong P$. Additionally note that since the map $P \to U$ is a surjective submersion then by 3.4 we have that π is also a stack epimorphism. $\square$

Note that the same proof works for the case $M \to [M/G]$. Now we will talk about an important example of a family of elliptic curves.

Example 3.16. We know that $\xi_\lambda = \{(x,y) \in \mathbb{C}^2 | \ y^2 = x(x-1)(x-\lambda) = 0\} \cup \{p_\infty\}$ is an elliptic curve defined in example 1.21. Based on the previous statement we can construct the following family of elliptic curves:

$$\begin{tikzcd} T := \{(x,y,\lambda) | (x,y) \in \xi_\lambda \text{ and } \lambda \in \mathbb{C} \setminus \{0,1\}\} \arrow[d, "pr_3"] \\ \mathbb{C} \setminus \{0,1\} \end{tikzcd} \tag{3.21}$$

Where $pr_3(x,y,\lambda) = \lambda$, since it is clear that the projection to the third coordinate is a holomorphic map with maximal rank and such that each fibre is an elliptic curve.

We remark that $\mathbf{SL}(2,\mathbb{Z})$ acts on the universal curve $\epsilon \to \mathfrak{h}$. This is a consequence of the fact that $\mathbf{SL}(2,\mathbb{Z})$ acts both on the group $\mathbb{Z}^2$ by right multiplication and on $\mathbb{C} \times \mathfrak{h}$ by the following action:

$$\gamma : (z, \tau) \to (z/(c\tau + d), \frac{a\tau + b}{c\tau + d}) \tag{3.22}$$

Since $(c\tau + d)^{-1}$ is clearly invertible and its inverse being a multiple of 1 and τ then we have that $(\mathbb{C}/\Lambda_\tau, 0) \cong (\mathbb{C}/\Lambda_{(c\tau+d)^{-1}\tau}, 0)$ thus this action of $\mathbf{SL}(2,\mathbb{Z})$ descends to the quotient ϵ. If we allow $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to act on an orbit of ϵ we get

$$\gamma : (z + m\tau + n, \tau) \to ((z + m\tau + n)/(c\tau + d), \frac{a\tau + b}{c\tau + d}) \tag{3.23}$$

But this is precisely the action of the semidirect product $\mathbf{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C} \times \mathfrak{h}$ since $\mathbf{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \cong \{\begin{bmatrix} \gamma & 0 \\ (mn) & 1 \end{bmatrix} \mid \gamma \in \mathbf{SL}(2,\mathbb{Z})\}$.

Proposition 3.17. *The map $\epsilon/\mathbf{SL}(2,\mathbb{Z}) \cong (\mathbb{C} \times \mathfrak{h})/\mathbf{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \to \mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$ is a fibration.*

Proof. Since the group $\mathbf{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ is discrete and the action of $\mathbb{Z}^2$ on $\mathbb{C} \times \mathfrak{h}$ is free, recall 3.12. Then we have that the action 3.23, is properly discontinuous. Consequently the projection map $\epsilon/\mathbf{SL}(2,\mathbb{Z}) \to \mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$ is proper and since it is also a surjective submersion then by Ehresmann fibration lemma [Ehr52] we get the result. □

We remark that the fibration just constructed does not solve the moduli problem (i.e. This is not a universal curve over $\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$). Since the fibration over the space $\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$ has as fibre over any of its points $[\tau]$ the space $(\mathbb{C}/\Lambda_\tau, 0)/\{(c\tau+d)^{-1} \mid c, d \in \mathbb{Z}\} \cong (\mathbb{C}/\Lambda_\tau, 0)/Aut((\mathbb{C}/\Lambda_\tau, 0))$ by 1.24. In the case where the group of automorphisms reduces to only two elements, the quotient has genus 0. Since the only possible 1-form which is invariant under $\{-Id, Id\}$ is the zero form, thus this bundle isn't even a family of elliptic curves.

Now we want to characterize the moduli stack $\mathscr{M}_{1,1}$, in order to do this we will give some preliminaries.

Let $g : P \to U$ be an $\mathbf{SL}(2,\mathbb{Z})$-bundle and $f : P \to \mathfrak{h}$ an equivariant map.
Since $((\mathbb{C} \times \mathfrak{h})/\mathbb{Z}^2) \to \mathfrak{h}$ is a family of marked elliptic curves and recall $\mathbf{SL}(2,\mathbb{Z})$ acts on P, we can define a family over the quotient $P/\mathbf{SL}(2,\mathbb{Z})$. Since f is an equivariant map see diagram 3.24, then two points in the same orbit have elliptic curves that lie in the same $\mathbf{SL}(2,\mathbb{Z})$-orbit (recall that they are isomorphic as elliptic curves 3.6), thus by forgetting the frame on them we get a family that descends to the quotient $P/\mathbf{SL}(2,\mathbb{Z})$ which is isomorphic to U.

$$\begin{tikzcd} f^*(\epsilon) \arrow[r] \arrow[d] & \epsilon \arrow[d] \\ P \arrow[r, "f"] \arrow[d, "g"] & \mathfrak{h} \\ U \cong P/\mathbf{SL}(2,\mathbb{Z}) & \end{tikzcd} \tag{3.24}$$

But note that equivalently we can define an action over $f^*(\epsilon)$ given by:

$$\begin{gathered} \alpha : f^*(\epsilon) \times \mathbf{SL}(2,\mathbb{Z}) \to f^*(\epsilon) \\ ((\mathbb{C}/\Lambda_{f(p)}, 0; 1, f(p)), p).\gamma \to ((\mathbb{C}/\Lambda_{f(p)}, 0; \gamma(1, f(p)), \gamma p) \end{gathered} \tag{3.25}$$

and since the $\mathbf{SL}(2,\mathbb{Z})$ action on P is free and proper then the action just defined inherits both properties. We can thus define the quotient and it will be a complex manifold that we will call $\overline{f^*\epsilon}$. Note that the quotient $\overline{f^*\epsilon}$ is equipped with a map $T\colon\overline{f^*\epsilon}\to U$, induced by the map $f^*(\epsilon)\to P$.

We first note that the map $T\colon\overline{f^*\epsilon}\to U$ is a family of elliptic curves over U since by construction the fibre above every point is an elliptic curve and the map T is a submersion since $f^*(\epsilon)\to P$ is.

Thus we define the following correspondence:

$$\Xi : [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U) \longrightarrow \mathscr{M}_{1,1}(U)$$

(3.26)
$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "g"] & \mathfrak{h} & \mapsto & \overline{f^*(\epsilon)} \arrow[d, "T"] \\
U & & & U
\end{tikzcd}$$

Note that Ξ is in fact a functor from $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$ and $\mathscr{M}_{1,1}(U)$ since if we have a morphism between two elements of $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$:

(3.27)
$$\begin{tikzcd}
 & \mathfrak{h} \\
P \arrow[ur, "f"] \arrow[r, "k"] \arrow[d] & Q \arrow[u, "g"'] \arrow[dl] \\
U, &
\end{tikzcd}$$

this induces a morphism between the two families of framed elliptic curves $f^*(\epsilon)$ and $g^*(\epsilon)$, see 2.2, which descends to the quotient since the maps where they come from are equivariant, by construction this assignment preserves composition.

We also define the following morphism:

if $l : V \to U \in \mathscr{A}n$ then we have the two families of elliptic curves over V which are:

(3.28)
$$\begin{tikzcd}
\Xi(l^*P) \arrow[rr, "\Xi_l"] \arrow[dr] & & l^*\Xi(P) \arrow[dl] \\
 & V &
\end{tikzcd}$$

and we define Ξ_l as the morphism given by the following pullback diagram:

(3.29)
$$\begin{tikzcd}
\overline{pr_2^* f^*\epsilon} = \Xi(l^*P) \arrow[dr, dotted, "\Xi_l"] \arrow[drr, bend left, "\overline{pr_3}"] \arrow[ddr, bend right] & & \\
 & l^*(\overline{f^*(\epsilon)}) \arrow[r] \arrow[d] & \overline{f^*\epsilon} \arrow[d] \\
 & V \arrow[r, "l"] & U
\end{tikzcd}$$

where $\overline{pr_3}$ is induced by:

(3.30)
$$\begin{tikzcd}
pr_2^* f^*\epsilon \arrow[r, "pr_3"] \arrow[d] & f^*\epsilon \arrow[d] & \\
l^*P \arrow[r, "pr_2"] & P \arrow[r, "f"] & \mathfrak{h}
\end{tikzcd}$$

Note that this morphism is in $\mathscr{M}_{1,1}(V)$ and defines a natural transformation. Since if we have a morphism in $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$ given by

$$\begin{tikzcd} & \mathfrak{h} \\ P \arrow[ur, "f"] \arrow[r, "\psi"] \arrow[d] & Q \arrow[u, "g"'] \arrow[dl] \\ U, & \end{tikzcd} \tag{3.31}$$

then we get the following diagram:

$$\begin{tikzcd} \Xi l^* P \arrow[r, "\Xi_l"] \arrow[d, "\Xi l^*(\psi)"'] & l^* \Xi P \arrow[d, "l^*\Xi(\psi)"] \\ \Xi l^* Q \arrow[r, "\Xi_l"'] & l^* \Xi Q. \end{tikzcd} \tag{3.32}$$

Which is commutative as a consequence of $f = g \circ \psi$.

Lemma 3.18. *Let $\Xi : \mathscr{M}_{1,1} \to [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ be the collection of functors defined in 3.26, together with the natural transformations defined in 3.28. The collection of this objects is a morphism of stacks.*

Now we can think of constructing a morphism in the other direction.
Let $X \to U$ be an elliptic curve. We can consider the universal cover of U given by $p : \hat{U} \to U$ and construct the following pullback diagram:

$$\begin{tikzcd} p^* X \arrow[r] \arrow[d] & X \arrow[d] \\ \hat{U} \arrow[r, "p"] & U \end{tikzcd} \tag{3.33}$$

Note that the family $p^*X \to \hat{U}$ has a simply connected base, thus it can be framed as was proved in theorem 3.13. Consequently there is a period map $\Phi : \hat{U} \to \mathfrak{h}$.

Note that both $X \to U$ and $p^*X \to \hat{U}$ have the same fibre over u and over each element of the fibre $p^{-1}(u)$. This means that the elliptic curve over an element of $p^{-1}(u)$ is the curve $(X_u, e(u))$. Moreover since any two elements in $p^{-1}(u)$ can be related by a deck transformation, i.e. an element $\gamma \in Aut(\hat{U}/U) \cong \pi_1(U)$, this means that the framings $(a(v), b(v))$ and $(a(\gamma v), b(\gamma v))$ of the curves above $v, \gamma v \in p^{-1}(u)$ must differ by an element of $\mathbf{SL}(2,\mathbb{Z})$ that we will call $\phi(\gamma)$.

This defines a group morphism:

$$\begin{aligned} \phi : \pi_1(U) \longrightarrow & \mathbf{SL}(2,\mathbb{Z}) \\ \gamma \longrightarrow & \phi(\gamma) \end{aligned} \tag{3.34}$$

Since:

$$\begin{aligned} \phi(\gamma * \delta)(a(v), b(v)) =& (a(\gamma * \delta(v)), b(\gamma * \delta(v))) \\ =& \phi(\gamma)(a(\delta(v)), b(\delta(v))) \\ =& \phi(\delta).\phi(\gamma)(a(v), b(v)). \end{aligned} \tag{3.35}$$

This means that we can construct an associated $\mathbf{SL}(2,\mathbb{Z})$-principal bundle over U in the following way. Let $\hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z})$ be the quotient by the diagonal action of $\pi_1(U)$ over $\hat{U}\times\mathbf{SL}(2,\mathbb{Z})$ then:

$$\begin{tikzcd} \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}) \arrow[d, "g"] \\ U \end{tikzcd} \tag{3.36}$$

is an $\mathbf{SL}(2,\mathbb{Z})$-principal bundle and note that the projection is $g([v,\gamma])=p(v)$. We note that the period map Φ is equivariant with respect to ϕ, since ϕ was precisely built to detect the change of framing induced by deck transformations, by 3.35 we have that if $c\in\pi_1(U)$ and $u\in\hat{U}$, then $\Phi(cu)=\phi(c)\Phi(u)$. Thus the period map induces the following equivariant map over the total space:

$$\begin{aligned} \Phi' : \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}) &\longrightarrow \mathfrak{h} \\ [u,\gamma] &\longmapsto \gamma(\Phi(u)) \end{aligned} \tag{3.37}$$

Which is well defined since if $c\in\pi_1(U)$ we have that:

$$\Phi'([c(u),\gamma\phi(c)^{-1}])=\gamma\phi(c^{-1})(\Phi(c(u)))=\gamma\phi(c^{-1})\phi(c)\Phi(u)=\gamma\Phi(u).$$

And since the action of $\mathbf{SL}(2,\mathbb{Z})$ on $\hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z})$ is just by multiplication on the second component this implies that $\Phi'(\gamma'.[u,\gamma])=\Phi'([u,\gamma'\gamma])=\gamma'\gamma(\Phi(u))=\gamma'(\Phi'([u,\gamma]))$. Thus this determines an element of $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$ hence we have:

$$\Xi' : \mathscr{M}_{1,1}(U) \longrightarrow [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$$

$$\begin{tikzcd} X \arrow[d] \arrow[r, phantom, "\longmapsto"] & \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}) \arrow[d] \arrow[r, "\Phi'"] & \mathfrak{h} \\ U & U & \end{tikzcd} \tag{3.38}$$

Which is a functor, since if we have a morphism between two elliptic curves:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & U & \end{tikzcd} \tag{3.39}$$

this will induce a morphism between the pullbacks $p^*X\to p^*Y$. The previous statement will induce two group homomorphisms $\pi_1(U) \overset{\phi}{\underset{\phi'}{\rightrightarrows}} \mathbf{SL}(2,\mathbb{Z})$. We define the following map between the two associated $\mathbf{SL}(2,\mathbb{Z})$-bundles:

$$\begin{aligned} \Xi'(f) : \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}) &\to (\hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}))' \\ [u,\gamma] &\to [u,\gamma]' \end{aligned} \tag{3.40}$$

This map is clearly equivariant and by construction it is a map of $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$. Additionally this correspondence respects the composition of maps in $\mathscr{M}_{1,1}(U)$.

Now we define the following natural transformation:

if $l : V \to U \in \mathscr{A}n$ then we have the two $\mathbf{SL}(2,\mathbb{Z})$-principal bundle over V (where we omit the equivariant maps to $\mathfrak{h}$ for simplicity) which are:

$$\begin{tikzcd} \Xi'(l^*P) \arrow[rr, "\Xi'_l"] \arrow[dr] & & l^*\Xi'(P) \arrow[dl] \\ & V & \end{tikzcd} \tag{3.41}$$

and we define Ξ'_l (similarly as we did for 3.29) as the morphism given by the pullback diagram:

$$\begin{tikzcd} l^*\Xi'(X) \arrow[r, "pr_2"] \arrow[d] & \Xi'(X) \arrow[d] \\ V \arrow[r, "l"] & U \end{tikzcd} \tag{3.42}$$

Lemma 3.19. *Let $\Xi' : [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})] \to \mathscr{M}_{1,1}$ be the collection of functors defined in 3.38, together with the natural transformations defined in 3.41. The collection of this objects is a morphism of stacks.*

Now we can consider both compositions of the previous morphisms. Firstly, the following composition:

$$\Xi \circ \Xi'(U) : \mathscr{M}_{1,1}(U) \to [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U) \longrightarrow \mathscr{M}_{1,1}(U)$$

$$\begin{tikzcd} X \arrow[d] & \mapsto & \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z}) \arrow[r, "\Phi'"] \arrow[d] & \mathfrak{h} & \mapsto & \overline{\Phi'^*\epsilon} \arrow[d] \\ U & & U & & & U \end{tikzcd} \tag{3.43}$$

Proposition 3.20. *There is an isomorphism of elliptic curves:*

$$\begin{tikzcd} X \arrow[rr, "\cong"] \arrow[dr] & & \overline{\Phi'^*(\epsilon)} \arrow[dl] \\ & U. & \end{tikzcd} \tag{3.44}$$

Where the second family of elliptic curves is obtained by the composition 3.43.

Proof. Note that there is an isomorphism of framed elliptic curves, as a consequence of theorem 3.10, which is in particular an isomorphism of families of elliptic curves, given by:

$$\begin{tikzcd} \Phi^*(\epsilon) \arrow[rr, "\cong"] \arrow[dr] & & p^*(X) \arrow[dl] \\ & \hat{U}. & \end{tikzcd} \tag{3.45}$$

And additionally we know that the map Φ' comes from the period map $\Phi : \hat{U} \to \mathfrak{h}$, induced by the family $p^*X \to \hat{U}$ (i.e. $\Phi'([u,\gamma]) = \gamma\Phi(u)$). This implies that the elliptic curve over a point $[u,\gamma]$ will be isomorphic to the elliptic curve over the point $u \in \hat{U}$, of the family $p^*X \to \hat{U}$, but their framings will differ by $\gamma \in \mathbf{SL}(2,\mathbb{Z})$.

Note that we have the morphism $g : \hat{U} \to \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z}), g(u) = [u, Id]$, which induces the following isomorphism of framed elliptic curves (since they both have the same period map i.e.

$\Phi = \Phi' \circ g$):

(3.46)
$$\begin{tikzcd} g^*\Phi'^*(\epsilon) \arrow[rr, "\cong"] \arrow[dr] & & p^*(X) \arrow[dl] \\ & \hat{U}. & \end{tikzcd}$$

But note that taking the $\mathbf{SL}(2,\mathbb{Z})$-orbit of the family $\Phi'^*(\epsilon) \to \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z})$ (where the action on the family was defined as in equation 3.22) is the same as taking the $\pi_1(U)$-orbit of the family $g^*\Phi'^*(\epsilon) \to \hat{U}$. Since for any $[u,\gamma] \in \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z})$ we know that the elements $\{[[l]u, \phi([l])\gamma] \mid \text{for every } [l] \in \pi_1(U)\}$ lay on the same $\mathbf{SL}(2,\mathbb{Z})$-orbit. Now, if two classes $[u,\gamma],[u',\gamma'] \in \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z})$ are in the same $\mathbf{SL}(2,\mathbb{Z})$-orbit then the curves $\Phi'^*(\epsilon)_{[u,\gamma]}$ and $\Phi'^*(\epsilon)_{[u',\gamma']}$ will only differ by the framing, but they will be isomorphic as elliptic curves.

This implies that for any two points $u, u' \in \hat{U}$ that lay on the same $\pi_1(U)$ orbit, then the curves $g^*\Phi'^*(\epsilon)_u$ and $g^*\Phi'^*(\epsilon)_{u'}$ will be isomorphic as elliptic curves and their only difference will be their framings.

Moreover since taking the $\pi_1(U)$-orbit of the family $p^*(X) \to \hat{U}$ is the family $X \to U$ and taking the $\mathbf{SL}(2,\mathbb{Z})$-orbit of the family $\Phi'^*(\epsilon) \to \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z})$ is the family $\overline{\Phi'^*\epsilon} \to U$. Then we have that, the isomorphism between $g^*\Phi'^*(\epsilon)$ and $p^*(X)$, defined in 3.46, induces an isomorphism

(3.47)
$$\begin{tikzcd} \overline{\Phi'^*\epsilon} \arrow[rr, "\cong"] \arrow[dr] & & X \arrow[dl] \\ & U. & \end{tikzcd}$$

∎

Now let us see the other composition:

$$\Xi' \circ \Xi(U) : [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U) \to \mathscr{M}_{1,1}(U) \to [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$$

(3.48)
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "q"] & \mathfrak{h} \arrow[r, mapsto, shorten=2mm] & \overline{f^*\epsilon} \arrow[d] \arrow[r, mapsto] & \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z}) \arrow[r, "\Phi'"] \arrow[d] & \mathfrak{h} \\ U & & U & U & \end{tikzcd}$$

where this time Φ' is induced by the period map of the framed family $p^*\overline{f^*\epsilon} \to \hat{U}$ and again $\overline{f^*\epsilon} \to U$ is the family induced by $f^*(\epsilon) \to P$.

Proposition 3.21. *There is an isomorphism of* $\mathbf{SL}(2,\mathbb{Z})$*-principal bundles:*

(3.49)
$$\begin{tikzcd} & \mathfrak{h} \\ P \arrow[ur, "f"] \arrow[r, "\cong"] \arrow[d] & \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2,\mathbb{Z}) \arrow[u, "\Phi'"'] \arrow[dl] \\ U. & \end{tikzcd}$$

Such that the diagram 3.49 commutes and where the second bundle is obtained by the composition 3.48.

Proof. In order to construct an isomorphism between the two bundles, we will exploit the fact that $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ is a stack. We will define an isomorphism between P and $\hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z})\in[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$ relative to a contractible open cover $\{V_\alpha\}$, and then glue together the isomorphisms through the property 2.12.

First note that, since $\{V_\alpha\}$ is a contractible cover then we have that $P|_{V\alpha}$ and $\hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z})|_{V_\alpha}$ are trivial bundles over V_α. Consequently let $s_\alpha : V_\alpha \to \hat{U}|_{V_\alpha}$[2] and $s'_\alpha : V_\alpha \to P|_{V_\alpha}$ be the sections that trivialize the corresponding principal bundles (that we will note s, s' when there is no risk of confusion).

Now note that, for each point $a\in P|_{V\alpha}$, the isomorphism class of the framed curve $f^*(\epsilon)_a$ can be identified with the value $f(a)$, as a consequence of theorem 3.46. On the other hand, we have that the isomorphism class of the framed curve above $s(q(a))$ is in the $\mathbf{SL}(2,\mathbb{Z})$-orbit of $f(a)$, since the isomorphism class of the curve $f^*(\epsilon)_a$ is preserved when it is carried to $\hat{U}$. In other words the only difference between the curves $f^*(\epsilon)_a$ and the curve $p^*\overline{f^*(\epsilon)}_{s(q(a))}$ is their framings. Moreover the framed elliptic curve above $s(q(a))$ is determined by the value $\Phi'(s(q(a)))$, then we have that the framings of the elliptic curves above a and $(s(q(a))$ differ by a unique element $\delta_a\in\mathbf{SL}(2,\mathbb{Z})$. In particular we have that $\delta_a\Phi'(s(q(a))) = f(a)$, by 3.19.

Thus we have the following map:

$$
\begin{aligned}
F_\alpha : P|_{V_\alpha} \longrightarrow & \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z})|_{V_\alpha} \\
a \mapsto & [s(q(a)),\delta_a]
\end{aligned}
\tag{3.50}
$$

Now we have that $\Phi'\circ F(a) = \Phi'([s(q(a)),\delta_a]) = \delta_a\Phi(s(q(a))) = f(a)$. Additionaly note that F is equivariant since if $\sigma\in\mathbf{SL}(2,\mathbb{Z})$ then $F(\sigma a) = [s(q(\sigma a)),\sigma\delta_a] = \sigma[s(q(a)),\delta_a] = \sigma F(a)$ this is because the elements $f(\sigma a) = \sigma f(a)$ and $\Phi(s(q(a)))$ differ by the element $\sigma\delta_a$. Moreover $q = g\circ F$ since $g\circ F(a) = p(s(q(a))) = q(a)$.

Note that this morphism is completely determined by its values on the set $\{s'(v)|v\in V_\alpha\}$ and the rest is obtained by the fact that the map is equivariant. This implies that we can easily define explicitly an inverse of this map, by defining how it behaves on the set $\{s(v)|v\in V_\alpha\}$ (we remark that since F_α is a morphism in the groupoid $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](V_\alpha)$ then it automatically is an isomorphism).

Thus we define:

$$
\begin{aligned}
F_\alpha^{-1} : \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z})|_{V\alpha} &\longrightarrow P|_{V\alpha} \\
[s(v),\alpha] &\mapsto \alpha\delta_{s'(v)}^{-1}\ s'(v)
\end{aligned}
\tag{3.51}
$$

Now we have that

$$
\begin{aligned}
F(F^{-1}([s(v),\alpha])) = F(\alpha\delta_{s'(v)}^{-1}\ s'(v)) &= \alpha\delta_{s'(v)}^{-1}F(s'(v)) \\
&= \alpha\delta_{s'(v)}^{-1}[s(q(s'(v))),\delta_{s'(v)}] \\
&= [s(v),\alpha\delta_{s'(v)}^{-1}\delta_{s'(v)}] \\
&= [s(v),\alpha]
\end{aligned}
\tag{3.52}
$$

and

$$
\begin{aligned}
F^{-1}F(\alpha s'(v)) = F^{-1}([s(q(s'(v))),\alpha\delta_{s'(v)}]) &= F^{-1}([s(v),\alpha\delta_{s'(v)}]) \\
&= \alpha\delta_{s'(v)}\delta_{s'(v)}^{-1}s'(v) \\
&= \alpha\delta_{s'(v)}\delta_{s'(v)}^{-1}s'(v) = \alpha s'(v).
\end{aligned}
\tag{3.53}
$$

[2]Note that a section $s : V_\alpha \to \hat{U}|_{V_\alpha}$ induces a section $S : V_\alpha \to \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z})|_{V_\alpha}$ with $S(v) = [s(v), Id]$.

Thus F^{-1} is the inverse of F. Additionally, note that F^{-1} is equivariant, is fibrewise (i.e $q \circ F^{-1} = g$). Additionally it satisfies that $f \circ F^{-1} = \Phi'$ since

$$f \circ F^{-1}[s(v), \alpha] = f(\alpha \delta_{s'(v)}^{-1} s'(v)) = \alpha \delta_{s'(v)}^{-1} f(s'(v)) = \alpha \Phi(s(v)) = \Phi'[s(v), \alpha].$$

Now, we have to show that $F_\alpha|_{V_{\alpha\beta}} = F_\beta|_{V_{\alpha\beta}}$. In order to do this let $a \in P|_{V_{\alpha\beta}}$. Then we have that $F_\alpha(a) = [s_\alpha(q(a)), \delta_a]$ where $\delta_a \Phi(s_\alpha(q(a))) = f(a)$ and $F_\beta(a) = [s_\beta(q(a)), \delta'_a]$ where $\delta'_a \Phi(s_\beta(q(a))) = f(a)$. Since the transition functions of the bundle $\hat{U} \times_{\pi_1(U)} \mathbf{SL}(2, \mathbb{Z}) \to U$ are induced by the transition functions of the bundle $\hat{U} \to U$ through the function ϕ. Then let $j_{\alpha\beta} : V_{\alpha\beta} \to \pi_1(U)$ be the transition functions of $\hat{U} \to U$, thus we have that $s_\alpha(q(a)) = j_{\alpha\beta}(q(a)) s_\beta(q(a))$ but this implies that:

$$\begin{aligned}[s_\alpha(q(a)), \delta_a] &= [j_{\alpha\beta}(q(a)) s_\beta(q(a)), \delta_a] \\ &= [s_\beta(q(a)), \delta_a \phi(j_{\alpha\beta}(q(a)))] \\ &= [s_\beta(q(a)), \delta'_a]\end{aligned} \tag{3.54}$$

as $\delta_a \phi(j_{\alpha\beta}(q(a))) \Phi(s_\beta(q(a))) = \delta_a \Phi(j_{\alpha\beta}(q(a)) s_\beta(q(a))) = \delta_a \Phi(s_\alpha(q(a))) = f(a)$. Thus using the property of gluing morphisms 2.12 we obtain the isomorphism:

$$\begin{tikzcd} P \arrow[rr, "F"] \arrow[dr, "q"'] & & \hat{U} \times_{\pi_1(U)} \mathbf{SL}(2, \mathbb{Z}) \arrow[dl, "g"] \\ & U & \end{tikzcd} \tag{3.55}$$

$\square$

Theorem 3.22. *The stack* $\mathscr{M}_{1,1}$ *is isomorphism to* $[\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})]$.

Proof. We will prove that the couple of morphisms $\Xi : \mathscr{M}_{1,1} \to [\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})]$ defined in 3.18 and $\Xi' : [\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})] \to \mathscr{M}_{1,1}$ defined in 3.19 form an equivalence of groupoids.

As a consequence of propositions 3.20 and 3.21 we have that for every $U \in \mathscr{A}n$ the functors $\Xi : \mathscr{M}_{1,1}(U) \to [\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})](U)$ and $\Xi' : [\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})](U) \to \mathscr{M}_{1,1}(U)$ form an equivalence of categories, and in particular an equivalence of groupoids.

Now we want to show that the composition of the natural transformations of Ξ and Ξ' are equivalent to the identity (natural transformation).

So let $l : V \to U$ and let $X \to U \in \mathscr{M}_{1,1}(U)$ then we have :

$$\begin{tikzcd} \Xi\Xi'(l^*X) \arrow[r, "\Xi'_l \circ (\Xi(\Xi'_l))"] \arrow[d, "\cong"'] & l^*\Xi\Xi'(X) \arrow[d, "\cong"] \\ l^*X \arrow[r, "id"] & l^*X \end{tikzcd} \tag{3.56}$$

that the above diagram commutes as a consequence of diagrams 3.24, 3.35 commuting. Note that for every $l : U \to V$ in $\mathscr{A}n$ we have that Ξ_l and Ξ'_l are isomorphisms in $\mathscr{M}_{1,1}(V)$ and $[\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})](V)$ respectively this implies that $\Xi'_l \circ (\Xi(\Xi'_l))$ is also an isomorphism. Additionally the composition of the natural transformations of Ξ' and Ξ are equivalent to the identity, and the prove is analogous.

In conclusion the stack $\mathscr{M}_{1,1}$ is isomorphic to the stack $[\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})]$. $\square$

Now that the isomorphism of the stack $\mathscr{M}_{1,1}$ and $[\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})$ has been established, we want to know explicitly what the universal elliptic curve over the stack $[\mathfrak{h}/\mathbf{SL}(2, \mathbb{Z})]$ looks like. Before we do this we firstly remark that the stacks $[(\mathbb{C} \times \mathfrak{h})/\mathbf{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2]$ and $[((\mathbb{C} \times \mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2, \mathbb{Z})]$ are clearly isomorphic.

Proposition 3.23. *The universal elliptic curve is the stack* $[((\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})]$ *and is given by the following morphism:*

$$\pi : [((\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})](U) \longrightarrow [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})](U)$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & (\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2 & \mapsto & X \arrow[r, "p\circ f"] \arrow[d] & \mathfrak{h} \\ U & & & U & \end{tikzcd} \tag{3.57}$$

Where $p : (\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2 \to \mathfrak{h}$ *is the universal framed elliptic curve and* $p\circ f$ *is clearly equivariant since both maps are equivariant. This map is the universal elliptic curve, this implies that* $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ *is the fine moduli space.*

Proof. Let $X \to U$ be a family of elliptic curves over U, by the stack isomorphism Ξ we automatically get the $\mathbf{SL}(2,\mathbb{Z})$-principal bundle $Y = \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}) \to U$ and an equivariant map $\Phi' : \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}) \to \mathfrak{h}$.

We can choose a trivializing cover $\mathfrak{U} = (U_i)$ of the bundle $g : \hat{U}\times_{\pi_1(U)}\mathbf{SL}(2,\mathbb{Z}) \to U$ by contractible sets, as they are all manifolds, and this cover also trivializes the family $X \to U$ seen as a C^∞-bundle. Since we can frame each $X_{|U_i} \to U_i$ then we have a period map for each U_i that we will call Φ_i and as $\epsilon \to \mathfrak{h}$ is universal we have that $X_{|U_i} \cong \Phi_i^*(\epsilon)$. At the same time we know that we have trivialisation of the $\mathbf{SL}(2,\mathbb{Z})$-bundle given by $g^{-1}(U_i) \cong U_i \times \mathbf{SL}(2,\mathbb{Z})$.

Moreover we have the following diagram

$$\begin{tikzcd} {[((\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})]\times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U} \arrow[r] \arrow[d] & {[((\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})]} \arrow[d, "\pi"] \\ U \arrow[r] & {[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} \end{tikzcd} \tag{3.58}$$

Where we identify the morphism $U \to [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ with the $\mathbf{SL}(2,\mathbb{Z})$-principal bundle $Y \to U$ together with the equivariant map $\Phi' : Y \to \mathfrak{h}$ thus we know that the fibred product $((\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})]\times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U(V)$ is equal to

$$\left\langle \psi : V \to U,\ \begin{matrix} T \xrightarrow{f} (\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2 \\ \downarrow \\ V \end{matrix}\ ,\ \begin{tikzcd} & \mathfrak{h} \\ \psi^*Y \arrow[ur, "h\circ pr"] \arrow[r, "\sim"] \arrow[d] & T \arrow[u, "p\circ f"'] \arrow[d] \\ V \arrow[r, "id"'] & V \end{tikzcd} \right\rangle.$$

Let us wonder what happens if $X \to U$ is a framed family of elliptic curves, this would imply that there is a map $U \to \mathfrak{h}$ which implies that $\psi^*X \to M$ is also a framed family and its period map is $M \to U \to \mathfrak{h}$ this imposes that $\Xi(T \to M)$ is a framed family this immediately implies that $T \cong M \times \mathbf{SL}(2,\mathbb{Z}) \to M$ is a trivial bundle[3] and $\psi^*Y \to M$ must also be trivial. Thus the fibred product has objects the set of trivializations of the bundle $T \to M$ which are the same thing as maps $M \to \mathbf{SL}(2,\mathbb{Z})$ and maps $\psi : M \to U$, which can be understood as maps $M \to U \times \mathbf{SL}(2,\mathbb{Z})$. Consequently $[((\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})]\times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U \cong U \times \mathbf{SL}(2,\mathbb{Z})$.

Note that, since restricting the family to the cover $\mathfrak{U}$ of U gives us framed families $X_{|U_i} \to U_i$ then $[((\mathbb{C}\times\mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})]\times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U_i \cong U_i \times \mathbf{SL}(2,\mathbb{Z})$. Then as we know that we are working inside the stack $\mathscr{M}_{1,1}$ or equivalentely $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ we can glue objects an isomorphisms using

[3] Since if the family is already framed then the pullback to its universal cover will inherit this framing thus changing elements in the fibre isomorphic to $\pi_1(U)$ will not change the frame of the curve above them.

the previously stated descent data (the trivialization of the $\mathbf{SL}(2,\mathbb{Z})$-bundle provides us with the transition maps that satisfy the cocycle condition) in order to glue together $U = \sqcup U_i/\sim$ and $Y = \sqcup Y_{|U_i}/\sim \quad \to U$ which implies that $[((\mathbb{C} \times \mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})] \times_{[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]} U \cong Y$.

Moreover so (between this objects we have "locally" the isomorphisms Φ_i given by the fact that $\epsilon \to \mathfrak{h}$ is an universal family, which coincide in double intersections) we can glue together this isomorphisms into

$$\begin{tikzcd} \sqcup\Phi_i^*(\epsilon)/\sim \arrow[r, "\sqcup\Phi_i"] \arrow[d] & \sqcup X_{|U_i}/\sim \arrow[dl] \\ U & \end{tikzcd} \tag{3.59}$$

Because each of them was an isomorphism that means that we can glue together the inverses thus $\sqcup\Phi_i$ is an isomorphism.
This exhibits $\pi : [((\mathbb{C} \times \mathfrak{h})/\mathbb{Z}^2)/\mathbf{SL}(2,\mathbb{Z})] \to [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ as the universal elliptic curve and $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ as a fine moduli space. $\square$

Theorem 3.24. *As Riemann surfaces we know that* $\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$ *is biholomorphic to* $\mathbb{C}$.

Proof. In the example 3.16 we saw that $\mathbb{C} \setminus \{0,1\}$ parametrizes a family of elliptic curves, by considering a complex torus as an algebraic variety. Seen in this way we know that the algebraic varieties defined by $y^2 - x(x-1)(x-\lambda) = 0$ and $y^2 - x(x-1)(x-\lambda') = 0$ are isomorphic if and only if $\lambda' = g(\lambda)$ where $g \in G$ and the group G is a group of order 6 generated by $\lambda \to 1/\lambda$ "inversion" and $\lambda \to 1-\lambda$ "translation" then the quotient of $\mathbb{C}\setminus\{0,1\}$ by this group is isomorphic to $\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$ since they both parametrize classes of isomorphism of elliptic curves.

Now consider the following map:

$$\begin{aligned} &\phi : \mathbb{C} \setminus \{0,1\} \longrightarrow \mathbb{C} \\ &\lambda \mapsto \phi(\lambda) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2} \end{aligned} \tag{3.60}$$

is a rational function (and is known as the modular lambda function) which is invariant under the substitutions $\lambda \to 1/\lambda$ and $\lambda \to 1-\lambda$ since:

$$\begin{aligned} \phi(1/\lambda) &= \frac{((1/\lambda)^2 - (1/\lambda) + 1)^3}{(1/\lambda)^2((1/\lambda) - 1)^2} \\ &= \left(\frac{1 - \lambda + \lambda^2}{\lambda^2}\right)^3 \left(\frac{\lambda^2}{1-\lambda}\right)^2 \\ &= \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2} \end{aligned} \tag{3.61}$$

and

$$\begin{aligned} \phi(1-\lambda) &= \frac{((1-\lambda)^2 - (1-\lambda) + 1)^3}{(1-\lambda)^2((1-\lambda) - 1)^2} \\ &= \frac{((1-\lambda)^2 + \lambda)^3}{(1-\lambda)^2\lambda^2} \\ &= \phi(\lambda) \end{aligned} \tag{3.62}$$

thus it is also invariant under the substitution that sends λ to $1/(1-\lambda), (\lambda-1)/\lambda, \lambda/(\lambda-1)$, thus ϕ is a six to one function since having more than six values that share the same value would contradict that the degree of the numerator is six. Thus this map descends to the quotient $(\mathbb{C} \setminus \{0,1\})/G$ and we have that $\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$ is isomorphic to the image of ϕ. but since ϕ is a non constant algebraic function, then it can only omit a finite set of points by the great Picard's theorem, but ϕ is a rational non-constant map then it is surjective because if we take any $a \in \mathbb{C}$ then we have that:

$$\begin{aligned}
\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} &= a \\
(\lambda^2-\lambda+1)^3 &= a\lambda^2(\lambda-1)^2 \\
(\lambda^2-\lambda+1)^3 - a\lambda^2(\lambda-1)^2 &= 0
\end{aligned} \tag{3.63}$$

but and by the fundamental theorem of algebra this last polynomial has roots, and they are clearly different from 0 and 1, consquently ϕ the image of ϕ is $\mathbb{C}$. Then $\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})$ is isomorphic to $\mathbb{C}$. $\square$

Remark 3.25. A different way of proving theorem 3.24 is through an invariant of plane algebraic curves called the j-invariant, a good reference for this proof is [Cle80], Section 3.12.

This implies that the moduli stack of elliptic curves (which is isomorphic to the stack $[\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$) is not only analytic but has an interesting underlying space, a complex manifold, we remark that this is a special case since not all moduli problems behave in this way.

The stack atlas $\mathfrak{h} \to [\mathfrak{h}/\mathbf{SL}(2,\mathbb{Z})]$ will allow us to talk about an even more interesting structure over the moduli space, that is the structure of orbifold.

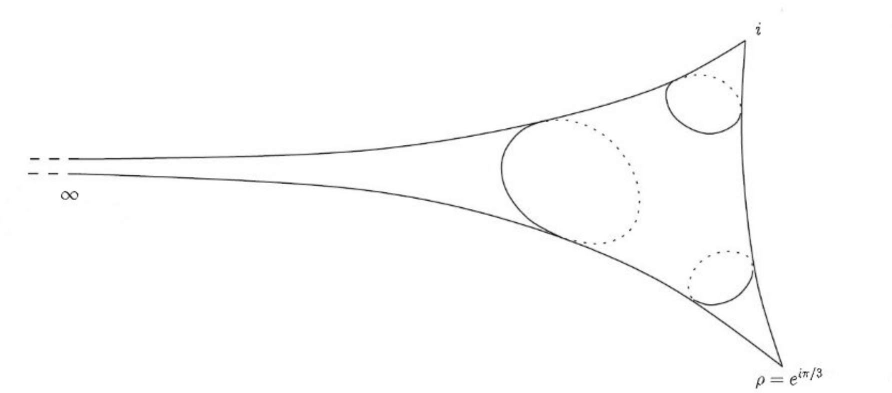

Figure 1. This image was obtained in [DFHH14].

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