Lie groups and definability

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Introduction

In the second half part of the 19th century, Sophus Lie studied group actions on smooth manifolds and observed that they were closely related with the study of certain vector space structures: "Lie algebras". From those observations Wilhelm Killing and Friedrich Engel began to classify some of those Lie algebras, simple complex Lie algebras. It is only in the late 19th century that Elie Cartan in his thesis completed the classification of simple complex Lie algebras. He also completed the classification in the real case using real forms theory.

We take here a particular approach to model theory which is usually interested in mathematical structures from a logical point of view. Model theory concentrates on some objects of this structure, the "definable sets". We will not be interested here in general and structural properties of first order theories but instead try to understand the distinction between definable and non-definable objects under the lights of first order logic. Specifically we will be interested in showing the strong connections between Lie groups and groups definable in certain expansions of the real field, the *o*-minimal ones.

Let us fix ideas and give some definitions in order to enunciate the main problem we will be treating here. We remind the reader that an *o*-minimal structure is a (densely) ordered structure in which definable sets in one variable are finite unions of points and intervals. First notice that having a dense ordering $<$ on a structure $\mathcal{M}$ gives us a topology on $\mathcal{M}$: the order topology. A direct consequence of the definition is that we cannot have an infinite discrete definable subset in $\mathcal{M}$ (see [Dri98] for a reference).

Fact 0.0.1. *Let $X \subseteq \mathcal{M}^k$ be a definable subset of the o-minimal structure $\mathcal{M}$. If X is discrete with respect to the order topology it is finite.*

By *definable* we mean definable (eventually with aprameters) in an *o*-minimal expansion of the real field. By *definable group* we mean a group whose underlying set, multiplication map and inverse map are definable.

As mentioned above definable groups will share a fair part of terrain with

Lie groups. More precisely, any definable group can be seen as a Lie group since A. Pillay proved the following in [Pil88].

Fact. *Let G be a definable group, G can be equipped with a topology making it a real Lie group.*

It is then natural to ask if the contrary is true.

Question (naive)**.** *Are all Lie groups definable groups?*

One first obstruction to this question lies in the fact that any definable group has finitely many connected component as a Lie group (think of the extreme case of $\mathbb{Z}$ which cannot be definable in $\mathbb{R}$ by Fact 0.0.1).

But even when restricted to connected Lie groups the two categories do not coincide. The easiest example to realize this might be the universal cover of $SL_2(\mathbb{R})$; it is a simply connected Lie group whose center (which would be definable if the group were) is infinite and discrete, which cannot happen in an o-minimal context as mentioned in Fact 0.0.1.

But this counterexample lies in a particular place among Lie groups, it is not linear (cannot be seen as a closed subgroup of any general linear group). That motivates the following question, which was the original question in this thesis project. It was raised by S. Starchenko in a private conversation with A. Onshuus.

Question 1. *Which linear real Lie groups are Lie-isomorphic to a group definable in an o-minimal expansion of the real field?*

In [COS18], A. Conversano, A. Onshuus and S. Starchenko noticed that if we let the additive group $(\mathbb{R}, +)$ act by rotation on the the plane $\mathbb{R}^2$ we get a Lie group $G = \mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ with $\varphi_\theta(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. This connected (actually simply connected) Lie group has a nice matrix presentation:

$$G = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} : \theta, x, y \in \mathbb{R} \right\}.$$

Suppose now that this group is definable, it acts naturally and definably on $\mathbb{R}^3$ but the (definable) stabilizer of $(1, 0, 0)$ is discrete (it is $2\pi\mathbb{Z}$) which is not possible by Fact 0.0.1. They give the following "triangular-by-compact" condition under which such behaviors do not occur.

Fact 4.0.1. *[Conversano, Onshuus, Starchenko, [COS18], Theorem* 5.4*] Let R be a connected solvable Lie group. Then the following are equivalent:*

- *R has a normal, connected, torsion-free and supersolvable subgroup T such that R/T is compact (we say that R is* triangular-by-compact*).*

- *R is Lie isomorphic to a group definable in an o-minimal expansion of the reals.*

Let us remind the reader that a connected Lie group G is said to be supersolvable if the eigenvalues of the operators $\mathrm{Ad}(g)$ of its adjoint representation are real for any element $g \in G$.

Remark. *We cannot avoid working up to Lie isomorphism as there are some presentations of groups that are not definable in any o-minimal expansion of the reals. For example,*

$$G = \left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix} : t \in \mathbb{R} \right\}$$

is not definable in any o-minimal expansion of $\mathbb{R}$ but it is Lie isomorphic to $(\mathbb{R}, +)$ which is definable.

In our treatment of Question 1 we begin by an improvement of Fact 4.0.1, proving in Chapter 4 that any connected solvable Lie group satisfying the first equivalent condition in Fact 4.0.1 is actually Lie isomorphic to a *definably linear* group, that is a definable subgroup of $GL_n(\mathbb{R})$ for some $n \in \mathbb{N}$.

Theorem 1. *Let G be a connected, triangular by compact, solvable Lie group. Then G is Lie-isomorphic to a definable matrix group.*

The main reason we took a particular interest in proving definable linearity of definable solvable Lie groups is because, associated to any Lie group there is a very interesting solvable subgroup: the solvable radical (see [Kna02],).

Fact. *Let G be a connected Lie group, there is a maximal connected normal solvable Lie subgroup R in G. (R contains any connected solvable Lie subgroup in G)*

It is a fact that if G is a definable group, its solvable radical R is definable, in particular it must be triangular by compact. Using Theorem 1 we gave a full answer to Question 1 in Chapter 5 and prove that this is also sufficient if G is linear.

Theorem 2. *Let G be a connected linear Lie group whose solvable radical is triangular by compact. Then G is Lie-isomorphic to a definable matrix group.*

Having dealt with the linear case we took a look at the general case of a connected Lie group without any assumption on its linearity.

To tackle this problem we first spot the impediment to definability of the universal cover of $SL_2(\mathbb{R})$. The reason we highlighted above was its infinite discrete center. We show in Section 3.5 that this is the only obstruction if the group is semisimple.

Theorem 3. *Let S be a connected semisimple Lie group. Then S is Lie isomorphic to a definable group if and only if its center $\mathcal{Z}(S)$ is finite.*

This result was achieved thanks to a careful study of finite coverings of definable Lie groups in Section 3.4. Along the way we proved that if a Lie group is definable, its *o*-minimal and topological universal cover coincide.

The reason we are in turn so interested in semisimple Lie groups is because of a well known decomposition for any connected Lie group and we want to use it in order to classify definable Lie groups.

Fact 1.2.30. *[Levi decomposition, [Lev97], Theorem 1] Let G be a connected Lie group and R its solvable radical. There is a unique (up to conjugacy) maximal connected and semisimple subgroup S of G such that $G = RS$ and $dim(R \cap S) = 0$. Any such S is called a* Levi subgroup of G *and they are all conjugate in G. If the center $\mathcal{Z}(S)$ of S is finite (which is the case whenever G is linear) or if G is simply connected then G is an almost semidirect product of R and S: $G = R(\rtimes)S$.*

With those tools in hand we managed to prove that finiteness of the center of the Levi subgroup together with triangularity by compactness of the solvable radical were a sufficient condition to get definability and prove the following.

Theorem 4. *Let G be a connected Lie group. Let R be the solvable radical of G and S a Levi subgroup of G. If S has finite center and R is triangular by compact then G is Lie-isomorphic to a definable group.*

Finiteness of the center of Levi subgroups is not a necessary condition since there are definable groups whose Levi subgroup are not definable (they have an infinite center). In particular in [CP12], A. Conversano and A. Pillay build a definable group whose Levi subgroup is isomorphic to the universal cover $\widetilde{SL_2}(\mathbb{R})$.

We structured the exposition in six chapters. The first one is a recompilation of well known facts in Lie theory: originally the document provided a proof of Fact 1.2.38 for which we could not find a complete proof in the literature, but since the jury suggested [AM07], Corollary 2.2.6 as a reference and hence it has been removed. In Chapter 2 we state basic facts of *o*-minimal theories but nothing of our own product. The third chapter is dedicated to a thorough study of definable groups and in Section 3.4 we give a proof of Theorem 3. The last three chapters present the main results we obtained. The first tools needed to answer Question 1 are presented in Chapter 4 where we prove that any definable solvable Lie group must be definably linear (Theorem 1). We continue the study of linear Lie groups in Chapter 5 where we fully characterized those which are definable (see Theorem 2). Finally in Chapter 6 we use the tools of definable covering from Section 3.4 and give a proof of Theorem 4.

Chapter 1

Lie theory

This thesis is about the relationship between Lie groups and definable groups. We will dedicate this chapter to Lie theory. Since the study of definable groups is so closely related to the study of Lie groups and algebraic groups the reader will also get a feeling of the results one should be able to obtain in Chapter 3.

1.1 Lie algebras

All results in this section, except for those that explicitly say otherwise, are taken from [Kna02]. We made the same achronological choice to present first Lie algebras and give the discussion about Lie group in a second section. This will allow us to have a more straightforward presentation later.

1.1.1 Basics

We begin with the definition of Lie algebra and the first interesting objects associated to it. We give the general definition over any field but in practice we will only use it for $\mathbb{C}$, $\mathbb{R}$ and more generally over any real closed field $\mathcal{R}$.

Definition 1.1.1. *A Lie algebra over a field $\boldsymbol{k}$ is a $\boldsymbol{k}$-vector space $\mathfrak{g}$ equipped with a bilinear map*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the Lie bracket of $\mathfrak{g}$ such that $(\mathfrak{g}, +, [\cdot,\cdot])$ is a k-algebra and

- $[X, X] = 0$ *for all* $X \in \mathfrak{g}$,
- $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$ *(Jacobi Identity).*

From now on, $\mathfrak{g}$ and $\mathfrak{h}$ will be Lie algebra over $\mathbf{k}$ and we write $[\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot]_{\mathfrak{h}}$ for their Lie brackets.

Definition 1.1.2. *A map $\varphi : \mathfrak{g} \to \mathfrak{h}$ is said to be a Lie algebra morphism if*

- *it is linear,*
- $\varphi([X,Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{h}}$ *for all $X, Y \in \mathfrak{g}$.*

The set of all of those morphisms is denoted $Hom(\mathfrak{g},\mathfrak{h})$ (the set of linear maps is denoted $Hom_{\boldsymbol{k}}(\mathfrak{g},\mathfrak{h})$). If $\mathfrak{g} = \mathfrak{h}$ we write $End(\mathfrak{g})$ (and $End_{\boldsymbol{k}}(\mathfrak{g})$ respectively).

The subset of $End_{\boldsymbol{k}}(\mathfrak{g})$ of maps D such that

$$D([X,Y]) = [X, D(Y)] + [D(X), Y]$$

is called the set of derivations of $\mathfrak{g}$ and we write $Der(\mathfrak{g})$.

Associated to any Lie algebra there is a particular and important map called the adjoint map.

Definition 1.1.3. *For any Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$ the adjoint map is defined as follows.*

$$\begin{array}{rccl} ad\colon & \mathfrak{g} & \longrightarrow & End_{\boldsymbol{k}}(\mathfrak{g}) \\ & X & \longmapsto & ad_X\colon Y \mapsto [X,Y]. \end{array}$$

The image $ad(\mathfrak{g})$ is contained in $Der(\mathfrak{g})$ which can be equipped with a Lie algebra structure as follow.

$$\forall D_1, D_2 \in Der(\mathfrak{g}) \quad [D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1.$$

We can be a bit more precise with the previous definition.

Fact 1.1.1. *Let $\mathfrak{g}$ be a Lie algebra, then $Der(\mathfrak{g})$ can be equipped with a natural Lie algebra structure, as in Definition 1.1.3. Moreover $ad : \mathfrak{g} \to Der(\mathfrak{g})$ is a Lie algebra morphism.*

From now on let us fix a Lie algebra $\mathfrak{g}$.

Definition 1.1.4. *We define here classical objects associated to the Lie algebra $\mathfrak{g}$.*

- *A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace that is stable under Lie bracket.*
- *An ideal $\mathfrak{h}$ in $\mathfrak{g}$ is a subspace such that $[\mathfrak{g},\mathfrak{h}] \subseteq \mathfrak{h}$. An ideal is automatically a subalgebra.*

- *The center of* $\mathfrak{g}$ *is the ideal*

$$\mathcal{Z}(\mathfrak{g}) := \{X \in \mathfrak{g} : \forall\, Y \in \mathfrak{g}\ [X,Y] = 0\}.$$

A Lie algebra $\mathfrak{g}$ *is said abelian if* $\mathcal{Z}(\mathfrak{g}) = \mathfrak{g}$.

- *The commutator of* $\mathfrak{g}$ *is the ideal*

$$[\mathfrak{g}, \mathfrak{g}] = Span(\{[X,Y] \colon X, Y \in \mathfrak{g}\}).$$

- *If* $\mathfrak{s}$ *is a subset of* $\mathfrak{g}$ *the centralizer of* $\mathfrak{s}$ *in* $\mathfrak{g}$ *is the subalgebra*

$$\mathcal{Z}_{\mathfrak{g}}(\mathfrak{s}) \colon = \{X \in \mathfrak{g} : \forall\, S \in \mathfrak{s}\ [X,S] = 0\}.$$

- *If* $\mathfrak{s}$ *is a subalgebra of* $\mathfrak{g}$ *the normalizer of* $\mathfrak{s}$ *in* $\mathfrak{g}$ *is the subalgebra*

$$\mathcal{N}_{\mathfrak{g}}(\mathfrak{s}) \colon = \{X \in \mathfrak{g} : \forall\, S \in \mathfrak{s}\ [X,S] \in \mathfrak{s}\}.$$

We have the following properties for ideals.

Fact 1.1.2. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of* $\mathfrak{g}$. *Then the direct sum (as vector space)* $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$ *and* $[\mathfrak{a}, \mathfrak{b}]$ *are ideals of* $\mathfrak{g}$.

With any good algebraic object comes the quotient object in the same category. Here we explain how to equip the vector space quotient with a Lie algebra structure.

Definition 1.1.5. *Let* $\mathfrak{a}$ *be an ideal of* $\mathfrak{g}$, *the vector space* $\mathfrak{g}/\mathfrak{a}$ *can be equipped with a Lie algebra structure with a natural bracket:*

$$[X + \mathfrak{a}, Y + \mathfrak{a}]_{\mathfrak{g}/\mathfrak{a}} := [X,Y] + \mathfrak{a}.$$

It is called the quotient Lie algebra of $\mathfrak{g}$ *by* $\mathfrak{a}$.

And things work pretty well since we have the following.

Fact 1.1.3. *Let* $\mathfrak{a}$ *be an ideal of* $\mathfrak{g}$, *the projection* $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ *is a morphism of Lie algebra.*

We also have a nice notion of direct product of Lie algebra.

Definition 1.1.6. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be Lie algebras, the vector space* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ *can be equipped with a Lie algebra structure retaining the Lie bracket on each component and defining* $[\mathfrak{a}, \mathfrak{b}] = 0$. *We call this Lie algebra the direct product of* $\mathfrak{a}$ *and* $\mathfrak{b}$ *and still write it* $\mathfrak{a} \oplus \mathfrak{b}$.

This definition can be twisted to obtain a notion of semidirect product of Lie algebras as follows.

Definition 1.1.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras and $\varphi : \mathfrak{a} \to Der(\mathfrak{b})$ be a Lie algebra morphism. There is a unique Lie algebra structure on the vector space $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ that retains the Lie bracket on $\mathfrak{a}$ and $\mathfrak{b}$ and such that*

$$[A, B] = \varphi(A)(B)$$

for $A \in \mathfrak{a}$ and $B \in \mathfrak{b}$. We say that $\mathfrak{g}$ is the semidirect product of $\mathfrak{a}$ and $\mathfrak{b}$ and write $\mathfrak{g} = \mathfrak{a} +_{\varphi} \mathfrak{b}$ or just $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ when it is not confusing.

1.1.2 Nilpotent, solvable and semisimple Lie algebras

Let us continue the exposition with the notions of nilpotency and solvability.

Definition 1.1.8. *The lower central series of $\mathfrak{g}$ is defined recursively as follows*

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n].$$

If $\mathfrak{g}_n = 0$ for some $n \in \mathbb{N}$ we say that $\mathfrak{g}$ is nilpotent.

Definition 1.1.9. *The commutator series of $\mathfrak{g}$ is defined recursively as follows*

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}^n].$$

If $\mathfrak{g}^n = 0$ for some $n \in \mathbb{N}$ we say that $\mathfrak{g}$ is solvable.

We also have the following alternative definition.

Fact 1.1.4. *A n-dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if there exists a sequence of subalgebras*

$$0 = \mathfrak{a}_n \subseteq \cdots \subseteq \mathfrak{a}_0 = \mathfrak{g}$$

such that, for each i, $\mathfrak{a}_{i+1}$ is an ideal in $\mathfrak{a}_i$ and $\dim(\mathfrak{a}_{\mathrm{i}}/\mathfrak{a}_{i+1}) = 1$.

Subalgebras and quotients of solvable (respectively nilpotent) Lie algebras are solvable (nilpotent respectively).

Fact 1.1.5. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Lie subalgebra and $\mathfrak{a}$ an ideal.*

- *If $\mathfrak{g}$ is nilpotent so are $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{a}$.*
- *If $\mathfrak{g}$ is solvable so are $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{a}$.*

- *If* $\mathfrak{g}/\mathcal{Z}(\mathfrak{g})$ *is nilpotent so is* $\mathfrak{g}$.

- *If* $\mathfrak{a}$ *and* $\mathfrak{g}/\mathfrak{a}$ *are solvable so is* $\mathfrak{g}$.

The next proposition is very important since the solvable radical of a Lie group (or definable group) will play an important role in the development of the corresponding theory.

Fact 1.1.6. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. There is a unique solvable ideal* $\mathfrak{r}$ *of* $\mathfrak{g}$ *containing all solvable ideals in* $\mathfrak{g}$*, it is called the solvable radical of* $\mathfrak{g}$.

Another interesting notion is simplicity. As the reader may know, we now have a complete characterization of finite simple groups; in the same fashion simple Lie algebras over the reals or complex numbers are completely characterized.

Definition 1.1.10. *A Lie algebra* $\mathfrak{g}$ *is said to be simple if it is nonabelian and it has no proper nonzero ideals. We say that* $\mathfrak{g}$ *is semisimple if it has no nonzero solvable ideals.*

Simple Lie algebras have a nice algebraic structure.

Definition 1.1.11. *A Lie algebra* $\mathfrak{g}$ *is said to be perfect if* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

One can easily deduce the following.

Fact 1.1.7. *Simple Lie algebras are perfect.*

The dual notion to solvability is semisimplicity as it is shown in the next proposition.

Fact 1.1.8. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra, then the quotient* $\mathfrak{g}/\mathfrak{r}$ *is semisimple (with* $\mathfrak{r}$ *being the radical of* $\mathfrak{g}$*).*

Finally there is a notion of nilpotent radical: the nilradical of $\mathfrak{g}$. First let us state that the sum of nilpotent ideals is a nilpotent ideal.

Fact 1.1.9. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be nilpotent ideals in a finite dimensional Lie algebra* $\mathfrak{g}$*. Then the direct sum* $\mathfrak{a} \oplus \mathfrak{b}$ *is a nilpotent ideal in* $\mathfrak{g}$.

Definition 1.1.12. *The maximal nilpotent ideal in* $\mathfrak{g}$ *is called the nilradical of* $\mathfrak{g}$*. It is an ideal in* $\mathfrak{r}$ *the solvable radical of* $\mathfrak{g}$.

We will need the following fact in Chapter 5.

Fact 1.1.10 ([OV94], Theorem 5.1)**.** *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra,* $\mathfrak{r}$ *its solvable radical and* $\mathfrak{n}$ *its nilradical. Then* $[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}$.

1.1.3 Linear Lie algebras

Let us first clarify what we mean by linear Lie algebra.

Definition 1.1.13. *Let V be a vector space over a field $\mathbb{K}$. We can equip the vector space $\mathfrak{gl}(V)$ of all linear endormorphisms of V with a Lie algebra structure defining the Lie bracket by*

$$[X, Y] := XY - YX$$

for each $X, Y \in End_{\mathbb{K}}(V)$.

A representation of a Lie algebra $\mathfrak{g}$ in a vector space V is a Lie algebra morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. We say that $\mathfrak{g}$ or $\rho(\mathfrak{g})$ is a linear Lie algebra if ρ is injective. Such injective representation ρ of $\mathfrak{g}$ is called faithful.

Example 1.1.1. *An important example for a finite dimensional Lie algebra $\mathfrak{g}$ is the adjoint representation already mentioned above. This is a representation of $\mathfrak{g}$ in $\mathfrak{g}$ and it is faithful if $\mathfrak{g}$ is centerless.*

Since the Lie algebra associated to a Lie group has always finite dimension, the following theorem lets us focus on linear Lie algebras.

Fact 1.1.11 (Ado's Theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Then $\mathfrak{g}$ admits a faithful representation.*

Any representation of a solvable Lie algebra in an algebraically closed field stabilizes a *flag* as shown in the following theorems.

Fact 1.1.12 (Lie's Theorem)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra, $V \neq 0$ be a finite dimensional $\mathbb{K}$-vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ in V. If $\mathbb{K}$ is algebraically closed there is a simultaneous eigenvector $v \in V$ for all $f \in \rho(\mathfrak{g})$.*

We can easily deduce from the previous result the following.

Fact 1.1.13. *Let $\mathfrak{g}$ be a Lie solvable Lie algebra, $V \neq 0$ be a finite dimensional $\mathbb{K}$-vector space and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ in V. If $\mathbb{K}$ is algebraically closed there exists a sequence of subspaces*

$$0 = V_m \subseteq \cdots \subseteq V_0 = V$$

such that each V_i is stable under $\rho(\mathfrak{g})$ and $\dim(V_{i+1}/V_i) = 1$.

Since we will be particularly interested in real Lie algebras the following definition will be useful.

Definition 1.1.14. *A solvable Lie algebra* $\mathfrak{g}$ *is said to be supersolvable if there is a sequence*

$$0 = \mathfrak{a}_n \subseteq \cdots \subseteq \mathfrak{a}_0 = \mathfrak{g}$$

in which each $\mathfrak{a}_i$ *is an ideal in* $\mathfrak{g}$ *and* $\dim(\mathfrak{a}_\mathfrak{i}/\mathfrak{a}_{i+1}) = 1$.

The following fact clarify why the notion is not interesting over the complex field.

Fact 1.1.14. *Let* $\mathfrak{g}$ *be a solvable Lie algebra over* $\boldsymbol{k}$*. Then* $\mathfrak{g}$ *is supersolvable if and only if the eigenvalues of all* ad_X *for* $X \in \mathfrak{g}$ *are in* $\boldsymbol{k}$*.*

Also, real supersolvable Lie algebras have a nice matrix representation.

Fact 1.1.15. *Let* $\mathfrak{g}$ *be a real supersolvable algebra of finite dimension. Then* $\mathfrak{g}$ *is isomorphic to a subalgebra of the upper-triangular matrices* $\mathfrak{t}_n(\mathbb{R})$ *for some* $n \in \mathbb{N}$*.*

Let us now take a look at nilpotent linear Lie algebras. Nilpotent Lie algebras behave in many ways like abelian Lie algebras.

Fact 1.1.16. *Let* $\mathfrak{g}$ *be a Lie algebra, then* $\mathfrak{g}$ *is nilpotent if and only if the Lie algebra* $ad(\mathfrak{g})$ *is nilpotent.*

This leads us to the following famous theorem due to Friedrich Engel.

Fact 1.1.17 (Engel's Theorem)**.** *Let* $V \neq 0$ *be a finite-dimensional vector space over* $\boldsymbol{k}$*, and let* $\mathfrak{g}$ *be a Lie algebra of nilpotent endomorpisms of* V*. Then*

- $\mathfrak{g}$ *is a nilpotent Lie algebra,*
- *there exists* $v \neq 0$ *in* V *with* $X(v) = 0$ *for all* $X \in \mathfrak{g}$*,*
- *in a suitable basis of* V*, all* $X \in \mathfrak{g}$ *are upper triangular with* 0*'s on the diagonal.*

Usually in the literature, Engel's Theorem is stated as the following corollary. Remember that a linear transformation φ is said to be nilpotent if $\varphi^\ell = 0$ for some $\ell \in \mathbb{N}$.

Fact 1.1.18. *Let* $\mathfrak{g}$ *be a Lie algebra such that* ad_X *is nilpotent for all* $X \in \mathfrak{g}$*. Then* $\mathfrak{g}$ *is a nilpotent Lie algebra.*

Combining this result with Lie's Theorem gives us the following useful property.

Fact 1.1.19. *Let* $\mathfrak{g}$ *be a finite dimensional solvable Lie algebra over a field of characteristic* 0*, then* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

1.1.4 Complexification and real forms

Let $\mathbf{k}$ be a field and let $\mathbf{K}$ be an extension field.

Definition 1.1.15. *If V is a vector space over $\boldsymbol{k}$ then $V \otimes_{\boldsymbol{k}} \boldsymbol{K}$ can be equipped with a vector space structure over $\boldsymbol{K}$, we write $V^{\boldsymbol{K}}$ for this new vector space. When $\boldsymbol{k} = \mathbb{R}$ and $\boldsymbol{K} = \mathbb{C}$ we call $V^{\mathbb{C}}$ the complexification of V.*

Definition 1.1.16. *If V is a vector space over $\boldsymbol{K}$ we can restrict its scalar multiplication to $\boldsymbol{k}$ and obtain a vector space over $\boldsymbol{k}$ and we write $V^{\boldsymbol{k}}$.*

Let V be a real vector space and W a complex vector space. It is worth noticing that these operations are not inverse to each other: $(V^{\mathbb{C}})^{\mathbb{R}} = V \oplus i\,V$ has twice the real dimension of V. In the right side of the equation by V we mean $V \otimes \{1\} \subseteq V \otimes \mathbb{C}$. On the other hand $(W^{\mathbb{R}})^{\mathbb{C}}$ has twice the complex dimension of W.

Definition 1.1.17. *When a complex vector space W and a real vector space V are related by $W^{\mathbb{R}} = V \oplus iV$ we say that V is a real form of W.*

Let us apply this notions to Lie algebras, we will equip those new vector spaces with a Lie algebra structure, that is with a Lie bracket.

Let $\mathfrak{g}_0$ be a Lie algebra over $\mathbf{k}$ and $\mathfrak{g} := (\mathfrak{g}_0)^{\mathbf{K}}$. Let us introduce the 4-linear map:

$$\begin{array}{rccl} F: & \mathfrak{g}_0 \times \mathbf{K} \times \mathfrak{g}_0 \times \mathbf{K} & \longrightarrow & \mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbf{k}} \mathbf{K} \\ & (X, a, Y, b) & \longmapsto & [X, Y] \otimes ab \end{array}$$

Since F defines a unique $\mathbf{k}$-linear map on $\mathfrak{g}_0 \otimes_{\mathbf{k}} \mathbf{K} \otimes_{\mathbf{k}} \mathfrak{g}_0 \otimes_{\mathbf{k}} \mathbf{K}$, by restricting it we get a $\mathbf{k}$ bilinear map:

$$[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

This map is $\mathbf{K}$-bilinear and gives us a Lie bracket on $\mathfrak{g}$.

Definition 1.1.18. *Let $\mathfrak{g}_0$ be a Lie algebra over $\boldsymbol{k}$ and $\mathfrak{g} := (\mathfrak{g}_0)^{\boldsymbol{K}}$. Then $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is a Lie algebra whose Lie bracket extends the original Lie bracket on $\mathfrak{g}_0$. If $\boldsymbol{k} = \mathbb{R}$ and $\boldsymbol{K} = \mathbb{C}$ we call $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$.*

Algebraic properties pass on well from a real algebra onto its complexification.

Fact 1.1.20. *Let $\mathfrak{g}$ be finite dimensional real Lie algebra, then $\mathfrak{g}$ is solvable if and only if its complexification $\mathfrak{g}^{\mathbb{C}}$ is solvable.*

On the other side it is easy to see that the restriction of scalar multiplication has no impact on the Lie bracket.

Definition 1.1.19. *If* $(\mathfrak{g}, [\cdot,\cdot])$ *is a Lie algebra over* $\boldsymbol{K}$ *then* $(\mathfrak{g}^{\boldsymbol{k}}, [\cdot,\cdot])$ *is a Lie algebra over* $\boldsymbol{k}$*. If* $\mathfrak{g}$ *is a complex Lie algebra and* $\mathfrak{g}_0$ *a real Lie algebra such that* $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ *as real vector spaces we say that* $\mathfrak{g}_0$ *is a real form of* $\mathfrak{g}$*.*

Definition 1.1.20. *We say that a Lie subalgebra* $\mathfrak{h}$ *of the complex Lie algebra* $\mathfrak{g}$ *is defined over* $\mathbb{R}$ *if* $\tau(\mathfrak{h}) = \mathfrak{h}$ *where* τ *is the complex conjugation.*

Again we understand well semisimplicity of a complex Lie algebra looking at its real forms.

Fact 1.1.21. *Let* $\mathfrak{g}$ *be finite dimensional complex Lie algebra and* $\mathfrak{g}_0$ *a real form of* $\mathfrak{g}$*. Then* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g}_0$ *is semisimple.*

1.1.5 Other tools

We present here a useful criterion to detect solvability and semisimplicity of Lie algebras: *the Killing form.*

Definition 1.1.21. *Let* $\mathfrak{g}$ *be a finite-dimensional* $\boldsymbol{k}$ *Lie algebra. We define the Killing form of* $\mathfrak{g}$ *as follows:*

$$B: \begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \boldsymbol{k} \\ (X,Y) & \longmapsto & Tr(ad_X \circ ad_Y) \end{array}.$$

It is a symmetric bilinear form on $\mathfrak{g}$ *and we notice that*

$$B([X,Y],Z) = -B(X,[Y,Z]) \quad \forall X,Y,Z \in \mathfrak{g}.$$

The Killing form gives us the following criterions for solvable Lie algebras and semisimple Lie algebras.

Fact 1.1.22. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra over a field* $\boldsymbol{k}$ *of characteristic* 0*. Then* $\mathfrak{g}$ *is solvable if and only if* $B(\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]) = 0$*.*

Fact 1.1.23. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra over a field* $\boldsymbol{k}$ *of characteristic* 0*. Then* $\mathfrak{g}$ *is semisimple if and only if the Killing form is non-degenerate.*

We have a nice decomposition theorem for semisimple algebras, this can be understood as the analogue of *Maschke's Theorem* for semisimple Lie algebras.

Fact 1.1.24. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra over a field of characteristic* 0*. Then* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \mathfrak{g}_m$ *with each* $\mathfrak{g}_i$ *ideals that are simple Lie algebras. This decomposition is unique and the only ideals of* $\mathfrak{g}$ *are sums of the* $\mathfrak{g}_i$*'s.*

A corollary from Definition 1.1.11 and Fact 1.1.24 is that semisimple Lie algebras are perfect.

Fact 1.1.25. *Let $\mathfrak{g}$ be a semisimple finite dimensional Lie algebra over a field of characteristic* 0*. Then $\mathfrak{g}$ is perfect.*

We finish this section with a very important decomposition theorem due to Levi. We will mostly use the group version presented in the next section but we deemed important to mention the Lie algebra version.

Fact 1.1.26 (Levi's Decomposition)**.** *Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra and $\mathfrak{r}$ its solvable radical. There is a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{r} +_{\varphi} \mathfrak{s}$ for some $\varphi : \mathfrak{s} \to Der(\mathfrak{r})$.*

1.2 Lie groups

In this section we present the basic definitions and facts about Lie groups and the tools we will need in the last chapters to study their definability. Here again we are using [Kna02] unless otherwise mentioned .

1.2.1 Lie groups and their Lie algebras

As we already mentioned above, there is, associated with a Lie group, a Lie algebra that contains the algebraic information close to the identity. This is why after giving the definitions of a Lie group we continue the exposition with the construction of its Lie algebra.

Basics of Lie groups

Let us begin with the definition of *topological* group and continue with the meat of this chapter: *Lie groups.*

Definition 1.2.1. *A topological group is a group G equipped with a topology such that the multiplication map $\mu : G \times G \to G$ and the inverse map $\iota : G \to G$ are continuous.*

Definition 1.2.2. *A (real) Lie group is a topological group G that is also a (real) smooth manifold and such that the multiplication map $\mu : G \times G \to G$ and the inverse map $\iota : G \to G$ are smooth maps.*

Example 1.2.1. *Any closed subgroup of $GL_n(\mathbb{R})$ (with the usual topology) is a Lie group with the induced topology.*

It is natural to put more than just algebraic condition on “subobjects” to have a coherent theory. With the following additional analytic condition we define Lie subgroups.

Definition 1.2.3. *Let G be a Lie group and H be a subgroup. If H is a Lie group (possibly with respect to a different topology) and the inclusion map is smooth we say that H is a Lie subgroup of G.*

And we have the useful following proposition to detect closed Lie subgroups due to J. von Neumann and E. Cartan.

Fact 1.2.1. *Let G be a Lie group and H a subgroup. If H is a submanifold of G then G is closed. Conversely if H is a closed subgroup of G, the trace topology of G makes H a Lie group.*

Now that we have a good notion of subgroup we can easily pass to the quotient.

Fact 1.2.2. *Let G be a Lie group and H a normal closed Lie subgroup. The group G/H equipped with the quotient topology is a Lie group. From now on it will always be implied that quotient are considered with the quotient topology.*

There is a particularly interesting Lie subgroup in every Lie group: its identity component.

Fact 1.2.3. *Let G be a Lie group, the connected component of the identity is a normal Lie subgroup of G. Moreover the quotient G/G^0 is discrete.*

Definition 1.2.4. *Let G be a Lie group, we say that G is connected if $G^0 = G$.*

We will mostly be interested in connected Lie groups since we have Chapters 4 to 6 in line of sight. Indeed, definable groups must have finitely many connected component so we will always assume $G = G^0$.

A last classical adaptation that we must make from group theory is the notion of morphism, here again we will ask extra analytic condition for a group morphism to be a Lie group morphism.

Definition 1.2.5. *Let G and H be Lie groups. A group morphism $\varphi : G \to H$ is said to be a Lie group morphism if it is smooth.*

It so happens that the group of automorphisms of a connected Lie group can be equipped with a “good” topology, making it a Lie group. We recall here the definition of the compact-open topology.

Definition 1.2.6. *Let X and Y be two topological spaces and $\mathcal{C}(X,Y)$ the set of all continuous maps between X and Y. Given a compact subset K of X and an open subset U of Y then let us denote $\mathcal{V}(K,U)$ the set of all $f \in \mathcal{C}(X,Y)$ such that $f(K) \subseteq U$. Then the collection of the $\mathcal{V}(K,U)$'s forms subbasis for a topology τ on $\mathcal{C}(X,Y)$ called the compact-open topology.*

Fact 1.2.4. *Let G be a connected Lie group. The group $(Aut(G), \circ)$ of Lie automorphism of G equipped with the compact-open topology is a Lie group.*

In the last three chapters we will be particularly interested in subgroups of a general linear group.

Definition 1.2.7. *Let G be a Lie group. A morphism of Lie groups $\rho : G \to GL_n(\mathbb{R})$ is called a representation of G. We say that the representation is faithful if ρ is injective. When it is the case we say that G is linear and that $\rho(G)$ is a matrix group.*

The direct product $G \times H$ of two Lie groups equipped with the product topology is also a Lie group. There is also a similar definition for semidirect product.

Definition 1.2.8. *Let K and H be Lie groups and $\varphi : K \times H \to H$ a smooth map such that for each $k \in K$, the map $\varphi_k : h \mapsto \varphi(k,h)$ is an automorphism of H (we say that K acts by automorphisms on H). Then the set $H \times K$ equipped with the product topology and group multiplication defined by*

$$(h_1, k_1) \cdot (h_2, k_2) := (h_1 \varphi(k_1, h_2), k_1 k_2)$$

is a Lie group called semidirect product of K and H and we write $H \rtimes_\varphi K$. The Lie subgroup $H \times \{1_K\}$ is closed and normal in $H \rtimes_\varphi K$.

Remark 1.2.1. *A special case is the inner semidirect product: consider a Lie group G with a normal Lie subgroup H and a Lie subgroup K. Suppose that $G = HK$ and that $H \cap K = \{1_G\}$. Then the action by conjugation of K on H let us see G as the semidirect product $H \rtimes K$. When $H \cap K$ is only finite we say that G is the almost semidirect product of H and K and we write $G = H(\rtimes)K$.*

Tangent space

We first recall the construction of the tangent space of a real manifold and actually give two equivalent constructions.

Tangent curves Let M be a real smooth manifold and $x \in M$. Pick a coordinate chart $\varphi : U \to \mathbb{R}^n$ with U open set in M containing x.

Let $\gamma_1, \gamma_2 \colon]-1, 1[\to M$ be two smooth curves such that $\gamma_1(0) = x = \gamma_2(0)$. We say that γ_1 and γ_2 are equivalent at x if the derivative at 0 of $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are equal.

This defines a relation equivalence on the set of all smooth curves γ with $\gamma(0) = x$; equivalence classes are called tangent vectors of M at x. The equivalence class of a smooth curve γ is denoted $\gamma'(0)$ and the set of all of those tangent vector is called the tangent space of M at x, denoted T_xM. This construction is independent of the choice of φ.

We can equip T_xM with more structure using the following bijection.

$$\begin{array}{rccl} d\varphi_x : & T_xM & \longrightarrow & \mathbb{R}^n \\ & \gamma'(0) & \longmapsto & \frac{d}{dt}(\varphi \circ \gamma)|_{t=0} \quad \text{with } \gamma \in \gamma'(0). \end{array}$$

Indeed, this allows us to transfer the real vector space structure from $\mathbb{R}^n$ to T_xM and the construction is again independent of the choice of φ.

Derivations Let M be a smooth real manifold. A function $f : M \to \mathbb{R}$ is said to be a smooth real valued function of M if for every chart $\varphi : U \to \mathbb{R}^n$ the map $f \circ \varphi^{-1}$ is smooth. The set of all smooth real valued functions of M is denoted $\mathcal{C}^\infty(M)$, it is a real vector space and also an associative algebra with respect to the pointwise product.

Now pick a point $x \in M$, a derivation at x is a linear map $D : \mathcal{C}^\infty(M) \to \mathbb{R}$ that satisfies the *Leibniz identity*:

$$\forall f, g \in \mathcal{C}^\infty(M) \quad D(f \cdot g) = D(f) \cdot g(x) + f(x) \cdot D(g).$$

We easily equip the set of derivation with a real vector space structure and call it the tangent space of M at x and write T_xM. The two notions coincide.

Fact 1.2.5. *Let $x \in M$ and γ a differentiable curve with $\gamma(0) = x$. Define for $f \in \mathcal{C}^\infty(M)$ the derivation $D_\gamma(f) \colon = (f \circ \gamma)'(0)$, equivalent curves yield the same derivation. Moreover the map that sends a tangent vector $\gamma'(0)$ to the derivation D_γ is a real vector space isomorphism.*

The Lie algebra of a Lie group Let G be a connected Lie group, we denote $\mathfrak{g}$ the tangent space of G at 1_G. We are going to define a Lie bracket on $\mathfrak{g}$ in order to make it a Lie algebra.

We can differentiate the multiplication and inverse maps μ and ι of G to obtain a group structure on $\mathfrak{g}$. This is actually the vectorial sum that we transferred from $\mathbb{R}^n$. This allows us to see $\mathfrak{g}$ as a Lie group.

Now for $g \in G$ consider the inner automorphism (conjugation) $\Psi_g : x \mapsto gxg^{-1}$, it is a Lie group automorphism. Since $\Psi_g(1) = 1$ we can differentiate at 1 on both sides and get a linear automorphism:

$$\begin{array}{rccl} \mathrm{Ad}_g : & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & X & \longmapsto & d_1(\Psi_g)(X). \end{array}$$

The map

$$\begin{array}{rccl} \mathrm{Ad} : & G & \longrightarrow & \mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}) \\ & g & \longmapsto & \mathrm{Ad}_g \end{array}$$

is called the adjoint representation of G in $\mathfrak{g}$. Note that the tangent space of $\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g})$ is $Der(\mathfrak{g})$. Since Ad is a Lie group morphism with $\mathrm{Ad}(1_G) = \mathrm{id}_{\mathfrak{g}}$ we can differentiate one more time and get

$$\begin{array}{rccl} \mathrm{ad} : & \mathfrak{g} & \longrightarrow & Der(\mathfrak{g}) \\ & X & \longmapsto & \mathrm{ad}_X = d_1(\mathrm{Ad})(X) \end{array}.$$

It allows us to give $\mathfrak{g}$ a Lie algebra structure:

$$\text{for } X, Y \in \mathfrak{g} \quad [X, Y] := \mathrm{ad}_X(Y).$$

Definition 1.2.9. *Let G be a Lie group. The tangent space $\mathfrak{g}$ of G at 1 equipped with the Lie algebra structure defined above is called the Lie algebra of G. We usually will write $\mathfrak{g} = Lie(G)$.*

The map ad is a Lie algebra morphism, we call it the *adjoint representation* of $\mathfrak{g}$.

The Lie algebra of a Lie group behaves well with the usual construction of (semi) direct product.

Fact 1.2.6. *Let G be a Lie group that is a direct product of two Lie groups $H_1 \times H_2$. Let $\mathfrak{g}$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ their respective Lie algebras. Then $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and h_1 and h_2 are ideals in $\mathfrak{g}$.*

Let $G = H \rtimes_\varphi K$ be a Lie semidirect product and $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ the Lie algebras of G, H and K respectively. Then $\mathfrak{g} = \mathfrak{h} +_{d\varphi_1} \mathfrak{k}$ with $d\varphi_1(k)$ defined as the differential of $\varphi(k, \cdot)$ at 1_H ($d\varphi_1(k)$ is a Lie algebra isomorphism).

We now continue with the definition of the exponential map that will allow us to pass information back from the Lie algebra to the Lie group. First let us introduce one parameter subgroups.

Fact 1.2.7. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $X \in \mathfrak{g}$ (seen as a tangent vector). There is a unique morphism of Lie group $\gamma_X : \mathbb{R} \to G$ such that $\gamma'_X(0) = X$. We call γ_X the one-parameter subgroup corresponding to X.*

We are now ready to define the exponential map.

Definition 1.2.10. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. We defined the exponential map as follows:*

$$\begin{array}{rccc} \exp : & \mathfrak{g} & \longrightarrow & G \\ & X & \longmapsto & \gamma_X(1) \end{array}$$

with γ_X the one-parameter subgroup corresponding to X.

Remark 1.2.2. *If G is a linear subgroup of $GL_n(\mathbb{R})$ then $\mathfrak{g}$ is a subspace of $\mathfrak{gl}_n(\mathbb{R})$ and the exponential map coincide with the matrix exponential:*

$$\exp(X) := \sum_{i=0}^{\infty} \frac{X^n}{n!}$$

Moreover $\mathfrak{g} := \{X \in \mathfrak{gl}_n(\mathbb{R}) \colon \exp(tX) \in G \ \forall t \in \mathbb{R}\}$.

Note that the Lie algebra of a Lie group contains information close to the identity.

Fact 1.2.8 (Lie's First Theorem)**.** *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, the exponential map $\exp : \mathfrak{g} \to G$ is smooth. Moreover there is an open neighborhood U of the origin in $\mathfrak{g}$ and an open neighborhood V of the identity in G such that $\exp|_U^V$ is a diffeomorphism.*

We will not worry about smoothness of group morphisms since we have the following.

Fact 1.2.9. *Let G and H be Lie groups and $\varphi : G \to H$ a continuous group morphism, then φ is smooth.*

A direct consequence is that given a topological group G, there is at most one Lie group structure on G.

1.2.2 Lie correspondence

All the properties that we defined for Lie algebras have a correspondent notion for Lie groups. We will start with a correspondence between ideals and normal subgroups.

Fact 1.2.10 (Lie's Third Theorem)**.** *If G is a connected Lie group and $\mathfrak{g}$ its Lie algebra. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, there is a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$. Moreover if $\mathfrak{h}$ is an ideal then H is normal in G.*

We can define nilpotent and solvable Lie groups as follow.

Definition 1.2.11. *A connected Lie group G is said to be nilpotent (respectively solvable) if it is nilpotent (respectively solvable) as an abstract group.*

And we get the nice characterization on the Lie algebra level for connected Lie groups.

Fact 1.2.11. *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Then G is nilpotent (respectively solvable) if and only if $\mathfrak{g}$ is nilpotent (respectively solvable).*

Definition 1.2.12. *If G is a connected Lie group and $\mathfrak{g}$ its Lie algebra, there is a unique connected normal subgroup N whose Lie algebra is the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. We say that N is the nilradical of G, it is nilpotent and contains every normal connected nilpotent subgroups of G.*

The same results goes for the solvable case.

Fact 1.2.12. *Let G be a connected Lie group. There is a unique largest solvable connected normal Lie subgroup R of G called the solvable radical of G. The Lie algebra of R coincides with $\mathfrak{r}$ the solvable radical of the Lie algebra of G. Moreover R is a closed Lie subgroup of G.*

We also need to define supersolvable Lie groups.

Definition 1.2.13. *A connected Lie group G is said to be supersolvable if there is a sequence of subgroups*

$$\{1\} = G_0 \leqslant \cdots \leqslant G_n = G$$

such that each G_i is normal in G and $\dim(G_{i+1}/G_i) = 1$. This is equivalent to ask the eigenvalues of $Ad(g)$ to be real for all $g \in G$.

And as expected we have the same criterion as Fact 1.2.11.

Fact 1.2.13. *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Then G is supersolvable if and only if $\mathfrak{g}$ is supersolvable.*

Example 1.2.2. *Let us consider the action φ by rotation of $\mathbb{R}$ on the plane $\mathbb{R}^2$. The Lie group $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ is connected solvable simply connected but it is not supersolvable.*

There are a lot of possible conditions that can make the exponential map of a Lie group surjective or even a diffeomorphism. We will not need such general results but the following fact will come of use in Chapter 4.

Fact 1.2.14 ([Dix57])**.** *Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ is supersolvable. Then the exponential map* $\exp$ *is surjective and if G is simply connected it is a global diffeomorphism.*

Finally let us introduce semisimple Lie groups.

Definition 1.2.14. *A connected Lie group G is said to be semisimple if it does not contain any non-trivial connected solvable normal Lie subgroup. Equivalently we can replace "solvable" by "abelian" in this definition.*

The reader is now probably used to have the corresponding notion equivalent in Lie algebras.

Fact 1.2.15. *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Then G is semisimple if and only if $\mathfrak{g}$ is semisimple.*

1.2.3 Simply connected Lie groups

We continue with simply connected Lie groups and we build the universal cover of a connected Lie group.

Let us begin with a discussion on smooth paths in the general context of a smooth connected manifold M. We will follow the discussion in [DK00].

Definition 1.2.15. *A path in M starting at $x_0 \in M$ is a continuous curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$.*

The space $P(x_0, M)$ of all paths in M that start at x_0 can be equipped with the topology of uniform convergence. Similarly the space $P(M)$ of all paths in M can be equipped with the same topology.

We say that two paths γ_0 and γ_1 in $P(M)$ such that $\gamma_0(0) = \gamma_1(0)$ are homotopically equivalent (and we write $\gamma_0 \sim \gamma_1$) if there is a homotopy from γ_0 to γ_1 leaving the end points fixed; that is a continuous function:

$$F: \begin{array}{ccc} [0,1] & \longrightarrow & P(M) \\ s & \longmapsto & \gamma_s \end{array}$$

such that $F(0) = \gamma_0$, $F(1) = \gamma_1$ and $s \mapsto \gamma_s(1)$ is constant on $[0,1]$.

Fact 1.2.16. *The relation $\sim$ defined above is an equivalence relation, if $\gamma \in P(M)$ we write $[\gamma]$ for its equivalence class. We define $\widetilde{M}$ as the collection of all the equivalence classes of path in M and we get a surjective map $\widetilde{\pi} : [\gamma] \mapsto \gamma(1)$ from $\widetilde{M}$ to M.*

Now let us take a look at the concatenation of two paths in M. If $\gamma_0 \in P(x_0, M)$ and $\gamma_1 \in P(\gamma_0(1), M)$ then we define

$$(\gamma_0 \dashv\vdash \gamma_1)(t) = \begin{cases} \gamma_0(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \gamma_1(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

If we consider loops at x_0 we can extend the concatenation to $\widetilde{M}$.

Fact 1.2.17. *Let $[\gamma_0], [\gamma_1] \in P(M)/\sim$ then if γ_0 and γ_1 are loops at x_0 then $[\gamma_0] \dashv\vdash [\gamma_1] := [\gamma_0 \dashv\vdash \gamma_1]$ is well defined. Hence $\dashv\vdash$ defines a group structurre on the space of classes of loops at x_0, this group is called the fundamental group $\pi_1(M, x_0)$ of M with base point x_0.*

Remark 1.2.3. *Notice that if x_1 is an other point in M we can pick any $\sigma \in P(x_0, M)$ such that $\sigma(1) = x_1$ and we get an isomorphim of groups:*

$$\begin{array}{rccc} \Psi: & \pi_1(M, x_0) & \longrightarrow & \pi_1(M, x_1) \\ & [\gamma] & \longmapsto & [\sigma] \dashv\vdash [\gamma] \dashv\vdash [\sigma]^{-1} \end{array}$$

Definition 1.2.16. *The space M is said simply connected if its fundamental group is trivial at any point (connected real manifolds are pathwise connected hence we will always find a σ as in Remark 1.2.3). When provided with the quotient topology, $\widetilde{M}$ is a simply connected real manifold.*

Since Lie groups are real manifolds the tools mentioned above apply; the careful reader will notice though that we are short of any smoothness so far. Fortunately we have the following.

Fact 1.2.18 ([DK00], Theorem 1.31.1)**.** *Let γ and σ be paths of class $\mathcal{C}^k$ in $P(M)$ such that $\gamma \sim \sigma$ then there is an homotopy of class $\mathcal{C}^k$ from γ to σ.*

Now we take a closer look to the special case of a connected real Lie group G. Let us recall first a few definitions on covering spaces.

Definition 1.2.17. *A Lie group covering of G is a Lie group H together with a Lie group morphism $p : H \to G$ such that for every $g \in G$ there is an open neighborhood U of g and $p^{-1}(U)$ is a union of open sets in H, each of which is mapped homeomorphically by p to U. If p has finite kernel we say the H is a finite covering of G.*

We have a notion of universal cover closely related to the previous constructions.

Definition 1.2.18. *If a covering group $\pi : H \to G$ is simply connected we say that it is a universal cover of G. The terminology universal comes from the fact that if $p_K : K \to G$ is a covering group of G then there is a covering morphism $p : H \to K$ of K such that $\pi = p_K \circ p$.*

Definition 1.2.19. *We can equip $P(1, G)$ with a group structure with the pointwise multiplication, that is for $t \in [0, 1]$ and γ_0, γ_1 in $P(1, G)$ we define $(\gamma_0 \cdot \gamma_1)(t) = \gamma_0(t) \cdot \gamma_1(t)$. Consider the closed normal subgroup $\Lambda(G)^\circ = \{\gamma \in P(1, G) \colon \gamma \sim 1\}$ where 1 in this definition is the constant path equal to 1_G.*

This will let us equip $\widetilde{G}$ with a Lie group structure.

Fact 1.2.19 ([DK00], Theorem 1.31.2)**.** *Let G be a connected Lie group, then $\widetilde{G} = P(1, G)/\Lambda(G)^\circ$ making $\widetilde{G}$ a Lie group. The morphism $\widetilde{\pi} : \widetilde{G} \to G$ is a Lie group covering, $\ker(\widetilde{\pi}) = \pi_1(1, G)$ and $\widetilde{G}$ is the universal cover of G. On $\ker(\widetilde{\pi})$ the group structure coincides with the one defined by concatenation and $\pi_1(1, G)$ is abelian.*

Remark 1.2.4. *In light of Fact 1.2.18, the previous theorem applies if we ask the paths to be smooth instead of just continuous.*

As we said in the beginning of this section simply connected Lie groups are entirely determined by their Lie algebra. Using the exponential map we can patch together the pieces of a Lie algebra morphism and lift it to their corresponding simply connected Lie groups.

Fact 1.2.20. *Let G and H be connected Lie groups with G simply connected and $\varphi : \mathfrak{g} \to \mathfrak{h}$ a morphism between their Lie algebras. There is a Lie group morphism $\Phi : G \to H$ such that $d\Phi_1 = \varphi$.*

Moreover any Lie algebra has a corresponding simply connected Lie group.

Fact 1.2.21. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. There is a simply connected Lie group G whose Lie algebra is precisely $\mathfrak{g}$.*

Connected Lie subgroups are always closed subgroups in simply connected Lie groups.

Fact 1.2.22. *Let G be a simply connected Lie group and H a connected Lie subgroup. Then H is closed and simply connected.*

We also get the converse of Fact 1.2.6 for simply connected Lie groups.

Fact 1.2.23. *Let H_1 and H_2 be connected Lie subgroups of the simply connected Lie group G. Let $\mathfrak{g}$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ their respective Lie algebras and suppose that $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Then $G = H_1 \times H_2$ as a Lie group.*

Let H and K be simply connected Lie groups with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$. If $\pi : \mathfrak{k} \to Der(\mathfrak{h})$ is a Lie algebra morphism then there exists a unique smooth action φ of K on H by automorphisms such that $d\varphi = \pi$. Moreover $G = H \rtimes_\varphi K$ is a simply connected Lie group whose Lie algebra is $\mathfrak{g} = \mathfrak{h} +_\varphi \mathfrak{k}$.

We also get a "nice" derived subgroup for simply connected Lie groups since it so happens to be closed.

Fact 1.2.24. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let H_1 and H_2 be connected Lie subgroups of G with Lie algebra $\mathfrak{h}_1$ and $\mathfrak{h}_2$ respectively. The subgroup $[H_1, H_2]$ is a connected Lie subgroup of G with Lie algebra $[\mathfrak{h}_1, \mathfrak{h}_2]$. If G is simply connected then H_1, H_2 and $[H_1, H_2]$ are closed in G. In particular the derived subgroup G' is a Lie subgroup of G whose Lie algebra is $[\mathfrak{g}.\mathfrak{g}]$ and it is closed when G is simply connected.*

1.2.4 Group representations

In this subsection we present usual material on group representations. Otherwise mentioned, all the basic results can be found in [FH04].

Definition 1.2.20. *A representation of a group G is a morphism $\rho : G \to GL(V)$ where V is a vector space over a field $\boldsymbol{k}$. We will usually require that V has finite dimension.*

An alternative way of seeing representations is via G-modules, we will use both point of view indifferently.

Definition 1.2.21. *Let G be a group and V a $\boldsymbol{k}$-vector space (usually finite dimensional). We say that (V, Φ) is a G-module if G acts linearly on V, that is an action $\Phi : G \times V \to V$ such that*

$$\forall v_1, v_2 \in V,\ \forall \lambda \in \boldsymbol{k},\ \forall g \in G \quad \Phi(g, v_1 + \lambda v_2) = \Phi(g, v_1) + \lambda \Phi(g, v_2).$$

Example 1.2.3. *Any matrix group $G \leq GL(V)$ allows us to see V as a G-module. This is the main use we will have of G-modules.*

Now let us give a few basic definitions and facts about G-modules.

Definition 1.2.22. *Let G be a group and (V, Φ), (W, Ψ) two G-modules. A linear mapping $f : V \to W$ is a G-module morphism (or sometimes simply G-morphism) if*

$$\forall g \in G,\ \forall v \in V \quad f(\Phi(g, v)) = \Psi(g, f(v)).$$

Definition 1.2.23. *A G-module* (V, Φ) *is said to be simple if it has no vector subspace that is G-stable. That is, there is no vector subspace* W *such that* $\Phi(g, W) \subseteq W$ *for all* $g \in G$*. In the language of representations we say that* $\rho : G \to GL(V)$ *is irreducible.*

A semisimple G-module is a direct sum of simple G-modules.

This let us state the following.

Fact 1.2.25 (Schur's Lemma)**.** *Let* V *and* W *be irreducible finite dimensional G-modules and* $\varphi : V \to W$ *a G-morphism. Then*

- *either* φ *is an isomorphism or* $\varphi = 0$*,*
- *if* $\boldsymbol{k}$ *is algebraically closed, there is a* $\lambda \in \boldsymbol{k}$ *such that* $\varphi = \lambda \cdot Id$*.*

Definition 1.2.24. *A decomposition series for a G-module* V *is a finite series of G-stable subspaces of* V *such that:*

$$\{0\} = V_0 \leq V_1 \leq \cdots \leq V_n = V,$$

with each quotient V_i/V_{i+1} *being simple.*

It is a theorem of Jordan and Hölder that two composition series must be equivalent.

Fact 1.2.26 (Jordan-Hölder's Theorem)**.** *If the G-module* V *has two composition series, they must have the same length and their factors ar G-isomorphics.*

We will be particularly interested in representations of Lie groups. Remember that in Definition 1.2.7 we defined a representation of Lie group, it is just a continuous representation with $GL(V)$ equipped with the usual topology.

Remark 1.2.5. *By Fact 1.2.9 it is the same as asking the representation to be smooth.*

Example 1.2.4. *The universal cover* $\widetilde{SL_2(\mathbb{R})}$ *of* $SL_2(\mathbb{R})$ *is not linear. This can be proved studying representation of* $\mathfrak{sl}_2(\mathbb{R})$ *and realizing that any representation (not necessarily faithful) of* $\widetilde{SL_2(\mathbb{R})}$ *must factor through* $SL_2(\mathbb{R})$*. An other proof require to see that this group has infinite discrete center and then use the following fact (see [Hoc65, Chapter XVIII, Proposition 4.1]).*

Fact 1.2.27. *Let* S *be a semisimple linear Lie group, then the center* $\mathcal{Z}(S)$ *is finite.*

Linearity of solvable Lie groups is well understood since we have the following decomposition.

Fact 1.2.28 ([OV94], Theorem 7.1)**.** *Let R be a connected solvable Lie group. Then the following are equivalent:*

- *R is linear,*
- *$[R, R]$ is simply connected,*
- *R can be decomposed as a semidirect product $T \rtimes K$ where T is simply connected and K is a torus (maximal compact group).*

Now let us state an important result about representations of compact Lie groups.

Fact 1.2.29 (Maschke's theorem)**.** *Let $\rho : G \to GL(V)$ be a continuous representation of a connected compact Lie group in a real vector space. If W is an irreducible G-stable subspace of V, it has an invariant G-stable complement. Hence any finite dimensional G-module is semisimple.*

Remark 1.2.6. *This is also true for finite group and the proof is the same since both finite and compact group have left invariant measure (counting measure and Haar measure).*

Finally we build representations of a connected Lie groups by considering the action on its space of *representative functions* defined below. This is a construction due to G. Hochschild and G. D. Mostow (see [Hoc65, Chapter XVIII] and [HM57]) and it will be applied several times in Chapter 4 and Chapter 5.

Let us fix a connected Lie group G and a representation $\rho : G \to GL(V)$ of G in a finite-dimensional vector space V. Let $(0) = V_k \leq \cdots \leq V_0 = V$ be a composition series for the G-module V (which exists because V is of finite dimension), then the direct sum of the V_i/V_{i+1} is a semi-simple G-module V' (two such G-modules differ by a G-isomorphism by *Jordan-Hölder's Theorem*).

We denote by ρ' the representation associated to ρ over V' and we say that ρ is *unipotent* if ρ' is trivial (which is equivalent to $N = \{\rho(g) - \mathrm{Id}_V : g \in G\}$ being nilpotent).

We will denote $\mathcal{C}^0(G)$ the set of continuous functions $f : G \to \mathbb{R}$. Notice that G acts on $\mathcal{C}^0(G)$ from the left as follows:

$$\forall g \in G, f \in \mathcal{C}^0(G) \qquad (g \cdot f) : x \mapsto f(xg).$$

On the other hand we have a right action of G on $\mathcal{C}^0(G)$ defined by

$$\forall g \in G, f \in \mathcal{C}^0(G) \qquad (f \cdot g) : x \mapsto f(gx).$$

We will consider two objects that are closely related. First, the space of representative functions of G defined as:

$$\mathcal{R}(G) = \{f \in \mathcal{C}^0(G) : \dim\Big(\mathrm{span}(\{g \cdot f\}_{g\in G})\Big) < \omega\}.$$

Remember that the dual space $V^\star$ of a real vector space V is defined as follows

$$V^\star = \{\varphi : V \to \mathbb{R} \mid \varphi \text{ is } \mathbb{R}\text{-linear}\}.$$

When the group G is linear we can fix a representation ρ of G and define a second space of representative functions **associated to** ρ as the space

$$S(\rho) = \{\varphi \circ \rho : \varphi \in End(V)^\star\}.$$

Claim 1. *$S(\rho) \subseteq \mathcal{R}(G)$, more particularly $S(\rho)$ is G-stable.*

Proof. Let us take $f = \varphi \circ \rho \in S(\rho)$ and $g \in G$. Then for any $x \in G$, $(g \cdot f)(x) = f(xg) = \varphi(\rho(x)\rho(g))$. Now let $\varphi_g : u \mapsto \varphi(u\,\rho(g))$, this is an element of $End(V)^\star$ and $g \cdot f = \varphi_g \circ \rho$. □

In this same context, we will now build a finite dimensional vector space W on which G acts linearly and such that W contains V and $S(\rho)$ as stable G-subspaces.

First consider the action of G on $W := S(\rho) \otimes_{\mathbb{R}} V$ defined by $g \cdot (f \otimes v) = (g \cdot f) \otimes v$. Notice that if $\dim(V) = n$ this representation is G-isomorphic to the action of G on $S(\rho)^n$.

We will now build a G-stable subspace $\widetilde{V}$ of W that is G-isomorphic to V. For all $x \in G$ define $\delta_x \in \mathrm{End}(S(\rho) \otimes V)$ as $\delta_x(f \otimes v) = (f \cdot x) \otimes v - f \otimes (x \cdot v)$ and let $\widetilde{V} := \bigcap_{x\in G} \mathrm{Ker}(\delta_x)$.

Claim 2. *The subspace $\widetilde{V}$ of W is G-stable. Moreover, let us define the linear map ε from W onto V by $\varepsilon(f \otimes v) = f(1)\,v$. Then the restriction σ of ε to $\widetilde{V}$ defines a G-isomorphism with V.*

Proof. A quick computation shows that $\widetilde{V}$ is G-stable: let $f \otimes v \in \widetilde{V}$ and $g \in G$, then for all $x \in G$

$$((g \cdot f) \cdot x) \otimes v = g \cdot ((f \cdot x) \otimes v) = g \cdot (f \otimes (x \cdot v)) = (g \cdot f) \otimes (x \cdot v).$$

Notice that $(x \cdot f)(1) = f(x) = (f \cdot x)(1)$, hence σ is a G-morphism.

Let us define for any $\lambda \in V^\star$ and $v \in V$ the function $\frac{\lambda}{v}$ on G given by $\frac{\lambda}{v}(x) = \lambda(x \cdot v)$ with $x \in G$. This can be seen as an element of $S(\rho)$: if $\lambda_v \in \mathrm{End}(V)^\star$ is the linear functional that sends φ to $\lambda(\varphi(v))$ then $\frac{\lambda}{v} = \lambda_v \circ \rho$.

Now let $\{v_1, \ldots, v_n\}$ be a basis for V and $\{v_1^\star, \ldots, v_n^\star\}$ its dual basis. We define the linear map τ on V as follows:

$$\begin{array}{rccl} \tau: & V & \to & W \\ & v & \mapsto & \sum_{i=1}^n \frac{v_i^\star}{v} \otimes v_i. \end{array}$$

Another quick computation gives us that $\tau(V) \subseteq \widetilde{V}$, $\tau \circ \sigma = \mathrm{Id}_{\widetilde{V}}$ and $\sigma \circ \tau = \mathrm{Id}_V$. □

We will use Claim 2 to see any representation ρ of a group G as a subrepresentation of the representation of G on finitely many copies of $S(\rho)$. In what follows we will use the word *extension* in this sense although it is not, strictly speaking, an extension. Notice that the construction above only deals with finite dimensional vector spaces so if the initial representation ρ is a definable action so is the action on W and its restriction to $\widetilde{V}$.

Remark 1.2.7. *We will also use the word extension in the following scenario. If $H \leq G$ and ρ is a representation of H in V, then a linear action τ of G on V that extends ρ, that is $\tau \restriction_H = \rho$, is also called an extension of ρ.*

Remark 1.2.8. *The construction presented above only uses the vector space struture of V, hence if G is definable and acts definably on V the extension will be in turn definable.*

1.2.5 Other tools

We finish the general discussion on Lie groups with a series of well known theorems that will be particularly useful in the last chapters.

First let us state the previously mentioned Levi decomposition lifted from the Lie algebra version (Fact 1.1.26).

Fact 1.2.30. *[Levi decomposition, [Lev97], Theorem 1] Let G be a connected Lie group and R its solvable radical. There is a unique (up to conjugacy) maximal connected and semisimple subgroup S of G such that $G = RS$ and $dim(R \cap S) = 0$. Any such S is called a* Levi subgroup of G *and they are all conjugate in G. If the center $\mathcal{Z}(S)$ of S is finite (which is the case whenever G is linear) or if G is simply connected then G is an almost semidirect product of R and S: $G = R(\rtimes)S$.*

Since we will be particularly interested in solvable Lie groups we give here a characterization of torsion-free solvable groups.

Fact 1.2.31 ([OV94], Chapter 2 Theorem 3.4). *Let G be a connected solvable Lie group of dimension n. The following are equivalent:*

- *G is torsion-free,*
- *G is simply connected,*
- *G is diffeomorphic to $\mathbb{R}^n$.*

Also remember that connected compact solvable Lie groups are abelian.

Fact 1.2.32 ([OV94], Chapter 2 Corollary 3). *Let G be a connected compact solvable Lie group. Then G is Lie isomorphic to $\mathbb{T}^k$ for some $k \in \mathbb{N}$ where $\mathbb{T}$ is the circle group.*

Finally we give the compact version of Fact 1.2.24.

Fact 1.2.33. *Let G be a compact connected Lie group and H_1, H_2 normal closed connected Lie subgroups of G. Then $[H_1, H_2]$ is a semisimple compact subgroup of G.*

1.2.6 Lie groups and linear algebraic groups

The discussion here will have as a purpose to establish the connection between linear Lie groups and algebraic groups. We will only be interested in algebraic groups over $\mathbb{C}$ and $\mathbb{R}$.

Definition 1.2.25. *We say that a group G is a complex linear algebraic group if it is a Zariski closed subset of $GL_n(\mathbb{C})$, that is if there exist polynomials $P_\alpha \in \mathbb{C}[X_{1,1}, X_{1,2}, \ldots, X_{n,n}]$ with $\alpha \in J$ such that*

$$G = \{g = (g_{i,j}) \in GL_n(\mathbb{C}) \colon P_\alpha(g_{i,j}) = 0 \ \forall \alpha \in J\}$$

In the same fashion we define real linear algebraic groups replacing the occurrences of $\mathbb{C}$ by $\mathbb{R}$.

Remark 1.2.9. *The polynomial requirement of G as a set is enough since multiplication is given by matrix multiplication.*

Also, real and complex linear algebraic groups are Lie groups when seen with the usual topology on the general linear group they are living in. In the complex case G is connected as an algebraic group if and only if it is connected as a Lie group.

Definition 1.2.26. *We say that a complex linear algebraic group G is defined over $\mathbb{R}$ if the polynomials of the previous definition are in $\mathbb{R}[X_{1,1}, X_{1,2}, \ldots, X_{n,n}]$.*

Fact 1.2.34. *If G is a complex linear algebraic group defined over $\mathbb{R}$ we can consider the real points satisfying the polynomials defining G. It is a real linear algebraic group with finitely many connected components as a Lie group.*

Remark 1.2.10. *Since any algebraic group is in turn a Lie group, there is a Lie algebra associated to it. Moreover if G is a complex algebraic group defined over $\mathbb{R}$ then $\mathfrak{g}$ its Lie algebra is defined over $\mathbb{R}$.*

Although any algebraic group can be seen as a Lie group, the contrary is not true; that encourages us to define algebraic Lie algebras.

Definition 1.2.27. *Let G be a linear algebraic group. A subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G is said to be an algebraic subalgebra of $\mathfrak{g}$ if it is the Lie algebra of some linear algebraic subgroup of G. A linear subalgebra $\mathfrak{h} \leq \mathfrak{gl}_n(K)$ is said to be algebraic if it is an algebraic subalgebra of $\mathfrak{gl}_n(K)$.*

Fact 1.2.35. *If $\mathfrak{h}$ is a subalgebra of a Lie algebra $\mathfrak{g}$, there is a smallest algebraic subalgebra $\mathfrak{h}^a$ of $\mathfrak{g}$ containing $\mathfrak{h}$. This Lie algebra is called the algebraic closure of $\mathfrak{h}$.*

We have a serie of nice properties on the algebraic closure of a Lie subalgebra.

Fact 1.2.36. *Let G be a linear algebraic group and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} = Lie(G)$, then*

- $[\mathfrak{h}^a, \mathfrak{h}^a] = [\mathfrak{h}, \mathfrak{h}]$ *and it is algebraic,*
- *if $\mathfrak{h}$ is an ideal then $\mathfrak{h}^a$ is an ideal,*
- *if $\mathfrak{h}$ is abelian then $\mathfrak{h}^a$ is abelian,*
- *if $\mathfrak{h}$ is solvable then $\mathfrak{h}^a$ is solvable.*

For complex Lie algebra we have an even better behavior of the commutator:

Fact 1.2.37. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ the Lie algebra of an algebraic group G, then $[\mathfrak{h}, \mathfrak{h}]$ is algebraic.*

The following is a classical result of C. Chevalley a complete proof can be found in [AM07] (Corollary 2.6.6).

Fact 1.2.38. *Let K be a connected compact subgroup of $GL_n(\mathbb{R})$. Then K is an algebraic subgroup of $GL_n(\mathbb{R})$.*

Chapter 2

o-minimal theories

We set here the logical context we will be working in and give the first relevant results to understand o-minimal groups.

First, $\mathcal{M}$ will always be a dense linearly ordered structure *i.e.* $(\mathcal{M}, <, ...)$ where $<$ defines a dense linear order and I will denote an open interval of $\mathcal{M}$. Unless otherwise mentioned, we will always use the language of ordered rings $\mathcal{L} = (0, 1, +, -, \cdot, <)$ and expansions of it.

Definition 2.0.1. *A dense linearly ordered structure is said o-minimal if its definable sets in one variable are finite unions of points and intervals.*

A theory is said o-minimal if one (equivalently all, see [KPS86]) of its models are o-minimal.

Example 2.0.1. *The easiest examples are dense linear orders. Let us consider the smaller language $\mathcal{L}_< = \{<\}$. We will denote this theory DLO, it can be axiomatized as follow.*

- $\forall x \, \neg(x < x)$
- $\forall x \forall y \forall z \, ((x < y \land y < z) \to x < z)$
- $\forall x \forall y \, (x < y \lor x = y \lor y < x)$
- $\forall x \forall y \, (x < y \to \exists z (x < z \land z < y))$

Example 2.0.2. *One of the theories we will be interested in is the theory of real closed fields. It is simply the complete theory of the usual real field, $RCF = Th(\mathbb{R}, +, \cdot, <)$. A full section will be dedicated to its study.*

We continue with the definition of interpretability of a structure inside another structure and bi-interpretability, from [PPS00b].

Definition 2.0.2. *We say that a structure $\mathcal{M}$ is interpretable in a structure $\mathcal{N}$ if $\mathcal{M}$ is isomorphic to a structure whose universe is a quotient of an $\mathcal{N}$-definable set by an $\mathcal{N}$-definable equivalence relation.*

Now let us take a look at the case where the two structures are interpretable in each other.

Definition 2.0.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be structure interpretable in each other via the isomorphisms f and g respectively. If the maps $f \circ g$ and $g \circ f$ are definable in $\mathcal{M}$ and $\mathcal{N}$ respectively, we say that $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.*

2.1 First definitions and theorems

We present here basic notions and theorems for general *o*-minimal structures. For proof references we would like to direct the reader to [Dri98].

Having an order in our structures allows us to speak about topology. The topology generated by all the open intervals is the *order topology.* We always consider the product topology on powers of $\mathcal{M}$.

First let us present some notions that are usual in topology and adapt well to the *o*-minimal setting. We begin with the closure.

Fact 2.1.1. *If $X \subseteq M^n$ is definable then the topological closure $\mathcal{C}l(X)$ of X is a definable set.*

We also have a good notion of connectedness.

Definition 2.1.1. *A definable set $X \subseteq \mathcal{M}^n$ is called definably connected if there are no definable open disjoint sets $U_1, U_2 \subseteq \mathcal{M}^n$ both intersecting X non trivially such that $X \subseteq U_1 \cap U_2$*

In a dense lineary ordered structure the notion of limit can be expressed at first order using the usual delta/epsilon definition (we let the reader imagine the multidimensional case) :

Definition 2.1.2. *If $f : I \longrightarrow \mathcal{M}$ is a definable function with $a \in I$ and $l \in \mathcal{M}$ we say that the limit of f at a is l and we write $\lim_{x \to a} f(x) = l$ if*

$$\mathcal{M} \models \forall \epsilon_1, \epsilon_2 \, (\epsilon_1 < l < \epsilon_2) \to [\exists \eta_1, \eta_2 : \forall x \, (\eta_1 < x < \eta_2) \to (\epsilon_1 < f(x) < \epsilon_2)]$$

With limits in hands we can define continuity in the usual sense.

Definition 2.1.3. *Let $f : I \longrightarrow \mathcal{M}$ be a definable function, we say that f is continuous at $a \in I$ if $\lim_{x \to a} f(x) = f(a)$. This can be expressed in first order.*

We finish with two important theorems, one transferred from real analysis, the second one gives us a "nice" decomposition for definable functions.

Fact 2.1.2 (Definable Intermediate Value Theorem). *Let $f, g : I \longrightarrow \mathcal{M}$ be continuous definable functions which are different at each point of I. Then either $f < g$ or $g < f$ on I.*

Fact 2.1.3 (Definable Monocity Theorem). *Let $f : I =]a, b[\longrightarrow \mathcal{M}$ be a definable function with possibly $a, b = \pm\infty$. Then there are $a_0 = a < a_1 < ... < a_n = b$ such that f is either constant or strictly monotone and continuous on each $]a_i, a_{i+1}[$.*

2.2 Model theoretical tools in *o*-minimality

2.2.1 Cell decomposition

The most important tool to understand definable sets in *o*-minimal structures is cell decomposition.

Definition 2.2.1 (Cells). *We define by induction k-cells in M^n:*

- *Every point in M is a 0-cell and every open interval in M is a 1-cell.*
- *A k-cell in M^{n+1} is a set X such that its projection C on M^n is a cell and satisfies either:*
    - *i) C is a k-cell and X is the graph of a definable function $f : C \to M$ or,*
    - *ii) C is a $k-1$-cell and there are definable functions $f, g : C \to M$ with $f(x) < g(x)$ for all $x \in C$ and X lies between the graphs of f and g. That is $X = \Big\{(x, y) : \Big(x \in C\Big) \wedge \Big(f(x) < y < g(x)\Big)\Big\}$.*

Remark 2.2.1. *The notion of a cell is defined with respect to a particular ordering of the coordinate axes and is not invariant under permutation of the coordinates.*

Definition 2.2.2 (Cell decomposition). *A cell decomposition of a definable set $X \subseteq M^n$ is a partition of X into finitely many pairwise disjoint cells such that for any two cells C_1 and C_2 their projection $\pi(C_1)$ and $\pi(C_2)$ under the projection map $\pi : M^n \to M^{n-1}$ on the first $n-1$ coordinates are equal or disjoint.*

A cell decomposition of X is compatible with $Y \subseteq X$ if every cell is contained in Y or disjoint from it.

Fact 2.2.1. *If $X \subseteq M^n$ is definable then there exist a cell decomposition on M^n compatible with X.*

Remark 2.2.2. *Cells are definably connected hence Fact 2.2.1 shows that definable sets have finitely many connected components.*

2.2.2 Dimension

Without going into the details (the reader might want to look at [TZ12] for a complete and general exposition on pregeometries) the following lemma makes *o*-minimal structures geometrical.

Fact 2.2.2 (Exchange Principle)**.** *If $A \subseteq M, b, c \in M$ and $c \in acl(Ab) \setminus acl(A)$ then $b \in acl(Ac)$*

Definition 2.2.3. *If $A, B \subseteq M$, the dimension of A over B, which will be denoted $\dim(A/B)$ is the cardinality of a maximal subset of B acl-independent over A. This does not depend on the chosen subset.*

Definition 2.2.4. *If $X \subseteq M^n$ is definable over A then a point $a \in X$ is said generic in X over A if $\dim(a/A) = \dim(X)$.*

Fact 2.2.3. *We give here basics properties of the o-minimal dimension:*

(i) *If $f : A \to B$ is a definable surjection then $\dim(B) < \dim(A)$. In particular dimension is preserved under definable bijection.*

(ii) *(Dimension formula): For every $\bar{a}, \bar{b}$ tuples in M and $A \subseteq M$ we have:*

$$\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A).$$

(iii) *A k-cell has dimension k.*

(iv) *If $X \subseteq M^n$ if a definable set and $X = \bigcup_i C_i$ is a cell decomposition then $\dim(X) = \max\Big\{ \dim(C_i)\Big\}$.*

(v) *If $X \subseteq M^n$ if a definable set then $\dim(\mathcal{C}l(X) \setminus X) < \dim(X)$.*

(vi) *If $X \subseteq M^n$ if a definable set then $\dim(X) \geq k$ if and only if there is a definable equivalence relation on X with infinitely many classes of dimension $\geq k$.*

The following definition will be of use when we enunciate *Pillay's Group Chunk Theorem* in Chapter 3.

Definition 2.2.5. *Let* $X \subseteq Y \subseteq \mathcal{M}^k$ *be definable sets. We say that* Y *is large in* X *if* $\dim(X - Y) < \dim(X)$.

An easy consequence is the following fact.

Fact 2.2.4. *Let* $X \subseteq Y$ *be definable sets. Then* Y *is large in* X *if and only if for every* A *over which* X *and* Y *are defined, every generic point of* X *over* A *is in* Y.

We finish with a useful tool that we will put to use in the study of torsion in definable groups. Again proofs can be found in [Dri98].

Definition 2.2.6. *If* X *is a definable set and* $\mathcal{C}$ *is a cell decomposition for* X *then the Euler characteristic of* X *is defined as:*

$$E_{\mathcal{C}}(X) := \Sigma_{C \in \mathcal{C}}(-1)^{\dim(C)}.$$

Euler characteristic does not depend on the choice of the decomposition.

Fact 2.2.5. *Let* X *be a definable set,* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *cell decomposition for* X*. Then* $E_{\mathcal{C}_1}(X) = E_{\mathcal{C}_2}(X)$ *and we will simply write* $E(X)$.

Moreover we also have the following.

Fact 2.2.6. *If* $f : X \to \mathcal{M}^k$ *is an injective definable map, then* $E(X) = E(f(X))$.

2.3 Real closed fields

2.3.1 The algebra and model theory of real closed fields

We give here some algebraic and model theoretic results on real close fields following [TZ12] and [Mar02]. Let us fix a field $(F, +, \cdot)$ and we denote by $\Sigma\square$ the set of all sums of squares in F.

Definition 2.3.1. *A linear ordering* $<$ *on* F *is said compatible with* F *if for all* $x, y, z \in F$

- $x < y \to x + z < y + z$,
- $(x < y \wedge 0 < z) \to x \cdot z < y \cdot z$.

We say that $(F, +, \cdot, <)$ *is an ordered field.*

We define the positive cone of a field, closely related to the notion of compatible ordering.

Definition 2.3.2. *A positive cone P of F is a subset of F such that:*

- $P + P \subseteq P$,
- $P \cdot P \subseteq P$,
- $P \cup (-P) = F$,
- $P \cap (-P) = \{0\}$.

Fact 2.3.1. *F admits a compatible linear ordering if and only if F contains a positive cone.*

We now turn to formally real fields and their equivalent definitions.

Definition 2.3.3. *We say that F is formally real if $-1 \notin \Sigma\Box$.*

Fact 2.3.2. *Let F be a field, the following are equivalent:*

- *F is formally real,*
- *there is a linear ordering $<$ on F making F an ordered field,*
- 0 *is not a sum of squared in F.*

With this definition in mind we can define the object we longed to define: real closed fields.

Definition 2.3.4. *A field F is real closed if it is formally real but has no proper formally real algebraic extension field.*

Fact 2.3.3 (Artin-Schreier)**.** *Let F be a field, the following are equivalent:*

- *F is real closed,*
- $\Sigma\Box$ *is a positive cone and every polynomial of odd degree has a root in F,*
- *$F(\sqrt{-1})$ is algebraically closed and a proper extension of F.*

We can see from the second equivalent condition that the theory of real closed fields can be axiomatized in first order logic in the language of rings, we write RCF for this theory.

Definition 2.3.5. *Let $(F, <_F)$ be an ordered field, a real closure of F is a real closed field $(K, <_K)$ which is an algebraic extension of F with $<_K$ extending $<_F$.*

Fact 2.3.4. *Any ordered field F has a real closure which is unique up to isomorphism of ordered fields.*

Let us recall the definition of quantifier elimination.

Definition 2.3.6. *A theory T in a language $\mathcal{L}$ is said to have quantifier elimination if for every $\mathcal{L}$-formula φ there is a quantifier free formula ψ such that $T \vdash (\varphi \leftrightarrow \psi)$.*

Having real closures let us do a back and forth argument and prove elimination of quantifiers.

Fact 2.3.5. *In the language of rings RCF does not have elimination of quantifiers but RCF has elimination of quantifiers in $\mathcal{L} = \{+, -, 0, \cdot, 1, <\}$.*

2.3.2 Analysis in real closed fields

When working inside a real closed field, differentiability can be written in first order. Let us fix a real closed field $(\mathcal{R}, +, \cdot, <)$ and I an interval in $\mathcal{R}$.

Definition 2.3.7. *If $f : I \longrightarrow \mathcal{R}$ is a definable function and $a \in I, b \in \mathcal{R}$. We say that f is $\mathcal{R}$-differentiable at a and write $f'(a) = b$ if*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = b \in \mathcal{R}.$$

If this limit exists for all $a \in I$ we say that f is $\mathcal{R}$-differentiable on I.

And we have naturally the notion of function of class $\mathcal{C}^k$ with $k \in \mathbb{N}$.

Definition 2.3.8. *We say that a definable function is of class $\mathcal{C}^k$ on $]a, b[$ if its n-th derivative exists and is continuous on $]a, b[$.*

We also have a smooth version of the Definable Monocity Theorem.

Fact 2.3.6. *Let $f \colon]a, b[\to \mathcal{R}$ a definable function. Then for every $k \in \mathbb{N}$ there are $a = a_{0,k}, \ldots, a_{r_k,k}$ such that f is of class $\mathcal{C}^k$ on each $]a_{i,k}, a_{i+1,k}[$.*

Last, we can transfer *Rolle's Theorem* in this context.

Fact 2.3.7 (Definable Rolle's Theorem)**.** *Let $f \colon [a, b] \to \mathcal{R}$ a definable function that is continuous on $[a, b]$, $\mathcal{R}$-differentiable on $]a, b[$ and such that $f(a) = f(b)$. There is a $c \in [a, b]$ such that $f'(c) = 0$.*

We will mainly use definable differentiability to define definable differentiable manifolds in Section 2.4.

There are two major differences between real analysis and analysis in arbitrary real closed field.

Remark 2.3.1. *As* $\mathbb{Z}$ *is not a definable subset of* $\mathcal{R}$ *we cannot quantify over integers and therefore we can not define the corresponding notions of sequences, infinite sums and integrals.*

Remark 2.3.2. *The behavior of an "exponential map" can be treacherous. Although the complete theory* T_{exp} *of* $(\mathbb{R}, +, \cdot, <, \exp)$ *is o-minimal (Wilkie and Khovanski), a real closed field equipped with an isomorphism of ordered groups between* $(\mathcal{R}, +, <)$ *and* $(\mathcal{R}_{>0}, \cdot, <)$ *need* ***NOT*** *be o-minimal nor a model of* T_{exp} *(see [Ber+ar])!*

2.4 Definable Manifolds and their tangent space

We want to emphasize here that all results in this section appear in [PPS98]. Here again we fix a real closed field $\mathcal{R}$.

2.4.1 Definable Manifolds and locally definable manifolds

Definable manifolds

Let $X \subseteq \mathcal{R}^n$ be a definable set and p a non zero natural number.

A *definable chart* on X is a triplet $\langle U, \varphi, n\rangle$ where U is a definable subset of X, $n \geq 0$ and φ is a definable bijection between U and some open subsets of $\mathcal{R}^n$ (n is called the dimension of the chart).

Two definable charts $\langle U', \varphi', n'\rangle$ and $\langle U, \varphi, n\rangle$ are $\mathcal{C}^p$*-compatible* if either $U \cap U' = \emptyset$ or if both $\varphi(U \cap U')$ and $\varphi'(U \cap U')$ are open and the transition maps $\varphi \circ \varphi'^{-1}$ and $\varphi' \circ \varphi^{-1}$ are of class $\mathcal{C}^p$.

A *definable* $\mathcal{C}^p$*-atlas* on X is a finite set $\mathfrak{C}$ of definable charts on X such that each pair is $\mathcal{C}^p$-compatible and whose domain cover X.

A *definable* $\mathcal{C}^p$*-manifold* (or simply a *definable manifold* when the class is not mentioned) is a pair $\mathbf{X}= \langle X, \mathfrak{C}\rangle$ where X is a definable set and $\mathfrak{C}$ a $\mathcal{C}^p$-atlas on X. If the order of differentiablity of the manifold is not mentioned we will simply call them *definable differentiable manifold* or even *definable manifold.*

Let $\mathbf{X}$ and $\mathbf{Y}$ be definable C^p-manifolds and $f : X \to Y$ a definable function. We say that f is a *definable morphism of* $\mathcal{C}^p$*-manifolds* if for any

$a \in X$ and for any charts $\langle U, \varphi, n\rangle$ on X with $a \in U$ and $\langle V, \psi, n'\rangle$ on Y with $f(a) \in V$ we have that $F := \psi \circ f\varphi^{-1}$ is of class $\mathcal{C}^p$.

Fact 2.4.1. *If* $\boldsymbol{X}$ *is a definable continuous manifold then we can equip* X *with a unique topology* τ *so that, for any definable chart* $\langle U, \varphi, n\rangle$ *on* X*, a subset* V *of* U *is* τ*-open if and only if* $\varphi(V)$ *is open (for the order topology).*

2.4.2 Tangent space of definable manifolds

Let **X,Y** be a definable $\mathcal{C}^p$-manifolds (with $p > 0$) and $f, g : X \to Y$ be definable $\mathcal{C}^1$ functions.

We say that f and g are tangent at $a \in X$ if $f(a) = g(a)$ and for any charts $\langle U, \varphi, n\rangle$ on **X** at a and $\langle V, \psi, n'\rangle$ on **Y** at $f(a)$ the differential at $\varphi(a)$ of $\psi \circ f \circ \varphi^{-1}$ and $\psi \circ g \circ \varphi^{-1}$ coincide.

We call tangent space of **X** at a and write $\mathcal{T}_a(X)$ the quotient of the space of all definable functions $f : \mathcal{R} \to X$ with $f(0) = a$ by the equivalence relation "f and g are tangent at 0".

For any chart $\langle U, \varphi, n\rangle$ we have a canonical bijection between $\mathcal{T}_a(X)$ and $\mathcal{R}^n$ mapping the class of f to $d(\varphi \circ f)(0)$. We can use this bijection to make $\mathcal{T}_a(X)$ a finite dimensional $\mathcal{R}$-vector space and thus definable. $\mathcal{T}_a(X)$ equipped with this vector space structure is called the tangent vector space of X at a.

Because tangency is preserved under composition by definable function, a morphism $X \to Y$ induces a mapping from $\mathcal{T}_a(X)$ to $\mathcal{T}_{f(a)}(Y)$. This mapping is linear and will be denoted $d_a(f)$.

A morphism of definable manifolds $f : Y \to X$ is said to be an immersion if for all $y \in Y$ the mapping $d_y(f)$ is injective.

Definition 2.4.1. *Let* $\boldsymbol{X}$ *be a definable manifold. A definable manifold* $\boldsymbol{Y}$ *is a definable submanifold of* $\boldsymbol{X}$ *if* $Y \subseteq X$ *and* $id : X \to Y$ *is an immersion. Also a subset* $X \subseteq Y$ *is a definable submanifold if it can be equiped with a definable manifold structure that makes it a submanifold in the previous meaning.*

Remark 2.4.1. *If* $\boldsymbol{Y}$ *is a submanifold of* $\boldsymbol{X}$ *and* $y \in Y$ *then* $d_y(id_Y)$ *is injective and we can see* $\mathcal{T}_y(Y)$ *as a subspace of* $\mathcal{T}_y(X)$*. This will be of importance when we will consider groups and subgroups with manifold structure.*

2.4.3 Locally Definable manifolds

We are usually concerned with definable objects. However, when defining the universal cover of a definable we will need to work in the locally definable category.

Now let us consider a set S such that:

- S=$\bigcup_{i\in\mathbb{N}} \mathcal{U}_i$.
- For each $i \in \mathbb{N}$ there is an injection map $\theta_i : \mathcal{U}_i \to \mathcal{R}^n$ for some $n \in \mathbb{N}$ such that $\theta_i(\mathcal{U}_i)$ is a definable open subset of $\mathcal{R}^n$.
- For each $i, j \in \kappa$ we have that $\theta_i(\mathcal{U}_i \cap \mathcal{U}_j)$ is an open definable subset of $\theta_i(\mathcal{U}_i)$ and the transition map $\theta_{i,j} : x \in \theta_i(\mathcal{U}_i \cap \mathcal{U}_j) \mapsto \theta_j \circ \theta_i^{-1}(x) \in \theta_j(\mathcal{U}_i \cap \mathcal{U}_j)$ is a definable morphism (of topological spaces).

We call the triplet $(S, (\mathcal{U}_i, \theta_i)_{i\in\mathbb{N}})$ a locally definable manifold and the $(\mathcal{U}_i, \theta_i)_{i\in\mathbb{N}}$ are the definable charts of S.

Fact 2.4.2. *If S is a locally definable manifold then we can equip S with a topology τ so that each $\mathcal{U}_i$ are open and the θ_i are morphisms. This is the unique topology such that any subset $U \subseteq S$ is τ-open if and only if $\theta_i(U \cap \mathcal{U}_i)$ is open for each $i \in \mathbb{N}$.*

As for the definable case a map f between locally definable manifolds $(X, (\mathcal{U}_i, \theta_i)_{i\in\mathbb{N}})$ and $(Y, (\mathcal{V}_i, \eta_i)_{i\in\mathbb{N}})$ is called a locally definable map if for every finite subset $I \subseteq \mathbb{N}$ there is a finite subset $J \subseteq \mathbb{N}$ such that:

- $f(\bigcup_{i\in I} \mathcal{U}_i) \subseteq \bigcup_{i\in J} \mathcal{V}_i$.
- the restriction $f : (\bigcup_{i\in I} \mathcal{U}_i) \to \bigcup_{i\in J} \mathcal{V}_i$ is a definable map between definable manifolds. That is for each $i \in I, j \in J$ with $f(\mathcal{U}_i) \cap \mathcal{V}_j \neq \emptyset$ the map $\eta_j \circ f \circ \theta_i^{-1} : \theta(\mathcal{U}_i) \to \mathcal{V}_j$ is a definable map.

Chapter 3

o-minimal groups

For the rest of this chapter let us fix an o-minimal expansion $(\mathcal{R}, <, \cdot, +, \dots)$ of a real closed field. When we say that something is definable we mean that it is definable in $(\mathcal{R}, <, \cdot, +, \dots)$. By semialgebraic we mean definable in $(\mathcal{R}, <, \cdot, +)$.

Definition 3.0.1. *Let G be a group, $\mu : G \times G \to G$ the multiplication in G and $\iota : G \to G$ the inverse map. We say that G is definable if G, μ and ι are definable.*

First we will need the classical definition of definable subgroup and morphism.

Definition 3.0.2. *Let H a subgroup of G. We say that H is a definable subgroup if it is also a definable group.*

Let G and H be definable groups and $\varphi : G \to H$ a group morphism. We say that φ is a definable morphism if its graph is definable.

This chapter's purpose is to recompile a large part of the work on definable groups that is scattered in several papers, published over the last thirty years. In the first section we will mention all the tools needed to prove the more advanced results of the second section. In the third section we talk about o-minimal homotopy. We continue understanding covers in the fourth section but with proofs of some statements that we could not find in the literature. Finally the last section is dedicated to the study of Levi subgroups in the definable context.

Except for results in the fourth section and a small corollary in the last section , the results presented here are not ours.

Remark 3.0.1. *Some of the results presented in this section are true in the more general context of groups definable in o-minimal expansions of ordered* *groups or even in general o-minimal structures. We will try to mention when the results apply to these cases.*

3.1 First results

Here we give the basics of definable groups. First we equip G with a topology and we get a handful of useful results from it. We continue with a presentation of definable groups of small dimension and a definable choice function theorem. Then we use the *o*-minimal version of *Euler characteristic* and apply it to definable groups (particularly torsion-free groups). Finally we extend the topological results to the context of definable manifolds and define the Lie algebra of a definable group.

3.1.1 The group chunk Theorem

The study of groups definable in *o*-minimal structures started with the following theorem.

Fact 3.1.1 (Pillay, [Pil88]). *Let G be a group definable in an o-minimal structure with $\dim(G) = n$. Then there are a large subset V of G and a topology t on G such that:*

- *G equipped with this topology is a topological group;*
- *V is a disjoint union of finitely many definable sets $U_1, ..., U_r$ such that each U_i is t-open and there is a definable homeomorphism between U_i and some open set $V_i \subseteq \mathcal{R}^n$ (with respect to the order topology).*

If G is definable in an o-minimal expansion of a real closed field, the homeomorphisms can be chosen to be diffeomorphisms making G a definable manifold such that the multiplication map is differentiable.

In [Hru86] E. Hrushovski shows that any constructible group can be equipped with a topology making it an algebraic group. In the previous theorem A. Pillay shows an analogous property for definable groups.

Now that G is equiped with a "definable" topological group structure we can get the usual *group-like* results.

Fact 3.1.2 (Pillay, [Pil88]). *Let G be a group definable in an o-minimal structure.*

1. *Any definable subgroup H of G is t-closed.*
2. *If H is a definable subgroup then the following are equivalent:*

- *H has finite index in G.*
- $\dim H = \dim G$*.*
- *H contains an open neighborhood of e_G.*
- *H is t-open in G.*

From now on when we use topological notions we always mean it with respect to the t-topology and will specify when are talking about other topologies (the order topology for example).

The following proposition will enable us to look only at the definability of morphism without being preoccupied about continuity in the same spirit as Fact 1.2.9, it is a direct consequence of Fact 3.1.1.

Fact 3.1.3. *Let $\varphi : G \to H$ be a definable morphism of definable groups, then φ is continuous.*

We will only be interested in connected definable groups as definable groups have finitely many connected components.

Fact 3.1.4 (Pillay, [Pil88])**.** *let G be a group definable in an o-minimal structure, there is a smallest definable subgroup G^0 of finite index in G called the identity component of G.*

This notion of connectedness coincides with Definition 2.1.1.

Fact 3.1.5 (Pillay, [Pil88])**.** *G^0 is actually the definable t-connected component of the identity in G. If $G = G^0$ we say that G is connected. For definable groups, definably connected and connected with respect to the t-topology are the same thing.*

From here we are ready to mention the very useful *descending chain condition on definable subgroups* for *o*-minimal groups.

Fact 3.1.6 (Pillay, [Pil88])**.** *Let G be a group definable in an o-minimal structure. There is no infinite strictly descending chain of definable subgroups in G. We will use the abbreviation* DCC *for this property.*

A direct consequence is the definibility of centralizers.

Fact 3.1.7. *Let G be a group definable in an o-minimal structure, centralizers in G are definable.*

Fact 3.1.6 also allows us to define the *definable hull* of a subset as in Section 5.5 of [BN94].

Definition 3.1.1. *Let X be a subset of a group definable in an o-minimal structure. There is a minimal definable subgroup of G containing X, it is called the definable hull of X and denoted $d(X)$.*

Next is a property on definable hulls that will be useful to define the solvable radical of a definable group.

Fact 3.1.8. *Let X and Y be subsets of a group definable in an o-minimal structure. If X is invariant under conjugacy, so is $d(X)$. Moreover we have $[d(X), d(Y)] \leq d([X, Y])$.*

Still following [BN94] we can extend the definition of connected component to non definabel subgroups.

Definition 3.1.2. *Let H be any subgroup of a group G definable in an o-minimal structure. We call the connected component of H the group $H^\circ := H \cap d(H)^\circ$.*

Fact 3.1.9. *The connected component of a subgroup H of a definable group G is a normal subgroup of finite index in H and $d(H^\circ) = d(H)^\circ$.*

3.1.2 Quotients and groups of small dimension

When working with groups, a very natural construction is the quotient by normal subgroups. In model theory one usually need to be flexible if one wants to work with quotients and relax "definability" to "interpretability". With *o*-minimal groups, one needs not to compromise at all since we have the following.

Fact 3.1.10 (Edmundo, [Edm03])**.** *Let G be a group definable in an o-minimal structure and let $\{T(x) : x \in X\}$ be a definable family of non-empty definable subsets of G. Then there is a definable function $t : X \to G$ called strong definable choice function such that*

- *for all $x \in X$, $t(x) \in T(x)$,*
- *for $x, y \in X$, if $T(x) = T(y)$ then $t(x) = t(y)$.*

From now on we will always assume, when talking about quotient groups, that we are actually working with a definable set of representatives of the quotient within the group.

We continue with three results for definable groups of small dimension, the first one is an analogue of the theorem of J. Reineke (in the case of strongly minimal groups) and is due to V. Ranzenj.

Fact 3.1.11 (Razenj,[Raz91]). *Let G be a connected 1-dimensional group definable in an o-minimal structure, then G is abelian. More precisely G is isomorphic (as a group) to either $\bigoplus_{p\in\mathcal{P}} \mathbb{Z}_{p^\infty} \oplus_\delta \mathbb{Q}$ or to $\oplus_\delta\mathbb{Q}$ for some δ and with $\mathcal{P}$ the set of primes number.*

The second theorem we present here deals with dimension 2.

Fact 3.1.12 (Nesin, Pillay, Razenj, [NPR91]). *Let G be a connected 2-dimensional group definable in an o-minimal expansion of an ordered group. Then G is solvable. Moreover, either G is abelian or it is centerless and it is definably isomorphic to a semi-product of the additive and multiplicative group of positive elements of a definable real closed field R.*

In the same paper the authors study the case of dimension 3.

Fact 3.1.13 (Nesin, Pillay, Razenj, [NPR91]). *Let G be a connected nonsolvable 3-dimensional group definable in an o-minimal expansion of an ordered group. Then $G/\mathcal{Z}(G)$ is definably isomorphic to either $PSL_2(R)$ or $SO_3(R)$ for some definable real closed field R.*

We finish this subsection with an interesting result about the algebraic rigidity that *o*-minimality imposes.

Fact 3.1.14 (Miller, Starchenko,[MS98]). *Let $(R, <, \dots)$ be an o-minimal structure. Up to definable isomorphism, there is at most two continuous (with respect to the order topology) definable groups with underlying set R.*

It follows that if we have an *o*-minimal expansion $(\mathcal{R}, < \cdot, +, \dots)$ of a real closed field and there is no definable isomorphism between $(\mathcal{R}, +)$ and $(\mathcal{R}^{>0}, \cdot)$, any continuous definable group $(\mathcal{R}, \oplus)$ must be definably isomorphic to either $(\mathcal{R}, +)$ or $(\mathcal{R}^{>0}, \cdot)$. We believe that it is still unknown whether there exists an *exponentional* real closed field that admits an other continuous definable group structure.

3.1.3 The Euler Characteristic

The *o*-minimal Euler characteristic is defined in Definition 2.2.6 and when it is applied to a definable group G it gives us a good idea of the algebraic structure of G. Using Euler Characteristic, A. Strzebonski made a thorough study of *o*-minimal Sylow subgroups in his thesis. Although we will not need these specific result, we mention some results Strzebonski proved along the way and that will be of use later.

The following was already mentioned above (in a more general context) but we deem important to write it again since it was proven without topological arguments.

Fact 3.1.15 (Strzebonski, [Str94])**.** *Let G be a group definable in an o-minimal expansion of an ordered group. There is no infinite descending chain of definable subgroups in G.*

Euler characteristic turned out to be a powerful tool to study torsion as it is shown in the following theorem.

Fact 3.1.16 (Strzebonski, [Str94])**.** *Let G be a group definable in an o-minimal expansion of an ordered group, then $E(G) = \pm 1$ if and only if G is torsion free.*

Since Euler characteristic is multiplicative, that is for H a normal definable subgroup of G then $E(G) = E(G/H) \cdot E(H)$, we can lift torsion from the quotient and the subgroup to the initial group.

Fact 3.1.17 (Torsion lifting)**.** *Let G be a group definable in an o-minimal expansion of an ordered group and H a definable torsion-free subgroup, then G/H is torsion-free if and only if G is torsion-free.*

Using those results on torsion Y. Peterzil and S. Starchenko proved the following.

Fact 3.1.18 (Peterzil, Starchenko, [PS05])**.** *Let G be a definable torsion-free group. Then G is definably connected and solvable.*

With more work on definable sections they prove a topological characterization of torsion-free definable groups.

Fact 3.1.19 (Peterzil, Starchenko, [PS05])**.** *Let G be definable group of dimension n. Then G is torsion-free if and only if G is definably diffeomorphic to $\mathcal{R}^n$.*

3.1.4 The $\mathcal{R}$-Lie algebra

Through this subsection we follow [PPS98] and present a Lie algebra theory for definable groups. We fix a positive integer $k \in \mathbb{N}$.

We begin with a strengthening of the Group Chunk Theorem of A. Pillay.

Definition 3.1.3. *Let G be a definable group, we say that $(G, \mathfrak{A})$ is a definable $\mathcal{C}^k$-group if $\mathfrak{A}$ is a definable $\mathcal{C}^k$-atlas on G and the group multiplication and inverse are $\mathcal{C}^k$ maps.*

Fact 3.1.20. *Let G be a definable group, there is a $\mathcal{C}^k$-atlas on G such that $(G, \mathfrak{A})$ is a definable $\mathcal{C}^k$-group.*

We will sometimes use the word *smooth* to refer to the "regularity" of functions instead of writing of class $\mathcal{C}^k$.

Now that we have this extra structure we deal with its unicity with a smooth version of Fact 3.1.3.

Fact 3.1.21. *Let $(G, \mathfrak{A})$ and $(H, \mathfrak{B})$ be definable $\mathcal{C}^k$-groups and $\varphi : G \to H$ a definable group morphism. Then φ is of class $\mathcal{C}^k$.*

An easy corollary is the uniqueness of the $\mathcal{C}^k$-group structure.

Fact 3.1.22. *Let G be a definable group, its $\mathcal{C}^k$-group structure is unique.*

Moreover the differential structure passes smoothly to definable subgroups.

Fact 3.1.23. *If H is a definable subgroup of G then it is a definable submanifold.*

Definable groups and Lie groups are alike in several aspects and we will develop the equivalent of the Lie algebra for definable groups. We start with the following definition.

Definition 3.1.4. *The tangent space $\mathcal{T}_e(G)$ of G at e_G is called the tangent space of G.*

Using Fact 3.1.23 we can see, via the differential of the identity map, $\mathcal{T}_e(H)$ as a subset of $\mathcal{T}_e(G)$. Since $\mathcal{T}_e(G)$ only depends on the manifold structure of G at e, the tangent space of G is equal to the tangent space of its connected component.

With a little bit of additional work we can get the following results.

Fact 3.1.24. *Let H_1, H_2 be definably connected definable subgroups of G. Then $H_1 = H_2$ if and only if $\mathcal{T}_e(H_1) = \mathcal{T}_e(H_2)$.*

Fact 3.1.25. *Let f be a definable automorphism of G, then $d_e(f)$ is a definable automorphism of the $\mathcal{R}$-vector space $\mathcal{T}_e(G)$.*

We are now ready to equip $\mathcal{T}_e(G)$ with a $\mathcal{R}$-Lie algebra structure (recall that a $\mathcal{R}$-Lie algebra is a $\mathcal{R}$-vector space together with a $\mathcal{R}$-bilinear form $[\ ,\ ]$ satisfying Jacobi's identity).

We will repeat here the construction mentioned in Section 1.2.1 noticing that it is definable. We start with inner automorphisms of G. For $g \in G$ we write $a(g) : x \mapsto x^g = g^{-1}xg$.

We let $Ad(g) = d_e(a(g))$ and we know by the previous fact that $Ad(g)$ is an automorphism of $\mathcal{T}_e(G)$ so we have the definable morphism:

$$Ad : G \to Aut(\mathcal{T}_e(G)).$$

Seeing $Aut(\mathcal{T}_e(G))$ as a definable manifold we can continue and consider

$$d_e(Ad) = ad : \mathcal{T}_e(G) \to End(\mathcal{T}_e(G)).$$

This allows us to define a binary operation on $\mathcal{T}_e(G)$:

$$[X, Y] : = ad(X)(Y).$$

Definition 3.1.5. $\langle \mathcal{T}_e(G), [\ ,\ ] \rangle$ *is a* $\mathcal{R}$*-Lie algebra called the* $\mathcal{R}$*-Lie algebra of* G*. We usually will write* $\mathfrak{g} = Lie(G)$ *without mentioning the* $\mathcal{R}$*-Lie structure.*

Now that we finished the construction we can translate all typical results on Lie groups and their Lie algebras to our context.

Fact 3.1.26. *Let us fix a definably connected definable group* G *and* $\mathfrak{g}$ *its* $\mathcal{R}$*-Lie algebra.*

- *If* H *is a definable subgroup of* G *then its* $\mathcal{R}$*-Lie algebra is a* $\mathcal{R}$*-Lie subalgebra of* $\mathfrak{g}$*.*
- G *is abelian if and only if* $\mathfrak{g}$ *is abelian.*
- *If* H *is a definable subgroup of* H*, then* H *is normal in* G *if and only if its* $\mathcal{R}$*-Lie algebra* $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$*.*

3.2 Other results

The reader will probably already have noticed the similarities between groups definable in *o*-minimal structures and Lie groups. In the same way that groups of finite Morley rank have been studied with algebraic group theory in sight, many results on definable groups take inspiration in Lie group theory.

We first take a particular interest in solvability and semisimplicity. We pursue our study with definably compact groups and finish this section with linearity considerations.

3.2.1 The solvable radical, definable compactness and the maximal torsion free subgroup

A very useful concept when studying Lie group is the solvable radical. If G is a connected Lie group it admits a maximal solvable normal connected subgroup R. It so happens that if G is a definable group, R is definable.

We begin with the construction of the solvable radical and then mention how it can be used to define a torsion-free radical.

There are two main ways to properly define R: it is possible to define a maximal solvable subgroup (non necessarily connected) and show that it is definable. Taking the connected component of this group will give us the subgroup we were looking for.

We take here a different and more direct approach and begin to define (definably) semisimple definable groups.

Definition 3.2.1. *Let S be a definable group. We say that S is semisimple if it has no infinite normal abelian subgroup.*

We mention a nice (and classical) characterization of semisimple definable groups using its Lie algebra.

Fact 3.2.1 (Peterzil, Pillay, Starchenko [PPS00a])**.** *Let S be a definable group. S is semisimple if and only if $\mathfrak{s} = Lie(S)$ is semisimple.*

The following fact is well known but we could not find a clear reference; it appears as Fact 1.6 in [Con14].

Fact 3.2.2. *Let G be a definable group. There is a unique normal definably connected subgroup R of G such that G/R is either semisimple or finite. We call R the (solvable) radical of G and it contains all normal connected solvable subgroups of G.*

Remark 3.2.1. *As mentioned above, in [BJO12] E. Baro, E. Jaligot and M. Otero proved the existence of a (non necessarily connected) solvable radical.*

When studying Lie groups, compact Lie group are of particular interest. In an *o*-minimal setting, Y. Peterzil and C. Steinhorn provided a nice analogue. We present here the case of an *o*-minimal expansion of a real field but the results hold in a general *o*-minimal structure.

Fact 3.2.3. *Let X be a definable set (equipped with the order topology). We say that X is definably compact if for every definable function $f : (0, 1) \to X$ the limit $\lim_{x\to 1^-} f(x)$ exists in X. A definable group is said definably compact if the same holds with respect to the t-topology of G.*

Using a variant of Zilber's *tangents at infinity* they were able to prove the following.

Fact 3.2.4 (Peterzil, Steinhorn, [PS99])**.** *Let G be a non definably compact o-minimal group. There is a definable 1-dimensional torsion-free subgroup of G.*

In the same paper, using topological/analytical methods they also give a real-like characterization of definably compact subsets of *o*-minimal structures.

Fact 3.2.5 (Peterzil, Steinhorn, [PS99])**.** *Let X be a definable subset of $\mathcal{R}^n$. Then X is definably compact if and only if X is closed and bounded.*

Using the previous results M. Edmundo was able to study the algebraic structure of definably compact groups.

Fact 3.2.6 (Edmundo, [Edm05])**.** *Let G be a definably compact definably connected definable group. Then there is a definably connected abelian subgroup T of G such that $G := \bigcup_{g \in G} T^g$ and G is divisible.*

Another interesting subgroup is the maximal normal definable torsion-free subgroup. With the help of Euler characteristic one can prove the following. We suspect that the result is older but could not find any clear reference; the reader can find proofs of the following results in [CP12].

Fact 3.2.7. *Let G be a definable group. There is a definable normal torsion-free subgroup T of G that contains every normal definable torsion-free subgroups of G. It is the unique normal definable torsion-free subgroup of G of maximal dimension.*

It is worth mentioning that T needs not contain all definable torsion-free subgroups of G, nonetheless we have the following.

Fact 3.2.8. *A normal definable torsion-free subgroup T of G contains every definable torsion-free subgroups of G if and only if G/T is definably compact.*

The solvable case is particularly interesting and we will use it strongly in Chapters 4 and 5.

Fact 3.2.9. *Let G be a solvable definable group and T its maximal normal definable torsion-free subgroup. If G is not definably compact then T is infinite and G/T is definably compact.*

Finally, as for Lie group, maximal (definably) compact subgroups are conjugate (in the quotient G/T).

Fact 3.2.10 (Conversano, [Con14])**.** *Let G be a definably connected definable group and T its maximal normal definable torsion-free subgroup. Then G/T has a maximal definably compact subgroup K (unique up to conjugacy) and G is definably homeomorphic to $K \times \mathcal{R}^\ell$ with $\ell = \dim(G) - \dim(K)$.*

3.2.2 Definably simple groups and considerations on linearity

The study of linear Lie groups (closed subgroup of $GL_n(\mathbb{R})$) is more pleasant than the general case of non linear Lie groups. With this in mind it is only natural to consider those definable groups that can be definably embedded in the general linear group $GL_n(\mathcal{R})$.

Definition 3.2.2. *We say that a definable group G is definably linear if there is an injective definable morphism $\rho : G \to GL_n(\mathcal{R})$ for some $n \in \mathbb{N}$, that is, G has a faithful definable representation of finite dimension. When dealing with definably linear groups we will usually work with the image $\rho(G)$ and see the group as a group of matrices, we say that $\rho(G)$ is a definable matrix group.*

Remark 3.2.2. *It is worth noticing that by Fact 3.1.3 we do not need to worry about regularity of the representation.*

We will sometimes refer to some properties as definable in G seen as a *pure group*. By that we mean that it is definable in the reduct $(G, \cdot)$. A definable group that is definable in the reduct $(\mathcal{R}, +, \cdot, <)$ is said to be semialgebraic.

Any simple (or centerless) Lie group is linear (it acts faithfully on its Lie algebra) so we can expect similar result in the category of definable groups. In [PPS00a] Y. Peterzil, A. Pillay and S. Starchenko give a deep study of the problem and prove an analogue in an *o*-minimal context of *Cherlin's Conjecture* on groups of finite Morley rank. First let us start with a classical definition.

Definition 3.2.3. *A definable group G is said to be definably simple if it has no proper nontrivial normal definable subgroup.*

Remark 3.2.3. *In [PPS02], the authors study the differences between definable simplicity and simplicity as an abstract group. Unlike definable connectedness and conectedness, the notions differ when the real field $\mathcal{R}$ strictly contains the real field $\mathbb{R}$.*

As mentioned above we have the following.

Fact 3.2.11 (Peterzil, Pillay, Starchenko.[PPS00a])**.** *Let G be a definable group. If G is definably simple (as a pure group) there is a definable real closed field $\mathcal{R}$ such that G is definably isomorphic to a semialgebraic matrix group over $\mathcal{R}$.*

Actually they proved something more general.

Fact 3.2.12 (Peterzil, Pillay, Starchenko.[PPS00a]). *Let G be a definably connected (as a pure group) definable group that has no nontrivial abelian normal subgroup. Then G is the direct product of definable subgroups $H_1, \ldots, H_k$ such that for each $i \in [\![1, k]\!]$ there are definable real closed fields $\mathcal{R}_i$ such that H_i is definably isomorphic to a semialgebraic matrix group over $\mathcal{R}_i$.*

This let us state the alternative definition for semisimple definable groups.

Fact 3.2.13 (peterzil, Pillay, Starchenko.[PPS00a]). *Let S be a definable group. Then S is semisimple if and only if it has no infinite normal solvable subgroups.*

In another paper, the same authors analyze definable groups as pure groups and prove the following enlightening theorem.

Fact 3.2.14 (Peterzil, Pillay, Starchenko.[PPS00b]). *Let G be an infinite, definably simple (as a pure group), o-minimal group. There is a real closed field $\mathcal{R}$ and its complexification $\mathcal{C} := \mathcal{R}(\sqrt{-1})$ such that one and only one of the following occur:*

- *$(G, \cdot)$ and $(\mathcal{C}, +, \cdot)$ are bi-interpretable. Moreover G is definably isomorphic to the $\mathcal{C}$-rational points of a linear algebraic group defined over $\mathcal{C}$.*
- *$(G, \cdot)$ and $(\mathcal{R}, +, \cdot)$ are bi-interpretable. Moreover G is definably isomorphic to the semialgebraic connected component of a group $\mathbb{G}(\mathcal{R})$ where $\mathbb{G}$ is an $\mathcal{R}$-simple algebraic group defined over $\mathcal{R}$.*

In a third paper, Y. Peterzil, A. Pillay and S. Starchenko study definably linear groups and in particular semisimple ones. They prove the following.

Fact 3.2.15 (Peterzil, Pillay, Starchenko,[PPS02]). *Let S a be definable, definably connected semisimple matrix group. Then S is semialgebraic.*

We mention an other interesting result about semialgebraicity that can be found in the same paper.

Fact 3.2.16 (Peterzil, Pillay, Starchenko,[PPS02]). *Let G be a definable matrix group that is bounded in the order topology. Then G is semialgebraic.*

They were also able to prove a definable Levi decomposition for definably linear groups.

Fact 3.2.17 (Peterzil, Pillay, Starchenko, [PPS02]). *Let G be a definably connected definable matrix group. Then G is the almost semidirect product of its solvable radical R and a definable semisimple subgroup S.*

In the last section we will discuss more in depth about the subgroup S (called Levi subgroup of G) in the non linear case.

They also give a characterization of definably simple definable groups.

Fact 3.2.18 (Peterzil, Pillay , Starchenko,[PPS02]). *Let G be an infinite group.Then G is definable and definably simple (as a pure group) group if and only if $(G, \cdot)$ is elementary equivalent to $(H, \cdot)$ a simple (as a Lie group) Lie group.*

We finish this section with a more recent algebraic result due to E. Baro. For groups of finite Morley rank, *ZilBer's Indecomposability Theorem* allow us to define groups generated by infinitely many "indecomposable" sets with finitely many of those indecomposable subsets. In *o*-minimal structures we don't have such a useful tool hence the definability of the commutator subgroup is not automatic.

Fact 3.2.19 (Baro, [Bar19]). *Let G be a definably connected, definable matrix group. Then the derived subgroup G' of G is semialgebraic and definably connected.*

In [Bar19] E. Baro also takes interest in definably simply connected definable groups (see Definition 3.3.11) and prove the following theorem.

Fact 3.2.20 (Baro,[Bar19]). *Let G be a definably simply connected definable group. Then the derived subgroup G' of G is definable and connected.*

3.3 Definable homotopy and covering maps

As we already mentioned several times, definable groups have a lot in common with Lie groups and in general *o*-minimality benefits from many tools that come from topology.

A powerful tool in the study of Lie groups (and in particular in the classification of simple Lie groups) is the universal cover. Hence if one wants to develop similar notion in the *o*-minimal context one should think about definable homotopy. To study covers of definable groups we will be need to work within the larger category of locally definable groups and we begin this section with their basic definitions and properties. In the second subsection we give the main results and definition about *o*-minimal homotopy together with new results. We finish providing a few results on definable Levi subgroups.

3.3.1 Locally definable groups

It might seem a bit out of context at this point but we will need the definitions and properties presented here to continue the analysis of definable covers. Let us fix an o-minimal expansion of a real closed field $(\mathcal{R}, <, +, \cdot, \dots)$ and $A \subseteq \mathcal{R}$ with $|A| < \aleph_1$. Let us begin with the definition of locally definable group.

Definition 3.3.1. *A group G is said locally definable over A if there is a collection $\{X_i : i \in I\}$ of definable subsets of $\mathcal{R}^n$ over A such that:*

- $G = \bigcup_{i \in I} X_i$,
- *for every $i, j \in I$ there is a $k \in I$ such that $X_i \cup X_j \subseteq X_k$,*
- *the restriction of the multiplication map to $X_i \times X_j$ is definable over A.*

Usually we will not mind the parameters A.

As for definable groups, we have a good notion of dimension for locally definable groups.

Definition 3.3.2. *Let $G = \bigcup_{i \in I} X_i$ be a locally definable group, we define* $\dim(G) := \max\{\dim(X_i) : i \in I\}$.

And in the same vein as the group chunk theorem we have a variant for locally definable groups.

Fact 3.3.1 (Peterzil, Starchenko [PS00])**.** *Let G be a locally definable group with $\dim(G) = n$. Then there is a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of G over A each containing 1_G and a unique topology τ on G such that:*

- *G with this topology is a topological group,*
- *$\{V_s : s \in S\}$ is a basis of τ-open neighborhoods of 1_G.*

This topology actually makes G an $\mathcal{R}$-Lie group (a Lie group over $\mathcal{R}$) and a locally definable manifold with smooth multiplication.

Definition 3.3.3. *A morphism of groups $\varphi : G \to H$ between locally definable groups over A is said to be locally definable if for every definable subset $X \subseteq G$ defined over A, the restriction $\varphi\restriction_X$ is definable over A.*

We also have the usual definitions of morphism and subgroup in the category of locally definable groups.

Definition 3.3.4. *We say that a locally definable group H is a locally definable subgroup of G if it is a subgroub of G.*

We say that H is a compatible locally definable subgroup if for every open definable subset U of G, the set $U \cap H$ is a definable subset of H.

The following definition will be particularly useful in the study of locally definable groups.

Definition 3.3.5. *Let G be a locally definable group and $D \subseteq G$. We say that D is discrete if for every definable subset X of G, $D \cap X$ is finite.*

Locally definable groups are a bit more trickier than definable groups but we still have a good notion of connectedness.

Definition 3.3.6 (Baro, Edmundo, [BE08])**.** *Let G be a locally definable group. We will say that a subset $X \subseteq G$ is connected if there is no subset $U \subseteq G$ such that*

- *the intersection of U with every definable subset of G is definable,*
- *$U \cap X$ is a non-empty proper subset of X which is closed and open in the topology induced on X by G.*

Again we have very similar result to Fact 3.1.2 for locally definable groups.

Fact 3.3.2 (Edmundo,[Edm03])**.** *Let G be a locally definable group and H a compatible locally definable subgroup of G. Then:*

- *the τ-topology on H is the induced topology (by the τ-topology) of G,*
- *H is τ-closed in G,*
- *H is τ-open if and only if $\dim(G) = \dim(H)$.*

Finally we can talk about the connected component of a locally definable group.

Fact 3.3.3 (Edmundo, [Edm03])**.** *let G be a locally definable group. There is a unique definably connected compatible locally definable normal subgroup G^0 of G with $\dim(G) = \dim(G^0)$. Moreover G^0 contains all definably connected locally definable subgroups of G and it is the smallest compatible locally definable subgroup of G of finite index.*

3.3.2 *o*-minimal homotopy and covers

We present here results about *o*-minimal homotopy and covering morphisms of definable groups. We then provide new results in the particular case where the real close field we are working on is $\mathbb{R}$.

Definitions and first results

Let us begin here with the definable variant of covering morphism.

Definition 3.3.7. *A locally definable morphism $p : G \to H$ over A (between locally definable groups over A) is called a locally definable covering if p is surjective and there is a family $\{U_\ell : \ell \in L\}$ of τ-open definable subsets of H such that $H = \bigcup_{\ell \in L} U_\ell$ and for each $\ell \in L$, $p^{-1}(U_\ell)$ is a disjoint union of τ-open definable subsets of G over A, each of which is homeomorphic to U_ℓ.*

With this definition comes two other definitions/properties.

Definition 3.3.8. *Let G be a locally definable group and let us denote by $Cov(G)$ the category whose objects are locally definable coverings of G and whose arrows between two objects $p : H \to G$ and $q : K \to G$ are surjective locally definable morphisms $r : H \to K$ satisfying $q \circ r = p$.*

Definition 3.3.9 (Edmundo, Eleftheriou, [EE07])**.** *If $r : H \to K$ is an arrow in $Cov(G)$ then it is a a locally definable covering morphism. Let $Cov(G)^0$ denote the full subcategory of $Cov(G)$ whose objects are locally definable covering morphisms $p : H \to G$ with H definably connected. Then $Cov(G)$ and $Cov(G)^0$ form inverse systems and the inverse limit $p : \widetilde{G} \to G$ of $Cov(G)^0$ is called the o-minimal universal covering morphism of G, its kernel $\pi(G)$ is called the o-minimal fundamental group of G.*

Now that we have defined the universal covering morphism we will take interest in the space of definable paths in G. The usual construction of the universal cover of a Lie group using continuous paths applies to the *o*-minimal context and the notion of universal cover mentioned above coincide.

Definition 3.3.10. *Let G be a connected locally definable group.*

- *A τ-path is a τ-continuous definable map $\sigma : [0, p] \to G$, it is called a loop if $\sigma(0) = \sigma(p)$. If $g \in G$ we will abuse notation and call g the constant path equal to g.*
- *A concatenation of τ-path $\sigma : [0, p] \to G$ and $\gamma : [0, q] \to G$ with $\sigma(p) = \gamma(0)$ is a τ-path $\sigma \dashv\vdash \gamma : [0, p + q] :\to G$ with*

$$(\sigma \dashv\vdash \gamma)(t) = \begin{cases} \sigma(t) & \text{if } t \in [0, p] \\ \gamma(t - p) & \text{if } t \in [p, p + q] \end{cases}$$

- *Let $f, g : Y \subseteq R^m \to G$ be two definable τ-continuous functions, we say that f and g are τ-homotopic (denoted $f \sim_\tau g$) if there is a definable τ-continuous map $F : Y \times [0, q] \to G$ such that $f = F(\ldots, 0)$ and $g = F(\ldots, q)$.*

- *We say that two τ-paths σ and γ such that $\sigma(0) = \gamma(0)$ and $\sigma(p) = \gamma(q)$ are τ-homotopic if they are as definable functions.*
- *Let $\mathbb{L}(G)$ denote the set of loops that start and end at 1_G. The restriction of $\sim_\tau$ to $\mathbb{L}(G) \times \mathbb{L}(G)$ is an equivalence relation and we set $\pi_1(G) := \mathbb{L}(G)/\sim_\tau$.*
- *The class in $\pi_1(G)$ of a loop σ is denoted $[\sigma]$ and $\pi_1(G)$ can be equipped with the group structure defined by $[\sigma] \cdot [\gamma] := [\sigma \dashv\vdash \gamma]$.*

All of these definitions let us define the concept of simply connectedness for any locally definable group.

Definition 3.3.11. *Let G be a connected locally definable group, G is called definably simply connected if $\pi_1(G) = \{[1_G]\}$.*

Remark 3.3.1. *Notice that definable groups are a particular case of locally definable groups and hence the previous definitions also apply in this context.*

As promised, the following states that the two objects coincide.

Fact 3.3.4 (Edmundo, Eleftheriou, [EE07])**.** *Let G be a definably connected locally definable group. The o-minimal universal covering morphism is a locally definable covering morphism. Moreover $\pi(G)$ the o-minimal fundamental group of G is isomorphic to $\pi_1(G)$.*

3.4 Definable covering of connected Lie groups

We assume in this section that we are working in an *o*-minimal expansion of the real field $\mathbb{R}$ and by definable we always mean definable in this expansion. We think that the results proved in this section also apply to $\mathcal{R}$-Lie groups, that is groups that are $\mathcal{R}$-manifolds (not necessarily definable) but it would require to prove all the classical theorems of Lie theory in this context.

We first prove that the topological universal cover of a definable Lie group coincides with the *o*-minimal universal cover from the previous section. Part of the content of the following is proved in [BO02] but we give here a proof for completeness.

Theorem 5. *Let G be a connected definable Lie group. The Lie universal cover $\widetilde{G}$ and the o-minimal universal cover $\widetilde{G}^{def}$ of G are Lie-isomorphic. Moreover there is a locally definable covering map $\pi_{def} : \widetilde{G}^{def} \longrightarrow G$ such that the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\sim} & \widetilde{G}^{\mathrm{def}} \\ {\scriptstyle \pi}\big\downarrow & \swarrow{\scriptstyle \pi_{\mathrm{def}}} & \\ G & & \end{array}$$

where $\pi : \widetilde{G} \twoheadrightarrow G$ is the usual universal covering map (in the Lie group category).

Proof. The construction of $\widetilde{G}^{\mathrm{def}}$ in [EE07] is based on the standard construction of $\widetilde{G}$ via continuous paths (quotienting by homotopy), requiring the paths and homotopies to be both definable and continuous. The isomorphism from $\widetilde{G}^{\mathrm{def}}$ to $\widetilde{G}$ will send a definable path to its homotopy class. We will need to show it is well defined (two definable maps which are homotopically equivalent have a definable homotopy between them) and that it is surjective, so that any path is homotopically equivalent to a definable one. We will prove the latter and sketch the proof of the former, which is essentially the same idea but the notation is much more complicated.

G is a definable Lie group we so can find (using cell decomposition) definable open sets $\mathcal{U}_1, \ldots, \mathcal{U}_k$ covering G, each definably homeomorphic to an open convex, simply connected subset $\mathcal{B}_i$ in some cartesian power $\mathbb{R}^\ell$.

Take a continuous path σ in G, and let σ be parametrized with $t \in I := [0, 1]$.

By *Lebesgue's Number Theorem*, there is some $\delta > 0$ such that the image under σ of any subinterval of I of size less than δ is contained in one of the $\mathcal{U}_i$'s. Let m be such that $\frac{1}{m} < \delta$ and so that if $I_j := \left[\frac{j}{m}, \frac{(j+1)}{m}\right]$, then the image of I_j under σ is fully contained in $\mathcal{U}_{i_j}$. We find a definable σ_{def} as follows. Let μ_j be the push forward of σ_j in B_n. By convexity, the straight line l_j joining $\mu_j(\frac{j}{m})$ and $\mu_j(\frac{(j+1)}{m})$ is fully contained in $\mathcal{B}_n$, and because B_n is simply connected μ_j is homotopic to l_j. We then pull back l_j to U_n and the resulting path is σ_j^{def}. By construction, the starting and endpoints of σ_j^{def} are precisely the same as those of σ_j which by definition define the connected path σ. The gluing of the σ_j^{def} defines a definable connected path homotopic to σ, as required.

The proof of the fact that two definable maps which are homotopically equivalent are definably homotopically equivalent is essentially the same: Let $F : [0, 1]^2 \mapsto G$ be an homotopy between definable paths $F(0, t)$ and $F(1, t)$. One uses the Lebesgue number theorem to find m such that for any $i, j \leq m$

such that the image

$$F\left(\left[\frac{i}{m}, \frac{(i+1)}{m}\right] \times \left[\frac{j}{m}, \frac{(j+1)}{m}\right]\right)$$

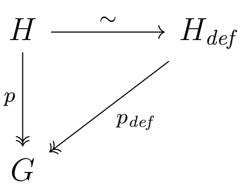
is fully contained in some $\mathcal{U}_k$.

As before, the pullback in U_k of the straight line segment $l_{i,j}$ in B_k joining the images of $F(\frac{i}{m}, \frac{j}{m})$ and $F(\frac{i}{m}, \frac{j+1}{m})$ is homotopically equivalent to $F\left(\frac{i}{m}, \left[\frac{j}{m}, \frac{j+1}{m}\right]\right)$, and the same thing happens when we replace i with $i+1$. But $l_{i,j}$ and $l_{i+1,j}$ belong to the convex subset B_k of $\mathbb{R}^\ell$, so they are definably equivalent via the straight-line homotopy. Taking the images in U_k, and doing this for every i, j we build a definable homotopy equivalent to F.

This concludes the proof. □

Recall that if G is a connected Lie group a covering map is a continuous surjective map $p : H \longrightarrow G$ where H is a connected Lie group such that for each point $g \in G$ there is an open neighborhood $\mathcal{U}$ of g such that $p^{-1}(\mathcal{U})$ is a disjoint union of open sets. We say that the covering is finite if p has finite kernel. We have the following theorem which appears to be a corollary of Theorem 1.4 in [EJP11] but we did not realize it in time; hence we provide a proof.

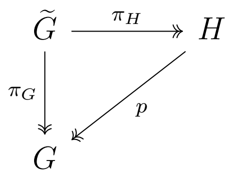
Theorem 6. *Let $p : H \longrightarrow G$ be a finite covering map of a connected definable Lie group G. Then H is Lie-isomorphic a definable Lie group H_{def} and there is a continuous definable covering map $p_{def} : H_{def} \longrightarrow G$ such that the following diagram commutes.*



Proof. First, since H is a finite cover of G we have a covering map $p : H \twoheadrightarrow G$ with finite kernel. Now consider the universal cover $\widetilde{G}$ of G and the definable universal cover $\widetilde{G}^{\mathrm{def}}$ of G. By Theorem 5, the two coincide.

From now on we will use the notation $\widetilde{G}$ when we talk about the locally definable cover of G.

The universal property of universal covers gives us the following commutating diagram:



We know that $\widetilde{G}$ is a locally definable group and that π_G is a continuous locally definable morphism. The morphisms π_H and π are only continuous but we are going to find a definable version of H and the corresponding morphisms will also be definable. Since π_G is locally definable and surjective we can find using (logic) compactness a definable subset D_G in $\widetilde{G}$ such that the restriction $\pi_G : D_G \twoheadrightarrow G$ is surjective. We may of course assume that D_G contains the identity.

Now $\mathrm{Ker}(\pi) = \{h_1, \ldots, h_k\}$ is finite with $h_1 = e_H$. We construct a definable set $D := \bigcup_{i=1}^{k} D_G \cdot g_i^{-1}$ contained in $\widetilde{G}$ where g_i is a preimage of h_i by π_H, choosing $e_{\widetilde{G}}$ as g_1. Notice that the restriction of π_H to D is surjective.

Since $\mathrm{Ker}(\pi_H) \subseteq \mathrm{Ker}(\pi_G)$ and the latter is locally definable and discrete, any intersection with a definable set must be finite. This implies the following claim which we will use several times.

Claim 3. *Ker(π_H) is a discrete (as in Definition 3.3.5) subset of $\widetilde{G}$.*

We can now define a group which is Lie-isomorphic to H. The universe is defined by taking $H_{\mathrm{def}} := D/\sim$ where $d_1 \sim d_2$ if $\pi_H(d_1) = \pi_H(d_2)$ (equivalently, $d_1 \cdot d_2 \in D \cdot D^{-1} \cap \mathrm{Ker}(\pi_H)$). This is a finite equivalence relation (and therefore definable) by the previous claim.

We will define group multiplication and inverse operations on D and check that they are compatible with $\sim$ and therefore pass nicely to H_{def}.

Define $d_1 \cdot d_2 = d_3$ whenever $\pi_H(d_1 \cdot d_2) = \pi_H(d_3)$. This is equivalent to $d_1 \cdot d_2 \cdot d_3^{-1} \in D^2 \cdot D^{-1} \cap \mathrm{Ker}(\pi_H)$, so it is a definable relation. Finally, we say that $d_1^{-1} = d_2$ if $d_1 \cdot d_2 \in D^2 \cap \mathrm{Ker}(\pi_H)$, so it is definable.

This defines a group structure on H_{def}, and it also inherits the Lie group structure from $\widetilde{G}$ which, as it is shown in [EEP13], it coincides with the usual τ-topology on definable groups. Moreover, everything was defined so that the pushforward of π_H to the $\sim$-quotient $H_{\mathrm{def}} \to H$ is a continuous group isomorphism. So H_{def} is a definable Lie group Lie-isomorphic to H, as required. □

Remark 3.4.1. *The construction above do not use any extra structure to define H_{def}. In particular if G is semialgebraic then H_{def} is also semialgebraic.*

3.5 Levi subgroups

When one studies Lie groups, a very useful tool is Fact 1.2.30. Although for some definable groups the Levi subgroup S will be definable, it is in general only locally definable. Let us first clarify what we mean by semisimple locally definable group.

Definition 3.5.1. *Let G be a definably connected locally definable group. G is called lcoally definable semisimple if $\mathcal{Z}(G)$ is discrete and $G/\mathcal{Z}(G)$ is definable and semisimple.*

A. Conversano and A. Pillay show the existence of a Levi subgroup as in the following.

Fact 3.5.1 (Conversano, Pillay,[CP13])**.** *Let G be a definably connected definable group. There is a unique (up to conjugacy) maximal locally definable semisimple subgroup S of G. Moreover,*

- *$G = RS$ with R the solvable radical of G,*
- *$\mathcal{Z}(S)$ is finitely generated and contains $R \cap S$.*

Since we are more interested in definable groups, the previous theorem motivates the following definition.

Definition 3.5.2. *Let G be a connected definable group. We say that G has a good Levi decomposition if its maximal locally definable semisimple subgroup S is definable.*

Here we give some cases of definable groups with a good Levi decomposition.

Fact 3.5.2 (Conversano, Pillay,[CP13])**.** *Let G be an $\mathcal{R}$-Lie group. Then G has a good Levi decomposition if:*

- *G/N is linear for some finite central N,*
- *G/R has finite o-minimal fundamental group.*

The simply connected case was treated by E. Baro in the same paper he studied derived subgroup of definably linear groups.

Fact 3.5.3 (Baro,[Bar19])**.** *Let G be a simply connected definable group and R its solvable radical. Then there is a semisimple simply connected definable subgroup S of G such that $G = R \cdot S$ and $R \cap S = \{1\}$.*

Remark 3.5.1. *We know that the Levi subgroup can be only locally definable. In [CP12] they build a definable amalgamated direct product of* $(\mathbb{R}, +)$ *and* $SL_2(\mathbb{R})$ *whose derived subgroup and Levi subgroup is* $\widetilde{SL_2(\mathbb{R})}$. *Since the Levi subgroup coincides in this case with the derived subgroup, this example shows that the conditions of Fact 3.2.20 and Fact 3.2.19 cannot be omitted.*

We finish with a nice corollary obtained from Theorem 6 that helps characterize definable Lie groups with good Levi decomposition.

Theorem 3. *Let* S *be a connected semisimple Lie group. Then* S *is Lie isomorphic to a definable group if and only if its center* $\mathcal{Z}(S)$ *is finite.*

Proof. Since connected semisimple Lie groups have discrete center, if S is definable it must have finite center. Now suppose that S has finite center, the quotient $S/Z(S)$ is centerless hence linear (it acts faithfully on its Lie algebra) and semialgebraic by Fact 5.1.1. The map $\pi : S \to S/Z(S)$ is a finite covering of $S/\mathcal{Z}(S)$. Hence by Theorem 6 it is definable. □

It follows that the definable Lie groups with definable Levi subgroups are precisely those with Levi subgroups with finite center. As we already mentioned this should probably generalize to Lie groups over arbitrary real closed field with the appropriate covering theory.

Chapter 4

Definable solvable Lie groups are definably linear

In order to answer the Question 1 we will take a closer look at the results already known when the Lie group is solvable. More precisely, in [COS18] A. Conversano, A. Onshuus and S. Starchenko give a full characterization for solvable Lie groups.

Fact 4.0.1. *[Conversano, Onshuus, Starchenko, [COS18], Theorem* 5.4*] Let R be a connected solvable Lie group. Then the following are equivalent:*

- *R has a normal, connected, torsion-free and supersolvable subgroup T such that R/T is compact (we say that R is* triangular-by-compact*).*
- *R is Lie isomorphic to a group definable in an o-minimal expansion of the reals.*

In our attempt to treat the problem we realized that we needed to understand well the implications of Fact 4.0.1. Since the proof of Fact 4.0.1 implies the semidirect condition of Fact 1.2.28 we know that definable solvable Lie groups are linear, it is natural to ask whether we can represent definably those groups.

The answer is positive and we will provide here the construction of a definable representation for those groups (*i.e.* a proof for Theorem 1).

4.1 The Extension Lemma

The general approach is to prove Lemma 1, the "Extension Lemma," and use it to extend definably a definable representation of the supersolvable part T of G which can be obtained using triangularity of T. This is essentially an adaptation of the proof of the analogous result in the real Lie group context. Particularly, our proof of the Extension Lemma essentially adapts the proof in the last chapter of [Hoc65] to the definable context.

We will strongly use the constructions of Section 1.2.4 and we will need the following lemma to ensure that the representation we build (in the proof of Theorem 1) is in a finite dimensional vector space.

Fact 4.1.1 ([Hoc65, Chap. XVIII, Lemma 2.1]). *Let G be a solvable Lie group and $\rho : G \to GL(V)$ be a faithful continuous representation in the finite-dimensional $\mathbb{R}$-vector space V. Let A be a set of automorphisms of G such that $\rho'(\alpha(x)x^{-1}) = \mathrm{Id}$ for all $x \in G$ and $\alpha \in A$. Then if $f \in S(\rho)$ is a representative function associated to ρ then the vector space generated by $\{f \circ \alpha\}_{\alpha \in A}$ is finite-dimensional.*

The following lemma is the main brick in the proof of Theorem 1.

Lemma 1 (Extension Lemma). *Let $G = K \ltimes H$ be a group definable in an o-minimal expansion of the reals with H and K definable subgroups with H normal and solvable. Suppose that H admits a faithful definable representation $\rho : H \to GL(V)$ where V is a finite-dimensional vector space over $\mathbb{R}$. Suppose moreover that ρ satisfies that for all $x \in G$ and $y \in H$ we have $\rho'(xyx^{-1}y^{-1}) = id$. Then there is a definable representation σ of G that extends ρ (and hence is faithful on H).*

Proof. Let us consider the following faithful action of H on the space $\mathcal{C}^0(H)$ of continuous functions from H to $\mathbb{R}$. For $h \in H, f \in \mathcal{C}^0(H)$ let $h{\cdot}f : x \mapsto f(xh)$.

We can extend the action to G as follows. For any $h \in H,\ k \in K$ with $g = kh$ and $f \in \mathcal{C}^0(H)$ define

$$g \cdot f : x \mapsto f(k^{-1}xkh).$$

Notice that we had to consider the right action of K on H to get a left action of G on $\mathcal{C}^0(H)$.

H acts on its space of representative functions $\mathcal{R}(H)$. We are going to show that actually G acts on $\mathcal{R}(H)$. Indeed, for $f \in \mathcal{R}(H)$, $h' \in H$ and $g = kh \in G$ with $h \in H$ and $k \in K$ then

$$h' \cdot (g \cdot f) = (h'kh) \cdot f = (kk^{-1}h'kh)) \cdot f = k \cdot ((k^{-1}h'kh) \cdot f) \in k \cdot (H \cdot f).$$

Since $(H \cdot f)$ is finite-dimensional, $g \cdot f \in \mathcal{R}(H)$.

We now use Fact 4.1.1 to find a finite dimensional subspace U of $\mathcal{R}(H)$ such that G acts on U and the restriction of this action to H is faithful. Let A

be the set of automorphisms $c_k : x \mapsto k^{-1}xk$ of H given by the conjugations in H by $k \in K$. For all $f \in \mathcal{C}^0(H)$ we have $k \cdot f = f \circ c_k$. Let us fix $f \in S(\rho)$, the hypothesis of Fact 4.1.1 are fulfilled, so that the vector space generated by the $k \cdot f = f \circ c_k$ for $k \in K$ is finite-dimensional. But we know that $S(\rho)$ is finite dimensional (it has the same dimension as $End(V)^\star$), so the vector subspace $U \leq \mathcal{R}(H)$ generated by $G \cdot S(\rho)$ has finite dimension. If $(f_1, f_2, \ldots, f_k)$ is a basis for U, by definition of U each $f_i = k_i \cdot (\varphi_i \circ \rho)$ for some $k_i \in K$ and $\varphi_i \in End(V)^\star$. Since these functions are definable and ρ is also definable we get a finite-dimensional definable representation of G but are missing faithfulness of the action on H. Now U contains $S(\rho)$ as a H-stable subspace and we know by the construction previous to this demonstration that V might be identified with a H-stable subspace in a direct sum of finite copies of $S(\rho)$. Taking W as the direct sum of the same number of copies of U yields a representation space that contains an H-stable subspace on which H acts faithfully (it is the same representation as ρ). This terminates the proof. □

4.2 Linearity implies definable linearity

We now apply the Extension Lemma to the decomposition of a solvable Lie group into its supersolvable and compact parts. In order to do so we need to find a representation of the supersolvable part where the nilradical is upper-triangular with 1's on the diagonal.

Proposition 4.2.1. *Let G be a simply connected supersolvable Lie group and N its nilradical. Then G is Lie isomorphic to a definable matrix group G_1 whose nilradical N_1 is unipotent.*

Proof. We first notice that since G is simply connected and solvable, Fact 1.2.14 tells us that the exponential map is a diffeomorphism between G and its Lie algebra $\mathfrak{g}$. As G is supersolvable by Fact 1.2.13 so is $\mathfrak{g}$; also, supersolvable Lie algebras have upper triangular representations by Fact 1.1.15. In any such representation $\text{Lie}(N) = \mathfrak{n}$ is an upper triangular nilpotent subalgebra, so it must be strictly upper triangular. The exponential map coincides with the matrix exponential, it is a finite sum on the strictly upper part and the diagonal coincides with the real exponential of the entries: it is definable in $\mathbb{R}_{\exp}$. Now the image of the exponential will be a matrix Lie group G_1 Lie-isomorphic to G whose nilradical $N_1 = \exp(\mathfrak{n})$ is unipotent, as required. □

We are finally ready to prove that definable solvable Lie groups are definably linear.

Theorem 1. *Let G be a connected, triangular by compact, solvable Lie group. Then G is Lie-isomorphic to a definable matrix group.*

Proof. Let G_1 be the definable Lie group isomorphic to G given by Fact 4.0.1, so that in particular G_1 is a definable semidirect product of a supersolvable subgroup H_1 and a compact group K_1. Take the definable representation of H_1 given by Proposition 4.2.1. This group is simply connected because it is solvable and torsion-free as in Fact 1.2.31.

In order to apply the Extension Lemma we have to check that this representation satisfies the commutator condition, and this can be checked on its Lie algebra by Fact 1.2.24. Since $[\mathfrak{g}_1, \mathfrak{h}_1]$ is nilpotent (by Fact 1.1.10) it is included in $\mathfrak{n} = Lie(N)$ (where N is the nilradical of H_1). But the representation we chose was unipotent on N so it automatically satisfies the commutator condition. Applying the Extension Lemma we get a definable representation ρ of G_1 which is faithful on H_1.

Let μ be any faithful continuous representation of K_1, which exists by the Peter-Weyl Theorem. By Fact 1.2.38, any faithful representation of a connected compact group is algebraic so $\mu(K_1)$ is algebraic, and hence definable.

The direct sum of ρ and μ will be a definable and faithful representation of G_1, as required. □

We actually just proved the following corollary.

Corollary 4.2.1. *Let R be a connected solvable definable Lie group. Then R is Lie-isomorphic to a definable matrix group.*

Chapter 5

Characterization of definable linear Lie groups

As we now understand well definability of solvable Lie groups we will continue the study towards linear Lie groups in general. Since the solvable radical of a definable Lie group is definable (see Fact 3.2.2) our strategy is to use Levi decomposition with the following in mind.

Fact 5.0.1 (Malcev, [Mal44])**.** *Let G be a connected Lie group. The G is linear if and only if its solvable radical R and Levi subgroup S are linear.*

Because it is true (see Fact 5.1.1) that any connected linear semisimple Lie group is semialgebraic we will get automatically definability of the Levi subgroup. The proof turned out to be more complicated since it was not as easy to glue the definable pieces together but the final result is:

Theorem 2. *Let G be a connected linear Lie group whose solvable radical is triangular by compact. Then G is Lie-isomorphic to a definable matrix group.*

5.1 Preliminary tools

In order to prove the criterion above we will need several tools that we present here in order to have a more straightforward proof in the end.

5.1.1 Definable representation of a quotient

Let us begin with an adaptation of a classical representation theory result to the definable context.

Lemma 2. *Let G be a definable Lie group with a definable and faithful representation over a finite dimensional vector space V. Let F be a central and finite subgroup of G. Then there is a definable and faithful representation of the quotient G/F on a finite dimensional vector space W.*

Proof. Because F is abelian and finite, it is enough to prove the lemma assuming that F has prime order since the general case will follow by induction. So we assume that F has prime order q.

We will need to extend the representation to a complex representation of G on $\widetilde{V} = V \otimes_{\mathbb{R}} \mathbb{C}$. By Maschke's Theorem for finite groups we can decompose $\widetilde{V}$ into $V_1 \oplus \cdots \oplus V_h$ where the V_i are F-stable and F-irreducible subspaces. By Schur's Lemma, the action of F on each V_i is an action by scalar multiplication, and because F has order q we know that the action of F on V_i will be multiplication by an integer power μ^{k_i} of a primitive q-root of unity μ. Fix a generator x of F. Then x acts by multiplication by μ^{k_i} on the V_i's and the vector $(k_1, \dots, k_h)$ is a non zero vector in the $(\mathbb{Z}/q\mathbb{Z})$-vector space $(\mathbb{Z}/q\mathbb{Z})^h$. The 1-dimensional subspace spanned by this vector is an intersection of $h-1$ hyperplanes. Let $\sum_{s=1}^{h} a_{r,s} x_s = 0$ for $s = 1, \dots, h-1$ be the linear equations of the hyperplanes. It follows that a vector $(x_1, \dots, x_h)$ is an integer multiple of $(k_1, \dots, k_h)$ if and only if it satisfies those equations.

Consider the complex vector space

$$W = \bigoplus_{i=1}^{h} V_i^{\otimes^q} \oplus \bigoplus_{r=1}^{h-1} V_1^{\otimes^{a_{r,1}}} \otimes \cdots \otimes V_h^{\otimes^{a_{r,1}}}$$

Each V_i is G-stable (because $g \cdot V_i$ is F-stable), so G acts on W by an action σ with F is in the kernel of σ. We are *a priori* acting on a complex vector space, but we can (and will) consider the real action on the real and imaginary components, thus obtaining a real action.

Claim 4. *The kernel σ is precisely F.*

Proof. Let g be any element in the kernel of the action. Let $v_i \in V_i$ be a non zero element such that $g \cdot v_i = \lambda v_i$ for some complex λ. Since g fixes $(v_i \otimes \cdots \otimes v_i) \in V_i^{\otimes^q}$ we get $\lambda^q = 1$ and hence $\lambda = \mu^{x_i}$. We will now show that g acts by multiplication by λ on V_i. Let $w_i \in V_i$ be linearly independent from v_i. Invariance of $(w_i \otimes v_i \otimes \cdots \otimes v_i) \in V_i^{\otimes^q}$ gives us:

$$w_i \otimes v_i \otimes \cdots \otimes v_i = \lambda^{q-1} g \cdot w_i \otimes v_i \otimes \cdots \otimes v_i$$

and by linear independence $g \cdot w_i = \lambda w_i$ so g acts by multiplication by λ on V_i.

Now fix $r \in [|1, h-1|]$. The element g acts by multiplication by μ^{x_i} on each V_i so the action of g on $V_1^{\otimes^{a_{r,1}}} \otimes \cdots \otimes V_h^{\otimes^{a_{r,1}}}$ is multiplication by μ^{K_r} where $K_r = \sum_{s=1}^{h} a_{r,s} x_s$. But the action is trivial so $K_r = 0$ for all r and $(x_1, \ldots, x_h)$ and $(x_1, \ldots, x_h) = m(k_1, \ldots, k_h)$. This implies that $g = x^m \in F$. □

Since F is the kernel of σ, the action of G/F on W is a faithful representation of G/F, as required. Notice that W is definable (any finite dimensional vector space is) and definability of ρ and V will imply definability of the natural action of G on W (since it is a finite sum of tensor products of restrictions of ρ).

□

5.1.2 Semisimple linear groups

In the same vein as Fact 1.2.38 we can deduce definable properties from semisimplicity.

Fact 5.1.1 (Peterzil, Pillay, Starchenko, [PPS02],Remark 4.4)**.** *If S is a connected semisimple matrix Lie group, then S is semialgebraic (i.e. definable in $(\mathbb{R}, 0, +, \cdot, <)$).*

Since it will be of major importance in the proof of Theorem 2 we provide a proof here.

Proof. Suppose that $S \leq GL_n(\mathbb{R})$ and let $\mathfrak{s}$ be the Lie algebra of S, it is semisimple and so is $\mathfrak{s}^{\mathbb{C}}$ its complexification. Semisimple Lie algebras are perfect, *i.e.* $[\mathfrak{s}^{\mathbb{C}}, \mathfrak{s}^{\mathbb{C}}] = \mathfrak{s}^{\mathbb{C}}$ and by Section 1.2.6 we know that $[\mathfrak{s}^{\mathbb{C}}, \mathfrak{s}^{\mathbb{C}}]$ is algebraic. Hence there is an affine algebraic group $\mathbb{H}$ defined over $\mathbb{R}$ (section 1.2.6) such that $\mathfrak{s}^{\mathbb{C}} = \text{Lie}(\mathbb{H})$ and $\mathbb{H}(\mathbb{R}) \leq GL_n(\mathbb{R})$. But then $\text{Lie}(\mathbb{H}(\mathbb{R})) = \text{Lie}(\mathbb{H}(\mathbb{R})^0) = \mathfrak{s}$ and since S is connected we must have $S = \mathbb{H}(\mathbb{R})^0$. This concludes the proof since the connected component of the real points of real algebraic group is semialgebraic. □

5.2 Classification in the linear case

As we already mentioned, the main idea behind the proof is to use the Levi decomposition and patch together Fact 4.0.1 and Fact 5.1.1. We would like to use the Extension Lemma after applying Fact 4.0.1 to the solvable radical. Unfortunately we will not be able to apply it directly to the Levi decomposition.

Theorem 2. *Let G be a connected linear Lie group whose solvable radical is triangular by compact. Then G is Lie-isomorphic to a definable matrix group.*

Proof. The proof can be understood in two parts; first we build a finer decomposition for G and then we apply the Extension Lemma and glue together the definable representations we obtained.

We know that a lot of pieces are definable and linear. The theorem's condition on the solvable radical R together with Fact 4.0.1 gives us that R is definably linear. Since G is a linear Lie group, Fact 5.0.1 tells us the any Levi subgroup must be linear and by Fact 5.1.1 it is semialgebraic.

We would like to invoke the Extension Lemma but the decomposition of G into its solvable radical R and a Levi subgroup S is actually not fine enough. We will need to refine the decomposition. Consider the decomposition of R into a simply connected (torsion free) normal subgroup T and a compact subgroup K as in Fact 1.2.28.

We first show that S can be chosen such that S and K commute making SK a subgroup of G. Consider the adjoint action of K on $\mathfrak{g} = \mathrm{Lie}(G)$, this lets us see $\mathfrak{g}$ as a semisimple K-module (since K is compact). Hence $\mathfrak{g}$ is a direct sum of simple K-modules $\mathfrak{g}_i$. Let us write $\mathfrak{k} = Lie(K)$, since $[\mathfrak{k}, \mathfrak{g}_\mathrm{i}]$ is a K-stable subspace we have either $[\mathfrak{k}, \mathfrak{g}_\mathrm{i}] = \{0\}$ either $[\mathfrak{k}, \mathfrak{g}] = \mathfrak{g}_i$. It follows that $\mathfrak{g} = [\mathfrak{k}, \mathfrak{g}] \oplus \mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ where $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ is centralizer the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$. Write the Levi decomposition for $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ as a semidirect sum $\mathfrak{z}_\mathfrak{g}(\mathfrak{k}) = \mathfrak{u}_R + \mathfrak{u}_S$ where $\mathfrak{u}_R$ is the solvable radical and $\mathfrak{u}_S$ is a maximal semisimple subalgebra. But $\mathfrak{r} + \mathfrak{u}_R$ is a solvable ideal of $\mathfrak{g}$ (here $\mathfrak{r} = Lie(R)$) and since $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$ we must have $\mathfrak{r} + \mathfrak{u}_R \subseteq \mathfrak{r}$ so $\mathfrak{u}_R \subseteq \mathfrak{r}$. So the Levi decomposition of $\mathfrak{g}$ can be written $\mathfrak{g} = \mathfrak{r} + \mathfrak{u}_S$ with $\mathfrak{u}_S$ is a maximal semisimple subalgebra. Let S_1 be the connected subgroup of G corresponding to the Lie algebra $\mathfrak{u}_S$. It is also a Levi factor (maximal semisimple subgroup) of G so it is in fact a conjugate of S. Since $[\mathfrak{u}_S, \mathfrak{k}] = (0)$ we get that S_1 commutes with K. By setting $S = S_1$ we may assume that S commutes with K and SK is a group.

Claim 5. *G is a semidirect product of T and $H := SK$*

Proof. We already know that $G = (SK)T$ so we are left to show that $H \cap T$ is trivial. Take an element $sk = t$ in the intersection with $s \in S$, $k \in K$ and $t \in T$. Then $s = tk^{-1} \in S \cap R$ which is finite. So we have finitely many choices for s and since $T \cap K$ is trivial, any fixed s determines exactly the possible choices of the pair (t, k). This shows that $H \cap T$ is finite. Since T has no torsion, the intersection must be trivial. □

We now use Proposition 4.2.1 to find a representation ρ of T whose image is definable. We extend the representation using [Hoc65, Chap. XVII, Theorem 2.2] (the Lie-version of the Extension Lemma) and obtain a representation $\widetilde{\rho}$ of G that is faithful on T and such that the image of T is definable. This is because $\widetilde{\rho}$ can be seen as an extension of ρ as mentioned in Section 1.2.4.

Both $\widetilde{\rho}(S)$ and $\widetilde{\rho}(K)$ are semialgebraic (by Fact 5.1.1 and Fact 1.2.38), so the full image of $\widetilde{\rho}$ is definable.

Notice that $\widetilde{\rho}$ is not necessarily faithful. To complete the proof we need a faithful and definable representation of $H = SK$. Since S and K commute we have $H \simeq (S \times K)/\Delta$ where $\Delta := \{(x, x) \colon x \in S \cap K\}$. Notice that $S \cap K \subseteq S \cap R$ hence it is central in H and finite. Since both K and S have definable and faithful representations ρ_K and ρ_S respectively, $\sigma = \rho_K \oplus \rho_S$ is a definable and faithful representation of $S \times K$. By Lemma 2 there is a definable representation of $K \times S$ on a definable space W, with kernel F. This is a definable faithful representation $\widetilde{\sigma}$ of H.

Now $G = T \rtimes H$ and H is Lie-isomorphic to G/T. Hence, we can use the quotient map to get a definable representation of G that is faithful on H; abusing notation, we still refer to this representation as $\widetilde{\sigma}$. Taking the direct sum of $\widetilde{\rho}$ and $\widetilde{\sigma}$ we get a faithful representation of G whose image is definable. □

In particular, for definable Lie groups, linearity and linear definability are the same.

Corollary 5.2.1. *Let G be a connected Lie group Lie-isomorphic to a definable group. Then G is linear if and only if it is Lie-isomorphic to a definable matrix group.*

This criterion and its proof also tell us that definable linear Lie groups must be definable in $\mathbb{R}_{exp}$.

Theorem 7. *Let G be a linear Lie group, the following are equivalent:*

- *The solvable radical of G is triangular by compact.*
- *G is Lie isomorphic to a group definable in an o-minimal expansion of $(\mathbb{R}, 0, +, 1, \cdot, <)$.*
- *G is Lie isomorphic to a matrix group definable in an o-minimal expansion of $(\mathbb{R}, 0, +, 1, \cdot, <)$.*
- *G is Lie isomorphic to a group definable in $\mathbb{R}_{\mathrm{exp}}$.*

- *G is Lie isomorphic to matrix group definable in* $\mathbb{R}_{\text{exp}}$.

Proof. We will prove that any definable linear Lie group is actually definable in $\mathbb{R}_{exp}$. To see this take a linear Lie group G satisfying the first condition. By Theorem 2 we get a matrix group G_1 that is definable and Lie-isomorphic to G. In the proof of Theorem 2 we obtained linearity and definability analysing G_1 into a semidirect product $(SK) \ltimes T$ where S was semialgebraic (it is semisimple and linear), K algebraic (it is compact and linear) and T definable in $\mathbb{R}_{exp}$ (it is the matrix exponential of an upper-triangular matrix). So G_1 as a set is definable in $\mathbb{R}_{exp}$; since the group law is given by matrix multiplication, it is polynomial in its coordinates, hence also definable in $\mathbb{R}_{exp}$. We notice that once we achieve linearity of the solvable radical $R = T \rtimes K$ of G we do not need the analytic functions which were needed in [COS18], since the compact part K acts by matrix multiplication on the supersolvable part T.

The other implications were proved in Theorem 2 and in [COS18]. □

Chapter 6

Lie groups with good Levi subgroups

In the proof of Theorem 2, things articulate well because of the definability of the Levi subgroup. As mentioned in Remark 3.5.1 there are definable (non linear) Lie groups that have non definable Levi subgroups. We proved in Theorem 3 that it is equivalent for the Levi subgroup to have finite center and to be definable. This incited us to take a look at the case where we don't assume linearity of the Lie group but assume finiteness of the center of the Levi subgroup. We prove the following.

Theorem 4. *Let G be a connected Lie group. Let R be the solvable radical of G and S a Levi subgroup of G. If S has finite center and R is triangular by compact then G is Lie-isomorphic to a definable group.*

Proof. Let G be a group satisfying the hypothesis. By Fact 4.0.1 and the linear case we may assume that the solvable radical R is Lie isomorphic to R_{def}, a definable matrix solvable group. Moreover we have the additional property that if we decompose $R_{\mathrm{def}} = T \rtimes K \subseteq GL_n(\mathbb{R})$ with T supersolvable and K compact, if $\mathfrak{r} = \mathrm{Lie}(R_{\mathrm{def}})$, $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{k} = \mathrm{Lie}(K)$ then $\mathfrak{r} = \mathfrak{t} + \mathfrak{k} \subseteq \mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{t}$ is upper triangular. Since S has finite center we know by the Levi decomposition that G is the almost semidirect product of R and S. Specifically, there is a morphism φ' such that

$$\begin{array}{rccc} \Psi : & R \rtimes_{\varphi'} S & \to & G \\ & (r, s) & \mapsto & r \cdot s \end{array}$$

is a Lie morphism with finite kernel.

In order to build a definable version of G we will need to construct a definable morphism $\varphi' : S \to \mathrm{Aut}(R_{\mathrm{def}})$. To do so we need the "appropriate" version of S.

Define the linearizer $\Lambda(G)$ of a connected lie group G as the intersection of the kernels of all its continuous finite dimensional representations. It is a closed and normal subgroup and as the adjoint representation is continuous it is central. It is a theorem of Goto ([HM57][Theorem 7.1]) that $G/\Lambda(G)$ has a faithful representation. That makes $\Lambda(G)$ the smallest normal closed subgroup P such that G/P is Lie isomorphic to a linear group.

In our case S has finite center and we do not need to invoke Goto's theorem, we just need the following lemma:

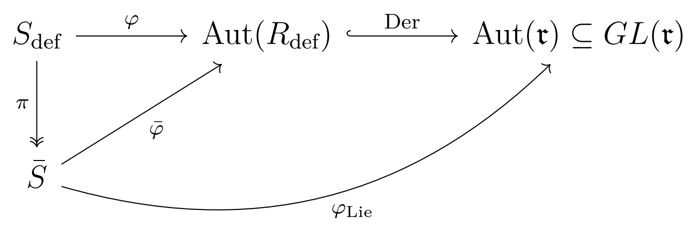
Lemma 3. *Let G be a Lie group and suppose that G/H_1 and G/H_2 (with H_1 and H_2 normal and closed subgroups) both have faithful representations ρ_1 and ρ_2. Then $G/(H_1 \cap H_2)$ has a faithful representation.*

Proof. Let us write $\widetilde{\rho_i} := \pi_i \circ \rho_i$ where $\pi_i : G \twoheadrightarrow G/H_i$ is the canocical projection. The direct sum $\widetilde{\rho_1} \oplus \widetilde{\rho_2}$ is a representation of G whose kernel is $H_1 \cap H_2$. □

Since S has finite center, $\Lambda(S) \subseteq Z(S)$, and $S/Z(S)$ is centerless and therefore linear, applying the lemma finitely many times gives us a faithful representation of $S/\Lambda(S)$. Let $\bar{S}$ be a matrix copy of $S/\Lambda(S)$.

$\bar{S}$ is linear and semisimple, hence semialgebraic. As S is a finite cover of $\bar{S}$, by Theorem 6 there is a group S_{def} that is definable, Lie isomorphic to S and a definable surjection $\pi : S_{\text{def}} \twoheadrightarrow \bar{S}$. Given that $\bar{S}$ is the biggest linear quotient of S any representation of S must factor through $\bar{S}$.

Finally, consider the action of S_{def} on R_{def} induced by the isomorphism from S to S_{def}. This is a morphism $\varphi : S_{\text{def}} \to \text{Aut}(R_{\text{def}}) \subseteq \text{Aut}(\mathfrak{r}_{\text{def}}) \subseteq \text{GL}(\mathfrak{r}_{\text{def}})$. This must therefore factor through $\bar{S}$ and we have the following commutating diagram:



The morphism φ_{Lie} maps a semisimple linear Lie group to a linear group, its graph Γ is Lie isomorphic to $\bar{S}$, so it must be semialgebraic (recall that any linear semisimple Lie group is semialgebraic by Fact 5.1.1).

We will prove that the action of $\bar{S}$ on R_{def} is definable. Let us write $R_{\text{def}} = T \rtimes K \subseteq GL_n(\mathbb{R})$ with T the maximal normal connected torsion-free subgroup of R_{def} and K a maximal compact subgroup. Because the maximal torsion-free subgroup T of R_{def} is characteristic it must be stable under the action. Recall that our construction of T implies that the Lie algebra $\mathfrak{t}$ is supersolvable and that the exponential restricted to $\mathfrak{t}$ is a definable diffeomorphism. For any $s \in \bar{S}$ and $t \in T$, we can choose $X \in \mathfrak{t}$ such that $\exp(X) = t$ and we have

$$\bar{\varphi}(s)(t) = \exp\restriction_{\mathfrak{t}} (\varphi_{Lie}(s)(X))$$

which implies that the restriction of the action to T is definable.

Understanding definability of $\bar{\varphi}$ on the maximal compact subgroup is a little harder.

Claim 6. *Let $\mathcal{K}$ be the set of maximal compact subgroups of R_{def} and $\mathbf{K} := R_{def}/T$. Then the following hold:*

(i) The natural action of $\bar{S}$ on $\mathbf{K}$ is well defined and trivial.

(ii) $\mathcal{K} = \{tK_0t^{-1} : t \in T\}$

(iii) The set $\mathcal{A} := \{(s,t) \in \bar{S} \times T : \bar{\varphi}(s)(K) = tKt^{-1}\}$ is definable.

(iv) The restriction of $\bar{\varphi}$ on K is definable.

Proof.

(i) Any automorphism of R_{def} must fix T which implies that $\bar{S}$ acts by continuous automorphisms on $\mathbf{K}$ in a natural way; this can be seen as a continuous map ψ from $\bar{S}$ to $\text{Aut}(\mathbf{K})$. Since $\mathbf{K}$ is a torus, $\text{Aut}(K) \simeq GL_m(\mathbb{Z})$ for some $m \in \mathbb{N}$ and it is discrete.[1] But $\bar{S}$ is connected, so ψ must be constant on $\text{Aut}(\mathbf{K})$, and the action is trivial.

(ii) It is known that all maximal tori of a linear group are conjugate, so $\mathcal{K}$ is the set of conjugates by elements of R_{def} of K_0. Since $R_{\text{def}} = TK$ we get that $\mathcal{K} = \{tKt^{-1} : t \in T\}$.

(iii) We begin by picking an open neighborhood $\mathfrak{u}_\circ$ of 0 in $\mathfrak{r}$ such that $\exp : \mathfrak{u}_\circ \to R_{\text{def}}$ is a diffeomorphism. If needed, we shrink $\mathfrak{u}_\circ$ such that we obtain a local diffeomorphism $\exp : \mathfrak{u} \to U$ with $\mathfrak{u}$ bounded and U open neighborhood of the identity. This restriction is an analytic function defined on a compact subset of $\mathfrak{r}$, and is therefore definable in

[1] $\mathbf{K} \simeq \mathbb{R}^m/\mathbb{Z}^m$ so any automorphism of $\mathbf{K}$ is an automorphism of $\mathbb{R}^m$ that leave $\mathbb{Z}^m$ fixed.

the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. Now we claim that $\bar{\varphi}_s(K) = tKT^{-1}$ if and only if

$$\exp(\mathfrak{u} \cap \varphi_{\text{Lie}}(s)(\mathfrak{k})) = U \cap tK_0t^{-1}.$$

The equivalence follows because if we define $\mathfrak{k}_t := \text{Lie}(K_t)$ with $K_t := t{\cdot}K_0{\cdot}t^{-1}$, then by the Lie correspondence any two distinct maximal compact subgroups K_{t_1} and K_{t_2} correspond to different Lie sub-algebras $\mathfrak{k}_{t_1}$ and $\mathfrak{k}_{t_2}$ of $\mathfrak{r}$. But Lie subalgebras are vector subspaces and $\mathfrak{u}$ is an open neighborhood of the identity, so $\mathfrak{k}_{t_1} \neq \mathfrak{k}_{t_2}$ if and only if $(\mathfrak{k}_{t_1} \cap \mathfrak{u}) \neq (\mathfrak{k}_{t_2} \cap \mathfrak{u})$ as required.

Since the latter is definable $\mathcal{A}$ is definable.

(iv) Let us pick $s \in \bar{S}$ and $k \in K$. Then for any $r \in R_{\text{def}}$ we have $\bar{\varphi}_s(k) = r$ if and only there is a $t \in T$ such that $(s,t) \in \mathcal{A}$ and $y = tkt^{-1} \in tKt^{-1}$.

□

Definability of $\bar{\varphi}$ now follows easily. Let $s \in \bar{S}$ and $r \in R_{\text{def}}$. Then we can write $r = t \cdot k$ with $t \in T$ and $k \in K$. Then we get:

$$\bar{\varphi}_s(r) := \bar{\varphi}_s(t) \cdot \bar{\varphi}_s(k)$$

so it is definable in $\mathbb{R}_{\text{an,exp}}$. Definability of $\varphi = \varphi_{\text{lin}} \circ \pi$ follows. This concludes the proof as we can set

$$G_{\text{def}} := (R_{\text{def}} \rtimes_\varphi S_{\text{def}})\,/F$$

with F a finite subgroup isomorphic to $R \cap S$ and the group operation given by

$$(r_1, s_1) \cdot (r_2, s_2) = (r_1 \varphi_{s_1}(r_2), s_1 s_2).$$

□

Remark 6.0.1. *We mention here two remarks from E. Baro and K. Peterzil that shorten and improve the proof.*

First Notice that inside G, the action of S on R is given by conjugation, hence the center $\mathcal{Z}(S)$ is contained in the kernel of φ. This means that we can factor φ through $S/\mathcal{Z}(S)$ and it is not needed to go through the linearizer of S to factor the action since $S/\mathcal{Z}(S)$ is centerless and hence linear.

Also notice that the only place we used analytic functions was to understand how $\bar{S}$ moved the compact subgroups of R_{def}. Using techniques from [PPS00a] one can understand definably this action without using analytical functions: the family of conjugates of K is a definable family of matrix groups and one can look directly at their associated Lie algebra.

This gives us the nicer following statement:

Theorem 8. *Let G be a connected Lie group whose Levi subgroups have finite center. The following are equivalent.*

- *The solvable radical R of G is triangular-by-compact.*
- *G is Lie-isomorphic to a group definable in o-minimal expansion of $\mathbb{R}$.*
- *G is Lie isomorphic to a group definable in $\mathbb{R}_{\exp}$.*

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