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Maximally entangled states from symmetric and alternating groups

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Para mi mamá y mi hermana, quienes estuvieron conmigo en todo este proceso.
Para mi padre, que me escuchó.
Y para Natalia, porque a pesar de todo, siempre he podido encontrarla.

Abstract

One of the main goals pursued by Quantum Information Theory is to provide secure protocols to share data between different parties, such that the information sent cannot be easily intercepted by an eavesdropper. Several of these protocols limit the information that can be obtained by one of the parties when they try to recover the original message individually, such as *Quantum Secret Sharing* schemes (QSS). In these schemes, any individual party or any selected subgroup of parties is incapable of recovering any portion of the original quantum state sent. Since locally maximally entangled states (LME states) serve these purposes, the search for this kind of quantum states have become one of the most explored areas in this field.

There is no canonical method to find the set of all possible maximally entangled states of N parties. In fact, finding SLOCC (stochastic local operations and classic comunication) orbits of entanglement classes is still an open task for $N \geq 4$. Fortunately, states of this type can be constructed by projecting any multipartite state over invariant one-dimensional subpaces of tensor products of irreducible representations of finite groups, such as the symmetric group S_n. Moreover, these can be found to have some symmetry properties given by how they transform under the action of the symmetric group S_N^P acting over its N parties.

In this sense, the current thesis proposes a more intuitive form of conceiving such states. It provides a form of constructing maximally entangled states from tensor products of representations of the symmetric group S_n and the alternating group $\mathcal{A}_n$. Each of these states can be labeled in terms of the irreducible action of the symmetric group S_N^P over the N indices of a multipartite state. In fact, taking this into account, this thesis provides a form of reducing the dimensionality of the problem when self-adjoint representations of S_n are used, given some relations between this group and the representations of the alternating group $\mathcal{A}_n$. In particular, it finds that in the case $S_5/\mathcal{A}_5$ these states can be geometrically interpreted, given some relations between some symmetric and alternating groups, and groups of rotations of regular polyhedra (or polytopes).

Contents

1. Introduction

Entanglement of multipartite quantum systems holds an special place in quantum mechanics given its several applications in this field. It is considered to be the most useful resource [4] found in any quantum state due to its role and its potential in diverse quantum information tasks, such as quantum teleportation [13], quantum computation [26] and quantum cryptography [22]. Moreover, quantum entanglement applications in quantum information codes have proven to be relevant in the domains of Quantum Field Theory and quantum gravitation, since it has led to propose exactly solvable models for the *anti-de Sitter/Conformal Theory correspondance* or AdS/CFT correspondance [27].

Given the previous motivations, and since not every entangled state can be used to perform the same task, several studies have been devoted to classifying and quantifying quantum entanglement of multipartite systems. The distinction between *entangled* and *unentangled* states is perhaps the most fundamental result that has been obtained [35]. From a mathematical point of view, given a *pure* state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$ made from tensor products of N spaces $\mathcal{H}_A$ and $\mathcal{H}_B$,..., $\mathcal{H}_N$, each with some dimension d_i or $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes ... \otimes \mathcal{H}_N$, then $|\psi\rangle$ is said to be *unentangled* if it can be decomposed as a tensor product of states in each of the $N-$ spaces, i.e., $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B \otimes ... \otimes |\psi\rangle_N$, with $|\psi\rangle_A \in \mathcal{H}_A$, $|\psi\rangle_B \in \mathcal{H}_B$,..., $|\psi\rangle_N \in \mathcal{H}_N$. This notion of entanglement can be extended to *mixed* states, which are quantum states composed of pure states labeled by density matrices ρ_i with some probability distribution $\{p_i\}$. In this case, a state ρ is said to be unentangled or *fully-separable* if it can written as follows:

$$\rho = \sum_i p_i \rho_i^1 \otimes ... \otimes \rho_i^N, \tag{1.1}$$

where $\rho_i^{(j)}$ are local density matrices. Otherwise, the quantum state is said to be entangled.

Entanglement is a non-local property of quantum states. Local operations on multipartite states can be used to classify entangled states in inequivalent entanglement classes [35], since entanglement cannot be created through local operations only. If these operators are unitary, then two states $|\psi\rangle$ and $|\phi\rangle$ of N parties are said to be LU equivalent if the following is true:

$$|\psi\rangle = U_1 \otimes U_2 \otimes ... \otimes U_N |\phi\rangle \,, \tag{1.2}$$

for some set of unitaries U_i, with $i \in \{1, 2, ..., N\}$. LU-equivalence classes are important since any two vectors in the same class can be used to perform the exact same quantum information task with the same rate of success. Representative states or *canonical forms* for every equivalence class under these operations in any Hilbert space of N parties has been a topic of research in quantum mechanics. These equivalence classes have been fully determined for the bipartite case, $N = 2$. In this sense, any two states are LU-equivalent if they have the same Schmidt decomposition [35], i.e. if they can be brought, using LU-operations, to the following state:

$$|\psi\rangle = \sum_{i}^{d} \sqrt{p_i}\,|\alpha_i\rangle \otimes |\beta_i\rangle\,, \tag{1.3}$$

where $|\alpha_i\rangle \in \mathcal{H}_A$, $|\beta_i\rangle \in \mathcal{H}_B$ and $d = min(dim\mathcal{H}_A, dim\mathcal{H}_B)$. Nevertheless, the Schmidt decomposition can not be generalized for $N > 2$. Just for the case of N qubits, i.e., $(d = 2)$, a total of $2(2^N - 1) - 3N$ parameters are needed to specify every equivalence class [35]. For greater dimensions $(d > 2)$, i.e., *qudit states*, this means that the number of parameters grows exponentially with N, making the problem of classifying LU-classes a computationally expensive one.

LU operations are not the only kind of local operations that can be used on multipartite states. These states can also be transformed according to *Stochastic Local Operations and classic Communication* [35] or *SLOCC*, which gives a broader criterion for classifying entangled states. In this sense, two states of N parties $|\phi\rangle$ and $|\psi\rangle$ are said to be equivalently entangled under SLOCC-operations if there exist *Kraus operators* $\{A_i\}$ for $i \in \{1, ..., N\}$ with unit determinant such that:

$$A_1 \otimes A_2 \otimes ... \otimes A_N\,|\psi\rangle = \lambda\,|\phi\rangle\,, \tag{1.4}$$

for some $\lambda \in \mathbb{C}$. In this case, $|\psi\rangle$ and $|\phi\rangle$ can deliver the same quantum information tasks, but with different probability of success.

A method for measuring entanglement in a multipartite state of N parties is by tracing out all but one of the parties. This is important beacuse each SLOCC orbit is represented by a canonical state, which is generally the state that carries the maximum degree of entanglement. According to Kempf-Ness theorem [34], if the SLOCC orbit is *stable*, i.e., if there is a non-zero multipartite state in the closure of the orbit, the representative of its class is a *locally maximally entangled* (LME) state [34], which carries the maximum degree of entanglement for each of its individual parties, identified by local density matrices ρ_i, for $i \in \{1, 2, ..., N\}$. This means that for a LME state every local density matrix is a multiple of the identity matrix, i.e., a totally mixed state:

$$\rho_i = \frac{1}{d_i} I, \tag{1.5}$$

where d_i is the dimension of the Hilbert space $\mathcal{H}_i$.

Similarly as for LU-equivalence, there have been several studies devoted to find SLOCC equivalence classes for multipartite states, as well as the canonical forms corresponding to these classes. These canonical states can be used to deliver certain set of quantum information tasks. In particular, it has been found that any two bipartite entangled states of arbitrary dimension are SLOCC equivalent if they have the same Schmidt rank [7]. This means that, in the case of two qubits, any entangled state is in the same SLOCC orbit of a so called *EPR* state [13], which is a maximally entangled state. In the three qubit case there is a total of 6 SLOCC classes, of which three correspond to states that are separable to respect to some bipartition of the state and two of them are genuinely entangled classes, represented by the well-known

W and the LME GHZ states, as found by Vidal and Cirac [10]. Moreover, it has been found that the SLOCC classes of qubit states of 4 parties are infinite, and that they can be arranged into 9 different families of classes, including a GHZ-like family of states [33]. Nevertheless, classifications of states under SLOCC orbits have not yet been achieved for more than 4 qubits and there is no complete classification of entangled states of N qudits of dimension greater than 3, although generalizations of permutationally symmetric maximally entangled states GHZ and W-like states and corresponding SLOCC classes of N parties have been revised by some authors [2],[25].

LME states are also relevant in quantum information because these can be used to successfully perform diverse quantum information tasks, including Quantum Error Correction codes [20]. In these protocols a quantum state is sent through a noisy channel that can modify the initial state of the full quantum system through some random operation, which can be modeled by Pauli operators in the qubit case, since these operators span a two-dimensional quantum space. In general, if a LME state is sent through a noisy channel, and an error on a single qudit occurs, then the fact that the local density matrices are totally mixed states can be exploited to recover the initial state using local unitary operations, but makes it impossible to retrieve it when an error in more than one qudit occurs. Nevertheless, if the density matrix is a multiple of the identity for any partition of the state of length $k > 1$, then it is possible to retrieve the state if an error on k qudits occur. In such cases, this $k-$uniform state serves as a code state that corrects k errors and detects $d = 2k + 1$ errors, or (N, k, d) QEC code [21].

LME states can be used to model systems of multiple coupled spins. Kitaev chains, and in particular quantum systems composed of N identical correlated fermions of spin $1/2$ interacting in a magnetic field, can be modeled using this notion. Let us consider a cyclic chain of size kL for which any L neighboring spins are maximally entangled. Then, a $L-$uniform state can be used to represent such systems. For instance, this means that in the case $L = 2$, any spin, and each nearest neighbor pair of spins will be in a maximally mixed state.

Since maximally entangled states can be used to deliver several quantum information schemes and given the fact that there is not a complete classification of SLOCC orbits for many-particle quantum systems, diverse methods for generating sets of maximally entangled states have been proposed. These methods generally involve the usage of tensor products of irreducible representations of finite groups, which yield the action of the group in a given complex vector vector space. Given some group G and a set of N irreducible representations α, β,..., ν, the tensor product of these representations is always a reducible representation, i.e., a direct sum of some other irreducible representations of the group, each which some multiplicity that can be greater than one [5]. If there is a one-dimensional irreducible representation in the tensor product decomposition of $\alpha \otimes \beta \otimes ...\nu$, then this irreducible subspace is generated by an operator that commutes with the set of all irreducible representations of G. According to Schur's Lemma of groups representations [30], this operator is an scalar multiple of the identity operator, which means that it represents the local density matrix of some LME state. Therefore, projecting any vector in a N-partite Hilbert space onto these one-dimensional subspaces can be used to obtain sets of locally maximally entangled states of N parties.

The construction of bases for the irreducible subspaces appearing from tensor products of irreducible representations of diverse groups, and in particular of locally maximally entangled states, have been reviewed by various authors. The well-known angular momentum coupling, which can be characterized by the Clebsch Gordan coefficients of tensor products of representations of SU(2) Lie algebra is the most relevant result of representations theory of finite groups in quantum mechanics [15].

Tensor products of irreducible representations of the symmetric group S_n, the group of permutations of n objects, have been more recently used to construct and classify LME states [4], [1]. The irreducible representations of S_n, as well as its conjugacy classes, are labeled by partitions of n, which are sequences of non-increasing integer numbers that add up to n, i.e.:

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_r),$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r$ and $\sum_i \lambda_i = n$, and each λ_i in λ is a *part* of the partition. These partitions can be represented using *Young diagrams*, which are collections of n boxes arranged such that λ_1 of these boxes are located in the first row, λ_2 of them in the second row, etc. Then, the coordinates of the boxes are tuples (j, k) such that $k \leq \lambda_j$ and $j \leq l(\lambda)$, where $l(\lambda)$ is the number non-zero parts of the partition λ. Using Young diagrams, it is possible to compute the dimension of the corresponding irreducible representation, by using the so-called *hook formula* [1], also denoted as f^λ. These representations are generally of dimension greater than one except for the trivial representation labeled by the partition (n), which maps every permutation to the identity, and the sign representation (1^n), which maps every permutation to 1 or -1, depending on whether the permutation is *even* or *odd*. This is important because LME states can be obtained by explicitly defining the projectors over these one-dimensional subspaces, in the cases where the multiplicity of the trivial and sign irreducible representations is not zero. Nevertheless, constructing LME states using this method is computationally expensive, since the projectors on the invariant subspaces of tensor product representations of the symmetric group S_n depend on the $n!$ elements of the group, which makes it difficult to construct multipartite maximally entangled states for$n > 7$. This substantially limits the capacity of such approach, and restricts the possibility of exploring states of large dimension.

Given the difficulties of finding maximally entangled states, and the importance of these states in quantum information, this thesis proposes a different approach for exploring maximally entangled states, which manages to reduce the computational cost of this processs. The fact that both the irreducible representations of the symmetric group S_n and its subgroup the alternating group $\mathcal{A}_n$ are labeled with partitions of n is used here to reduce the dimensionality of the problem of constructing LME states, in the cases that the partitions of n are *self-adjoint* [28]. This occurs if transposing the boxes of its corresponding Young diagram about its main diagonal, composed of the boxes in coordinates (k, j) where $k = j$, leaves the Young diagram invariant. Then, if λ corresponds to a self-adjoint diagram with dimension f^λ, as calculated using the so-called hook formula for the irreducible representations of the symmetric group, the representation $[\lambda]$ can be interpreted as the direct sum of two irreducible representations under $\mathcal{A}_n$. This occurs because every self-adjoint representation $[\lambda]$ splits into two irreducible representations of dimension $f^\lambda/2$ under the alternating group, i.e. $[\lambda] \downarrow \mathcal{A}_n = [\lambda^+] \oplus [\lambda^-]$. It is shown in the present document that multipartite LME states of multiple qudits with $d = f^\lambda$ made from tensor products of multiple copies self-adjoint representations labeled by some partition λ can be obtained from tensor products of the irreducible representations $[\lambda^\pm] \downarrow \mathcal{A}_n$, i.e., from qudit states of dimension $d = f^\lambda/2$, which can be used to correct errors in one qudit. Moreover, it is shown here that some of these states are symmetric under permutations of the qudits, which is a property that can be used to perform quantum information tasks using identical bosons, such as quantum protocols in linear optics.

In order to present the results concerning the construction of maximally entangled states from tensor product representations of symmetric and alternating groups, this document is organized as follows: In the second chapter, some fundamental definitions and axioms of group representation theory are presented, as well as the irreducible representations of the symmetric group S_n and the alternating group $\mathcal{A}_n$. After that, in the third chapter, maximally entangled states, as well as their relevance in some quantum information protocols is introduced. In the fourth chapter, a method of constructing maximally entangled states is presented, using tensor products of irreducible representations of the symmetric group. Subsequently, in chapter 5, the connection between self adjoint irreducible representations of the symmetric group $S_3$and the alternating group $\mathcal{A}_n$ with maximally entangled states of qubits is presented, as well as constructions of some of these states. Finally in the sixth chapter, generalization of constructions of maximally entangled qudit states for any symmetric and alternating group is shown.

2. Representation Theory and Symmetric group

The main purpose of this chapter is to expose the basic axioms and concepts of group representation theory, and to introduce an algorithm to construct irreducible representations of the symmetric group S_n. Firstly, the concept of group and group action is presented. Secondly, representations of groups are introduced, as well as some fundamental tools relating to it, such as Schur-Weyl duality and tensor products of representations. These definitions will be useful to construct the ordinary matrix representations of symmetric group, which shall later serve the purposes of this thesis.

2.1 Groups and group action

A set G equipped with an operation $\cdot$ is a group if it satisfies the following conditions[32]:

- **Closure**: For any $a, b \in G$, $a \cdot b$ is also an element of G.
- **Associativity**: For any $a, b, c \in G$ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- **Identity element**: There is an unique element e such that $a \cdot e = e \cdot a = a$, for every $a \in G$
- **Inverse element**: For every $a \in G$ there exists an element $b \in G$ such that $a \cdot b = e$. This element is frequently denoted as a^{-1}.

The operation $\cdot$ is called *group multiplication*, and the multiplication of two elements $a, b \in G$ will be simplified as ab henceforth.

Now let G be a group and M a set. Then the (left) *action* of G on M is a function ψ

$$\psi : G \times M \longrightarrow M$$
$$(g, m) \longrightarrow \psi((g, m)),$$

if it satisfies the following conditions:

(i) For every $m \in M$,$\psi(e, m) = m$,where e is the identity in G.

(ii) For every $a, b \in G$ and $m \in M$,$\psi(ab, m) = \psi(a, \psi(bm))$

If the group G consists of a finite number of elements, then G is a finite group and the cardinality of its underlying set is called the *order* of the group, or $|G|$. Finite groups can be obtained from a so called *generating set*. This set refers to a subset of the group from which any other element of the group can be obtained by finitely combinating the elements of this subset, under the group multiplication.

Defining equivalence classes is useful in group theory. Two elements g and g' are equivalent under conjugacy if there exists another element $x \in G$ such that $g' = xgx^{-1}$. Conjugacy classes are important because there exist functions that are constant for all the elements of such classes, which are called *class functions*.

2.2 Representations of groups

A vector space is a set equipped with an inner product. The action of any group on a vector space V, or the *representation* of a group G can always be defined. A representation D of a finite group G on a vector space V is a homomorphism from G to the general linear $GL(V)$ acting on V [30] :

$$\begin{aligned} D : G &\longrightarrow GL(V) \\ g &\longrightarrow D(g), \end{aligned}$$

where $D(g)$ is a linear operator on V. Thus, an explicit construction of such operators allows to define the action of the group on V. In this sense, representations preserve a group structure:

$$D(g_1, g_2) = D(g_1)D(g_2) \quad \forall g_1, g_2 \in G. \tag{2.1}$$

In group representation theory there is always a *trivial* representation, which maps every element $g \in G$ to the identity matrix in V ($g \longrightarrow 1_V$).

The following are some relevant definition and theorems relating representations of groups.

Definition: Let $D(g)$ be a representation of a group G in a vector space V, equipped with an arbitrary inner product$\langle , \rangle$. Then, $D(g)$ is **unitary** if it preserves the inner product for any $v_1, v_2 \in V$ [29]:

$$\langle v_1, v_2 \rangle = \langle D(g)v_1, D(g)v_2 \rangle \quad \forall g \in G. \tag{2.2}$$

Definition: Two representations D_1 and D_2 of a group G are equivalent if there exist an isomorphism $T : V_1 \longrightarrow V_2$ such that:

$$T \circ D_1(g) = D_2(g)T \quad \forall g \in G. \tag{2.3}$$

Definition: Let $D(g)$ be a linear representation of a group in a vector space V and let W be a subspace of V ($W \leq V$). If $D(g)w \in W$ for every $w \in W$ and $g \in G$, then W is an *invariant* subspace of V [29].

Definition: A representation $D(g)$ on V is *irreducible* if its only invariant subspaces are 0 and V itself. Otherwise, the representation is *reducible* and it can be decomposed as *direct sums* of irreducible representations. A direct sum of two representations D_1 and D_2 associated to vector spaces V_1 and V_2 respectively is represented as $D_1 \oplus D_2$ and can be interpreted as a block matrix:

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix},$$

where the block D_1 is a matrix of dimension $dim(V_1)$ and D_2 is a matrix of dimension $dim(D_2)$. Thus, any reducible representation D in V can be decomposed in direct sums of invariant spaces V_is such that $V = V_1 \oplus V_2 ... \oplus V_k$.

Theorem: Orthogonality relations [16]: Let $D^{(\alpha)}$ and $D^{(\beta)}$ be two irreducible representations of some group G. Then:

$$\sum_{g \in G} D^{(\alpha)}_{il}(g) D^{\beta *}_{jm}(g) = \frac{|G|}{n_a} \delta_{\alpha\beta} \delta_{lm} \delta ij, \tag{2.4}$$

where D^{α}_{il} specifies the (i,l) entries of matrix representation D^{α}, and n_α the dimension of such representation.

Definition: Let D_1 and D_2 be two representations of a group in vector spaces V_1 and V_2, respectively, with dimensions $dim(V_1) = n_1$ and $dim(V_2) = n_2$. Then, the *tensor product* $D_1 \otimes D_2$ is a representation of G in $V_1 \otimes V_2$ with dimension $n_1 \times n_2$. In general, tensor products of irreducible representations decompose as direct sums of other representations:

$$D_\alpha \otimes D_\beta ... \otimes D_\nu = \bigoplus_\lambda \mathbb{C}^{g_{\alpha\beta...\nu\lambda}} \otimes D_\lambda, \tag{2.5}$$

where $g_{\alpha\beta...\nu\lambda}$ is the *multiplicity* of the representation labeled by λ, i.e., the number of times the irreducible representation λ appears in the decomposition of the tensor product. This means that tensor products of irreducible representations are generally reducible.

One of the main results of group representations theory is Schur's Lemma, which is introduced in the next section.

2.3 Schur's Lemma

Definition: Two representations $D_1(g)$ an $D_2(g)$ on vector spaces V_1 and V_2 respectively are **equivalent** if and only if there is a map $T : V_1 \longrightarrow V_2$ such that $T \circ D_1(g) = D_2(g) \circ T$ for all $g \in G$. Thus, if D_1 and D_2 are equivalent, the following commutative diagram is satisfied:

$$\begin{array}{ccc} V_1 & \xrightarrow{D_1(g)} & V_1 \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ V_2 & \xrightarrow{D_2(g)} & V_2. \end{array}$$

Schur's Lemma[32]: Let $D_1 : G \longrightarrow GL(V_1)$ and $D_2 : G \longrightarrow GL(V_2)$ be two irreducible representations of a group and let T be a homomorphism from V_1 to V_2. Thus:

(i) If $T \neq 0$, then $D_1 \sim D_2$

(ii) If $D_1 = D_2$, then T is a scalar multiple of the identity 1_V.

Thus, any operator that commutes with every irreducible representation of a group is proportional to the identity operator. This result is fundamental in Quantum Mechanics, since operators satisfying this property serve diverse purposes in this field, as it will be further explored in this document.

2.4 Characters of representations

Let D be a representation of a group G. The *character* χ of D is defined as the trace of the representation matrix:

$$\begin{aligned} D : G &\longrightarrow \mathbb{C} \\ g &\longrightarrow \chi(g). \end{aligned}$$

Let D_1 and D_2 be two representations of some group in vector spaces V_1 and V_2, and let χ_1 and χ_2 be its characters. Since characters are functions defined on the elements of a group, it is natural to introduce the notion of inner product of two such functions χ_1 and χ_2 as[30]:

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_g \chi_1(g) \chi_2^{\dagger}(g), \tag{2.6}$$

where $\chi_2^{\dagger}(g)$ refers to the conjugate of the character $\chi_2^{\dagger}(g)$, sinnce characters are complex numbers.

Characters have special properties:

- **Class functions**: Characters are constant under group conjugation, i.e., $\chi(h) = \chi(ghg^{-1})$ $\forall g, h \in G$. This occurs because the trace of a matrix is cyclic and then $\chi(ghg^{-1}) = \chi(gg^{-1}h) = \chi(h)$.

- **Direct sums of representations**: Let D_1 and D_2 be two representations of G on a vector space V. Then, $\chi^{D_1 \oplus D_2} = \chi^{D_1} + \chi^{D_2}$.

- **Equivalent representations**: If $\chi^{D_1} = \chi^{D_2}$, then $D_1 \sim D_2$, i.e., D_1 and D_2 are equivalent.

- **Inner products of inequivalent characters**: If χ_1 and χ_2 are the characters of two irreducible and inequivalent representations D_1 and D_2, then $\langle \chi_1, \chi_2 \rangle = 0$.

- **Irreducible representations**: If $D(g)$ is an irreducible representation of G, then its character satisfies $\langle \chi, \chi \rangle = 1$.

The character of any reducible representation D decomposes as the sum of characters of irreducible representations D_i, while taking into account their multiplicities m_i ($\chi_D = m_1\chi_1 + m_2\chi_2...$), i.e., the number of times the representation D_i appears in the direct sum decomposition of D. Since tensor products of representations are reducible, the previous properties of characters provide a tool to determine the multiplicities of irreducible representations in such products. This means that the following statement is true:

$$\langle \chi_i, \chi_D \rangle = m_i = \text{Multiplicity of irreducible representation } D_i \text{ in } D. \tag{2.7}$$

Using the fact that $\langle \chi_i, \chi_j \rangle = \delta_{i,j}$ and the previous equation it is straightforward to see that the squared character of a reducible D is the sum of the squares of the multiplicities of the irreducible representations it decomposes in:

$$\langle \chi_D, \chi_D \rangle = \chi_D^2 = m_1^2 + m_2^2... \quad . \tag{2.8}$$

Since the identity element e in a group G maps all the elements of the group to themselves, the matrix representation of e in a given vector space V is the $dim(V)$-dimensional identity operator. Then, the character of the element e yields the dimension of the vector space V.

In group representation theory it is important to determine how many inequivalent irreducible representations there exist. This number can be obtained from the properties of the characters of irreducible representations. These characters are orthonormal, since these representations are inequivalent. In fact, they can be shown to generate an orthonormal basis for the space of class functions. The latter means that the dimension of this space coincides with the number of inequivalent irreducible representations, which confirms the given statement. A more detailed proof can be found in [31].

Characters are usually arranged in a square table, called the *character table*. Here, each entry (i, j) of the table corresponds to the character of the ith conjugacy class in the jth irreducible representation.

The finite group of interest in the current thesis is the symmetric group S_n of order n. Since irreducible representations and conjugacy classes of this group are labeled by *partitions* of n, this notion is defined in the following section. The group action, its irreducible representations and character tables are presented in subsequent sections, as well as its connection with Quantum Information Theory.

2.5 Partitions

A partition λ is a sequence of numbers arranged in a non-increasing order:

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_r, ...),$$

where $\lambda_1 \geq \lambda_2 ... \geq \lambda_r, ...$ If the *parts* λ_i satisfy $\sum_i \lambda_i = n$ for some $n \in \mathbb{N}$, then λ *is a partition of* n, which is denoted as:

$$\lambda \vdash n.$$

The number of non-zero parts of the partition is the *length* of λ, also denoted as $l(\lambda)$.

The partitions of a number n are frequently arranged in a *reverse lexicographic ordering.* In this arrangement, λ precedes ν, denoted as $\lambda > \nu$, if and only if the first non-zero difference $\lambda_i - \nu_i$ is positive. For instance, under this convention the partitions of 4 are ordered as follows:

$$(4), \quad (3,1), \quad (2,2), \quad (1^4).$$

2.6 The symmetric group

The symmetric group S_n of order n is the group of all possible $n!$ permutations of n objects. A permutation is defined as a bijection from an ordered set C into itself, where the object in position i is switched to position $\pi(i)$ as follows:

$$\pi : C \longrightarrow C$$
$$[1, 2, 3, ..., n] \longrightarrow [\pi(1), \pi(2), \pi(3), ..., \pi(n)].$$

The identity element of S_n is denoted as $[1, 2, ..., n]$.

It is possible to describe any permutation in terms of products of cycles. A permutation $\pi \in S_n$ is said to be *cyclic* if there are integers $i_1, i_2, ...i_k$ such that $\pi(i_1) = i_2$, $\pi(i_2) = i_3$,..., $\pi(i_k) = i_1$. Such cycle is denoted as $(i_1, i_2, ..., i_k)$.

Two cyclic permutations π and π' are *disjoint* if none of the integers in the cycle π is repeated in the cycle π'. This definition is useful because it is common to find any permutation written as a product of disjoint cycles. For example, the permutation $\sigma = [5, 3, 2, 4, 7, 6, 1] \in S_7$ can be visualized as the product of cycles $(5, 1, 7)$, $(3, 2)$,(4) and (6). Therefore,

$$\sigma = (5, 1, 7)(3, 2)(4)(6).$$

Cycles with one element can be omitted in this construction, because their positions remain unchanged. Thus, the latter permutation can also be written as $\sigma = (5, 1, 7)(3, 2)$.

The number k of elements in a cycle $(i_1, i_2, ..., i_k)$ is the *length* of the cycle. This means that the permutation σ described above has a *cycle type* consisting of the *3-cycle* $(5, 1, 7)$, the *2-cycle* $(3, 2)$, and two *1-cycles* (4) and (6).

Cycle types of S_n are important because conjugacy classes in S_n are composed of elements with the same cycle structure. Since any element of S_n can be written in terms of products of disjoint cycles, then the sum of the lengths of these cycles yield n. Therefore, partitions can be used to label these equivalence classes. Each partition $\rho \vdash n$ refers to one of these classes. For instance, this means that the permutation $[5,3,2,4,7,6,1] \in S_7$ and its conjugate elements are labeled by the partition $(3,2,1,1)$, according to its cycle structure.

The number of elements in a conjugacy class ρ of S_n, h_ρ, is given by the following formula [1]:

$$h_\rho = n! z_\rho^{-1}. \tag{2.9}$$

Here z_ρ is an integer number that can be calculated taking into account the length r of each of the cycles in the cycle structure of ρ, and the number of disjoint cycles of such lengths, given by n_r:

$$z_\rho = \prod r^{n_r} n_r!. \tag{2.10}$$

The number z_ρ satisfies $\sum_{\rho \vdash n} n! z_\rho^{-1} = 1$. Then, for S_3 one obtains that the number of elements in $\rho = (3)$ is 2, $z_{(1^3)} = 1$ and $z_{(2,1)} = 3$, which yields a total number of $3! = 6$ elements, as expected.

Any permutation can be defined as a product of transpositions, i.e., permutations that interchange two elements and keep the others fixed. Furthermore, any permutation can be expressed as products of *adjacent* transpositions $T_j = (j, j+1)$, which swap two consecutive elements j and $j+1$. Taking this into account and the condition of conjugacy for any two elements in S_n, it is possible to construct any cyclic permutation $(i_1, i_2, ..., i_k)$ using the following rules:

$$\begin{aligned} (i_1, i_2, ..., i_k) &= (i_1, i_2)(i_2, i_3)...(i_{k-1}, i_k) \\ (i, j) &= (i, ..., j-1)(j-1, j)(i, ..., j-1)^{-1}. \end{aligned} \tag{2.11}$$

Since every permutation can be defined as a product of disjoint cycles, this gives an algorithm to find all the $n!$ permutations of S_n. Moreover, the elements of this group can be generated using any transposition and a length-n cycle.

The conjugacy classes of S_n can be further classified by introducing the *sign* of a permutation $\pi \in S_n$, denoted as $sgn(\pi)$. The sign of a permutation is $+1$ if the number of transpositions that generate π is *even* and -1 if this number is *odd*. Therefore, $sgn(\pi) = 1$ indicates that the permutation π is even, and odd otherwise.

The set of all even permutations constitutes a subgroup of S_n, called the *alternating group* $\mathcal{A}_n$, since multiplying two or more of such permutations always results in an even permutation. The order of this group is half the order of S_n, i.e., $|A_n| = n!/2$. In a similar form as in S_n, every permutation in A_n can be generated from every permutation of the form $(1,2)(r, r+1)$, for $r \in \{2, 3, ...n\}$.

It is important to note that the quotient $S_n/\mathcal{A}_n$ does not constitute a subgroup of $\mathcal{A}_n$. This occurs because the multiplication of two odd permutations is always even.

2.7 Schur functions

The symmetric group S_n acts on the set of polynomials on n independent variables $x_1, x_2, ..., x_n$ by permuting the arguments. A function is said to be *symmetric* if it remains invariant under such action. Such functions are labeled by partitions $\lambda \vdash n$ and are homogeneous in their variables of order n, i.e., $f_\lambda(tx) = t^n f_\lambda(x)$ [1].

A particular set of symmetric functions, referred to as Schur functions, are important in the construction of the representations of S_n [1]. The Schur functions s_λ in k variables $x_1, ..., x_k$ and homogeneity order n labeled by partitions $\lambda \vdash_k n$ are given by [1]:

$$s_\lambda(x_1, ..., x_k) = \frac{det(x_i^{\lambda_i+k-j})_{i,j=1}^k}{det(x_i^{k-j})_{i,j=1}^k}, \tag{2.12}$$

where the denominator $det(x_i^{k-j})_{i,j=1}^k$ is called the Vandermonde determinant [1], which is denoted to $\Delta(x_1, ..., x_k)$. The term in the numerator is the determinant of the following matrix:

$$(x_i^{k-j})_{i,j}^k = \begin{bmatrix} x_1^{\lambda_1+k-1} & x_2^{\lambda_1+k-1} & ... & x_k^{\lambda_1+k-1} \\ x_1^{\lambda_1+k-2} & x_2^{\lambda_1+k-2} & ... & x_k^{\lambda_1+k-2} \\ . & ... & & \\ . & ... & & \\ 1 & 1 & ... & 1 \end{bmatrix}. \tag{2.13}$$

Note that $s_\lambda(x_1, ..., x_k) = 0$ for $l(\lambda) > k$ and that $\lambda_i = 0$ if $i > l(\lambda)$.

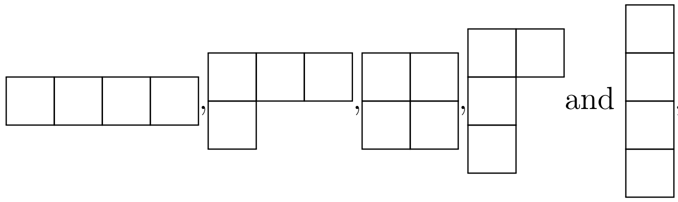
In this section the case where all k variables assume the same value 1 and $l(\lambda) \leq k$ will be reviewed. If this happens, it is possible to show using L'Hôpital rule that the Schur function s_λ yields [3]:

$$s_\lambda(1^k) = \frac{\Delta(\lambda_1 + k - 1, \lambda_2 + k - 2, ..., \lambda_k)}{\Delta(k-1, k-2, ..., 0)} \tag{2.14}$$

Schur functions are closely related to the theory of representations of the symmetric and General Linear groups. This occurs as a consequence to the fact that the irreducible representations of both groups are labeled by partitions of some integer n. The following section gives further insight on the relations between these functions and partitions.

2.8 Young diagrams

Conjugacy classes and irreducible representations of S_n are labeled with partitions $\lambda \vdash n$, i.e. $\lambda = (\lambda_1, \lambda_2, ...)$. Each partition is associated with a *Young diagram*, a collection of n boxes arranged such that λ_1 of these boxes are located in the first row, λ_2 of them in the second row, etc. To illustrate this, the Young diagrams of the partitions of 4 are shown:



which corresponds to the partitions (4), (3, 1), (2, 2), (2, 1, 1) and (1, 1, 1, 1). Notice that the Young diagrams corresponding to the partitions (4) and (1, 1, 1, 1), as well as (3, 1) and (2, 1, 1), can be obtained from each other by transposing the boxes about the main diagonal of the diagram, i.e., the diagonal composed by the entries (i, i). This means that they are *associated* or *conjugate* partitions. This is important because the dimensions of associated representations in S_n are the same.

A Young diagram with n boxes filled with integers from 1 to n is called a *Young tableau*. A Young tableau is *standard* or , also referred as *Standard Young Tableau* or SYT, if the numbers appear increasingly in every row from left to right and increasingly in every column downwards, such that no number can be repeated in any box.

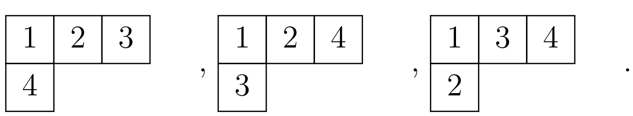
The number f^λ of SYTs is given by the hook formula [1]:

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{ij}}, \tag{2.15}$$

where h_{ij} is the *hook length* of the hook H_{ij} of the box located at the coordinates (i, j), or in other words, the number of boxes at the right of the box (i, j) plus the number of boxes under the box (i, j) plus one (the box itself). Therefore, the hook H_{ij} of the box at coordinates (i, j) consists of the boxes at coordinates (i', j') satisfying the following condition:

$$(i' = i \wedge j' \geq j) \vee (i' \geq i \wedge j' = j).$$

In order to illustrate this, let us visualize the partition (3, 1). There are 3 Standard Young tableaux corresponding to this partition, according to the equation (2.15):



The latter Young tableaux are ordered in a lexicographic order. The criteria for such order is as follows: Compare the labels of each row from left to right of different Young tableaux associated to the same partition $\lambda \vdash n$, say$[\lambda_1]$ and $[\lambda_2]$, such that the first row is compared to the second, etc. If the first filled different digit between these diagrams is greater for the tableau $[\lambda_2]$, then this tableau is said to be larger than $[\lambda_1]$. Thus, by writting the Young tableaux from the smallest to largest the so called *dictionary order* is found.

Besides the standard form, there are many ways of filling the Young diagrams. In particular, a Young tableau is *semi-standard* when the labels are filled such that they occur *non-decreasingly* from left to right in every row and *increasingly* in every column from top to bottom. This means that any label can appear more than once in a semi-standardad Young Tableau (SSYT), and that the number of different labels, say k of them, satisfies $k \leq n$.

The number of SSYTs is exactly the value of the Schur function given by $s_{(n)}(1^k)$. In fact the Schur functions can also be expressed as follows, in terms of Young Tableaux:

$$s_\lambda(x_1, x_2, ..., x_k) = \sum_T x^T = \sum_{t_i} x_1^{t_1} x_2^{t_2} ... x_k^{t_k},$$

where T refers to a semi-standard Young Tableau. Here, the sum is performed over all possible SSYTs on k labels, and the sum over t_i gives the weigth of T, i.e., the number of times the label i appears in the tableau T. For instance, the following is one of the possible 15 semi-standard Young Tableaux on 3 objects corresponding to the diagram $\lambda = (3,1) \vdash 4$:

1	1	2
3		

.

Schur functions are important since they can be used to determine the dimensions of the irreducible representations of $GL(d, \mathbb{C})$. Each of these is represented by some partition λ, and their dimensions are given by $s_\lambda(1^d)$. On the other hand, the dimensions of the irreducible representations of S_n are given by the number of SYTs f^λ, which can be calculated using equation (2.15). The fact that both representations are labeled by partitions is further reviewed in next section, where Weyl-Schur duality is introduced.

2.9 Representations of the symmetric group

2.9.1 The character table of the symmetric group

The elements of the matrix representations of the symmetric group are all real. In order to find explicit expressions for the matrix elements of the irreducible representations of S_n, it is necessary to state some facts about the characters of those representations. One of the most fundamental properties is that the dimension of some representation λ is given by the character of the identity element e. This character coincides with the number of SYTs f^λ of the Young diagram labeled by λ. Then:

$$\chi^\lambda(e) = f^\lambda. \tag{2.16}$$

The most fundamental representation of S_n is the trivial representation, labeled by the partition $[n]$. It is trivial because it maps every element of S_n to the identity in a one dimensional vector space, i.e., $\chi^\lambda(\pi) = 1$ for all $\pi \in S_n$.

Besides the trivial representation, there is always one more one-dimensional irreducible representation, known as *alternating representation*. This representation, labeled by the partition

$\lambda = (1^n)$, maps every permutation $\pi \in S_n$ to $+1$ or -1 according to its sign. Therefore:

$$S^{(1)^n}(\pi) = \chi^{(1)^n}(\pi) = sgn(\pi).$$

Another important property of the characters of S_n is that for the equivalence class given by all possible n-cycles, denoted by the partition $\rho = (n)$, the following is true:

$$\pi : C \longrightarrow C$$
$$[1, 2, 3, ..., n] \longrightarrow [\pi(1), \pi(2), \pi(3), ..., \pi(n)].$$

For instance, the following tables give the character table of the symmetric groups S_3 and S_4:

$\lambda \setminus \rho$	(3)	(2,1)	1^3
(3)	1	1	1
(2,1)	-1	0	2
(1^3)	1	-1	1

Table 2.1: Character table of S_3. Here, conjugacy classes are referred as $\rho \vdash n$ and irreducible representations are denoted by partitions $\lambda \vdash n$

.

$\lambda \setminus \rho$	(4)	(3,1)	2^2	$(2,1^2)$	e
(4)	1	1	1	1	1
(3,1)	-1	0	-1	1	3
(2^2)	0	-1	2	0	2
$(2,1^2)$	1	0	-1	-1	3
(1^4)	-1	1	1	-1	1

Table 2.2: Character table of S_4. Here, conjugacy classes are referred as $\rho \vdash n$ and irreducible representations are denoted by partitions $\lambda \vdash n$

.

From the properties of the characters of irreducible representations, which were given in section 2.4, the following orthogonality relation is always valid:

$$(n!)^{-1} \sum_{\pi \in S_n} \chi^{\lambda}(\pi)\chi^{\lambda'}(\pi) = \delta_{\lambda\lambda'}. \tag{2.17}$$

From the latter relation, and using equation (2.15), the following expression can be obtained straightforwardly:

$$\sum_{\lambda \vdash n} (f^{\lambda})^2 = \sum_{\lambda \vdash n} \chi^{\lambda}(e)\chi^{\lambda}(e) = n! \tag{2.18}$$

and since $\sum_{\lambda\vdash n}(f^\lambda)^2/n! = 1$, the numbers $f^{\lambda 2}/n!$ form a distribution over the Young diagrams called the *Plancherel measure of the symmetric group* [1].

The most common choice of matrix elements for representations of the symmetric group found in literature is given by the so called *Young's orthogonal form* [19]. In this form, each irreducible representation labeled by a partition $\lambda \vdash n$ are unitary (orthogonal) and their matrix elements are constructed by identifying a basis in which the *ith* basis vector corresponds to the *ith* SYT in the dictionary order. A form of labeling and constructing these basis vectors is determined by the *Yamanouchi symbols*, which are explained in the next section.

2.9.2 The Young-Yamanouchi algorithm

The Young-Yamonuchi algorithm gives an ordered basis for irreducible representations of S_n. In this algorithm, the SYTs of a given partition $\lambda \vdash n$ are set in the dictionary order. Then, it is possible to identify any SYT by specifying in which row each integer from 1 to n appears. This labeling method generates the Yamanouchi symbols denoted by $M = (M_1, ..., M_n)$, where M_i is the row in which the *ith* label appears. Doing this gives an ordered basis e_M which can be used to generate the f^λ-dimensional irreducible matrices. For instance, this convention indicates that the basis elements for the irreducible representation for $(3, 1) \vdash 4$ are $e_{1112}, e_{1121}, e_{1211}$.

The Young-Yamanouchi algorithm operates by first establishing how the representation acts on adjacent transpositions, and then constructing the representations of the other permutations following the steps established in section 2.6. Since every adjacent transposition $T_r := (r, r+1)$ in S_n interchanges the positions of the r and $r+1$ elements in $\{1, ..., n\}$, this group acts likewise on the labels M_i composing the Yamanouchi symbol M of some SYT. The transposition operation gives another tableau labeled by a symbol M', which does not necessarily represent a SYT.

The matrix elements of an irreducible representation acting on a permutation T_r are defined in terms of the *axial distances* between two adjacent labels r and $r+1$, for every $r \in \{1, ..., n-1\}$. For a given SYT labeled by the Yamanouchi symbol M, the axial distance from label x to label y $\rho_M(x, y)$ is defined as follows:

$$\rho_M(x, y) = \rho_M((i_x, j_x), (i_y, j_y)) = (i_y - i_x) + (j_x - j_y), \tag{2.19}$$

where (i_x, j_x) and (i_y, j_y) are the coordinates of labels x and y. If $a_{j,M}$ is the inverse of the axial distance between labels r and $r + 1$, i.e., $a_{j,M} = 1/\rho_M(j + 1, j)$, then the Young Yamanouchi algorithm establishes that the matrix elements of an irreducible representation labeled by the partition λ acting on a permutation T_r can be obtained as follows:

$$S_\lambda(T_j)e_M = a_{j,M}e_M + \sqrt{1 - a_{j,M}^2}e_{T_j(M)}. \tag{2.20}$$

Since the representations of the transpositions can be obtained using this formula, the representations of the remaining permutations can also be obtained, as stated before. For instance, this algorithm can be used to determine the matrix elements associated to the representation of S_3 labeled by the partition $(2,1) \vdash 3$. Its corresponding SYTs, in dictionary order, are the following:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad .$$

Using equation (2.20) and the latter SYTs, which can be labeled with the Yamanouchi symbols (1,1,2) and (1,2,1), respectively, the following matrices are obtained for the transpositions T_1 and T_2:

$$S_{(2,1)}(T_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.21}$$

and

$$S_{(2,1)}(T_2) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}. \tag{2.22}$$

Using these transpositions, the remaining representations of S_3 can be obtained.

For the purposes of the current thesis, the Young-Yamanouchi algorithm was implemented using PYTHON. This code is shown in Appendix A.

The explicit form of the matrix elements of the irreducible representations of the symmetric group can be used to decompose any tensor product space in terms of these representations. This is reviewed in the next section, where Weyl-Schur duality is explored, which also takes account the fact that irreducible representations of the General Linear group can also be labeled by partitions of some integer number n.

2.10 Weyl-Schur duality

Quantum systems of dimension d are described in Hilbert spaces $\mathcal{H} = \mathbb{C}^d$, defined as vector spaces over the complex field $\mathbb{C}$. A common problem in quantum information theory is to describe a bigger system which includes N copies of such d-dimensional subsystems. Then the complete Hilbert space containing the N copies is described as the N-fold tensor product of $\mathcal{H}$, or $(C^d)^{\otimes N}$.

Consider a tensor product of vectors in $\mathcal{H}^{\otimes N}$: $\psi_1 \otimes \psi_2 \otimes ... \otimes \psi_N$. Then, the symmetric group S_N acts on this object by permuting the factors of the tensor product:

$$\begin{aligned} \pi : \mathcal{H}^{\otimes N} &\longrightarrow \mathcal{H}^{\otimes \backslash} \\ \psi_1 \otimes \psi_2 \otimes ... \otimes n &\longrightarrow \psi_{\pi(1)} \otimes \psi_{\pi(2)} \otimes ... \otimes \psi_{\pi(n)}. \end{aligned}$$

The action of S_N on $\mathcal{H}^{\otimes N}$ yields a representation of S_N called the *index permutation matrix* U_π, which permutes the labels of the subsystems in the composite state $\psi_1 \otimes \psi_2 \otimes .. \otimes \psi_N$. This means that the space of rank-N tensors serves as representation space for S_N, since $U_\pi U_{\pi'} = U_{\pi\pi'}$. The representation U_π is a reducible representation of S_n.

$\mathcal{H}^{\otimes N}$ is also a representation space for the group of all linear and invertible transformations, the General Linear Group $GL(d,\mathbb{C})$. Let $A \in GL(d,\mathbb{C})$ be a d-dimensional non-singular transformation on $\mathcal{H} : \psi \longrightarrow A\psi$. Then A induces the following transformation in the N-fold tensor product of vectors in $\mathcal{H}$:

$$\psi^{(N)} \longrightarrow A^{\otimes N}\psi^{(N)}, \quad \forall \psi^{(N)} \in \mathcal{H}^{\otimes N},$$

where the N-fold tensor product of A is also a representation of $GL(d,\mathbb{C})$, since $A^{\otimes N}B^{\otimes N} = (AB)^{\otimes n}$. $A^{\otimes N}$ is known as a *polynomial representation*, because its matrix elements are polynomials of the matrix elements of A.

The representations of the groups $GL(d,C)$ and S_N in the tensor product space $\mathcal{H}^{\otimes N}$ are each other commutant, which means that $A^{\otimes N}U_\pi = A^{\otimes N}U_\pi$. This implies that any operator defined on $\mathcal{H}^{\otimes N}$ that commutes with every $A^{\otimes N}$, for all $A \in GL(d,C)$, is a linear combination of index permutation matrices $U_\pi(\mathcal{H})$, and vice versa. Their commutant relation is called the *Weyl-Schur duality* [1]. In fact, the Hilbert space $\mathcal{H}^{\otimes N}$ decomposes as:

$$\mathcal{H}^{\otimes N} = \bigoplus_{\lambda \vdash_d N} \mathcal{R}_\lambda \otimes \mathcal{S}_\lambda.$$

Such a decomposition of $\mathcal{H}^{\otimes N}$ is called the *Weddeburn decomposition*. Here, $\lambda \vdash_d N$ stands for partitions of N of maximum length d, and $\mathcal{R}_\lambda$, $\mathcal{S}_\lambda$ corresponds to irreducible representations of $GL(d,C)$ and S_N respectively. Both representations are labeled by Young tableaux of λ.

The reducible representations U_π and $A^{\otimes N}$ of the Hilbert space $\mathcal{H}^{\otimes N}$ can be decomposed in terms of irreducible representations of S_N and $GL(d,\mathbb{C})$ as follows:

$$U_\pi = \bigoplus_{\lambda \vdash N} 1_{r_\lambda} \otimes S_\lambda(\pi)$$
$$A = \bigoplus_{\lambda \vdash N} R_\lambda(A) \otimes 1_{s_\lambda},$$

and since the space $\mathcal{H}^{\otimes N}$ is fully described by those two representations, it is possible to obtain the following relation:

$$U_\pi A^{\otimes N} = A^{\otimes N} U_\pi = \bigoplus_{\lambda \vdash N} R_\lambda(A) \otimes S_\lambda(\pi)$$

From the previous equations, it is straightforward to see that the dimensions of the representations of $GL(d,\mathbb{C})$ and S_N in this rank-N tensor product space, denoted by r_λ and s_λ respectively, are given by the traces of the corresponding identity representations. In general, for any

$A \in GL(d, \mathbb{C})$ and $\pi \in S_N$, one obtains the following simple characters:

$$tr(R_\lambda(A)) = s_\lambda(a_1, ..., a_d)$$
$$tr(S_\lambda(\pi)) = \chi^\lambda(\pi),$$

where $\{a_1, ..., a_d\}$ is the set of eigenvalues of the matrix A. When applied to the identity elements of $GL(d, \mathbb{C})$ and S_N, these characters yield the dimension of the respective subspaces:

$$r_\lambda := tr(R_\lambda(1_d)) = s_\lambda(1^{\times d})$$
$$s_\lambda := tr(S_\lambda(e)) = f^\lambda,$$

where f^λ is the number of Standard Young Tableaux and $s_\lambda(1^d)$ the number of semi-standard Young Tableaux of d objects.

The identity matrices appearing in the decompositions of both U_π and $A^{\otimes N}$ indicate that each irreducible representation occurs with some multiplicity. Thus, r_λ is both the dimension of the invariant subspace $\mathcal{R}_\lambda$ and the multiplicity of the irreducible representation $\mathcal{S}_\lambda$ in the representation P_π. Similarly, s_λ is the dimension of the matrix $S_\lambda(\pi)$ and also the multiplicity of the irreducible representation $R_\lambda(A)$. Then, any tensor product space of the form $\mathcal{H}^{\otimes N}$ can be visualized as a direct sum of irreducible subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$. These subspaces can be further reduced into direct sums of $s(\lambda)$ subspaces of dimension $r(\lambda)$ which are invariant under the action of N-fold tensor product of matrices $A \in GL(d, \mathbb{C})$, but that are not invariant under the action of the symmetric group S_N. These subspaces are commonly known as the *symmetry classes* of $\mathcal{H}^{\otimes N}$. These classes are important in physics, because they are used to classify systems of N identical particles according to how they transform under permutations of the symmetric group. For instance, the most used of such classes are the *totally symmetric* and *totally antisymmetric* classes, which are labeled by partitions $\lambda = (N)$ and $\lambda = (1^N)$, respectively. The wave function of a system of N identical bosons remain invariant under any permutation in S_N, whereas the wave function of a system consisting of N identical fermions is antisymmetric under such action, i.e., its sign depends on whether the permutation is even or odd.

The next section provides a form of obtaining the projectors on the λ subspaces of a N-fold tensor product space.

2.11 Projectors on invariant subspaces of $\mathcal{H}^{\otimes N}$

Quantum states lying in the subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$ of the N-rank tensor product space $\mathcal{H}^{\otimes N}$ serve diverse purposes in quantum information theory. In this sense, it is important to construct projectors on such subspaces. These projectors, referred as P^λ, can be expressed in terms of the index permutation matrices U_π:

$$P^\lambda = (n!)^{-1} f^\lambda \sum_{\pi \in S_N} \chi^\lambda(\pi) U_\pi. \tag{2.23}$$

These projectors are orthogonal, idempotent and add up to the identity in the full $\mathcal{H}^{\otimes N}$ space:

- $P^\lambda P^{\lambda'} = \delta_{\lambda\lambda'} P^\lambda$.
- $\sum_{\lambda \vdash N} P^\lambda = 1$.
- $Tr(P^\lambda) = f^\lambda s_\lambda(1^{\times d})$ (dimension of ithe irreducible subspace λ).

The representation U_π decomposes as a direct sum of irreducible representations of the symmetric group, each with a certain multiplicity $r_\lambda = s_\lambda(1^d)$. Then, the projector can be used to decompose irreducibly any state under any permutation of the indices of the N-fold tensor product space. This means that any state can be interpreted as a linear combination of states in diverse symmetry classes, which are labeled by partitions of n.

2.12 Young projectors

The projectors P_λ described in the previous section map a vector to the irreducible subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$ of $\mathcal{H}^{\otimes N}$. Every subspace $\mathcal{R}_\lambda$ appears with a multiplicity f^λ, as derived from the Schur-Weyl decomposition. This means that the subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$ can be further reduced into direct sums of f^λ subspaces of dimension $s_\lambda(1^d)$ which are invariant under tensor products of elements of $GL(d,\mathbb{C})$ but are no longer invariant under P_π, i.e., $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda = \oplus \mathcal{R}_{\lambda,a}$, where the label a refers to each of the f^λ Young tableaux. Each of these subspaces constitute the *symmetry classes* of $\mathcal{H}^{\otimes N}$, each one labeled by a standard Young tableau of the partition λ. The projectors defined on these subspaces are called the *Young projectors*.

Let $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda = \oplus_a \mathcal{R}_{\lambda,a}$, where a labels each of the SYTs of λ. It can be shown that the Young projectors $P^{\lambda,a}$ can be constructed from the projectors on the totally symmetric $P^{(n)}$ and totally antisymmetric subspaces $P^{(1^n)}$ of $\mathcal{H}^{\otimes k}$, for $1 \leq k \leq n$. Using the shorthand notation S^k and A^k for these subspaces, the following expressions are obtained:

$$S^k := k! P^{(k)} = \sum_{\pi \in S_k} U_\pi,$$

$$A^k := k! P^{(1^k)} = \sum_{\pi \in S_k} (-1^{sgn(\pi)}) U_\pi.$$

In terms of S^k and A^k the Young projectors $P^{\lambda,a}$ can be defined as:

$$P^{\lambda,a} = \frac{f^\lambda}{n!} \prod_{k\in Col(\lambda,a)} A^k \prod_{l\in Row(\lambda,a)} S^l, \tag{2.24}$$

where the term $Col(\lambda, a)$ refers to the set of columns in tableau (λ, a), and the term $Row(\lambda, a)$ is the set of rows in that tableau. Then, the second factor in the expression indicates the product of one A^k per column in the diagram λ, and each A^k acts on the indices of the corresponding column, which means that the column is anti-symmetrised. On the other hand, the third factor indicates that the product is performed in one S^l per row in the diagram λ, and each S^l operates on the indices contained in the boxes of its row, symmetrising the indices. Define the following set of operators:

$$P^{\lambda,ij} = \frac{f^\lambda}{n!} \sum_{\pi\in S_n} (S_\lambda(\pi))_{ij} U_\pi,$$

where the indices i and j run through the f^λ SYTs for the Young diagram λ. By this means, one finds that the Young projectors $P^{\lambda,a}$ are related to the projectors P^λ as $\sum P^{\lambda,a} = P^\lambda$ which means that they also can be expressed in a simpler form:

$$P^\lambda = \sum_i P^{\lambda,ii} = \frac{f^\lambda}{n!} \sum_{\pi\in S_n} (S_\lambda(\pi))_{ii} U_\pi, \tag{2.25}$$

where $S_\lambda(\pi)_{ij}$ is the (i, j) entry of the permutation π in the irreducible representation λ. This means that the construction of these projectors can be explicitly obtained through the Young-Yamanouchi algorithm.

2.13 Tensor products of irreducible representations of the symmetric group

The permutation matrices U_π are not the only representations of S_n on Hilbert spaces. If α, β,..., ν is a set of irreducible representations of S_n, then their tensor product is a reducible representation of this group. The action of the symmetric group in this case differs from the permutation matrix U_π in the sense that here the symmetric group acts on each of the spaces $\alpha, \beta, ..., \nu$, whereas U_π acts by permuting each of the $\mathcal{H}^i$ that compose the N-fold tensor product space. Tensor products of representations decompose as direct sums of other irreducible representations of S_n, as shown in section 2.2. Then, it is possible to construct projectors of the irreducible subspaces of tensor products of representations of S_n as follows:

$$P^\lambda(S_\alpha \otimes S_\beta \otimes ... \otimes S_\nu) = (n!)^{-1} f^\lambda \sum_{\pi\in S_n} \chi^\lambda(\pi) S_\alpha(\pi) \otimes S_\beta(\pi) ... \otimes S_\nu(\pi). \tag{2.26}$$

As seen in section 2.2, the tensor product of irreducible subspaces splits in a direct sum of other irreducible subspaces as $[\alpha] \otimes [\beta] = \underset{\gamma}{\oplus} [\gamma] \otimes \mathbb{C}^{g_{\alpha\beta\gamma}} := \underset{\gamma}{\oplus} g_{\alpha\beta\gamma}[\gamma]$, where each subspace $[\nu]$ occurs with some multiplicity $g_{\alpha\beta\ldots\nu}$, commonly known as the *kronecker coefficient*, given by:

$$g_{\alpha\beta\gamma\ldots\lambda} := (n!)^{-1} \sum_{\pi \in S_n} \chi^{\alpha}(\pi)\chi^{\beta}(\pi)\chi^{\gamma}(\pi)\ldots\chi^{\lambda}(\pi). \tag{2.27}$$

As a matter of fact, there is no known combinatorial construction of this coefficient[24]. Such a task is an open problem in combinatorics and group theory.

Since the character of the trivial representation $\lambda = (n)$ equals 1 for every permutation in S_n and since this representation is one-dimensional, the projector on this subspace in a tensor product space simplifies as:

$$P^{(n)}((S_\alpha \otimes S_\beta \otimes \ldots \otimes S_\nu) = (n!)^{-1} \sum_{\pi \in S_n} S_\alpha(\pi) \otimes S_\beta(\pi)\ldots \otimes S_\nu(\pi). \tag{2.28}$$

Similarly, the Kronecker coefficient $g_{\alpha\beta\ldots\nu\lambda}$ simplifies as $g_{\alpha\beta\ldots\nu}$ for the trivial irreducible representation of S_n. The definition of the projector on this one-dimensional subspace is of paramount importance in terms of fulfilling the purposes of this thesis, since this mapping can generate multipartite quantum states whose local density operators satisfy the Schur's Lemma, which are *maximally entangled states*. A similar statement can be made about projectors defined on the alternating subspace, labeled by the representation $\lambda = (1^n)$.

2.14 Representations of the Alternating group $\mathcal{A}_n$

After specifying how to construct explicit expressions for the matrix components of irreducible representations of S_n it is interesting to determine whether a similar construction is achievable for the alternating group $\mathcal{A}_n$. It turns out that these representations are also labeled by partitions of n [12]. Let $[\alpha]$ denote a Young diagram for a irreducible representation of S_n, and denote with $[\alpha']$ its associated diagram, i.e., the diagram obtained by transposing the boxes about its main diagonal. Then, $[\alpha]$ and $[\alpha']$ are equivalent under $\mathcal{A}_n$:

$$[\alpha] \downarrow \mathcal{A}_n = [\alpha'] \downarrow \mathcal{A}_n.$$

This equivalence suggests that the number of irreducible representations and conjugacy classes reduces to the half for $\mathcal{A}_n$ in comparison to S_n. Nevertheless, it might occur that reflecting some diagram $[\alpha]$ about its main diagonal leaves the diagram invariant. In such case, the representation $[\alpha]$ is said to be *self-associated*. The classification of Young diagrams on whether they are *self-associated* or not is relevant at the construction of explicit expressions for the representations of $\mathcal{A}_n$, as the following theorem shows:

Theorem [19]: Let α be some partition of n. Then:

- (a) If $\alpha \neq \alpha'$, then $[\alpha] \downarrow A_n = [\alpha'] \downarrow A_n$ is irreducible and its construction inherits from S_n.
- (b) If $\alpha = \alpha'$, then $[\alpha]$ is reducible and splits in two irreducible representations of equal dimension $f^\lambda/2$, which are denoted as $[\alpha]^+$ and $[\alpha]^-$.

This theorem implies that the number of conjugacy classes and irreducible representation is more than half the value h_ρ, given by equation 2.7. For instance, if $n = 3$, the set of irreducible representations is given by the set $\{[3] \downarrow \mathcal{A}_n = [1^3] \downarrow A_n, [2,1]^+, [2,1]^-\}$.

As the theorem above suggests, the construction of the irreducible representations of $\mathcal{A}_n$ differs from that of S_n. In fact, whereas the entries of the matrix representations of the latter are always real, for self-associated representations, their matrix elements can be complex. Thus, the construction of these representations require a special description.

It is important to notice that the sequence $h(\alpha) = (h_{11}^\alpha, h_{22}^\alpha, ..., h_{kk}^\alpha)$,containing the hook lengths of the hooks in the main diagonal of a Young diagram $[\alpha]$, is also a valid partition of n (Here k is the length of the main diagonal of the diagram, i.e. the number of boxes in the diagonal). If α is a self-associated partition of n, then the Young diagram $[\alpha]$ is symmetric with respect to its main diagonal. This means that the *arms* and *legs* of the hooks H_{ii}^α have the same lengths. Thus, the elements in the sequence $h(\alpha)$ are all different and odd.

Let $SA(n)$ be the set of all self-associated partitions of n and let $SP(n)$ be the set of all partitions whose non-zero elements are all different and odd. From the arguments stated above, it is always possible to find a bijection between these sets. And since the irreducible representations of S_n differ from those of A_n in the case that these are labeled by self-associated partitions, it is relevant to give special attention to the characters $\chi_{h(\alpha)}^{\alpha\pm}$, which are the characters of the representations $[\alpha^\pm]$ evaluated in the elements of the class $h(\alpha)$. On this respect, it is important to address the following theorem:

Theorem: If α is a self-associated partition of n, for $n > 1$, then the value of the characters $\chi^{\alpha\pm}$ of the irreducible representations are given by:

$$\begin{aligned} \chi_{h(\alpha)+}^{\alpha\pm} &= \frac{1}{2}\left(\chi_{h(\alpha)}^\alpha \pm \sqrt{\chi_{h(\alpha)}^\alpha \prod_i h_{ii}^\alpha}\right). \\ \chi_{h(\alpha)-}^{\alpha\pm} &= \frac{1}{2}\left(\chi_{h(\alpha)}^\alpha \mp \sqrt{\chi_{h(\alpha)}^\alpha \prod_i h_{ii}^\alpha}\right), \end{aligned} \tag{2.29}$$

where $\chi_{h(\alpha)}^\alpha = -1^{(n-k)/2}$ is the character of the reducible representation α in the conjugacy class $h(\alpha)$. In case the conjugacy class is not given by partitions in $SP(n)$, i.e., $\gamma \neq h(\alpha)$, the characters $\chi^{\alpha+}$ and $\chi^{\alpha-}$ are the same:

$$\chi_\gamma^{\alpha\pm} = \frac{1}{2}\chi_\gamma^\alpha. \tag{2.30}$$

A way of constructing orthogonal representations for $\mathcal{A}_n$ for self-associated Young diagrams is given by Puttaswamaiah and Robinson in [28] , and is similar to the Young-Yamanouchi algorithm. The aim in this case is to first generate the elements $(1,2)(r,r+1)$ of A_n. This construction requires first to arrange the f^α SYTs in dictionary order and to assign the first half of them to the representation $[\alpha]^+$ and the remaining half to $[\alpha]^-$. Let us associate each Young tableau t_u^α to a basis element e_M, as in the Young-Yamanouchi algorithm, and the same Young tableau t_v^α after the transposition of the labels r and $r+1$ to the element $e_{T_r(M)}$. If t_u^α lies in $[\alpha]^+$(or $[\alpha]^-$) and t_v^α also lies in $[\alpha]^+$(or $[\alpha]^-$), then the construction of the matrix elements is identical as for the representations of S_n. However, if t_u^α lies in the set $[\alpha]^+$ and t_v^α in $[\alpha]^-$, the basis element $e_{T_r}(M)$ must be associated to a tableau also lying in $[\alpha]^+$. This tableau, denoted by $t_{u_1}^\alpha$ is the one in which the part of the tableau in $n-2$ letters is *conjugate* to the same part in $t_{u_1}^\alpha$. In this particular case, setting the following quadratic 2x2 matrix:

$$\begin{bmatrix} -\rho & \epsilon\sqrt{1-\rho^2} \\ \epsilon^{-1}\sqrt{1-\rho^2} & \rho \end{bmatrix}$$

at the intersection of rows and columns corresponding to t_u^α and $t_{u_1}^\alpha$ and zeros in every other entry gives the representation of the permutation $(1,2)(r,r+1)$. Here, ϵ is a complex number given by $\epsilon = (i)^{\frac{1}{2}p_1p_2\dots p_k - 1}$ and $p_i = 2\alpha_i - (2i-1)$. Interchanging ϵ and ϵ^{-1} gives the corresponding matrix representation of $[\alpha]^-$.

2.15 Summary

This chapter emphasized in specifying the construction of the matrix elements of the irreducible representations of symmetric and alternating groups. It was shown that the representations of S_n and $\mathcal{A}_n$ could be labeled by partitions of n. In particular, it was shown that any $N-$fold tensor product space could be decomposed in terms of irreducible representations of the symmetric group S_N and the General Linear group, according to Weyl-Schur duality. This is useful because it allows to decompose any Hilbert space in terms of its corresponding symmetry classes.

It was also shown that any tensor products of irreducible representations of some group can be reduced into different irreducible subspaces. This is relevant because if there is a one-dimensional subspace in the direct sum decomposition of such tensor product, it is possible to find multipartite locally maximally entangled states within these subspaces, which will serve the purposes of this thesis.

Tensor products of representations of the symmetric and alternating groups can be used to construct maximally entangled states, which are useful for the elaboration of Quantum Error Correction codes. These protocols are reviewed in the next chapter, as well as some fundamental concepts in Quantum Information theory.

3. Quantum Entanglement and quantum operations

The main purpose of this chapter is to introduce local operations, as well as the concept of quantum entanglement. This serves to define the concept of locally maximally entangled states, which will further be explored throughout the remaining of the document. It is also shown that these states can be used to construct some quantum information protocols such as Quantum Error Correction codes and Quantum Secret Sharing schemes.

3.1 Quantum states and the density operator

The first postulate of quantum mechanics [15] establishes that any physical quantum system can be fully described by a vector in a complex vector space endowed with an inner product, often called the Hilbert space $\mathcal{H}$. Thus, the information of a quantum system in $\mathcal{H}$ at a given time is contained in a norm-1 column vector denoted as $|\phi\rangle$, such that $\langle\phi|\phi\rangle = 1$, i.e., the vector is normalized.

The simplest quantum state is the *qubit*. Its name is related to the fact that, in analogy to classical information theory, the possible results that can be obtained upon a measurement in the computational basis are $|0\rangle$ or $|1\rangle$, which are the possible states of a classical bit. However, in contrast to classical information theory, 0s and 1s are not certain, since they occur with probabilities p_0 and p_1 (satisfying $p_0 + p_1 = 1$). In this sense, generally any qubit can be written as a superposition of $|0\rangle$ and $|1\rangle$ as follows:

$$|\phi\rangle = a\,|0\rangle + b\,|1\rangle\,, \tag{3.1}$$

such that $a, b \in C$ and $p_0 = |a|^2$, $p_1 = |b|^2$. The generalization of a qubit for a system with $d > 2$ levels is commonly called a *qudit*.

A quantum system can alternatively be described using **density operators**, which used when there is uncertainty with respect to the quantum state. This means that this formalism is used to distinguish pure from mixed states. A quantum system is said to be *mixed* if, given a set of quantum states $\{|\psi_i\rangle\}_i$, each state $|\psi_i\rangle$ occurs with a given probability p_i. Then, the density operator ρ of a mixed state is given by:

$$\rho = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|\,. \tag{3.2}$$

This means that the density operator of a pure state $|\phi\rangle$ simplifies as $|\phi\rangle\,\langle\phi|$.

Every density operator satisfies the positivity condition, which means that its eigenvalues are all greater or equal than zero($\lambda \geq 0$). Besides that, its trace must be equal to 1, i.e., $\mathrm{tr}(\rho) = 1$.

A form of distinguishing a mixed from a pure state is through the purity operator, ρ^2. If $\rho^2 = \rho$, the density matrix describes a pure state, whereas if that does not happen, i.e., if

$tr(\rho^2) < 1$, then the operator describes a mixed state. The maximum degree of mixture of a quantum state occurs when each of the pure states $|\psi_i\rangle$ occurs with the same probability. In a d-dimensional quantum mixture, the density matrix reduces to $\rho = \frac{1}{d}\sum_i |\psi_i\rangle\langle\psi_i|$ and then the operation $tr(\rho^2)$ yields the minimum possible value, $tr(\rho^2) = 1/d$.

It is important in multipartite quantum systems to know the states of its constituents. For instance, if a quantum system is composed of subsystems A and B, then the state of the isolated subsystem A can be obtained by computing the *partial trace* over the subsystem B. Then, the *reduced density operator* of A is:

$$tr_B(\rho^{AB}) = \rho^A. \tag{3.3}$$

If $\{|a_i\rangle\}$ is a basis of $\mathcal{H}_A$ and $|b_i\rangle$ a basis for $\mathcal{H}_B$, then there are always coefficients c_{ijkl} such that the composite state ρ^{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as $\rho^{AB} = \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l|$. Then, the partial trace over B can be computed as follows [26]:

$$tr_B(\rho^{AB}) = \sum_{ijkl} c_{ijkl} |a_i\rangle\langle a_j| \otimes \langle b_k|a_l\rangle\,. \tag{3.4}$$

3.2 Generalized Measurements

Quantum operations are important since they model the evolution of quantum systems in time and the effect of quantum measurements in the state of such systems, among other quantum processes. For instance, if the state of a system at a time t_1 is given by ρ and it evolves to ρ' at t_2 there is always possible to find an unitary matrix U ($UU^\dagger = 1$) relating the two operators as follows:

$$\rho' = U\rho U^\dagger. \tag{3.5}$$

When quantum systems are observed, its state is changed. In this sense, quantum measurements constitute an important set of quantum operations. Since these operations affect the state ρ of a quantum system by collapsing its corresponding wave function, the collection of measurement operators $\{M_m\}$ must satisfy the completeness condition [26], i.e.:

$$\sum_m M_m^\dagger M_m = I$$
$$1 = \sum_m tr(M_m M_m^\dagger \rho),$$

where $p(m) = tr(M_m M_m^\dagger \rho)$ is the probability that outcome m occurs when a measurement is performed on the system. If m effectively occurs, then the state of the system after the measurement is given by:

$$\rho_m = \frac{M_m \rho M_m^\dagger}{tr(M_m M_m^\dagger \rho)}. \tag{3.6}$$

Positive Operator-Valued Measure formalism or POVM formalism is frequently used in quantum measurements, especially in the cases where the outcome probabilities are more relevant than the post-measurement state itself. In this case, instead of using the set of measurement operators $\{M_m\}$, one defines a positive operator $E_m = M_m^\dagger M_m$, such that $\sum_m E_m = I$ and $p(m) = \langle\psi| E_m |\psi\rangle$. Thus, the operators $\{E_m\}$ are sufficient to determine the outcome probabilities.

One special set of POVM measurements are the projective measurements. The corresponding generating operators satisfy $P_m P_{m'} = \delta_{mm'}$. This means that the probability that m occurs reduces to $p(m) = \langle\phi| P_m |\phi\rangle$. The resulting normalized state is then:

$$\rho_m = \frac{P_m \rho P_m^\dagger}{tr(P_m \rho)}$$

POVM measurements are used in *Local Operators and Classical Communication* protocols or *LOCC* protocols, which are introduced in subsequent sections.

3.3 Entanglement and Local Unitary Equivalence

In quantum mechanics and, in particular, in quantum information theory, the study of the interactions of multiple quantum systems is a common object of interest, because of its many applications in quantum computation. The space of systems consisting of N interacting constituents is described by tensor products of Hilbert spaces $\mathcal{H}_i$ ($\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$), where i labels the space corresponding to the i-th individual constituent.

If the composite state describing the N parts of the system is a pure state and can be described as a tensor product of the individual states, then the system is said to be *unentangled* [35], which means that there is no correlation between the states of its N constituents. This corresponds to having a *product state*, which is a vector of the form:

$$|\psi\rangle = \left|\psi^{(1)}\right\rangle \otimes ... \otimes \left|\psi^{(N)}\right\rangle .$$

If a multipartite system cannot be written in such form in no basis, then the pure state is *entangled*.

In the case of multipartite mixed states the notion of *separability* can be introduced to distinguish between entangled and unentangled states [35]. A N-partite mixed states is fully separable if it can be written as follows:

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes ... \otimes \rho_i^{(N)},$$

for local density matrices $\rho_i^{(j)}$ and probability distribution p. If a mixed state is not fully separable, then it is entangled.

Two entangled multi-partite states $|\psi\rangle$ and $|\phi\rangle$ are considered to be equivalently entangled if they differ by an N-fold tensor product of unitaries $\{U_i\}$. This means that $|\psi\rangle$ and $|\phi\rangle$ are local unitary equivalent (LU-equivalent) if and only if:

$$|\psi\rangle = U_1 \otimes ... \otimes U_N |\phi\rangle . \tag{3.7}$$

Since $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent, they can be used for performing the exact same Quantum Information theory tasks.

A form of determining whether two bipartite states are equivalently entangled under LU operations or not is by finding their *Schmidt decomposition.* For simplicity, say $d_1 = d_2 = d$. In this case, such decomposition indicates that for a state vector $|\psi\rangle$ there are two orthonormal bases $\{|\alpha_i\rangle\}_{i=1}^d\}$, $\{|\beta_i\rangle\}_{i=1}^d\}$, and d non-negative real numbers p_i such that:

$$|\psi\rangle = \sum_i^d \sqrt{p_i} |\alpha_i\rangle \otimes |\beta_i\rangle . \tag{3.8}$$

Then, if $|\psi\rangle$ and $|\phi\rangle$ are LU-equivalent, the state $|\phi\rangle$ can be written in terms of the Schmidt coefficients p_i of $|\psi\rangle$ as follows:

$$|\phi\rangle = \sum_i^d \sqrt{p_i}(U_1 |\alpha_i\rangle) \otimes (U_2 |\beta_i\rangle), \tag{3.9}$$

for some d-dimensional unitaries U_1 and U_2. Then, two bipartite states are LU-equivalent if and only if their Schmidt coefficients coincide. Nevertheless, this notion cannot be extended to N-partite states with $N > 2$.

3.4 Equivalence under LU and SLOCC operations

Classifying vectors according to the equivalence under LU operations is justified because it is not possible to create entanglement from local operations only. A way of finding whether two vectors are LU-equivalent is by determining a complete set of *invariants* under this operation. Any vector could be brought to a *canonical form* if all its parameters are given for every equivalence class, which is the state that can be used to distinguish inequivalent classes. Then, any state is LU equivalent to another if their canonical forms coincide . Nevertheless, the number of invariants required for a Hilbert space of N qudits is not trivial and large for any d. This is the reason why it is necessary to introduce wider notions of equivalence of multipartite states, such as *Local Unitary Equivalence and Classical Communication* operations or *LOCC*.

In a *LOCC*-protocol, N particles are held at distant laboratories. The joint state $|\phi\rangle$ of the N particles can be entangled, if the particles interacted in the past. These protocols occur in different rounds, such that in each of these, a POVM measurement is performed in one of the laboratories. Then, the result of this operation is shared between the parties through classical channels and the round finalizes. If $|\phi\rangle$ is LOCC-equivalent to another state $|\psi\rangle$ then these two can be approximated to each other after r rounds, even if r tends to infinity. Nevertheless, two states cannot be transformed into each other through LOCC protocols canonically [35].

Loosening the LOCC-equivalence condition leads to another equivalence notion, denoted as stochastic LOCC equivalence or SLOCC. Under SLOCC protocols, the transformation from a vector $|\phi\rangle$ to a vector $|\psi\rangle$ in the same class occurs with a probability that is not necessarily one. Then, two states are SLOCC equivalent if and only if there are N operators $\{A_i\}$ with unit determinant $\det A_i = 1$ (Kraus operators) and a complex number λ such that:

$$(A_1 \otimes ... \otimes A_N |\psi\rangle) = \lambda |\phi\rangle . \tag{3.10}$$

Kraus operators belong to the General Linear Group . Two vectors are SLOCC equivalent if they belong to the same $GL(C^{d_1} \times ... \times GL(C^{d_N}))$ orbit. In the $N-$qubit case, the lower bound for the number of parameters is $2^{N+1} - 6N - 2$ [35]. For larger dimensions, this lower bound also grows exponentially.

It is certainly known that any two entangled bipartite states are SLOCC-equivalent if they have the same Schmidt-rank. This fact can be used in order to explain some equivalence classes for $N = 3$. Since entanglement cannot be created through these operations, it is straightforward to see that the product states $|\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$ form a single class themselves. Furthermore, since two bipartite states are SLOCC-equivalent, the vectors of the form $|\phi_1\rangle \otimes |\psi_{2,3}\rangle$ form another equivalence class. This means that the tripartite states that can be written as a bipartition of the form $2|13$ or $3|12$ also constitute two inequivalent classes. Nevertheless, vectors which cannot be written as product states of two parts for no bipartition can be in two different entanglement classes under SLOCC, the classes labeled by the so called W and GHZ states, which as shown be Vidal and Cirac [10], are given by the following expressions:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{3.11}$$

and

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{3.12}$$

In general, multipartite states that cannot be written as a product state in no possible way are called *genuinely entangled states* and they do not necessarily constitute an unique SLOCC class, i.e., any two genuinely entangled states could belong to two inequivalent classes.

Even if the equivalence classes under SLOCC have been completely described for two, three and four qubits, as Verstraete et al. show in [33], and canonical method for classifying SLOCC-classes for qudits and even for more than 4 qubits does not currently exist, these methods are not efficient [23]. Nevertheless, defining a set of covariants under SLOCC-operations is useful in order to solve this task, and this has been achieved at least for some values of N. For instance, in the case of three qubits, the concept of *hyperdeterminant* and *tensor rank* serve these purposes [35].

Caley's hyperdeterminant is the generalization of the determinant for states in the tensor product $(C^2) \otimes (C^2) \otimes (C^2)$. In the bipartite case, consider two vectors $|\psi\rangle$ and $|\phi\rangle$ that are SLOCC equivalent, i.e., $A_1 \otimes A_2 |\psi\rangle = |\phi\rangle$, where $A_{1,2}$ are Kraus operators. Then, they can be decomposed as:

$$|\psi\rangle = \sum_{i,j=1}^{d} T_{i,j} |i\rangle \otimes |j\rangle$$

and

$$|\phi\rangle = \sum_{i,j=1}^{d} T'_{i,j} |i\rangle \otimes |j\rangle .$$

It can be shown that $\lambda T' = A_1 T A_2^T$. This means that the determinant of the matrix T is left invariant under SLOCC:

$$det(\lambda T') = det A_1 T A_2^T = det(A_1) det(T) det(A_2) = det(T).$$

For the case $N = 3$, any state vector can be written similarly:

$$|\psi\rangle = \sum_{i,j,k} \alpha_{i,j,k} |i, j, k\rangle .$$

For this case, the hyperdeterminant can be defined as [35]:

$$Det(\psi) = \alpha_{i1,j1,k1}\alpha_{i2,j2,k2}\alpha_{i3,j3,k3}\alpha_{i4,j4,k4}\epsilon_{i1,i2}\epsilon_{i3,i4}\epsilon_{j1,j2}\epsilon_{j3,j4}\epsilon_{k1,k2}\epsilon_{k3,k4},$$

where $\epsilon_{i,j}$ is the completely anti-symmetric or Levi-Civita tensor. Using this quantity it is easy to distinguish between non-equivalent 3 qubit-SLOCC classes. For example, it can be used to distinguish between the so called W and GHZ-states, whose hyperdeterminants are $Det(GHZ) = 0 \neq Det(W)$, which means that they are inequivalent classes, even if the W states can be approximated to arbitrary precision to a GHZ state [35]. In fact their equivalence classes also have different *tensor rank* or R_{min} (2 for the GHZ-class and 3 for the W-class), a quantity determines the minimal number of coefficients c_i needed to describe a state of a given equivalence class according to the decomposition:

$$|\psi\rangle = \sum_{i=1}^{R} c_i \left|\psi_i^{(1)}\right\rangle \otimes ... \otimes \left|\psi_i^{N}\right\rangle .$$

3.5 Measures of entanglement

A measure of entanglement quantifies the degree of entanglement of a multipartite pure state ρ. Any entanglement measure E must be an *entanglement monotone*[35]. This means that it must be a non-negative function ($E(\rho) \geq 0$, $E(\rho) = 0$ if the state is unentangled) and it must not increase on average under LOCC operations. If the LOCC protocol consists of POVMs operators M_j, then this means that the entanglement measure E must satisfy the following condition:

$$E(\rho) = \sum_j p_j E(\rho_j), \tag{3.13}$$

where p_j is the probability of outcome j, i.e., $p_j = tr(M_j \rho M_j^\dagger)$.

One measure of tripartite qubit entanglement is the *3-tangle*, which was introduced by Coffman, Kundu and Wooters [9]. This measure is used to distinguish between W and GHZ states. The 3-tangle is calculated as the norm of the hyperdeterminant of the state, defined in the previous section:

$$\tau_3(\psi) = |Det(\psi)| \tag{3.14}$$

One common entanglement measure is the *geometric measure of entanglement*[35]. This is an entanglement monotone and can be interpreted as the minimum distance from the state ρ to the set of product states, which is the set of unentangled states. This measure is given by:

$$E_{geometric}(\psi) = min||\rho - \sigma||_2, \tag{3.15}$$

where $|| * ||_2$ is the Hilbert-Schmidt norm, and σ is a separable state. Then, the minimum is taken from the set of all product states.

Another entanglement measure is the *Schmidt measure*. This is defined as the log of the tensor rank of a quantum state, i.e., $log(R_{min})$.

3.5.1 Entanglement Entropy

The most common methods to establish the degree of bipartite entanglement is to compute the Entanglement Entropy of a quantum density operator ρ, defined as [26]:

$$E(\psi_{AB}) = S(\rho) = -tr(\rho log \rho) \tag{3.16}$$

If the spectrum of ρ is given by the set of scalars $\{\lambda_i\}$, then the latter expression simplifies as:

$$S(\rho) = -\sum_i \lambda_i log \lambda_i. \tag{3.17}$$

In a similar fashion as in classical information theory, this measure can be interpreted as the degree of *uncertainty* of a quantum state to be obtained [26]. Then, if the density operator ρ

specifies a pure state, one finds that $\lambda = 1$ and the entanglement entropy vanishes, since there is complete certainty that the state is in a pure state. In contrast, if every λ_i is the same, it is found that $\lambda_i = 1/d$ for a d-dimensional quantum state. In this case, since $\sum_i \lambda_i = 1$, this quantity is exactly equal to the logarithm of the dimension of the state, $S(\rho) = log(d)$. Then, the density matrix corresponds to a totally mixed state. Therefore, the entanglement entropy can be described as the degree of mixture of a quantum state.

Determining whether a multipartite quantum state is more or less entangled is important since the efficiency of Quantum Information protocols depends hugely on this property. Taking this into account, a class of states that is *maximally entangled* is introduced in the next section.

3.6 Maximally Entangled States

A multipartite state of N qudits is said to be *locally maximally entangled* (LME) or totally entangled if the partial trace in each of its N parties corresponds to the density matrix of a totally mixed state. This means that the entanglement entropy maximizes for each of the parts [35]:

$$S(\rho_i) = log(d). \tag{3.18}$$

Then, the local density matrices are multiples of the identity. For instance, the GHZ state is a maximally entangled state.

Locally maximally entangled states are used as canonical forms for SLOCC orbits. This occurs in the case where the orbits are stable, i.e., if there is a non-zero multipartite state in the closure of the orbit, which turns out to be a LME state. This is a consequence of the so-called Kempf-Ness theorem [34], which states that any closed orbit under the action of a compact Lie group contains a non-zero vector of minimal length, i.e., a vector with the minimal possible norm, which in this case means that the trace of the unnormalized density operator is minimized under SLOCC operations. In order to see this, let $\rho = |\phi\rangle \langle\phi|$ and let the SLOCC operations generate an orbit of the form $SL \times SL \times ...SL$, i.e. operators with unit determinant, which belong to the *Special Linear* group. One can show that the trace can be minimized by applying iteratively the operation $\rho_i^{'} = X\rho_i X^\dagger$ to each of the local density matrices ρ_i, with $X = |det(\rho_1)|^{1/d_i}(\sqrt{\rho_1})^{-1}$, and d_i the dimension of the local party $i \in \{1, 2.., N\}$. This brings each of the local density matrices to a totally mixed state, i.e. a multiple of the identity. By proceeding this way, the following inequality holds:

$$tr(\rho') = d_i(det(\rho_1)^{1/d_1}) = tr(\rho_i), \tag{3.19}$$

with equality when ρ_i is a totally mixed state. This procedure is done iteratively for all of the N parts, and is repeated until the trace of the density operator does not diminishes further. Then, by this procedure one finds the normal form, which is an unique LME state, up to local unitary operations.

There are multipartite states that maximize entropy for more than one subsystem. In particular, a multipartite pure state of N parties that maximizes entropy after tracing any $N-k$ subsystems is said to be *k-uniform* [14]. If $k = [N/2]$ the state is *absolutely maximally entangled* or *AME*, since it is the type of state that maximizes entropy in every bipartition of the system. More precisely, an *AME(N,d)* state $|\psi\rangle \in \mathcal{H} = (C^d)^N$ with N qudits is a state such that all its reduced density matrices in any subspace $\mathcal{A} = (C^d)^{\otimes[N/2]}, \mathcal{H} = \mathcal{A} \otimes -\mathcal{A}$ carry maximal entropy [17]:

$$S(\rho_A) = \frac{N}{2} log(d). \tag{3.20}$$

As in a locally maximally entangled state, this means that for any k qudits, the reduced density matrix must be equal to a multiple of the identity:

$$\rho_k = \frac{1}{d^k} I_{d^k}, \quad \forall k \leq \frac{N}{2}. \tag{3.21}$$

Locally maximally entangled states and in general $k-$uniform states are useful to correct quantum codes. This is further explained in the next section.

3.7 Quantum Error Correction codes

The interaction between open quantum systems and its surrounding environment often make their initial states to change. This process results in *decoherence* an in the decay of the quantum information stored in a quantum computer. In classical information, errors in codes can be corrected by using redundancy. If a code consisting of a single bit $|0\rangle$ is implemented, in some environmental setting the probability that the bit flips to $|1\rangle$ is p, whereas the probability that the bit remains the same is $1-p$. Then, this probability can be diminished substantially if N copies of the same bit are arranged in the code. This is related to the fact that the probability that the N bits flip is now $(p)^N$, which rapidly tends to zero when $p < 1/2$. Nevertheless, due to the no-cloning theorem [21], which states that no machine can replicate every quantum pure state $|\psi\rangle \in \mathcal{H}$ and the fact that quantum states are changed upon measurements, the problem of quantum error correction cannot be solved by making the quantum states redundant, i.e., by making various copies of the quantum state.

Given a multipartite state of N qubits, every possible error in each of the local states can be modeled using the Pauli matrices, which form a basis for a two-dimensional complex vector space. The first of these errors is the so called *bit flip* map, which switches $|0\rangle$ by $|1\rangle$ and vice versa. A bit flip map for a qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ is represented by the X-Pauli operator:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.22}$$

On the other hand, the *phase flip* map, which changes $|\psi\rangle$ to $|\psi'\rangle = \alpha\,|0\rangle - \beta\,|1\rangle$, is represented by the Z-Pauli operator:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.23}$$

The remaining Pauli operator, Y, can be obtained by $Y = iXZ$. This means that the set of all possible errors occurring on any multipartite qubit state can be modeled with the following set of Pauli operators :

$$\{\mathbf{E}_a\} = \{I, X, Y, Z\}^{\otimes N}. \tag{3.24}$$

Given a subset $\mathcal{E} \in \{E_a\}$ of these errors, it is important to determine sufficient and necessary conditions for which these can be corrected. Knill and Laflamme [20] determined that, for some code space $\{|i\rangle\}$ these conditions can be simplified by the following expression:

$$\langle j| \mathbf{E}_b^\dagger \mathbf{E}_a |i\rangle = C_{ba}\delta_{ij} \tag{3.25}$$

where C_{ba} is the (b, a) entry of a Hermitian matrix, and $\mathbf{E}_{b,a} \in \mathcal{E}$. Then, if these conditions are satisfied, it is possible to find a protocol which corrects every error in $\mathcal{E}$.

The latter procedures can be extended for any multipartite qudit system. In this case, the error ocurring in each of the local states can be modeled using the so-called *Gellman Matrices*, which are generalizations of the Pauli matrices for greater dimensions.

In general, for a given code it is only possible to correct some subset of the total possible errors. Each error $\mathbf{E}_a$ has a weight k, i.e., the number of qubits in which it acts non-trivially. If it is possible to correct any error in at most k qubits for the code, then the code subpace of the full Hilbert space is a $(N, k, 2k + 1)$ *Quantum Error Correction* code, and the density matrices on k qubits are totally mixed states. Given the previous arguments, sets of $k-$uniform states can be used to construct $(N, k, 2k + 1)$ QEC codes. Here, the term $2k + 1$ indicates that it can detect errors in $2k + 1$

3.7.1 Quantum Secret Sharing

Secret sharing schemes are special cases of error correction codes [8]. These are important since they allow to distribute data between parties in such a way that each of these have some share of the original information, but some of them have to gather in order to retrieve the whole original data. In Quantum Secret Sharing schemes (QSS), a dealer $\mathcal{D}$ chooses one state from a set of pure states $\mathcal{X} = \{|X_1\rangle, ... |X_N\rangle\}$ and distributes it to a set $\mathcal{P}$ of players according to some access structure Γ [18], which defines the set of players that can recover the secret state. Then, each of the players has access to one *share* of the secret, which means that they posses some quantum state labeled by a density matrix ρ_P. Any set of players in the access structure Γ is capable of recovering the secret state through some protocol, whereas any individual possessing a state not in Γ cannot obtain any information. This means that the density operators ρ_P are multiple of the identity, i.e., each local state is maximally entangled with the secret.

Denote by $\mathcal{H}_i$ the Hilbert space associated to the share of the player i, and denote the space that describes the set of shares of some subset A in $\mathcal{P}$ by the tensor product $\mathcal{H}_A = \otimes_{a\in A} H_a$. Then the dealer obtains the distribution of shares through a map from the space of the secret to the space of shares as follows:

$$\Lambda_D : S(\mathcal{S}) \longrightarrow S(\mathcal{H}_1 \otimes ...\mathcal{H}_n). \tag{3.26}$$

A QSS scheme with (n, t) threshold is a scheme where n players participate and any set of p players with $p \geq t$ can always recover the original quantum state, but any set of $t-1$ or fewer players is uncapable of recovering any information. This means that a every QSS scheme is also a QEC code, but it does not indicate that every QEC code is a secret sharing scheme. This is because there are error correction codes for which less some information can be recovered accounting for less than t parties. For instance, consider the following four-qutrit LME state.

$$|\phi\rangle = \alpha(|0000\rangle + |1111\rangle) + \beta(|0011\rangle + |1100\rangle)$$

Since tracing all the qubits except for one leaves the state in a totally mixed state, it is possible to correct an error on any qubit. Nevertheless, tracing two qubits do not leave the remaining two in such a state. This means that some information can be retrieved from the secret, and it is not possible to use it as a $(3, 4)$ threshold scheme, even if it can be used as a $(4, 1, 3)$ code.

3.8 Summary

In this section, the basic notions of quantum entanglement were presented. In particular, LOCC and SLOCC equivalence were introduced, as well as maximally entangled state and some applications of these states in quantum information protocols. The construction of such states is not canonical, and given its relevance, the search of these states is of interest in quantum mechanics.

In the next chapter, given the definitions provided in this chapter and the previous one, where notions of group representations theory of the symmetric an alternating groups were introduced, a method for constructing maximally entangled states is presented. This method relies on finding the projectors on the one-dimensional subspaces of the tensor product of irreducible representations of the symmetric group, where these states can be found, according to the Schur's Lemma.

4. Maximally entangled states and the group representation theory

In this chapter a method for constructing maximally entangled states is introduced, based on tensor products of representations of the symmetric group. The method is as follows: first the projectors on invariant subspaces of reducible representations of groups are introduced. Secondly, the connection between these subspaces and maximally entangled states is presented. Finally, two particular subspaces of the symmetric group are introduced, as well as a method for finding these states, based on projectors onto these subspaces.

4.1 Projectors on irreducible subspaces

In the second chapter representations of groups were extensively studied. It was seen that any reducible representation of a group G on a carrier space V decomposes into direct sums of irreducible representations of the group, associated to distinct subspaces W_i of the vector space V, i.e., $V = W_1 \oplus W_2 \oplus ....$ Now, the emphasis relies on finding basis vectors for each of these irreducible subspaces. A way of achieving this is by defining projectors onto these subspaces. In this section, we present an explicit construction of these operators for any finite group G.

Let $|\phi\rangle$ be a state vector in a Hilbert space $\mathcal{H}$, and let $\mathbf{P}^\alpha$ be a projector onto the irreducible subspace α. Then $\mathbf{P}^\alpha|\phi\rangle$ yields the component of $|\phi\rangle$ lying in the subspace α, which will be a linear combination of some basis vectors that span this subspace, i.e., $\{|\phi^\alpha\rangle\}$. This means that, given a representation of some group G in a vector space V, $U_V(g)$, these basis vectors transform according to the irreducible representation $D^{(\alpha)}(g)$ of the group G, i.e.:

$$U_V(g)\left|\phi_i^{(\alpha)}\right\rangle = \sum_{j=1}^{n_a} D_{ij}^{(\alpha)}(g)\left|\phi_j^{(\alpha)}\right\rangle, \tag{4.1}$$

where $D_{ij}^{(\alpha)}(g)$ is the entry (i,j) of the representation matrix $D^{(\alpha)}$, and n_α is its dimension, for $g \in G$. Multiplying both sides of the latter equation by $D_{kl}^{(\beta)*}$ and using the orthogonality theorem given in equation (2.4) for irreducible representations, it is possible to obtain the following expression:

$$\sum_{g\in G} D_{kl}^{(\beta)*}(g)U_V^{(\alpha)}(g)\left|\phi_i^{(\alpha)}\right\rangle = \frac{|G|}{n_\alpha}\delta_{il}\delta_{\alpha\beta}\left|\phi_k^{(\alpha)}\right\rangle, \tag{4.2}$$

which allows to obtain the expression of the k,l-th component of the projector $\mathbf{P}^\alpha$:

$$\mathbf{P}_{kl}^\alpha = \frac{n_\alpha}{|G|}\sum_{g\in G} D_{kl}^{(\alpha)*}(g)U_V(g). \tag{4.3}$$

In order to obtain the projector on the full subspace, all components most be added:

$$\mathbf{P}^{\alpha} = \sum_{i}^{n_\alpha} \mathbb{P}_{ii}^{\alpha} = \sum_{i}^{n_\alpha} \frac{n_\alpha}{|G|} \sum_{g \in G} D_{ii}^{(\alpha)*}(g) U_V(g). \tag{4.4}$$

From the definition of a character of a representation, $\chi^{\alpha}(g) = \sum_i D_{ii}^{\alpha}$. Then, [16]:

$$\mathbb{P}^{\alpha} = \frac{n_\alpha}{|G|} \sum_{g \in G} \chi^{(\alpha)}(g) U_V(g). \tag{4.5}$$

Then, any reducible space can be decomposed into irreducible subspaces of a group $|G|$ using these projectors. The eigenvalues of these projectors are either 1s or 0s. Eigenvectors with eigenvalue 1 span each of these subspaces, i.e., finding these eigenvectors gives a basis for the irreducible subspace α. An orthonormal basis can be found by applying the Gram-Schmidt orthonormalization process on the columnn vectors of these projectors, as it will be shown in subsequent sections for tensor products of representations of the symmetric group.

4.2 One-dimensional representations and maximally entangled states

Let $D(g)$ be a one dimensional unitary representation of some group G. Given some vector $|v\rangle \in V$, then this vector transforms as follows under the action of such representation:

$$D(g) |\phi\rangle = e^{i\psi} |\phi\rangle ,$$

where ψ is a complex phase and $g \in G$. This means that any vector remains invariant up to a global phase under the action of a one-dimensional representation.

Let $[\alpha]$, $[\beta]$,...$[\nu]$ be N irreducible representations of some group G. As stated in previous sections, the tensor product of irreducible representations of a group is generally reducible, as it decomposes into a direct sum of irreducible representations $[\lambda]$, each with a certain multiplicity given by the Kronecker coefficient $g_{\alpha\beta\ldots\nu\lambda}$. If there is a one-dimensional representation in the tensor product decomposition, then the action of this representation on this subspace will leave its vector invariant,. This means that if the multiplicity of such one-dimensional subspace is not zero, then it is possible to find a vector $|\phi\rangle \in V$ that is left invariant under the action of the tensor product $[\alpha] \otimes [\beta] \otimes \ldots[\nu]$. This means that vectors in such subspaces satisfy the following condition for any $g \in G$ [29]:

$$S^{\alpha}(g) \otimes S^{\beta}(g) \otimes \ldots.S^{\nu} |\phi\rangle = \mathbf{S}(g) |\phi\rangle = e^{i\psi} |\phi\rangle . \tag{4.6}$$

Here, $S^{\alpha}(g) \otimes S^{\beta}(g)\ldots \otimes S^{\nu}(g) = \mathbf{S}(g)$. Following the latter equation, and taking the transpose conjugate, the density matrix ρ of these states can be written as follows:

$$\rho := |\psi\rangle\langle\psi| = \mathbf{S}(g)\,|\psi\rangle\langle\psi|\,\mathbf{S}^{\dagger}(g). \tag{4.7}$$

Since the representation $\mathbf{S}(g)$ of G commutes with the density operator ρ, it is straigthforward to see that for any local subsystem α, the commutation condition is also satisfied, i.e., $S^{\alpha}(\pi)\rho_{\alpha}S^{\alpha\dagger}(\pi) = \rho_{\alpha}$ [4]. Then, for every irreducible representation in the tensor product the following is true:

$$[\mathbf{D}^{\alpha}, \rho^{\alpha}] = 0.$$

Therefore, from the Schur's lemma, which was presented in section 2.3, one finds that the local density matrix ρ_{α} is a scalar multiple of the identity, which means that the system is a maximally entangled state:

$$\rho_{\alpha} = \frac{1}{d_{\alpha}} I_{d_{\alpha}}, \tag{4.8}$$

where d_{α} is the dimension of the representation α. Then, by projecting any state on such one dimensional subspaces one obtains a maximally-entangled state, up to normalization.

In the next section, the one-dimensional irreducible representations of the symmetric group are introduced.

4.3 Invariant subspaces and the symmetric group S_n

Irreducible Representations of S_n are labeled by partitions of n. If $[\alpha]$, $[\beta]$,..., $[\nu]$ are irreducible representations of S_n, then the tensor product of these representations decomposes into direct sums of some irreducible representations, labeled by some partitions $\lambda \vdash n$. In the case of the symmetric group, the Kronecker coefficient $g_{\alpha\beta\ldots\lambda}$ gives the multiplicity of $[\lambda]$ in $[\alpha]\otimes[\beta]\otimes\ldots\otimes[\nu]$:

$$g_{\alpha\beta\gamma\ldots\lambda} := (n!)^{-1} \sum_{\pi\in S_n} \chi^{\alpha}(\pi)\chi^{\beta}(\pi)\chi^{\gamma}(\pi)\ldots\chi^{\nu}(\pi)\chi^{\lambda}(\pi). \tag{4.9}$$

This means that it is possible to find a total of $g_{\alpha\beta\gamma\ldots\lambda}$ of orthogonal $[\lambda]$ subspaces of dimension f^{λ} by defining the projector $\mathbb{P}^{\lambda}$, following equation (4.6):

$$\mathbb{P}^{\lambda} = \frac{f^{\lambda}}{n!} \sum_{\pi\in S_n} \chi^{\lambda}(\pi) S^{\alpha}(\pi) \otimes S^{\beta}(\pi)\ldots \tag{4.10}$$

From the second chapter, we know that any symmetric group S_n contains two one-dimensional irreducible representations. These representations are the *trivial* representation, labeled by partition $[n]$, and the *alternating* or *sign* representation, labeled by $[1^n]$. This means that if a tensor product of irreducible representations of the symmetric group contains one or both representations in its direct sum decomposition, then it is possible to obtain one or more maximally entangled states from these maps.

The projector on the trivial subspace $\lambda = [n]$ is given from equation (4.10) by:

$$\mathbb{P}^{(n)} = (n!)^{-1} \sum_{\pi \in S_n} S^{\alpha}(\pi) \otimes S^{\beta}(\pi) \otimes ..., \tag{4.11}$$

since $\chi^{(n)}(\pi) = 1$ for any $\pi \in S_n$, and the multiplicity reduces similarly to:

$$g_{\alpha\beta\gamma...(n)} := (n!)^{-1} \sum_{\pi \in S_n} \chi^{\alpha}(\pi)\chi^{\beta}(\pi)\chi^{\gamma}(\pi)...\chi^{(n)}(\pi). \tag{4.12}$$

It is also possible to give an expression for the projector for the remaining one-dimensional subspace of S_n, the sign subspace. This projector is defined as follows:

$$\mathbf{P}^{(1^n)} = (n!)^{-1} \sum_{\pi \in S_n} sgn(\pi) S^{\alpha}(\pi) \otimes S^{\beta}(\pi) \otimes ... \tag{4.13}$$

It is important to note that the multiplicities of the trivial and sign subspaces do not generally coincide. In fact, the multiplicity of the sign subspace can never be greater than that of the trivial subspace. Nevertheless, there is an special case where both are the same: If $[\alpha] = [\beta] = ... = [\nu]$ and $[\alpha]$ is an self-associated Young diagram, then both multiplicities are equivalent. Then, the total number of maximally entangled states that can be found in the tensor product space is $2g_{\alpha\beta...\nu}$. This occurs because the characters of odd permutations of self associated partitions are always zero.

Projectors on invariant subspaces of tensor products of representations of the alternating group can also be computed, following the equation (4.6). In this case, it is important to take into account that trivial and sign subspaces coincide, since odd permutations are not taken into account, and that self-associated representations have half the dimension of the irreducible representations of self-associated partitions in $\mathcal{A}_n$. These facts can be exploited to reduce the dimensionality of the problem of finding maximally entangled states, as it will be seen in the next chapter.

4.4 Construction of maximally entangled states of 3 qubits using representations of S_3

In order to illustrate the construction of maximally entangled states, in this section a explicit construction of maximally entangled states of three qubits from tensor products of representations of S_3 is presented. This is possible because the irreducible representation labeled by partition $[2, 1]$ corresponds to a Young diagram of dimension two, according to the Hook formula in equation (2.15). Its corresponding SYTs are:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad , \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

The symmetric group S_3 contains a total of six permutations, including three transpositions and two 3-cycles. Taking this into account and the character table 2.1 of this group, one obtains the following matrices representations of the elements of the group, using Young-Yamanouchi algorithm in equations (2.19) and (2.20):

$$S_{[2,1]}(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_{[2,1]}((1,2,3)) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}, \tag{4.14}$$

$$S_{[2,1]}((2,3)) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \quad S_{[2,1]}((1,3)) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, \tag{4.15}$$

$$S_{[2,1]}((1,2)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_{[2,1]}((1,3,2)) \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}. \tag{4.16}$$

Using these matrices, and computing the projector on the trivial subspace $\lambda = [n]$ of the tensor product $[2,1] \otimes [2,1] \otimes [2,1]$ according to the equation (4.11), one finds the following matrix:

$$\mathbf{P}^{[n]} = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.17}$$

Since all the columns are linearly dependent, it is straightforward to see that the projector can be simplified as follows using the *bra-ket* notation:

$$\mathbf{P}^{[n]} = \frac{1}{4}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)(\langle 000| - \langle 011| - \langle 101| - \langle 110|), \tag{4.18}$$

which means that the vector on the trivial subspace is:

$$|\phi\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle). \tag{4.19}$$

The multiplicity of the trivial subspace, given by the Kronecker coefficient, is one in this case, which coincides with the number of vectors found. Nevertheless, in general the multiplicity of such subspace is greater than one. In such case, Gram-Schmidt orthonormalization can be used on the eigenvectors with eigenvalue 1 of the corresponding projection matrix, which gives a basis of states for the trivial subspace, according to its multiplicity. The set of eigenvectors and eigenvalues of a matrix can be easily be found using the function *numpy.linalg.eig()* in PYTHON. Then, if $\{|v_1\rangle, |v_2\rangle, ...\}$ is the eigenspace of the matrix, an orthonormal set of vectors $\{|u_1\rangle, |u_2\rangle, ...\}$ can be found iteratively as follows:

$$|u_k\rangle = |v_k\rangle - \sum_{j=1}^{k-1} \frac{\langle v_k|u_j\rangle}{\langle u_j|u_j\rangle} |u_j\rangle . \tag{4.20}$$

Then, by sorting the eigenvectors with eigenvalue 1, one finds the set of maximally entangled states that span the trivial subspace in a tensor product of irreducible representations of the symmetric group, according to its multiplicity. This algorithm was implemented in PYTHON, as shown in Appendix B.

4.5 Summary

This chapter presented a method for constructing maximally entangled states using tensor products of irreducible representations of the symmetric group. These states can be obtained using projectors on the trivial and sign invariant subspaces of those tensor products.

In the next chapter, some examples of maximally entangled states constructed using this method are shown. In particular, it is shown that maximally entangled states of several particles obtained from tensor products of self-adjoint representations of the symmetric group can be constructed from irreducible representations of the alternating group, which reduces substantially the dimensionality of the problem. It also provides a way of classifying maximally entangled states according to their symmetry properties, which can be used for specific quantum information tasks, as it will be shown.

5. Construction of generic maximally entangled states of qubits

In the previous chapter, a method to construct maximally entangled states using tensor products of representations of the symmetric group was presented. This method relied on constructing projectors onto the trivial and sign subspace of the tensor product decomposition, which are spanned by states with the property that all their local density matrices were multiples of the identity. Nevertheless, the construction of such states is computationally hard when the number N of irreducible representations in the tensor product is large. Besides that, the symmetric group S_n contains $n!$ elements, which means that the number of matrix representations to be constructed for each irreducible representation in the tensor product goes as n^n, according to the Stirling formula. Given the difficulties of such construction, this thesis proposes to simplify the computations of those states. In particular, this chapter proposes an explicit and canonical construction of maximally entangled states of N qubits, based on tensor products of irreducible representations of both the symmetric group S_3 and its subgroup, the alternating group $\mathcal{A}_3$. We show here that it is possible to find a change of basis between the representations of these two groups, and that the problem of finding maximally entangled states for any N entangled qubits is simplified by proceeding this way.

As it was seen in the second chapter, it is possible to decompose an $N-$fold tensor product space in terms of irreducible representations of the symmetric group and the General Linear group, as a consequence of the Schur-Weyl duality. This is useful because it shows how state vectors decompose under the action of the symmetric group $S_N^{parties}$ over its N *parties*. By applying such decomposition on the trivial subspaces generated by tensor products of N copies of some irreducible representation $[\alpha]$ of S_n for $\alpha \vdash n$, i.e. $[\alpha]^{\otimes N}$, it is possible to classify maximally entangled states in different symmetric group sectors labeled by irreducible representations $\lambda \vdash N$ of $S_N^{parties}$. In this chapter, it is shown in particular that by decomposing the trivial subspace of the tensor product space $[2,1]^{\otimes N}$ according to Schur-Weyl duality, where $[2,1]$ is a two-dimensional representation of S_3, some symmetric maximally entangled qubit states can be found. The importance of this relies on the fact that this procedure can be used to find totally symmetric maximally entangled states, which are states of copies of N entangled and identical bosons. Note here that tensor products of representations of S_n can be used to find locally maximally entangled states, whereas the symmetric group $S_N^{parties}$ acts on these states by permuting its parties. This means the actions of these two symmetric groups are distinct and they should not be confused.

5.1 Geometric construction of qubit Kronecker states

Multipartite locally maximally entangled qubit states (LME qubit states) of N parties can be obtained by projecting any arbitrary N-partite qubit state onto the trivial and sign invariant subspaces of tensor products of two dimensional-irreducible representations of S_n, for some n.

Each of these one-dimensional subspaces can appear with a certain multiplicity in the tensor product decomposition, given by the Kronecker coefficient, as defined in equation (4.10). The only irreducible representations of dimension $d = 2$ for any symmetric group are labeled by the partitions $[2, 1]$ of $n = 3$ and $[2, 2]$ of $n = 4$. Therefore, N-partite LME qubit states or qubit *Kronecker* states can be obtained using N-fold tensor products of $[2, 1]$ or $[2, 2]$. Table 5.1 summarizes the Kronecker coefficients corresponding to the multiplicity of the trivial invariant subspace of the reducible tensor product $[2, 1]^{\otimes N}$, for N between 2 and 8. These coefficients always coincide with the coefficients of $[2, 2]^{\otimes N}$, which means that they can be used indistinctly to generate the same set of maximally entangled states. In order to see this equivalence, it is necessary to take into account the character table of the symmetric group S_3, given in table 2.1 of chapter 2. From the definition of the Kronecker coefficient, the multiplicity of the trivial subspace in the tensor product representation $[2, 1]^{\otimes N}$ has the following form for any N:

$$g_{\alpha\beta\gamma\ldots\nu} = \frac{1}{6}[(2)^N + 2(-1)^N], \tag{5.1}$$

where $[\alpha] = [\beta] = [\gamma] = \ldots = [\nu] = [2, 1]$. On the other hand, from the character table 2.2 of the symmetric group S_4, the multiplicity of the trivial subspace in the tensor product representation $[2, 2]^{\otimes N}$ is:

$$g_{\alpha\beta\gamma\ldots\nu} = \frac{1}{24}[(2^N) + 8(-1)^N + 3(2^N)], \tag{5.2}$$

which can be straightforwardly reduced to equation (5.1). This means that both representations generate the same set of locally maximally entangled states, and can be used indistinctly.

System	$(g_{\alpha\alpha\ldots\alpha})$
$2 \otimes 2$	1
$2 \otimes 2 \otimes 2$	1
$2 \otimes 2 \otimes 2 \otimes 2$	3
$2 \otimes 2 \otimes 2 \otimes 2 \otimes 2$	5
$2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 \otimes 2$	11
$2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 \otimes 2$	21
$2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 \otimes 2 \otimes 2$	43

Table 5.1: Kronecker coefficients of N copies of two-dimensional representations of S_3, for $N =\in \{2, 3, 4, 5, 6, 7, 8\}$ over the trivial invariant subspace, labeled by the partition [3]. These values indicate the dimension of the trivial invariant subspace of the N-fold tensor product space.

Using the projector over the trivial invariant subspace, and using Gram-Schmidt orthonormalization on its eigenvectors with eigenvalue 1, it is straightforward to find that the only bipartite qubit Kronecker(LME) state is the *Bell state* ϕ^+:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Similarly, it is possible to find another bipartite Kronecker state if the projection is performed over the invariant sign subspace, which maps every permutation to its sign [1]. The explicit form of such vector is:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

which is another Bell state. Similar statements can be made for the three-qubit maximally-entangled states, for which the trivial and sign subspaces are generated by the following vectors, respectively:

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle), \\ |\phi_2\rangle &= \frac{1}{2}(|111\rangle - |001\rangle - |010\rangle - |100\rangle). \end{aligned} \tag{5.3}$$

In the case $N = 2$, the two maximally entangled states found through projectors of the symmetric group are familiar ones, since they are two of the states that constitute the so called *Bell basis* [13]. The first pair, $|\psi_1\rangle$, is symmetric under transposition of the qubits, whereas $|\psi\rangle_2$ is antisymmetric and is well-known as the *singlet* spin 1/2 state. In the case $N = 3$, the form of these vectors is still simple to analyze, since these states are just symmetric linear combinations of product states with the same coefficient (1/2), just differing by a sign. In fact, both states are LU-equivalent, i.e., $|\phi_1\rangle = X^3 |\phi_2\rangle$, where X is the *Pauli-X* operator or *Quantum NOT gate* [26], which maps $|0\rangle$ to $|1\rangle$ and vice versa. Nevertheless, the generalization of these states for $N > 3$ is not that simple to interpret, since they are not necessarily neither symmetric nor antisymmetric linear combinations of product states.

The construction of N-partite qubit states can be simplified by taking into account that the representations $[2, 1]$ and $[2, 2]$ are both labeled by self-adjoint Young diagrams. This means that Kronecker coefficients over the trivial subspace, like the ones shown in table 5.1, coincide with the Kronecker coefficients over the sign subspace of $S_3(S_4)$, as stated in section 4.3. Moreover, as discussed, the representation $[2, 1]$ splits into two one-dimensional representations under the alternating group ($\mathcal{A}_3$, i.e., $[2, 1]^+$ and $[2, 1]^-$. Thus, $N-$fold tensor products of $[2, 1]$ can be interpreted as direct sums of one-dimensional representations in the $\mathcal{A}_3$ representations basis. This means that he dimensionality of the problem can be reduced by half when changing from the computational basis, i.e., the Young-Yamanouchi basis of S_3, to the basis generated by the representation $[2, 1] \downarrow \mathcal{A}_3$. The irreducible representations of $\mathcal{A}_3$ are one-dimensional. This means that the tensor product of any of these representations yield another one-dimensional representation of the group, according to the following rules:

$$\begin{aligned} [2, 1]^{\pm} \otimes [2, 1]^{\mp} &= [3], \\ [2, 1]^{\pm} \otimes [2, 1]^{\pm} &= [2, 1]^{\mp}. \end{aligned}$$

According to these rules, one can combine N of these one-dimensional representations such that the tensor product yields the trivial representation, labeled by the partition [3]. Since the irreducible representations $[2,1]^{\pm}$ are one-dimensional, the reducible representation $[2,1] \downarrow \mathcal{A}_3$ acting on a permutation $\pi \in \mathcal{A}_3$ can be interpreted as a direct sum of one-dimensional subspaces, which can be represented using a two-dimensional diagonal matrix where the entry $(1,1)$ corresponds to the representation $[2,1]^+$ and the entry (2,2) to $[2,1]^-$ acting on some permutation π. This means that the cycles$(1,2,3)$ and $(1,3,2)$ transform from the Young-Yamanouchi basis of $\mathcal{S}_3$ to the basis of representations of $\mathcal{A}_3$ as follows:

$$S_{[12]}((1,2,3)) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} e^{\frac{i2\pi}{3}} & 0 \\ 0 & e^{-\frac{i2\pi}{3}} \end{bmatrix}, \tag{5.4}$$

$$S_{[1,2]}((1,3,2)) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} e^{-\frac{i2\pi}{3}} & 0 \\ 0 & e^{\frac{i2\pi}{3}}. \end{bmatrix} \tag{5.5}$$

Since the two dimensional representation $[2,1] \downarrow \mathcal{A}_3$ is diagonal for every $\pi \in A_3$, its eigenvectors are $|+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Here signs $+$ and $-$ refer to the eigenvalues $e^{+i\phi}$ and $e^{-i\phi}$ of the matrix representation of the permutation $\pi = (123)$, with $\phi = 2\pi/3$. These eigenvalues, together with the identity permutation, are the phases that correspond to the rotation symmetries of an equilateral triangle, that is, the three roots of the unity. Then, these eigenvectors identify the operations that leave the spatial configuration of an equilateral triangle invariant.

Since the representation $[2,1]$ can be interpreted as a two-dimensional diagonal matrix as seen in the $\mathcal{A}_3$ basis, the form of the Kronecker vectors in this basis is simpler, and the dimensionality of the problem reduces. In order to construct the projectors in such a basis one still needs to find the expression for the representations of the transpositions. Since there is no natural way of constructing odd permutations in this basis, given that this construction is only meant for elements of the alternating group, it is important to find an explicit change of basis between the Young-Yamanouchi basis and the alternating group representation basis. One can show that the representations of the 3-cycles given in equations (5.4) and (5.5) are related through a similarity transformation given by:

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \tag{5.6}$$

or equivalently, $T = \frac{1}{\sqrt{2}}(1 + iX)$, where X is the Pauli X operator. Using this similarity transformation, it is possible to find that the representations of the transpositions $(1,2)$, $(2,3)$ and $1,3$ have only off-diagonal elements in the new basis:

$$S_{[1,2]}((1,2)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \tag{5.7}$$

$$S_{[2,1]}((2,3)) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & e^{\frac{i2\pi}{3}} \\ e^{\frac{-i2\pi}{3}} & 0 \end{bmatrix}, \tag{5.8}$$

$$S_{[2,1]}((1,3)) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & e^{\frac{-i2\pi}{3}} \\ e^{\frac{i2\pi}{3}} & 0 \end{bmatrix}. \tag{5.9}$$

Using the matrix representations in this new basis, and by definining the projection operator over the trivial invariant subspace using equation (4.12), one obtains the following three orthogonal Kronecker states in terms of the vectors $|+\rangle$ and $|-\rangle$ for the four-qubit case:

$$|1,2,+\rangle = \frac{1}{\sqrt{2}}(|++--\rangle + |--++\rangle), \tag{5.10}$$

$$|1,3,+\rangle = \frac{1}{\sqrt{2}}(|+-+-\rangle + |-+-+\rangle), \tag{5.11}$$

$$|1,4,+\rangle = \frac{1}{\sqrt{2}}(|+--+\rangle + |-++-\rangle). \tag{5.12}$$

And for the sign subspace:

$$|1,2,-\rangle = \frac{1}{\sqrt{2}}(|++--\rangle - |--++\rangle), \tag{5.13}$$

$$|1,3,-\rangle = \frac{1}{\sqrt{2}}(|+-+-\rangle - |-+-+\rangle), \tag{5.14}$$

$$|1,4,-\rangle = \frac{1}{\sqrt{2}}(|+--+\rangle - |-++-\rangle), \tag{5.15}$$

where the notation $|i,j,\pm\rangle$ is used taking into account that these states are linear combinations of product states where the qubits in positions i and j are labeled with the same phase $\pm$, and the remaining qubits are labeled by the opposite phase $\mp$. Note that every Kronecker state is an equal-coefficient addition or substraction of two product states, and that the product states in each Kronecker states are related by a 4-fold tensor product of the X quantum gate. Then, every Kronecker state is of the form:

$$|i,j,\pm\rangle = \frac{1}{\sqrt{2}}(|i,j\rangle \pm X^{\otimes 4}\,|i,j\rangle), \tag{5.16}$$

for $i \in \{1,2,3\}$. This is related to the fact that the cycles of length 3, which are elements of $\mathcal{A}_3$, can be obtained from each other when they are conjugated with any transposition in $\mathcal{S}_3$, i.e., $(123) = \sigma(213)\sigma^{-1}$ with $\sigma \in \mathcal{S}_3/\mathcal{A}_3$.

After computing the projector over the trivial and sign invariant subspaces, one obtains similar expressions for the 5-qubit Kronecker states, related to:

$$|5,\pm\rangle = \frac{1}{\sqrt{2}}(|++++-\rangle \pm i\,|----+\rangle), \tag{5.17}$$

$$|4,\pm\rangle = \frac{1}{\sqrt{2}}(|+++-+\rangle \pm i\,|---+-\rangle), \tag{5.18}$$

$$|3,\pm\rangle = \frac{1}{\sqrt{2}}(|++-++\rangle \pm i\,|--+--\rangle), \tag{5.19}$$

$$|2,\pm\rangle = \frac{1}{\sqrt{2}}(|+-+++\rangle \pm i\,|-+---\rangle), \tag{5.20}$$

$$|1,\pm\rangle = \frac{1}{\sqrt{2}}(|+----\rangle \pm i\,|-++++\rangle), \tag{5.21}$$

where the notation $|j,\pm\rangle$, for $j \in \{1,2,3,4,5\}$ is used noticing that each of these states are linear combinations of product states whose local phases are all the same except for the j-th qubit. This means that, in the $\{|+\rangle, |-\rangle\}$ basis, the maximally entangled states found using the formalism of tensor products of irreducible representations reduce to GHZ-like states, as seen for the 4 and 5-qubit cases. These states are of the form $1/\sqrt{2}(|a\rangle \pm |b\rangle)$, where $|a\rangle$ and $|b\rangle$ are product states and the sign $\pm$ in the linear combination depends on whether the state belongs to the trivial or the invariant subspace, i.e., $+$ or $-$, respectively. Furthermore, recalling that the local states $|\pm\rangle$ indicate a phase $e^{i\pm 2\pi/3}$, it is straightforward to see that the product states in the linear combination must be such that the multiplication of the local phases is equal to 1, such that the state remains invariant under the action of the N-fold tensor product space of representations. For instance, this means that for the case described above, $N = 5$, the following *phase sum condition* is necessary:

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 = 0 \mod 2\pi \quad \phi_i \in \{2\pi/3, -2\pi/3\}. \tag{5.22}$$

In the five-qubit case, there are a total of 5 combinations of $|+\rangle$s and $|-\rangle$s that satisfy the above condition, giving a total of 10 Kronecker states: 5 from the trivial representation, and 5 from the sign representation. It can be observed that this can be generalized to any number of qubits, since these states arise as sums or differences of two product states with the same coefficient. The combinations allowed are the ones where for the i-th qubit, with $i \in \{1, 2, ..., N\}$, if the first product state contains $|\pm\rangle$, the second product state contains $|\mp\rangle$ in the i-th qubit. This means that, for any N-qubit state, the product states in these linear combinations must satisfy the following conditions:

$$n_+ + n_- = N, \tag{5.23}$$

and

$$\frac{2\pi}{3}n_+ - \frac{2\pi}{3}n_- = 0 \mod 2\pi, \tag{5.24}$$

where n_+ indicates the number of parties with phase $2\pi/3$ and n_- the number of parties with phase $-2\pi/3$, or, equivalently, the number of local parties with associated eigenvectors $|+\rangle$ and $|-\rangle$, respectively.

This construction of Kronecker states suggests that qudit maximally entangled states of N parties can be constructed using a more geometrical intuition. It suggests that Kronecker states could be generated by identifying some geometrical symmetry in d dimensions; for example, rotations of some regular object. In the qubit case, this object is an equilateral triangle. These symmetries must be related to d different phases, such that each of the elements of the basis is labeled in bijection to these phases. In this sense, the generalization of Kronecker states for d dimensional N-fold tensor product spaces must be some kind of linear combination of *product states* whose N local phases add up to $0 \mod 2\pi$, in addition to some other restrictions.

In the next section, further analysis of qubit maximally entangled states is given, based on their symmetry properties.

5.2 Symmetric group classification of maximally entangled states

N-partite symmetric states have the property that the entanglement distributes equivalently for any partition of M parties, for $M < N$. This means that tracing out any $N - M$ of the qudits, always yields the same partial density matrix. In the tripartite qubit case there are two symmetric generic states: the W and GHZ states. The latter is a LME tripartite qubit state. In fact, the states $|\phi_1\rangle$ and $|\phi_2\rangle$ in equation (5.3) are GHZ states as seen in the $\{|+\rangle, |-\rangle\}$ basis:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle), \tag{5.25}$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle). \tag{5.26}$$

These states have the property that the 3-partite entanglement is maximized, whereas tracing out any of the three qubits leaves the remaining two qubits in a separable state, i.e., an unentangled state. On the other hand, W states are not LME but tracing out any single qubit leaves the remaining two qubits in an entangled state.

In the 4-qubit case the locally maximally entangled states do not appear to be symmetric or antisymmetric, provided the expressions in equations (5.10)-(5.15). Nevertheless, Schur-Weyl duality establishes that any Hilbert space $\mathcal{H}^{\otimes N}$ decomposes as direct sums of irreducible representations of the General Linear group and the symmetric group $S_N^{parties}$, which acts on the N parties by permuting them. Then, this duality can be used in order to find the decomposition of any Kronecker state in terms of irreducible subspaces under the action of $S_N^{parties}$. In the case of 4 qubits, the full Hilbert space $\mathcal{H}^{\otimes 4} = (\mathbf{C}^2)^{\otimes 4}$ decomposes into irreducible subspaces of

$GL(2, \mathbf{C})$ and $S_4^{parties}$ as follows:

$$\mathcal{H}^{\otimes 4} = (\mathcal{R}_{[4]} \otimes \mathcal{S}_{[4]}) \oplus (\mathcal{R}_{[3,1]} \otimes \mathcal{S}_{[3,1]}) \oplus (\mathcal{R}_{[2,2]} \otimes \mathcal{S}_{[2,2]}). \tag{5.27}$$

The dimensions of the invariant subspaces of $S_4^{parties}$ can be computed using the Hook formula, defined in equation (2.11). Then, one obtains that the subspace of the trivial or totally symmetric representation $S_[4]$ is one-dimensional, $S_[3,1]$ is three-dimensional and the space labeled by the partition $S_[2,2]$ is two-dimensional. In fact, recalling that any permutation matrix U_π acting on a N-fold tensor product Hilbert space decomposes into irreducible $S_4^{parties}$ sectors whose multiplicity $s_\lambda(1^2)$ coincides with the dimension of the corresponding $GL(2, \mathbf{C})$ associated irreducible subspace, any permutation matrix on the Hilbert space can be interpreted as a direct sum of irreducible representations of $S_4^{parties}$ as follows:

$$U_\pi = [4]^5 \oplus [2,2] \oplus [3,1]^3. \tag{5.28}$$

Using equation (2.17) to define the projectors onto these irreducible subspaces, one can decompose the three states in equations (5.10)-(5.12) in terms of vectors lying in these subspaces. One finds that these three vectors, which appear in the trivial subspace of the direct sum decomposition of $[2,1]^{\otimes 4}$, can be written as linear combinations of one symmetric state:

$$|[3],[4],1,1\rangle = \frac{1}{\sqrt{6}}(|++--\rangle + |+-+-\rangle + |+--+\rangle + |--++\rangle + |-+-+\rangle + |-++-\rangle), \tag{5.29}$$

and two states that span a two-dimensional subspace that is invariant under the action of the symmetric group $S_4^{parties}$ on the $\mathcal{H}^{\otimes 4}$ space:

$$|[3],[2,2],1,1\rangle = \frac{1}{\sqrt{3}}(|++--\rangle - \frac{1}{2}|+-+-\rangle - \frac{1}{2}|+--+\rangle - \frac{1}{2}|-++-\rangle - \frac{1}{2}|-+-+\rangle + |--++\rangle), \tag{5.30}$$

and

$$|[3],[2,2],2,1\rangle = \frac{1}{2}(|+-+-\rangle - |+--+\rangle - |-++-\rangle + |-+-+\rangle), \tag{5.31}$$

and that these states are again LME states. The notation of these states is further explained in the next paragraphs, but some insight onto these notations is given by the decomposition of the 4-fold tensor product space in equation (5.27). According to equation (5.28), five linearly independent symmetric states can be found in $\mathcal{H}^{\otimes 4}$. Nonetheless, there is only one symmetric Kronecker state, i.e., the one in equation (5.27). In contrast, the states (5.30) and (5.31) generate the only two-dimensional invariant subspace of the full Hilbert space. This means that any linear combination of these two states is again a maximally entangled state, and that any state that transforms as the representation $[2,2]$ under the action of the symmetric group $Sparties_4$ on the parties is a LME state.

Any symmetric state has the property that tracing out any M qubits, with $M < N$ always yields the same density operator for the remaining N-M qubits. For instance, tracing out any qubit, always yields the following density matrix for the symmetric state found, given in equation (5.29):

$$tr_i(|[3],[4],1,1\rangle\langle[3],[4],1,1|) = \frac{1}{2}|W\rangle\langle W| + \frac{1}{2}|W'\rangle\langle W'|, \quad i = \{1,2,3,4\}, \tag{5.32}$$

where $|W'\rangle = U|W\rangle$, and the operator U is defined as $U = X^{\otimes 4}$, and X is the quantum NOT gate, which transforms $|\pm\rangle$ to $|\mp\rangle$.

If two out of the four qubits are traced out, the remaining state is:

$$tr_{i,j}(|[3],[4],1,1\rangle\langle[3],[4],1,1|) = \frac{1}{6}|{++}\rangle\langle{++}| + \frac{1}{6}|{--}\rangle\langle{--}| + \frac{2}{3}|\psi^+\rangle\langle\psi^-|, \quad i \neq j, \tag{5.33}$$

where $|\psi^+\rangle = 1/\sqrt{2}(|{+-}\rangle + |{-+}\rangle)$.

Tracing out one of the qubits of the density matrices of states (5.30) and (5.31) leaves different partial density matrices depending on the qubits that are traced. Nevertheless, they are all of the following form:

$$tr_i(|[3],[2,2],1,1\rangle\langle[3],[2,2,],2,1|) = \frac{1}{2}|\psi^W\rangle\langle\psi^W| + \frac{1}{2}|\psi'^W\rangle\langle\psi'^W|, \tag{5.34}$$

where $|\psi^W\rangle$ and $|\psi'^W\rangle)$ are SLOCC-equivalent to $|W\rangle$. In particular, if the first qubit is traced one obtains that $|\psi^W\rangle = \sqrt{2/3}|001\rangle + \sqrt{1/6}|010\rangle + \sqrt{1/6}|100\rangle$ and $|\psi'^W\rangle = \sqrt{2/3}|110\rangle + \sqrt{1/6}|101\rangle + \sqrt{1/6}|011\rangle$.

The states in equations (5.13)-(5.15), which appear in the sign subspace of the tensor product representation $[2,1]^{\otimes 4}$, generate one of three orthogonal invariant subspaces of dimension three that appear in the decomposition of the space $\mathcal{H}^{\otimes 4}$ under the action of $S_4^{parties}$ on the qubits, which is labeled by the representation $[3,1]$. This means that the states $|1,2,-\rangle$, $|1,3,-\rangle$ and $|1,4,-\rangle$ can be relabeled as $|[1^3],[3,1],1,1\rangle$, $|[1^3],[3,1],2,1\rangle$ and $|[1^3],[3,1],3,1\rangle$ in accordance to the invariant subspace they belong to and they do not need to be further decomposed. These states have the property that, tracing out any of the qubits, leaves the remaining three qubits in separable states, i.e., unentangled.

The previous arguments show that the basis of 6 four-qubit LME states given in equations (5.10)-(5.15) can be used to generate another basis of 6 LME states which can be arranged into irreducible subspaces under the action of S_4^P on the parties, and suggest that the trivial and sign subspaces of the tensor product $[2,1]^{\otimes 4}$ can be further reduced into irreducible representations of the symmetric group $S_4^{parties}$ and $GL(2,\mathbf{C})$. This could be used to classify LME states according to how they transform under the action of the symmetric group on the parties. This suggests that the cardinality of the set of LME qubit states generated by the tensor product space $[2,1]^{\otimes N}$, which is twice its Kronecker coefficient on the trivial subspace (since its multiplicity coincides with the multiplicity of the sign subspace), is equal to the sum of dimensions of some subset of irreducible representations of the symmetric group $S_N^{parties}$.

The previous statements can be further explored by using some definitions that arise from the theory of group representations. Since each of the irreducible subspaces labeled by partitions ν of $n = 3$ in the direct sum decomposition of $[2,1]^{\otimes N}$ can be further reduced into irreducible subspaces of $GL(2,\mathbf{C})$, which are labeled as $\mathcal{R}_\lambda$, with $\lambda \vdash N$, then the multiplicity of $\mathcal{R}_\lambda$ in $[\nu]$ can be computed as follows, according to equation (2.6):

$$m_{R_\lambda,\nu} = \frac{1}{|\mathcal{S}_3|}\sum_{\pi\in\mathcal{S}_3}\chi_\nu(\pi)\chi_\lambda(D_\nu(\pi)), \tag{5.35}$$

where $D_\nu(\pi)$ is the matrix representation of $[\nu]$ evaluated in the permutation π, χ_ν is the character of the representation D_ν, $\chi_\lambda(D_\nu(\pi))$ is the character of the representation R_λ of $GL(2,\mathbf{C})$ evaluated for the matrix $D_\nu(\pi)$ and $|S_3| = 3! = 6$. The previous equation can be generalized for any tensor product of irreducible representations of S_n, for any integer n. From the latter equation, and equation (5.35), it is possible to decompose the Hilbert space $\mathcal{H}^{\otimes 4}$ in terms of direct sums of representations of $GL(2,C)$ and the symmetric group S_3 as follows:

$$\mathcal{H}^{\otimes 4} = \oplus_{\lambda,\nu} m_{R_\lambda,\nu}[\nu]\otimes\mathcal{R}_\lambda = 2([1^3]\otimes[3,1])\oplus([3]\otimes[2,2])\oplus([3]\otimes[4])\oplus([2,1]\otimes[4])\oplus([2,1]\otimes[3,1]), \tag{5.36}$$

which confirms that the 6 Kronecker states of 4 qubits, which lie in the one-dimensional subspaces of the tensor product space $[2,1]^{\otimes 4}$, are generators of irreducible subspaces of dimensions 1, 2 and 3 under the action of the symmetric group $S_4^{parties}$. Taking into account that this tensor product decomposes into direct sums of irreducible representations of S_3, labeled by some partition $\lambda \vdash 3$, and that each of these representations is associated with a subspace of $\mathcal{H}^{\otimes 4}$ labeled by some partition $\nu \vdash 4$, then the 6 Kronecker states of 4 qubits could be relabeled using the following convention:

$$|\nu,\lambda,i,m_i\rangle\,, \quad 1\le i\le f^\lambda, \quad 1\le m_i\le m_{R_\lambda,\nu}, \quad \lambda\vdash N, \tag{5.37}$$

where ν is the partition $\nu = [n]$ (in this case n=3) related to the trivial subspace or $\nu = [1^n]$ if the alternating subspace is being visualized, and f^λ is the dimension of the irreducible representation λ. Again, this can be generalized to any N-fold tensor product of an irreducible representation of S_n, for any integer n. This notation is used to label the states in equations (5.29)-(5.31) and (6.13)-(6.15).

Similar decompositions can be obtained for the 10 Kronecker states found for the 5-qubit case. Nevertheless, the states in the trivial subspace are LU-equivalent to the ones in the sign subspace, which reduces the number of inequivalent LME states to five. The states in equations (5.15)-(5.19) can be found to be linear combinations of the following symmetric LME state, using the given convention:

$$\begin{aligned}|[3],[5],1,1\rangle = \frac{1}{\sqrt{10}}(&|++++-\rangle + |----+\rangle + |+++-+\rangle + |---+-\rangle + \\ &+|++-++\rangle + |--+--\rangle + |+-+++\rangle + |-+---\rangle + \\ &+|-++++\rangle + |+----\rangle),\end{aligned} \tag{5.38}$$

and the following four LME states, which generate a irreducible subspace of dimension four under the action of the symmetric group $S_5^{parties}$ on the parties, which can be labeled with the partition $[4,1] \vdash 5$:

$$\begin{aligned}|[3],[4,1],1,1\rangle = \frac{1}{\sqrt{40}}(&4|++++-\rangle + 4|----+\rangle - |+++-+\rangle - |---+-\rangle - \\ &-|++-++\rangle - |--+--\rangle - |+-+++\rangle - \\ &-|-+---\rangle - |-++++\rangle - |+----\rangle),\end{aligned} \tag{5.39}$$

$$\begin{aligned}|[3],[4,1],2,1\rangle = \frac{1}{\sqrt{24}}(&3|+++-+\rangle + 3|---+-\rangle - |++-++\rangle - |--+--\rangle - \\ &-|+-+++\rangle - |-+---\rangle - |-++++\rangle + 3|+----\rangle),\end{aligned} \tag{5.40}$$

$$\begin{aligned}|[3],[4,1],3,1\rangle = \frac{1}{\sqrt{12}}(&2|++-++\rangle + 2|--+--\rangle - |+-+++\rangle - |-+---\rangle - \\ &-|-++++\rangle - |+----\rangle),\end{aligned} \tag{5.41}$$

$$|[3],[4,1],4,1\rangle = \frac{1}{2}(|+-+++\rangle + |-+---\rangle - |-++++\rangle - |+----\rangle). \tag{5.42}$$

The states in the previous four equations generate a 4-dimensional irreducible representation $[4,1]$ of the symmetric group $S_5^{parties}$ that is invariant under the action of the symmetric group on the indices of the qubits, whereas state (5.38) is symmetric under any permutation in S_5.

It can be shown that for greater N, the set of maximally entangled states found using the projector on the trivial and sign subspaces of tensor products of copies of the representation $[2,1]$ can be decomposed different irreducible sectors of $\mathcal{H}^{\otimes \mathcal{N}}$ according to Weyl-Schur decomposition. This means that not only these states are LME, but they also transform according to some irreducible representation of $S_N^{parties}$ when a permutation U_π on the parties is applied. To see this, to equation (4.1). These facts can be used to find totally symmetric LME states, such as the ones in equations (5.29) and (5.38), which can be interpreted as maximally entangled states of identical bosons.

5.3 Summary

In this chapter, a construction of LME states was presented, based on tensor products of copies of the two-dimensional irreducible representation of S_3. Since this representation is self-adjoint, it is reducible into two one-dimensional representations,as seen from the alternating group $\mathcal{A}_3$, this was used to show that every LME state found using this method can be written as a sum of just two product states in the alternating group representation basis that satisfy the so-called phase sum condition, which simplifies the construction of any $N-$ qubit LME state. Moreover, this construction was used to classify LME states of N parties, according to how they transform under the action of the symmmetric group $S_N^{parties}$. It was shown that these vectors transformed according to some irreducible representation of $S_N^{parties}$. In particular, there were found expressions for LME states of 4 and 5 qubits, and they were shown to be associated to irreducible spaces of $\mathcal{H}^{\otimes N}$ according to Weyl-Schur duality, including the totally symmetric space.

In the next chapter, and as a generalization of the LME qubit states, a construction of LME states of N qudits of dimension $d = 6$ based on self associated diagrams of the alternating group $\mathcal{A}_5$. By this means, it is shown that these states can be obtained from LME qutrit states ($d = 3$), and that these states can be classified according to how they transform under any permutation of its N parties.

6. Maximally entangled states of dimensions $d = 3$ and $d = 6$ from representations of the Icosahedral rotation group

In this chapter, we show the construction of locally maximally entangled states of qutrit and qudit states of dimension $d = 6$ from tensor products of irreducible representations of the symmetric and alternating groups S_5 and $\mathcal{A}_5$. The latter group is isomorphic to the group of rotations of the *icosahedron*, which is a regular three-dimensional platonic figure. By taking advantage of this isomorphism a more intuitive approach to maximally entangled states of qutrits can be provided. In this order of ideas, first the icosahedral group is presented, as well as its irreducible representations. After this, a construction of maximally entangled qutrit states is provided, based on the notions of the icosahedral group representations. Thereafter, a construction of maximally entangled states of dimension $d = 6$ based on states of qutrit states is provided, which is shown to be similar to the construction of qubit states given in the previous chapter. Finally, a generalized construction of locally maximally entangled states from N-fold tensor products of self-adjoint representations of symmetric groups is presented.

6.1 The rotational icosahedral group $\mathcal{I}$ and $\mathcal{A}_5$

The irreducible representation matrices of the alternating group $\mathcal{A}_5$ can be interpreted geometrically by noticing that this group is isomorphic to the rotational icosahedral group $\mathcal{I}$, which contains all rotations of the icosahedron that take vertices to vertices. This isomorphism is possible since their conjugacy classes coincide and $|\mathcal{A}_5| = |\mathcal{I}| = 5!/2 = 60$. In this section, an approach for the construction of matrix elements of irreducible representations of $\mathcal{I}$ and tensor products of these representations is shown, based in a paper by Lisa Everett and Alexander Stuart in which they explore applications of these representations to give family symmetries for solar neutrino mixing [11].

The icosahedron is a platonic solid consisting of 20 equilateral triangles, such that it has 12 vertices and 30 edges. There is a total of 60 rotations that leave the spatial configuration of this solid invariant. In particular, one finds the rotation by 0 (the identity) radians, the possible 12 rotations of $2\pi/5$ and $4\pi/5$ about an axis passing through each vertex and its antipode, 20 rotations of $2\pi/3$ about an axis through the center of each face, and 15 different rotations of π, which are illustrated in figure 6.1. This means that there is a total of 5 disjoint conjugacy classes labeled by C_n^k, referring to the rotations by $2\pi k/n$. Then, there are exactly five irreducible representations, with dimensions 1, 3, 3, 4 and 5, which satisfy the relation $1+3^2+3^2+4^2+5^2 = 60 = |\mathcal{I}|$ [11]. Each of these representations are labeled using the following informal notation, according to its dimension: $\mathbf{1}, \mathbf{3}, \mathbf{3'}, \mathbf{4}, \mathbf{5}$. In terms of the representations of $\mathcal{A}_5$, these correspond to the irreducible representations $[1^5]$, $[3, 1, 1]^+$, $[3, 1, 1]^-$, $[4, 1]$ and $[3, 2]$, respectively.

One possible parametrization in cartesian coordinates of the vertices of an icosahedron can

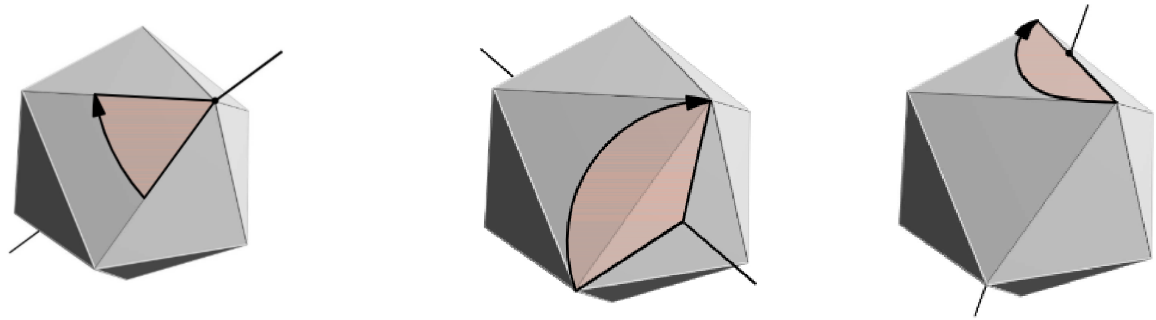


Figure 6.1: Representatives of conjugacy classes of $\mathcal{I}$. Left: Rotation of $\pi/5$ about a vertex. Center: Rotation of $\pi/3$ about the center of a face. Right: Rotation of $\pi/2$ about the center of an edge. The remaining non-trivial conjugacy class, obtained by rotating a vertex an angle of $2\pi/5$ is obtained by rotating once again by $\pi/5$ the figure on the left [6].

be achieved by applying cyclic permutations of the coordinate $(0, \pm 1, \pm\phi)$, where ϕ refers to the so-called *golden ratio*, i.e., $\phi = (1+\sqrt{5})/2$ [11]. Then, it is convenient to express the characters of the irreducible representations of $\mathcal{I}$ in terms of ϕ, as it can be seen in table 6.1. From this character table, it is possible to obtain the tensor product decomposition of the irreducible representations of $\mathcal{I}$, which is explicitly given in Table 6.2.

$\mathcal{I}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3'}$	$\mathbf{4}$	$\mathbf{5}$
e	1	3	3	4	5
$12C_5$	1	ϕ	1-ϕ	-1	0
$12C_5^2$	1	1-ϕ	ϕ	-1	0
$20C_3$	1	0	0	1	-1
$15C_2$	1	-1	-1	0	1

Table 6.1: Character table of $\mathcal{I}$.

$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$
$\mathbf{3'} \otimes \mathbf{3'} = \mathbf{1} \oplus \mathbf{3'} \oplus \mathbf{5}$
$\mathbf{3} \otimes \mathbf{3'} = \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3} \otimes \mathbf{4} = \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3'} \otimes \mathbf{4} = \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{3'} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}$
$\mathbf{4} \otimes \mathbf{5} = \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5} \oplus \mathbf{5}$
$\mathbf{5} \otimes \mathbf{5} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{4} \oplus \mathbf{5} \oplus \mathbf{5}$

Table 6.2: Tensor product decomposition of the irreducible representations of $\mathcal{I}$.

The elements of $\mathcal{I}$ can be generated by two elements a and b of the group having the following properties:

$$\langle a, b | a^2 = b^3 = (ab)^5 = e \rangle,$$

which means that the element a must be an element of the conjugacy class C_2 and b an element of C_3. This notation means that by multiplying these two elements iteratively all elements of the group can be generated. These elements can alternatively be constructed through an order-two generator $S = a$, and an order-five generator $T = bab$ such that:

$$\langle S, T | S^2 = T^5 = (T^2 S T^3 S T^{-1} S T S T^{-1})^3 = e \rangle$$

A convinient basis for these representations is the Shirai basis, presented in [11], which is a a basis that uses matrix generators of order two and order five with the form of S and T. Using this basis, it is possible to find the irreducible decomposition of the tensor product representations. According to table 6.2, the tensor product $\mathbf{3} \otimes \mathbf{3}$ decomposes as a direct sum of a one-dimensional, a three-dimensional and a five-dimensional irreducible space, which are left invariant under the action of the tensor product $\mathbf{3} \otimes \mathbf{3}$. Then, each of these spaces can be identified with a singlet, a triplet and a fiveplet state. Given any three-dimensional basis $\{|0\rangle, |1\rangle, |2\rangle\}$, the singlet state, associated with the irreducible representation $\mathbf{1}$ is given by:

$$\mathbf{1} = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \tag{6.1}$$

whereas the three dimensional representation $\mathbf{3}$ is spanned by the following three vectors:

$$\mathbf{3} = \frac{1}{\sqrt{2}} \begin{bmatrix} |21\rangle - |12\rangle \\ |02\rangle - |20\rangle \\ |10\rangle - |01\rangle \end{bmatrix}, \tag{6.2}$$

and the following five states generate a five-dimensional representation, labeled by the number 5:

$$\mathbf{5} = \frac{1}{\sqrt{2}} \begin{bmatrix} |11\rangle - |00\rangle \\ |10\rangle + |01\rangle \\ |21\rangle + |12\rangle \\ |02\rangle + |20\rangle \\ -\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle - 2\,|22\rangle) \end{bmatrix}. \tag{6.3}$$

It is important to take into account that there is an abuse of notation in the latter equations. The left-hand side of these equations are not really equal to the right-side. The meaning of these expressions is that the right-hand side of equations (6.1)-(6.3) are the vectors that span the irreducible subspaces in the left-hand side. This notation is used for convenience and in a similar manner as in the paper this section is based on [11].

According to table 6.2, the tensor product $\mathbf{3'} \otimes \mathbf{3'}$ has a similar decomposition as $\mathbf{3} \otimes \mathbf{3}$. In fact, its direct sum decomposition also has one-dimensional and three-dimensional subspaces generated by the vectors shown in equations (6.1) and (6.2), respectively. Nevertheless, one finds a different symmetric combination for the five-dimensional irreducible subspaces that differs from that in equation (6.3):

$$\mathbf{5} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{2}(\frac{-1}{\phi}\left|00\right\rangle - \phi\left|11\right\rangle + \sqrt{5}\left|22\right\rangle) \\ \left|10\right\rangle + \left|01\right\rangle \\ -\left|20\right\rangle - \left|02\right\rangle \\ \left|12\right\rangle + \left|21\right\rangle \\ -\frac{1}{2\sqrt{3}}((1-3\phi)\left|00\right\rangle + (3\phi-2)\left|11\right\rangle + \left|22\right\rangle) \end{bmatrix}. \tag{6.4}$$

On the other hand, the tensor product $\mathbf{3} \otimes \mathbf{3'}$ decomposes into a four-dimensional subspace and a five-dimensional subspace, which are generated by vectors of the form:

$$\mathbf{4} = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{\phi}\left|21\right\rangle - \phi\left|02\right\rangle \\ \phi\left|20\right\rangle + \frac{1}{\phi}\left|12\right\rangle \\ -\frac{1}{\phi}\left|00\right\rangle + \phi\left|11\right\rangle \\ \left|10\right\rangle - \left|01\right\rangle + \left|22\right\rangle \end{bmatrix}, \tag{6.5}$$

$$\mathbf{5} = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{2}(\phi^2\left|10\right\rangle - \frac{1}{\phi^2}\left|01\right\rangle - \sqrt{5}\left|22\right\rangle) \\ -(\phi\left|00\right\rangle + \frac{1}{\phi}\left|11\right\rangle) \\ \frac{1}{\phi}\left|20\right\rangle - \phi\left|12\right\rangle \\ \phi\left|21\right\rangle + \frac{1}{\phi}\left|02\right\rangle \\ \frac{3}{2}(\frac{1}{\phi}\left|10\right\rangle + \phi\left|01\right\rangle + \left|22\right\rangle) \end{bmatrix}. \tag{6.6}$$

Given a four dimensional basis, say $\{\left|0\right\rangle, \left|1\right\rangle, \left|2\right\rangle, \left|3\right\rangle\}$, then the one-dimensional invariant subspace of the tensor product space $\mathbf{4} \otimes \mathbf{4}$ is generated by the following state:

$$\mathbf{1} = \frac{1}{2}(\left|00\right\rangle + \left|11\right\rangle + \left|22\right\rangle + \left|33\right\rangle), \tag{6.7}$$

and the tensor product $\mathbf{5} \otimes \mathbf{5}$ has a one-dimensional invariant subspace with its unique vector of the following form:

$$1 = \frac{1}{\sqrt{5}}(\left|00\right\rangle + \left|11\right\rangle + \left|22\right\rangle + \left|33\right\rangle + \left|44\right\rangle). \tag{6.8}$$

6.2 Qutrit-states: Icosahedral rotation group $\mathcal{I}$

In section 5.1 the construction of qubit Kronecker states using irreducible representations of the symmetric group S_3 was simplified by taking into account that the representation $[2,1]$

is self-adjoint. Similar constructions of maximally entangled states can be performed from tensor products of N copies of irreducible representations of other symmetric groups. In this section, such construction is focused on the self-adjoint diagram $[3,1,1]$, which corresponds to a six-dimensional representation of S_5 and is self-adjoint. It will be shown in this section that these states can be obtained from Kronecker states of dimension $d=3$, arising from two inequivalent irreducible representations of $\mathcal{A}_5$. By proceeding this way, the construction of maximally entangled states can be simplified, because instead of dealing with states with local dimension $d=6$, LME states arise as linear combinations of just two product states of N qutrits, which reduces the complexity of the problem. This is further shown to generalize for any tensor product of self-adjoint representations for S_n, for any n, and gives a simpler way of interpreting LME states.

It is known that S_5 representation $[3,1,1]$ is six-dimensional, but under the alternating group it decomposes into two inequivalent three-dimensional representations, i.e., $[3,1,1]^+$ and $[3,1,1]^-$, which can equivalently be labeled as $\mathbf{3}$ and $\mathbf{3'}$, given the isometry between this group and the icosahedral group $\mathcal{I}$. It is possible to construct maximally entangled states from tensor products of the two inequivalent three-dimensional irreducible representations of the icosahedral rotation group $\mathcal{I}$ by following the rules given by equations (6.1)-(6.8) in section 6.1. One could couple N copies of one of these inequivalent three-dimensional representations and find the explicit expressions for the Kronecker states in a N-fold tensor product, or combine N_1 copies of the irreducible representation $\mathbf{3}$ and N_2 copies of the representation $\mathbf{3'}$ and find its trivial invariant subspace. Nevertheless, these states, as obtained using such construction, are not easy to analyse. Here, a different approach is discussed.

The rotational icosahedral group is a finite subgroup of the Special Orthogonal group $SO(3)$, the group of all orthogonal matrices of dimension 3 with determinant 1, which characterize rotations in the real 3-dimensional space. Thus, the matrix representations of $\mathcal{I}$ can be labeled by some phase ϕ, since they have eigenvalues $e^{i\phi}$,$e^{-i\phi}$ and 0. This phase indicates the angle by which some coordinate is rotated. As mentioned in the previous section, each of these phases is related to the respective conjugacy classes of $\mathcal{I}$. Furthermore, it has been seen that the entire group can be generated by two elements: one from the class C_2 and one from C_3. With the elements S_3 and T_3 of the so-called Shirai basis, as defined in [11], the elements of $\mathcal{I}$ can be generated.

Let the basis vectors $|0\rangle$, $|1\rangle$, $|2\rangle$ label the x,y and z axes in a 3-dimensional vector space, and denote this as the *computational basis*. In this space, the vertices of the icosahedron can be obtained from all possible cyclic permutations of the coordinate $(0,\pm1,\pm\phi)$. It is possible to show that the eigenvectors of the representative of the C_3 class, whose eigenvalues are related to the phases $2\pi/3$,$-2\pi/3$ and 0, are the following, respectively:

$$|+\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} -e^{i\pi/6} \\ 1 \\ -e^{-i\pi/6} \end{bmatrix}, \quad |-\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} -e^{-i\pi/6} \\ 1 \\ -e^{i\pi/6} \end{bmatrix}, \quad |0\rangle = \frac{-1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{6.9}$$

Using the notation (5.37), it is possible to find that, in the $\{|+\rangle, |-\rangle, |0\rangle\}$ basis, the only invariant state under the action of the 2-fold tensor product of the representation **3** of $\mathcal{I}$, which is equivalent to the representation $[3,1,1]^+$ of $\mathcal{A}_5$, is:

$$|[5],[2],1,1\rangle = \frac{1}{\sqrt{3}}(|+-\rangle + |-+\rangle + |00\rangle), \tag{6.10}$$

which is an equal-weight linear combination of product states whose local phases add up to 0 module 2π, and is symmetric under the interchange of any two qubits. If $N=3$, the Kronecker state takes the form:

$$\left|[5],[1^3],1,1\right\rangle = \frac{1}{\sqrt{6}}(|+-0\rangle - |-+0\rangle - |+0-\rangle + |0+-\rangle + |-0+\rangle - |0-+\rangle). \tag{6.11}$$

Again, this state is a sum of product states whose local phases add up to zero modulo 2π. It can also be checked that any permutation $\pi \in S_3^{parties}$ on the parties maps the state to $|[5],[1^3],1,1\rangle$ or $-|[5],[1^3],1,1\rangle$ depending on whether the permutation is even or odd, respectively. This means that this state is antisymmetric under the action of $S_3^{parties}$ over the indices. So far the states for $N=2$ and $N=3$ are similar to the qubit case, since they must satisfy the local phase sum condition and they are either symmetric or antisymmetric.

The analysis of LME states for $N \geq 4$ in terms of the symmetric group $S_4^{parties}$ is not that straightforward. In this case, there is a set of 3 orthogonal qutrit Kronecker states that can be found in the tensor product space $[3,1,1]^+$. These states are not necessarily symmetric or antisymmetric. Nevertheless, as in the qubit case, the tensor product space can be decomposed in terms of invariant subspaces of $GL(3,\mathbf{C})$ and $S_4^{parties}$, using Schul-Weyl duality, and this can be used to classify the states according to how they transform under the action of $S_4^{parties}$ on the parties. Then, the Hilbert space $\mathcal{H}^{\otimes 4}$ decomposes as follows:

$$\mathcal{H}^{\otimes 4} = (\mathcal{R}_{([4]} \otimes \mathcal{S}_{[4]}) \oplus (\mathcal{R}_{[3,1]} \otimes \mathcal{S}_{[3,1]}) \oplus (\mathcal{R}_{[2,2]} \otimes \mathcal{S}_{[2,2]}) \oplus (\mathcal{R}_{[2,1,1]} \otimes \mathcal{S}_{[2,1,1]}). \tag{6.12}$$

Therefore, the LME states found can be labeled by irreducible subspaces under the action of $GL(3,\mathbf{C})$ of $\mathcal{H}^{\otimes 4}$. Using the labeling convention introduced in the previous chapter in equation (5.37), the following three states are obtained when computing the projectors onto the irreducible $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$ sectors, in terms of the $\{|+\rangle, |-\rangle, |0\rangle\}$ basis:

$$\begin{aligned} |[5],[4],1,1\rangle \propto\ & 2\,|++--\rangle + 2\,|+--+\rangle + 2\,|+-+-\rangle + 2\,|-+-+\rangle + 2\,|-++-\rangle + \\ & +2\,|--++\rangle + |+-00\rangle + |-+00\rangle + |+0-0\rangle + |-0+0\rangle + \\ & + |+00-\rangle + |-00+\rangle + |00-+\rangle + |00+-\rangle + |0-0+\rangle + \\ & + |0+0-\rangle + |0-+0\rangle + |0+-0\rangle + 3\,|0000\rangle , \end{aligned} \tag{6.13}$$

which is a symmetric state and:

$$
\begin{aligned}
|[5],[2,2],1,1\rangle \propto (|-++-\rangle - |+-+-\rangle) + (|+--+\rangle - |-+-+\rangle) + (|+0-0\rangle - \\
- |0+-0\rangle) + (|0+0-\rangle - |+00-\rangle) + (|-0+0\rangle - |0-+0\rangle) + \\
+ (|0-0+\rangle - |-00+\rangle),
\end{aligned}
\tag{6.14}
$$

$$
\begin{aligned}
|[5],[2,2],2,1\rangle \propto -2\,|++--\rangle + |+-+-\rangle + |+--+\rangle + 2\,|+-00\rangle - |+0-0\rangle \\
- |+00-\rangle + |-++-\rangle + |-+-+\rangle + 2\,|-+00\rangle - 2\,|--++\rangle \\
- |-0+0\rangle - |-00+\rangle - |0+-0\rangle - |0+0-\rangle - |0-+0\rangle \\
- |0-0+\rangle + 2\,|00+-\rangle + 2\,|00-+\rangle ,
\end{aligned}
\tag{6.15}
$$

which generate an irreducible two-dimensional subspace under the action of the symmetric group $S_4^{parties}$, labeled by the partition $[2,2]$. In all of these states the norm is omitted. The state (6.13) is symmetric under permutations of its parties. According to the decomposition of the tensor product space in equation (6.12), any permutation on the indices of the qutrits of a state in this space decomposes as follows:

$$
P_\pi = [4]^{15} \oplus [2,2]^{6} \oplus [3,1]^{15} \oplus [2,1,1]^{3}. \tag{6.16}
$$

This indicates that apart from the symmetric Kronecker state in (6.13) there are 14 orthogonal states that are symmetric but not maximally entangled. Similarly, there is a total of 6 two-dimensional irreducible subspaces under the action of S_4^P, and only one of these is composed of maximally entangled states. This suggests that, even if Kronecker states generate subspaces that are invariant under the symmetric group, its construction is not canonical since there are states with similar symmetric group structures that are not maximally entangled.

The latter statements suggest that any qutrit LME state generated using tensor products of representations of $\mathcal{A}_5$ can be classified according to their symmetric properties. This generalizes from the qubit case, and therefore suggests that can be used for any dimension. The LME states, again, can be identified through the action of the permutation matrix P_π over its N qutrits, and its decomposition according to Schur-Weyl duality. In order to give further insight into these ideas, it is necessary to visualize what happens for greater N. For instance, the 5-fold tensor product of the irreducible representation $\mathbf{3}$ of $\mathcal{I}$ reduces into the following direct sum of irreducible representations of $\mathcal{I}$:

$$
\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1}^{6} \oplus \mathbf{3}^{16} \oplus \mathbf{3'}^{11} \oplus \mathbf{4}^{14} \oplus \mathbf{5}^{20},
$$

which means that the multiplicity of the invariant trivial subspace of $\mathcal{I}$ is 6, i.e., there are six linearly independent Kronecker states. Then, again, the task here is to check whether the six states preserve some symmetry property. It can be shown that any permutation matrix P_π for $\pi \in S_5^P$ decomposes in a direct sum of representations labeled by the partitions $[5]$, $[4,1]$, [3, 2],[3, 1, 1], with dimensions one, four, five and six, respectively. It can be shown that the six different Kronecker vectors:

$$
\begin{aligned}
|[5],[3,1,1],1,1\rangle \propto (&|+-+-0\rangle - |+-+0-\rangle - |+--+0\rangle + |+--0+\rangle + \\
&+|+-0+-\rangle - |+-0-+\rangle + |-++-0\rangle - |-++0-\rangle - \\
&-|-+-+0\rangle + |-+-0+\rangle + |-+0+-\rangle - |-+0-+\rangle + \\
&+|00+-0\rangle - |00+0-\rangle - |00-+0\rangle + |00-0+\rangle + \\
&+|000+-\rangle - |000-+\rangle,
\end{aligned}
\tag{6.17}
$$

$$
\begin{aligned}
|[5],[3,1,1],2,1\rangle \propto (&|++--0\rangle - |++-0-\rangle + |+-+-0\rangle - |+-+0-\rangle \\
&+|+00-0\rangle - |+000-\rangle - |-+-+0\rangle + |-+-0+\rangle - \\
&-|--++0\rangle + |--+0+\rangle - |-00+0\rangle + |-000+\rangle + \\
&+|0+-+-\rangle - |0+--+\rangle + |0-++-\rangle - |0-+-+\rangle + \\
&+|000+-\rangle - |000-+\rangle,
\end{aligned}
\tag{6.18}
$$

$$
\begin{aligned}
|[5],[3,1,1],3,1\rangle \propto (&|+-+0-\rangle + |+--0+\rangle - |+-0+-\rangle - |+-0-+\rangle - \\
&-|+0+--\rangle + |+0-+-\rangle + |+0-00\rangle - |+00-0\rangle - \\
&-|-++00\rangle - |-+-0+\rangle + |-+0+-\rangle + |-+0-+\rangle - \\
&-|-0+-+\rangle - |-0+00\rangle + |-0-++\rangle + |-00+0\rangle + \\
&+|0++--\rangle - |0+-+-\rangle - |0+-00\rangle + |0+0-0\rangle + \\
&+|0-+-+\rangle + |0-+00\rangle - |0--++\rangle - |0-0+0\rangle),
\end{aligned}
\tag{6.19}
$$

$$
\begin{aligned}
|[5],[3,1,1],4,1\rangle \propto (&6|++-0-\rangle - 6|++0--\rangle - 2|+-+-0\rangle - |+-+0-\rangle + \\
&+2|+--+0\rangle + |+--0+\rangle + |+-0+-\rangle - |+-0-+\rangle + \\
&+3|+0+--\rangle - 3|+0-+-\rangle + 3|+0-00\rangle - 3|+00-0\rangle - \\
&-2|-++-0\rangle - |-++0-\rangle + 2|-+-+0\rangle + |-+-0+\rangle + \\
&+|-+0+-\rangle - |-+0-+\rangle - 6|--+0+\rangle + 6|--0++\rangle + \\
&+3|-0+-+\rangle - 3|-0+00\rangle - 3|-0-++\rangle + 3|-00+0\rangle + \\
&+3|0++--\rangle - 3|0+-+-\rangle + 3|0+-00\rangle - 3|0+0-0\rangle + \\
&+3|0-+-+\rangle - 3|0-+00\rangle - 3|0--++\rangle + 3|0-0+0\rangle + \\
&+4|00+-0\rangle + 2|00+0-\rangle - 4|00-+0\rangle - 2|00-0+\rangle - \\
&-2|000+-\rangle + 2|000-+\rangle),
\end{aligned}
\tag{6.20}
$$

$$
\begin{aligned}
|[5],[3,1,1],5,1\rangle \propto (&-2\,|{+}-{+}-0\rangle + 3\,|{+}-{+}0-\rangle - 2\,|{+}--{+}0\rangle + 3\,|{+}--0+\rangle + \\
&+3\,|{+}-0+-\rangle + 3\,|{+}-0-+\rangle + 4\,|{+}-000\rangle - |{+}0+--\rangle - \\
&-|{+}0-+-\rangle - 6\,|{+}0--+\rangle - 3\,|{+}0-00\rangle - 3\,|{+}00-0\rangle + \\
&+2\,|{+}000-\rangle + 2\,|{-}++-0\rangle - 3\,|{-}++0-\rangle + 2\,|{-}+-{+}0\rangle - \\
&-3\,|{-}+-0+\rangle - 3\,|{-}+0+-\rangle - 3\,|{-}+0-+\rangle - 4\,|{-}+000\rangle + \\
&+6\,|{-}0++-\rangle + |{-}0+-+\rangle + 3\,|{-}0+00\rangle + |{-}0-++\rangle + \\
&+3\,|{-}00+0\rangle - 2\,|{-}000+\rangle + |0++--\rangle + |0+-+-\rangle + \\
&+6\,|0+--+\rangle + 3\,|0+-00\rangle + 3\,|0+0-0\rangle - 2\,|0+00-\rangle - \\
&-6\,|0-++-\rangle - |0-+-+\rangle - 3\,|0-+00\rangle - |0--++\rangle - \\
&-3\,|0-0+0\rangle + 2\,|0-00+\rangle),
\end{aligned}
\tag{6.21}
$$

$$
\begin{aligned}
|[5],[3,1,1],6,1\rangle \propto (&7\,|{+}+--0\rangle - 3\,|{+}+-0-\rangle - 3\,|{+}+0--\rangle - 2\,|{+}-{+}0-\rangle + \\
&+2\,|{+}--0+\rangle - 2\,|{+}-0+-\rangle + 2\,|{+}-0-+\rangle + 2\,|{+}0+--\rangle + \\
&+2\,|{+}0-+-\rangle - 3\,|{+}0--+\rangle + 2\,|{+}0-00\rangle + 2\,|{+}00-0\rangle - \\
&-3\,|{+}000-\rangle - 2\,|{-}++0-\rangle + 2\,|{-}+-0+\rangle - 2\,|{-}+0+-\rangle + \\
&+2\,|{-}+0-+\rangle - 7\,|{-}-++0\rangle + 3\,|{-}-+0+\rangle + 3\,|{-}-0++\rangle + \\
&+3\,|{-}0++-\rangle - 2\,|{-}0+-+\rangle - 2\,|{-}0+00\rangle - 2\,|{-}0-++\rangle - \\
&-2\,|{-}00+0\rangle + 3\,|{-}000+\rangle + 2\,|0++--\rangle + 2\,|0+-+-\rangle - \\
&-3\,|0+--+\rangle + 2\,|0+-00\rangle + 2\,|0+0-0\rangle - 3\,|0+00-\rangle + \\
&+3\,|0-++-\rangle - 2\,|0-+-+\rangle - 2\,|0-+00\rangle - 2\,|0--++\rangle - \\
&-2\,|0-0+0\rangle + 3\,|0-00+\rangle + 3\,|00+0-\rangle - 3\,|00-0+\rangle + \\
&+3\,|000+-\rangle - 3\,|000-+\rangle),
\end{aligned}
\tag{6.22}
$$

generate a six-dimensional $[3,1,1]$ irreducible subspace under the action of a permutation matrix P_π on the qutrits. Thus, any permutation $\pi \in S_5^P$ acting over the five qutrits of these states leaves this subspace of vectors invariant, i.e., $\sum_i^6 c_i\,|[5],[3,1,1],i,1\rangle \in [3,1,1]$ for any set of complex numbers $\{c_i\}_{i=1}^6$.

It is possible to analyze bigger tensor product spaces and show that the set of all Kronecker vectors in the $\mathbf{3}^{\otimes N}$ representation can be decomposed into direct sums of invariant subspaces under the action of $S_N^{parties}$ appearing from the decomposition of the tensor product space $\mathcal{H}^{\otimes N}$. For instance, there are 16 different Kronecker states in the $3^{\otimes 6}$ tensor product space, and one can show that any of these states decomposes as a linear combination of Kronecker states lying the following S_6^P irreducible subspaces:

$$(3^{\otimes 6})^{Kronecker} = 2[6] \oplus [4,2] \oplus [2,2,2].$$

The dimensions of the subspaces [6], [4, 2] and [2, 2, 2] of S_6 are 1, 9 and 5, respectively, yielding a total dimension of $2*1+9+5 = 16$ that coincides with the multiplicity of the trivial subspace. Notice that, in this case, there are two orthogonal symmetric maximally entangled states.

From the previous examples of LME states that can be obtained from tensor product representations of the form $\mathbf{3}^{\otimes N}$, and from Weyl-Schur duality, it seems that any set of LME states found using this method can be arranged into irreducible subspaces of $\mathcal{H}^{\otimes N}$ under the action of the symmetric group $S_N^{parties}$ on its N parties. This was already observed in the qubit case. In the next subsection, the connection between these subspaces and the set of LME states found is further explored

6.2.1 Symmetric properties of LME qutrit states

As stated in the previous paragraphs, it should be possible to construct a basis of N-partite maximally entangled qutrit states that and arrange them into irreducible subspaces under the action of the symmetric group $S_N^{parties}$ on the Hilbert space $\mathcal{H}^{\otimes N}$. Similarly as in equation (5.35) for Kronecker states of qubits, the multiplicity of an irreducible representation $\lambda \vdash N$ in an invariant subspace I associated to the tensor product space $\mathbf{3}^{\otimes N}$ is given by:

$$m_{I,\lambda} = \frac{1}{|\mathcal{A}_5|} \sum_{g \in \mathcal{A}_5} \chi_I(g) \chi_\lambda(I(g)), \tag{6.23}$$

where $m_{I,\lambda}$ is the multiplicity of the irreducible representation I of the group $\mathcal{I}$ in the representation R_λ of $GL(3, \mathbf{C})$. In the expression, $|A_5| = |\mathcal{I}| = 60$, $\chi_I(g)$ is the character of the representation I for $g \in \mathcal{I}$ and $\chi_\lambda(I(g))$ is the character of the representation R_λ for the matrix $I(g)$. In order to compute these multiplicities one only needs to have the character table 6.1 for each of the conjugacy classes of $\mathcal{I}$. The term $\chi_\lambda(I(g))$ can be obtained taking into account that the characters of the representations of $GL(3, \mathbf{C})$ are given by the Schur polynomials, which were introduced in section 2.7. Since the matrices for the irreducible representations of dimension three of the icosahedron belong to the $SO(3)$ group, in this case the characters of R_λ take the form:

$$\chi_\lambda(I(g)) = s_\lambda(1, e^{i\phi}, e^{-i\phi}), \tag{6.24}$$

where the phase ϕ depends on the conjugacy class of the element $g \in \mathcal{I}$. Taking these facts into account, any tensor product of the form $\mathbf{3}^{\otimes N}$ can be decomposed in terms of irreducible subspaces of $S_N^{parties}$. For instance, the simplest non-trivial case $\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$ gives the following decomposition in terms of irreducible representations of $S_2^{parties}$:

$$\mathbf{3} \otimes \mathbf{3} = (\mathbf{1} \otimes [2]) \oplus (\mathbf{3} \otimes [1^2]) \oplus (\mathbf{5} \otimes [2]).$$

Here, the terms of the form $A \otimes B$ in the direct sum indicate the irreducible representation A of $\mathcal{I}$ and the irreducible representation B of $GL(3, \mathbf{C})$ in the tensor product space $\mathcal{H}^{\otimes N}$. According to the Schur-Weyl duality, the multiplicity of A is the dimension of B and the multiplicity of B is the dimension of A. This means that the subspaces associated to the irreducible representations **1** and **5** of the icosahedral group are symmetric and **3** is anti-symmetric, as expected [11]. One can also see that the 3 and 4-fold tensor product of the irreducible representation **3** decomposes in terms of the irreducible representations of $\mathcal{I}$ and $GL(3, \mathbf{C})$ as follows:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (1 \otimes [1^3]) \oplus (\mathbf{3} \otimes ([3] \oplus [2,1])) \oplus (\mathbf{3'} \otimes [3]) \oplus (\mathbf{4} \otimes [3]) \oplus (\mathbf{5} \otimes [2,1]), \tag{6.25}$$

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{1} \otimes ([4] \oplus [2,2])) \oplus (\mathbf{3} \otimes ([3,1] \oplus [2,1,1])) \oplus (\mathbf{3'} \otimes [3,1]) \oplus \\ \oplus (\mathbf{4} \otimes ([4] \oplus [3,1])) \oplus (\mathbf{5} \otimes (2[4] \oplus [3] \oplus [2,2])), \end{aligned} \tag{6.26}$$

The tables 6.3 and 6.4 contain the multiplicity of the irreducible representations of $\mathcal{I}$ in the irreducible representations of $GL(3, \mathbf{C})$ for tensor products of the form $\mathbf{3}^{\otimes N}$, for the cases $N = 5$ and $N = 6$. These multiplicities were obtained using the code in Appendix C, implemented in Mathematica [1]. Similar decompositions can be obtained for the tensor product spaces of the form $\mathbf{3'}^{\otimes N}$, for greater N by simply interchanging the representations **3** and **3'** in the decompositions. Furthermore, one can also find the trivial subspace in direct sum decomposition of tensor products of the form $\mathbf{3}^{\otimes N_1} \otimes 3^{\otimes N_2}$. However, the decomposition of these spaces under the Schur-Weyl duality is not that straightforward. In the next subsection, this latter statement is further reviewed.

	1	3	3'	4	5
(5)	0	1	1	2	2
(4,1)	0	2	2	1	1
(3,2)	0	1	1	1	1
(3,1,1)	1	0	1	0	0
(2,2,1)	0	0	0	1	0

Table 6.3: Table with the multiplicities of irreducible representations of $\mathcal{I}$ in the irreducible representations of $GL(3, \mathbf{C})$ in the tensor product space $\mathbf{3}^{\otimes 5}$. The entry (i,j) of this table indicates the multiplicity of the representation$j$ of $\mathcal{I}$ in the representation i of $GL(3, C)$.

[1]The code was implemented by professor Alonso Botero

	1	3	3'	4	5
(6)	2	2	3	1	0
(5,1)	0	2	3	2	2
(4,2)	1	2	3	0	1
(4,1,1)	0	1	0	1	1
(3,3)	0	1	0	1	1
(3,2,1)	0	0	1	1	0
(2,2,2)	1	0	0	0	0

Table 6.4: Table with the multiplicities of irreducible representations of $\mathcal{I}$ in the irreducible representations of $GL(3,C)$ in the tensor product space $\mathbf{3}^{\otimes 6}$. The meaning of these entries is the same as for table 6.3

6.2.2 LME states from tensor products of the form $\mathbf{3}^{\otimes N_1} \otimes \mathbf{3'}^{\otimes N_2}$

Tensor products containing N_1 copies of the representation $\mathbf{3}$ and N_2 copies of the remaining inequivalent 3-dimensional representation, known as $\mathbf{3'}$, can also be used to find LME states. The task here is to find the direct sum irreducible decomposition of tensor products of the form $\mathbf{3}^{\otimes N_1} \otimes \mathbf{3'}^{\otimes N_2}$, and, in particular, the multiplicity of its trivial subspace. According to Schur-Weyl duality, each of the irreducible subspaces arising in these decompositions further decompose into irreducible representations of $GL(3,\mathbf{C})$ and $S_N^{parties}$, which are now labeled by partitions $\lambda_1 \vdash N_1$ and $\lambda_2 \vdash N_2$. Then, the maximally entangled states found in this tensor product space can be labeled in terms of irreducible subspaces of the form $[\lambda_1] \otimes [\lambda_2]$, which are invariant under the simultaneous action of the groups $S_{N_1}^{parties}$ and $S_{N_2}^{parties}$.

From the tensor product decomposition rules given in table 6.1, it is straightforward to see that the multiplicity of the trivial subspace $\mathbf{3}^{\otimes N_1} \otimes \mathbf{3'}^{\otimes N_2}$ for $N_1 + N_2 = 2$ or $N_1 + N_2 = 3$ for $N_1 > 0$ and $N_2 > 0$ is zero, so that it is not possible to find further LME states. In contrast, for $N_1 + N_2 \geq 4$ there can be found LME states such that $N_1 > 0$ and $N_2 > 0$. For instance if $N_1 + N_2 = 4$ the solutions $N_1 = N_2 = 2$ and $N_1 = 1$, $N_2 = 3 (N_1 = 3,\ N_2 = 1)$ give two and one Kronecker states each. Taking into account the tensor product decomposition of $\mathbf{3} \otimes \mathbf{3}$ and $\mathbf{3'} \otimes \mathbf{3'}$, it is possible to find that the invariant one dimensional subspace decomposes as follows in terms of the irreducible representations of $GL(3,\mathbf{C})$:

$$(\mathbf{3}^{\otimes 2} \otimes (\mathbf{3'}^{\otimes 2}))^{trivial} = 2([2] \otimes [2']), \tag{6.27}$$

where [2] and [2'] correspond to the same trivial representation of $S_2^{parties}$ but are labeled distinctly in order to distinguish the product spaces $\mathbf{3}^{\otimes 2}$ and $\mathbf{3'}^{\otimes 2}$, respectively. The superscript ^{trivial} in the expression refers to the fact that only the trivial subspace of the tensor product representation is taken into account. This decomposition indicates that these states are symmetric under transpositions on the first two qutrits and the last two qutrits. In fact, one of these two states can be written in terms of the vectors $\{|+\rangle, |-\rangle, |0\rangle, |+'\rangle, |-'\rangle, |0'\rangle\}$ as follows:

$$|[2]\otimes[2']\rangle = \frac{1}{3}(|+-+'-'\rangle + |+--'+'\rangle + |+-0'0'\rangle + |-++'-'\rangle + |-+-'+'\rangle + \\ + |-+0'0'\rangle + |00+'-'\rangle + |00-'+'\rangle + |000'0'\rangle), \tag{6.28}$$

and can easily be shown to be invariant under a permutation over the first or the last two indices. Here is important to note that the vectors $|+'\rangle$, $|-'\rangle$ and $|0'\rangle$ are the eigenvectors of the representative of the C_3 class of the representation $\mathbf{3'}$. This basis is used on the qutrits where this latter representation acts, which in this case coincides with the third and fourth qutrit. The form of the vector (6.28) indicates that tracing out the first or the last two indices generates a symmetric two-dimensional state. Remarkably, this state is a product state between the first two qutrits and the last two.

The latter symmetric properties of Kronecker states in the tensor product $\mathbf{3}^{\otimes 2}\otimes\mathbf{3'}^{\otimes 2}$ suggests that any Kronecker state in tensor products of the form $\mathbf{3}^{\otimes N_1}\otimes\mathbf{3'}^{\otimes N_2}$ can be written in terms of bases of irreducible subspaces under the simultaneous action of some irreducible representation $\lambda_1 \vdash N_1$ of $S_{N_1}^{parties}$ over the N_1 parties where the representation $\mathbf{3}$ acts, and some other representation $\lambda_2 \vdash N_2$ over the remaining N_2 parties. This means that they are invariant under $[\lambda_1]\otimes[\lambda_2]$, up to permutations of the irreducible representations in the tensor product. In fact, there are $\binom{N_1+N_2}{N_1}$ possible different tensor product combinations of tensor products of N_1 copies of the representation $\mathbf{3}$ and N_2 copies of the representation $\mathbf{3'}$. To illustrate this, it is important to note that there are $\binom{2+2}{2} = 6$ possible forms of reorganizing the terms in the tensor product. The resulting Kronecker states are obtained by interchanging these indices according to each of these six inequivalent combinations. For instance, the first of the two possible Kronecker states obtained in the tensor product $\mathbf{3}\otimes\mathbf{3'}\otimes\mathbf{3'}\otimes\mathbf{3}$ is shown:

$$|[2]\otimes[2']\rangle' = |++'-'-\rangle + |+-'+'-\rangle + |+0'0'-\rangle + |-+'-'+\rangle + |--'+'+\rangle + \\ + |-0'0'+\rangle + |0+'-'0\rangle + |0-'+'0\rangle + |00'0'0\rangle\,, \tag{6.29}$$

which is only a permutation of the indices of the state given in the equation (6.28). Therefore, the form of a Kronecker state in a tensor product representation of N_1 copies of the representation 3 and N_2 copies of $\mathbf{3'}$ depends on how these states transform under the action of irreducible representations $[\lambda_1]$ and $[\lambda_2]$ of $S_{N_1}^P$ and $S_{N_2}^P$, the dimension of these representations, and the multiplicity of the different subspaces $[\lambda_1]\otimes[\lambda_2]$ in which the tensor product decomposes. Taking this into account, one could make use of a total of five labels in order to identify these states for a tensor product combination of 3-dimensional irreducible representations of $\mathcal{I}$:

- Given a tensor product representation composed of N_1 copies of representation $\mathbf{3}$ and N_2 copies of representation $\mathbf{3'}$ such that $N_1+N_2 = N$, the first two labels identify the Kronecker states according to how the N_1 qutrits corresponding to representation $\mathbf{3}$ transform under the action of $S_{N_1}^P$ and how the remaining qutrits transform under $S_{N_2}^P$. Therefore, these labels correspond to partitions $\lambda_1 \vdash S_{N_1}^P$ and $\lambda_2 \vdash S_{N_2}^P$.

- The third label is a quantum number i_1 that identifies each of the possible f^{λ_1} standard Young tableaux of λ_1.

- The fourth label is a quantum number i_2 that identifies each of the possible f^{λ_2} standard Young tableaux of λ_2.

- The fifth label is a quantum number m that distinguishes the Kronecker states according to the multiplicity of $[\lambda_1]\otimes[\lambda_2]$ in the trivial subspace of the tensor product representation.

Therefore LME states in a N-fold tensor product combination of inequivalent irreducible three-dimensional representations of the icosahedral group can be labeled as follows:

$$|[\lambda_1]_{a,b,c..}, [\lambda_2]_{d,e,f..}, i_1, i_2, m\rangle\,, \quad 1 \leq i_1 \leq f^{\lambda_1}, \quad 1 \leq i_2 \leq f^{\lambda_2}, \quad 1 \leq m \leq m_{1,\lambda_1,\lambda_2}, \tag{6.30}$$

where the subscripts $_{a,b,c...}$ and $_{d,e,f..}$ indicate the qutrits in which the representations $\mathbf{3}$ and $\mathbf{3'}$ act, respectively. Then, the Kronecker states in equations (6.28) and (6.29) can be simplified as, correspondingly:

$$|[2]_{1,2}, [2]_{3,4}, 1, 1, 1\rangle\,, \tag{6.31}$$

$$|[2]_{1,4}, [2]_{2,3}, 1, 1, 1\rangle\,. \tag{6.32}$$

One can find other 4-qutrit Kronecker states using other combinations of the inequivalent 3-dimensional representations. For example:

$$\begin{aligned} \mathbf{3}^{\otimes 3} \otimes \mathbf{3}'^{trivial} &= [3] \\ \mathbf{3}'^{\otimes 3} \otimes \mathbf{3}^{trivial} &= [3]. \end{aligned}$$

In this case, the only possible states are of the form $|[3]_{a,b,d}, [1]_d, 1, 1, 1\rangle$ which are symmetric under the action on the qutrits a, b and d, where $a, b, c, d \in \{1, 2, 3, 4\}$ and $a \neq b \neq c \neq d$.

The same is valid for greater N, and it can be easily checked. For combinations of five of these irreducible 3-dimensional representations ($N_1 + N_2 = 5$) one finds that the one-dimensional trivial subspace decomposes into the following subspaces under the action of $S_{N_1}^{parties}$ and $S_{N_2}^{parties}$:

$$\mathbf{3}^{\otimes 4} \otimes \mathbf{3'} = [3, 1]$$

$$\begin{aligned} \mathbf{3}^{\otimes 3} \otimes \mathbf{3}'^{\otimes 2} &= ([1^3] \otimes [2']) \oplus ([2, 1] \otimes [2']) \oplus ([3] \otimes [1^2]') \\ \mathbf{3}^{\otimes 2} \otimes \mathbf{3}'^{\otimes 3} &= ([2] \otimes [1^3]') \oplus ([2] \otimes [2, 1]') \oplus ([1^2] \otimes [3]'). \end{aligned}$$

This kind of construction can be generalized for any N qutrits. Different sets of LME states can be generated from tensor product spaces containing N_1 copies of **3** and N_2 copies of **3**$'$, and can be classified into $[\lambda_1] \otimes [\lambda_2]$ irreducible sectors according to Weyl-Schur duality, with $\lambda_1 \vdash N_1$ and $\lambda_2 \vdash N_2$. This classification will be shown to be important for constructing maximally entangled states of N qudits, with $d = 6$ in the next section. In particular, it will be shown that any Kronecker state in the tensor product space $[3, 1, 1]^{\otimes N}$ is a linear combination of just two qutrit LME states that transform similarly under the simultaneous action of the groups $S_{N_1}^{parties}$ and $S_{N_2}^{parties}$, with $N_1 + N_2 = N$. We show that this classification can be extended to any $N-$ fold product space of self-adjoint irreducible representation of any symmetric group, and that such construction substantially reduces the computational complexity of the problem of finding Kronecker states.

6.3 Kronecker states for $d = 6$ from representations of $\mathcal{I}$

The icosahedral rotational group $\mathcal{I}$ is isomorphic to the alternating group $\mathcal{A}_5$, as previously commented, with the irreducible representations **3** and **3**$'$ corresponding to the representations $[3, 1, 1]^+ \downarrow \mathcal{A}_5$ and $[3, 1, 1]^- \downarrow \mathcal{A}_5$, respectively. Nevertheless, as seen from S_5 the partition $[3, 1, 1]$ is associated to an irreducible representation of dimension 6. The proposal here is to find whether a construction of the Kronecker states in the tensor product $[3, 1, 1]^{\otimes N}$ can be achieved from the representations of $\mathcal{A}_5$ by choosing an appropriate change of basis between these representations. The matrix that generates this transformation is defined as follows:

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, the transposition $(1, 2) \in S_5$, which is a diagonal matrix of the form $diag(1, 1, 1, -1, -1, -1)$, can be seen in the basis $\{|+\rangle, |-\rangle, |0\rangle, |+'\rangle, |-'\rangle, |0'\rangle\}$ that is generated from the direct sum of the representations $[3, 1, 1]^+$ and $[3, 1, 1]^-$ of $\mathcal{A}_5$ as follows:

$$X = (12)|_{\mathcal{A}_5} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Any other element of $S_5/\mathcal{A}_5$ is an off-diagonal block matrix with blocks of dimension 3 as seen in the basis of the representations of $\mathcal{A}_5$.

It is well known that the symmetric group can be generated by any transposition and a length-n cycle. Then, the six-dimensional irreducible representations of S_5 can be generated with the (12) transposition and the (12345) cycle. The matrix for this last cycle can be easily obtained since this is an even permutation and therefore belongs to $\mathcal{A}_5$, so that its matrix representations are naturally given by the construction of the alternating group. In this basis, the representation $[3,1,1]$ of S_5 is a diagonal block matrix of two blocks of dimension 3 for the even permutations and an off-diagonal block matrix for the odd permutations, as stated before. Then, it is possible to write the Kronecker vectors of the tensor product $[3,1,1]^{\otimes N}$ in terms of the basis vectors $\{|+\rangle, |-\rangle, |0\rangle, |+'\rangle, |-'\rangle, |0'\rangle\}$.

For the case $N = 2$, there are just two Kronecker states, one in the trivial subspace and the other from the sign subspace. These Kronecker vectors are found to be:

$$|x, \pm\rangle = \frac{1}{\sqrt{6}}(|+-\rangle + |-+\rangle + |00\rangle \pm |+'-'\rangle \pm |-'+'\rangle \pm |0'0'\rangle),$$

which can be interpreted as linear combinations of the Kronecker states corresponding to the $\mathbf{3}^{\otimes 2}$ and the $\mathbf{3}'^{\otimes 2}$ tensor product representations of $\mathcal{I}$. Recall from section 3.16 that the tensor products $\mathbf{3}\otimes\mathbf{3}'$ and $\mathbf{3}'\otimes\mathbf{3}$ have no one-dimensional trivial subspace, so that there are no additional Kronecker states.

It is straightforward to see that the vectors $|x, \pm\rangle$ are linear combinations of orthogonal symmetric states under the action of $S_2^{parties}$ over the parts. Therefore, these states can be found to be linear combinations of states of the form $|\nu, \lambda, i, m_i\rangle$, as given by the notation (5.37):

$$\begin{aligned} \left|[2], [3,1,1]^+, 1, 1, +\right\rangle &= \frac{1}{\sqrt{2}}(\left|[2], [3,1,1]^+, 1, 1\right\rangle + \left|[2'], [31,1]^-, 1, 1\right\rangle, \\ \left|[2], [3,1,1]^+, 1, 1, -\right\rangle &= \frac{1}{\sqrt{2}}(\left|[2], [3,1,1]^+, 1, 1\right\rangle - \left|[2'], [3,1,1]^-, 1, 1\right\rangle), \end{aligned} \tag{6.33}$$

where the state $|[2'], [3,1,1]^-, 1, 1\rangle$ is obtained by just interchanging the vectors $|+\rangle, |-\rangle$ and $|0\rangle$ with the vectors $|+'\rangle, |-'\rangle$ and $|0'\rangle$, respectively.

In the tripartite case, one finds that the Kronecker coefficients over the trivial invariant subspace and the sign subspace give a total of 2 maximally entangled states, one from each subspace. The construction of the two Kronecker states for $[3,1,1]^{\otimes 3}$ is very similar to the bipartite case. These Kronecker states can be obtained from the state of equation (6.11) as follows:

$$\left|[5], [1^3], 1, 1, +\right\rangle = \frac{1}{\sqrt{2}}(\left|[5], [1^3], 1, 1\right\rangle \pm X^{\otimes 3}\left|[5], [1^3], 1, 1\right\rangle), \tag{6.34}$$

which is the same as having:

$$\left|[5], [1^3], 1, 1, +\right\rangle = \frac{1}{\sqrt{2}}(\left|[1^3], 1, 1\right\rangle \pm \left|[1^3]', 1, 1\right\rangle). \tag{6.35}$$

Note that in this case $|[5],[1^3],1,1\rangle$ is the only vector of the invariant subspace from the tensor product $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$ and the operation $T^{\otimes 3}$ transforms this vector to the Kronecker state in the tensor product $\mathbf{3}'\otimes\mathbf{3}'\otimes\mathbf{3}'$. Furthermore, it is important to note that the representations $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}'$ and $\mathbf{3}'\otimes\mathbf{3}'\otimes\mathbf{3}$ and any permutation of the indices in these tensor products have no invariant one-dimensional subspaces. This means that $|[5],[1^3],1,1,+\rangle$ and $|[5],[1^3],1,1,-\rangle$ are the only Kronecker states in this case.

The construction of Kronecker states of N qudits with $d=6$ for $N=2$ and $N=3$ suggest that these kind of states could be constructed from linear combinations of invariant states as the ones introduced in equation (6.30), which are obtained from the tensor products of the two inequivalent three dimensional representations of the alternating group $\mathcal{A}_5$. These states have the following form:

$$\begin{aligned} |[\lambda_1]_{a,b,c,\ldots},[\lambda_2]_{d,e,f,\ldots},i_1,i_2,m,\pm\rangle = \frac{1}{\sqrt{2}}(&|[\lambda_1]_{a,b,c,\ldots},[\lambda_2]_{d,e,f,\ldots},i_1,i_2,m\rangle \pm \\ &\pm |[\lambda_2]_{d,e,f,\ldots},[\lambda_1]_{a,b,c,\ldots},i_1,i_2,m\rangle), \\ 1\le i_1\le f^{\lambda_1},\quad 1\le i_2\le f^{\lambda_2},\quad &1\le m\le m_{1,\lambda_1,\lambda_2}. \end{aligned} \tag{6.36}$$

In order to further understand this construction, it could be useful to ilustrate the form of these states for N greater than 3.

For $N=4$, calculating the Kronecker coefficient for the invariant one dimensional subspaces, which are known as the trivial and sign representations of the tensor $[3,1,1]\otimes[3,1,1]\otimes[3,1,1]\otimes[3,1,1]$ gives a multiplicity of 26 linearly independent Kronecker states. This total is the sum of the 13 Kronecker states from the trivial subspace and 13 from the sign subspace. This number of states coincide with the sum of the multiplicities of the one-dimensional subspaces generated from every possible combination of the irreducible representations $\mathbf{3}$ and $\mathbf{3}'$ in a 4-fold tensor product. One can easily see that the first six states are given in terms of vectors of the form (6.13)-(6.15) as follows:

$$|[5],[4]_{1,2,3,4},1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[5],[4],1,1\rangle \pm |[5],[4]',1,1\rangle), \tag{6.37}$$

$$|[5],[2,2]_{1,2,3,4},1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[5],[2,2],1,1\rangle \pm |[5],[2,2]',1,1\rangle), \tag{6.38}$$

$$|[5],[2,2]_{1,2,3,4},2,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[5],[2,2],2,1\rangle \pm |[5],[2,2]',2,1\rangle). \tag{6.39}$$

(6.40)

Other states appear as combinations of representations $\mathbf{3}$ and $\mathbf{3}'$, such that the total number of representations in the tensor product is 4. The tensor products $\mathbf{3}^{\otimes 3}\otimes\mathbf{3}'$ and $\mathbf{3}'^{\otimes 3}\otimes\mathbf{3}$, and any possible distinct combination of the factors in these tensor products, yields one qutrit maximally entangled state each. Using the notation (6.36) There is a total of 8 distinct combinations of these factors, which gives eight different qutrit states:

$$|[3]_{2,3,4},[1]_1,1,1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[3]_{2,3,4},[1]_1,1,1,1\rangle \pm |[1]_1,[3]_{2,3,4},1,1,1\rangle), \tag{6.41}$$

$$|[3]_{1,3,4},[1]_2,1,1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[3]_{1,3,4},[1]_2,1,1,1\rangle \pm |[1]_2,[3]_{1,3,4},1,1,1\rangle), \tag{6.42}$$

$$|[3]_{1,2,4},[1]_3,1,1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[3]_{1,2,4},[1]_3,1,1,1\rangle \pm |[1]_3,[3]_{1,2,4},1,1,1\rangle), \tag{6.43}$$

$$|[3]_{1,2,3},[1]_4,1,1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[3]_{1,2,3},[1]_4,1,1,1\rangle \pm |[1]_4,[3]_{1,2,3},1,1,1\rangle). \tag{6.44}$$

The explicit expressions of such states can be obtained in terms of the computational basis by the invariant subspaces generated by $\mathbf{3}' \otimes \mathbf{3}'$ with the subspaces in $\mathbf{3} \otimes \mathbf{3}$, following the rules given in section 6.1.

Tensor products of three-dimensional representations of $\mathcal{I}$ containing two copies of the representation 3 and two copies of the representation 3', such as $3^{\otimes 2} \otimes 3'^{\otimes 2}$, as well as any other inequivalent combination of these factors, yield two qutrit Kronecker states each. These can be constructed by coupling the one dimensional invariant subspace appearing in the tensor products $3^{\otimes 2}$ and $3'^{\otimes 2}$ and the five dimensional subspaces appearing in each of these two tensor products using the rules given by the equations (6.3), (6.4) and (6.6) (since coupling two identical subspaces always guarantees a one-dimensional subspace in its direct sum decomposition). Similar coupling rules apply for the tensor product $3'^{\otimes 2} \otimes 3^{\otimes 2}$. One can find the following four Kronecker states with $d = 2$ in terms of those coupling coupling rules:

$$|[2]_{1,2},[2]_{3,4},1,1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[2]_{1,2},[2]_{3,4},1,1,1\rangle \pm |[2]_{3,4},[2]_{1,2},1,1,1\rangle), \tag{6.45}$$

$$|[2]_{1,2},[2]_{3,4},1,1,2,\pm\rangle = \frac{1}{\sqrt{2}}(|[2]_{1,2},[2]_{3,4},1,1,2\rangle \pm |[2]_{1,2},[2]_{3,4},1,1,2\rangle), \tag{6.46}$$

where $|[2]_{1,2},[2]_{3,4},1,1,1,\pm\rangle$ corresponds to the maximally entangled states appearing from coupling two bipartite one dimensional invariant subspaces and $|[2]_{1,2},[2]_{3,4},1,1,2,\pm\rangle$ appear from coupling two bipartite 5-dimensional invariant subspaces.

Other states appear from linear combinations of qutrit Kronecker states in $(\mathbf{3}\otimes\mathbf{3}')\otimes(\mathbf{3}\otimes\mathbf{3}')$ and $(\mathbf{3}'\otimes\mathbf{3})\otimes(\mathbf{3}'\otimes\mathbf{3})$, as well as linear combinations of the product spaces $(\mathbf{3}\otimes\mathbf{3}')\otimes(\mathbf{3}'\otimes\mathbf{3})$ and $(\mathbf{3}'\otimes\mathbf{3})\otimes(\mathbf{3}\otimes\mathbf{3}')$, which appear from coupling the 4 and 5-dimensional bipartite subspaces of tensor products of the form $\mathbf{3}\otimes\mathbf{3}'$, following the rules for coupling $\mathbf{4}\otimes\mathbf{4}$ and $\mathbf{5}\otimes\mathbf{5}$ given by equations (6.7) and (6.8). Thus, the following states are obtained:

$$|[2]_{1,3},[2]_{2,4},1,1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[2]_{1,3},[2]_{2,4},1,1,1\rangle \pm |[2]_{2,4},[2]_{1,3},1,1,1\rangle), \tag{6.47}$$

$$|[2]_{1,3},[2]_{2,4},1,1,2,\pm\rangle = \frac{1}{\sqrt{2}}(|[2]_{1,3},[2]_{2,4},1,1,2\rangle \pm |[2]_{2,4},[2]_{1,3},1,1,2\rangle), \tag{6.48}$$

$$|[2]_{1,4},[2]_{2,3},1,1,1,\pm\rangle = \frac{1}{\sqrt{2}}(|[2]_{1,4},[2]_{2,3},1,1,1\rangle \pm |[2]_{2,3},[2]_{1,4},1,1,1\rangle), \tag{6.49}$$

$$|[2]_{1,4},[2]_{2,3},1,1,2,\pm\rangle = \frac{1}{\sqrt{2}}(|[2]_{1,4},[2]_{2,3},1,1,2\rangle \pm |[2]_{2,3},[2]_{1,4},1,1,2\rangle). \tag{6.50}$$

Thus, the states of the form $|[\lambda_1]_{a,b,c},[\lambda_2]_{d,e,f},i_1,i_2,m,\pm\rangle$ give a total of 26 LME states, which coincides with the sum of the multiplicities of the trivial and sign subspaces in $[3,1,1]^{\otimes 4}$.

The states found suggest that Kronecker states arising from N-fold tensor products of self-adjoint irreducible representations of dimension f^λ can be constructed from linear combinations of just two multipartite states of dimension $f^\lambda/2$, which appear from the irreducible representations of the alternating group $\mathcal{A}_n$, for $\lambda \vdash n$, which reduces the dimensionality of the problem. The next section gives a generalization of such expressions, and it also provides a method to find these Kronecker states according to the multiplicity of the trivial and sign subspaces of N-fold tensor prodcuts of such representations.

6.4 Construction of Kronecker states from self-adjoint Young diagrams

Let $[\nu]$ be an irreducible representation of S_n, such that that $\nu \vdash n$ is a partition corresponding to a self-adjoint Young diagram, and let f^ν be its dimension, which can be calculated using the Hook formula, defined in equation (2.15). Then, the N-fold tensor product of copies of $[\nu]$ decomposes into a direct sum of some irreducible representations of S_n, each with certain multiplicity, There, it is possible to find the one-dimensional trivial and alternating representations, labeled by partitions $[n]$ and $[1^n]$. Since $[\nu]$ is a self-adjoint diagram, the multiplicity of the $[n]$ and $[1^n]$ representations coincide, because the characters of the odd permutations of self-adjoint representations are all zero. Since both representations are one-dimensional, their associated subspaces are generated by Kronecker or LME states, i.e., states whose local density matrices are multiples of the identity, according to Schur's lemma. Then, if the multiplicity of the trivial subspace is $g_{\alpha\beta\ldots\nu}$ ($[\alpha]=[\beta]=\ldots=[\nu]$), as calculated using the Kronecker coefficient formula in equation 4.10, it is possible to find a total of $2g_{\alpha\beta\ldots\nu}$ Kronecker states in the tensor product space $[\nu]^{\otimes N}$. It is possible to find all of these states by computing the projector onto these one-dimensional subspaces, according to equations 4.12 and 4.13.

Self-adjoint irreducible representations of S_n are reducible from the point of view of the alternating group $\mathcal{A}_n$. Then, $[\nu]\downarrow\mathcal{A}_n = [\nu]^+ \oplus [\nu]^-$, where $[\nu]^\pm$ are irreducible representations with half the dimension of $[\nu]$, i.e.,$f^\nu/2$. Since the matrix representations of $[\nu]\downarrow\mathcal{A}_n$ can be interpreted as block matrices composed of two diagonal blocks of dimension $f^\nu/2$, the tensor product space $[\nu]^{\otimes N}$ can be interpreted as a direct sum of tensor products of these blocks. It is possible to find a similarity transformation that produces a change of basis from the irreducible matrix representations $[\nu]$ of S_n to the basis associated to the matrix representations $[\nu]^{\pm}$ of $\mathcal{A}_n$. By performing such change of basis, the Kronecker states of N qudits of dimension $d = f^{\nu}$ generated from the tensor product space $[\nu]^{\otimes N}$ can be visualized as linear combinations of two Kronecker states of dimension $d = f^{\nu}/2$, each of them associated with a different tensor product of blocks. Concretely, given a Kronecker state with $d = f^{\nu}/2$ in a N-fold tensor product space generated by N_1 copies of the representation $[\nu]^{+}$ and N_2 copies of $[\nu]^{-}$, such that $N_1 + N_2 = N$, then a Kronecker state of dimension f^{ν} of N parties can be generated by combining it with another maximally entangled state generated by a tensor product of N_2 copies of $[\nu]^{+}$ and N_1 copies of $[\nu]^{-}$. This means that the N-partite Kronecker states of dimension f^{ν} lye in subspaces of the form:

$$([\nu^{+}]^{\otimes N_1} \otimes [\nu^{-}]^{\otimes N_2}) \oplus ([\nu^{-}]^{\otimes N_2} \otimes [\nu^{+}]^{\otimes N_1}), \tag{6.51}$$

and any other combination of the factors in the product spaces of the form $[\nu^{+}]^{\otimes N_1} \otimes [\nu^{-}]^{\otimes N_2}$, of which there are $\binom{N_1+N_2}{N_1}$. Any tensor product space with N_1 copies of $[\nu^{\pm}]$ and N_2 copies $[\nu^{\mp}]$ has the same multiplicity of Kronecker states. Then, it is possible to see that the total number of N-partite Kronecker states of dimension f^{ν} satisfies the following condition:

$$\#\text{ of Kronecker states} = 2g_{\alpha\beta\ldots\nu} = \sum_{N_1=0, N_1+N_2=N}^{N} \binom{N_1+N_2}{N_1} m_{N_1,N_2}, \tag{6.52}$$

where m_{N_1,N_2} is the multiplicity of the trivial subspace in the tensor product of N_1 copies of $[\nu^{\pm}]$ and N_2 of $[\nu^{\mp}]$. This means that the complete set of Kronecker states in $[\nu^{\otimes N}]$ can be generated from Kronecker states appearing in tensor products of representations of half the dimension under the alternating group by finding the appropriate change of basis, which reduces the complexity and dimensionality of the problem.

One can use the fact that any N-fold tensor product space can be decomposed in terms of direct sums of irreducible representations of both the symmetric group $S_N^{parties}$ and the General Linear group in order to classify the Kronecker states found using the previously mentioned method, in terms of how they transform under the action of the symmetric group $S_N^{parties}$ on their N parties. These states can be shown to generate irreducible subspaces under the action of this group, which are labeled by partitions of N. Since the tensor product spaces of the form $[\nu]^{\otimes N}$ were decomposed in terms of tensor products of N_1 copies of $[\nu^{\pm}]$, and tensor products of N_2 copies of $[\nu^{\mp}]$ adding up to N, then the Hilbert space can be decomposed by the action of the groups $S_{N_1}^{P}$ and $S_{N_2}^{P}$ on the corresponding parties. These actions are labeled in terms of partitions $\lambda_1 \vdash N_1$ and $\lambda_2 \vdash N_2$, according to the irreducible subspace they belong. Following these ideas, and taking into account that each of the composite $[\lambda_1] \otimes [\lambda_2]$ subspaces could be associated to some Kronecker state, and that this subspace could occur with some multiplicity m_{λ_1,λ_2}, it is possible to label and generate the complete set of N-partite Kronecker states of dimension f^ν for any self-associated partition ν as follows:

$$\begin{aligned}\frac{1}{\sqrt{2}}(|[\lambda_1]_{a,b,c,...},[\lambda_2]_{d,e,f,...},i_1,i_2,m\rangle \pm |[\lambda_2]_{d,e,f,...},[\lambda_1]_{a,b,c,...},i_1,i_2,m\rangle\,,\\ 1\leq i_1\leq f^{\lambda_1},\quad 1\leq i_2\leq f^{\lambda_2},\quad 1\leq m\leq m_{1,\lambda_1,\lambda_2}),\end{aligned} \tag{6.53}$$

where the convention $[\lambda_1]_{a,b,c,...}$ indicates that the representation $[\lambda_1]$ of $S^{P_{N_1}}$ acts on the qudits $a,b,c,..$, and $[\lambda_2]_{d,e,f,...}$ indicates that $[\lambda_2]$ acts on the qudits $d,e,f,....$ This means that these states are linear combination of just two states of half the dimension, as mentioned in previous paragraphs. In fact, the two states in equation (6.53) are related by an $N-$fold tensor product of a matrix X with the following entries:

$$\begin{aligned}X_{i,j}=\delta_{i,j+f^\nu/2},\quad i\leq f^\nu/2,\\ X_{i,j}=X_{j,i},\end{aligned} \tag{6.54}$$

which turns out to be the matrix of the matrix for the transposition $(1,2)$ for the irreducible representation $[\nu]$, as seen in the basis of the alternating group $\mathcal{A}_n$ representations. This means that the expression (6.53) can be rewritten as follows.

$$\begin{aligned}\frac{1}{\sqrt{2}}(|[\lambda_1]_{a,b,c,...},[\lambda_2]_{d,e,f,...},i_1,i_2,m\rangle \pm X^{\otimes N}\,|[\lambda_1]_{a,b,c,...},[\lambda_1]_{d,e,f,...},i_1,i_2,m\rangle\,,\\ 1\leq i_1\leq f^{\lambda_1},\quad 1\leq i_2\leq f^{\lambda_2},\quad 1\leq m\leq m_{1,\lambda_1,\lambda_2})\end{aligned} \tag{6.55}$$

This gives a standardized form of representing maximally entangled states constructed from $N-$fold tensor copies of self-adjoint diagrams of dimension f^ν. Basically, any such state can be constructed using LME states of N qudits of dimension $d=f^\nu/2$. This is important because simplifies the construction of LME states, which is the purpose of this thesis.

6.5 Summary

In this chapter it was shown that maximally entangled states of N qudits with $d=6$ could be constructed from maximally entangled states of N qutrits, i.e., $d=3$. This was possible because the irreducible representation of dimension 6 associated to the symmetric group S_5 is reducible into two representations of dimension 3, as seen from the point of view of the alternating group $\mathcal{A}_5$. Moreover, it was shown that this could be done for any irreducible representation of any symmetric group, and this provide a simpler form of interpreting such states.

It was also shown that the set of all Kronecker states in a N-fold tensor product of identical irreducible representations could be classified according to how they transform under permutations in its N parties. and that this set could be arranged in subsets of states that generate invariant subspaces under the action of the symmetric group S_N^P, which are labeled by partitions $\lambda\vdash N$.

7. Conclusions and discussion

Since locally maximally entangled are important in Quantum Error Correction codes and Quantum Secret Sharing schemes, as well as in other protocols, the search for these states have been relevant in the field of quantum mechanics and quantum information theory. Moreover, these states are representatives of SLOCC equivalence classes, as motivated by Kempf-Ness theorem, and they are used to characterize such classes. However, there is no canonical form of obtaining such states, and therefore many methods have been introduced in order to find this kind of vectors. Yet, for large N-partite systems, finding LME states becomes a computationally hard problem, given that the dimension of these states grows exponentially with N. Moreover, the number of parameters for describing each SLOCC class grows as fast as such dimension. For instance, for the simple case of N qubits, one needs a lower bound of $2^{N+1} - 6N - 2$ real numbers in order to parametrize the sets of inequivalent pure quantum states [35].

Given the well-founded motivations for the construction of LME states, some methods have been proposed to find some subsets of such vetors . The most common method used to find these states is by computing the projectors on one-dimensional invariant subspaces of tensor products of irreducible representations of finite groups, since the local density matrices of the vectors that span such subspaces are multiples of the identity. The problem with this method relies on the fact the larger the group, the computational cost tends to increment substantially. In the case of tensor products of irreducible representations of the symmetric group S_n, the number of parameters needed to achieve such construction is $n!$, which coincides with the order of the group. According to Stirling formula [1], this term grows faster than n^n which means that the construction of irreducible representations of this group becomes non-viable for roughly $n > 8$. Then, there is a need of reducing the computational cost of this process.

In this work, a method to find locally maximally entangled states of N qudits based on tensor products of N copies of irreducible representations of the symmetric group S_n was presented. These states were computed by initially finding the eigenvectors of the projectors onto the one-dimensional trivial and sign subspaces arising from the direct sum decomposition of these tensor products. It was shown that if the representations involved in the tensor product were associated to self-adjoint Young diagrams, then these states could be interpreted as an equal-coefficient linear combination of two maximally entangled states of N qudits with half the dimension of the diagram, given that these representations were reducible under the alternating group. This means that if the Kronecker states were associated to $N-$fold tensor products of some irreducible self-adjoint representation $[\lambda]$ with dimension f^λ, then these states could be interpreted as a sum or substraction of two LME states of total dimension $(f^\lambda/2)^N$, given that $[\lambda] \downarrow \mathcal{A}_n = [\lambda^+] \oplus [\lambda^-]$. In fact, it was shown that the two LME states involved in the construction of such Kronecker state could be obtained from each other by using a LU transformation of the form $X^{\otimes N}$, where X is the transposition $(1, 2)$ as seen in the alternating group basis. This means that the construction of such states using this interpretation reduces the computational cost of the problem to the half for each of the N parties.

The states obtained from $N-$fold tensor products of self-adjoint Young diagrams were labeled in terms of its decomposition under the action of the symmetric group $S_N^{parties}$ on its N parties, according to Weyl-Schur duality. It was found that the multiplicity of such LME states was such that they could be arranged into subspaces that were irreducible under such action. By performing such decomposition, some totally symmetric maximally entangled states of four and five qubits were found, as well as some qutrit LME states. These kind of states have the potential to be applied in quantum optics experiments involving entangled photons.

It was also found that the local parties of some Kronecker states obtained from $N-$fold tensor products of irreducible representations could be associated to phases of rotation of some regular objects. By realizing such interpretation, the construction of such states is eased. In the qubit case, it was shown that the basis associated to the alternating group irreducible representation was associated to the phases of the possible rotations that left the spacial configuration of an equilateral triangle invariant, and that any Kronecker state was composed of a sum of two states whose local phases added up to $0 \mod 2\pi$. In the qutrit case, these states were found to be linear combinations of states whose local bases where associated to eigenvectors of irreducible representations of the rotational icosahedral group I, and that qudit Kronecker states of dimension $d = 6$ could be construct constructed from these previous states.

LME states can be applied in diverse fields of physics, and then serve diverse purposes. For instance, the method presented here where N_1 copies of some irreducible representations are coupled with N_2 copies of other inequivalent irreducible representations of the same dimension of the same symmetric group, labeled with self-adjoint partitions, can be used to represent systems where a chain of N_1 spins are being affected by some magnetic field, and N_2 spins by another magnetic field. Then, the construction introduced here for LME states could be further applied in condensed matter systems, such as Kitaev chains.

Locally maximally entangled states, such as the ones found and shown throughout the current document, are natural Quantum Error Correction codes for any error in just one party. More robust error codes require that errors in more than one qudit could be corrected. Taking this into account, we explored the possibility of finding subspaces of Kronecker states for each of the cases reviewed throughout this thesis for which errors in at least two qudits were corrected. In order to do this, we arranged the Kronecker states found in tensor products of irreducible representations of the symmetric group S_n into pairs according to the multiplicity of the trivial and sign subspaces, and tried to verify whether the Knill-Laflamme conditions for quantum error correction, given in equation (3.25), were satisfied. Nevertheless, it turned out to be that, at least for the cases reviewed, i.e. three, four and five qubits, and three, four and five qutrits, no pair of Kronecker states provided a QEC code capable to correct errors in more than one party. It remains to be explored whether such codes can be achieved for tensor products of irreducible representations of other symmetric groups.

LME maximally entangled states can be used as representatives of stable SLOCC orbits. Nevertheless, we were not able to specify whether the set of Kronecker states found throughout this document could be used as canonical forms for different SLOCC orbits. These canonical forms are unique, up to local unitary equivalence. In this sense, the results provided here can be extended by finding canonical forms for SLOCC orbits in the cases of study presented, as well as the respective quotient spaces SLOCC/LU, which is represented by the quotient of the tensor product space of operators in the special linear group and tensor products of operators in the unitary group, i.e.:

$$\frac{SL(d) \times SL(d) \times SL(d) \times ... \times SL(d)}{U(d) \times U(d) \times U(d) \times ...U(d)}.$$

This implicates finding the space of SLOCC operations that are locally unitary. This is a *nondeterministic polynomial time*-hard problem or *NP-hard* problem, and therefore goes beyond the time scope of the current thesis. Then, this is left as task to be solved.

A. Appendix A

PYTHON code computed to find the matrix representations of all permutations of S_n for some partititon $\lambda \vdash n$.

```
import numpy as np
import math
import sys
np.set_printoptions(threshold=sys.maxsize)

#Returns the identity matrix of dimension n
def identidad(n):
    ide=np.zeros((n,n))
    for i in range(n):
        ide[i][i]=1
    return ide

#Returns the multiplication of two matrices A and B.
def matrixmult(A,B):
    matrix=[]
    Bp=np.array(B)
    for i in range(len(A)):
        fila=[]
        for j in range(len(B[0])):

            elem=np.sum(np.multiply(A[i],Bp[:,j]))

            fila.append(elem)
        matrix.append(fila)
    return matrix

#Returns the multiplication of the matrices in a list.
def matrixmultN(matrices):
    resultado=matrices[0]
    for i in range(len(matrices)-1):
        resultado=matrixmult(resultado,matrices[i+1])
    return resultado

#Returns the matrix A except for the row i and column j.
```

```
def filaycolumna(A,i,j):
    lista=[]
    length=len(A[0])
    for i2 in range(len(A)):

        if(i!=i2):
            lista2=[]
            for j2 in range(length):
                if(j2!=j):
                    lista2.append(A[i2][j2])
            lista.append(lista2)
    return lista

#Returns the determinant of matrix A.
def determinant(A):

    if(type(A)==int):
        return A
    elif(len(A[0])==1):
        return A[0][0]
    det=0
    for i in range(len(A[0])):
        if(len(A[0])==16):
            print(i)
        if(i%2==0):
            det+=A[0][i]*determinant(filaycolumna(A,0,i))
        else:
            det+=-A[0][i]*determinant(filaycolumna(A,0,i))
    return det

#Returns the adjoint matrix of A.
def adjoint(A):
    if(type(A)==int):
        return [[1]]
    elif(len(A[0])==1):
        return [[1]]
    length=len(A[0])
    adj=[]
    for i in range(length):
        lista=[]
```

```
        for j in range(length):
            lista.append(((-1)**(i+j))*determinant(filaycolumna(A,j,i)))
        adj.append(lista)
    return adj

#Returns the inverse of A.
def inverse(A):
    inv=[]
    adj=adjoint(A)
    invdet=determinant(A)**(-1)
    for i in range(len(A[0])):
        lista=[]
        for j in range(len(A[0])):
            lista.append(adj[i][j]*invdet)
        inv.append(lista)
    return inv

#Returns an array with its elements i and j  transposed.
def transposition(array, i, j):
    if(i==j-1):
      array2=array.copy()
      temp=array[i]
      array2[i]=array[j]
      array2[j]=temp
    return array2

#Standard Yamanouchi symbol associated to the partition lambd of number n.
#The standard Yamanouchi symbol can be read as (1**lambd1,2**lambd2...)
#It also returns the Yamanouchi symbol associated to the columns of the
respective Young tableau.
def stdyamanouchi1(n,lambd):
    stdyamanouchi=[]
    stdcolumn=[]
    if(type(lambd)==int):
        for j in range(n):
            stdyamanouchi.append(1)
            stdcolumn.append(j+1)
    else:
        for i in range(len(lambd)):
```

```
            for j in range(lambd[i]):
                stdyamanouchi.append(i+1)
                stdcolumn.append(j+1)
    return stdyamanouchi,stdcolumn

#Iterative function that returns every Yamanouchi symbol that can be
found from  the standard Yamanouchi symbol.
#Param n: The number of entries of the Yamanouchi symbol.
#Param a: Is the first part of the partition  lambda for which  it can
be obtained the Yamanouchi symbols.
#Param stdyam: It is obtained by computing the standard Yamanouchi symbol
for some partition.
#Param stdcol: Yamanouchi symbol obtained from the columns of the Young
tableau associated to stdyam.
#Param itera: Current iteration.
#Param iteramax: Determines the number of maximum iterations, which
corresponds to the number of parts of the partition.
def yamanouchi(n,a,stdyam,stdcol,itera,iteramax):
    stdyamanouchi=stdyam
    stdcolumn=stdcol
    yamanou=[]
    cols=[]
    permYamanouchi=stdyamanouchi.copy()
    permColumn=stdcolumn.copy()
    yamanou.append(stdyamanouchi)
    cols.append(stdcolumn)
    if(a==n and itera==0):
        return yamanou,cols,0
    boo=0
    a2=a
    if(itera==0):
        while(a2+2<=n-1):
            if(permYamanouchi[a2+2]-permYamanouchi[a2+1]!=0 and
            stdcolumn[a2+2]-stdcolumn[a2+1]!=0):
                itera2=itera+1
                yama=yamanouchi(n,a2+2,stdyamanouchi,stdcolumn,itera2,
                iteramax-1)
                for i in range(len(yama[0])-1):
                    yamanou.append(yama[0][i+1])
                    cols.append(yama[1][i+1])
```

```
            a2+=1
        a2=a
    for p in range(a):
        if(permYamanouchi[a-p]-permYamanouchi[a-p-1]==0 or
        permColumn[a-p]-permColumn[a-p-1]==0):
            break
        else:
            temp=permYamanouchi[a-p]
            permYamanouchi[a-p]=permYamanouchi[a-p-1]
            permYamanouchi[a-p-1]=temp
            temp2=permColumn[a-p]
            permColumn[a-p]=permColumn[a-p-1]
            permColumn[a-p-1]=temp2
            symbol=permYamanouchi.copy()
            yamanou.append(symbol)
            scolumn=permColumn.copy()
            cols.append(scolumn)
            permY2=permYamanouchi.copy()
            permC2=permColumn.copy()
            while (a2+1<n):
                if(permY2[a2]-permY2[a2+1]!=0 and permC2[a2]-permC2[a2+1]!=0):
                    itera2=itera+1
                    if(itera2>iteramax):
                        boo=1
                        break
                    yama=yamanouchi(n,a2+1,permY2,permC2,itera2,iteramax)
                    for i in range(len(yama[0])-1):
                        yamanou.append(yama[0][i+1])
                        cols.append(yama[1][i+1])
                a2+=1
            a2=a
            if(itera==0):
                iteramax+=1
    return yamanou,cols,boo
#Irreducible representations of all possible adjacent transpositions
of n elements.
#Param lambd: Irreducible representation of Sn.
def irreps(n,lambd):

    k=0
```

```
    if(type(lambd)==int):
        k=lambd
    else:
        k=lambd[0]
    tupla=yamanouchi(n,k,stdyamanouchi1(n,lambd)[0],stdyamanouchi1(n,lambd)[1],
    symbols=tupla[0]
    columns=tupla[1]
    irreps=[]
    for j in range(n-1):
        matrix=[]
        for i in range(len(symbols)):
            gentransp=np.zeros(len(symbols))
            axial=symbols[i][j]-symbols[i][j+1]+columns[i][j+1]-columns[i][j]
            inv=1/float(axial)
            gentransp[i]=inv
            actual=symbols[i]
            transp=transposition(actual,j,j+1)
            if(transp==actual):
                gentransp[i]=gentransp[i]+math.sqrt(1.0-(inv**2))
            elif((symbols[i][j]!=symbols[i][j+1])):
                for k in range(len(symbols)):
                    if(transp==symbols[k]):
                        gentransp[k]=math.sqrt(1.0-(inv**2))
                        break
            matrix.append(gentransp)
        if(j==0):
            irreps.append(identidad(len(matrix)))
        irreps.append(matrix)
    return irreps

#Determines the factorial of n.
def factorial(n):
    res=n
    for i in range(n-2):
        res=res*(i+2)
    return res

#Cyclic permutations of the form (ini,ini+1,...,fin).
#Param irreps: A list containing all the adjacent transpositions.
def ciclicas(ini,fin,irreps):
```

```
    return matrixmultN(irreps[ini:fin-1])

#Returns all the transpositions of Sn.
#Param irreps: defined as the list of adjacent transpositions of Sn.
def permutaciones2(n,irreps):
    perm=[]
    if(n==2):
        return irreps
    for i in range(n-2):

        perm.append(irreps[0])
        perm.append(irreps[i+1])
        for j in range(i+2,n):
            cicl=ciclicas(i+1,j+1,irreps)
            uno=matrixmult(cicl,irreps[j])
            dos=matrixmult(uno,inverse(cicl))
            perm.append(dos)


    perm.append(irreps[0])
    perm.append(irreps[n-1])


    return perm

#Returns all the permutations of Sn.
#Param irreps: Contains all the adjacent permutations of Sn for some irrep.
#Param param: Iterative parameter.
def permutaciones(n,irreps,param):
    if(n==2):
        return irreps,[1,-1]
    perms=[]
    perms2=[]
    signos=[]
    signosperms2=[]
    if(param==0):
        perms2=permutaciones2(n,irreps)
    else:
        perms2=irreps.copy()
    for t in range(len(perms2)):
```

```
    signosperms2.append(-1)

permut,signospermut=permutaciones(n-1,perms2[n:],1)
length=len(permut)
for k in range(n):
    for j in range(length):
        oli=permut[j]
        perms.append(matrixmult(perms2[k],oli))
        signos.append(signosperms2[k]*signospermut[j])
return perms,signos
```

B. Appendix B

Python functions where projectors on trivial and sign subspaces of tensor products of irreducible representations of the symmetric group are computed.

```
#Particiones de un n mero n con longitud l

def partitionsL(n,l):
    if(l==1):
        return [[n]]

    lista=[]
    lista2=[]
    lista4=[]
    lista2.append(n-(l-1))
    for j in range(l-1):
        lista2.append(1)
    lista.append(lista2)
    lista4=lista2.copy()
    for i in range(lista2[0]-2):
        lista3=lista4.copy()
        lista3[0]=lista2[0]-(i+1)
        for k in range(l-1):
            lista5=lista3.copy()
            prueba=lista3[k+1]+1
            if(lista5[k]>=prueba ):
                lista5[k+1]=prueba
                lista.append(lista5)
                if(prueba==2):
                    lista4[0]=lista5[0]
                    lista4[k+1]=2
                    break
    return lista

#Trace of matrix A.
def trace(A):
    trace=0
    length=len(A)
    for i in range(length):
```

```
        trace+=A[i][i]
    return trace

#Projector onto the trivial irreducible subspace of a tensor product of
irreps of Sn.
#Param p: Contains a certain number of partitions of n. Then, the tensor
product is of the form:
# p[0]x p[1]x p[2]....
def invprojector(n,p):
    fact=factorial(n)
    proj=0.0
    transp2=irreps(n,p[0])
    perms2=permutaciones(n,transp2,0)[0]
    #dim=dimension(n,p[0])
    for i in range(fact):
        suma=perms2[i].copy()
        for j in range(len(p)-1):
            transp=irreps(n,p[j+1])
            perms=permutaciones(n,transp,0)[0]
            if(j==0):
                suma=np.kron(suma,[[(float(k))/fact for k in l] for l
                in perms[i]])
            else:
                suma=np.kron(suma,[[(float(k)) for k in l] for l
                in perms[i]])
        proj+=suma


    return proj

#Projector onto the sign subspace of a tensor product of irreducible
representations of Sn
def invprojectorsign(n,p):
    fact=factorial(n)
    proj=0.0
    transp2=irreps(n,p[0])
    perms2=permutaciones(n,transp2,0)[0]
    dim=dimension(n,p[0])
    sign=partitionsL(n,n)
    permsign=permutaciones(n,irreps(n,sign[0]),0)[0]
```

```
pprime=[]
for i in range(len(p)):
    pprime.append(p[i])
pprime.append(sign[0])

for i in range(fact):
    suma=perms2[i].copy()
    for j in range(len(p)-1):
        transp=irreps(n,p[j+1])
        perms=permutaciones(n,transp,0)[0]
        if(j==0):
            suma=np.kron(suma,[[(float(k)*trace(permsign[i]))/(fact)
            for k in l] for l in perms[i]])
        else:
            suma=np.kron(suma,[[(float(k)) for k in l] for l
            in perms[i]])
    proj+=suma
return proj
```

C. Appendix C

Mathematica code used to compute the multiplicities of the icosahedral irreducible representations in the $GL(3,C)$ representations for a tensor product spaces of the form $\mathbf{3}^{\otimes N}$.

```
validPartitionQ[part_]:=
VectorQ[part,(IntegerQ[#]&& NonNegative[#]&]&&Apply [GreaterEqual, part]
```

```
SchurPolynomial[part_?validPartitionQ,vars_List]/;Length[part]        ≤
Length[vars]:=Module[{n = Length[vars],tmp},tmp=C/@Range[n]; (* computes the
Schur polynomial for valid partitions.*)
```

```
Cancel[ |Outer[Power,tmp,PadLeft[Reverse[part],n]+Range[0,n−1]]| / |LinearAlgebraVandermondeMatrix(tmp)| ] /. Thread[tmp :→ vars]
```

$$\texttt{evs} = \begin{pmatrix} 1 & \{1,1,1\} \\ 15 & \{1,-1,-1\} \\ 20 & \left\{1, e^{\frac{i2\pi}{3}}, e^{\frac{1}{3}(-i)2\pi}\right\} \\ 12 & \left\{1, e^{\frac{i2\pi}{5}}, e^{\frac{1}{5}(-i)2\pi}\right\} \\ 12 & \left\{1, e^{\frac{i4\pi}{5}}, e^{\frac{1}{5}(-i)4\pi}\right\} \end{pmatrix};$$

$$\texttt{chartab} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & \frac{1}{2}\left(\sqrt{5}+1\right) & \frac{1}{2}\left(1-\sqrt{5}\right) \\ 3 & -1 & 0 & \frac{1}{2}\left(1-\sqrt{5}\right) & \frac{1}{2}\left(\sqrt{5}+1\right) \end{pmatrix}$$

```
mult[par_,irr_]:=
1/60 Plus@@Table[evs[[i,1]]chartab[[irr,i]]SchurPolynomial(par,evs[[i,2]]),{i,Length[evs]}];
(* computes the multiplicity of the irreducible representation irr_ of
I in the representation of GL(3,ℂ) labeled by partition par_.*)
```

```
multtab[parts_]:=Module[{tab},
tab  =  Table[FullSimplify[TrigExpand[mult(parts[[i]],j)]],{i,Length[parts]},
{j,Length[chartab]}];
```

```
multtab({{4},{3,1},{2,2},{2,1,1}})
```

```
multtab({{5},{4,1},{3,2},{3,1,1},{2,2,1}})
```

```
multtab({{6},{5,1},{4,2},{4,1,1},{3,3},{3,2,1},{2,2,2}})
```

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