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Spin Motion in General Relativity

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Movimiento de Spin en
Relatividad General

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To Ramón.

He never got to see what I became.

I hope I make him proud.

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Abstract

In direct analogy to the quantum-mechanical study of a particle's spin when said particle is placed in an external magnetic field, we herein present a brief exploration of the general-relativistic behavior of a particle's spin when said particle is placed in an external gravitational field. The concept of geodetic effects is explained as motivation through a literature review before explicit calculations are presented for a few of the most commonly encountered metrics using two separate formalisms, finally returning to the geodetic effects for some closing remarks.

Resumen

En analogía directa con el estudio del spin de una partícula en mecánica cuántica cuando dicha partícula se coloca en un campo magnético externo, aquí presentamos una breve exploración del comportamiento general-relativista del spin de una partícula cuando dicha partícula se coloca en un campo gravitacional externo. El concepto de efectos geodéticos se explica como motivación a través de una revisión bibliográfica antes de presentar cálculos explícitos para algunas de las métricas más comúnmente encontradas usando dos formalismos distintos, finalmente volviendo a los efectos geodéticos para algunas observaciones finales.

Contents

Introduction

Particles in quantum physics are not only characterized by a mass and an electric charge, they also posses an intrinsic angular momentum s, called spin, which is an invariant property of the particle and thus completely independent of its state of motion[1][p. 224]. The physical manifestation of spin is usually taken to be the particle's magnetic moment, which is proportional to its spin. It is then natural to consider the behavior of a particle's spin when said particle is introduced in an external magnetic field, leading to the well-known Larmor precession.

[1] F. Scheck, *Quantum Physics*, 2nd ed. (Springer-Verlag, Berlin, 2007)

Spin is itself, however, a vector, and as such it is subject to relativistic corrections. Indeed the placement of a particle in an external magnetic field, in the special-relativistic case, requires the use of the Bargmann-Michel-Telegdi (BMT) equation, which results not only in Larmor precession but also a small correction known as Thomas precession.

But what of general-relativistic corrections? We should expect there to be some effect on the spin when a particle is placed in an external gravitational field. This is in fact known as the geodetic effect and can be further subdivided depending on the source of the gravitational field. When simply considering the effect due to a central mass we arrive at what is known as de Sitter precession; meanwhile, if the central mass happens to itself be rotating, there is an additional frame dragging effect which results in so-called Lense-Thirring precession[2][p. 252–254]

[2] W. Rindler, *Relativity: Special, General, and Cosmological*, 2nd ed. (Oxford University Press, Oxford, 2006)

What follows is a brief exploration of spin motion in the general-relativistic case. De Sitter and Lense Thirring precessions, collectively the geodetic effects, are further explained as motivation after a literature review, and some explicit calculations are then carried out for the most commonly encountered metrics, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric used in cosmology and the Schwarzschild metric for a static central mass, in both cases using a formalism for point particles, called the geodesic formalism, and a formalism suited to extended bodies, through the Mathisson-Papapetrou-Dixon (MPD) equations. We finally return to the geodetic effects and their experimental confirmation for some closing remarks.

Part I

Spin Motion in External Fields

Some Formalisms for Spin

While defining spin can be conceptually simple, as has been done in the introduction following Scheck[3][p. 224], the way it is treated mathematically can take several forms.

[3] F. Scheck, *Quantum Physics*, 2nd ed. (Springer-Verlag, Berlin, 2007)

The most general for quantum mechanical applications is to consider a spin operator $\hat{\mathbf{S}}$, which, following the Heisenberg picture of quantum mechanics, will have a time-evolution. Indeed, in the Heisenberg picture of quantum mechanics, it is the operators that change with time according to the relation [4]

[4] B. Zwiebach, *Quantum Dynamics*, Cambridge, MA, 2013

$$\frac{\partial}{\partial t}\hat{A}(t) = \frac{i}{\hbar}[\hat{H}, \hat{A}], \tag{1}$$

where $\hat{A}(t)$ is any time-dependent operator, $\hat{H}$ is the Heisenberg Hamiltonian (this is not necessarily the same as the Schrödinger Hamiltonian), and $[\hat{H}, \hat{A}]$ is their commutator.

In particular for the spin operator, $\hat{\mathbf{S}}$, we have

$$\frac{\partial}{\partial t}\hat{\mathbf{S}}(t) = \frac{i}{\hbar}[\hat{H}, \hat{\mathbf{S}}]. \tag{2}$$

Following this it is also possible to turn the spin operator into a spin function by simply taking the operator's expectation value, we then have

$$\boldsymbol{S}(t) = \langle\psi|\hat{\mathbf{S}}(t)|\psi\rangle. \tag{3}$$

For relativistic treatments of spin one must consider it as a tensor. This may well be a (1,0) tensor—the spin vector—S^{α}, analogous to the spin function $\boldsymbol{S}(t)$, albeit dependent now on *proper* time τ. It may also be a (2,0) tensor—the spin bivector—$S^{\alpha\beta}$, which arises naturally from another tensor describing the rotational motion of particles in spacetime through Noether's theorem.

It can be shown for arbitrary translations that

$$\partial_\nu T^{\mu\nu} = 0, \tag{4}$$

with $T^{\mu\nu}$ the energy-momentum tensor. $T^{\mu\nu}$ is then called the *Noether current* for translations and it leads to a conserved quantity upon spatial integration, the four-momentum

$$\begin{aligned} P^\mu &= \int d^3x T^{\mu 0} \\ \partial_t P^\mu &= 0. \end{aligned} \tag{5}$$

It may also be shown [5][p. 19] for arbitrary Lorentz transformations without translations that

[5] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, 1st ed. (McGraw-Hill, Inc., New York, 1965)

$$\begin{aligned} \partial_\mu \mathfrak{M}^{\mu\nu\lambda} &= 0 \\ \mathfrak{M}^{\mu\nu\lambda} &= \left(x^\nu T^{\mu\lambda} - x^\lambda T^{\mu\nu}\right) + \frac{\partial \mathcal{L}}{\partial\left(\partial\phi_r/\partial x^\mu\right)} \sum_{rs}^{\nu\lambda} \phi_s \end{aligned} \tag{6}$$

with $\mathfrak{M}^{\mu\nu\lambda}$ the corresponding Noether current for rotations—itself the sum of two terms, one again for energy-momentum, and one for infinitesimal rotations. This then leads to a conserved quantity upon spatial integration, the angular momentum

$$\begin{aligned} S^{\mu\nu} &= \int d^3x \mathfrak{M}^{0\mu\nu} \\ \partial_t S^{\mu\nu} &= 0, \end{aligned} \tag{7}$$

which, although containing both spin and orbital terms, may be separated such that one obtains only the spin part according to the relation

$$\begin{aligned} S^\mu &= \frac{1}{2}\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma} u_\nu S_{\rho\sigma} \\ S^\mu &= \frac{1}{2}\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma} g_{\nu\alpha} u^\alpha g_{\rho\beta} g_{\sigma\gamma} S^{\beta\gamma}. \end{aligned} \tag{8}$$

Spin in External Electromagnetic Fields

In order to study the spin motion in an external magnetic field, $\boldsymbol{B}$, without relativistic corrections, it is most convenient to use the spin function $\boldsymbol{S}(t)$, together with the magnetic dipole moment, defined similarly to the classical case as

$$\begin{aligned} \hat{\boldsymbol{\mu}} &= \gamma \hat{\boldsymbol{S}} \\ \boldsymbol{\mu}(t) &= \gamma \boldsymbol{S}(t), \end{aligned} \tag{9}$$

for some constant γ, the form of which is irrelevant for the study at hand, and finally the Hamiltonian for the problem, which takes the form

$$\begin{aligned} \hat{H} &= -\hat{\boldsymbol{\mu}} \cdot \boldsymbol{B} \\ &= -\gamma \boldsymbol{B} \cdot \hat{\boldsymbol{S}}. \end{aligned} \tag{10}$$

It can then be shown [6], using (2), that the spin function will behave as

$$\frac{d\boldsymbol{S}}{dt} = -\gamma \boldsymbol{B} \times \boldsymbol{S}, \tag{11}$$

corresponding to the well-known Larmor precession, valid in the non-relativistic case.

[6] L. Landau and E. Lifshitz, *Quantum Mechanics Non-relativistic Theory*, 2nd ed. (Pergamon Press, Oxford, 1974), B. Zwiebach, *Two State Systems*, Cambridge, MA, 2013

For the special-relativistic case, the so-called Bargmann-Michel-Telegdi (bmt) equation may be used, which in the simplest case takes the form [7]

$$\frac{DS^\alpha}{d\tau} = \frac{e}{m}\left[\frac{g}{2}F^{\alpha\beta}S_\beta + \left(\frac{g}{2} - 1\right)u^\alpha(S_\lambda F^{\lambda\mu}u_\mu)\right] \tag{12}$$

with S^α being the spin vector and $F^{\alpha\beta}$ the electromagnetic tensor for the external field. Much like in the non-relativistic version we see a first term that couples the spin to the external field resulting in Larmor precession. Unlike the non-relativistic case, however, we

[7] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, Inc., New York, 1999)

now find an additional term where the external field and the spin get further coupled to the particle's four-velocity u^α, this term accounts for a small relativistic correction, an additional precession called Thomas precession.

Spin in Gravity

It is well known that classical spin, $\boldsymbol{S}$, defined as in (3), reacts to external fields, in the simplest, non-relativistic case, according to (11), and in the special-relativistic case according to the BMT equation (12). However, $\boldsymbol{S}$ also interacts with gravitational fields leading to some general-relativistic equations of motion, of which we may now consider two versions.

The first, and simplest of them, which we may call the geodesic formalism, starts from the non-relativistic fact that a point particle not under the influence of any force will have constant velocity and spin

$$\begin{aligned} \frac{du^\alpha}{d\tau} &= 0 \\ \frac{dS^\alpha}{d\tau} &= 0 \end{aligned} \tag{13}$$

and then uses the Principle of General Covariance to arrive at the two geodesic equations [8][p.122]

[8] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, 1st ed. (John Wiley & Sons, Inc., New York, 1972)

$$\begin{aligned} \frac{Du^\alpha}{d\tau} &= \frac{\partial u^\alpha}{\partial \tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0 \\ \frac{DS^\alpha}{d\tau} &= \frac{\partial S^\alpha}{\partial \tau} + \Gamma^\alpha_{\beta\gamma} S^\beta u^\gamma = 0 \end{aligned} \tag{14}$$

where we again find a coupling of the spin, S^α, and the four-velocity, u^α, as in the BMT equation.

here the differential operator $\frac{D}{d\tau}$ is the operator of proper-time-derivative, defined as

$$\frac{D}{d\tau} = u^\mu \nabla_\mu \tag{15}$$

with ∇_μ the ordinary covariant derivative (see appendix A.)

The geodesic formalism quite simply results in the parallel transport of the spin vector which in general should result in some precession.

The second formalism we are to consider is one that is adapted to extended bodies, it makes use instead of the spin bivector in what are called the Mathisson-Papapetrou-Dixon (henceforth referred to simply as MPD) equations [9]. They require the use of the bivector because they are meant to describe the motion of spinning massive extended bodies and thus their angular momentum should include both the spin and orbital parts. We can, regardless, ultimately consider only the spin by using (8). Several equivalent forms of the MPD equations exist, depending on how each of them is written, whether in terms of the 4-velocity or the 4-momentum of the particle, but the form we use in the calculations that follow will be

$$\frac{D}{d\tau}\left(mu^{\lambda} + u_{\mu}\frac{D}{d\tau}S^{\lambda\mu}\right) + \frac{1}{2}u^{\pi}S^{\rho\sigma}R^{\lambda}{}_{\pi\rho\sigma} = 0$$
$$\frac{D}{d\tau}S^{\mu\nu} + u^{\mu}u_{\sigma}\frac{D}{d\tau}S^{\nu\sigma} - u^{\nu}u_{\sigma}\frac{D}{d\tau}S^{\mu\sigma} = 0. \tag{16}$$

[9] A. Papapetrou, "Spinning Test-Particles in General Relativity. I", Proceedings of the Royal Society A **209**, 248–258 (1951)

One final consideration regarding the MPD equations is that they are underdetermined and require an additional constraint so as to be solvable, what is known as a spin supplementary condition (SSC). Once again, there exist several possible SSCs, the discussion of which we leave for later. Suffice it to say, for the time being, that the chosen SSC in what follows is the so-called Frenkel condition

$$u_{\alpha}S^{\alpha\beta} = 0. \tag{17}$$

Thomas, de Sitter, and Lense-Thirring Precessions

Larmor precession was first described mathematically by Joseph Larmor in 1897[10], decades before the concept of spin was even introduced, in trying to explain the results of experiments described by Pieter Zeeman. In 1916, Willem de Sitter calculated the precession of the Earth-moon gyroscope orbiting the sun due to general relativity, an effect which now bears his name. Only a few years later, around 1918[11], Josef Lense and Hans Thirring studied the weak-field approximation for a spinning spherical body of uniform density and found a similar precession affecting gyroscopes orbiting said body, an effect that came to be named after them[12]. Finally, in 1927, Llewellyn Thomas studied the relativistic effects on spin for flat spacetime, i.e. special-relativistic effects, for use in atomic physics (the gyroscope here would be the electron orbiting the atomic nucleus), a precession effect named after him[13][*p.*1119].

[10] A. S. Eddington, "Joseph Larmor, 1857-1942", Obituary Notices of Fellows of the Royal Society **4**, 197–207 (1942)

[11] H. Pfister, "On the history of the so-called Lense-Thirring effect", General Relativity and Gravitation **39**, 1735–1748 (2007)

[12] Pfister, in a historical review of the Lense-Thirring effect, discovered that in fact Einstein may have been first to suggest such an effect, as far back as 1917, to Thirring himself, and argues that it should be renamed the Einstein-Thirring-Lense effect.

[13] C. W. Misner et al., *Gravitation*, 1st ed. (W. H. Freeman and Co., San Francisco, 1973)

These four account for the known precession of gyroscopes (or spin, as it were) due to an external magnetic field (Larmor), special-relativistic effects (Thomas), or an external gravitational field (de Sitter and Lense-Thirring). For an extended gyroscope orbiting, say, Earth, only the lattermost three apply; Thomas precession applies when the gyroscope is close to the surface and the other two when it is placed far from the surface (the total contribution is a sum of them both, not just any one of them on its own). Specifically, Thomas precession gives the special-relativistic correction near the surface of the body, de Sitter precession accounts for the general-relativistic effect of a central mass, and Lense-Thirring precession accounts for the general-relativistic effect of the central mass itself spinning (so-called frame-dragging).

Finally, it must be noted just how weak these effects can be. When considering the specific case of the precession of a gyroscope orbiting Earth, they turn out to be about, respectively, 8/3, 8, and 0.1 seconds of arc per year.

All told, we can quickly see how hard these effects are to detect.

Indeed, an orbiting gyroscope away from Earth's surface would have to stay there for some 450 years to precess one degree, and a gyroscope near the surface would have to do so for over 13000 years.

Nevertheless, experiments have been carried out that confirm both Thomas and de Sitter precessions, and observational data shows what would be expected from Lense-Thirring precession. Further discussion of said experiments and observations will be left for later.

Part II

The Friedmann-Lemaître-Robertson-Walker (FLRW) Metric

The Metric

The development of cosmology in the early twentieth century led to the abandonment of the theretofore prevalent assumption of "perfect" symmetry in the universe, that is, symmetry throughout both space and time. Instead, the preferred assumption, the one that was consistent with observation, became that of a spatially homogeneous and isotropic universe that would nonetheless evolve in time. In general-relativistic terms, spacetime is to be seen as having a real temporal dimension and spatial components represented by a maximally symmetric three-manifold, i.e. the spacetime manifold takes the form $\mathbb{R} \times \Sigma$, with Σ being the aforementioned maximally symmetric three-manifold[14][*p.*329]. We now derive the spacetime metric that arises from such an assumption.

[14] S. M. Carroll, *Spacetime and Geometry*, 1st ed. (Pearson Education Limited, Harlow, 2014)

For simplicity's sake we may begin by considering the more general form[15][p. 403]

[15] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, 1st ed. (John Wiley & Sons, Inc., New York, 1972)

$$ds^2 = -dt^2 + U(r,t)dr^2 + V(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{18}$$

so the non-vanishing Christoffel symbols are

$$\begin{gathered}
\Gamma^t_{rr} = \frac{\dot{U}(r,t)}{2}, \qquad \Gamma^t_{\theta\theta} = \frac{\dot{V}(r,t)}{2}, \qquad \Gamma^t_{\phi\phi} = \frac{\dot{V}(r,t)}{2}\sin^2\theta, \\
\Gamma^r_{tr} = \Gamma^r_{rt} = \frac{\dot{U}(r,t)}{2U(r,t)}, \qquad \Gamma^r_{rr} = \frac{U'(r,t)}{2U(r,t)}, \\
\Gamma^r_{\theta\theta} = -\frac{V'(r,t)}{2U(r,t)}, \qquad \Gamma^r_{\phi\phi} - \frac{V'(r,t)}{2U(r,t)}\sin^2\theta, \\
\Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \frac{\dot{V}(r,t)}{2V(r,t)}, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{V'(r,t)}{2V(r,t)}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} = \frac{\dot{V}(r,t)}{2V(r,t)}, \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{V'(r,t)}{2V(r,t)}, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta,
\end{gathered} \tag{19}$$

where the convention is a dot for $\frac{\partial}{\partial t}$ and a prime for $\frac{\partial}{\partial r}$.

We may now take the following general form for a spherically symmetric homogeneous spacetime

$$ds^2 = -g(v)dv^2 + f(v)\left(d\boldsymbol{u}^2 + \frac{k(\boldsymbol{u}\cdot d\boldsymbol{u})^2}{1-k\boldsymbol{u}^2}\right), \tag{20}$$

with one v coordinate and 3 u coordinates. From here we may define the four coordinates t, r, θ, ϕ, as well as the function $R(t)$ by

$$\begin{aligned} \int (-g(v))^{1/2} dv &\equiv t, \\ u^1 &\equiv r\sin\theta\cos\phi, \\ u^2 &\equiv r\sin\theta\sin\phi, \\ u^3 &\equiv r\cos\theta, \\ f(v) &\equiv R^2(t), \end{aligned} \tag{21}$$

so that the metric takes the form

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right), \tag{22}$$

where we can easily read off $U(r,t) = \frac{R^2(t)}{1-kr^2}$ and $V(r,t) = R^2(t)r^2$.

Taking the partial derivatives of these and making the appropriate substitutions into (19) we arrive at the Christoffel symbols

$$\begin{aligned} \Gamma^t_{rr} &= \frac{R(t)\dot{R}(t)}{1-kr^2}, & \Gamma^t_{\theta\theta} &= R(t)\dot{R}(t)r^2, & \Gamma^t_{\phi\phi} &= R(t)\dot{R}(t)r^2\sin^2\theta, \\ \Gamma^r_{tr} = \Gamma^r_{rt} &= \frac{\dot{R}(t)}{R(t)}, & \Gamma^r_{\theta\theta} &= (kr^2-1)r, & \Gamma^r_{\phi\phi}&(kr^2-1)r\sin^2\theta, \\ \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} &= \frac{\dot{R}(t)}{R(t)}, & \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} &= \frac{1}{r}, & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta, \\ \Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} &= \frac{\dot{R}(t)}{R(t)}, & \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= \frac{1}{r}, & \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \cot\theta. \end{aligned} \tag{23}$$

In the special case of a flat universe, where $k = 0$, however, it is more convenient to go back to (20) and simply consider Cartesian coordinates such that it takes the form

$$ds^2 = -(dx^0)^2 + R^2(t)\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2\right), \tag{24}$$

which leads to the far simpler Christoffel symbols

$$\begin{gathered}
\Gamma^0_{11} = R(t)\dot{R}(t), \qquad \Gamma^0_{22} = R(t)\dot{R}(t), \qquad \Gamma^0_{33} = R(t)\dot{R}(t), \\
\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{R}(t)}{R(t)}, \\
\Gamma^2_{02} = \Gamma^2_{20} = \frac{\dot{R}(t)}{R(t)}, \\
\Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{R}(t)}{R(t)}.
\end{gathered} \tag{25}$$

Point Particles in the FLRW Metric

We now consider particle kinematics in the FLRW metric[16][p.36]. The geodesic equations, as well as the constraints, that must be satisfied by the velocity u^μ and spin S^μ are

[16] E. W. Kolb and M. S. Turner, *The early universe*, 1st ed. (CRC Press, Boca Raton, 2018)

$$\begin{aligned}
&\frac{du^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} u^\mu u^\nu = 0 \\
&g_{\mu\nu} u^\mu u^\nu = u_\nu u^\nu = -1, \\
&\frac{dS^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} S^\mu u^\nu = 0 \\
&g_{\mu\nu} S^\mu u^\nu = S_\nu u^\nu = 0.
\end{aligned} \tag{26}$$

The given magnitude for u^μ[17][p.37] is consistent with only a comoving observer truly perceiving the universe as isotropic, it does not, however, determine what each of the components is.

[17] E. W. Kolb and M. S. Turner, *The early universe*, 1st ed. (CRC Press, Boca Raton, 2018)

The easiest case is to take the comoving frame, $u^\mu = (1, 0, 0, 0)$, we may thus find for the spin's temporal part

$$\begin{aligned}
\frac{dS^0}{d\tau} + \Gamma^0_{\mu\nu} S^\mu u^\nu &= 0 \\
\frac{dS^0}{d\tau} + \cancelto{0}{\dot{R}(t) R(t) \delta_{ij} S^i u^j} &= 0 \\
S^0 &= \text{const.} \\
S_\mu u^\mu = 0 \Rightarrow S^0 &= 0,
\end{aligned} \tag{27}$$

and for the spatial part

$$
\begin{aligned}
\frac{dS^i}{d\tau} + \Gamma^i_{\mu\nu} S^\mu u^\nu &= 0 \\
\frac{dS^i}{d\tau} + \frac{\dot{R}(t)}{R(t)} \left(\cancelto{0}{S^0 u^i} + S^i u^0 \right) &= 0 \\
\frac{dS^i}{S^i} &= -\frac{dR}{R(t)} \\
S^i &= \frac{R_0}{R(t)} S^i_0,
\end{aligned} \tag{28}
$$

for some initial conditions R_0 and S^i_0. This is in particular valid whether in spherical coordinates, $i \in \{r, \theta, \phi\}$, or in Cartesian coordinates, $i \in \{1, 2, 3\}$.

From now on, a dot will indicate differentiation with respect to τ, while a prime will indicate differentiation with respect to t.

On the other hand we may consider a test-particle, taking only the assumption that $u^\theta = u^\phi = 0$ (in order for $R_{\theta\phi} = R_{\phi\theta} = 0$ to be satisfied according to the Einstein equations with a perfect fluid as a source), the geodesic equations take the form

$$
\begin{aligned}
&\frac{du^t}{d\tau} + \Gamma^t_{rr} u^r u^r = 0, && \frac{du^r}{d\tau} + 2\Gamma^r_{tr} u^t u^r = 0; \\
&\frac{du^t}{d\tau} + R'(t) R(t) u^r u^r = 0, && \frac{du^r}{d\tau} + 2\frac{R'(t)}{R(t)} u^t u^r = 0.
\end{aligned} \tag{29}
$$

Now the second of these may be solved directly,

$$
\begin{aligned}
\frac{du^r}{d\tau} + 2\frac{R'(t)}{R(t)} \frac{dt}{d\tau} u^r &= 0 \\
\frac{du^r}{d\tau} + 2\frac{R'(t)}{R(t)} u^r &= 0 \\
\frac{d\tau}{u^r} \left(\frac{du^r}{d\tau} + 2\frac{R'(t)}{R(t)} u^r \right) &= 0 \\
\frac{du^r}{u^r} + 2\frac{dR}{R(t)} &= 0 \\
\int_{u^r_0}^{u^r} \frac{d\tilde{u}^r}{\tilde{u}^r} &= -2 \int_{R_0}^{R} \frac{d\tilde{R}}{\tilde{R}(t)} \\
\log \frac{u^r}{u^r_0} &= -2 \log \frac{R(t)}{R_0} \\
u^r &= u^r_0 \left(\frac{R_0}{R(t)} \right)^2,
\end{aligned} \tag{30}
$$

for some initial conditions u^r_0, R_0. Putting this solution into the other geodesic equation we get

$$\begin{aligned}
\frac{du^t}{d\tau} + R'(t)R(t)u^r_0u^r_0\left(\frac{R_0}{R(t)}\right)^4 &= 0 \\
\frac{dt}{d\tau}\left(\frac{du^t}{d\tau} + R'(t)\cancel{R(t)}u^r_0u^r_0\frac{R_0^4}{R^{\cancelto{3}{4}}(t)}\right) &= 0 \\
d\tau\left(u^t\frac{du^t}{d\tau} + R'(t)u^r_0u^r_0\frac{R_0^4}{R^3(t)}\right) &= 0 \\
u^t du^t &= -u^r_0u^r_0R_0^4\frac{dR}{R^3(t)} \\
\int_{u^t_0}^{u^t}\tilde{u}^t d\tilde{u}^t &= -u^r_0u^r_0R_0^4\int_{R_0}^{R}\frac{d\tilde{R}}{\tilde{R}^3(t)} \\
\cancel{1/2}(u^tu^t - u^t_0u^t_0) &= \cancel{1/2}u^r_0u^r_0R_0^4\left(\frac{1}{R^2(t)} - \frac{1}{R_0^2}\right) \\
u^tu^t - u^t_0u^t_0 &= u^r_0u^r_0R_0^2\left(\frac{R_0^2}{R^2(t)} - 1\right) \\
u^t &= \sqrt{u^r_0u^r_0R_0^2\left(\frac{R_0^2}{R^2(t)} - 1\right) + u^t_0u^t_0}.
\end{aligned} \tag{31}$$

We may finally use (30) and (31) to check the constraint

$$\begin{aligned}
-1 &= u_\mu u^\mu \\
-1 &= g_{\mu\nu}u^\mu u^\nu \\
-1 &= -u^tu^t + R^2(t)u^ru^r \\
-1 &= -\left(u^r_0u^r_0R_0^2\left(\frac{R_0^2}{R^2(t)} - 1\right) + u^t_0u^t_0\right) + R^2(t)\left(u^r_0u^r_0\left(\frac{R_0}{R(t)}\right)^4\right) \\
-1 &= \cancel{-u^r_0u^r_0\frac{R_0^4}{R^2(t)}} + u^r_0u^r_0R_0^2 - u^t_0u^t_0 + \cancel{u^r_0u^r_0\frac{R_0^4}{R^2(t)}} \\
-1 &= -u^t_0u^t_0 + R_0^2u^r_0u^r_0,
\end{aligned} \tag{32}$$

and we see that we may choose any u^t, u^r, $R(t)$ such that the constraint is satisfied for the initial conditions u^t_0, u^r_0, R_0, i.e. it is enough for the initial conditions to satisfy the constraint to have it satisfied for any time t thereafter.

We may also perform analogous calculations in Cartesian coordinates, starting with the geodesic equations

$$
\begin{aligned}
\frac{du^0}{d\tau} + \Gamma^0_{ii} u^i u^i = 0 \qquad & \qquad \frac{du^i}{d\tau} + 2\Gamma^i_{0i} u^0 u^i = 0 \\
\frac{du^0}{d\tau} + R'(t)R(t)\delta_{ij} u^i u^j = 0 \qquad & \qquad \frac{du^i}{d\tau} + 2\frac{R'(t)}{R(t)} u^0 u^i = 0,
\end{aligned}
\tag{33}
$$

then solving for the spatial components u^i, all of which take the same form

$$
\begin{aligned}
\frac{du^i}{d\tau} + 2\frac{R'(t)}{R(t)}\frac{dx^0}{d\tau} u^i &= 0 \\
\frac{du^i}{d\tau} + 2\frac{R'(t)}{R(t)} u^i &= 0 \\
\frac{d\tau}{u^i}\left(\frac{du^i}{d\tau} + 2\frac{R'(t)}{R(t)} u^i\right) &= 0 \\
\frac{du^i}{u^i} + 2\frac{dR}{R(t)} &= 0 \\
\int_{u^i_0}^{u^i} \frac{d\tilde{u^i}}{\tilde{u^i}} &= -2\int_{R_0}^{R} \frac{d\tilde{R}}{\tilde{R}(t)} \\
\log\frac{u^i}{u^i_0} &= -2\log\frac{R(t)}{R_0} \\
u^i &= u^i_0 \left(\frac{R_0}{R(t)}\right)^2,
\end{aligned}
\tag{34}
$$

and putting this into the other geodesic equation to solve for u^0

$$
\begin{aligned}
\frac{du^0}{d\tau} + R'(t)R(t)\delta_{ij} u^i_0 u^j_0 \left(\frac{R_0}{R(t)}\right)^4 &= 0 \\
\frac{dx^0}{d\tau}\left(\frac{du^0}{d\tau} + R'(t)\cancel{R(t)}\delta_{ij} u^i_0 u^j_0 \frac{R_0^4}{R^{\cancel{4}\,3}(t)}\right) &= 0 \\
d\tau\left(u^0\frac{du^0}{d\tau} + R'(t)\delta_{ij} u^i_0 u^j_0 \frac{R_0^4}{R^3(t)}\right) &= 0
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
u^0 du^0 &= -\delta_{ij} u_0^i u_0^j R_0^4 \frac{dR}{R^3(t)} \\
\int_{u_0^0}^{u^0} \tilde{u^0} d\tilde{u^0} &= -\delta_{ij} u_0^i u_0^j R_0^4 \int_{R_0}^{R} \frac{d\tilde{R}}{\tilde{R}^3(t)} \\
\cancel{1/2}(u^0 u^0 - u_0^0 u_0^0) &= \cancel{1/2}\delta_{ij} u_0^i u_0^j R_0^4 \left(\frac{1}{R^2(t)} - \frac{1}{R_0^2} \right) \\
u^0 u^0 - u_0^0 u_0^0 &= \delta_{ij} u_0^i u_0^j R_0^2 \left(\frac{R_0^2}{R^2(t)} - 1 \right) \\
u^0 &= \sqrt{\delta_{ij} u_0^i u_0^j R_0^2 \left(\frac{R_0^2}{R^2(t)} - 1 \right) + u_0^0 u_0^0}.
\end{aligned}
$$

We finally check the constraint

$$
\begin{aligned}
-1 &= u_\mu u^\mu \\
-1 &= g_{\mu\nu} u^\mu u^\nu \\
-1 &= -u^0 u^0 + R^2(t) \delta_{ij} u^i u^j \\
-1 &= -\left(\delta_{ij} u_0^i u_0^j R_0^2 \left(\frac{R_0^2}{R^2(t)} - 1 \right) + u_0^0 u_0^0 \right) + R^2(t) \delta_{ij} u_0^i u_0^j \left(\frac{R_0}{R(t)} \right)^4 \\
-1 &= \cancel{-\delta_{ij} u_0^i u_0^j \frac{R_0^4}{R^2(t)}} + \delta_{ij} u_0^i u_0^j R_0^2 - u_0^0 u_0^0 + \cancel{\delta_{ij} u_0^i u_0^j \frac{R_0^4}{R^2(t)}} \\
-1 &= -u_0^0 u_0^0 + R_0^2 \delta_{ij} u_0^i u_0^j,
\end{aligned}
\tag{36}
$$

where the conclusion is, similarly, that we may choose any u^0, u^i, $R(t)$ such that the constraint is satisfied for the initial conditions u_0^0, u_0^i, R_0.

NEXT WE TURN OUR ATTENTION to the spin motion of a test particle, in Cartesian coordinates with $k = 0$, we have the solutions, for the temporal part

$$
\begin{aligned}
\frac{dS^0}{d\tau} + \Gamma^0_{\mu\nu} S^\mu u^\nu &= 0 \\
\frac{dS^0}{d\tau} + R' R \delta_{ij} S^i u^j &= 0 \\
S_\mu u^\mu = 0 \Rightarrow \delta_{ij} S^i u^j &= \frac{1}{R^2} S^0 u^0 \\
\frac{dS^0}{d\tau} + \frac{R'}{R} S^0 u^0 &= 0 \\
\frac{dS^0}{S^0} &= -\frac{dR}{R} \\
S^0 &= \frac{R_0}{R} S_0^0,
\end{aligned}
\tag{37}
$$

for some initial conditions R_0 and S^0_0, and for the spatial part

$$
\begin{aligned}
\frac{dS^i}{d\tau} + \Gamma^i_{\mu\nu} S^\mu u^\nu &= 0 \\
\frac{dS^i}{d\tau} + \frac{R'}{R}\left(S^0 u^i + S^i u^0\right) &= 0 \\
\frac{dS^i}{d\tau} + \frac{R'}{R}\left(\frac{R_0}{R} S^0_0 u^i + S^i u^0\right) &= 0 \\
\frac{dS^i}{d\tau} + \frac{R'}{R^2} R_0 S^0_0 u^i + \frac{dR}{d\tau}\frac{1}{R} S^i &= 0,
\end{aligned}
\tag{38}
$$

where the second, inhomogeneous term might mean this is not a readily solvable differential equation.

Switching between $u^i = \frac{dx^i}{d\tau}$ and $v^i = \frac{dx^i}{dt}$ is possible using the relations

$$
\begin{aligned}
u^i &= \frac{v^i}{\sqrt{1 - v_j v^j}} \\
v^i &= \frac{u^i}{\sqrt{1 + u_j u^j}} \\
&= \frac{u^i}{\sqrt{1 + h_{jk} u^j u^k}},
\end{aligned}
\tag{39}
$$

with the spatial metric being $h_{ij} = -g_{ij} = -R^2(t)\delta_{ij}$.

such that (38) may be rewritten in the form

$$
\begin{aligned}
\frac{dS^i}{d\tau} + \frac{dR}{d\tau}\frac{1}{R(t)}\left(\frac{1}{R(t)} R_0 S^0_0 v^i + S^i\right) &= 0 \\
\frac{dS^i}{d\tau} + \frac{dR}{d\tau}\frac{1}{R(t)}\left(\frac{1}{R(t)} R_0 S^0_0 \frac{u^i}{\sqrt{1 - R^2(t)\delta_{jk} u^j u^k}} + S^i\right) &= 0 \\
\frac{dS^i}{d\tau} + \frac{dR}{d\tau}\frac{1}{R(t)}\left(\frac{1}{R(t)} R_0 S^0_0 \frac{u^i}{\sqrt{1 - R^2(t)(u^1u^1 + u^2u^2 + u^3u^3)}} + S^i\right) &= 0.
\end{aligned}
\tag{40}
$$

Now inserting the solution we previously found for u^i for a test particle, and applying the constraint on u^μ to the initial conditions we have

$$\frac{dS^i}{d\tau} + \frac{dR}{d\tau}\frac{1}{R(t)}\left(\frac{1}{R(t)}R_0 S_0^0 \frac{u^i}{\sqrt{1-\frac{R_0^4}{R^2(t)}(u_0^1u_0^1+u_0^2u_0^2+u_0^3u_0^3)}} + S^i\right) = 0$$
$$\frac{dS^i}{d\tau} + \frac{dR}{d\tau}\frac{1}{R(t)}\left(\frac{1}{R(t)}R_0 S_0^0 \frac{u^i}{\sqrt{1-\frac{R_0^4}{R^4(t)}(u_0^0u_0^0-1)}} + S^i\right) = 0, \tag{41}$$

and inserting the solution we found for u^i once more

$$\frac{dS^i}{d\tau} + \frac{dR}{d\tau}\frac{1}{R(t)}\left(\frac{1}{R(t)}\frac{R_0 S_0^0}{\sqrt{1-\frac{R_0^4}{R^4(t)}(u_0^0u_0^0-1)}}u_0^i\left(\frac{R_0}{R(t)}\right)^2 + S^i\right) = 0$$
$$\frac{dS^i}{d\tau} + \frac{dR}{d\tau}\frac{1}{R(t)}\left(\frac{1}{R^3(t)}\frac{R_0^3 S_0^0 u_0^i}{\sqrt{1-\frac{R_0^4}{R^4(t)}(u_0^0u_0^0-1)}} + S^i\right) = 0 \tag{42}$$
$$dS^i + \frac{dR}{R(t)}\left(\frac{1}{R(t)}\frac{R_0^3 S_0^0 u_0^i}{\sqrt{R^4(t)-R_0^4(u_0^0u_0^0-1)}} + S^i\right) = 0.$$

Finally defining the parameters $\varepsilon^i \equiv R_0^3 S_0^0 u_0^i$ and $\kappa \equiv R_0^4(u_0^0u_0^0-1)$ allows us to write the differential equations in the form

$$dS^i + \frac{dR}{R(t)}\left(\frac{1}{R(t)}\frac{\varepsilon^i}{\sqrt{R^4(t)-\kappa}} + S^i\right) = 0, \tag{43}$$

whence a solution may be found, perhaps numerically.

Extended Bodies in the FLRW Metric

PRESENTLY WE MOVE ON to the study of cosmological kinematics, this time for extended bodies. In order to use the MPD equation, the components of the Riemann tensor for the given metric must be calculated. Starting with the simpler $k = 0$ case, with Cartesian coordinates, the non-zero components of the Riemann tensor are

$$\begin{aligned}
R^0{}_{101} = -R^0{}_{110} = R(t)R''(t), \qquad R^0{}_{202} = -R^0{}_{220} = R(t)R''(t), \qquad R^0{}_{303} = -R^0{}_{330} = R(t)R''(t),\\
R^1{}_{001} = -R^1{}_{010} = \frac{R''(t)}{R(t)}, \qquad R^1{}_{212} = -R^1{}_{221} = R'^2(t), \qquad R^1{}_{313} = -R^1{}_{331} = R'^2(t),\\
R^2{}_{002} = -R^2{}_{020} = \frac{R''(t)}{R(t)}, \qquad R^2{}_{121} = -R^2{}_{112} = R'^2(t), \qquad R^2{}_{323} = -R^2{}_{332} = R'^2(t),\\
R^3{}_{003} = -R^3{}_{030} = \frac{R''(t)}{R(t)}, \qquad R^3{}_{131} = -R^3{}_{113} = R'^2(t), \qquad R^3{}_{232} = -R^3{}_{223} = R'^2(t).
\end{aligned} \tag{44}$$

AT THIS POINT we may write down the MPD equations, for the co-moving frame, $u^\mu = (1, 0, 0, 0)$, the first equation involving S^{00} (indeed we could have skipped this step and noted that, having a skew-symmetric tensor, immediately $S^{00} = 0$) is

$$\begin{aligned}
\frac{D}{d\tau}\left(mu^0 + u_0\frac{D}{d\tau}S^{00}\right) + \frac{1}{2}u^0 S^{\rho\sigma}R^0{}_{0\rho\sigma} = 0\\
\frac{D}{d\tau}\left(m - \frac{D}{d\tau}S^{00}\right) = 0,
\end{aligned} \tag{45}$$

the first equation involving $S^{0i} = -S^{i0}$ is

$$\begin{aligned}\frac{D}{d\tau}\left(mu^i + u_0\frac{D}{d\tau}S^{i0}\right) + \frac{1}{2}u^0 S^{\rho\sigma}R^i{}_{0\rho\sigma} &= 0\\ \frac{D}{d\tau}\left(-\frac{D}{d\tau}S^{i0}\right) + \frac{1}{2}S^{i0}R^i{}_{0i0} &= 0\\ \frac{D}{d\tau}\left(-\frac{D}{d\tau}S^{i0}\right) - \frac{1}{2}\frac{R''(t)}{R(t)}S^{i0} &= 0\\ \frac{D}{d\tau}\left(\frac{D}{d\tau}S^{0i}\right) + \frac{1}{2}\frac{R''(t)}{R(t)}S^{0i} &= 0,\end{aligned} \tag{46}$$

while the first equations involving Sii and S^{ij} are identically zero (never mind the fact that $Sii = 0$ all the same seeing as how this is a skew-symmetric tensor.)

The second equation involving S^{00} (once again we note that this step is ultimately unnecessary) is

$$\begin{aligned}\frac{D}{d\tau}S^{00} + u^0u_\sigma\frac{D}{d\tau}S^{0\sigma} - u^0u_\sigma\frac{D}{d\tau}S^{0\sigma} &= 0\\ \frac{D}{d\tau}S^{00} + u^0u_0\frac{D}{d\tau}S^{00} - u^0u_0\frac{D}{d\tau}S^{00} &= 0\\ \frac{D}{d\tau}S^{00} &= 0,\end{aligned} \tag{47}$$

the second equation involving $S^{0i} = -S^{i0}$ is

$$\begin{aligned}\frac{D}{d\tau}S^{0i} + u^0u_\sigma\frac{D}{d\tau}S^{i\sigma} - u^iu_\sigma\frac{D}{d\tau}S^{0\sigma} &= 0\\ \frac{D}{d\tau}S^{0i} + u^0u_0\frac{D}{d\tau}S^{i0} &= 0\\ \frac{D}{d\tau}S^{0i} - \frac{D}{d\tau}S^{i0} &= 0\\ \frac{D}{d\tau}S^{0i} + \frac{D}{d\tau}S^{0i} &= 0\\ \frac{D}{d\tau}S^{0i} &= 0,\end{aligned} \tag{48}$$

the second equation involving S^{ii} (once again we note that this step is ultimately unnecessary; henceforth these will all be supressed) is

$$\begin{aligned}\frac{D}{d\tau}S^{ii} + u^iu_\sigma\frac{D}{d\tau}S^{i\sigma} - u^iu_\sigma\frac{D}{d\tau}S^{i\sigma} &= 0\\ \frac{D}{d\tau}S^{ii} &= 0,\end{aligned} \tag{49}$$

and the second equation involving $S^{ij} = -S^{ji}$ is

$$
\begin{aligned}
\frac{D}{d\tau}S^{ij} + u^i u_\sigma \frac{D}{d\tau}S^{j\sigma} - u^j u_\sigma \frac{D}{d\tau}S^{i\sigma} &= 0 \\
\frac{D}{d\tau}S^{ij} &= 0.
\end{aligned}
\tag{50}
$$

Now, (47) means (45) is trivially satisfied and (48) lets us write (46) as

$$
\begin{aligned}
\frac{1}{2}\frac{R''(t)}{R(t)}S^{0i} &= 0 \\
S^{0i} &= 0,
\end{aligned}
\tag{51}
$$

and putting everything together we get that $S^{\mu\nu}$ satisfies two conditions

$$
\begin{aligned}
\frac{D}{d\tau}S^{\mu\nu} &= 0 \\
S^{0i} &= 0,
\end{aligned}
\tag{52}
$$

explicitly

$$
\begin{aligned}
\frac{D}{d\tau}S^{\mu\nu} &= 0 \\
u^\sigma \nabla_\sigma S^{\mu\nu} &= 0 \\
u^0 \nabla_0 S^{\mu\nu} &= 0 \\
\partial_0 S^{\mu\nu} + \Gamma^\mu_{0\sigma}S^{\sigma\nu} + \Gamma^\nu_{0\sigma}S^{\mu\sigma} &= 0.
\end{aligned}
\tag{53}
$$

We now solve for S^{00}

$$
\begin{aligned}
\partial_0 S^{00} + \cancelto{0}{\Gamma^0_{0\sigma}S^{\sigma 0}} + \cancelto{0}{\Gamma^0_{0\sigma}S^{0\sigma}} &= 0 \\
\partial_0 S^{00} &= 0 \\
S^{00} &= \text{const.},
\end{aligned}
\tag{54}
$$

for $S^{0i} = -S^{i0}$

$$
\begin{aligned}
\partial_0 S^{0i} + \cancelto{0}{\Gamma^0_{0\sigma}S^{\sigma i}} + \Gamma^i_{0\sigma}S^{0\sigma} &= 0 \\
\partial_0 S^{0i} + \Gamma^i_{0i}S^{0i} &= 0 \\
\partial_0 S^{0i} + \frac{R'}{R}S^{0i} &= 0 \\
\frac{dS^{0i}}{S^{0i}} &= -\frac{dR}{R} \\
S^{0i} &= \frac{R_0}{R}S^{0i}_0,
\end{aligned}
\tag{55}
$$

for some initial conditions R_0 and S_0^{0i}, for S^{ii}

$$\begin{aligned}
\partial_0 S^{ii} + \cancelto{0}{\Gamma^i_{0\sigma} S^{\sigma i} + \Gamma^i_{0\sigma} S^{i\sigma}} &= 0 \\
\partial_0 S^{ii} &= 0 \\
S^{ii} &= \text{const.},
\end{aligned} \tag{56}$$

and for $S^{ij} = -S^{ji}$

$$\begin{aligned}
\partial_0 S^{ij} + \Gamma^i_{0\sigma} S^{\sigma j} + \Gamma^j_{0\sigma} S^{i\sigma} &= 0 \\
\partial_0 S^{ij} + \Gamma^i_{0i} S^{ij} + \Gamma^j_{0j} S^{ij} &= 0 \\
\partial_0 S^{ij} + 2\frac{R'}{R} S^{ij} &= 0 \\
\frac{dS^{ij}}{S^{ij}} &= -2\frac{dR}{R} \\
S^{ij} &= \left(\frac{R_0}{R}\right)^2 S_0^{ij},
\end{aligned} \tag{57}$$

for some initial conditions R_0 and S_0^{ij}.
Finally, using (8) we find

$$\begin{aligned}
S^0 &= \frac{1}{2}\frac{1}{\sqrt{-g}}\epsilon^{00\rho\sigma} g_{00} u^0 g_{\rho\beta} g_{\sigma\gamma} S^{\beta\gamma} \\
&= 0 \\
S^1 &= \frac{1}{2}\frac{1}{\sqrt{-g}}\epsilon^{10\rho\sigma} g_{00} u^0 g_{\rho\beta} g_{\sigma\gamma} S^{\beta\gamma} \\
&= \frac{1}{2}\frac{1}{R^3(t)} R^4(t)\left(S^{32} - S^{23}\right) \\
&= R(t) S^{32} \\
&= \frac{R_0^2}{R(t)} S_0^{32} \\
S^2 &= \frac{1}{2}\frac{1}{\sqrt{-g}}\epsilon^{20\rho\sigma} g_{00} u^0 g_{\rho\beta} g_{\sigma\gamma} S^{\beta\gamma} \\
&= \frac{1}{2}\frac{1}{R^3(t)} R^4(t)\left(S^{13} - S^{31}\right) \\
&= R(t) S^{13} \\
&= \frac{R_0^2}{R(t)} S_0^{13}
\end{aligned} \tag{58}$$

$$
\begin{aligned}
S^3 &= \frac{1}{2}\frac{1}{\sqrt{-g}}\epsilon^{30\rho\sigma}g_{00}u^0 g_{\rho\beta}g_{\sigma\gamma}S^{\beta\gamma} \\
&= \frac{1}{2}\frac{1}{R^3(t)}R^4(t)\left(S^{21}-S^{12}\right) \\
&= R(t)S^{21} \\
&= \frac{R_0^2}{R(t)}S_0^{21}
\end{aligned}
$$

in particular this must hold for $t = 0$, allowing us to switch between the initial conditions S_0^{ij} and the initial conditions S_0^i, thus we finally write

$$
\begin{gathered}
S^0 = 0 \\
S^1 = \frac{R_0}{R(t)}S_0^1 \qquad S^2 = \frac{R_0}{R(t)}S_0^2 \qquad S^3 = \frac{R_0}{R(t)}S_0^3
\end{gathered}
\tag{59}
$$

which turn out to be exactly the same solutions we found using the geodesic formalism.

For a test particle, on the other hand, the equations don't take quite such an easy form, we have for S^{0i}

$$
\begin{aligned}
\frac{D}{d\tau}\left(mu^0 + u_i\frac{D}{d\tau}S^{0i}\right) + \frac{1}{2}u^\pi S^{\rho\sigma}R^0{}_{\pi\rho\sigma} &= 0 \\
\frac{D}{d\tau}\left(mu^0 + u_i\frac{D}{d\tau}S^{0i}\right) + \frac{1}{2}u^j S^{0k}\delta_{jk}R^0{}_{j0k} &= 0 \\
\frac{D}{d\tau}\left(mu^0 + u_i\frac{D}{d\tau}S^{0i}\right) + \frac{1}{2}R''(t)R(t)\delta_{jk}u^j S^{0k} &= 0,
\end{aligned}
\tag{60}
$$

together with

$$
\begin{aligned}
\frac{D}{d\tau}S^{0i} + u^0u_\sigma\frac{D}{d\tau}S^{i\sigma} - u^iu_\sigma\frac{D}{d\tau}S^{0\sigma} &= 0 \\
\left(1 - u^0u_0 - u^iu_i\right)\frac{D}{d\tau}S^{0i} + u^0u_j\frac{D}{d\tau}S^{ij} &= 0 \\
\left(1 - u^\mu u_\mu\right)\frac{D}{d\tau}S^{0i} + u^0u_j\frac{D}{d\tau}S^{ij} &= 0 \\
2\frac{D}{d\tau}S^{0i} + u^0u_j\frac{D}{d\tau}S^{ij} &= 0,
\end{aligned}
\tag{61}
$$

where in the last equality we recall the constraint on u^μ, $u^\mu u_\mu = -1$, and for S^{ij}

$$\begin{aligned}
\frac{D}{d\tau}\left(mu^i + u_j \frac{D}{d\tau} S^{ij}\right) + \frac{1}{2} u^\pi S^{\rho\sigma} R^i{}_{\pi\rho\sigma} = 0 \\
\frac{D}{d\tau}\left(mu^i + u_j \frac{D}{d\tau} S^{ij}\right) + \frac{1}{2} u^\pi S^{\rho i} R^i{}_{\pi\rho i} = 0 \\
\frac{D}{d\tau}\left(mu^i + u_j \frac{D}{d\tau} S^{ij}\right) + \frac{1}{2}\left(\frac{R''(t)}{R(t)}\delta_{0\rho} u^0 - R'^2(t)\delta_{k\rho} u^k\right) S^{\rho i} = 0,
\end{aligned} \tag{62}$$

together with

$$\begin{aligned}
\frac{D}{d\tau} S^{ij} + u^i u_\sigma \frac{D}{d\tau} S^{j\sigma} - u^j u_\sigma \frac{D}{d\tau} S^{i\sigma} = 0 \\
\frac{D}{d\tau} S^{ij} - u_\sigma \left(u^i \frac{D}{d\tau} S^{\sigma j} + u^j \frac{D}{d\tau} S^{i\sigma}\right) = 0.
\end{aligned} \tag{63}$$

Putting everything together we get the following system of equations

$$\begin{aligned}
\frac{D}{d\tau}\left(mu^0 + u_i \frac{D}{d\tau} S^{0i}\right) + \frac{1}{2} R''(t) R(t) \delta_{jk} u^j S^{0k} = 0, \\
2\frac{D}{d\tau} S^{0i} + u^0 u_j \frac{D}{d\tau} S^{ij} = 0, \\
\frac{D}{d\tau}\left(mu^i + u_j \frac{D}{d\tau} S^{ij}\right) + \frac{1}{2}\left(\frac{R''(t)}{R(t)}\delta_{0\rho} u^0 - R'^2(t)\delta_{k\rho} u^k\right) S^{\rho i} = 0, \\
\frac{D}{d\tau} S^{ij} - u_\sigma \left(u^i \frac{D}{d\tau} S^{\sigma j} + u^j \frac{D}{d\tau} S^{i\sigma}\right) = 0,
\end{aligned} \tag{64}$$

and using (A.2), (A.4), and (A.5) these become

$$
\begin{aligned}
&u_i\ddot{S}^{0i} + \left(\dot{u}_i + u_i u^0 \Gamma^i_{0i}\right)\dot{S}^{0i} + \left[u_i u^j u^j \Gamma^0_{jj}\Gamma^j_{j0} + \frac{d}{d\tau}\left(u_i u^0 \Gamma^i_{0i}\right)\right] S^{0i} - \left(u_i u^i u^j \Gamma^0_{jj}\Gamma^i_{i0} - \frac{1}{2}\delta_{jk}u^k R''(t)R(t)\right) S^{0j} \\
&\quad - 2u_i u^j \Gamma^0_{jj}\dot{S}^{ij} - \left[u_i u^j u^0 \Gamma^0_{jj}\Gamma^j_{0j} + u_i u^j u^0 \Gamma^0_{jj}\Gamma^i_{0i} + \frac{d}{d\tau}\left(u_i u^j \Gamma^0_{jj}\right)\right] S^{ij} + m\dot{u}^0 + m u^j u^j \Gamma^0_{jj} = 0,
\end{aligned}
$$

$$
2\dot{S}^{0i} + \left(2u^0\Gamma^i_{0i} - u_j u^j u^0 \Gamma^j_{j0}\right)S^{0i} + u_j u^i u^0 \Gamma^i_{i0} S^{0j} + u_j u^0 \dot{S}^{ij} - \left[2u^j\Gamma^0_{jj} - u_j u^0 u^0\left(\Gamma^i_{0i} + \Gamma^j_{0j}\right)\right] S^{ij} = 0,
$$

$$
\begin{aligned}
&u_j\ddot{S}^{ij} + \left[\dot{u}_j + u_j u^0\left(2\Gamma^i_{0i} + \Gamma^j_{0j}\right)\right]\dot{S}^{ij} + \left[u_j u^0 u^0 \Gamma^i_{0i}\left(\Gamma^i_{0i} + \Gamma^j_{0j}\right) + \frac{d}{d\tau}\left(u_j u^0\left(\Gamma^i_{0i} + \Gamma^j_{0j}\right)\right)\right] S^{ij} \\
&\quad - u_j u^i u^k \Gamma^i_{0i}\Gamma^0_{kk} S^{jk} + \delta_{k\rho} u^k R'^2(t) S^{i\rho} \\
&\quad - u_j u^j \Gamma^j_{j0}\dot{S}^{0i} + 2u_j u^i \Gamma^i_{i0}\dot{S}^{0j} \\
&\quad - \left[u_j u^j u^0 \Gamma^i_{0i}\Gamma^j_{j0} + \frac{d}{d\tau}\left(u_j u^j \Gamma^j_{j0}\right)\right] S^{0i} + \left[u_j u^i u^0 \Gamma^i_{i0}\left(\Gamma^i_{0i} + \Gamma^j_{0j}\right) + \frac{d}{d\tau}\left(u_j u^i \Gamma^i_{i0}\right)\right] S^{0j} \\
&\quad + m\dot{u}^i + m u^0 u^i\left(\Gamma^i_{0i} + \Gamma^i_{i0}\right) = 0,
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
&\dot{S}^{ij} + u^0\left(\Gamma^i_{0i} + \Gamma^j_{0j}\right)S^{ij} + u_k u^i \dot{S}^{jk} - u_k u^j \dot{S}^{ik} \\
&\quad + \left(u_0 u^i u^k \Gamma^0_{kk} + u_k u^0 u^i\left(\Gamma^j_{0j} + \Gamma^k_{0k}\right)\right) S^{jk} - \left(u_0 u^j u^k \Gamma^0_{kk} + u_k u^0 u^j\left(\Gamma^i_{0i} + \Gamma^k_{0k}\right)\right) S^{ik} \\
&\quad + u_0 u^j \dot{S}^{0i} - u_0 u^i \dot{S}^{0j} \\
&\quad + \left(u_0 u^0 u^j \Gamma^i_{0i} + u_k u^k u^j \Gamma^k_{0k} - u^j \Gamma^j_{j0}\right) S^{0i} - \left(u_0 u^0 u^i \Gamma^j_{0j} + u_k u^k u^i \Gamma^k_{0k} - u^i \Gamma^i_{i0}\right) S^{0j} \\
&\quad + u_k u^i u^j\left(\Gamma^j_{j0} - \Gamma^i_{i0}\right) = 0.
\end{aligned}
$$

Finally, considering the Christoffel symbols we previously found we may write

$$
\begin{aligned}
&u_i\ddot{S}^{0i} + \left(\dot{u}_i + u_i u^0 \frac{R'(t)}{R(t)}\right)\dot{S}^{0i} + \left[\delta_{jk} u_i u^j u^k R'^2(t) + \frac{d}{d\tau}\left(u_i u^0 \frac{R'(t)}{R(t)}\right)\right] S^{0i} - \delta_{jk}\left(u_i u^i R'^2(t) - \frac{1}{2}R''(t)R(t)\right) u^k S^{0j} \\
&\quad - 2\delta_{jk} u_i u^k R'(t)R(t)\dot{S}^{ij} - \delta_{jk}\left[2u_i u^k u^0 R'^2(t) + \frac{d}{d\tau}\left(u_i u^k R'(t)R(t)\right)\right] S^{ij} + m\dot{u}^0 + \delta_{jk} m u^j u^k R'(t)R(t) = 0,
\end{aligned}
\tag{66}
$$

$$
2\dot{S}^{0i} + \left(2 - u_j u^j\right) u^0 \frac{R'(t)}{R(t)} S^{0i} + u_j u^i u^0 \frac{R'(t)}{R(t)} S^{0j} + u_j u^0 \dot{S}^{ij} - 2\left[\delta_{jk} u^k R'(t)R(t) - u_j u^0 u^0 \frac{R'(t)}{R(t)}\right] S^{ij} = 0, \tag{67}
$$

$$
\begin{aligned}
&u_j\ddot{S}^{ij} + \left[\dot{u}_j + 3u_ju^0\frac{R'(t)}{R(t)}\right]\dot{S}^{ij} + 2\left[u_ju^0u^0\left(\frac{R'(t)}{R(t)}\right)^2 + \frac{d}{d\tau}\left(u_ju^0\frac{R'(t)}{R(t)}\right)\right]S^{ij} \\
&\quad - \delta_{kl}u_ju^iu^lR'^2(t)S^{jk} + \delta_{kl}u^lR'^2(t)S^{ik} \\
&\quad - u_ju^j\frac{R'(t)}{R(t)}\dot{S}^{0i} + 2u_ju^i\frac{R'(t)}{R(t)}\dot{S}^{0j} \\
&\quad - \left[u_ju^ju^0\left(\frac{R'(t)}{R(t)}\right)^2 + \frac{d}{d\tau}\left(u_ju^j\frac{R'(t)}{R(t)}\right)\right]S^{0i} + \left[2u_ju^iu^0\left(\frac{R'(t)}{R(t)}\right)^2 + \frac{d}{d\tau}\left(u_ju^i\frac{R'(t)}{R(t)}\right)\right]S^{0j} \\
&\quad + m\dot{u}^i + 2mu^0u^i\frac{R'(t)}{R(t)} = 0,
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
&\dot{S}^{ij} + 2u^0\frac{R'(t)}{R(t)}S^{ij} + u_ku^i\dot{S}^{jk} - u_ku^j\dot{S}^{ik} \\
&\quad + \left(\delta_{kl}u_0u^iu^lR'(t)R(t) + 2u_ku^0u^i\frac{R'(t)}{R(t)}\right)S^{jk} - \left(\delta_{kl}u_0u^ju^lR'(t)R(t) + 2u_ku^0u^j\frac{R'(t)}{R(t)}\right)S^{ik} \\
&\quad + u_0u^j\dot{S}^{0i} - u_0u^i\dot{S}^{0j} \\
&\quad + \left(u_0u^0 + u_ku^k - 1\right)u^j\frac{R'(t)}{R(t)}S^{0i} - \left(u_0u^0 + u_ku^k - 1\right)u^i\frac{R'(t)}{R(t)}S^{0j} = 0.
\end{aligned}
\tag{69}
$$

WE NOW ARGUE that because the FLRW metric arises from considering a maximally symmetric space, every spatial component of any given tensor should be equivalent and thus we need only find 4 solutions here, u^0, u^i, S^{0i}, S^{ij}. With this simplification we immediately see from (69)

$$
\begin{aligned}
&\dot{S}^{ij} + 2u^0\frac{R'(t)}{R(t)}S^{ij} + \cancel{u_ku^i\dot{S}^{jk}} - \cancel{u_ku^j\dot{S}^{ik}} \\
&\quad + \cancel{\left(\delta_{kl}u_0u^iu^lR'(t)R(t) + 2u_ku^0u^i\frac{R'(t)}{R(t)}\right)S^{jk}} - \cancel{\left(\delta_{kl}u_0u^ju^lR'(t)R(t) + 2u_ku^0u^j\frac{R'(t)}{R(t)}\right)S^{ik}} \\
&\quad + \cancel{u_0u^j\dot{S}^{0i}} - \cancel{u_0u^i\dot{S}^{0j}} \\
&\quad + \cancel{\left(u_0u^0 + u_ku^k - 1\right)u^j\frac{R'(t)}{R(t)}}S^{0i} - \cancel{\left(u_0u^0 + u_ku^k - 1\right)u^i\frac{R'(t)}{R(t)}}S^{0j} = 0 \\
&\dot{S}^{ij} + 2u^0\frac{R'(t)}{R(t)}S^{ij} = 0 \\
&S^{ij} = \left(\frac{R_0}{R(t)}\right)^2 S^{ij}_0
\end{aligned}
\tag{70}
$$

for some initial conditions R_0, S^{ij}_0. It turns out that this is exactly the same solution we found in the comoving frame. Again using the simplification, plus the Frenkel condition (17), we may then write the remaining 3 ODEs as

$$u_i\ddot{S}^{0i} + \dot{u}_i\dot{S}^{0i} - \delta_{jk}u_iu^kR'(t)R(t)\dot{S}^{ij} + m\dot{u}^0 + \delta_{ij}mu^iu^jR'(t)R(t) = 0, \tag{71}$$

$$2\dot{S}^{0i} + \left(2 - u_ju^j\right)u^0\frac{R'(t)}{R(t)}S^{0i} + u_ju^0\dot{S}^{ij} - 2\left[\delta_{jk}u^kR'(t)R(t) - u_ju^0u^0\frac{R'(t)}{R(t)}\right]S^{ij} = 0, \tag{72}$$

$$\begin{aligned} u_j\ddot{S}^{ij} &+ \left[\dot{u}_j + 3u_ju^0\frac{R'(t)}{R(t)}\right]\dot{S}^{ij} + 2\left[\frac{1}{2}\delta_{jk}u^kR'^2(t) + u_ju^0u^0\left(\frac{R'(t)}{R(t)}\right)^2 + \frac{d}{d\tau}\left(u_ju^0\frac{R'(t)}{R(t)}\right)\right]S^{ij} \\ &- \left[u_ju^ju^0\left(\frac{R'(t)}{R(t)}\right)^2 + \frac{d}{d\tau}\left(u_ju^j\frac{R'(t)}{R(t)}\right)\right]S^{0i} + m\dot{u}^i + 2mu^0u^i\frac{R'(t)}{R(t)} = 0, \end{aligned} \tag{73}$$

whose form is still relatively complicated for the work at hand.
Their analytical or numerical treatment is left for some future work.

Part III

The Schwarzschild Metric

The Metric

Beyond cosmological considerations, the first kind of spacetime one might wish to study is one with a spherically symmetric gravitational field, one caused by central masses as is generally the case with stars, planets, and black holes. This kind of spacetime, as it turns out, has both internal and external solutions. While the former are extremely important in the study of black holes—these solutions concern the space inside the event horizon—, it is the latter which interest us currently, for example for a free falling particle, or one orbiting the central mass[18][p.193]. We now derive such a spacetime metric.

[18] S. M. Carroll, *Spacetime and Geometry*, 1st ed. (Pearson Education Limited, Harlow, 2014)

We may start by considering the general form of the metric for a static isotropic spacetime[19][p. 179]

[19] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, 1st ed. (John Wiley & Sons, Inc., New York, 1972)

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{74}$$

from which we may find the only non-vanishing Christoffel symbols

$$\begin{gathered}
\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'(r)}{2B(r)}, \\
\Gamma^r_{tt} = \frac{B'(r)}{2A(r)}, \qquad \Gamma^r_{rr} = \frac{A'(r)}{2A(r)}, \\
\Gamma^r_{\theta\theta} = -\frac{r}{A(r)}, \qquad \Gamma^r_{\phi\phi} - \frac{r\sin^2\theta}{A(r)}, \\
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta,
\end{gathered} \tag{75}$$

and thus the only non-vanishing components of the Ricci tensor

$$\begin{aligned} R_{tt} &= -\frac{B''(r)}{2A(r)} + \frac{1}{4}\frac{B'(r)}{A(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\frac{B'(r)}{A(r)}, \\ R_{rr} &= \frac{B''(r)}{2B(r)} - \frac{1}{4}\frac{B'(r)}{B(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\frac{A'(r)}{A(r)}, \\ R_{\theta\theta} &= -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)}, \\ R_{\phi\phi} = \sin^2\theta R_{\theta\theta} &= -\sin^2\theta + \frac{r\sin^2\theta}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{\sin^2\theta}{A(r)}. \end{aligned} \tag{76}$$

We may now consider $\frac{R_{tt}}{B} + \frac{R_{rr}}{A}$, along with the equations for empty space, $R_{\mu\nu} = 0$, to get

$$-\frac{1}{rA(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) = 0, \tag{77}$$

hence $\frac{A'(r)}{A(r)} = -\frac{B'(r)}{B(r)}$ and so we must have

$$A(r) \propto \frac{1}{B(r)}, \tag{78}$$

indeed the fact that for $r \to \infty$ we get $g_{\mu\nu} \to \eta_{\mu\nu}$ guarantees that this must be an equality

$$A(r) = \frac{1}{B(r)}. \tag{79}$$

We finally rewrite $R_{\theta\theta}$ using this last relation and set it to zero so as to satisfy its corresponding equation for empty space

$$\begin{aligned} R_{\theta\theta} &= -1 + rB'(r) + B(r) = 0 \\ rB'(r) + B(r) &= \frac{d}{dr}rB(r) = 1, \end{aligned} \tag{80}$$

so that the function $B(r)$ takes the form

$$B(r) = 1 + \frac{C}{r} \tag{81}$$

for some constant of integration C —considering the Newtonian limit we should find this constant to be $-2M$ using units such that $G = c = 1$— and thus the metric will finally take the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{82}$$

and (75) will look like

$$
\begin{aligned}
\Gamma^t_{tr} = \Gamma^t_{rt} &= -\frac{M}{2Mr - r^2}, \\
\Gamma^r_{tt} = -\frac{M(2M-r)}{r^3}, &\qquad \Gamma^r_{rr} = \frac{M}{2Mr - r^2}, \\
\Gamma^r_{\theta\theta} = 2M - r, &\qquad \Gamma^r_{\phi\phi} = (2M - r)\sin^2\theta, \\
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, &\qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, &\qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.
\end{aligned}
\tag{83}
$$

Point Particles in the Schwarzschild Metric

The geodesic equations, as well as the constraints, that must be satisfied by the velocity u^μ and spin S^μ are

$$\begin{aligned} \frac{du^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} u^\mu u^\nu = 0 \\ g_{\mu\nu} u^\mu u^\nu = u_\nu u^\nu = -1, \\ \\ \frac{dS^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} S^\mu u^\nu = 0 \\ g_{\mu\nu} S^\mu u^\nu = S_\nu u^\nu = 0. \end{aligned} \tag{84}$$

We may now find the 4-velocity for a test particle falling radially inward from a distance $r_0 \gg 2M$[20], that is, we solve the geodesic equations for the 4-velocity subject to the initial conditions

[20] M. Ross, *The Schwarzschild Solution and Timelike Geodesics*, Logan, 2016

$$\begin{aligned} x^\alpha_0 = (0, r_0, \tfrac{\pi}{2}, 0) \\ u^\alpha_0 = (u^t_0, 0, 0, 0). \end{aligned} \tag{85}$$

Considering the constraint $g_{\mu\nu} u^\mu u^\nu = -1$ we find u^t_0 to be

$$u^t_0 = \frac{1}{\sqrt{1 - \frac{2M}{r_0}}}. \tag{86}$$

We may write the geodesic equations

$$\begin{aligned} \frac{du^t}{d\tau} + \Gamma^t_{\beta\gamma} u^\beta u^\gamma = 0 \\ \frac{du^t}{d\tau} - \frac{2M}{2Mr - r^2} u^t u^r = 0, \end{aligned} \tag{87}$$

$$\begin{aligned} \frac{du^r}{d\tau} + \Gamma^r_{\beta\gamma} u^\beta u^\gamma = 0 \\ \frac{du^r}{d\tau} - \frac{GM(2M-r)}{r^3}(u^t)^2 + \frac{M}{2Mr - r^2}(u^r)^2 + (2M-r)(u^\theta)^2 + (2M-r)\sin^2\theta(u^\phi)^2 = 0, \end{aligned} \tag{88}$$

$$\begin{aligned} \frac{du^\theta}{d\tau} + \Gamma^\theta_{\beta\gamma} u^\beta u^\gamma = 0 \\ \frac{du^\theta}{d\tau} + \frac{1}{r} u^r u^\theta - \sin\theta\cos\theta (u^\phi)^2 = 0, \end{aligned} \tag{89}$$

and

$$\begin{aligned} \frac{du^\phi}{d\tau} + \Gamma^\phi_{\beta\gamma} u^\beta u^\gamma = 0 \\ \frac{du^\phi}{d\tau} + \frac{1}{r} u^r u^\phi + \cot\theta u^\theta u^\phi = 0. \end{aligned} \tag{90}$$

Immediately we can integrate (87) to get

$$u^t = u_0^t \left(\frac{1 - \frac{2M}{r_0}}{1 - \frac{2M}{r}} \right), \tag{91}$$

as well as (89) and (90) to get

$$\begin{aligned} u^\theta = 0 \\ u^\phi = 0. \end{aligned} \tag{92}$$

Finally, integrating (88) is not as straightforward, so we may instead turn back to the constraint $g_{\mu\nu} u^\mu u^\nu = -1$, along with the solutions we just found, to find

$$u^r = \sqrt{\frac{2M}{r_0} - \frac{2M}{r}}. \tag{93}$$

FOR A FREE FALLING TEST PARTICLE, $u^\mu = (u_0^t \left(\frac{1 - \frac{2M}{r_0}}{1 - \frac{2M}{r}} \right), \sqrt{\frac{2M}{r_0} - \frac{2M}{r}}, 0, 0)$, we may find the following solutions for the spin, for the temporal part

$$\begin{aligned} \frac{dS^t}{d\tau} + \Gamma^t_{\mu\nu} S^\mu u^\nu = 0 \\ \frac{dS^t}{d\tau} + \Gamma^t_{tr} S^t u^r + \Gamma^t_{rt} S^r u^t = 0 \\ \frac{dS^t}{d\tau} - \frac{M}{2Mr - r^2} \left(S^t u^r + S^r u^t \right) = 0, \end{aligned} \tag{94}$$

and for the spatial part

$$\begin{aligned} \frac{dS^r}{d\tau} + \Gamma^r_{\mu\nu} S^\mu u^\nu = 0 \\ \frac{dS^r}{d\tau} + \cancelto{0}{\Gamma^r_{\mu\mu} S^\mu u^\mu} = 0 \\ S^r = \text{const.} \\ g_{\mu\nu} S^\mu u^\nu = 0 \Rightarrow S^r = 0, \end{aligned} \tag{95}$$

$$\begin{aligned} \frac{dS^\theta}{d\tau} + \Gamma^\theta_{\mu\nu} S^\mu u^\nu &= 0 \\ \frac{dS^\theta}{d\tau} + \cancelto{0}{\Gamma^\theta_{r\theta} S^r u^\theta} + \Gamma^\theta_{\theta r} S^\theta u^r + \cancelto{0}{\Gamma^\theta_{\phi\phi} S^\phi u^\phi} &= 0 \\ \frac{dS^\theta}{d\tau} + \frac{1}{r} S^\theta \frac{dr}{d\tau} &= 0 \\ S^\theta &= \frac{r}{r_0} S^\theta_0, \end{aligned} \tag{96}$$

and

$$\begin{aligned} \frac{dS^\phi}{d\tau} + \Gamma^\phi_{\mu\nu} S^\mu u^\nu &= 0 \\ \frac{dS^\phi}{d\tau} + \cancelto{0}{\Gamma^\phi_{r\phi} S^r u^\phi} + \cancelto{0}{\Gamma^\phi_{\theta\phi} S^\theta u^\phi} + \Gamma^\phi_{\phi r} S^\phi u^r + \cancelto{0}{\Gamma^\phi_{\phi\theta} S^\phi u^\theta} &= 0 \\ \frac{dS^\phi}{d\tau} + \frac{1}{r} S^\phi \frac{dr}{d\tau} &= 0 \\ S^\phi &= \frac{r}{r_0} S^\phi_0. \end{aligned} \tag{97}$$

We may finally return to (94), and using (95) we get

$$\begin{aligned} \frac{dS^t}{d\tau} - \frac{M}{2Mr - r^2} (S^t u^r + \cancelto{0}{S^r u^t}) &= 0 \\ \frac{dS^t}{d\tau} - \frac{M}{2Mr - r^2} S^t \frac{dr}{d\tau} &= 0 \\ S^t &= \sqrt{\frac{r(r_0 - 2M)}{r_0(r - 2M)}} S^t_0. \end{aligned} \tag{98}$$

Extended Bodies in the Schwarzschild Metric

In order to use the MPD equation, the components of the Riemann tensor for the given metric must be calculated. For the Schwarzschild metric the non-zero components of the Riemann tensor are

$$\begin{aligned}
&R^t{}_{rrt} = -R^t{}_{rtr} = \frac{2M}{(2M-r)r^2} \qquad R^t{}_{\theta\theta t} = -R^t{}_{\theta t\theta} = \frac{M}{r} \qquad R^t{}_{\phi\phi t} = -R^t{}_{\phi t\phi} = \frac{M\sin^2\theta}{r}\\
&R^r{}_{trt} = -R^r{}_{ttr} = \frac{2M(2M-r)}{r^4} \qquad R^r{}_{\theta\theta r} = -R^r{}_{\theta r\theta} = \frac{M}{r} \qquad R^r{}_{\phi\phi r} = -R^r{}_{\phi r\phi} = \frac{M\sin^2\theta}{r}\\
&R^\theta{}_{tt\theta} = -R^\theta{}_{t\theta t} = \frac{M(2M-r)}{r^4} \qquad R^\theta{}_{r\theta r} = -R^\theta{}_{rr\theta} = \frac{M}{(2M-r)r^2} \qquad R^\theta{}_{\phi\theta\phi} = -R^\theta{}_{\phi\phi\theta} = \frac{2M\sin^2\theta}{r}\\
&R^\phi{}_{tt\phi} = -R^\phi{}_{t\phi t} = \frac{M(2M-r)}{r^4} \qquad R^\phi{}_{r\phi r} = -R^\phi{}_{rr\phi} = \frac{M}{(2M-r)r^2} \qquad R^\phi{}_{\theta\phi\theta} = -R^\phi{}_{\theta\theta\phi} = \frac{2M}{r}
\end{aligned} \tag{99}$$

At this point we may write down the MPD equations, for the free-falling test particle, $u_0^\mu = (u_0^t, 0, 0, 0)$. We first notice that the non-vanishing components of the Riemann tensor in this metric happen to be the same as in the FLRW metric—they, of course, take different values, but nevertheless we can quickly get from (64)

$$\begin{aligned}
&\frac{D}{d\tau}\left(mu^t + u_r\frac{D}{d\tau}S^{tr}\right) - \frac{1}{2}\frac{2M}{(2M-r)r^2}u^r S^{tr} = 0,\\
&2\frac{D}{d\tau}S^{ti} + u^t u_r \frac{D}{d\tau}S^{ir} = 0.
\end{aligned} \tag{100}$$

Now the second of these reads for $i = r$

$$
\begin{aligned}
\frac{D}{d\tau}S^{tr} &= 0 \\
\frac{d}{d\tau}S^{tr} + \Gamma^{t}_{\sigma\lambda}u^{\sigma}S^{\lambda r} + \Gamma^{r}_{\sigma\lambda}u^{\sigma}S^{t\lambda} &= 0 \\
\frac{d}{d\tau}S^{tr} + \Gamma^{t}_{rt}u^{r}S^{tr} + \Gamma^{r}_{rr}u^{r}S^{tr} &= 0 \\
\frac{d}{d\tau}S^{tr} + \left(\Gamma^{t}_{rt} + \Gamma^{r}_{rr}\right)u^{r}S^{tr} &= 0 \\
\frac{d}{d\tau}S^{tr} + \cancel{\left(-\frac{M}{2Mr-r^2} + \frac{M}{2Mr-r^2}\right)}u^{r}S^{tr} &= 0 \\
\frac{d}{d\tau}S^{tr} &= 0 \\
S^{tr} &= \text{const.},
\end{aligned}
\tag{101}
$$

and indeed putting this back into the first equation yields $S^{tr} = 0$. Using this fact we further get from (64)

$$
\begin{aligned}
\frac{D}{d\tau}\left(u_r\frac{D}{d\tau}S^{ir}\right) + \frac{1}{2}\left(\frac{M(2M-r)}{r^4}\delta_{t\rho}u^{t} - \frac{M}{(2M-r)r^2}\delta_{r\rho}u^{r}\right)S^{\rho i} &= 0, \\
\frac{D}{d\tau}S^{ij} - u_{\sigma}\left(u^{i}\frac{D}{d\tau}S^{\sigma j} + u^{j}\frac{D}{d\tau}S^{i\sigma}\right) &= 0,
\end{aligned}
\tag{102}
$$

where once again we immediately see from the second one, taking $i = \theta, j = \phi$,

$$
\begin{aligned}
\frac{D}{d\tau}S^{\theta\phi} &= 0 \\
\frac{d}{d\tau}S^{\theta\phi} + \Gamma^{\theta}_{\sigma\lambda}u^{\sigma}S^{\lambda\phi} + \Gamma^{\phi}_{\sigma\lambda}u^{\sigma}S^{\theta\lambda} &= 0 \\
\frac{d}{d\tau}S^{tr} + \Gamma^{\theta}_{r\theta}u^{r}S^{\theta\phi} + \Gamma^{\phi}_{r\phi}u^{r}S^{\theta\phi} &= 0 \\
\frac{d}{d\tau}S^{\theta\phi} + \left(\Gamma^{\theta}_{r\theta} + \Gamma^{\phi}_{r\phi}\right)u^{r}S^{\theta\phi} &= 0 \\
\frac{d}{d\tau}S^{\theta\phi} + \left(\frac{1}{r} + \frac{1}{r}\right)u^{r}S^{\theta\phi} &= 0 \\
\frac{d}{d\tau}S^{\theta\phi} &= -2\frac{u^{r}}{r}S^{\theta\phi} \\
\frac{dS^{\theta\phi}}{S^{\theta\phi}} &= -2\frac{dr}{r} \\
S^{\theta\phi} &= \left(\frac{r_0}{r}\right)^2 S_0^{\theta\phi}, \\
S^{\phi\theta} &= \left(\frac{r_0}{r}\right)^2 S_0^{\phi\theta},
\end{aligned}
\tag{103}
$$

which, once again going from the spin bivector to the spin vector using (8), yields the two solutions

$$S^\theta = \frac{r}{r_0} S_0^\theta,$$
$$S^\phi = \frac{r}{r_0} S^\phi, \tag{104}$$

these being exactly the same as we previously found using the geodesic formalism. The remaining differential equations, allowing us to find S^t upon applying (8), take on a more complicated form and, just as we did with the FLRW metric, we choose to stop at this point, having obtained two equivalent solutions, and leave the analytical or numerical treatment of the others for future work.

Part IV

Final Remarks

On the Results from the Different Formalisms

THE PRECEDING CALCULATIONS showed us that, at least as far as the cases we considered and managed to fully solve, using either formalism will result in the same motion. We are left wondering why and the answer to that goes back to the SSCs, mentioned at the end of the chapter *Spin in Gravity*. As is also explained there, the MPD equations happen to be adapted to extended bodies, as opposed to the geodesic formalism which is specifically for point particles, the choice of SSC is then a matter of choosing the reference observer with respect to which the body's center of mass is defined[21]. It turns out that when using the Frenkel condition (also found in the literature as Frenkel-Mathisson-Pirani condition, or simply Pirani condition), and changing from the spin bivector to the spin vector with the help of (8), one finds that the MPD equations reduce to simply[22]

$$\frac{DS^\alpha}{d\tau} = u^\alpha a_\beta S^\beta, \tag{105}$$

which is nothing more than the spin vector being Fermi-Walker transported. Then, considering how for negligible accelerations Fermi-Walker transport reduces to parallel transport, it is easy to see how the MPD equations, given the chosen SSC and the chosen metrics, would yield the same results as the geodesic formalism.

Moreover, it is also worth noting that in the weak field approximation all the different commonly-used choices of SSC happen to be equivalent and no current experiments or observations would be able to choose amongst them[23]. In practice this means that the choice of SSC, at least for the time being, is primarily made for computational reasons. Indeed, future work to expand upon this one could consider the MPD equations for those cases deemed here too difficult to solve analytically and attempt a solution under different SSCs.

[21] O. Semerák, "Spinning particles in vacuum spacetimes of different curvature types: Natural reference tetrads and massless particles", Physical Review D - Particles, Fields, Gravitation and Cosmology **92**, 064036 (2015)

[22] Z. Keresztes and B. Mikóczi, "Evolutions of spinning bodies moving in rotating black hole spacetimes", 2019

[23] F. Cianfrani and G. Montani, "Spinning particles in general relativity", in Nuovo cimento della societa italiana di fisica b (2007)

De Sitter and Lense-Thirring Precessions, Revisited

As previously mentioned, de Sitter precession takes place as a result of gravitational fields due to the presence of mass while Lense-Thirring precession accounts for frame-dragging, the effect of the source of the gravitational field itself spinning. Naturally, having chosen the FLRW and Schwarzschild metrics, we are only concerning ourselves with gravitational fields that result from static masses and thus the results of these, or any analogous calculations, would only correspond to de Sitter precession. On the other hand, if we were now to choose a metric for a spinning central mass and perform the calculations with respect to it, the results would contain a sum of both de Sitter and Lense-Thirring precessions. Appropriate choices for such a metric would include the one studied by Lense and Thirring themselves back in 1918, itself not an exact solution, or the Kerr metric, which is an exact solution, although it wouldn't have been known by Lense and Thirring.

It is important to note, however, that such calculations involving more complicated metrics, including those for a spinning central mass, would greatly complicate the resulting equations, which even here were not always readily solvable. One possible workaround for this, and one which has in fact been used before[24], is to go into a tetrad formalism such as the Newman-Penrose formalism, these being better adapted to such computations. Even after all of this, analytical solutions might still not be possible and one may need to use numerical methods, also an approach which has been previously used with great success[25].

Lastly we consider experiments and observations confirming de Sitter and Lense-Thirring precessions, these happen to be of great importance to the theory of general relativity insofar as they serve as experimental confirmation for it. By comparing the data to the predictions that may be made using calculations such as the ones presented here, especially in the weak field approximation, we have a very direct way of confirming the validity of the theory and some

[24] O. Semerák, "Spinning particles in vacuum spacetimes of different curvature types: Natural reference tetrads and massless particles", Physical Review D - Particles, Fields, Gravitation and Cosmology **92**, 064036 (2015), O. Semerák and M. Šrámek, "Spinning particles in vacuum spacetimes of different curvature types", Physical Review D - Particles, Fields, Gravitation and Cosmology **92**, 064032 (2015)

[25] Z. Keresztes and B. Mikóczi, "Evolutions of spinning bodies moving in rotating black hole spacetimes", 2019, R. M. Plyatsko et al., "Mathisson-Papapetrou-Dixon equations in the Schwarzschild and Kerr backgrounds", Classical and Quantum Gravity **28**, 195025 (2011), N. Velandia and J. M. Tejeiro, "Numerical Solutions of Mathisson-Papapetrou-Dixon equations for spinning test particles in a Kerr metric", Momento, 60–85 (2018)

of its assertions, specifically some that have no counterpart in Newtonian gravity.

The first tests of de Sitter precession[26] were made for the Earth-Moon gyroscope orbiting the sun, mostly using Lunar Laser Ranging (LLR), that is, using retroreflectors planted on the moon during the Apollo missions. First in 1987 by Bertotti, Ciufolini, and Bender using LLR, as well as Very Long Baseline Interferometry (VLBI), to obtain data with an accuracy of the order of 10%; next in 1988 by Shapiro, Reasenberg, Chandler, and Babcock, and in 1989 by Dickey, Newhall, and Williams, both teams using LLR and obtaining data with an accuracy of about 2%; finally in 1991 by Müller, Schneider, Soffel, and Ruder, and in 1994 by Dickey, Bender, Faller, Newhall, Ricklefs, Ries, Shelus, Veillet, Whipple, Wiant, Williams, and Yoder, both teams once again using LLR and obtaining data with an accuracy of about 1%.

As for Lense-Thirring precession, the first proposed experiments[27] came in 1977 from Cugusi and Proverbio and involved using the Laser Geodynamics Satellite (LAGEOS) launched by NASA in 1976. A different method[28] was proposed in 1986 by Ciufolini using two satellites and the first tests were performed in 1996 by Ciufolini, Lucchesi, Vespe, and Mandiello using LAGEOS, together with LAGEOS 2 launched in 1992. Tests using these two satellites, albeit a somewhat different method, were once again performed[29] in 2004 by Ciufolini and Pavlis, and in 2006 by Ciufolini, Pavlis, and Peron. Finally, similar experiments are being performed by the Italian Space Agency under the direction of Ciufolini[30], the Laser Relativity Satellite (LARES) launched in 2012 with the aim of measuring the effect with an accuracy of 1%, and the LARES 2 is expected to launch in December of this year, hopefully being able to improve the accuracy to 0.2%. It should be noted that the accuracy of all experiments performed so far is nevertheless still up for debate[31]. One final possible future experiment worth mentioning would use data from the Juno spacecraft launched in 2011[32], finally testing the effect with a source different from Earth.

Lastly, an experiment that managed to test both effects was Gravity Probe B, launched in 2004 and collecting data for about a year with an error of about 19% and the predicted values lying at the center of the confidence interval[33].

[26] B. Bertotti et al., "New test of general relativity: Measurement of de sitter geodetic precession rate for lunar perigee", Physical Review Letters **58**, 1062 (1987), I. I. Shapiro et al., "Measurement of the de sitter precession of the moon: A relativistic three-body effect", Physical Review Letters **61**, 2643 (1988), J. O. Dickey et al., "Investigating relativity using lunar laser ranging: Geodetic precession and the Nordtvedt effect", Advances in Space Research **9**, 75–78 (1989), J. Mueller et al., "Testing Einstein's theory of gravity by analyzing Lunar Laser Ranging data", The Astrophysical Journal **382**, L101 (1991), J. O. Dickey et al., "Lunar laser ranging: A continuing legacy of the apollo program", Science **265**, 482–492 (1994)

[27] L. Cugusi and E. Proverbio, "Relativistic Effects on the Motion of Earth's Artificial Satellites", Astronomy and Astrophysics **69**, 321–325 (1978)

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[29] I. Ciufolini and E. C. Pavlis, "A confirmation of the general relativistic prediction of the Lense-Thirring effect", Nature **431**, 958–960 (2004), I. Ciufolini et al., "Determination of frame-dragging using Earth gravity models from CHAMP and GRACE", New Astronomy **11**, 527–550 (2006)

[30] I. Ciufolini et al., "Towards a one percent measurement of frame dragging by spin with satellite laser ranging to LAGEOS, LAGEOS 2 and LARES and GRACE gravity models", Space Science Reviews **148**, 71–104 (2009), I. Ciufolini et al., "A new laser-ranged satellite for General Relativity and space geodesy: I. An introduction to the LARES2 space experiment", European Physical Journal Plus **132**, 336 (2017)

[31] G. Renzetti, "Some reflections on the Lageos frame-dragging experiment in view of recent data analyses", New Astronomy **29**, 25–27 (2014)

[32] L. Iorio, "Juno, the angular momentum of Jupiter and the Lense-Thirring effect", New Astronomy **15**, 554–560 (2010)

[33] C. W. Everitt et al., "Gravity probe B: Final results of a space experiment to test general relativity", Physical Review Letters **106**, 221101 (2011)

Conclusion

As we set out to do in the introduction, the precession of a gyroscope due to both a central mass and a rotating central mass was briefly explained as motivation, followed by the calculations of spin motion in the FLRW metric and the Schwarzschild metric, both of them using the geodesic formalism as well as the MPD equations with the Frenkel condition, returning finally to the two types of precession, how they serve as confirmation of the general theory of relativity, and the experiments which have been carried out to test them. As far as the calculations made, although a number of papers have somewhat similar treatments for the Schwarzschild metric and the Kerr metric, we tried here to provide something novel with the extended particle kinematics in the FLRW metric, including a distinction between a particle part of the cosmological flow and a test particle separate from it.

The results from the calculations were not always able to be presented in full, at times arriving only at simplified differential equations which may or may not have analytical solutions, and the treatment of which, analytical or numerical, is left for future work. Where solutions were found they turned out to be equivalent in both formalisms, this being finally explained by the choice of SSC which in certain specific cases, and in the weak field approximation, reduce the MPD equations to simple parallel transport. Moreover, we explained how in the weak field approximation all choices of SSC are equivalent and thus the choice is ultimately to be made for ease of computation, future treatments of this subject could attempt to find those solutions here deemed too difficult by trying different SSCs.

Something that was ultimately missing from all the calculations was Lense-Thirring precession, having chosen metrics where the source of the gravitational field is not rotating. Future work could attempt calculations similar to the ones here for metrics that incorporate a rotating central mass such as the one studied by Lense and Thirring or the complete Kerr solution. Where these treatments might need to differ is in the use of a tetrad formalism such as the

Newman-Penrose formalism to ease the computations; this may ultimately expand, not only this work, but also that of Daniel Rozo carried out in 2019 under the supervision of Professor Pedro Bargueño, Ph.D.[34], in which he provided an overview of these formalisms and how they can become a powerful computational tool in some cases for which they are well adapted.

[34] D. F. Rozo Oviedo, *Introducción a la formulación de Newman-Penrose y al formalismo de tétradas con aplicación en cálculos en Relatividad General*, 2019

Overall, what is presented here is an overview of the motion of spin in external gravitational fields, the importance of studying it as far as experimental confirmation of general relativity is concerned, and different tools that may be used to do so, along with some seemingly new treatment in the extended particle kinematics. It certainly leaves a lot open for future work on the subject, taking different approaches and considering different cases of great interest.

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[37] D. F. Rozo Oviedo, *Introducción a la formulación de Newman-Penrose y al formalismo de tétradas con aplicación en cálculos en Relatividad General*, 2019.

Appendix A: Operator of Proper-Time-Derivative

The differential operator $\frac{D}{d\tau}$ is the operator of proper-time-derivative, defined as

$$\frac{D}{d\tau} = u^\mu \nabla_\mu, \tag{A.1}$$

with ∇_μ the ordinary covariant derivative. When we have this operator acting on a 4-vector, for instance, we have

$$\begin{aligned} \frac{D}{d\tau} A^\nu &= u^\mu \nabla_\mu A^\nu \\ &= \frac{dx^\mu}{d\tau}\left(\partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda\right). \end{aligned} \tag{A.2}$$

The RHS in particular may not be applied to position, x^μ, since it is not a 4-vector in curved space-time (however displacement, dx^μ, is) but the operator by itself applied to position actually specifies 4-velocity, $u^\mu \equiv \frac{Dx^\mu}{D\tau}$, and likewise a second application specifies 4-acceleration, $a^\mu \equiv \frac{Du^\mu}{D\tau}$, in this case taking the RHS yields

$$\begin{aligned} \frac{D}{d\tau} u^\nu &= \frac{dx^\mu}{d\tau}\left(\partial_\mu u^\nu + \Gamma^\nu_{\mu\lambda} u^\lambda\right) \\ &= \frac{dx^\mu}{d\tau}\frac{\partial}{\partial x^\mu}\frac{dx^\nu}{d\tau} + \frac{dx^\mu}{d\tau}\Gamma^\nu_{\mu\lambda} u^\lambda \\ &= \frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\lambda} u^\mu u^\lambda, \end{aligned} \tag{A.3}$$

where we see how imposing $a^\mu = 0$ leads directly to the geodesic equations—as it should be, since a free-falling body in a gravitational field must follow a geodesic.

The operator may also act on (2,0) tensors as follows

$$\begin{aligned} \frac{D}{d\tau} A^{\mu\nu} &= u^\lambda \nabla_\lambda A^{\mu\nu} \\ &= \frac{dx^\lambda}{d\tau}\left(\partial_\lambda A^{\mu\nu} + \Gamma^\mu_{\lambda\rho} A^{\rho\nu} + \Gamma^\nu_{\lambda\rho} A^{\mu\rho}\right), \end{aligned} \tag{A.4}$$

which, in addition to the application given here for spin, finds use in certain applications in electrodynamics.

FINALLY, as may be seen from the first of the MPD equations, we would like to be able to readily expand terms of the form $\frac{D}{d\tau}\left(u_\mu \frac{D}{d\tau} S^{\lambda\mu}\right)$. We then have for arbitrary $A^{\mu\nu}$

$$\begin{aligned}
\frac{D}{d\tau}\left(u_\nu \frac{D}{d\tau} A^{\mu\nu}\right) &= u^\rho \nabla_\rho \left(u_\nu u^\sigma \nabla_\sigma A^{\mu\nu}\right) \\
&= u^\rho \partial_\rho \left(u_\nu u^\sigma \nabla_\sigma A^{\mu\nu}\right) + \Gamma^\mu_{\rho\alpha} u_\nu u^\rho u^\sigma \nabla_\sigma A^{\alpha\nu} \\
&= u^\rho \partial_\rho \left(u_\nu u^\sigma \partial_\sigma A^{\mu\nu} + \Gamma^\mu_{\sigma\beta} u_\nu u^\sigma A^{\beta\nu} + \Gamma^\nu_{\sigma\beta} u_\nu u^\rho A^{\mu\beta}\right) \\
&\quad + \Gamma^\mu_{\rho\alpha} u_\nu u^\rho u^\sigma \partial_\sigma A^{\alpha\nu} + \Gamma^\mu_{\rho\alpha} \Gamma^\alpha_{\sigma\beta} u_\nu u^\rho u^\sigma A^{\beta\nu} + \Gamma^\mu_{\rho\alpha} \Gamma^\nu_{\sigma\beta} u_\nu u^\rho u^\sigma A^{\alpha\beta},
\end{aligned} \tag{A.5}$$

thus yielding second order derivatives that make the MPD equations considerably harder to solve than geodesic equations.