Quantum Fields with Dynamical Boundary Conditions

Quantum Field Theory and Mathematical Physics

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Abstract

In this work it is exposed the main characteristics of quantum field theory in flat spacetime and curved spacetime focusing on the non uniqueness of the vacuum state in the later theory. This fact involves many datails on the flat spacetime theory that can not be generalized in an easy way to curved spacetime. Therefore the observed physical phenomena looks quite different in both theories.

Such differences between theories were studied in the context of the Casimir effect with several modifications: change in topology of spacetime, introduction of moving boundaries, in a gravitational collapse and in different types of motion for the observers on spacetime. In the study of the physical phenomena it was shown that the expectation value of the energy-momentum tensor and the expectation value of the number of particles on vacuum changes when introducing such modifications. The research directed mostly on the observation of changes on the observable of number of particles in vacuum as it depends directly on acceleration motion and spacetime curvature.

In brief, changes on vacuum introduce changes on observables of the physical system. Therefore one would think about relating those observables with topological invariants.

This, in order to have a well defined problem which aims to solve the latter idea. It is claimed that the physical phenomena studied have a great degree of universality and they can be modeled as quantum phase transitions by defining an order parameter which is going to be the topological invariant.

As a first step for future work, some of the mathematical tools that are going to be needed in order to solve such a problem in the context of fermionic and bosonic systems are exposed. Aspects concerning the vacuum condition are mentioned and their close relation with orthogonal complex structures and irreducible representations of the CAR algebras are highlighted.

Resumen

En este trabajo se exponen las características principales de la teoría cuántica de campos en espacio-tiempo plano y en espacio-tiempo curvo enfocándose en la no unicidad del vacío. Este hecho involucra muchos detalles de la teoría que son la causa de que no exista una generalización simple y, por lo tanto, se observan diferencias en los fenómenos físicos desde cada una de las teorías.

Dichas diferencias entre ambas teorías fueron estudiadas en el contexto del efecto Casimir con modificaciones tales como; cambios en la topología del espacio-tiempo, introducción de fronteras dinámicas, en un colapso gravitacional y para observadores con distintos tipos de movimiento en el espacio-tiempo. En el estudio del fenómeno físico se mostró que el valor esperado del tensor de energía-momento y el valor esperado del número de partículas en el vacío cambiaba cuando se introducían dichas modificaciones. En este orden de ideas, la investigación se orientó en la observación de los cambios en el observable del número de partículas en el vacío ya que este depende del movimiento acelerado y la curvatura del espacio-tiempo.

En resumen, los cambios en el vacío introducen cambios en los observables del sistema físico, luego se puede pensar en relacionar dichos cambios con invariantes topológicos.

Para tener un problema bien definido que busque solucionar la idea anterior cabe resaltar que los fenómenos físicos estudiados tienen un alto grado de universalidad y se pueden modelar como transiciones de fase cuánticas después de definir un parámetro de orden que va a hacer las veces de invariante topológico.

Como primer paso para un trabajo futuro, se van a presentar las herramientas matemáticas necesarias para resolver dicho problema en el contexto de sistemas fermiónicos y bosónicos. Se mencionan aspectos relacionados con las condiciones de vacío y su estrecha relación con estructuras complejas ortogonales, y se resaltan aspectos relevantes de las representaciones irreducibles de algebras CAR en este contexto.

Dedication

Para Papa, Mama, Lauris, Abuelitos Biffi, Abuelitos Papi y Titina. Lo han hecho todo posible. Los quiero mucho!

Declaration

I declare that the present work uses theoretical results from other investigations that are properly cited. This dissertation does not have to pass to the ethical committee of the Science Faculty.

Acknowledgements

I want to thank my advisor Andrés Reyes for his quality as a researcher, academical professor and as a person, and the research group of Mathematical Physics and Quantum Field Theory.

"¿Lo intentaste? ¿Fallaste?
No importa!
Falla otra vez... Falla mejor."

SAMUEL BECKETT

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Chapter 1

Introduction

Using the Quantum Field theory formalism in flat spacetime we can expand the field in an infinite collection of decoupled oscillators in terms of the usual covariant creation and annihilation operators that define the vacuum at spacetime. This can be thought intuitively as the physical vacuum a state where all oscillators from the field decomposition are in its ground state. This is Minkowski space vacuum, it has the special property of being invariant under the action of the Poincaré group, this implies its uniqueness in the context of flat spacetime [2]. Therefore the definition of particle and vacuum depend on the field decomposition in positive and negative modes and they depend on the existence of a killing vector, it is evident that in effect the existence of killing vectors depend on the existence of spacetime symmetries, in particular time traslation symmetries [22].

The generalization from Quantum Field Theory in flat spacetime to curved spacetime is nontrivial. The Poincaré group is no longer a symmetry group of spacetime, in general, curved spacetimes do not have groups of symmetries because of the non existence of the killing vectors. There exists special classes of spacetime symmetries under transformation restriction with their associated coordinate system equiped with the killing vectors, but taking into account the general relativity principle of general covariance in consequence it is not possible to decompose the field in modes [2]. Then, in the context of curved spacetimes it does not exist the natural notion of particles and vacuum, this issue considered in several different physical situations is going to be the main subject of this work [22].

In particular we will illustrate the ideas discussed in the context of the Casimir effect, one of the few physical phenomena that relates Quantum Field theory with the experiment. The effect consists in the observable attraction of two conducting plates as a result of electromagnetic perturbations induced by the plates when located at a distance much smaller than their size. In this work we will take DeWitt proposal, and consider nontrivial geometrical effects such as modifications in the global topology of flat spacetime and introduction of boundaries with imposed conditions to observe interesting problems arising in the theory [12].

The main physical situations to be treated are the cylinder $\mathbb{R}^1 \times \mathbb{S}^1$ (this is the simplest topological modification of flat spacetime), the moving mirror model as a manifestation of the dynamical Casimir effect considering a family of accelerated trajectories for the spacetime boundary which is going to gives us intuition for more complicated systems through the exact correspondence with the black hole formation from collapse, and a Rindler spacetime where the observers hyperbolic motion in the presence of horizons results in the Unruh effect. In all cases it will be proved the ambiguity of the vacuum definition as a result of particle production.

It is natural to expect subtle differences while formulating such physical problems from the point of view of the quantum field theory in flat spacetime and its generalization to curved spacetime.

In this spirit one would ask for variations in the observables of physical systems under the change of the physical system conditions.

In particular we are concerned with the vacuum state, physically these state is the lowest energy state of the system which one should expect to have no energy neither particles. But when imposing new conditions that modify the physical phenomena this idea of the vacuum state it is not true anymore. Then the focus of this research will end up in some way on searching for vacuum conditions.

In this order of ideas there is new physics behind this fact, this observable changes are closely related to a high degree of universality of the system which relates with quantum phase transitions in a non conventional way of its use, since the later has been used traditionally to describe phenomena on statistical physics and condensed matter physics.

It is important to get a claer conceptual context about the problem in order to leave any possible ambiguity behind.

The approach to describe such changes in this class of physical systems reduces the problem to be mathematical since very advanced mathematical tools are needed in order to construct a language that generalizes this obervations.

This will be a very important achievement, in some way it could generalize the language used in theoretical physics.

Chapter 2

Elements of Quantum Field Theory

2.1 The Formulation of Quantum Field Theory in Flat Spacetime

Consider Minkowski spacetime, a 2−manifold (2.1) with a line element (2.2), for simplicity we will deal with $(1+1)-$dim spacetime.

$$M = \mathbb{R}^2 \tag{2.1}$$

$$ds^2 = -dt^2 + dx^2 \tag{2.2}$$

The Riemann tensor associated to his line element is nonzero, this means we are dealing with special relativity [4]

2.1.1 Lagrangian Formulation

Consider the scalar field $\phi(x)$ such that the mass of the field quanta is m. The field has a Lagrangian density is (2.3) where $\xi R(x)\phi^2(x)$ is the coupling of the scalar and gravitational field.

$$\mathcal{L}(x) = \frac{1}{2}[-g(x)]^{1/2}\left(g^{\mu\nu}\phi(x)_{,\mu}\phi(x)_{,\nu} - [m^2 + \xi R(x)]\phi^2(x)\right) \tag{2.3}$$

The action in an n-dimensional spacetime is given by (2.4), imposing the variational condition for the lagrangian density (2.5) we have the scalar field equation in a general case (2.6).

$$S = \int \mathcal{L} d^n x \tag{2.4}$$

$$\frac{\delta \mathcal{L}}{\delta \phi} = 0 \tag{2.5}$$

$$[\Box_x + m^2 + \xi R(x)]\phi(x) = 0 \tag{2.6}$$

2.1.2 Use of Fourier Transformation

Definition 1 *(Killing vector, Positive frequency modes) In Minkowski spacetime a killing vector ∂_t is such that the modes $\phi_\omega \sim e^{i\omega(t-x)}$ are positive frequency modes, this means $i\partial_t\phi_\omega = \omega\phi_\omega$ such that $\omega > 0$.*

In flat spacetime there exist killing vectors ∂_t so we will use Fourier transform to decompose the solutions of the field equations into positive and negative frequency parts. The Fourier transformation of the field is given by (2.7).

$$\phi(x) = \frac{1}{(2\pi)^2}\int d^2\bar{\phi(k)}e^{i(kx-\omega t)} \tag{2.7}$$

By replacing the Fourier Transform into the field equation (2.6) without the gratation coupling constant we get that it has to be satisfied the condition (2.8).

$$\omega_k^2 := k^2 + m^2 \tag{2.8}$$

Then the integral has support in the hyperoboloid of mass m. And the field in Fourier space is given by (2.9), at the moment we will call the operators $\hat{a}(k)$ Fourier coefficients.

$$\bar{\phi(k)} = \delta(-\omega_k^2 + \omega^2)\hat{a}(k) \tag{2.9}$$

Lets define a scalar product in the most general form (3.4), where Σ is a spacelike Cauchy hypersurface in the globally hyperbolic spacetime and n^μ is a future-directed unit vector orthogonal to the spacelike hypersurface Σ.

$$(\phi_1, \phi_2) = -i\int \phi_1(x)\overleftrightarrow{\partial_\mu}\phi_2^*(x)\left[-g_\Sigma(x)\right]^{1/2} d\Sigma^\mu \tag{2.10}$$

Considering $\phi_i(x)$ the mode solutions for the eqaution (2.6) and the scalar product (3.4) we have a complete set $\{\phi_i, \phi_i^*\}$ of mode solutions which satisfy the conditions (2.11).

$$(\phi_i, \phi_j) = \delta_{ij} \quad (\phi_i^*, \phi_j^*) = -\delta_{ij} \quad (\phi_i, \phi_j^*) = 0 \tag{2.11}$$

Then the field can be expanded on an infinite collection of decoupled oscillators (2.12), where $\hat{a_k}^\dagger$ and $\hat{a_k}$ are the creation and annihilation operators, respectively that satisfy the usual commutation relations in the covariant quantization of the theory (2.13) and which define de vacuum according to (2.14).

$$\phi(t, x) = \sum_i \left(\phi_i\hat{a}_i + \phi_i^*\hat{a_i}^\dagger\right) \tag{2.12}$$

$$[\hat{a}_i, \hat{a}_j] = 0 = \left[\hat{a_i}^\dagger, \hat{a_j}^\dagger\right] \quad , \quad \left[\hat{a}_i, \hat{a_j}^\dagger\right] = \delta_{ij} \tag{2.13}$$

So after quantization, the annihilation and creation operators will take the operator character of the field.

$$a_i|0\rangle = 0 \quad \forall i \tag{2.14}$$

The vacuum ($|0\rangle$) defined according to (2.14) can be thought intuitively as the physical vacuum, a state where all the oscillators from the field decomposition (2.12) are in its ground state. This is Minkowski space vacuum which has the special property of being invariant under the action of the Poincaré group, action that leaves the Minkowski line element unchanged. This vacuum property implies its uniqueness in the context of flat spacetime. [2]

2.1.3 Positive Frequency Decomposition and Construction of the Hilbert

According to the ideas exposed, the definition of particle and vacuum depend on the field decomposition in positive and negative modes which depend on the existence of a killing vector. By definition (1) it is evident that in effect, the existence of killing vectors depend on the existence of symmetries, in particular time traslation symmetries. [22]

Let V be the real vector space of solutions to the Klein-Gordon equation and consider its complexification $V^{\mathbb{C}}$. The Poincaré symmetries permits us to use the Fourier transform so that we can write any real solution in the complexified vector space as a sum of positive and negative frequency parts, $\phi = \phi^+ + \phi^-$. We have a covariant decomposition $V^{\mathbb{C}} = V^+ \oplus V^-$, we are intersted in the positive frequency solutions (V^+).

In order to give V^+ the structure of a Hilbert space we define an inner product on this space (2.15), it comes canonical from the scalar product defined before.

$$\langle \phi_1^+, \phi_2^+ \rangle := \frac{i}{\hbar} \int_\Sigma dx (\bar{\phi}_1^+ \nabla_\mu \phi_2^+ - \phi_2^+ \nabla_\mu \bar{\phi}_1^+) n^\mu \tag{2.15}$$

It follows that (2.16) is a $1-$particle Hilbert space and (2.17) is the $n-$particle Hilbert space called the Fock space.

$$\mathcal{H} := (V^+, \langle , \rangle) \tag{2.16}$$

$$\mathcal{F} := \mathcal{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus ... \oplus (\otimes_s^n \mathcal{H}) \oplus ... \tag{2.17}$$

2.2 The Formulation of Quantum Field Theory in Curved Spacetime

Curved spacetime refers to consider arbitrary manifolds of the form $M^{(2)}$, but in this section we will consider globally hyperbolic spacetimes endowed with a metric of Minkowski sigature $(-,+)$, this is a large class of curved spacetimes. We are dealing topologically with a $2-$manifold of the form (2.18).

$$M^{(2)} = \Sigma^{(1)} \times \mathbb{R} \tag{2.18}$$

Definition 2 *(Cauchy Surface) In a globally hyperbolic spacetime (Lorentzian Manifold) (X, g), a Cauchy surface is an embedded submanifold $\Sigma \hookrightarrow X$ sich that every timelike curve in X intersects Σ exactly in one point.*

Globally hyperbolic spacetimes have the structure of a foliation, where the submanifold Σ is a Cauchy surface in the sense of Definition 2.

The generalization from Quantum Field Theory in flat spacetime to curved spacetime is nontrivial. The Poincaré group is no longer a symmetry group of spacetime, in general curved spacetimes dont have groups of symmetries because of the non existence of the killing vectors. There exists special classes of spacetime symmetries under transformation restriction with the existence of its associated coordinate system equiped with the killing vectors, but taking into account the general relativity principle of general covariance in consequence it is not possible to decompose the field in modes. [2]

Then, in the context of curved spacetimes it doesnt exist the natural notion of particles and vacuum, this issue considered in several different physical situations is going to be one of the main subjects of this work. [22]

2.2.1 Bogolubov Transformations and Ambiguity of the Physical Vacuum

To see this in a general way consider another complete orthonormal set $\{\bar{\phi}_i, \bar{\phi}_i^*\}$ of solutions to the equation (2.6) respect to the scalar product (3.4). The decomposition of the field is now (2.19).

$$\phi(t, x) = \sum_i \left(\bar{a}_i \bar{\phi}_i(x) + \bar{a}_i^\dagger \bar{\phi}_i^*(x) \right) \tag{2.19}$$

According to the election of the set of mode solutions we define canonically the vacuum $\bar{a}_i|\bar{0}\rangle = 0 \ \forall i$. As both sets of solutions are complete we can get an analytical expression for one set of modes in terms of the other (2.20) using Bogolubov transformations.

$$\begin{aligned}\bar{\phi}_j &= \sum_i (\alpha_{ji}\phi_i + \beta_{ji}\phi_i^*) \\ \phi_i &= \sum_j \left(\alpha_{ji}^*\bar{\phi}_j - \beta_{ji}\bar{\phi}_j^*\right)\end{aligned} \tag{2.20}$$

Where α_{ij}, β_{ij} are the Bogolubov coefficients. Using the resutls (2.20) and the scalar product (3.4) we have an expression for the Bogolubov coefficients (2.21).

$$\alpha_{ij} = (\bar{\phi}_i, \phi_j) \qquad \beta_{ij} = -(\bar{\phi}_i, \phi_j^*) \tag{2.21}$$

Now, using the field mode expansions (2.12) and (2.19) with the generalized scalar product (3.4) we obtain expressions for the expansion of the creation and annihilation operators (2.22).

$$\begin{aligned}a_i &= \sum_j \left(\alpha_{ji}\bar{a}_j + \beta_{ji}^*\bar{a}_j^\dagger\right) \\ \bar{a}_j &= \sum_j \left(\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger\right)\end{aligned} \tag{2.22}$$

It follows from the previous ideas the action of the annihiliation operator from the first set of solutions over the vacuum defined in the field expansion with the second set of solutions (2.23) and the expected number of ϕ_i-particles (of the first set of solutions) in the vacuum state of the second set of solutions (2.23).

$$a_i|\bar{0}\rangle = \sum_j \beta_{ji}^*|\bar{1}_j\rangle \neq 0 \tag{2.23}$$

We can also calculate the expected value of the number of particles operator, which tells s the amount of p_i-particles present in the $|\bar{0}\rangle$ vacuum (2.24)

$$\langle\bar{0}|N_i|\bar{0}\rangle \equiv \langle\bar{0}|a_i^\dagger a_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \neq 0 \tag{2.24}$$

It is proved that the vacuum definition is ambiguous as the vacuum state depends on the choice of solutions to the equation (2.6) and the vacuum of the $\bar{\phi}_i$ modes contains a nonzero number of particles in the ϕ_i mode.

This is a phylosophical issue as the physical vaccum we were thinking about was a well defined vacuum in the sense that there is absence of particles ant energy, as in flat spacetime.

Approaches for generalizing the notion of vacuum state have been done with the notion of quasi-free Hadamard state [15], which are not going to be considered in the present work. Then the problem with this approach is the non uniqueness of the vacuum state.

2.2.2 Energy-Momentum Tensor

It is constructive to show the derivation of the energy-momentum tensor via Noether's theorem.

Let φ a massive real (for simplicity) scalar field that satisfies the Klein-Gordon equation (2.25).

$$\left(\partial_\mu \partial^\mu + m^2\right) \varphi(x) = 0 \tag{2.25}$$

Consider the Klein-Gordon equation for a massive field (2.25), associated to a classical dynamical system. And let $\mathcal{L}(\varphi, \partial_\mu \varphi)$ be its associated Lagrangian density, following the ansatz for the Lagrangian of the Klein-Gordon theory, we have (2.26).

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = \frac{1}{2}\left[\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2\right] \tag{2.26}$$

It is straightforward to check the Lagrangian ansatz makes sense, by replacing (2.26) into the Euler-Lagrange equations (2.27).

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{2.27}$$

Let $S[\varphi]$ given by (2.28) be the action associated to the Klein-Gordon Lagrangian. And lets impose the principle of least action (2.29),

$$S[\varphi] = \int dx^4 \mathcal{L}(\varphi, \partial_\mu \varphi) \tag{2.28}$$

$$\delta S[\varphi] = 0 \tag{2.29}$$

In the most general case, let G be the group of continous transformations with an action over $\mathcal{L}$ such that $\forall g \in G$ near 1_G we have the variation of the Lagrangian of the form $\delta\mathcal{L} = \mathcal{L} - g\mathcal{L}$ and suppose $S[\varphi]$ is invariant under G.

In particular, considering an homogeneous spacetime suppose $S[\varphi]$ is invariant under the group of continous traslation transformations of the form $x^{\mu'} = x^\mu + a^\mu$.

By hypothesis the Lagrangian $\mathcal{L}(\varphi, \partial_\mu \varphi)$ is invariant modulo $4-$divergence terms of the form J^μ. Considering the lagrangian $\mathcal{L}' = \mathcal{L} - J^\mu$ and using the variation definition $(\delta x := (\partial_\mu x/\partial\varphi)\partial\varphi)$ we obtain the form of $\delta\mathcal{L}$ (2.30).

$$\begin{aligned}
&S'[\varphi] = \int dx^4 \mathcal{L}'(\varphi, \partial_\mu \varphi) = \int dx^4 \mathcal{L} - J^\mu \\
&\Rightarrow \delta S'[\varphi] = \int dx^4 \delta\mathcal{L}'(\varphi, \partial_\mu \varphi) = \int dx^4 \delta\mathcal{L} - \delta J^\mu = 0 \\
&\Rightarrow \delta\mathcal{L} - \delta J^\mu = 0 \Rightarrow \delta\mathcal{L} = \delta J^\mu := \left(\frac{\partial_\mu J^\mu}{\partial \varphi}\right)\partial\varphi \\
&\Rightarrow \delta\mathcal{L} = \partial_\mu J^\mu
\end{aligned} \tag{2.30}$$

But we would like to have an explicit expression for the term $J^\mu(\delta\varphi)$, using Euler-Lagrange equations (2.27) and the fact (2.31), then we get the expression (2.32).

$$\begin{aligned}\delta(\partial_\mu\varphi) &:= (\partial(\partial_\mu\varphi)/\partial\varphi)\partial\varphi = (\partial(\partial_\mu\varphi)/\partial\varphi\partial x^\mu)\partial\varphi\partial x^\mu = (\partial(\partial_\mu\varphi)/\partial\varphi)(\partial\varphi/\partial x^\mu)\partial x^\mu \\ &= \partial_\mu(\partial\varphi/\partial\varphi)(\partial\varphi/\partial x^\mu)\partial x^\mu = \partial_\mu\delta\varphi\end{aligned} \tag{2.31}$$

$$\begin{aligned}&\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) = \partial_\mu J^\mu \\ &\Rightarrow \delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) = \partial_\mu J^\mu \\ &\Rightarrow \delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu(\delta\varphi) = \partial_\mu J^\mu \\ &\Rightarrow \delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi = \partial_\mu J^\mu \\ &\Rightarrow \delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi - J^\mu\right) = 0 \\ &\Rightarrow J^\mu(\delta\varphi) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\end{aligned} \tag{2.32}$$

By construction we selected an element $g \in G$ near the group identity (1_G) we can expand in Taylor series the following expressions, $\delta\varphi = \partial_\nu\varphi + \partial_\nu^{(2)}\varphi + ... \Rightarrow \delta\varphi = \partial_\nu\varphi$ and $\delta\mathcal{L} = \partial_\nu\mathcal{L} + \partial_\nu^{(2)}\mathcal{L} + ... \Rightarrow \delta\mathcal{L} = \partial_\nu\mathcal{L} = \partial_\nu g_{\mu\nu}\mathcal{L}$. So we have the following expression (2.33).

$$\begin{aligned}&\delta\mathcal{L} = \partial_\mu J^{\mu\nu}(\delta\varphi) = \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi = \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\left(\frac{\partial_\mu\varphi}{\partial x^\mu}\right)\partial x^\mu = \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\varphi \\ &\delta\mathcal{L} = \partial_\nu g_{\mu\nu}\mathcal{L} \\ &\Rightarrow \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi - g_{\mu\nu}\mathcal{L}\right) = \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\varphi)}\partial_\mu\varphi - g_{\mu\nu}\mathcal{L}\right) = 0\end{aligned} \tag{2.33}$$

The expression (2.33) suggests to define (2.34) as the Noether's current.

$$j_{\nu\mu} := \frac{\partial\mathcal{L}}{\partial(\partial^\nu\varphi)}\delta\varphi - g_{\nu\mu}\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial^\nu\varphi)}\partial_\mu\varphi - g_{\nu\mu}\mathcal{L} \tag{2.34}$$

By Noether's theorem, each symmetry has a conserved quantity associated. In particular the symmetry of translations is associated with the energy-momentum conservation. Therefore the conserved current (2.34) is the quantity associated to the conservation of the energy-momentum. So we define the energy-momentum tensor as (2.35).

$$T_{\nu\mu} := j_{\nu\mu} = \frac{\partial \mathcal{L}}{\partial(\partial^\nu \varphi)} \partial_\mu \varphi - g_{\nu\mu}\mathcal{L} \tag{2.35}$$

It is clear by the calculation (2.33) that the energy momentum tensor (2.35) satisfies the continuity equation (2.36).

$$\partial^\mu T_{\nu\mu} = 0 \tag{2.36}$$

Replacing the Lagrangian density (2.26) on the general expression (2.35), we get the general expression for the energy-momentum tensor in Klein-Gordon theory (2.37).

$$\begin{aligned} T_{\nu\mu} &= \frac{\partial \mathcal{L}}{\partial(\partial^\nu \varphi)} \partial_\mu \varphi - g_{\nu\mu}\mathcal{L} = (\partial_\nu \varphi)(\partial_\mu \varphi) - \frac{1}{2}\left[g_{\nu\mu}\partial_\mu \varphi \partial^\mu \varphi - g_{\nu\mu} m^2 \varphi^2\right] \\ \Rightarrow T_{\nu\mu} &= (\partial_\nu \varphi)(\partial_\mu \varphi) - \frac{1}{2}\left[g_{\nu\mu}\partial_\alpha \varphi \partial^\alpha \varphi - g_{\nu\mu} m^2 \varphi^2\right] \\ \Rightarrow T_{\nu\mu} &= (\partial_\nu \varphi)(\partial_\mu \varphi) - \frac{1}{2} g_{\nu\mu}\left[\partial_\alpha \varphi \partial^\alpha \varphi - m^2 \varphi^2\right] \end{aligned} \tag{2.37}$$

where we have the matrix components (2.38) and (2.39) for the energy and momentum, respectively.

$$T_{tt} = T_{xx} = \frac{1}{2}\left[(\partial_t \phi)^2 + \textstyle\sum_{i=1}^{n-1} (\partial_i \phi)^2 + m^2 \phi^2\right] \tag{2.38}$$

$$T_{ix} = T_{xi} = \partial_t \phi \partial_i \phi \tag{2.39}$$

2.2.3 Energy-Momentum Tensor and Casimir Effect on a Cylinder $S^1 \times R^1$

To illustrate the theory of the previous section we treat the case of the cylinder $S^1 \times R^1$ which is the simplest topological modification of flat spacetime, it is a two dimensional spacetime with compactified spatial sections and which is locally flat.

The expectation value of the energy-momentum tensor in the vacuum is going to be computed, this expression is the energy at the vacuum state of the cylinder spacetime. This procedure will requiere the use of renormalization and regularization techniques as the defining the zero-point energy, use of two-point functions and cut-off factors. [12]

Consider a dim$-(1+1)$ cilinder as shown in the Figure 2.2.3 , with circunference L.

Let ϕ be a scalar, non-massive field. For this theory we have the Lagrangian density (2.40), where $\eta^{\alpha\beta}$ is the Minkowski metric.

$$\mathcal{L}(x) = \frac{1}{2}\left(\eta^{\alpha\beta}\phi_{.\alpha}\phi_{.\beta} - m^2\phi^2\right) \tag{2.40}$$

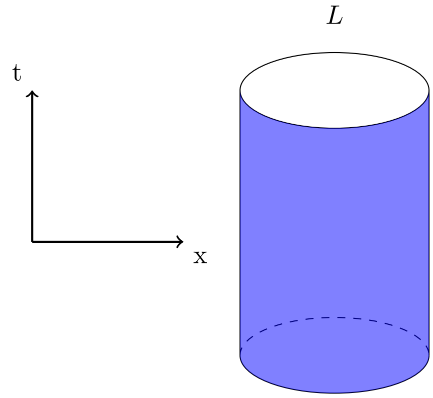

Figure 2.1: Cylinder spacetime ($S^1 \times \mathbb{R}^1$)

By constructing the action using the Lagrangian density (2.40) and imposing the variational condition for the Lagrangian density, we get that tha field satisfies the equation (2.41) which is just the Klein-Gordon equation for a non-massive field.

$$\Box\phi = 0 \ , \ \Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \overrightarrow{\nabla}^2 \tag{2.41}$$

We look for solutions of the form $\phi_k \sim e^{i(k\dot{x}-\omega t)}$ such that the modes ϕ_k are positive frequency modes respect to time. Using the scalar product (3.4) we get the normalization constant so that the modes get the form (2.42), and satisfy the conditions (2.11). Lets restrict the solutions to the interior of the cylinder by imposing periodic boundary conditions $\phi_k(t, \overrightarrow{x}) = \phi_k(t, \overrightarrow{x} + nL) \ \forall n \in \mathbb{Z}$, by this assumption we are discretizing the solutions space (2.42).

$$\phi_k(t, \overrightarrow{x}) = \frac{1}{\sqrt{2\omega L}} e^{i(k\dot{x}-\omega t)} \ , \ k = \frac{2\pi n}{L} \ \ \forall n \in \mathbb{Z} \tag{2.42}$$

Until now we have a space of mode solutions of positive frequency which is orthonormal and complete $\{\phi_k, \phi_k^*\}$.

As the final objective is to calculate the energy of the quantum vacuum. Lets proced with the expressions for the energy-momentum tensor, its most general expression is given by (2.39). In particular, for this case we have the matrix components (2.43) and (2.44).

$$T_{tt} = T_{xx} = \frac{1}{2}\left[(\partial_t\phi)^2 + (\partial_x\phi)^2\right] \tag{2.43}$$

$$T_{tx} = T_{xt} = \partial_t\phi\partial_x\phi \tag{2.44}$$

We are going to define two different vacuum states: $|0\rangle$ the Minkowski space vacuum state which is clearly a physical vacuum in the sense of energy absence and $|0_L\rangle$ the vacuum of the cylinder spacetime it is the vacuum associated to the modes ϕ_k such that $|0_L\rangle \to |0\rangle$ when $L \to \infty$.

We are looking for an expression of the form $\langle T_{\alpha\beta}\rangle_{|0\rangle} \equiv \langle 0_L|T_{\alpha\beta}|0_L\rangle$. Starting by using the fact that $\{\phi_k, \phi_k^*\}$ is complete and orthonormal base, the field can be expanded in terms of this basis (2.45), where the creation and annihilation operators satisfy the usual commutation relations (2.46) and the following condition holds $\hat{a_k}|0\rangle \equiv \langle 0|\hat{a_k}^\dagger = 0$ $\forall k$.

$$\phi(t, \overrightarrow{x}) = \sum_k [\hat{a_k} u_k(t, \overrightarrow{x}) + \hat{a_k}^\dagger u_k^*(t, \overrightarrow{x})] \tag{2.45}$$

$$[\hat{a_k}, \hat{a_{k'}}] = 0 = [\hat{a_k}^\dagger, \hat{a_{k'}}^\dagger] \ , \ [\hat{a_k}, \hat{a_{k'}}^\dagger] = \delta_{kk'} \tag{2.46}$$

In general we have the following expression for the quantum vacuum energy (**??**). Using the canonical commutation relations (2.46), the general expression for the quantum vacuum energy simplifies to the compact form (2.47), where $T_{\alpha\beta} \equiv T_{\alpha\beta}[\phi, \phi]$ denotes a bilinear form.

$$\begin{aligned} \langle 0_L|T_{\alpha\beta}|0_L\rangle =& \langle 0_L|\phi_{.\alpha}\phi_{.\beta}|0_L\rangle - \frac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\langle 0_L|\phi_{.\lambda}\phi_{.\delta}|0_L\rangle + \frac{1}{2}m^2\langle 0_L|\phi^2|0_L\rangle\eta_{\alpha\beta} \\ =& \sum_k [\phi_{k.\alpha}\phi^*_{k.\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\lambda\delta}\phi_{k.\lambda}\phi^*_{k.\delta} + \frac{1}{2}m^2\phi_k\phi_k^*\eta_{\alpha\beta}] \equiv \sum_k T_{\alpha\beta}[\phi_k, \phi_k^*] \end{aligned} \tag{2.47}$$

In particular, for the cylinder case the quantum vacuum energy (2.48) is given by applying the later to the Hamiltonian density (2.43).

$$\langle 0_L|T_{tt}|0_L\rangle \equiv \sum_k T_{tt}[u_k, u_k^*] = \frac{\pi}{L^2}\sum_{n=0}^{\infty} n \tag{2.48}$$

This last quantity diverges, so we will proceed in a stadard manner applying regularization followed by renormalization. Define a normal order (::) with a regularization of the form point-separation (2.49) subject to the condition of renormalizing the zero-point energy $\langle 0|T_{tt}|0\rangle = 0$.

$$\langle 0_L| :T_{tt}: |0_L\rangle \equiv \langle 0_L|T_{tt}|0_L\rangle - \langle 0|T_{tt}|0\rangle = \langle 0_L|T_{tt}|0_L\rangle - \lim_{L'\to\infty}\langle 0_{L'}|T_{tt}|0_{L'}\rangle \tag{2.49}$$

At the moment, the last expression does not have sense as both terms diverge. Using the fact that in quantum field theory results after renormalization are independent of the regularization procedures for various types of procedures.

In this case it is going to be used the cut-off factor regularization procedure, it consists on introducing a term of the form $e^{-\alpha k}$ on the divergent sum and after, taking the limit when $\alpha \to 0$. This can be thought of as writing the divergent quantity in terms of the Riemann zeta function, its analytic continuation in one point its a finite number that can be interpreted as the renormalized value of the divergent quantity.

For the first term on (2.49), using the geometrical series and assuming $|e^{-\alpha/L}| < 1$ we have its convergence and expanding in Taylor series we have (2.50).

$$\begin{aligned}\langle 0_L|T_{tt}|0_L\rangle =&\frac{\pi}{L^2}\sum_{n=0}^{\infty} ne^{-n\alpha/L} = \frac{\pi}{L}\partial_\alpha\left[\sum_{n=0}^{\infty}(e^{-\alpha/L})^n\right] \to \frac{\pi}{L}\partial_\alpha\left[\frac{1}{1-e^{-\alpha/L}}\right]\\ =&\frac{\pi}{L^2}\frac{e^{-\alpha/L}}{(1-e^{-\alpha/L})^2} \approx \frac{\pi}{L^2}\left[-\frac{L^2}{\alpha^2}-\frac{1}{12}+O(\alpha^2)\right]\\ \approx&\frac{\pi^2}{\alpha^2}-\frac{\pi}{12L^2}+O(\alpha^2)\end{aligned} \tag{2.50}$$

Applying the same procedure for the second term on (2.49) we get (2.51).

$$\lim_{L'\to\infty}\langle 0_{L'}|T_{tt}|0_{L'}\rangle \approx \lim_{L'\to\infty}\frac{\pi}{\alpha^2}-\frac{\pi}{12L'^2}+O(\alpha^2) \approx \frac{\pi}{\alpha^2}+O(\alpha^2) \tag{2.51}$$

Finally taking the limit when $\alpha \to 0$ on both terms (2.50) and (2.51). To obtain the quantum vacuum energy on the cylinder (2.52).

$$\langle 0_L|\ {:}T_{tt}{:}\ |0_L\rangle = -\frac{\pi}{12L^2} \tag{2.52}$$

2.2.4 Field Representations, Positive Frequency Decomposition and Construction of Hilbert Space

As said at the beginning of the section our discussion of curved spacetimes is restricted to a large class called globally hyperbolic spaces, we are talking abput manifolds of the form (2.18) [4].

Let $\psi(\bar{x})$ and $\pi(\bar{x})$, $\bar{x} \in \Sigma^2$ the initial data for the equation of motion (2.25), therefore we have a unique solution $\phi(x)$, $x \in M^2$ such that (2.53).

$$\phi(x)|_{\Sigma^2} = \psi(\bar{x}) \ ,\ n^a\nabla_a\phi(x)|_{\Sigma^2} = \pi(\bar{x}) \tag{2.53}$$

We will focus on the structure of infinite dimensional vector spaces, where there are many inequivalent representations. It is a very common situation in physics and in particular it is relevant for the study of scalar quantum fields in curved spacetimes [4].

Let Γ be the vector space of real solutions of (2.25) and Γ_c the vector space of initial data. The previous spaces are related by an isomoprhism (2.54),

$$\begin{aligned}I_\Sigma : \Gamma &\longrightarrow \Gamma_c\\ \phi &\longmapsto (\psi,\pi)\end{aligned} \tag{2.54}$$

we can define a symplectic form on Γ_c (2.55) which defines the equivalent symplectic form on Γ (2.56) by means of a pull-back.

$$\bar{\Omega}((\psi_1,\pi_1),(\psi_2,\pi_2)) := \int_\Sigma (\psi_1\pi_2 - \psi_2\pi_1)dV_\Sigma \tag{2.55}$$

$$\Omega(\phi_1,\phi_2) := \int_\Sigma (\phi_1\nabla_a\phi_2 - \phi_2\nabla_a\phi_1)dV_\Sigma \ , \ \forall\phi_1,\phi_2 \in \Gamma \tag{2.56}$$

As said before, in general we can not use Fourier transform to decompose the field in positive and negative frequency modes. So we will use polarization defined on $\Gamma_{\mathbb{C}}$, we want to express the field as (2.57) [4].

$$\phi = \phi^+ + \phi^- \quad \forall\phi \in \Gamma \ and \ \bar{\phi^-} = \phi^+ \tag{2.57}$$

By construction, Γ is a real vector space of solutions, so we can use a complex structure, (Definition 3)

Definition 3 *(Complex Structure) A complex structure J is a linear map, $J : \Gamma \to \Gamma$ such that $J^2 = -1$*

to define a polarization on $\Gamma_{\mathbb{C}}$ on the following way (2.58),

$$\phi_+ := \frac{1}{2}(\phi - iJ\phi) \ , \ \phi_- := \frac{1}{2}(\phi + iJ\phi) \tag{2.58}$$

using the above definitions (3) and (2.58) we have (2.59).

$$J\phi^\pm := \frac{1}{2}J(\phi \mp iJ\phi) = \frac{1}{2}(J\phi \mp iJ^2\phi) = \frac{1}{2}(J\phi \mp (-1)i\phi) = \frac{1}{2}i(-iJ\phi \pm \phi) = \frac{1}{2}i(\pm\phi - iJ\phi)$$
$$= \pm i\frac{1}{2}i(\phi \mp iJ\phi) \Rightarrow J\phi^\pm = \pm i\phi^\pm \tag{2.59}$$

To construct an inner product on the complexified vector space Γ_J (where Γ_J is the complexification of Γ using J), we must require the following compatibility conditions (2.60) between the symplectic form and the complex structure used [4],

$$\begin{aligned} &(I) \quad \Omega(J\phi_1, J\phi_2) = \Omega(\phi_1,\phi_2) \ , \ \forall\phi_1,\phi_2 \in \Gamma \\ &(II) \quad \Omega(\phi, J\phi) \geqslant 0 \ , \ \forall\phi \in \Gamma \ and \ \Omega(\phi, J\phi) = 0 \Leftrightarrow \phi = 0 \end{aligned} \tag{2.60}$$

when satisfied the conditions (2.60) we can proceed by defining the inner product on Γ_J by (2.61).

$$\langle\phi_1,\phi_2\rangle_J := \frac{1}{2\hbar}\left[\Omega(\phi_1, J\phi_2) + i\Omega(\phi_1,\phi_2)\right] \tag{2.61}$$

Or instead we can only use $\Gamma_{\mathbb{C}}$ and restricting the inner product (2.15) to $\Gamma^+ := \{\phi^+ = 1/2(1-iJ)\phi | \phi \in \Gamma\}$ we get (2.62) [4].

$$\begin{aligned}\langle \phi_1^+, \phi_2^+ \rangle &= \frac{1}{2\hbar}\left[\Omega(\bar{\phi_1^+}, J\phi_2^+) + i\Omega(\bar{\phi_1^+}, \phi_2^+)\right] = \frac{1}{2\hbar}\left[\Omega(\bar{\phi_1^+}, i\phi_2^+) + i\Omega(\bar{\phi_1^+}, \phi_2^+)\right] \\ &= \frac{1}{2\hbar}\left[i\Omega(\bar{\phi_1^+}, \phi_2^+) + i\Omega(\bar{\phi_1^+}, \phi_2^+)\right] = \frac{i}{\hbar}\Omega(\bar{\phi_1^+}, \phi_2^+) = \frac{i}{\hbar}\int_\Sigma dV_\Sigma n^a(\bar{\phi_1^+}\nabla_a\phi_2^+ - \phi_2^+\nabla_a\bar{\phi_1^+})\end{aligned} \tag{2.62}$$

using the result (2.62) and the axioms of the inner product space we have the following equality (2.63),

$$\begin{aligned}\langle \phi_1^+, \phi_2^+ \rangle &= \frac{i}{\hbar}\Omega(\bar{\phi_1^+}, \phi_2^+) = \frac{i}{\hbar}\Omega(\frac{1}{2}(\bar{\phi_1^+} - iJ\bar{\phi_1^+}), \frac{1}{2}(\phi_2^+ - iJ\phi_2^+)) = \frac{i}{4\hbar}\Omega((\bar{\phi_1^+} - iJ\bar{\phi_1^+}), (\phi_2^+ - iJ\phi_2^+)) \\ &= \frac{i}{4\hbar}\left[\Omega(\bar{\phi_1^+}, (\phi_2^+ - iJ\phi_2^+)) - \Omega(iJ\bar{\phi_1^+}, (\phi_2^+ - iJ\phi_2^+))\right] \\ &= \frac{i}{4\hbar}\left[\Omega(\bar{\phi_1^+}, \phi_2^+) - \Omega(\bar{\phi_1^+}, iJ\phi_2^+) - \Omega(iJ\bar{\phi_1^+}, \phi_2^+) - \Omega(iJ\bar{\phi_1^+}, iJ\phi_2^+)\right] \\ &= \frac{i}{4\hbar}\left[\Omega(\bar{\phi_1^+}, \phi_2^+) - i\Omega(\bar{\phi_1^+}, J\phi_2^+) - i\Omega(\bar{\phi_1^+}, J\phi_2^+) - i^2\Omega(J\bar{\phi_1^+}, J\phi_2^+)\right] \\ &= \frac{i}{4\hbar}\left[\Omega(\bar{\phi_1^+}, \phi_2^+) - 2i\Omega(\bar{\phi_1^+}, J\phi_2^+) + \Omega(\bar{\phi_1^+}, \phi_2^+)\right] = \frac{i}{4\hbar}\left[2\Omega(\bar{\phi_1^+}, \phi_2^+) - 2i\Omega(\bar{\phi_1^+}, J\phi_2^+)\right] \\ &= \frac{1}{2\hbar}\left[\Omega(\bar{\phi_1^+}, J\phi_2^+) + i\Omega(\bar{\phi_1^+}, \phi_2^+)\right] \Rightarrow \langle \phi_1^+, \phi_2^+ \rangle = \langle \phi_1, \phi_2 \rangle_J\end{aligned} \tag{2.63}$$

which give us a Hilbert space isomoprhism between $(\Gamma_J, \langle, \rangle_J)$ and $(\Gamma^+, \langle, \rangle)$. Let $\bar{\Gamma^+}$ denote the Cauchy completion of Γ^+, it follows that (2.64) is the 1-particle Hilbert space with its associated Fock space (2.65) [4].

$$\mathcal{H} := (\bar{\Gamma^+}, \langle, \rangle) \tag{2.64}$$

$$\mathcal{F} := \mathcal{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus ... \oplus (\otimes_s^n \mathcal{H}) \oplus ... \tag{2.65}$$

Finally, to give a field representation start by choosing the appropriate complex structure J. Choose a basis of $\mathcal{H}$, say $\{e_n(x)\}_n$ solutions of (2.25) and require (2.66) [4],

$$\begin{aligned}&(I) \quad Je_n = ie_n \\ &(II) \quad \langle e_n, e_m \rangle = \delta_{nm}\end{aligned} \tag{2.66}$$

notice that (I) in (2.66) makes consistent to call the basis selected "positive frequency" [4]. And we can represent the field operator choosing the appropriate complex structure in the selected basis (2.67), where $\hat{a}_n$, $\hat{a}_n^\dagger$ are the annihilation and creation operators, respectively.

$$R_J(\hat{\phi}(x)) := \sum_n \left[e_n(x)\hat{a}_n + \bar{e_n}(x)\hat{a}_n^\dagger \right] \tag{2.67}$$

Chapter 3

The Casimir Effect in Quantum Field Theory

The Casimir Effect is a physical phenomena which is one of the few that relate Quantum Field Theory with the experiment. It consists in the observable attraction of two conducting plates as a result of electromagnetic perturbations induced by the plates when located at a distance much smaller than their size. When considering a flat spacetime with its Minkowski vacuum associated, the number of particles and energy at this state is zero, but as we showed in the previous section when considering a curved spacetime the vacuum definition depends on the election of mode solutions and the number of particles at the vacuum state is different from zero, this leads us to conclude that there is quanta at the vacuum state for curved spacetime, giving rise to a nonzero value of energy at the ground state.

DeWitt proposed very interesting problems may arise when nontrivial geomterical effects (modifications in the global topology of flat spacetime, acceleration and introduction of boundaries) are considered in the context of the Casimir effect [12]. The main result we are concerned with through this work is, when taking DeWitt's considerations within the physical problem we find that in contrast to flat spacetime Casimir effect, in curved spacetime the number of particles and the energy at the quantum vacuum state is different to zero which arise as nontrivial geometrical effects and which is going to be studied in the following particular cases. [2]

3.1 Moving Mirrors in Expanding Minkowski Spacetime

This physical phenomena is a manifestation of the dynamical Casimir effect, it consists on the disturbance of a field in flat spactime by an accelerated boundary through a family of trajectories, results in particle production and a flux of energy. [8] In partiular we will find the energy and the number of particle operator at the vacuum state for the moving boundary model is different from zero for different type of mirror trajectories.

The particle production process gives us an heuristic model of more complicated systems, in particular, as will be proved later exists an exact correspondence between black hole formation from collapse and late time particle production in the moving mirror model. [9]

Consider a massles scalar field in dim1 + 1 flat spacetime that obeys Dirichlet boudary conditions on a perfectly reflecting boundary (mirror), such that it satisfies Klein-Gordon equation $\Box\phi = 0$, Figure 3.1.

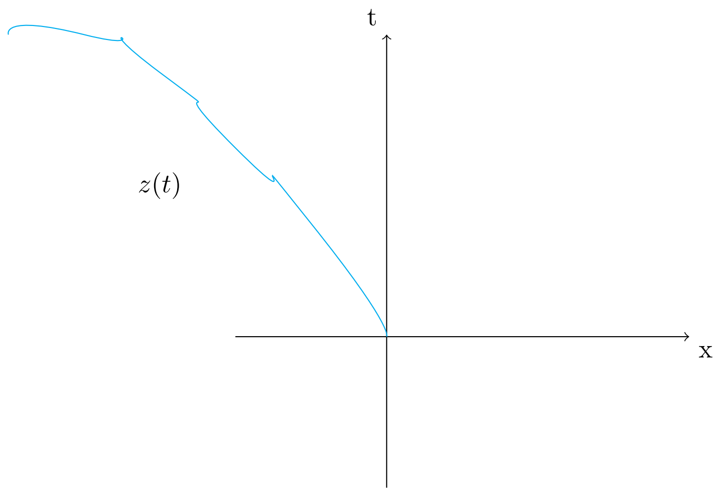

Figure 3.1: Moving Mirror in flat spacetime

We are going to denote ϕ_ω the field expansion in mode functions parametrized by the frequency ω. Lets introduce the light cone coordinates (3.1),

$$\begin{cases} u \equiv t - x \\ v \equiv t + x \end{cases} \tag{3.1}$$

therefore, the Klein-Gordon equation can be writen in terms of this new coordinates as (3.2).

$$(-\partial_t^2 + \partial_x^2)\phi_\omega = -\partial_u \partial_v \phi_\omega = 0 \tag{3.2}$$

Lets think of a general solution of the form (3.3), with g and h arbitrary functions.

$$\phi_\omega = g(v) + h(u) \tag{3.3}$$

Using the scalar product (3.4), where Σ is a Cauchy surface, η^μ is a future-directed unit normal to the surface and $\phi_1(x)\overleftrightarrow{\partial_\mu}\phi_2^*(x) \equiv \phi_1(x)\partial_t\phi_2^*(x) - \phi_2^*(x)\partial_t\phi_1(x)$. Now have a complete set of orthonormal solutions such that the canonical relations hold (3.5).

$$(\phi_1, \phi_2) = -i \int \eta^\mu \left[\phi_1(x) \overleftrightarrow{\partial_\mu} \phi_2^*(x)\right] d\Sigma \tag{3.4}$$

$$(\phi_\omega(x), \phi_{\omega'}(x)) = -(\phi_\omega * (x), \phi_{\omega'} * (x)) = \delta(\omega - \omega') \ , \ (\phi_\omega(x), \phi_{\omega'} * (x)) = 0 \tag{3.5}$$

Lets start with the simple case, consider Minkowski spacetime with no boundaries. We can choose the normalized modes (3.6), then the field ca be expanded in that set of modes (3.7), with $\hat{a_{\omega u}}$, $\hat{a_{\omega v}}$, $\hat{a_{\omega u}}^\dagger$, $\hat{a_{\omega v}}^\dagger$ the usual annihilation and creation operators.

$$\phi_\omega = \phi_{\omega u} + \phi_{\omega v} = \frac{1}{\sqrt{4\pi\omega}} \left[e^{-i\omega u} + e^{-i\omega v}\right] \tag{3.6}$$

$$\phi = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left[\hat{a_{\omega u}}\phi_{\omega u} + \hat{a_{\omega v}}\phi_{\omega v} + \hat{a_{\omega u}}^\dagger \phi_{\omega u} * + \hat{a_{\omega v}}^\dagger \phi_{\omega v}\right] \tag{3.7}$$

Lets consider a mirror trajectory $z(t)$ so that at any time t the mirror is at the position $x = z(t)$. This way Minkowski spacetime has a reflecting boundary and we are interested on the solutions to the right of the mirror.

Dirichlet boundary coditions are imposed, such that the scalar field ϕ vanishes at the mirror surface. This condition is quantified by the functions (3.9) and (3.9), which give the values of the light-cone coordintates at the location of the mirror for any given time, and both relations have inverses $t \equiv t_m(u)$ and $\bar{t} \equiv \overline{t_m}(v)$, respectvely.

$$u \equiv u_m(t) = t - z(t) \tag{3.8}$$

$$v \equiv v_m(t) = t + z(t) \tag{3.9}$$

In the presence of the mirror the set of solutions is quite a bit different from the solutions set in Minkowski spacetime (3.6). We have now to consider mode functions that are positive frequency either at ρ^- or ρ^+ corresponding to the "in" and "out" vacuum states, respectiely.

It is useful to simplify the general expression for the scalar product (3.4) using different Cauchy surfaces (3.10) for ρ^- and (3.11) for ρ^+.

$$(\phi_1, \phi_2) = -i \int \left[\phi_1(u = -\infty, v) \overleftrightarrow{\partial_v} \phi_2^*(u = -\infty, v)\right] dv \tag{3.10}$$

$$\begin{aligned}(\phi_1, \phi_2) = &- i \int \left[\phi_1(u, v = \infty) \overleftrightarrow{\partial_v} \phi_2^*(u, v = \infty)\right] dv \\ &- i \int \left[\phi_1(u = \infty, v) \overleftrightarrow{\partial_v} \phi_2^*(u = \infty, v)\right] dv\end{aligned} \tag{3.11}$$

At ρ^- we have the corresponing solutions (3.12), that are normalized using (3.10). The parameter $p(u)$ is determined by the Dirichlet boundary conditions as follows, we imposed the field has to vanish at the surface of the mirror this means it has to be satisfied $v = p(u)$, using the inverse relation of (3.9) $t \equiv t_m(u)$ and the relation (3.9) we get (3.13). Physically the term (3.13) is due to the Doppler effect generated by the mirror movement.

$$\phi^{in}_{\omega'} = \frac{q}{\sqrt{4\pi\omega}}\left[e^{-i\omega v} - e^{-i\omega p(u)}\right] \tag{3.12}$$

$$p(u) = t_m(u) + z(t_m(u)) \tag{3.13}$$

At ρ^+ we have a horizon at $v = v_0$ so there are two sets of positive frequency mode functions: ϕ^R_ω and ϕ^L_ω. The set of modes $\phi^{R,out}_\omega$ is nonzero at ρ^+_R and zero at ρ^+_L, they are given by the solutions (3.14) and are properly normalized using (3.11). As we did before the parameter $f(v)$ is also determined by the Dirichlet boundary conditions, this time the condition to be satisfied is $u = f(v)$ for the modes to vanish at the surface of the mirror (this condition can be thought of as the inverse of $v = p(u)$). Using the inverse relation of (3.9) $t \equiv \overline{t_m}(v)$ and the relation (3.9)we get (3.15). For completeness, when $v_0 = \infty$ there are no additional set of modes and if $v = v_0$ the set of modes ϕ^L_ω has to be added, but as when they reach ρ^+_L never interact with the mirror so they are irrelevant for our study of the physical phenomena.

$$\phi^{R,out}_\omega = \frac{q}{\sqrt{4\pi\omega}}\left[e^{-i\omega f(v)} - e^{-i\omega u}\right]\ ,\ v < v_0 \tag{3.14}$$

$$f(v) = \overline{t_m}(v) - z(\overline{t_m}(v)) \tag{3.15}$$

As well as the Dirichlet conditions where imposed in the mode solutions such that the field vanishes at the surface of the mirror, we have that the vacuum associated to the solutions (3.12) and (3.14), reduces to the usual physical vacuum. When $t \leqslant 0$ we have $t_m(u) = u$, $z(t_m(u)) = 0$ in (3.13) and $\overline{t_m}(v) = v$, $z(\overline{t_m}(u)) = 0$ in (3.15). Then the solutions (3.12) and (3.14) reduce to the Minkowski spacetime mode solutions (3.6).

We will proceed with the vacuum energy calculation in the presence of an accelerating mirror using the Bologubov transformations. Following the idea that the positive frequency modes at ρ^- (ϕ^{in}_ω) form a complete set we can expand the modes in ρ^+ in terms of them as in (3.16), with $\alpha^J_{\omega\omega'}$ and $\beta^J_{\omega\omega'}$ the Bogolubov coefficients.

$$\phi^J_\omega = \int_0^\infty d\omega'\left[\alpha^J_{\omega\omega'}\phi^{in}_{\omega'} + \beta^J_{\omega\omega'}\phi^{in*}_{\omega'}\right]\ ,\ J \in \{R, L\} \tag{3.16}$$

The Bogolubov coefficients are calcualted as follows, using the canonical relations (3.5) we get the expressions for the coefficient (3.17) and with a similar procedure we get (3.18).

$$(\phi^J_\omega, \phi^{in}_{\omega'}) = \int_0^\infty d\omega' \left[\alpha^J_{\omega\omega'}(\phi^{in}_{\omega'}, \phi^{in}_{\omega'}) + \beta^J_{\omega\omega'}(\phi^{in*}_{\omega'}, \phi^{in}_{\omega'})\right] = \int_0^\infty d\omega' \alpha^J_{\omega\omega'}\delta(\omega - \omega') = \alpha^J_{\omega\omega'}$$
$$\Rightarrow \alpha^J_{\omega\omega'} = (\phi^J_\omega, \phi^{in}_{\omega'}) \tag{3.17}$$

$$(\phi^J_\omega, \phi^{in*}_{\omega'}) = \int_0^\infty d\omega' \left[\alpha^J_{\omega\omega'}(\phi^{in}_{\omega'}, \phi^{in*}_{\omega'}) + \beta^J_{\omega\omega'}(\phi^{in*}_{\omega'}, \phi^{in*}_{\omega'})\right] = \int_0^\infty d\omega' \beta^J_{\omega\omega'}\left(-\delta(\omega - \omega')\right) = \beta^J_{\omega\omega'}$$
$$\Rightarrow \beta^J_{\omega\omega'} = -(\phi^J_\omega, \phi^{in*}_{\omega'}) \tag{3.18}$$

We can now write the field in terms of the mode functions in ρ^- and ρ^+ as in (3.19), where b^J_ω and $b^{J\dagger}_\omega$ are the annihilation and creation operators in the region J satisfying the usual commutation relations (3.5) and defining the vacuum as usual.

$$\begin{aligned} \phi &= \int_0^\infty d\omega' \left[\hat{a}^{in}_{\omega'}\phi^{in}_{\omega'} + \hat{a}^{in\dagger}_{\omega'}\phi^{in*}_{\omega'}\right] \\ &= \sum_J \int_0^\infty d\omega' \left[\hat{b}^J_{\omega'}\phi^J_{\omega'} + \hat{b}^{J\dagger}_{\omega'}\phi^{J*}_{\omega'}\right] \ , \ J \in \{R, L\} \end{aligned} \tag{3.19}$$

We can compute the expressions for the new annihilation (3.20) and creation (3.21) operators as follows.

$$\begin{aligned} b^J_\omega = (\phi, \phi^J_\omega) = \int_0^\infty d\omega' \int_0^\infty d\omega' \hat{a}^{in}_{\omega'}(\alpha^J_{\omega\omega'})^*(\phi^{in}_{\omega'}, \phi^{in}_{\omega'}) + \hat{a}^{in}_{\omega'}(\beta^J_{\omega\omega'})^*(\phi^{in}_{\omega'}, \phi^{in*}_{\omega'}) \\ + \hat{a}^{in\dagger}_{\omega'}(\alpha^J_{\omega\omega'})^*(\phi^{in*}_{\omega'}, \phi^{in}_{\omega'}) + \hat{a}^{in\dagger}_{\omega'}(\beta^J_{\omega\omega'})^*(\phi^{in*}_{\omega'}, \phi^{in*}_{\omega'}) \\ = \int_0^\infty d\omega' \int_0^\infty d\omega' \left[\hat{a}^{in}_{\omega'}(\alpha^J_{\omega\omega'})^*\delta(\omega - \omega') + \hat{a}^{in\dagger}_{\omega'}(\beta^J_{\omega\omega'})^*(-\delta(\omega - \omega'))\right] \\ \Rightarrow b^J_\omega = \int_0^\infty d\omega' \left[\hat{a}^{in}_{\omega'}(\alpha^J_{\omega\omega'})^* - \hat{a}^{in\dagger}_{\omega'}(\beta^J_{\omega\omega'})^*\right] \end{aligned} \tag{3.20}$$

$$\begin{aligned} (b^J_\omega)^\dagger = \int_0^\infty d\omega' \left[\hat{a}^{in\dagger}_{\omega'}((\alpha^J_{\omega\omega'})^*)^* - \hat{a}^{in\dagger\dagger}_{\omega'}((\beta^J_{\omega\omega'})^*)^*\right] \\ \Rightarrow (b^J_\omega)^\dagger = \int_0^\infty d\omega' \left[\hat{a}^{in\dagger}_{\omega'}\alpha^J_{\omega\omega'} - \hat{a}^{in}_{\omega'}\beta^J_{\omega\omega'}\right] \end{aligned} \tag{3.21}$$

If the field is in the "in" vacuum state specified by the positive frequency modes at ρ^-, the operator $N^J_\omega \equiv (b^J_\omega)^\dagger b^J_\omega$ gives the average number of particles with frequency ω

that reach ρ^+, so that the total number of particles that reach ρ^+ is (3.22).

$$\begin{aligned}\langle N^J\rangle_{0_{in}} &= \int_0^\infty d\omega \langle N^J_\omega\rangle_{0_{in}} \equiv \int_0^\infty d\omega \langle (b^J_\omega)^\dagger b^J_\omega\rangle_{0_{in}} \\ &= \int_0^\infty d\omega \int_0^\infty d\omega' \int_0^\infty d\omega' \alpha^J_{\omega\omega'}\alpha^{J*}_{\omega\omega'}\langle \hat{a}^{in\dagger}_{\omega'}\hat{a}^{in}_{\omega'}\rangle_{0_{in}} - \alpha^J_{\omega\omega'}\beta^J_{\omega\omega'}\langle \hat{a}^{in\dagger}_{\omega'}\hat{a}^{in\dagger}_{\omega'}\rangle_{0_{in}} \\ &- \beta^J_{\omega\omega'}\alpha^{J*}_{\omega\omega'}\langle \hat{a}^{in}_{\omega'}\hat{a}^{in}_{\omega'}\rangle_{0_{in}} + \beta^J_{\omega\omega'}\beta^{J*}_{\omega\omega'}\langle \hat{a}^{in}_{\omega'}\hat{a}^{in\dagger}_{\omega'}\rangle_{0_{in}} \\ &= \int_0^\infty d\omega \int_0^\infty d\omega' \int_0^\infty d\omega' \beta^{J*}_{\omega\omega'}\beta^J_{\omega\omega'}\delta(\omega-\omega') = \int_0^\infty d\omega \int_0^\infty d\omega' \beta^{J*}_{\omega\omega'}\beta^J_{\omega\omega'} \\ &\Rightarrow \langle N^J\rangle_{0_{in}} = \int_0^\infty d\omega \int_0^\infty d\omega' \mid \beta^J_{\omega\omega'} \mid^2\end{aligned} \tag{3.22}$$

Where we use the expressions (3.23) and (3.24) associated to the Cauchy surfaces ρ^- and ρ^+, respectively [8].

$$\beta_{\omega\omega'} = \frac{1}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{\infty} dv e^{i\omega' v}e^{i\omega f(v)}(\omega' - \omega f'(v)) \tag{3.23}$$

$$\beta_{\omega\omega'} = \frac{1}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{\infty} du e^{i\omega u}e^{i\omega' p(u)}(\omega' p'(u) - \omega) \tag{3.24}$$

3.1.1 Static Trajectory

Lets start with the simplest mirror trajectory (3.25), the static case can be thought to be trivial but it is very useful when considering more complicated trajectories, one would expect to recover the static case theory when collapsing an arbitrary trajectory into this particular case.

$$z(t) = 0 \tag{3.25}$$

Using (3.9) we obtain $t_m(u) = u$ and using (3.9) we obtain $\overline{t_m}(v) = v$, replacing this results into (3.13) and (3.15) respectively, we get the ray-tracing functions (3.26).

$$\begin{aligned} p(u) &= u \\ f(v) &= v \end{aligned} \tag{3.26}$$

To compute Bogolubov coefficient $\beta_{\omega\omega'}$ (3.27), we use the past Cauchy surface ρ^- associated to the equation (3.23).

$$\begin{aligned}\beta_{\omega\omega'} &= \frac{1}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{\infty} dv e^{-i(\omega'+\omega)v}(\omega'-\omega) = \frac{(\omega'-\omega)}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{\infty} dv e^{-i(\omega'+\omega)v} = \frac{(\omega'-\omega)}{2\sqrt{\omega\omega'}}\delta(\omega'+\omega) \\ &\Rightarrow \beta_{\omega\omega'} = 0\end{aligned} \tag{3.27}$$

As it was expected the number of ”out” particles present at the ”in” vacuum is null for a static mirror trajectory. We conclude that the presence of a static mirror does not imply particle production in vacuum.

3.1.2 Inertial Trajectories and Doppler Effect Argument for Particle Production

Lets consider the family of inertial trajectories for the moving mirror (3.28), where v_0 is a constant parameter of speed.

$$z(t) = -v_0 t \ , \ v_0 \ constant \tag{3.28}$$

In this particular case is easier to see the Doppler shift due to reflection of the moving mirror. We will use the relativistic Doppler effect equations. First consider an observer (O, ω_O) moving away from an emitter (E, ω_E), this will lead to a redshift (3.29). [?]

$$\omega_O = \sqrt{\frac{1-v_0}{1+v_0}}\omega_E \tag{3.29}$$

Second consider that the observer has a mirror that reflects all the redshift frequency back to the emitter, then the initial emitter will oberve a second redshift $((E, \omega_{E'}))$ given by (3.30). [?]

$$\begin{aligned} \omega_{E'} &= \sqrt{\frac{1-v_0}{1+v_0}}\omega_O = \sqrt{\frac{1-v_0}{1+v_0}}\sqrt{\frac{1-v_0}{1+v_0}}\omega_E \\ \Rightarrow \omega_{E'} &= \frac{1-v_0}{1+v_0}\omega_E = D\omega_E \ , \ with \ D \equiv \frac{1-v_0}{1+v_0} \end{aligned} \tag{3.30}$$

We can also calculate the ray-tracing functions following the same procedure as in previous cases, from (3.9) and (3.9) we obtain the expressions $t_m(u) = u/1+v_0$ and $\overline{t_m}(v) = v/1-v_0$, by replacing in (3.13) and (3.15) we get (3.31).

$$\begin{aligned} p(u) &= \frac{1-v_0}{1+v_0}u \\ f(v) &= \frac{1+v_0}{1-v_0}v \end{aligned} \tag{3.31}$$

To see the result for the number of particles we select the Cauchy surface ρ^- associated to the equation (3.23).

$$\beta_{\omega\omega'} = \frac{1}{4\pi\sqrt{\omega\omega'}} \int_{-\infty}^{\infty} dve^{-i(\omega+\omega' D^{-1})v} (\omega' D^{-1} - \omega) = \frac{(\omega' D^{-1} - \omega)}{4\pi\sqrt{\omega\omega'}} \int_{-\infty}^{\infty} dve^{-i(\omega+\omega' D^{-1})v}$$
$$= \frac{(\omega' D^{-1} - \omega)}{2\sqrt{\omega\omega'}} \delta(\omega + \omega' D^{-1}) \Rightarrow \beta_{\omega\omega'} = 0 \tag{3.32}$$

Then there is no particle creation for inertial trajectories.

3.1.3 Uniformly Accelerated Mirror Trajectory and Acceleration Free Parameter

We know that acceleration curves spacetime and in this case considering an acelerated mirror trajectory induces particle creation in the quantum vacuum. To understand better acceleration effects in a possible phase transition associated to the particle creation in vacuum we treat the simplest case for an accelerated trajectory (3.33).

$$z(t) = k^{-1} - \sqrt{k^{-2} + (-v_0 t)^2} \tag{3.33}$$

The trajectory considered has the advantages that it can be collapsed into the static case trajectory to check calculations and to the inertial trajectory where there is no particle creation (3.34). Moreover, it has a free parameter k that tunes acceleration, thus k is inversely proportional to acceleration and to curvature of spacetime, see Figure 3.2.

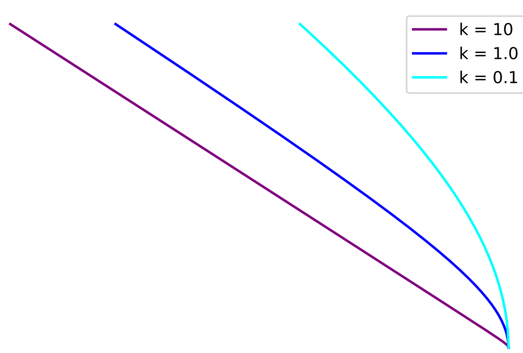

Figure 3.2: Variaton of the k parameter

$$\begin{aligned} &\textit{Static Case: } \lim_{k\to\infty} z(t)|_{t=0} = 0 \\ &\textit{Inertial Case: } \lim_{k\to\infty} z(t) = -v_0 t \end{aligned} \tag{3.34}$$

Selecting the Cauchy surface ρ^- we get for the Bogolubov coefficient (3.35), where K_1 is the modified Bessel function.

$$\beta_{\omega\omega'} = \frac{1}{k\pi} e^{i(\omega'-\omega)/k} K_1 \left(\frac{2\sqrt{\omega\omega'}}{k} \right) \tag{3.35}$$

We find that, as $K_1(2\sqrt{\omega\omega'}/k)$ is inversely proportional to $2\sqrt{\omega\omega'}/k$ ($K_1(2\sqrt{\omega\omega'}/k) \sim (2\sqrt{\omega\omega'}/k)^{-1}$) then $\beta_{\omega\omega'}$ is directly proportionl to k ($\beta_{\omega\omega'} \sim k$). That is, the number of particles in vacuum are proportional to the acceleration of the reflecting boundary.

3.1.4 The Causal Diamond and the Sorkin-Johnston state

This subsection is a particular case of the moving mirror on flat spacetime, we apply the formalism learned in the previous section to the causal diamond by modeling the flat spacetime with two mirror trajectories, each of them traveling from ρ^- to ρ^+ at the right and at the left (Figure 3.3).

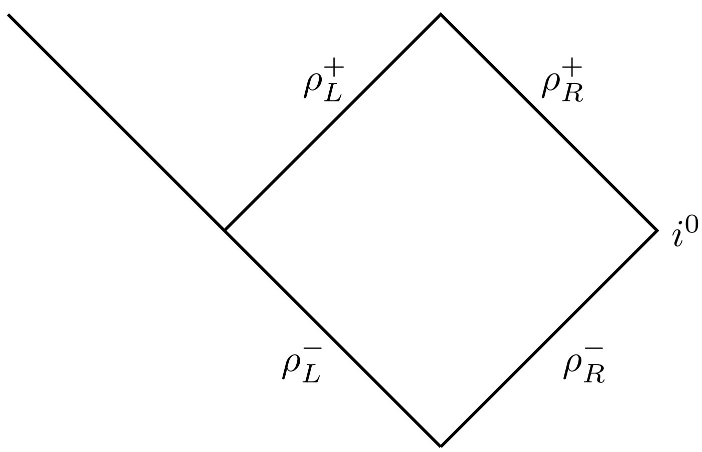

Figure 3.3: Causal Diamond modeled es flat $1+1-$dim spacetime with two mirror tajectories

Consider a Klein-Gordon scalar field ϕ in the $1+1-$dim causal diamond, defining the light cone coordinates by (3.1), in Minkowski spacetime we construct the casual diamond as the set of points (3.36),

$$\mathcal{M} = \{-l \leqslant u \leqslant l \ , \ -l \leqslant v \leqslant l\} \tag{3.36}$$

the field satifies the Klein-Gordon equation for a massles field (3.37) and by confining the field into the set $\mathcal{M}$ (the causal diamond) with the borders of the diamond being two reflecting boundaries or two moving mirror trajectories, we are considering Dirichlet boundary conditions of the form (3.38).

$$\Box\phi = 0 \tag{3.37}$$

$$\phi(u = \pm l, v) = \phi(u, v = \pm l) = 0 \tag{3.38}$$

We propose two sets of mode solutions (3.39) and (3.40). Defining the ray-tracing functions $p(u)$ and $f(v)$ in the interior $\mathcal{M}^0$ and the border of the set $\mathcal{M}$ such that the set of solutions satisfy the Dirichlet boundary conditions (3.38) using the same normalization constant as in the most general case of the moving mirror.

$$\phi^u_\omega = \frac{1}{\sqrt{4\pi\omega}}\left[e^{-i\omega p(u)} - e^{-i\omega v}\right] \ , \ p(u = \pm l) = v \ and \ p(u) = u \ \ \forall u \in \mathcal{M}^0 \tag{3.39}$$

$$\phi^v_\omega = \frac{1}{\sqrt{4\pi\omega}}\left[e^{-i\omega u} - e^{-i\omega f(v)}\right] \ , \ f(v = \pm l) = u \ and \ f(v) = v \ \ \forall v \in \mathcal{M}^0 \tag{3.40}$$

Physically the set of modes $\{\phi^u_\omega\}$ (3.39) can be thought as modes traveling to the right when $t < 0$ and to the left when $t > 0$, the set of modes $\{\phi^v_\omega\}$ (3.40) can be thought as modes traveling to the right when $t > 0$ and to the left when $t < 0$.

As before, both set of solutions are complete and orthonormal relative to the positive-definite scalar product (3.4). Then the field can be expanded in both set of modes (3.41).

$$\phi = \sum_{\omega>0}\left[\hat{a}^u_\omega\phi^u_\omega + \hat{a}^{u\dagger}_\omega\phi^{u*}_\omega\right] = \sum_{\omega>0}\left[\hat{a}^v_\omega\phi^v_\omega + \hat{a}^{v\dagger}_\omega\phi^{v*}_\omega\right] \tag{3.41}$$

where $\hat{a}^u_\omega, \hat{a}^v_\omega$, and $\hat{a}^{u\dagger}_\omega, \hat{a}^{v\dagger}_\omega$ the annihilation and creation operators for the sets od solutions $\{\phi^u_\omega\}$ (3.39) and $\{\phi^v_\omega\}$ (3.40), respectively. And both sets of modes have their associated vacuum states defined by (3.42).

$$\begin{aligned} &\hat{a}^u_\omega|0_u\rangle \quad \forall\omega \\ &\hat{a}^v_\omega|0_v\rangle \quad \forall\omega \end{aligned} \tag{3.42}$$

Now we can compute the Wightman function [2].This is the expected value of the two-point function of the set of $u-$solutions at the $u-$vacuum state (3.43). For this calculation we used the approximation $\sum_{n=1}^{\infty} e^{inx} \approx -\log(1-e^{ix})$ if $|x| \leqslant 1$ and the Taylor series expansion for the exponential function to first order $e^x \approx 1 + x$.

$$
\begin{aligned}
&W_{\mathcal{M}} \equiv \langle 0_u|\phi(u,v)\phi(u',v')|0_u\rangle \\
&= \frac{1}{4\pi\omega}\sum_{\omega>0}\sum_{\omega'>0}\left[\langle \hat{a}^u_\omega \hat{a}^u_{\omega'}\rangle_{0_u}\phi^u_\omega\phi^u_{\omega'} + \langle \hat{a}^u_\omega \hat{a}^{u\dagger}_{\omega'}\rangle_{0_u}\phi^u_\omega\phi^{u*}_{\omega'} + \langle \hat{a}^{u\dagger}_\omega \hat{a}^u_{\omega'}\rangle_{0_u}\phi^{u*}_\omega\phi^u_{\omega'} + \langle \hat{a}^{u\dagger}_\omega \hat{a}^{u\dagger}_{\omega'}\rangle_{0_u}\phi^{u*}_\omega\phi^{u*}_{\omega'}\right] \\
&= \frac{1}{4\pi\omega}\sum_{\omega>0}\sum_{\omega'>0}\delta_{\omega\omega'}\phi^u_\omega\phi^{u*}_{\omega'} = \frac{1}{4\pi}\sum_{\omega>0}\phi^u_\omega\phi^{u*}_\omega = \frac{1}{4\pi\omega}\sum_{\omega>0}\left(e^{-i\omega p(u)} - e^{-i\omega v'}\right)\left(e^{i\omega p(u')} - e^{i\omega v'}\right) \\
&= \frac{1}{4\pi}\left[\sum_{\omega>0}\frac{e^{-i\omega(p(u)-p(u'))}}{\omega} - \sum_{\omega>0}\frac{e^{-i\omega(p(u)-v')}}{\omega} - \sum_{\omega>0}\frac{e^{-i\omega(v-p(u'))}}{\omega} + \sum_{\omega>0}\frac{e^{-i\omega(v-v')}}{\omega}\right] \\
&= \frac{1}{4\pi}\left[-\log(1-e^{-i\omega(p(u)-p(u'))}) + \log(1-e^{-i\omega(p(u)-v')}) + \log(1-e^{-i\omega(v-p(u'))}) - \log(1-e^{-i\omega(v-v')})\right] \\
&= \frac{1}{4\pi}\left[-\log\left((1-e^{-i\omega(p(u)-p(u'))})(1-e^{-i\omega(v-v')})\right) + \log\left((1-e^{-i\omega(p(u)-v')})(1-e^{-i\omega(v-p(u'))})\right)\right] \\
&= \frac{1}{4\pi}\left[\frac{(1-e^{-i\omega(p(u)-v')})(1-e^{-i\omega(v-p(u'))})}{(1-e^{-i\omega(p(u)-p(u'))})(1-e^{-i\omega(v-v')})}\right] = \frac{1}{4\pi}\left[\frac{(p(u)-v')(v-p(u'))}{(p(u)-p(u'))(v-v')}\right]
\end{aligned}
\tag{3.43}
$$

Finally if we restrict to the interior of $\mathcal{M}$ we can simplify the ray-tracing functions using the construction (3.39). Then we have the Sorkin-Johnston state for the field in the causal diamond (3.44) and we verify the result (W^{SJ}_{box}) in [21] and [1] without the "correction term" ϵ.

$$
W^{SJ}_{\mathcal{M}} = \frac{1}{4\pi}\left[\frac{(u-v')(v-u')}{(u-u')(v-v')}\right] \tag{3.44}
$$

3.2 Moving Mirrors in a Two Dimensional Cavity

In this section we will study a particular case of the dynamical Casimir effect, where we have a one dimensional cavity bounded by two-mirrors, one mirror is static (at $x=0$) and the other (initially at $x=l$) oscillates during a period of time (T), Figure 3.2.

Let ϕ be a scalar field in $(1+1)-$dim that satsfies the Klein-Gordon equation (3.45) with Dirichlet boundary conditions (3.46), the field vanishes at the surface of the perfectly refletcing plates.

$$
\Box\phi = 0 \ , \ \Box \equiv -\partial_x^2 + \partial_t^2 \tag{3.45}
$$

$$
\phi(t,0) = \phi(t,L) = 0 \tag{3.46}
$$

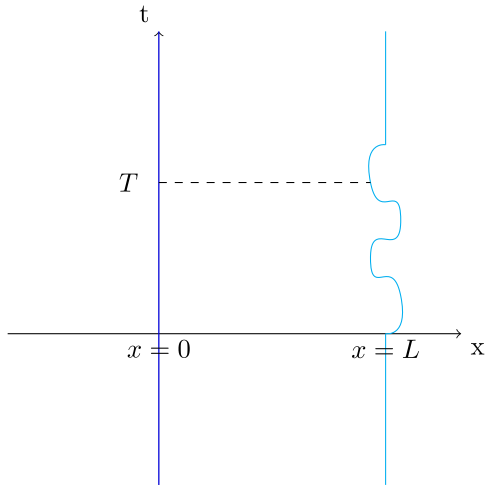

Figure 3.4: Moving Mirrors in Two-Dimensional Cavity

The mirror from the right moves arbitrarily for a finite period of time T and after returns to the initial position (3.47). Under this restriction in quantum theory we expect a finite particle production on vacuum [19].

$$L(t) = L \ , \ for \ t \leqslant 0 \ and \ t \geqslant 0 \tag{3.47}$$

Let V be the vector space of solutions to (3.45) with boundary conditions (3.46). V has an invariant bilinear form (3.48), where Σ is a Cauchy surface. This bilinear form is not positive definite, this means it has positive and negative eigenvalues.

$$(\phi, \psi) = i \int_{\Sigma} dx^2 \, [\phi^*(\partial_t \psi) - (\partial_t \phi^*)\psi] \tag{3.48}$$

We can proceed by decomposing the space of solutions into a direct sum of subspaces (3.49). The elements of V^+ are solutions that have positive squared norm, this means $(\phi, \phi) > 0$ and the complex conjugate of the positive-energy solutions are negative-energy solutions, $V^- = \overline{V^+}$. Notice that this decomposition is not unique, distinct decomposition will correspond to distinct choices of vacuum in the theory [19].

$$V = V^+ \oplus V^- \tag{3.49}$$

By construction we have that the bilinear form (3.48) is positive definite in V^+, then its restriction to this vector space turns to be an inner product. Therefore, we have the Hilbert space (3.50).

$$\mathcal{H} := (V^+, (,)) \tag{3.50}$$

Let $\{\phi_n(t, x)\}_n$ be an orthonormal basis of $\mathcal{H}$, where $\phi_n(t, x)$ are normal modes. For any choice of basis we can expand the field operator in terms of the normal modes

(3.51), where $\hat{a}_n$ and $\hat{a}_n^\dagger$ are the annihilation and creation operators, respectively and satisfy the usual commutation relations (3.52).

$$\hat{\phi}(t,x) = \sum_n \left[\phi_n(t,x)\hat{a}_n + \phi_n^*(t,x)\hat{a}_n^\dagger\right] \tag{3.51}$$

$$[\hat{a}_m, \hat{a}_n^\dagger]\delta_{mn} \ , \ [\hat{a}_m, \hat{a}_n] = [\hat{a}_m^\dagger, \hat{a}_n^\dagger] = 0 \tag{3.52}$$

The vacuum state $|0\rangle$ is defined by (3.53), it is important to notice that the vacuum definition depends on the vector space decomposition (choice of positive frequency modes $\mathcal{H} = V^+$) and is independent of the normal modes basis $\{\phi_n(t,x)\}_n$.

$$\hat{a}_n|0\rangle = 0 \quad \forall n \tag{3.53}$$

We need to construct the space of solutions to the equation (3.45) with boundary conditions (3.46). But first we will start with the simplest cases "*in*" ($t \leqslant 0$) and "*out*" ($t \geqslant T$) , that is before and after the mirror starts to move and which will give us two different representations of solutions: "*in*"-solutions and "*out*"-solutions, respectively. In this situations we can propose the following ansatz (3.54) ($e^{i(k_n x - \omega_n t)} = (\cos(k_n x) + i\sin(k_n x))\, e^{-i\omega_n t}$).

$$\begin{aligned}
&\textit{"in": } \phi_n^{in}(t,x) \sim (A_n \cos(k_n x) + B_n i \sin(k_n x))\, e^{-i\omega_n t} \ , \textit{ for } t \leqslant 0 \textit{ with } A_n, B_n \in \mathbb{C} \\
&\textit{"out": } \phi_k^{out}(t,x) \sim (A_k \cos(k_k x) + B_k i \sin(k_k x))\, e^{-i\omega_k t} \ , \textit{ for } t \geqslant 0 \textit{ with } A_k, B_k \in \mathbb{C}
\end{aligned} \tag{3.54}$$

Applying the boundary conditions (3.46) to the "*in*"-solutions ansatz, we get (3.55),

$$\begin{aligned}
&\phi_n^{in}(t,0) = 0 \Rightarrow A_n = 0 \\
&\phi_n^{in}(t,L) = 0 \Leftrightarrow \sin(k_n L) = 0 \Leftrightarrow k_n L = n\pi \ , \ n \in \mathbb{Z}
\end{aligned} \tag{3.55}$$

and to get an expression for ω_n we replace the $"in" - solutions$ ansatz into the wave equation (3.45) and use the expression for k_n in (3.55) to get (3.56).

$$\begin{aligned}
&\Box\phi_n^{in}(t,x) = (-\partial_x^2 + \partial_t^2)\sin(k_n x)e^{-i\omega_n t} = (k_n^2 - \omega_n^2)\phi_n(t,x) = 0 \\
&\Rightarrow \omega_n = k_n = \frac{n\pi}{L}
\end{aligned} \tag{3.56}$$

So we have the explicit expression for the "*in*"-solutions (3.57) where N is a normalization constant.

$$\phi_n^{in}(t,x) = N\sin(k_n x)e^{-i\omega_n t} \tag{3.57}$$

To find the normalization constant we apply (3.58).

$$\begin{aligned}
&\int_0^L dx|\phi_n^{in}(t,x)|^2 = N^2\int_0^L dx\sin^2\left(\frac{n\pi}{L}x\right) = 1\\
&\Rightarrow \frac{N^2}{2}\int_0^L dx\left[1-\cos\left(2\frac{n\pi}{L}x\right)\right] = \frac{N^2}{2}\left[x|_0^L - \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)|_0^L\right]\\
&= \frac{N^2}{2}\left[L-\frac{L}{n\pi}\left(\sin\left(2n\pi\right)-\sin\left(0\right)\right)\right] = 1\\
&\Rightarrow N = \sqrt{\frac{2}{L}}
\end{aligned} \tag{3.58}$$

Therefore, for the "*in*"-solutions ($t \leqslant 0$) we have the explicit expression (3.60),

$$\phi_n^{in}(t,x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)e^{-i\omega_n t}\ ,\ with\ \omega_n = \frac{n\pi}{L} \quad \forall n\in\mathbb{Z} \tag{3.59}$$

and analogously we have the explicit expression for the "*out*"-solutions ($t \geqslant 0$), expression (3.60).

$$\phi_k^{out}(t,x) = \sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)e^{-i\omega_k t}\ ,\ with\ \omega_n = \frac{k\pi}{L} \quad \forall k\in\mathbb{Z} \tag{3.60}$$

Now we have that the "*in*" and "*out*" -solution sets are orthonormal (3.61).

$$\int_0^L dx\phi_k^{in/out}(t,x)\phi_n^{in/out}(t,x) = \frac{2}{L}\int_0^L dx\sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) = \frac{2}{L}\begin{cases}L/2\ ,\ k=n\\ 0\ ,\ k\neq n\end{cases} = \delta_{kn} \tag{3.61}$$

In this way we have two different bases of the space of solutions V: $B_{in} = \{\phi_n^{in}, \phi_n^{in*}\}$ and $B_{out} = \{\phi_k^{out}, \phi_k^{out*}\}$. Therefore we have two different expansions for the field operator (3.62), where $\hat{a}_n^{in}, \hat{a}_n^{in\dagger}$ are annihilation and creation operators in the "*in*"-representation and $\hat{a}_n^{out}, \hat{a}_n^{out\dagger}$ are annihilation and creation operators in the "*out*"-representation, both satisfying (3.52).

$$\begin{aligned}
\hat{\phi}(t,x) &= \sum_n\left[\phi_n^{in}(t,x)\hat{a}_n^{in} + \phi_n^{in*}(t,x)\hat{a}_n^{in\dagger}\right]\\
&= \sum_k\left[\phi_n^{out}(t,x)\hat{a}_n^{out} + \phi_n^{out*}(t,x)\hat{a}_n^{out\dagger}\right]
\end{aligned} \tag{3.62}$$

Both bases define different decompositions into positive and negative energy subspaces [19], then they define two different vacuum states (**??**) and (3.64).

$$\hat{a}_n^{in}|0_{in}\rangle = 0 \quad \forall n \tag{3.63}$$

$$\hat{a}_k^{out}|0_{out}\rangle = 0 \quad \forall k \tag{3.64}$$

Understanding the "*in*" and "*out*"-solutions we can look for a solution while the mirror is moving ($t \in [0, T]$), in this situation the modes evolve in a non-trivial way so for each time t we can expand the modes in a basis of instantaneous Fourier modes satisfying the boundary conditions (3.46). Notice that we will need to fix each time t in the closed interval $[0, T]$ because now $L \equiv L(t)$, in other words fixing the time is the same as selecting a Cauchy surface and solving the problem as in the Figure .

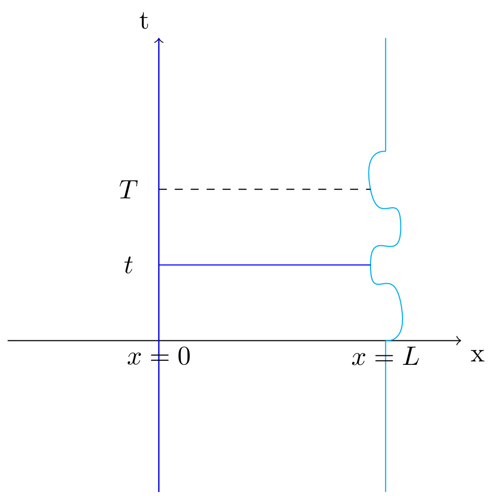

Figure 3.5: Moving Mirrors in Two-Dimensional Cavity with selected Cauchy Surface

Fix a time t, consider the Fourier mode basis $\{\phi_m(t, x)\}$ ("instantaneous" set of solutions) that satisfy the eigenvalue problem (3.65) with the boundary conditions (3.46),

$$-\partial_x^2 \phi_k(t, x) = \omega_k^2 \phi_k(t, x) \tag{3.65}$$

so we propose the following ansatz (3.66),

$$\phi_k(t, x) \sim e^{ik_k x} = A_k \cos(k_k x) + B_k i \sin(k_k x) \tag{3.66}$$

evaluating (3.66) in the boundary conditions (3.46) we get (3.67),

$$\begin{aligned} &\phi_k(t, 0) = 0 \Rightarrow A_k = 0 \\ &\phi_k(t, L(t)) = 0 \Leftrightarrow \sin(k_k L(t)) = 0 \Leftrightarrow k_k L(t) = k\pi \ , \ k \in \mathbb{Z} \end{aligned} \tag{3.67}$$

So we have the explicit expression for the "*instantaneous*" -solutions (3.68) where N is a normalization constant.

$$\phi_k(t, x) = N \sin(k_k L(t)) \tag{3.68}$$

To find the normalization constant we apply (3.69).

$$\begin{aligned}
&\int_0^{L(t)} dx|\phi_k(t,x)|^2 = N^2 \int_0^{L(t)} dx \sin^2\left(\frac{k\pi}{L(t)}x\right) = 1 \\
&\Rightarrow \frac{N^2}{2}\int_0^{L(t)} dx \left[1 - \cos\left(2\frac{k\pi}{L(t)}x\right)\right] = \frac{N^2}{2}\left[x|_0^{L(t)} - \frac{L(t)}{k\pi}\sin\left(\frac{k\pi}{L(t)}x\right)|_0^{L(t)}\right] \\
&= \frac{N^2}{2}\left[L(t) - \frac{L(t)}{k\pi}\left(\sin\left(2k\pi\right) - \sin\left(0\right)\right)\right] = 1 \\
&\Rightarrow N = \sqrt{\frac{2}{L(t)}}
\end{aligned} \tag{3.69}$$

Therefore, for the Fourier mode basis while the mirror is moving ($t \in [0, T]$) we have the explicit expression (3.70) [6],

$$\phi_k(t,x) = \sqrt{\frac{2}{L(t)}}\sin\left(\frac{k\pi}{L(t)}x\right) \ , \ \textit{with}\ \omega_k = \frac{k\pi}{L(t)} \quad \forall k \in \mathbb{Z} \tag{3.70}$$

We have that the "*instantaneous*" mode solutions form an orthonormal set (3.71) [13].

$$\begin{aligned}
&\int_0^{L(t)} dx\phi_k(t,x)\phi_n(t,x) = \frac{2}{L(t)}\int_0^{L(t)} dx \sin\left(\frac{k\pi}{L(t)}x\right)\sin\left(\frac{n\pi}{L(t)}x\right) \\
&= \frac{2}{L(t)}\begin{cases} L(t)/2 \ , \ k = n \\ 0 \ , \ k \neq n \end{cases} = \delta_{kn}
\end{aligned} \tag{3.71}$$

But (3.70) is not the whole solution for the time interval where the mirror is moving. By completeness of the Fourier mode basis we can expand the field $\phi(t,x)$ to get a solution of the form (3.72), with $C_{km}(t)$ time-dependent Fourier amplitudes [14], [6].

$$\phi_m^i(t,x) = \sum_k C_{km}(t)\phi_k(t,x) = \sum_k C_{km}(t)\sqrt{\frac{2}{L(t)}}\sin\left(\frac{k\pi}{L(t)}x\right) \tag{3.72}$$

Using the fact that the Fourier mode basis is orthonormal (3.71), one can obtain an expression for the Fourier amplitudes (3.73) [14].

$$\begin{aligned}
&\sqrt{\frac{2}{L(t)}}\int_0^{L(t)} dx \sin\left(\frac{n\pi}{L(t)}x\right)\phi(t,x) = \sum_k C_{km}(t)\frac{2}{L(t)}\int_0^{L(t)} dx \sin\left(\frac{n\pi}{L(t)}x\right)\sin\left(\frac{m\pi}{L(t)}x\right) \\
&= \sum_k C_{km}(t)\frac{2}{L(t)}\frac{L(t)}{2}\delta_{nk} = C_{km}(t) \\
&\Rightarrow C_{km}(t) = \sqrt{\frac{2}{L(t)}}\int_0^{L(t)} dx\phi(t,x)\sin\left(\frac{n\pi}{L(t)}x\right)
\end{aligned} \tag{3.73}$$

We are interested on describing completeley the Fourier amplitudes $C_{km}(t)$. To do so we will impose the initial conditions (3.74) and boundary conditions (3.75) to the Fourier amplitudes [6], taking into account that the first derivative of the soution has to be continous at the boundary.

$$\begin{aligned}
&\phi^{i}_{m}(0,x)=\phi^{in}_{m}(0,x)\\
&\Rightarrow \sum_k C_{km}(0)\sqrt{\frac{2}{L(0)}}\sin\left(\frac{k\pi}{L(0)}x\right)=\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right)e^{-i\omega_m(0)}\\
&\Rightarrow \sum_k C_{km}(0)\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)=\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right)\Leftrightarrow C_{km}(0)=\delta_{km}
\end{aligned}\tag{3.74}$$

$$\begin{aligned}
&\phi^{i'}_{m}(t,x)\quad\Big|_{t=0}=\phi^{in'}_{m}(t,x)\quad\Big|_{t=0}\\
&\Rightarrow \sum_k C^{'}_{km}(t)\sqrt{\frac{2}{L(t)}}\sin\left(\frac{k\pi}{L(t)}x\right)\quad\Bigg|_{t=0}=\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right)(-i\omega_m)e^{-i\omega_m t}\quad\Bigg|_{t=0}\\
&\Rightarrow \sum_k C^{'}_{km}(0)\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}x\right)=(-i\omega_m)\sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right)\\
&\Rightarrow C^{'}_{km}(0)=-i\omega_m\delta_{km}
\end{aligned}\tag{3.75}$$

In order to obtain the equation of motion for the Fourier amplitudes (coefficients C_{km}) we start by substituing the general solution (3.72) on the wave equation (3.45) and use the eigenvalue equation (3.65) to obtain (3.76),

$$\begin{aligned}
&\Box\phi^{i}_{m}(t,x)=(-\partial_x^2-\partial_t^2)\sum_k C_{km}(t)\phi_k(t,x)=0\\
&\Rightarrow -\sum_k C_{km}(t)\partial_x^2\left(\phi_k(t,x)\right)-\sum_k\partial_t^2\left(C_{km}(t)\phi_k(t,x)\right)\\
&=\sum_k C_{km}(t)\left(\omega_k^2\phi_k(t,x)\right)-\sum_k\left[\partial_t^2\left(C_{km}(t)\right)\phi_k(t,x)+2\partial_t\left(C_{km}(t)\right)\partial_t\left(\phi_k(t,x)\right)+C_{km}(t)\partial_t^2\left(\phi_k(t,x)\right)\right]\\
&=\sum_k\left[\omega_k^2C_{km}\phi_k-C^{''}_{km}\phi_k-2C^{'}_{km}\partial_t\left(\phi_k\right)-C_{km}\partial_t^2\left(\phi_k\right)\right]=0\\
&\Rightarrow \sum_k\left[-C^{''}_{km}+\omega_k^2C_{km}\right]\phi_k=\sum_k\left[2C^{'}_{km}\partial_t\left(\phi_k\right)+C_{km}\partial_t^2\left(\phi_k\right)\right]
\end{aligned}\tag{3.76}$$

we will use the orthonormality (3.71) of the space of solutions (3.72), so we proceed by multiplying a field term on both sides and integrating over the whole Cauchy surface of the equality (3.76). We made use of the notation $\partial_t L \equiv L'$ and similarly for higher order time-derivatives of L, the chain rule and the second derivative of a composite function to obtain (3.77),

$$
\begin{aligned}
&\Rightarrow \int_0^{L(t)} dx \sum_k \left[-C''_{km} + \omega_k^2 C_{km}\right] \phi_k \phi_j = \int_0^{L(t)} dx \sum_k \left[2C'_{km}\partial_t\left(\phi_k\right) + C_{km}\partial_t^2\left(\phi_k\right)\right]\phi_j \\
&\Rightarrow \sum_k \left[-C''_{km} + \omega_k^2 C_{km}\right]\delta_{kj} = \int_0^{L(t)} dx \sum_j \left[2C'_{jm}\phi_j\partial_t\left(\phi_k\right) + C_{jm}\phi_j\partial_t^2\left(\phi_k\right)\right] \\
&\Rightarrow -C''_{km} + \omega_k^2 C_{km} = \sum_j \int_0^{L(t)} dx 2C'_{jm}\phi_j \left(\frac{\partial \phi_k}{\partial t}\right) + \sum_j \int_0^{L(t)} dx C_{jm}\phi_j \left(\frac{\partial^2 \phi_k}{\partial t^2}\right) \\
&= 2\sum_j \int_0^{L(t)} dx\phi_j \left(\frac{\partial L}{\partial t}\frac{\partial \phi_k}{\partial L}\right) C'_{jm} + \sum_j \int_0^{L(t)} dx\phi_j \left(\frac{\partial^2 L}{\partial t^2}\frac{\partial \phi_k}{\partial L} + \left(\frac{\partial L}{\partial t}\right)^2 \frac{\partial^2 \phi_k}{\partial L^2}\right) C_{jm} \\
&= 2\sum_j \int_0^{L(t)} dx\phi_j \left(L'\frac{\partial \phi_k}{\partial L}\right) C'_{jm} + \sum_j \int_0^{L(t)} dx\phi_j \left(L''\frac{\partial \phi_k}{\partial L} + L'^2\frac{\partial^2 \phi_k}{\partial L^2}\right) C_{jm} \\
&= 2L'\sum_j \int_0^{L(t)} dx\phi_j \frac{\partial \phi_k}{\partial L} C'_{jm} + L''\sum_j \int_0^{L(t)} dx\phi_j \frac{\partial \phi_k}{\partial L} C_{jm} + L'^2 \sum_j \int_0^{L(t)} dx\phi_j \frac{\partial^2 \phi_k}{\partial L^2} C_{jm} \\
&= 2\frac{L'}{L}\sum_j \left(q\int_0^{L(t)} dx\phi_j \frac{\partial \phi_k}{\partial L}\right) C'_{jm} + \frac{L''}{L}\sum_j \left(L\int_0^{L(t)} dx\phi_j \frac{\partial \phi_k}{\partial L}\right) C_{jm} + \frac{L'^2}{L^2}\sum_j \left(L^2\int_0^{L(t)} dx\phi_j \frac{\partial^2 \phi_k}{\partial L^2}\right) C_{jm}
\end{aligned}
\tag{3.77}
$$

at this stage it is useful to define the coefficient (3.78) and as a consequence get the relation (3.79),

$$
g_{kj} := L\int_0^{L(t)} dx\phi_j \frac{\partial \phi_k}{\partial L} \tag{3.78}
$$

$$
\begin{aligned}
&\sum_k g_{jk}g_{lk} := \sum_k \left(L\int_0^{L(t)} dx\phi_k \frac{\partial \phi_j}{\partial L}\right)\left(L\int_0^{L(t)} dx\phi_k \frac{\partial \phi_l}{\partial L}\right) \\
&= \sum_k L^2 \left(\int_0^{L(t)} dx \frac{\partial \phi_j}{\partial L}\frac{\partial \phi_l}{\partial L}\right)\left(\int_0^{L(t)} dx \int_0^{L(t)} dx\phi_k\phi_k\right) = \sum_k L^2 \left(\int_0^{L(t)} dx \frac{\partial \phi_j}{\partial L}\frac{\partial \phi_l}{\partial L}\right)\delta_{kk} \\
&\Rightarrow \sum_k g_{jk}g_{lk} = L^2 \int_0^{L(t)} dx \frac{\partial \phi_j}{\partial L}\frac{\partial \phi_l}{\partial L}
\end{aligned}
\tag{3.79}
$$

where the g_{kj} coefficients are dimensionless elements of a matrix gien by (3.80),

$$\begin{aligned} &If\ k=j \Rightarrow g_{kk} = L\int_0^{L(t)} dx\phi_k \frac{\partial \phi_k}{\partial L} = L\int_0^{L(t)} dx \frac{\partial \phi_k^2}{\partial L} = L\partial_L \int_0^{L(t)} dx \partial \phi_k^2 = L\partial_L \partial \delta_{kk} = 0 \\ &If\ k \neq j \Rightarrow g_{kj} = L\int_0^{L(t)} dx\phi_j \frac{\partial \phi_k}{\partial L} \end{aligned} \tag{3.80}$$

then we have (3.81),

$$g_{kj} = \begin{cases} 0\ ,\ k=j \\ (-1)^{k+j}\dfrac{2kj}{j^2-k^2}\ ,\ k\neq j \end{cases} \tag{3.81}$$

until now we can replace the first two terms of the last line of (3.77) by the matrix coefficients, it only remains the last one that could be very tricky. Using the fact that for fixed kj, g_{kj} is a constant coefficient we have that (3.82)

$$\begin{aligned} &\partial_L(g_{kj}) = \partial_L\left(L\int_0^{L(t)} dx\phi_j\partial_L\phi_k\right) = \partial_L\left(L\right)\left(\int_0^{L(t)} dx\phi_j\partial_L\phi_k\right) + L\partial_L\left(\int_0^{L(t)} dx\phi_j\partial_L\phi_k\right) \\ &= \int_0^{L(t)} dx\phi_j\partial_L\phi_k + L\int_0^{L(t)} dx\partial_L\left(\phi_j\partial_L\phi_k\right) = \int_0^{L(t)} dx\phi_j\partial_L\phi_k + L\int_0^{L(t)} dx\left(\partial_L\phi_j\partial_L\phi_k + \phi_j\partial_L^2\phi_k\right) \\ &\Rightarrow 0 = \int_0^{L(t)} dx\phi_j\partial_L\phi_k + L\int_0^{L(t)} dx\partial_L\phi_j\partial_L\phi_k + L\int_0^{L(t)} dx\phi_j\partial_L^2\phi_k \\ &\Rightarrow L\int_0^{L(t)} dx\phi_j\frac{\partial^2\phi_k}{\partial_L^2} = -\int_0^{L(t)} dx\phi_j\frac{\partial\phi_k}{\partial_L} - L\int_0^{L(t)} dx\frac{\partial\phi_j}{\partial_L}\frac{\partial\phi_k}{\partial_L} \\ &\Rightarrow L^2\int_0^{L(t)} dx\phi_j\frac{\partial^2\phi_k}{\partial_L^2} = -L\int_0^{L(t)} dx\phi_j\frac{\partial\phi_k}{\partial_L} - L^2\int_0^{L(t)} dx\frac{\partial\phi_j}{\partial_L}\frac{\partial\phi_k}{\partial_L} \end{aligned} \tag{3.82}$$

replacing the result (3.82) on our calculations for the equation of motion (3.77) we have the expression (3.83),

$$\begin{aligned} &= 2\frac{L'}{L}\sum_j\left(q\int_0^{L(t)} dx\phi_j\frac{\partial\phi_k}{\partial L}\right)C'_{jm} + \frac{L''}{L}\sum_j\left(L\int_0^{L(t)} dx\phi_j\frac{\partial\phi_k}{\partial L}\right)C_{jm} \\ &- \frac{L'^2}{L^2}\sum_j\left(L\int_0^{L(t)} dx\phi_j\frac{\partial\phi_k}{\partial_L}\right)C_{jm} - \frac{L'^2}{L^2}\sum_j\left(L^2\int_0^{L(t)} dx\frac{\partial\phi_j}{\partial_L}\frac{\partial\phi_k}{\partial_L}\right)C_{jm} \end{aligned} \tag{3.83}$$

but we still dont have an expression in terms of matrix coefficients for the last sumand at the last line in (3.82). We can find it as a particular case of (3.79). Notice the matrix elements g_{kj} are anti-symmetric ($g_{kj} = g_{jk}$), then we have (3.84),

$$
\begin{aligned}
&-\sum_j L^2 \int_0^{L(t)} dx \frac{\partial \phi_j}{\partial_L} \frac{\partial \phi_k}{\partial_L} C_{jm} = -\sum_{j,l} L^2 \int_0^{L(t)} dx \frac{\partial \phi_j}{\partial_L} \frac{\partial \phi_k}{\partial_L} \delta_{jl} C_{lm} = -\sum_{j,l} L^2 \int_0^{L(t)} dx \frac{\partial \phi_j}{\partial_L} \frac{\partial \phi_k}{\partial_L} \int_0^{L(t)} dx \phi_j \phi_l C \\
&= -\sum_{j,l} \left(L \int_0^{L(t)} dx \phi_j \frac{\partial \phi_k}{\partial_L} \right) \left(L \int_0^{L(t)} dx \phi_l \frac{\partial \phi_j}{\partial_L} \right) C_{lm} = \sum_{j,l} (-g_{kj}) g_{jl} C_{lm} = \sum_{j,l} g_{jk} g_{jl} C_{lm}
\end{aligned}
\tag{3.84}
$$

finally, using the definition (3.78) and the particular case relationship (3.84) we get the equation of motion for the Fourier amplitude coefficients (3.85).

$$
\begin{aligned}
&- C''_{km} + \omega_k^2 C_{km} = \\
&= 2\frac{L'}{L} \sum_j \left(q \int_0^{L(t)} dx \phi_j \frac{\partial \phi_k}{\partial L} \right) C'_{jm} + \frac{L''}{L} \sum_j \left(L \int_0^{L(t)} dx \phi_j \frac{\partial \phi_k}{\partial L} \right) C_{jm} \\
&- \frac{L'^2}{L^2} \sum_j \left(L \int_0^{L(t)} dx \phi_j \frac{\partial \phi_k}{\partial_L} \right) C_{jm} - \frac{L'^2}{L^2} \sum_j \left(L^2 \int_0^{L(t)} dx \frac{\partial \phi_j}{\partial_L} \frac{\partial \phi_k}{\partial_L} \right) C_{jm} \\
&= 2\frac{L'}{L} \sum_j g_{kj} C'_{jm} + \frac{L'' L}{L^2} \sum_j g_{kj} C_{jm} - \frac{L'^2}{L^2} \sum_j g_{kj} C_{jm} + \frac{L'^2}{L^2} \sum_{j,l} g_{jk} g_{jl} C_{lm} \\
&\Rightarrow -C''_{km} + \omega_k^2 C_{km} = 2\frac{L'}{L} \sum_j g_{kj} C'_{jm} + \frac{L'' L - L'^2}{L^2} \sum_j g_{kj} C_{jm} + \frac{L'^2}{L^2} \sum_{j,l} g_{jk} g_{jl} C_{lm}
\end{aligned}
\tag{3.85}
$$

The equation of motion for the Fourier amplitudes (3.85) determines completely the dynamics of the coefficients as a function of time. There is a wide class of forms for the coefficients C_{km} that satisfy (3.85) and one can simplify the physical situation to get different solutions. For example, if the moving mirror moves during a period T and returns to its initial position then the right-hand side of (3.85) is going to be zero and one gets the solution $C_{km}(t) = A_{km} e^{-i\omega_k t} + B_{km} e^{i\omega_k t}$, this situation is just the simplest case [6], is an example that we are not particularly interested on.

The method just stuied solves generally the the Klein-Gordon equation with dynamical boundary conditions for a fixed time Cauchy surface, it is very useful to use when we have dynamical boundaries. We use the fact that spacetime has a foliation structure such that for each fixed time we have a solution. We first saw the use of this method on [19], to see more details about the method we studied the articles [14], [13] and [6].

3.3 Rindler Spacetime

Considering the vacuum state in the inertial frame of a Rindler observer in Minkowski spacetime with a trajectory described by hyperbolic motion in the presence of horizons. The number of particles in vacuum can be computed with the formalism used in the previous cases, this results in the Unruh effect. Then, uniform acceleration of the observer results in a thermal bath radiation usually known as Hawking radiation with its Hawking temperature associated.

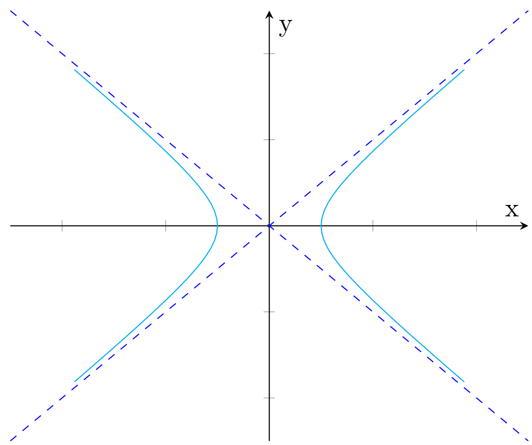

Figure 3.6: Rindler Spacetime

Consider Minkowski dim$1+1$ flat spacetime with line element (3.86), if we locate a Rindler observer, we now have cartesian (x,t) and Rindler (η,ζ) coordinates as seen in Figure 3.3, where ζ constant denotes the Rindler observers trajectory.

$$ds^2 = d\bar{u}d\bar{v} = dt^2 - dx^2 \tag{3.86}$$

$$\begin{cases} t = \dfrac{e^{a\zeta}}{a}\sinh a\eta \\ x = \dfrac{e^{a\zeta}}{a}\cosh a\eta \end{cases} \Leftrightarrow \begin{cases} \bar{u} = -\dfrac{1}{ae^{av}} \\ \bar{v} = -\dfrac{e^{av}}{a} \end{cases} \quad ,\ a\ constant\ ,\ -\infty < \eta,\zeta < \infty\ and\ u = \eta - \zeta, v = \eta + \zeta \tag{3.87}$$

Taking the coordinate transformations (3.87) we get a new expression for the line element (3.88).

$$ds^2 = d\bar{u}d\bar{v} = e^{2a\zeta}dudv = e^{2a\zeta}(d\eta - d\zeta)(d\eta + d\zeta) = e^{2a\zeta}(d\eta^2 - d\zeta^2) \tag{3.88}$$

According to the Penrose diagram, the null rays (u,v) act as a event horizon, the regions L and R represent casualy disjoint universes tht can only be joint by a space-like trajectory, the regions F and P can be joint to L and R by null rays. It is important to take into account that the regions labeled R, L, F, P have to be covered by separate sets of solutions.

Consider a massles scalar field in dim$1+1$ flat spacetime such that it satisfies the Klein-Gordon equation (3.89).

$$\Box\phi \equiv (\partial_t^2 - \partial_x^2)\phi = \partial_{\bar{u}}\partial_{\bar{v}}\phi = 0 \tag{3.89}$$

The field equation (3.89) has an orthonormal set of mode solutions (3.90), by construction this set of modes are of positive frequency respect to the time killing vector as we wanted.

$$\phi_k = \frac{1}{\sqrt{4\pi\omega}} e^{i(kx-\omega t)} \tag{3.90}$$

This solution can be interpreted in the sense that the modes with $|k|>0$ are right-moving waves along ($\bar{u}$ constant) $(4\pi\omega)^{-1/2}e^{-i\omega\bar{u}}$ and modes with $|k|<0$ are left-moving waves ($\bar{v}$ constant) $(4\pi\omega)^{-1/2}e^{-i\omega\bar{v}}$.

As the set of solutions (3.90) is orthonormal and complete we can expand the field as (3.91), where $\hat{a_k}^\dagger$ and $\hat{a_k}$ are the creation and annihilation operators satisfying the canonical commutation relations (2.46), and the vacuum is defined as usual $\hat{a_k}|0_M\rangle = 0$ $\forall k$.

$$\phi_a = \sum_k [\hat{a_k}\phi_k + \hat{a_k}^\dagger \phi_k^*] \tag{3.91}$$

It is interesting to think about this phenomena from another point of view, the Rindler line element (3.88) is conformal to the Minkowski spacetime by the transformation $g_{\mu\nu} \mapsto e^{-2a\zeta}g_{\mu\nu} = \bar{g_{\mu\nu}}$, consequently we have the expression (3.92) for the line element under the conformal transformation used.

$$ds^2 = e^{2a\zeta}(d\eta^2 - d\zeta^2) \mapsto d\bar{s}^2 = d\eta^2 - d\zeta^2 \ , \ with \ -\infty < \eta\zeta < \infty \tag{3.92}$$

And as the wave equation is conformally invariant, we an write it in the Rindler basis (3.93).

$$e^{2a\zeta}\Box\phi \equiv (\partial_\eta^2 - \partial_\zeta^2)\phi = \partial_{\bar{u}}\partial_{\bar{v}}\phi = 0 \tag{3.93}$$

The last equation has positive frequency mode solutions respect to the time-like killing vectors ∂_η and ∂_ζ (3.94) , where $\omega \equiv |k| > 0$ and $-\infty < k < \infty$. The $+$ sign in the argument of the exponential is related to the solutions in the region R amd the $-$ sign is related to solutions in the region L.

$$\phi_k = \frac{1}{\sqrt{4\pi\omega}} e^{i(k\zeta \pm \omega\eta)} \tag{3.94}$$

As we need a complete set of orthonormal solutions that cover the whole of the region R and L we use (3.95), such that $\{\phi_k^R, \phi_k^{R*}\}$ complete set of solutions in the region R and $\{\phi_k^L, \phi_k^{L*}\}$ is a complete set of solutions in the region L.

$$\phi_k^R = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{i(k\zeta+\omega\eta)} \ in \ R \\ 0 \ in \ L \end{cases} \quad , \ \phi_k^L = \begin{cases} 0 \ in \ R \\ \frac{1}{\sqrt{4\pi\omega}} e^{i(k\zeta-\omega\eta)} \ in \ L \end{cases} \tag{3.95}$$

The last idea leads us to construct the set of solutions $\{\phi_k^R, \phi_k^{R*}, \phi_k^L, \phi_k^{L*}\}$ that is complete on the whole of Minkowski spacetime and is orthonormal respect to the scalar product (3.4), they satisfy the canonical commutation relations (3.96).

$$(\phi_i^J, \phi_j^J) = \delta_{i,j} = -(\phi_i^{J*}, \phi_j^{J*}) \ , \ (\phi_i^J, \phi_j^{J*}) = 0 \ , \ with\ fixed\ J \in \{R, L\} \tag{3.96}$$

Following the same procedure as we applied before we can expand the field in this basis (3.97), where $\hat{b_k}^J$ and $\hat{b_k}^{J\dagger}$ are the annihilation and creation operators that satisfy the canonical commutation relations (2.46) with $J \in \{R, L\}$.

We define the Rindler vacuum ($|0_R\rangle$) such that $\hat{b_k}^R|0_R\rangle = \hat{b_k}^L|0_R = 0 \ \forall k$. It is important to note that this set of mode solutions associated to the Rindler vacuum $|0_R\rangle$ mix positive and negative frequncy mode solutions, this can be seen by its non-analyticity given the sign interchange in the exponential of (3.94) at $\bar{u} = 0$ and $\bar{v} = 0$.

$$\phi_b = \sum_k [\hat{b_k}^L \phi_k^L + \hat{b_k}^{L\dagger} \phi_k^{L*} + \hat{b_k}^R \phi_k^R + \hat{b_k}^{R\dagger} \phi_k^{R*}] \tag{3.97}$$

Taking into account the later arguments and the fact that the solution (3.91) is analytical on $\bar{u} = 0$ and $\bar{v} = 0$, it is claer that the vacuum associated to Minkowski space ($|0_M\rangle$) is not equivalent to the one associated to the Rindler space ($|0_R\rangle$). As we have now two field expansions in terms of Minkowski and Rindler modes with its associated vacuums we can find a relationship between both vacuums and give an answer to the question whether Rindler particles are present in $|0_M\rangle$ or not. As it is not surprise we will use Bogolubov transformations between both complete sets of mode solutions ((3.90) and (3.95)).

For this we can use the method used by Unruh, we have ϕ_k^R and ϕ_k^L that aro non-analytic but the combinations (3.99) and (3.98) are analytic on the complex plane, by construction they are of positive frequency and they share the same vacuum state which is $|0_M\rangle$.

$$\phi_k^R + e^{-\pi\omega/a} \phi_{-k}^L \tag{3.98}$$

$$\phi_{-k}^{R*} + e^{\pi\omega/aL} \phi_k^L \tag{3.99}$$

The linear combinations (3.98) and (3.98) form an orthonormal and complete set of mode solutions, so we can expand the field in terms of this set (3.100) (h.c denotes $(...)^\dagger$), where $\hat{d}_k^{(1)\dagger}, \hat{d}_k^{(2)\dagger}, \hat{d}_k^{(1)}, \hat{d}_k^{(2)}$ are the creation and annihilation operators that satisfy the canonical commutation relations (2.46) and the vacuum is defined as usual by $\hat{d}_k^{(1)}|0_M\rangle = \hat{d}_k^{(2)}|0_M\rangle = 0 \ \forall k$.

$$\begin{aligned}
\phi_d &= \sum_k \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(1)} \left(e^{\pi\omega/2a}\phi_k^R + e^{-\pi\omega/2a}\phi_{-k}^{L*} \right) + \hat{d}_k^{(2)} \left(e^{-\pi\omega/2a}\phi_{-k}^{R*} + e^{\pi\omega/2a}\phi_k^L \right) + \ h.c \ \right] \\
&= \sum_k \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(1)} \left(e^{\pi\omega/2a}\phi_k^R + e^{-\pi\omega/2a}\phi_{-k}^{L*} \right) + \hat{d}_k^{(2)} \left(e^{-\pi\omega/2a}\phi_{-k}^{R*} + e^{\pi\omega/2a}\phi_k^L \right) \right] \\
&+ \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(1)\dagger} \left(e^{\pi\omega/2a}\phi_k^{R*} + e^{-\pi\omega/2a}\phi_{-k}^{L} \right) + \hat{d}_k^{(2)\dagger} \left(e^{-\pi\omega/2a}\phi_{-k}^{R} + e^{\pi\omega/2a}\phi_k^{L*} \right) \right]
\end{aligned} \tag{3.100}$$

All the previous work results in two different field expansions; ϕ_d in terms of Rindler modes on the linear combinations, this basis has solutions of positive frequency so its associated vacuum is $|0_M\rangle$ and the expansion ϕ_b on the canonical basis of the Rindler solutions space with the associated vacuum $|0_R\rangle$. So that we can get analytical expressions relating both vacuums ((3.101) and (3.102)).

$$\begin{aligned}
&(\phi_b, \phi_k^L) = (\phi_d, \phi_k^L) \\
&\Rightarrow \sum_k \hat{b_k}^{(1)} (\phi_k^L, \phi_k^L) = \sum_k \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(2)} e^{\pi\omega/2a} (\phi_k^L, \phi_k^L) + \hat{d}_k^{(1)\dagger} e^{-\pi\omega/2a} (\phi_{-k}^L, \phi_k^L) \right] \\
&\Rightarrow \sum_k \hat{b_k}^{(1)} \delta_{k,k} = \sum_k \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(2)} e^{\pi\omega/2a} \delta_{k,k} + \hat{d}_k^{(1)\dagger} e^{-\pi\omega/2a} \delta_{-k,k} \right] \\
&\Rightarrow \hat{b_k}^{(1)} = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(2)} e^{\pi\omega/2a} + \hat{d_{-k}}^{(1)\dagger} e^{-\pi\omega/2a} \right]
\end{aligned} \tag{3.101}$$

$$\begin{aligned}
&(\phi_b, \phi_k^R) = (\phi_d, \phi_k^R) \\
&\Rightarrow \sum_k \hat{b_k}^{(2)} (\phi_k^R, \phi_k^R) = \sum_k \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(1)} e^{\pi\omega/2a} (\phi_k^R, \phi_k^R) + \hat{d}_k^{(2)\dagger} e^{-\pi\omega/2a} (\phi_{-k}^R, \phi_k^R) \right] \\
&\Rightarrow \sum_k \hat{b_k}^{(2)} \delta_{k,k} = \sum_k \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(1)} e^{\pi\omega/2a} \delta_{k,k} + \hat{d}_k^{(2)\dagger} e^{-\pi\omega/2a} \delta_{-k,k} \right] \\
&\Rightarrow \hat{b_k}^{(2)} = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \left[\hat{d}_k^{(1)} e^{\pi\omega/2a} + \hat{d_{-k}}^{(2)\dagger} e^{-\pi\omega/2a} \right]
\end{aligned} \tag{3.102}$$

With this two expressions we can calculate the number of Rindler particles in the vacuum $|0_M\rangle$, following the idea of considering an Rindler observer with acceleration ζ constant and with its assoiated vacuum $|0_R\rangle$. According to the field expansion (3.97), this observer perceives $\hat{N} \equiv \hat{b_k}^{(1)\dagger}\hat{b_k}^{(1)}$ particles in L and $\hat{N} \equiv \hat{b_k}^{(2)\dagger}\hat{b_k}^{(2)}$ particles in R. Then, if the field is in the state $|0_M\rangle$ the Rindler observer will see $\hat{N}_{|0_M\rangle} \equiv \langle\hat{b_k}^{(1,2)\dagger}\hat{b_k}^{(1,2)}\rangle_{|0_M\rangle}$ Rindler particles in the mode k present at Minkowski vacuum (3.103).

$$
\begin{aligned}
\hat{N}_{|0_M\rangle} &\equiv \left\langle \hat{b_k}^{(2)\dagger}\hat{b_k}^{(2)} \right\rangle_{|0_M\rangle} \\
&= \left(\frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\right)^2 \left\langle \left[\hat{d_k}^{(1)}e^{\pi\omega/2a} + \hat{d_{-k}}^{(2)\dagger}e^{-\pi\omega/2a}\right]^\dagger \left[\hat{d_k}^{(1)}e^{\pi\omega/2a} + \hat{d_{-k}}^{(2)\dagger}e^{-\pi\omega/2a}\right] \right\rangle_{|0_M\rangle} \\
&= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left\langle \hat{d_k}^{(1)\dagger}\hat{d_k}^{(1)} \right\rangle_{|0_M\rangle} e^{\pi\omega/2a} + \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left\langle \hat{d_k}^{(1)\dagger}\hat{d_{-k}}^{(2)\dagger} \right\rangle_{|0_M\rangle} \\
&+ \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left\langle \hat{d_{-k}}^{(2)}\hat{d_k}^{(1)} \right\rangle_{|0_M\rangle} + \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left\langle \hat{d_{-k}}^{(2)}\hat{d_k}^{(2)\dagger} \right\rangle_{|0_M\rangle} e^{-\pi\omega/a} \\
&\Rightarrow \hat{N}_{|0_M\rangle} = \frac{e^{-\pi\omega/a}}{\sqrt{2\sinh(\pi\omega/a)}} = \frac{1}{e^{2\pi\omega/a} - 1}
\end{aligned} \tag{3.103}
$$

This proves that the Rindler accelerated observer perceives particles in the Minkowski vacuum. In other words, the particle bath is obtained by the application of a conformal transformation from an observer in Minkowski spacetime into an acelerated observer in Rindler spacetime.

3.4 Collapsing null shell

In this section we study the particle production in vacuum for a null shell collapsing to form a black hole in $(1+1)-$ dimnesional curved spacetime and prove the equivalence with a perfectly reflecting accelerating boundary trajectory in $(1+1)-$ dimnesional flat spacetime, via calculating the same Bogolubov coefficients that connect the "in" and "out" states for a massless minimally coupled scalar field. [9]

Consider a massless minimally coupled scalar field ϕ that satisfies the Klein-Gordon equation, in this case this is just the wave equation $\Box\phi = 0$.

The line element inside the shell is given by (3.104) and outside the shell and the horizon the line element is given by (3.105), where the null coordinates are defined as usual (3.1).

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_s^2 + \left(1 - \frac{2M}{r}\right) dr^2 \tag{3.104}$$

$$\begin{aligned} u_s &\equiv t_s - r_* \\ v &\equiv t_s + r_* \\ r_* &\equiv r + 2M \log\left(\frac{r - 2M}{2M}\right) \end{aligned} \tag{3.105}$$

Given the coordinates systems associated to flat and Schwarchild spacetime we will proceed as it is usually done in General Relativity. We need to match both coordinate systems across the null shell, this means the coordiantes v and r are continous across the surface $v = v_0$, but the coordinates t and u are not continous across the surface.

$$r = \frac{1}{2}(v_0 - u) \tag{3.106}$$

Now we are looking for a relation between the coordinates u and u_s and its inverse relation. At the surface and outside the event horizon we have $r = 1/2(v_0 - u_s)$ given (3.106) and it is imposed the condition $r = r_*$ to get the relationship (3.107).

$$r_* = \frac{1}{2}(v_0 - u_s) = r + 2M \log\left(\frac{r - 2M}{2M}\right) \tag{3.107}$$

According to the las relation found, we substitute the (3.106) on (3.107) and this gives us a relation of u_s in terms of u (3.108).

$$\begin{aligned} & r_* = \frac{1}{2}(v_0 - u_s) = \frac{1}{2}(v_0 - u) + 2M \log\left(\frac{1/2(v_0 - u) - 2M}{2M}\right) \\ & \Rightarrow \frac{1}{2}u_s = \frac{1}{2}u - 2M \log\left(\frac{v_0 - u - 4M}{4M}\right) \ , \ v_H \equiv v_0 - 4M \\ & \Rightarrow u_s = u - 4M \log\left(\frac{v_H - u}{4M}\right) \end{aligned} \tag{3.108}$$

It is possible to find an inverse to equation (3.108) by writing it in a more practical way such that after we can apply the Lambert W function ($z = W(z)\exp(W(z)) = W(z\exp(z))$) and obtain a relation of u in terms of u_s (3.109), whih is the inverse of

(3.108).

$$
\begin{aligned}
&\frac{u_s - u}{4M} = -\log\left(\frac{v_H - u}{4M}\right) \\
&\Rightarrow \exp\left(\frac{u - u_s}{4M}\right) = \frac{v_H - u}{4M} \\
&\Rightarrow \exp\left(\frac{u - u_s + v_H - v_H}{4M}\right) = \exp\left(\frac{v_H - u_s}{4M}\right)\exp\left(-\frac{v_H - u}{4M}\right) = \frac{v_H - u}{4M} \\
&\Rightarrow \exp\left(\frac{v_H - u_s}{4M}\right) = \frac{v_H - u}{4M}\exp\left(\frac{v_H - u}{4M}\right) \\
&\Rightarrow W\left[\exp\left(\frac{v_H - u_s}{4M}\right)\right] = W\left[\frac{v_H - u}{4M}\exp\left(\frac{v_H - u}{4M}\right)\right] = \frac{v_H - u}{4M} \\
&\Rightarrow u = v_H - 4M \;\; W\left[\exp\left(\frac{v_H - u_s}{4M}\right)\right]
\end{aligned}
\tag{3.109}
$$

Having both relations (3.108) and (3.109) implies that any solution on the flat region is a solution in the Schwarchild region and it is also true all the way around.

We start by finding the solutions at the past, we have a set of "in" modes normalized on ρ^- (3.110) and (3.111) associated to ρ_R^- and ρ_R^-, respectively.

$$\phi^{in}_{\omega,R} = \frac{e^{-i\omega v}}{\sqrt{4\pi\omega}} \tag{3.110}$$

$$\phi^{in}_{\omega,L} = \frac{e^{-i\omega u}}{\sqrt{4\pi\omega}} \tag{3.111}$$

To proceed, we need solutions for the late-time behaviours of the field. There are modes that pass the horizon and end up at the singularity (ϕ^{sing}_ω), others end up on ρ_L^+ ($\phi^{out}_{\omega,L}$) and others end up on ρ_R^+ (ϕ^{out}_ω). In particular, we are concerned with the modes that end up on ρ_R^+ (3.112), as in the case of the moving mirror.

$$\phi^{out}_\omega = \frac{e^{-i\omega u_s}}{\sqrt{4\pi\omega}} \tag{3.112}$$

In this order of ideas we have two different complete set of solutions, so we can expand the field in both of them (3.113), the first equality corresponds to the past set of modes and the second to the future set of modes with its associated annihilation and creation operators defining the vacuum at the past and at the future.

$$
\begin{aligned}
\phi &= \int_0^\infty d\omega \left[a^{in}_{\omega,R}\phi^{in}_{\omega,R} + a^{in\dagger}_{\omega,R}\phi^{in*}_{\omega,R} + a^{in}_{\omega,L}\phi^{in}_{\omega,L} + a^{in\dagger}_{\omega,L}\phi^{in*}_{\omega,L}\right] \\
&= \int_0^\infty d\omega \left[a^{out}_\omega\phi^{out}_\omega + a^{out\dagger}_\omega\phi^{out*}_\omega + a^{left}_\omega\phi^{left}_\omega + a^{left\dagger}_\omega\phi^{left*}_\omega + a^{sing}_\omega\phi^{sing}_\omega + a^{sing\dagger}_\omega\phi^{sing*}_\omega\right]
\end{aligned}
\tag{3.113}
$$

The next step is to calculate the number of "out" particles present at the "in" vacuum, this will be done with the particle number operator $\hat{N_\omega^{out}} = a_\omega^{out\dagger} a_\omega^{out}$. For calculating the "out" annihilation and creation operators it is useful to know that the modes in ρ_R^+ vanish in ρ_R^- so $(\phi_{\omega,R}^{in}, \phi_\omega^{out})$.

$$a_\omega^{out} = (\phi, \phi_\omega^{out}) = \int_0^\infty d\omega \left[a_{\omega,L}^{in} (\phi_{\omega,L}^{in}, \phi_\omega^{out}) + a_{\omega,L}^{in\dagger} (\phi_{\omega,L}^{in*}, \phi_\omega^{out}) \right] \tag{3.114}$$

To get a more elegant expression for the annihilation operator (3.114). We proceed as follows, as $\{\phi_{\omega,L}^{in}, \phi_{\omega,L}^{in*}\}$ is a complete set of solutions the field ϕ_ω^{out} can be expanded in this set (3.115).

$$\phi_\omega^{out} = \int_0^\infty d\omega' \left[\alpha_{\omega\omega'} \phi_{\omega',L}^{in} + \beta_{\omega\omega'} \phi_{\omega',L}^{in*} \right] \tag{3.115}$$

Where $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are the Bogolubov coefficients which can be easily calculated as follows ((3.116) and (3.117)).

$$\alpha_{\omega\omega'} = (\phi_\omega^{out}, \phi_{\omega',L}^{in}) \tag{3.116}$$

$$\beta_{\omega\omega'} = -(\phi_\omega^{out}, \phi_{\omega',L}^{in*}) \tag{3.117}$$

The expression we want for the "out" annihilation operator (3.118) is given by replacing (3.115) on (3.114), its associated expression for the annihilation operator is (3.118) and the expression for the creation operator is (3.119).

$$\begin{aligned}
a_\omega^{out} &= \int_0^\infty d\omega' \int_0^\infty d\omega' a_{\omega,L}^{in} \left((\phi_{\omega,L}^{in}, \alpha_{\omega\omega'} \phi_{\omega,L}^{in}) + (\phi_{\omega,L}^{in}, \beta_{\omega\omega'} \phi_{\omega,L}^{in*}) \right) \\
&+ a_{\omega,L}^{in\dagger} \left((\phi_{\omega,L}^{in*}, \alpha_{\omega\omega'} \phi_{\omega,L}^{in}) + (\phi_{\omega,L}^{in*}, \beta_{\omega\omega'} \phi_{\omega,L}^{in*}) \right) \\
&= \int_0^\infty d\omega' \int_0^\infty d\omega' a_{\omega,L}^{in} \left(\alpha_{\omega\omega'}^* (\phi_{\omega,L}^{in}, \phi_{\omega,L}^{in}) + \beta_{\omega\omega'}^* (\phi_{\omega,L}^{in}, \phi_{\omega,L}^{in*}) \right) \\
&+ a_{\omega,L}^{in\dagger} \left(\alpha_{\omega\omega'}^* (\phi_{\omega,L}^{in*}, \phi_{\omega,L}^{in}) + \beta_{\omega\omega'}^* (\phi_{\omega,L}^{in*}, \phi_{\omega,L}^{in*}) \right) \\
&= \int_0^\infty d\omega' \int_0^\infty d\omega' a_{\omega,L}^{in} \alpha_{\omega\omega'}^* \delta(\omega - \omega') + a_{\omega,L}^{in\dagger} \beta_{\omega\omega'}^* (-\delta(\omega - \omega')) \\
&\Rightarrow a_\omega^{out} = \int_0^\infty d\omega' \left[a_{\omega,L}^{in} \alpha_{\omega\omega'}^* - a_{\omega,L}^{in\dagger} \beta_{\omega\omega'}^* \right]
\end{aligned} \tag{3.118}$$

$$\begin{aligned}
(a_\omega^{out})^\dagger &= \int_0^\infty d\omega' \left[a_{\omega,L}^{in\dagger} \alpha_{\omega\omega'}^* * - a_{\omega,L}^{in\dagger\dagger} \beta_{\omega\omega'}^* * \right] \\
&\Rightarrow (a_\omega^{out})^\dagger = \int_0^\infty d\omega' \left[a_{\omega,L}^{in\dagger} \alpha_{\omega\omega'} - a_{\omega,L}^{in} \beta_{\omega\omega'} \right]
\end{aligned} \tag{3.119}$$

With this expressions we can calculate the number of "out" particles $\hat{N_\omega^{out}}$ present at the "in" vacuum ($|0_{in}\rangle$), we obtain a known general result (3.120).

$$\begin{aligned}
\langle N^{out}\rangle_{0_{in}} &\equiv \langle (a_\omega^{out})^\dagger a_\omega^{out}\rangle_{0_{in}} \\
&= \int_0^\infty d\omega' \int_0^\infty d\omega' \alpha^J_{\omega\omega'} \alpha^*_{\omega\omega'} \langle \hat{a}^{in\dagger}_{\omega'} \hat{a}^{in}_{\omega'}\rangle_{0_{in}} - \alpha_{\omega\omega'}\beta^*_{\omega\omega'}\langle \hat{a}^{in\dagger}_{\omega'} \hat{a}^{in\dagger}_{\omega'}\rangle_{0_{in}} \\
&- \beta_{\omega\omega'}\alpha^*_{\omega\omega'}\langle \hat{a}^{in}_{\omega'}\hat{a}^{in}_{\omega'}\rangle_{0_{in}} + \beta_{\omega\omega'}\beta^*_{\omega\omega'}\langle \hat{a}^{in}_{\omega'}\hat{a}^{in\dagger}_{\omega'}\rangle_{0_{in}} \\
&= \int_0^\infty d\omega' \int_0^\infty d\omega' \beta^*_{\omega\omega'}\beta_{\omega\omega'}\delta(\omega-\omega') \\
&\Rightarrow \langle N^{out}\rangle_{0_{in}} = \int_0^\infty d\omega \int_0^\infty d\omega' \mid \beta_{\omega\omega'} \mid^2
\end{aligned} \tag{3.120}$$

The expression (3.120) is the same as the expression for the number of "out" particles present at the "in" vacuum state at the accelerated moving mirror, they also have the same Bogolubov coefficients, so we conclude the equivalence between the the black hole and the accelerated moving mirror.

Chapter 4

Outlook

This section is devoted to the formulation of a problem which aims at understanding the previously studied phenomena in the context of quantum phase transitions, understood in a broad sense. It connects all the previous work with what we plan to do in the future and in some sense it changes the conventional path of the resarch one would expect to follow from the conventional point of view into a new and unstudied subject.

Thinking about quantum phase transitions in Quantum Field Theory and General Relativity could sound very ambiguous, quantum phase transitions is a language commonly used in statistical mechanics and in particular applied to physical phenomena in solid state physics. As a first attempt aiming at a well defined notion of quantum phase transitions in this context we will start by talking about quantum phase transitions to introduce the concept of universality which induces a partition in the set of physical systems by relating their general properties independently of their microscopic properties. After understanding the previous concept it is useful to illustrate these ideas with the particular case of topological materials, it is a well studied subject and for our purpose is a fermionic model of quantum phase transitions which is going to help us approach some of the mathematical ideas we will need to solve our problem. Finally we will treat the bosonic case following an analogous procedure.

4.1 Quantum Phase Transitions and Universality

4.1.1 Quantum Phase Transitions

We will start this section by studying the quantum phase transitions from the point of view of [20].

Consider a Hamiltonian $H(g)$ where is a dimensionless coupling and with each point of the lattice a degree of fredom, we assume the finite lattice has N dgrees of freedom.

For the finite lattice we would have a ground state that is a smooth function of the parameter g, Figure 4.1.1. Except the case when the dimensionless quantity is coupled to a conserved quantity ($H(g) = H_0 + gH_1$) and hence we would have a level

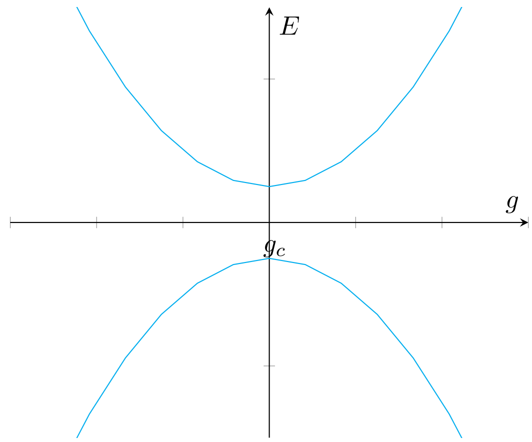

Figure 4.1: Analyticity in the curve, energy as a function of the dimensionless parameter g.

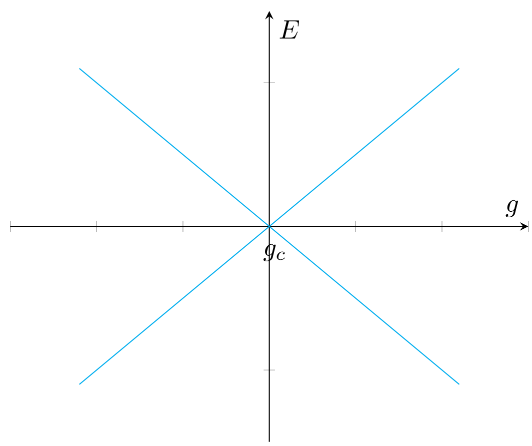

Figure 4.2: Non-analyticity in the curve, energy as a function of the dimensionless parameter g.

crossing, this means the grond state becomes an excited state and some excited state becomes the ground state. This results in the fact that it is going to appear a point of non-analyticity of the ground state energy as a function of g.

The infinite lattice can be considered as the continuum limit of the finit lattice ($N \to \infty$). If we have a finite lattice there is going to exist an energy gap between the ground state and an excited state as in Figure 4.1.1, this is an avoided level-crossing zone frequently known as the energy gap that becomes sharper as the lattice size increases leading to a point of non-analyticity in the critical point of the dimensionless parameter g_c, in the infinite lattice limit as shown in Figure 4.1.1.

In this order of ideas we shall identify the quantum phase transition with the Definition 4, physically the the non-analyticity could be the limiting case of an avoided level crossing or a level crossing.

Definition 4 *(Quantum Phase Transition) The points of non-analyticity in the ground state energy of the infinite lattice system are identified with quantum phase transitions [20].*

The phase transitions as defined in Definition 4, are usually accompanied by a qualitative change in the nature of the correlations in the ground state. This is really important in terms of our purpose, since the expected number of particles in vacuum calculated previously for several physical phenomena is a correltion function between two sets of modes in the ground state of one of them. As we have phsysical systems where it has been observed that change in the expected number of particles in vacuum, then according to [20] it has sense to conjecture a phase transition related to the expected number of particles in vacuum.

Focusing our discussion to second order phase transitions, these are a class of phase transitions in which we have the vanishing of the energy scale of fluctuations (Δ, the lowest energy excitation level above the ground state) when the parameter g approaches its critical point g_c. The behavior of the energy scale is given by (4.1), where J is the energy scale of a characteristic microscopic coupling and $z\nu$ is its associated critical exponent.

$$\Delta \sim J|g - g_c|^{z\nu} \tag{4.1}$$

The value of the critical exponent $z\nu$ is usually universal and does not depend on microscopic properties of the physical system, therefore this critical exponent can determine a universality class as defined in the previous subsection.

According to the behavior of the energy scale (4.1) it is worth to give a word with the analyticity of the energy as a function of the dinmensionless parameter.

If $\Delta \neq 0$ there exists an energy gap and as in the finite lattice case we would have analyticity in the energy as a function of the parameter g and hence no phase transition.

Otherwise if $\Delta = 0$, obviously there is not an energy gap and there will be excitations at low energies provided by a non-analyticity point which results in a phase transition.

4.1.2 Universality

Now we will like to make clear the concept of universality (Definition 5) according to [7].

Definition 5 *(Universality) Independence between general properties of the physical system from microscopic details [7].*

For example different systems with different microscopic structure have quantitavely identical long-range behaviour near the phase transition exhibit a high degree of universality [7].

In this work we have studied a large class of physical phenomena that exhibit features of universality, which are the analog of continous phase transitions in critical phenomena. So we can expect the studied systems to present phase transitions in the sense of defining a topological invariant (order parameter).

Explictitly, the change of vacuum state is related to changes on observables in the system so we are looking forward to relate those observable changes with topological invariants. In other words we conjecture the analogy between vacuum changes and continous phase transitions using topological invariants.

4.1.3 Schwinger Pair Production and Universality

The Schwinger pair production was investigated in spatially inhomogeneous backgrounds in [7].

According to [7], the sufficient amount of electrostatic energy results in a long-range electron-positron fluctuation that becomes a real pair. Using the pair-production rate defined as an order parameter it is observed features of universality close to the critial point, that are analogous to continous phase transitions.

In particular, the electric backgrounds induce a partition of the physical system into universality classes each determined by the pair production.

The research on Schwinger pair production in [7], concluded on the discovery of an analogy between pair production and continous phase transitions. The scaling laws showed a high degree of universality as their corresponding critical exponents only depend on the large-scale properties of the electric background disregarding microscopic properties of the system, hence the mentioned partition into universality classes is determined by pair production if it exists or not.

4.1.4 Chern Numbers and Quantum Phase Transitions

Following the discussion of the last two sections we have studied quantum phase trasnitions and universality, in particular we have only mentioned the case of Schwinger pair production which is a physical phenomena in which we observe quantum phase transitions via universality. In this section we are going to consider the situation all the way around, we will study universality via quantum phase transitions.

Following [5], where the relation between Chern numbers and Quantum Phase Transitions is studied in the XY spin-chain model. A spin chain can be coupled to a single spin, such that one can define the Hamiltonian of the system which has topological invariants (Chern numbers) associated to it. It is shown in [5] that the topological invariants are label different phases of the system, therefore these invariants are closely related to Quantum Phase Transitions.

Additionally, the Chern numbers contain information about the general properties of the system, they are topological invariants and in particular they can be expressed as an integral in the parameter space. In this order of ideas the topological invariants are order parameters that reveal the high degree of universality in the system.

4.2 Topological Materials and The Fermionic Case

The mathematical formalism used to describe fermions on topological materials is closely related to the one we look forward to use in order to describe such critical phenomena in Quantum Field Theory and General Relativity. In this section we will study fermionic systems described by orthogonal complex structures, the self-dual formalism and its Hilbert space relation with a $\mathbb{Z}_2-$index and some calculations about the two-site Kitaev chain, following [3].

4.2.1 Fermionic Systems

Consider a $1-$particle Hilbert space with its fermionic Fock space associated (4.2).

$$(\mathcal{H}, \langle , \rangle) \ , \ \mathcal{F} = \bigwedge \mathcal{H} \tag{4.2}$$

Following the standard formalism of second quantization a fermionic system is described by the canonical anticommutation relations (4.3), in particular in the quantum field theory limit (when the degrees of freedom of the system is infinite) the CAR algebra can be realized in many inequivalent ways. Therefore the problem changes to characterize the algebraic properties of the CAR relations for any realization through a Hilbert space representation [3].

$$\{a_i, a_j^\dagger\} = \delta_{ij} \ , \ \{a_i^\dagger, a_j^\dagger\} = \{a_i, a_j\} = 0 \tag{4.3}$$

We will use the following notation, for any $1-$particle Hilbert space (4.2) it is associated a CAR algebra denoted by (4.4) with generators: $a(u)$ and $a^\dagger(u)$ $\forall u \in \mathcal{H}$ and that satisfy the canonical anticommutation relations (4.5).

$$\mathcal{A}_{CAR}(\mathcal{H}, \langle , \rangle) \tag{4.4}$$

$$\{a(u), a^\dagger(v)\} = \langle u, v\rangle \ , \ \{a^\dagger(u), a^\dagger(v)\} = \{a(u), a(v)\} = 0 \tag{4.5}$$

In the context of Majorana fermions it is conveninent to diagonalize quadratic Hamiltonians with orthogonal complex structures. Neverthelless this is just an alternative form, usuallly such diagonalization is done using Bogolubov transformations and a very useful one when working with CAR-algebras in the quantum field theory limit is the self-dual formalism which we will study.

4.2.2 Orthogonal Complex Structures

Let V be a real vector space such that $dim_{\mathbb{R}} = 2n$ for $n \in \mathbb{Z}^+$ and let $g(.,.)$ be a positive, symmetric bilinear form on V according to Definition 6.

Definition 6 *(Positive, Symmetric, Bilinear Form on V) Let $g : V \times V \to \mathbb{R}$ be a function that is bilinear respect to each of its variables, such that is:*

$$\begin{array}{ll} (I) & Positive\ g(v,v) > 0\ \ \forall v \in V \diagdown \{0\} \\ (II) & Symmetric\ g(u,v) = g(v,u)\ \ \forall u,v \in V \end{array} \tag{4.6}$$

The usual complexification of V is $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, but we will construct a complexification of V using orthogonal complex structures, Definition 7 (a detailed discussion of orthogonal complex structures is treated in [10]). Let V_J denote the complex vector space obtained from V via complexification. The orthogonal complex structure reveals the meaning of scalar multiplication by i, in V_J we have the rule $iv := Jv$, $\forall v \in V$ then the rule for multiplication by complex scalars given by $(\alpha + i\beta)v := \alpha v + \beta Jv$, $\forall v \in V$ and $\alpha, \beta \in \mathbb{R}$.

Definition 7 *(Orthogonal Complex Structures) An orthogonal complex structure is a real linear operator $J : V \to V$ such that:*

$$\begin{array}{ll} (I) & J^2 = -1 \\ (II) & g(Ju, Jv) = g(u,v)\ \ \forall u,v \in V \end{array} \tag{4.7}$$

Lets define the inner product in V_J by (4.8). In order to obtain a complex Hilbert space (4.9), such that $dim_{\mathbb{C}} = n$. It is important to have in mind that each complex structure J has a vacuum associated $|0_J\rangle$.

$$\langle u, v\rangle_J := g(u,v) + ig(Ju,v) \tag{4.8}$$

$$(V_J, \langle .,. \rangle_J)\ ,\ \mathcal{F}_J(V) := \bigwedge V_J \tag{4.9}$$

$$\mathcal{A}_{CAR}(V_J, \langle , \rangle_J) \tag{4.10}$$

One of the advantages to work with orthogonal complex structures is that we obtain a complex Clifford algebra $\mathbb{C}l(V)$ that results in irreducible representation on the Fock space (4.9), while if we consider the usual complexification we obtain a complex Clifford algebra $\mathbb{C}l(V)$ that acts canonically on the exterior algebra $\bigwedge V^{\mathbb{C}}$ but the representation is not irreducible.

Because of the close relationship between the CAR algebras and the Clifford algebras, by using the complexification via orthogonal complex structures we obtain an irreducible representation of the CAR algebra (4.10).

Then we can have in mind that as every complex structure J has a vacuum $|0_J\rangle$ associated and every vacuum on the Fock space $\mathcal{F}$ has an irreducible representation of the CAR algebra, then each J has associated an irreducible representation of the CAR algebra.

For completeness, in the representation (4.10) the creation and annihilation operators $a_J(v)$ and $a_J^\dagger(v)$ acting on $\mathcal{F}_J(V)$ are defined by (4.11) and satisfy the canonical anticommutation relations (4.12).

$$\begin{aligned} a_J^\dagger(v)(u_1 \wedge ... \wedge u_k) &= v \wedge u_1 \wedge ... \wedge u_k \ , \ \forall v \in V \ \ \forall u_1, ..., u_k \in V_J \\ a_J(v)(u_1 \wedge ... \wedge u_k) &= \sum_{j=1}^{k} (-1)^{j-1} \langle v, u_j \rangle_J u_1 \wedge ... \wedge \hat{u}_j \wedge ... \wedge u_k \ , \ \forall v \in V \ \ \forall u_1, ..., u_k \in V_J \end{aligned} \tag{4.11}$$

$$\{a_J(u), a_J^\dagger(v)\} = \langle u, v \rangle_J \ , \ \{a_J^\dagger(u), a_J^\dagger(v)\} = \{a_J(u), a_J(v)\} = 0 \tag{4.12}$$

The creation and annihilation operators defined in (4.11) result in a representation of the real Clifford algebra $Cl(V)$ on $\mathcal{F}_J(V)$, and its vacuum can be described in terms of a two-point function (4.13).

$$\langle 0_J | a_J(u) a_J^\dagger(v) | 0_J \rangle := \langle u, v \rangle_J \tag{4.13}$$

We can construct the Clifford generators (4.14).

$$\pi(v) := a_J^\dagger(v) + a_J(v) \tag{4.14}$$

To get a useful representation of the vacuum state $|0_J\rangle$ in the representation induced by J we need to extend the operators from the vector space V to its usual complexification $V^{\mathbb{C}}$. The idea is to obtain a representation of the Clifford generators (4.14), the creation and annihilation operators (4.11) as linear maps (4.15), where $\mathcal{L}(\mathcal{F}_J(V))$ is the space of bounded linear operators on the Fock space.

$$\overline{\pi}_J, \overline{a}_J, \overline{a}_J^\dagger : V^{\mathbb{C}} \to \mathcal{L}(\mathcal{F}_J(V)) \tag{4.15}$$

For the operators extension we obtain the identities (4.16).

$$\begin{aligned} J a_J^\dagger(v) &:= i a_J^\dagger(v) = \overline{a}_J^\dagger(iv) \ , \ \forall v \in V \\ J a_J(v) &:= i a_J(v) = \overline{a}_J(iv) \ , \ \forall v \in V \end{aligned} \tag{4.16}$$

For this to work we need also to consider a linear extension of the orthogonal complex structure J from the space V to $V^{\mathbb{C}}$. If we consider the eigenvalue problem for J we would have eigenvalues $\pm i$ that define two disjoint subspaces of $W_J^+, W_J^- \geqslant V^{\mathbb{C}}$. It is clearer if we define the projection operator (4.17).

$$P_{\pm J} := \frac{1}{2}(1 \mp iJ) \tag{4.17}$$

We will use the notation for the projected subspaces $W_{\pm} := P_{\pm J}(V^{\mathbb{C}})$. We define an inner product (4.18) on $V^{\mathbb{C}}$, using the extension of the positive symmetric bilinear form.

$$\langle\langle w, z\rangle\rangle := 2\overline{g}(\overline{w}, z) \tag{4.18}$$

With the inner product (4.18) on $V^{\mathbb{C}}$, we have the property $W_J^- = W_J^{\perp}$. Therefore, we have a vector spae decomposition (4.19). Which is analogous to the vector space decomposition in positive and negative frequencies we mentioned when treating the physical phenomena of the Casimir effect.

$$V^{\mathbb{C}} = W_J \oplus W_J^{\perp} \tag{4.19}$$

Finally, considering the restriction of the inner product (4.18) to the subspace $W_J \geqslant V^{\mathbb{C}}$ we get the a Hilbert space (4.20) and additionally a Hilbert space isomorphism (4.21).

$$(W_J, \langle\langle ., .\rangle\rangle) \tag{4.20}$$

$$(V_J, \langle ., .\rangle_J) \cong (W_J, \langle\langle ., .\rangle\rangle) \tag{4.21}$$

With this representation we get a complete description of vacuum (4.22) [3].

$$\overline{\pi}(u)|0_J\rangle = 0 \Leftrightarrow u \in W_J^{\perp} \tag{4.22}$$

4.2.3 The Role of The Complex Structure in The Diagonalization Problem

As mentioned before, the complex structures can be used in an alternative procedure to diagonalize quadratic Hamiltonians, it is very useful since the argument presented in the finite dimensional case remain valid in the quantum field theory limit. In this section we present the mentioned procedure, concerning the vacuum state following [10].

Consider a quadratic Hamiltonian (4.23) or (4.24) describing an N-site fermionic system ($N < \infty$), where A is an Hermitian matrix and B is a skew-symmetric matrix.

$$H = \sum_{i,j=1}^{N} \left[a_i^\dagger A_{ij} a_j + \frac{1}{2} \left(a_i^\dagger B_{ij} a_j^\dagger - a_i^\dagger \overline{B}_{ij} a_j \right) \right] \tag{4.23}$$

$$\frac{1}{2} \begin{pmatrix} a^\dagger & a \end{pmatrix} \begin{pmatrix} A & B \\ -\overline{B} & -\overline{A} \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \tag{4.24}$$

Let V be the space of solutions to Dirac equation and let g be the symmetric bilinear form over V given by Dirac equation (Definition 6).

For the Hamiltonian (4.24) we have annihilation and creation operators a_k and $a_k^\dagger$ that that generate a CAR algebra by satysfing (4.25) and define a vacuum $|0_a\rangle$ by (4.26).

$$\{a_k, a_l^\dagger\} = \delta_{kl} \ , \ \{a_k^\dagger, a_l^\dagger\} = \{a_k, a_l\} = 0 \tag{4.25}$$

$$a_k|0_a\rangle = 0 \ , \ \forall k \tag{4.26}$$

One way to diagonalize the Hamiltonian is to introduce new operators c_k and $c_k^\dagger$ of the form (4.27), with $g, h \in M_{N\times N}$ such that they satisfy the CAR relations (4.28).

$$c_k = \sum_{i=1}^{N} (g_{ki} a_i + h_{ki} a_i^\dagger) \ , \ c_k^\dagger = \sum_{i=1}^{N} (\overline{g}_{ki} a_i^\dagger + \overline{h}_{ki} a_i) \tag{4.27}$$

$$\{c_k, c_l^\dagger\} = \delta_{kl} \ , \ \{c_k^\dagger, c_l^\dagger\} = \{c_k, c_l\} = 0 \tag{4.28}$$

The above conditions result in the equations (4.29), that completely determine $g \in M_{N\times N}$ and $h \in M_{N\times N}$. But as our aim of this section is to focus on what happens to vacuum we are going to omit such detailed alculations.

$$gg^\dagger + hh^\dagger = 1 \ , \ gh^\dagger + hg^\dagger = 0 \tag{4.29}$$

But we will do it using the orthogonal complex structures. Fix $\{e_1, ..., e_N\}$ an orthogonal basis of V such that (4.30) and use the orthogonal complex structure J such that $Je_k := e_{N+k}$ and hence $Je_{N+k} = -e_k$. The complex structures induces canonically a vector space complexification resulting on V_J, thenwe have a $1-$particle Hilbert space $(V_J, \langle ., . \rangle_J)$ and its associeted CAR algebra $\mathcal{A}_{CAR}(V_J, \langle ., . \rangle_J)$ generated by the new creation and annihilation operators (4.31).

$$V = span\{e_k\rangle\}_{k=1,...,N} \tag{4.30}$$

$$c_k := c_J(e_k) \ , \ c_k^\dagger := c_J^\dagger(e_k) \ , \ k = 1, ..., N \tag{4.31}$$

Such that the relation between basis vectors and generators of the CAR algebra holds (4.32) and therefore our hilbert space can be written (4.33) and these annihilation and creation operators satisfy the CAR relations (4.28).

$$e_k = c_k^\dagger|0_J\rangle \ , \ k = 1, ..., N \tag{4.32}$$

From now on we will use the notation $\mathcal{H}$ for the 1−particle Hilbert space V_J.

$$\mathcal{H} = span\{c_k^\dagger|0_J\rangle\}_{k=1,...,N} \tag{4.33}$$

In the two different ways we considered to arrive to the operators c_k and $c_k^\dagger$ we arrive to the diagonalized hamiltonian (4.34).

$$H = \sum_k \lambda_k c_k^\dagger c_k \tag{4.34}$$

In this order of ideas we can define a vacuum $|0_c\rangle$ for the new creation and annihilation operators by (4.28).

$$c_k|0_c\rangle = 0 \ , \ \forall k \tag{4.35}$$

Recall that the Dirac equation permits positive and negative energies, if $|\psi\langle\in V_J$ then $H|\psi\langle= \lambda_k|\psi\langle$ where $\lambda_k \geqslant 0$ or $\lambda_k < 0$. So the problem reduces to find the ground state for the Hamiltonian (4.34).

Let $|\Omega\rangle$ be the state of minimum energy (ground state) of the Hamiltonian (4.34) and supose the energies are given by $\lambda_1 < ... < \lambda_k < 0 \leqslant \lambda_{k+1} < ... < \lambda_N$, which is just a reordering of the energies in increasing order for a clearer notation in what follows.

Define the subspaces (4.36) using the relation (4.32) we can write these subspace in terms of the creation operators (4.37).

$$V_J^+ \leqslant \mathcal{H} \ such \ that \ \lambda_k \geqslant 0 \ , \ V_J^- \leqslant \mathcal{H} \ such \ that \ \lambda_k < 0 \tag{4.36}$$

$$V_J^+ = span\{c_i^\dagger\}_{i=k+1,...,N} \ , \ V_J^- = span\{c_i^\dagger\}_{i=1,...,k} \tag{4.37}$$

Let P_+ and P_- be projector operators defined by (4.38).

$$P_+ \equiv P : V_J \longrightarrow V_J^+ \ , \ P_- \equiv 1 - P : V_J \longrightarrow V_J^- \tag{4.38}$$

And define the orthogonal complex structure (4.39) such that (4.40).

$$J := i(P_+ - P_-) \tag{4.39}$$

$$\begin{aligned} J^2 &= i^2(P_+ - P_-)^2 = -(P_+^2 - P_+P_- - P_-P_+ + P_-^2) = -(P_+^2 + P_-^2) \\ &= -(P_+ + P_-) = -1 \end{aligned} \tag{4.40}$$

We have that (4.41).

$$P_+(V_J) = V_J^+ \ , \ P_-(V_J) = V_J^- \tag{4.41}$$

According to (4.37) we have a vector space decomposition (4.43) that can be written in alternative way (4.43) by letting $V_J^+ \equiv V_J$ and using (4.22).

$$\mathcal{H} = V_J^+ \oplus V_J^- \tag{4.42}$$

$$\mathcal{H} = V_J \oplus V_J^\perp = V_J \oplus \overline{V_J} \tag{4.43}$$

Finally, we can define formally the Hamiltonian (4.34) ground state by a very intuitive procedure given the context presented "filling the Dirac sea" (4.44).

$$|\Omega\rangle := \prod_{\lambda_k<0} c_k^\dagger |0_c\rangle \tag{4.44}$$

4.2.4 2-site Fermionic System and $\mathbb{Z}_2$ Index

In this section it is considered a 2-site fermionic, the orthogonal complex structures introduced before are used in order to calculate a topological invariant of such a system which in this case is a $\mathbb{Z}_2$ index, following [18].

Consider the following vector space and bilinear symmetric form (4.45).

$$V = \mathbb{R}^2 \ , \ g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.45}$$

Let $\mathcal{J}$ be the set of orthogonal complex structures (4.46).

$$\mathcal{J} := \{J : \mathbb{R}^2 \to \mathbb{R}^2 | J^2 = -1\} \tag{4.46}$$

Lets consider now the action of the set $O(2)$ over the set $\mathcal{J}$, this action is given by (4.47) and is transitive.

$$\begin{aligned} O(2) \times \mathcal{J} &\longrightarrow \mathcal{J} \\ (h, J) &\longmapsto hJh^{-1} \end{aligned} \tag{4.47}$$

Lets see that if J is a complex structure and $h \in O(2)$ then $J_h := hJh^{-1}$ is a complex structure (4.48), to prove it we need to verify that J_h satisfy (I) $J_h^2 = -1$ and (II) the compatibility condition. For the following let g be an arbitraty symmetric bilinear form.

$$\begin{aligned} &\textit{(I) } J_h^2 = (hJh^{-1})^2 = hJh^{-1}hJh^{-1} = hJ^2h^{-1} = -1 \\ &\textit{(II) } g(J_hu, J_hv) = g(hJh^{-1}u, hJh^{-1}v) = g(Jh^{-1}u, Jh^{-1}v) = g(h^{-1}u, h^{-1}v) = g(u,v) \ , \ \forall u, v \in V \end{aligned} \tag{4.48}$$

As the relation between $O(2)$ and $\mathcal{J}$ is not one to one it could be the case that $h_1 \neq h_2$ but $h_1 J h_1^{-1} = h_2 J h_2^{-1}$. So we can define the equivalence relation in $O(2)$ (4.49) [11].

$$h_1 \sim h_2 \Leftrightarrow h_1 J h_1^{-1} = h_2 J h_2^{-1} \ , \ then \ h_1 \sim h_2 \Leftrightarrow h_2^{-1} h_1 \in U(1) \tag{4.49}$$

So by making the following partition, we obtain an isomorphism (4.50), in general we would have $V = \mathbb{R}^{2m} \to \mathcal{J} \cong O(2m) \diagup U(m)$.

$$V = \mathbb{R}^2 \to \mathcal{J} \cong O(2) \diagup U(1) \tag{4.50}$$

Lets consider the most general element we can have from $O(2)$ is given by (4.51).

$$h := \begin{pmatrix} \cos\alpha & \sigma\sin\alpha \\ -\sin\alpha & \sigma\cos\alpha \end{pmatrix} \in O(2) \ , \ 0 \leqslant \alpha < 2\pi \ , \ \sigma \in \{+1, -1\} \tag{4.51}$$

To see the form of such a space $\mathcal{J}$ lets fix $J_0 \in \mathcal{J}$ (4.52) and alculate the action of $O(2)$ on $\mathcal{J}$ (4.53).

$$J_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.52}$$

$$\begin{aligned} hJ_0h^{-1} &= \begin{pmatrix} \cos\alpha & \sigma\sin\alpha \\ -\sin\alpha & \sigma\cos\alpha \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sigma\sin\alpha & \sigma\cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha & \sigma\sin\alpha \\ -\sin\alpha & \sigma\cos\alpha \end{pmatrix} \begin{pmatrix} -\sigma\sin\alpha & -\sigma\cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix} \\ &= \begin{pmatrix} \sigma(-\cos\alpha\sin\alpha + \cos\alpha\sin\alpha) & -\sigma(\cos^2\alpha + \sin^2\alpha) \\ \sigma(\sin^2\alpha + \cos^2\alpha) & \sigma(\sin\alpha\cos\alpha - \cos\alpha\sin\alpha) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} = \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sigma J_0 \end{aligned} \tag{4.53}$$

As $\sigma \in \{+1, -1\}$, according to (4.53) we can determine the set of orthogonal complex structures (4.54).

$$\mathcal{J} = \{J_0, -J_0\} \tag{4.54}$$

Now we would like to decompose $h \in O(2)$ into a linear and antilinear part. Define (4.55) such that we can write $h \in O(2)$ as (4.56).

$$p_h := \frac{1}{2}(h - JhJ) \ , \ q_h := \frac{1}{2}(h + JhJ) \tag{4.55}$$

$$h = p_h + q_h \tag{4.56}$$

Lets see that p_h is linear (4.57) and q_h is antilinear (4.58).

$$Jp_h = \frac{1}{2}(Jh - J^2hJ) = \frac{1}{2}(Jh + hJ) \wedge p_hJ = \frac{1}{2}(hJ - JhJ^2) = \frac{1}{2}(hJ + Jh) \\ \Rightarrow Jp_h = p_hJ \text{ , then } p_h \text{ is a linear} \tag{4.57}$$

$$Jq_h = \frac{1}{2}(Jh + J^2hJ) = \frac{1}{2}(Jh - hJ) \wedge q_hJ = \frac{1}{2}(hJ + JhJ^2) = \frac{1}{2}(hJ - Jh) \\ \Rightarrow Jq_h = -q_hJ \text{ , then } q_h \text{ is antilinear} \tag{4.58}$$

To get some insight of the problem we would like to calculate the kernel of the operator p_h. Applying h to the right and J to the left of (4.53) we have (4.59).

$$JhJh^{-1}h = J\sigma Jh \Rightarrow JhJ = -\sigma h \tag{4.59}$$

Replacing the property (4.59) on the definition of the linear operator we get the alternative way to write the linear operator (4.60).

$$p_h = \frac{1}{2}(1 + \sigma)h \tag{4.60}$$

The dependence of σ on the expression (4.60) makes evident the kernel of p_h (4.61) with dimensions (4.62), taking into acount that the complexification reduces the diemensions of the space by a half. If $\sigma = 1$, $Kerp_h$ is trivial and if $\sigma = 1$, $Kerp_h$ is all of the space.

$$Kerp_h = \begin{cases} \{0\} \text{ , } \sigma = 1 \\ \mathbb{R}^2 \text{ , } \sigma = -1 \end{cases} \tag{4.61}$$

$$dim_{\mathbb{C}}(Kerp_h) = \begin{cases} 0 \text{ , } \sigma = 1 \\ 1 \text{ , } \sigma = -1 \end{cases} \tag{4.62}$$

$h \in O(2)$ induces a Bogolubov automorphism θ_h therefore we can define a Bogolubov transformation (4.63).

$$c(v) := a(p_hv) + a^\dagger(q_hv) \tag{4.63}$$

Using (4.12) and (4.8) we obtain that such a Bogolubov transformation satisfies the CAR algebra relations (4.64).

$$\{c(u), c(v)\} = \langle u, v\rangle_J \tag{4.64}$$

In order to continue with some calculations lets make explicit some details. The complexified space which is the 1−particle Hilbert space and its associated Fock space are given by (4.65).

$$V_J \cong \mathbb{C}\ ,\ \mathcal{F}(V_J) \equiv \bigwedge V_J = \mathbb{C} \oplus V_J \tag{4.65}$$

For our vector space V we have the following basis (4.66).

$$Basis(V) \equiv \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \tag{4.66}$$

Notice that the complexification satisfy the following relations among the basis of V (4.67).

$$Je_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_2 \tag{4.67}$$

Taking into account (4.67), e_1 generates all the space $V_J \cong \mathbb{C}$ therefore its basis is given by (4.68)

$$Basis(V_J) \equiv \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \tag{4.68}$$

We are going to use a ket notation for the following, the generator of the complex numbers (1, using the sum) is going to be associated with the generator of the state space V_J ($|0\rangle$, the ground state by applying creation operators), that is $|0\rangle \equiv 1$ such that $e_1 \equiv |1\rangle = a^\dagger(e_1)|0\rangle \in V_J$. Therefore we have a basis for the fock space (4.69).

$$Basis\mathcal{F}(V_J) \equiv \{|0\rangle\} \tag{4.69}$$

Notice that the linear and antilinear operators can be expressed in terms of (4.70) by (4.71) and (4.72), respectively.

$$h_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \in O(2) \tag{4.70}$$

$$\begin{aligned} p_h &= \frac{1}{2}(h_\sigma - Jh_\sigma J) = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \\ &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} - \begin{pmatrix} -\sigma & 0 \\ 0 & -1 \end{pmatrix} \right) = \frac{1+\sigma}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4.71}$$

$$\begin{aligned} q_h &= \frac{1}{2}(h_\sigma + Jh_\sigma J) = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \\ &= \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} + \begin{pmatrix} -\sigma & 0 \\ 0 & -1 \end{pmatrix} \right) = \frac{1-\sigma}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{4.72}$$

Lets see how does it look the Bogolubov transformation of the basis element of V_J (4.73), using the previous calcultions.

$$\begin{aligned} c_1 \equiv c(e_1) &= a(p_h e_1) + a^\dagger(q_h e_1) = a\left(\frac{1+\sigma}{2}e_1\right) + a^\dagger\left(\frac{1-\sigma}{2}e_1\right) \\ &= \left(\frac{1+\sigma}{2}\right)a(e_1) + \left(\frac{1-\sigma}{2}a^\dagger\right)(e_1) = \left(\frac{1+\sigma}{2}\right)a_1 + \left(\frac{1-\sigma}{2}\right)a_1^\dagger \end{aligned} \tag{4.73}$$

All the previous calculations have been done thinking about how to get a description of the vacuum state $|\Omega\rangle$ such that $c_k|\Omega\rangle = 0 \ \forall k$, in particular we have the case where $k = 1$. According to the Bogolubov transformation of the basis element (4.73) we get to the following conclusion (4.74).

$$\begin{aligned} &\mathit{If}\ \sigma = 1 \Rightarrow c_1 = a_1\ ,\ \mathit{then}\ |\Omega_1\rangle = |0\rangle \\ &\mathit{If}\ \sigma = -1 \Rightarrow c_1 = a_1^\dagger\ ,\ \mathit{then}\ |\Omega_{-1}\rangle = a_1^\dagger|0\rangle \end{aligned} \tag{4.74}$$

The later tells us that σ characterizes the vacuum, in particular the vacuum parity. Finally we can define the exact topological invariant, a $\mathbb{Z}_2-$index (4.75), associated to each of the conditions of the vacuum (4.74).

$$\begin{aligned} index : \mathcal{J} &\longrightarrow \mathbb{Z}_2 \\ p_h &\longmapsto dim_{\mathbb{C}} Ker p_h \end{aligned} \tag{4.75}$$

4.3 Bosonic Formulation

Until now we focused (Section 3) on the solution of Klein-Gordon equation, we had a vector space of solutions equipped with a canonical bilinear form which could be turned to be positive definite and in effect we could call it a symplectic covariant scalar product, and we have showed the mathematical formalism used to describe systems of fermions on relevant physical situations, the mathematical formalism developed in the last section can be straightforwardly applied to a system of bosons, maybe with some subtle modifications.

Consider a real vector space V, this is the space of solutions to Klein-Gordon equation (in the previous section it was the space of solutions to Dirac equation). The subtle modification comes from the bilinear form we are considering, in the fermionic case we had a positive, symmetric bilinear form (Definition 6) that ame from the invariance of Dirac equation, analogously the invariance of the Klein-Gordon equation results in symplectic bilinear form (Defnition 8).

Definition 8 *(Symplectic Bilinear Form) Let $\sigma : V \times V \to \mathbb{R}$ be a mapping that is bilinear respect to each of its variables, such that is:*

$$\begin{aligned} &(I) \quad \textit{Alternating } \sigma(v,v) = 0 \;\; \forall v \in V \\ &(II) \quad \textit{Symmetric If } \sigma(u,v) = 0 \;\; \forall v \in V \Rightarrow u = 0 \end{aligned} \tag{4.76}$$

4.3.1 Scalar field in dim$-(0+1)$: The Harmonic Oscillator

We will start by studying the procedure of determining the vector space of solutions for a scalar field in dim$-(0+1)$ which is the harmonic oscillator confined in a spatial point, it is a toy model for further applications of the procedure to more general physical situations. The notes below are taken from a Mathematical Physics and QFT seminar in our research group [17].

Consider a scalar field q in dim$-(0+1)$ that satisfies the equation (4.77).

$$(\partial_t^2 + \mu^2)q(t) = 0 \Rightarrow q^{''} + \mu^2 q = 0 \tag{4.77}$$

We get a general two-dimensional solution (4.78).

$$q(t) = q_0 \cos \mu t + \frac{p_0}{\mu} \sin \mu t \tag{4.78}$$

The canonical symplectic form analogous to the one discussed in Sections 2 and 3 is (4.79).

$$\begin{aligned} \sigma(q^{(1)}, q^{(2)}) &= q^{'(1)}(t) q^{(2)}(t) - q^{(1)}(t) q^{'(2)}(t) = q_0^{(2)} p_0^{(1)} - q_0^{(1)} p_0^{(2)} \\ &= \begin{pmatrix} q_0^{(1)} & p_0^{(1)} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_0^{(2)} \\ p_0^{(2)} \end{pmatrix} \end{aligned} \tag{4.79}$$

Let $\mathcal{S}$ be the space of solutions defined (4.80),

$$\mathcal{S} := \{q(t) | q^{''} + \mu^2 q = 0\} \tag{4.80}$$

formally we have for the symplectic form (4.81).

$$\begin{aligned} \sigma :&\mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R} \\ &(I) \quad \sigma \textit{ well defined (does not depend on time} \\ &(II) \quad \textit{Anti-symmetric, } \sigma(x,y) = -\sigma(y,x) \\ &(III) \quad \textit{Non-degenerte, } \sigma(x,y) = 0 \Rightarrow x = 0 \;\; \forall y \end{aligned} \tag{4.81}$$

Until now we have a vector space of solutions equipped with a symplectic form, which gives us the structure of a symplectic vector space $(\mathcal{S},\sigma)$. This symplectic vector space determines uniquely a Weyl $*-$algebra $\mathcal{W}(\mathcal{S},\sigma)$ up to $*-$isomorphism [16].

Define the Pauli-Jordan function (4.82) and define the function which we are going to use (4.83),

$$\langle 0|[q(t),q(t')]|0\rangle \equiv i\Delta(t-t') = i\frac{\sin(\mu(t-t'))}{2\mu} \tag{4.82}$$

$$\xi_\mu(t) := \frac{\sin(\mu t)}{2\mu} \tag{4.83}$$

define the integral operator (4.84),

$$\begin{aligned} \mathbb{E} : C_0^\infty(\mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}) \\ f &\longmapsto (\mathbb{E}f)(t) := \int_{\mathbb{R}} dt'\Delta(t-t')f(t') = (\xi_\mu * f)(t) \end{aligned} \tag{4.84}$$

we have that $\mathbb{E}f \in \mathcal{S}\ \forall f \in C_0^\infty(\mathbb{R})$, then $Im\mathbb{E} \subset \mathcal{S}$. Collapsing null soutions $\mathbb{E}f = 0$ we have (4.89).

$$C_0^\infty(\mathbb{R}) \diagup Ker\mathbb{E} \cong \mathcal{S} \tag{4.85}$$

We want to know what is $ker\mathbb{E}$, in order to do so we use Fourier transformation (4.86),

$$\begin{aligned} &(\hat{\mathbb{E}f})(\omega) = (\xi_\mu \hat{*} f)(\omega) = \hat{\xi_\mu}(\omega)\hat{f}(\omega) = \int_{-\infty}^{\infty} dt \frac{e^{-i\omega t}}{4\mu i}\left(e^{i\mu t} - e^{-i\mu t}\right)\hat{f}(\omega) \\ &= \frac{i\pi}{2\mu}\left(\delta(\mu+\omega) - \delta(\mu-\omega)\right)\hat{f}(\omega) = \frac{i\pi}{2\mu}\left(\hat{f}(-\mu) - \hat{f}(\mu)\right) = \frac{i\pi}{2\mu}\left(-\overline{\hat{f}(\mu)} - \hat{f}(\mu)\right) \\ &\Rightarrow (\hat{\mathbb{E}f})(\omega) = \frac{-\pi}{2\mu}\left(i\overline{\hat{f}(\mu)}\delta(\mu+\omega) + i\hat{f}(\mu)\delta(\mu-\omega)\right) \end{aligned} \tag{4.86}$$

Now we can calculate the Fourier transform (4.87) of the solution (4.78),

$$\begin{aligned} \hat{q}(\omega) &= \int_{\mathbb{R}} dt q_0 \frac{\left(e^{i\mu t} + e^{-i\mu t}\right)}{2} e^{-i\omega t} + \int_{\mathbb{R}} dt \frac{p_0}{\mu}\frac{\left(e^{i\mu t} - e^{-i\mu t}\right)}{2i} e^{-i\omega t} \\ &= \left(\frac{q_0}{2} - i\frac{p_0}{2\mu}\right)\int_{\mathbb{R}} dt e^{i(\mu-\omega)t} + \left(\frac{q_0}{2} + i\frac{p_0}{2\mu}\right)\int_{\mathbb{R}} dt e^{i(\mu+\omega)t} \\ &= \left(\frac{q_0}{2} - i\frac{p_0}{2\mu}\right)\delta(\mu-\omega) + \left(\frac{q_0}{2} + i\frac{p_0}{2\mu}\right)\delta(\mu+\omega) \end{aligned} \tag{4.87}$$

comparing (4.86) and (4.87) we get (4.88) which defines the equivalence clases in the space of solutions. And now we can define the kernel of the integral operator, $Ker\mathbb{E} := \{f \in C_0^\infty(\mathbb{R}) | \hat{f}(\mu) = 0\}$.

$$\hat{f}(\mu) = \frac{1}{\pi}\left(p_0 + i\mu q_0\right) \tag{4.88}$$

Now we would like to give the space $C_0^\infty(\mathbb{R})/Ker\mathbb{E}$ a symplectic structure, for that we construct a vector space isomorphism (4.89),

$$\begin{aligned} \psi : C_0^\infty(\mathbb{R}) \diagup Ker\mathbb{E} &\longrightarrow \mathcal{S} \subset C^\infty(\mathbb{R}) \\ [f] &\longmapsto \psi([f]) \end{aligned} \tag{4.89}$$

for each solution f we have initial conditions as in (4.88), so for $f, g \in \mathcal{S}$ we have $\pi\hat{f}(\mu) = p_0^{(1)} + i\mu q_0^{(1)}$ and $\pi\hat{g}(\mu) = p_0^{(2)} + i\mu q_0^{(2)}$. We can express the initial conditions in terms of the solutions (4.90),

$$\begin{aligned} q_0^{(1)} &= \frac{\pi}{2\mu i}\left(\hat{f}(\mu) - \hat{f}(-\mu)\right) \ , \ q_0^{(2)} = \frac{\pi}{2\mu i}\left(\hat{g}(\mu) - \hat{g}(-\mu)\right) \\ p_0^{(1)} &= \frac{\pi}{2}\left(\hat{f}(\mu) + \hat{f}(-\mu)\right) \ , \ p_0^{(2)} = \frac{\pi}{2}\left(\hat{g}(\mu) + \hat{g}(-\mu)\right) \end{aligned} \tag{4.90}$$

we calculate now (4.79) to obtain a symplectic form in terms of solutions (4.91), let $\overline{\sigma}$ be the inner product on $C_0^\infty(\mathbb{R})/Ker\mathbb{E}$.

$$\begin{aligned} &\sigma(q^{(1)}, q^{(2)}) = q_0^{(2)} p_0^{(1)} - q_0^{(1)} p_0^{(2)} = 2\pi^2 \int dt dt' f(t') f(t) \Delta(t', t) g(t) = \langle f, \mathbb{E}g\rangle_{L^2} \\ &\Rightarrow \overline{\sigma}([f], [g]) \equiv \langle f, \mathbb{E}g\rangle_{L^2} = \sigma(q^{(1)}, q^{(2)}) \end{aligned} \tag{4.91}$$

Continuing with the field quantization we have canonical commutation relations (4.92),

$$[\phi(f), \phi(g)] = -i\sigma(f, g) = -i\langle f, \mathbb{E}g\rangle_{L^2} \ , \ \phi(f) \equiv \int dt q(t) f(t) \tag{4.92}$$

in particular, for the oscillator we have $\phi(f) = \int dt q(t) f(t)$ and replacing on (4.92) we get (4.93)

$$[\phi(f), \phi(g)] = \int dt \int dt' f(t) g(t') [q(t), q(t')] = i \int dt f(t) (\mathbb{E}g)(t) = i\langle f, \mathbb{E}g\rangle_{L^2} \tag{4.93}$$

We can write the Pauli-Jordan function using a technique of renormalization (4.94), that makes evident that it depends on the election of the vacuum. Notice that the first term at the right does not depend on the choice of vacuum but the second term on the right does depend on the choice of vacuum and so it depends on the selection of positive and negative frequency modes.

$$\langle 0|q(t)q(t')|0\rangle = \langle 0|\frac{1}{2}[q(t),q(t')] + \frac{1}{2}\{q(t),q(t')\}|0\rangle = \frac{i}{2}\Delta(t-t') + \frac{1}{2}\{q(t),q(t')\}|0\rangle \tag{4.94}$$

Lets focus our attention on the decomposition into positive and negative frequency modes, until now we have a real symplectic vector space $(\mathcal{S}_{\mathbb{R}}, \sigma)$ we can consider its complexitication (4.95),

$$\mathcal{S}_{\mathbb{C}} = \mathcal{S}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \equiv span\left(e^{i\mu t}, e^{-i\mu t}\right) \tag{4.95}$$

with elements of the form $u = u_- e^{i\mu t} + u_+ e^{-i\mu t}$ and $v = v_- e^{i\mu t} + v_+ e^{-i\mu t}$, $\forall u, v \in \mathcal{S}_{\mathbb{C}}$. Where we have an inner product defined by $(u,v) = i\sigma(\overline{u}, v) = 2\mu(\overline{u_+}v_+ - \overline{u_-}v_-)$, that is not positive definite as we would like.

In order to obtain an appropriate Hilbert space we have to restrict the solution space to the positive frequency modes of the form $u = u_+ e^{-i\mu t}$ and now we have a positive definite inner product $(u,v) = 2\mu\overline{u_+}v_+$.

Let $\mathcal{H} = span\left(e^{-i\mu t}\right) \simeq \mathcal{C}$ such that $\mathcal{H}_- = \overline{\mathcal{H}}_+$. Therefore we can write the positive and negative frequency mode decomposition as a vector space decomposition (4.96), where the subspace $\mathcal{H}_+$ is defined such that the restriction of the inner product on $\mathcal{S}_{\mathbb{C}}$ is positive definite ($(,)|_{\mathcal{H}_+}$ positive definite).

$$\mathcal{S}_{\mathbb{C}} = \mathcal{H}_+ \oplus \mathcal{H}_- = \mathcal{H}_+ \oplus \overline{\mathcal{H}_+} \tag{4.96}$$

4.3.2 Scalar Klein-Gordon field in dim$-(1+1)$

Let (M, g) be a globally hyperbolic spacetime and let ϕ be a scalar field satisfying Klein-Gordon equation (4.97).

$$(\Box + m^2)\phi = 0 \tag{4.97}$$

In analogy with the previous case, fix Σ_0 the Cauchy surface with the initial conditions for the solutions of Klein-Gordon equation and define the following vector space of solutions (4.98).

$$\mathcal{S} := \{\phi \in C^\infty(M) | (\Box + m^2)\phi = 0 \wedge \phi|_{\Sigma_0} \in C_0^\infty(\Sigma_0)\} \tag{4.98}$$

As mentioned before the field equation (4.97) comes equipped with a canonical symplectic structure (4.99).

$$\sigma(\phi_1, \phi_2) := \int_\Sigma \left(\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1\right) n_\mu dV_g \tag{4.99}$$

Let A and R be the advanced and retarded solutions of the Klein-Gordon equation, then for $f \in C_0^\infty(M)$ and (4.100).

$$(\Box + m^2)Af = f \ , \ (\Box + m^2)Rf = f \tag{4.100}$$

Define the following map (4.101), that give us the solutions of the Klein-Gordon equation (4.102).

$$\begin{aligned} \mathbb{E} : C_0^\infty &\longrightarrow \mathcal{S} \\ f &\longmapsto A - R \end{aligned} \tag{4.101}$$

$$\begin{aligned} (\Box + m^2)Ef &= (\Box + m^2)Af - (\Box + m^2)Rf = (\Box + m^2)f - (\Box + m^2)f \\ &= (\Box + m^2)(f - f) = 0 \end{aligned} \tag{4.102}$$

In analogy with the previous section we can find the CCR for the quantum field (4.103).

$$[\hat{\phi}(f), \hat{\phi}(g)] = -i\sigma(\mathbb{E}f, \mathbb{E}g) \tag{4.103}$$

As we are dealing with bosons, the operators that satisfy the CCR (4.103) are unbounded it is conveninent to consider the Weyl algebra $\mathcal{W}(\mathcal{S}, \sigma)$ of $(\mathcal{S}, \sigma)$ [16]. It only remains to decompose the space of solutions in positive and negative frequencies as we did in the previous section and which is the same of choosing a vacuum state.

Chapter 5

Conclusions

A contrast between quantum field theory and quantum field theory in curved spacetime was presented, focusing on the main aspects concerning the lack of vacuum uniqueness in the later theory and its consequences on the difficulty to develop the theory such as the nontrivial way of decomposing the solution space into positive and negative frequencies, having in mind the physical situations discussed.

The state of the art of the Casimir effect phenomena was rigorously studied from the most relevant sources in the subject. The physical phenomena was presented in different contexts ranging from: topological modifications, the introduction of dynamical boundaries which represent the dynamical Casimir effect with the moving mirror case and cavity quantum field theory with the moving mirror in a two dimensional cavity case, phenomena from general relativity as the Rindler spacetime case and the collapsing of a null shell. Additionally it was verified the expression of the Sorkin-Johnston state for the causal diamond using a quantum field thoery model that followed an analogous procedure to the other physical phenomena.

As a result of the study of the Casimir effect phenomena and of most importance is the observation that change of a vacuum state is related to changes on observables in the system, as the number of particles on vacuum. This leads the research forward to relate those observable changes with topological invariants, which is the most general language employed to formulate them.

In agreement with the observations made from the study of the physical phenomena and the ambiguity of something like a phase transition on quantum field theory or general relativity phenomena. It was thought as a first step to have a well defined context for the future work, in order to do so, some similar approaches were studied and leaded the investigation into the concepts of universality and quantum phase transitions.

In particular we have physical phenomena that present high degree of universality, therefore we can model them as quantum phase transitions by defining an order parameter which is going to work as a topological invariant.

In this order of ideas, the mathematical formalism used to decribe topological invarints on physical situations was presented for fermionic and bosonic systems related to physical situations.

Finally, the language used to formulate the physical problems in order to talk about topological invariants suggests to use orthogonal complex structures to complexify the space of solutions. The main physical advantage of this procedure is basically that as for every complex structre we have a physical vacuum and for every vacuum we have a CAR algebra. Then, for each othogonal complex structure we obtain an irreducible representation of the CAR algebra and we can look forward to find vacuum characterizations based on the CAR algebra representation properties.

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