The Slack Model in the Study of Polytopes

THESIS
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Abstract

In this thesis we present some of the applications of studying polytopes via their slack matrices and slack ideals. We prove that McMullen's operations on polytopes, which includes the join, the vertex sum, the vertex splitting and their dual operations, preserve graphicality in the same way it was known it preserves projective uniqueness. We use this result to identify a large class of projectively unique order polytopes; namely, we prove that every ranked finite poset with no 3-antichain has a graphic and thus projectively unique order polytope. We give a complete characterization of complex psd-minimal polygons; we prove that the complex psd-minimal polygons are precisely the triangles, the quadrialterals and a special class of hexagons which we call Pappus' hexagons. Using this, it can be seen that complex psd-minimal 3-polytopes must have vertices of degree 3, 4 or 6 and facets with 3, 4 or 6 sides. We identify all the combinatorial classes of 3-polytopes with a vertex of degree 6 or a hexagonal facet that has a complex psd-minimal realization. We do the same for combinatorial classes of 3-polytopes with at most 7 vertices or facets. It is well known that the matching polytope is equal to the fractional matching polytope for bipartite graphs, and we prove the analogous result for k-matching polytopes. We also prove that k-matching polytopes of bipartite graphs are normal, that is, every integer point in its n-dilate is the sum of n integers points of the original polytope, generalizing the fact that Birkhoff polytopes are normal.

Keywords: Birkhoff polytopes, matching polytopes, matchings, McMullen's operations, normal polytopes, order polytopes, projectively unique polytopes, psd-minimal polytopes, slack ideals, slack matrices, spectrahedral lifts

Resumen en español: En esta tesis presentamos algunas aplicaciones de estudiar politopos a través de sus matrices de holgura e ideales de holgura. Demostramos que las operaciones de McMullen sobre politopos, las cuales incluyen la unión, la suma sobre un vértice, la ruptura de un vértice y sus operaciones duales, preservan la graficalidad de la misma manera que se sabía preservaban la unicidad projectiva. Usamos este resultado para identificar una clase grande de politopos de orden proyectivamente únicos; de manera más precisa, demostramos que todo poset finito con rango y sin 3-anticadenas tiene un politopo de orden gráfico y por ende proyectivamente único. Damos una caracterización completa de los polígonos psd-minimales complejos; demostramos que los polígonos psd-minimales complejos son precisamente los triángulos, los cuadriláteros y una clase especial de hexágonos que llamamos hexágonos de Pappus. Usando esto podemos ver que los 3-politopos psd-minimales complejos deben tener vértices de grado 3, 4 o 6 y facetas con 3, 4 o 6 lados. Identificamos todas las clases combinatoriales de 3-politopos con un vértice de grado 6 o una faceta hexagonal que tienen una realización psd-minimal compleja. Hacemos lo mismo para las clases combinatoriales de 3-politopos con a lo sumo 7 vértices o facetas. Es bien conocido que el politopo de emparejamientos es igual al politopo fraccional de emparejamientos para grafos bipartitos, y demostramos el resultado análogo para politopos de k-emparejamientos. También demostramos que los politopos de k-emparejamientos de grafos bipartitos son normales, es decir, cada punto entero en su n-dilatación es la suma de n puntos enteros del politopo original, generalizando así el hecho de que los politopos de Birkhoff son normales.

Palabras clave: emparejamientos, ideales de holgura, levantamientos espectrales, matrices de holgura, operaciones de McMullen, politopos de Birkhoff, politopos de emparejamientos, politopos de orden, politopos normales, politopos poyectivamente únicos, politopos psd-minimales

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Contents

Introduction

In recent years there have been important advances in the study of polytopes by studying their slack matrices and the related concept of slack ideal introduced in [12] and further developed in [9] and [10]. These objects encode the combinatorial structure of the polytope and they are useful for the study of its realizations. For example, they are useful for the study of the realization space of polytopes up to projective or affine transformations, to determine if a polytope is projectively unique, if it has a rational realization, or even if there exists a polytope with a given combinatorial type. By studying polytopes through slack ideals, we can use techniques coming from computational commutative algebra and algebraic geometry.

We can also apply this approach in optimization. When we have to do linear optimization over a polytope, it is useful to ask if we can find a lift of that polytope over which we can optimize faster. For example if we can find a polyhedral lift that projects onto the original with fewer facets, then optimization over the lift is preferable. It is also useful to ask for spectrahedral lifts; this is because semidefinite optimization, which optimizes linear funtions over spectrahedra, finds an optimal solution in polynomial time up to a given precision. In [11], these approaches are unified via the notion of a K-lift, where K is a closed convex cone (a polyhedron is an affine slice of a positive orthant, and a spectrahedron is an affine slice of the cone of positive-semidefinite matrices of a given size). Here the existence of a K-lift can be determined by whether the slack matrix of the polytope can be factored in a certain way.

Related to spectrahedral lifts is the concept of (real) psd-minimality. A spectrahedron is constructed from positive-semidefinite matrices of a given size, which determine the size of the spetrahedron. A natural question is how small a spectahedral lift could be in terms of the dimension d of the polytope. It happens that the size of any spectrahedral lift is at least $d + 1$. Those polytopes with a spectrahedral lift of size $d + 1$ are called psd-minimal. In [13] and [12], psd-minimal polytopes up to dimension 4 are completely characterized using the slack model approach.

This vast range of applications makes this approach to polytopes of special interest for researchers in combinatorics, algebra and optimization. In this thesis we explore new applications of what we call the *slack model* for representing polytopes. After a first chapter of preliminaries, where we introduce the notation and the background, we devote four chapters to new applications and the last one to the related topic of matching polytopes.

In Chapter 2, motivated by the work of McMullen in [19] in which he determined under what conditions the operations of dualization, join, vertex sum and vertex splitting preserve projective uniqueness among polytopes, we do the same for graphicality. Graphic polytopes are a specific group of polytopes for which its slack ideal is equal to a certain toric ideal, and whose properties can be studied from the bipartite graph of non-incidences between the vertices and facets of the polytope. Graphicality is a sufficient condition for projective uniqueness.

In Chapter 3, we apply the results in Chapter 2 to order polytopes. This is a class of interesting polytopes constructed from finite posets, whose vertices and facets are easy to describe in terms of properties of the poset. Here we see how common operations in the posets correspond to (McMullen's) operations in their order polytopes which will help us to identify a large class of graphic order polytopes. More precisely, we will prove that every ranked finite poset with no 3-antichain has a graphic and thus projectively unique order polytope.

Chapter 2 and Chapter 3, along with the sections in Chapter 1 needed to understand them, are in essence our article [3] with minor changes done in order to fit the overall structure of the thesis.

Chapter 4 is devoted to complex psd-minimality. Here we use complex positive-semidefinite matrices to construct our spectrahedral lifts. Again the size of the spectrahedron can be defined in terms of the size of the matrices we use to constructed it. Since we allow complex matrices, the size of the minimal lifts is never greater than the real case. Again it happens that $d + 1$ is a lower bound for the smallest size of a spectrahedral lift of a d-polytope, and the complex psd-minimal ones are those who have a spectrahedral lift of this size. What we know about complex psd-minimality is less than in the real case. In [7], it was proved that pentagons are not complex psd-minimal but the regular hexagon is (the only real psd-minimal polygons are the triangles and quadrilaterals). This gives an example of a polygon that is complex psd-minimal but not real psd-minimal. It was not even known what happens with polygons with more than six vertices. In our work we provided a complete characterization of complex psd-minimal polygons. Namely, we prove that the complex psd-minimal polygons are precisely the triangles, the quadrilaterals, and a special group of hexagons which we call Pappus hexagons. We also present a complex psd-minimal lift for the regular hexagon, which to our knowledge had not previously been constructed explicitly.

In Chapter 5, we ask the next natural question: which are the complex psd-minimal 3-polytopes? Although we do not present a complete answer to this question, we make substantial progress. From our result on polygons, it can be derived that complex psd-minimal 3-polytopes must have only vertices of degree 3, 4 or 6 and facets with 3, 4 or 6 sides. We characterize those combinatorial classes of 3-polytopes with at a vertex of degree 6 or a hexagonal facet that have a complex psd-minimal realization. We also identify the combinatorial classes of 3-polytopes with at most 7 vertices or 7 facets which have a complex psd-minimal realization. It might appear that there is still an infinite amount of work to do, but we know that 3-polytopes with sufficiently many vertices and facets cannot be complex psd-minimal since there is only finitely many of them. When we pass from the world of real psd-minimal polygons to that of complex psd-minimal polygons, the only new habitants are the Pappus hexagons. In contrast, many new cases appear in the 3-dimensional case which makes the characterization of complex psd-minimal 3-polytopes a substantially more difficult endeavor.

In the last chapter, Chapter 6, we make a small combinatorial epilogue. In it we will study a variation of the well-known matching polytope of a graph. Recall that the matching polytope is the convex hull of the incidence vectors of all matching of the graph. We considered the k-matching polytope of a graph which is the one obtained if we restrict to matchings of size k. Analogous to what happens in the case of the matching polytope, for bipartite graphs we prove that the k-matching polytope is equal to the k-fractional polytope, so it has a nice H-representation. The Birkhoff polytopes can be seen as n-matching polytopes of the complete bipartite graphs $\mathrm{K}_{n,n}$. It is known that the Birkhoff polytope is normal, that is, every integer point in its n-dilate is the sum of n integer points in the original. We will extend this result and prove that every k-matching polytope of a bipartite graph is normal.

Our study of matching polytopes originated from the fact that the slack ideal of a d-polytope is constructed from the $(d+2)$-minors of what is called the symbolic slack matrix, and the monomials that appear in this minors correspond to $(d+2)$-matchings of the bipartite graph constructed from the non-incidences between vertices and facets. There is also a connection of normality with toric algebra, and lifts on matching polytopes are studied in [15] and [23].

The richness of studying polytopes via their slack representations is not only reflected in the aforementioned applications and results, including the ones presented in this thesis, but also in the open problems and new research directions it presents. We hope the readers of this thesis will gain an interest in these topics and appreciate the potential for future research.

Chapter 1

Preliminaries

In this chapter we will introduce the notation and the background required to understand the material presented in the next chapters.

1.1 Notation

As usual in combinatorics, for a positive integer n, $[n] := \{1, \dots, n\}$. The sets of nonnegative and positive real numbers are denoted as $\mathbb{R}_+$ and $\mathbb{R}_{++}$, respectively. Points in $\mathbb{R}^k$ are thought as column vectors, where $\mathbf{0}$ and $\mathbf{1}$ are, respectively, the all-zeros and the all-ones column vectors. For the all-zeros and all-ones row vectors we use the notations $\mathbf{0}^T$ and $\mathbf{1}^T$, respectively. The sizes of these vectors will be clear from the context. The set of real matrices of size $m \times n$ is denoted by $\mathbb{R}^{m \times n}$, and the zero matrix is denoted by O. A positive diagonal matrix is a real diagonal matrix with positive entries in the diagonal. Finally, we will use $\preceq$ for the order relation of an arbitrary finite poset and the symbol $\prec$ for the cover relation.

The set of $k \times k$ real symmetric matrices is denoted by $\mathcal{S}^k$, and the set of $k \times k$ Hermitian matrices is denoted by $\mathcal{H}^k$. The subsets of $k \times k$ real and complex positive semidefinite matrices are denoted respectively by $\mathcal{S}^k_+$ and $\mathcal{H}^k_+$. As is standard, $A \succeq 0$ will indicate that the real (complex) matrix A is a real (complex) positive semidefinite matrix. Although the notation is the same as the one used for the order relation of a poset, the context will make clear which one we are using.

1.2 Projective uniqueness and McMullen's operations

When studying polytopes we usually do not want to consider a specific geometric realization of a polytope, but instead some equivalence class that preserves the properties we are interested in. We begin by recalling three different types of equivalences between polytopes.

Definition 1.2.1. Let $P, Q \subseteq \mathbb{R}^d$ be two full-dimensional polytopes.

1. We say that P and Q are *combinatorially equivalent* if their face lattices are isomorphic as posets.

2. We say that P and Q are *projectively equivalent* if there is a projective transformation $\phi : \mathbb{R}^d \dashrightarrow \mathbb{R}^d$ such that $\phi(P) = Q$.

3. We say that P and Q are *affinely equivalent* if there is an affine transformation $\psi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\psi(P) = Q$.

Recall that a projective transformation $\phi : \mathbb{R}^d \dashrightarrow \mathbb{R}^d$ is defined as

$$\phi(t) = \frac{A\mathbf{t} + \mathbf{b}}{\mathbf{c}^T\mathbf{t} + \gamma}$$

where $A \in \mathbb{R}^{d\times d}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$, and $\gamma \in \mathbb{R}$ with

$$\det \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^T & \gamma \end{bmatrix} \neq 0.$$

An affine transformation $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined as $\psi(\mathbf{t}) = A\mathbf{t} + \mathbf{b}$ where $A \in \mathbb{R}^{d\times d}$ and $\mathbf{b} \in \mathbb{R}^d$. If $Q = \psi(P)$, then A must be invertible due to the full-dimensionality of P and Q.

Observation. For $P, Q \subseteq \mathbb{R}^d$ full-dimensional polytopes,

$$\text{affine equivalence} \Rightarrow \text{projective equivalence} \Rightarrow \text{combinatorial equivalence.}$$

When studying geometric realizations of polytopes, projective transformations are the largest canonical class of maps from $\mathbb{R}^d$ to $\mathbb{R}^d$ that preserve the combinatorics of a polytope, so it is natural to consider realizations of polytopes up to projective equivalence. Occasionally, there is only one such realization.

Definition 1.2.2. We say that a full-dimensional polytope P is *projectively unique* if any full-dimensional polytope that is combinatorially equivalent to P is also projectively equivalent to P.

Example 1.2.3. The case of polygons illustrates the distinction between these three notions of equivalence. Any pair of triangles (or more generally, d-simplices) are affinely and thus projectively equivalent. Since affine transformations preserve parallel lines, a quadrilateral Q is affinely equivalent to the square if and only Q is a parallelogram. However, all quadrilaterals are projectively equivalent; that is, the square is projectively unique. For $m \geq 5$, the m-gon is not even projectively unique.

One of the biggest problems in the study of projectively unique polytopes is that we have few ways of constructing new examples. One of the most well-known ways, and the one we will apply in this paper, is to use certain operations proposed by McMullen [19] that preserve projective uniqueness. The simplest operation we will consider is taking the dual of a polytope.

Definition 1.2.4. Two polytopes P and P^* are *duals* of each other if their face lattices are antisomorphic, that is, if there is an order-reversing bijection between these lattices.

Note that a polytope $P \subseteq \mathbb{R}^d$ containing the origin in its interior and its polar $P^\circ := \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1, \text{ for all } \mathbf{y} \in P\}$ are dual to each other. It can be shown that the dual of a projectively unique polytope is projectively unique. Apart from the dual, McMullen considers three additional constructions.

Definition 1.2.5. Let P and Q be polytopes of respective dimensions d and e.

1. Let $\hat{P}$ and $\hat{Q}$ be embeddings of P and Q in $\mathbb{R}^{d+e+1}$ with nonintersecting affine spans whose underlying linear spaces intersect trivially. We define the *join* of P and Q to be $P \vee Q := \operatorname{conv}(\hat{P} \cup \hat{Q})$.

2. Let $\mathbf{v}$ and $\mathbf{w}$ be vertices of P and Q, respectively. Let $\hat{P}$ and $\hat{Q}$ be embeddings of P and Q in $\mathbb{R}^{d+e}$ whose affine spans intersect in a single point $\mathbf{p}$ which is the image of both $\mathbf{v}$ and $\mathbf{w}$. Then $P \oplus_{(\mathbf{v},\mathbf{w})} Q := \operatorname{conv}(\hat{P} \cup \hat{Q})$ is called the *vertex sum* of P and Q along the pair $(\mathbf{v}, \mathbf{w})$. We will also write $P \oplus_{\mathbf{p}} Q := P \oplus_{(\mathbf{v},\mathbf{w})} Q$.

3. Let $\mathbf{p}$ be a vertex of P. The polytope

$$P_{\mathbf{p}} := \operatorname{conv}(\{(\mathbf{w}, 0) : \mathbf{w} \in \operatorname{Vert}(P), \mathbf{w} \neq \mathbf{p}\} \cup \{(\mathbf{p}, 1), (\mathbf{p}, -1)\})$$

   is called the *vertex split* of P along $\mathbf{p}$.

Note that we are sometimes identifying the constructions with specific embeddings for brevity of exposition, although we are interested in the combinatorial equivalence classes of these constructions. The combinatorial structure of all of these constructions is well known.

Observation. Let P and Q be polytopes with dimensions d and e (appropriately embedded, depending on the operation), vertex sets V and W, and facet sets $\mathcal{F}$ and $\mathcal{G}$ respectively. We then have the following structure.

Polytope	dimension	vertex set	facet set
$P \vee Q$	$d+e+1$	$V \cup W$	$\{F \vee Q : F \in \mathcal{F}\}$ $\cup$ $\{P \vee G : G \in \mathcal{G}\}$
$P \oplus_{\mathbf{p}} Q$	$d+e$	$(V \setminus \mathbf{p})$ $\cup$ $(W \setminus \mathbf{p})$ $\cup$ $\{\mathbf{p}\}$	$\{F \oplus_{\mathbf{p}} Q : \mathbf{p} \in F \in \mathcal{F}\}$ $\cup$ $\{P \oplus_{\mathbf{p}} G : \mathbf{p} \in G \in \mathcal{G}\}$ $\cup$ $\{F \vee G : \mathbf{p} \notin F \in \mathcal{F}, \mathbf{p} \notin G \in \mathcal{G}\}$
$P_{\mathbf{p}}$	$d+1$	$(V \setminus \mathbf{p}) \cup \{\widehat{\mathbf{p}}, \overline{\mathbf{p}}\}$	$\{\operatorname{conv}(F \cup \{\widehat{\mathbf{p}}, \overline{\mathbf{p}}\}) : \mathbf{p} \in F \in \mathcal{F}\}$ $\cup$ $\{\operatorname{conv}(F \cup \{\widehat{\mathbf{p}}\}) : \mathbf{p} \notin F \in \mathcal{F}\}$ $\cup$ $\{\operatorname{conv}(F \cup \{\overline{\mathbf{p}}\}) : \mathbf{p} \notin F \in \mathcal{F}\}$

Note that $\widehat{\mathbf{p}}$ and $\overline{\mathbf{p}}$ denote $(\mathbf{p}, -1)$ and $(\mathbf{p}, 1)$.

In [19], McMullen shows that these three operations preserve projective uniqueness, under certain mild conditions for the vertex splitting operation.

Theorem 1.2.6 ([19]). *Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ be two full-dimensional polytopes. Then:*

1. *$P \vee Q$ is projectively unique if and only P and Q are projectively unique,*

2. *if P and Q are projectively unique, then so is $P \oplus_{(\mathbf{v},\mathbf{w})} Q$ for any vertices $\mathbf{v}$ of P and $\mathbf{w}$ of Q, and*

3. *if P is projectively unique and is not the vertex sum of two polytopes at $\mathbf{p}$, then $P_{\mathbf{p}}$ is projectively unique.*

These operations are enough to construct from direct sums of simplices (easily shown to be projectively unique) all the 11 known projectively unique 4-polytopes, but necessarily cannot produce all projectively unique polytopes, as they generate only a finite list of d-dimensional examples for every fixed d. In fact they are not enough to generate even all projectively unique 5-polytopes from the lower dimensional ones [32, Theorem 4.4.1]. Nevertheless, they are a very useful tool to construct new examples from existing ones.

For some of the objects we will be studying, it will be useful to introduce the dual operations to vertex splitting and vertex sum that we will call, respectively, *facet wedging* and *facet product*. We will start by defining the facet wedge.

Definition 1.2.7. Let $P \subseteq \mathbb{R}^d$ be a full-dimensional polytope and F be a facet of P. Then the polytope

$$P_F := \operatorname{conv}(\{(\mathbf{v}, 0) : \mathbf{v} \in \operatorname{Vert}(P)\} \cup \{(\mathbf{v}, \operatorname{dist}(\mathbf{v}, L)) : \mathbf{v} \in \operatorname{Vert}(P)\})$$

where L is the affine space spanned by the facet F is called the *facet wedge* of P along F.

Note that if P and P^* are dual, then the vertex split of P at $\mathbf{p}$ is dual to the facet wedge of P^* along the facet F that is dual to $\mathbf{p}$. Thus we can translate Theorem 1.2.6 (3) into a result about facet wedges.

We now turn our attention to facet products. This is a special case of a *subdirect product*, introduced in [19] as a dual to the *subdirect sum*, an operation that generalizes vertex sum.

Definition 1.2.8. Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ be two full-dimensional polytopes and $\hat{F}$ and $\bar{F}$ facets of P and Q, respectively. We define the facet product of P and Q with respect to $\hat{F}$ and $\bar{F}$, which we denote by $P \otimes_F Q$, as $(P^* \oplus_{\mathbf{p}} Q^*)^*$ where the vertex sum is with respect to the vertices of P^* and Q^* that are dual to $\hat{F}$ and $\bar{F}$, respectively.

In practical terms, a precise geometric description of this polytope is not needed, as we are mostly concerned with its combinatorial structure.

1.3 Slack matrices and slack ideals

Gouveia, Pashkovich, Robinson, and Thomas [12] introduced the notion of the slack ideal of a polytope in order to study its positive semidefinite lifts. The first and last of these authors, along with Macchia and Wiebe [9, 10] then applied slack ideals to give an algebraic criterion for projective uniqueness. We now review the key definitions and results from these papers that form the starting point of our own work.

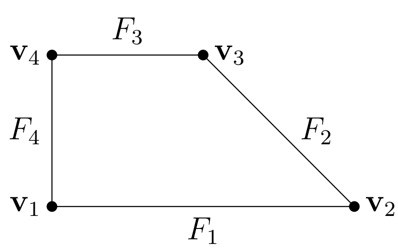
Definition 1.3.1. Let $P \subseteq \mathbb{R}^d$ be a full-dimensional polytope with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and facets $F_1, \ldots, F_m$. Then $P = \{\mathbf{t} \in \mathbb{R}^d : A\mathbf{t} + \mathbf{b} \geq \mathbf{0}\}$ for some $A = [a_{ij}]_{m\times d} \in \mathbb{R}^{m\times d}$, $\mathbf{b} \in \mathbb{R}^m$, and such that $F_i = \{\mathbf{t} \in P : a_{i1}t_1 + \cdots + a_{id}t_d + b_i = 0\}$ for all $i = 1, \ldots, m$. If $h_i(t) := a_{i1}t_1 + \cdots + a_{id}t_d + b_i$ for $i = 1, \ldots, m$, then the matrix defined as $[h_i(\mathbf{v}_j)]_{m\times n}$ is called a *slack matrix* of P. If we take a slack matrix of P and replace each non-zero entry with a distinct variable, we obtain the *symbolic slack matrix* of P.

Observations.

- Slack matrices and the symbolic slack matrix of a polytope depend on the ordering of the vertices and the facets. So whenever we talk about these matrices, we will implicitly fix orderings on the vertices and the facets.

- If we scale h_i by a positive real number, we still obtain the same embedded polytope P. Thus if S is a given slack matrix of P, then so is DS, where $D \in \mathbb{R}^{m\times m}$ is a positive diagonal matrix. Furthermore, all of the slack matrices of P are of this form.

Example 1.3.2. Consider the polytope $P := \text{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ where $\mathbf{v}_1 = (0, 0)$, $\mathbf{v}_2 = (2, 0)$, $\mathbf{v}_3 = (1, 1)$ and $\mathbf{v}_4 = (0, 1)$. Its facets are $F_1 = [\mathbf{v}_1, \mathbf{v}_2]$, $F_2 = [\mathbf{v}_2, \mathbf{v}_3]$, $F_3 = [\mathbf{v}_3, \mathbf{v}_4]$ and $F_4 = [\mathbf{v}_4, \mathbf{v}_1]$.



Facets F_1, F_2, F_3 and F_4 are defined, respectively, by the inequalities

$$
\begin{aligned}
t_2 &\geq 0, \\
t_1 + t_2 &\leq 2, \\
t_2 &\leq 1, \\
t_1 &\geq 0.
\end{aligned}
$$

By taking

$$\begin{aligned} h_1(t_1,t_2) &:= t_2, \\ h_2(t_1,t_2) &:= 2 - t_1 - t_2, \\ h_3(t_1,t_2) &:= 1 - t_2, \\ h_4(t_1,t_2) &:= t_1, \end{aligned}$$

we obtain the slack matrix

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}.$$

Thus, the symbolic slack matrix is

$$S_P(\mathbf{x}) = \begin{bmatrix} 0 & 0 & x_1 & x_2 \\ x_3 & 0 & 0 & x_4 \\ x_5 & x_6 & 0 & 0 \\ 0 & x_7 & x_8 & 0 \end{bmatrix}.$$

It is not hard to see that any slack matrix of a d-dimensional polytope has rank $d+1$. It is also not hard to check that this is the minimum rank of any matrix with the same support.

Lemma 1.3.3 ([9])**.** *If $S_P(x_1,\dots,x_k)$ is the symbolic slack matrix of a d-polytope P and $\boldsymbol{\zeta} \in (\mathbb{C}^*)^k$, then* $\operatorname{rank} S_P(\boldsymbol{\zeta}) \geq d+1$.

In fact the rank plays a very important role in characterizing slack matrices.

Definition 1.3.4. Given a polytope P, we call any slack matrix of a polytope combinatorially equivalent to P a *true slack matrix* of P. Any matrix that can be obtained from a true slack matrix by scaling columns by positive scalars is called a *generalized slack matrix* of P.

Theorem 1.3.5 ([8], [9])**.** *Let P be a d-polytope with symbolic slack matrix $S_P(x_1,\dots,x_k)$, and $\boldsymbol{\alpha} \in \mathbb{R}^k_{++}$. Then*

1. *$S_P(\boldsymbol{\alpha})$ is a generalized slack matrix of P if and only if* $\operatorname{rank} S_P(\boldsymbol{\alpha}) = d+1$
2. *$S_P(\boldsymbol{\alpha})$ is a true slack matrix of P if and only if* $\operatorname{rank} S_P(\boldsymbol{\alpha}) = d+1$ *and $\mathbf{1}^T$ belongs to the row span of $S_P(\boldsymbol{\alpha})$.*

Generalized slack matrices have a more natural description than true slack matrices. Moreover, scaling rows and columns is a natural thing to do when studying polytopes up to projective equivalence. Two polytopes P and Q are projectively equivalent if $S_P = DS_QD'$, where S_P and S_Q are slack matrices of P and Q and D and D' are positive diagonal matrices. We can use this to give a characterization for projective uniqueness.

Theorem 1.3.6 ([9]). *Let P be a full-dimensional polytope.*

1. *There is a bijection between true slack matrices up to row scaling by positive scalars with realizations of P up to affine equivalences.*

2. *There is a bijection between generalized slack matrices up to row and column scaling by positive scalars with realizations of P up to projective equivalences.*

In particular, P is projectively unique if and only if P has only one generalized slack matrix up to row and column scaling by positive scalars.

We can extract from this geometric picture an algebraic version. To do that, for any matrix $M(\mathbf{x})$ of constants and variables and any natural number e, denote the determinantal ideal of e-minors of $M(\mathbf{x})$ by $\text{Minors}_e\left(M(\mathbf{x})\right) := \langle e - \text{minors of } M(x)\rangle$. Also, if J is an ideal of $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_k]$, we define $J : \left(\prod_{i=1}^{k} x_i\right)^\infty$ to be the ideal generated by all polynomials f for which a monomial multiple of f lies in J. This ideal is called the *saturation* of J with respect to all variables. The condition on the rank of a slack matrix suggests consideration of the following ideal.

Definition 1.3.7 ([12]). Let P be a d-polytope with symbolic slack matrix $S_P(\mathbf{x}) = S_P(x_1, \ldots, x_k)$. We define the *slack ideal* of P as

$$I_P := \text{Minors}_{d+2}\left(S_P(\mathbf{x})\right) : \left(\prod_{i=1}^{k} x_i\right)^\infty .$$

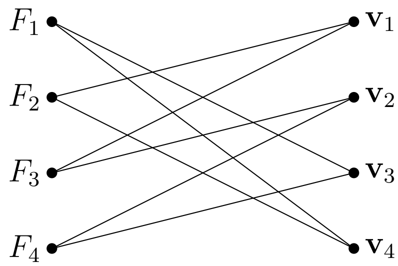
Observation. Saturation by the product of all variables removes common factors from the terms of a polynomial. More precisely, $\mathbf{m}\,f \in I$ for some monomial $\mathbf{m}$ if and only if $f \in I : (x_1x_2\cdots x_k)^\infty$. A small observation is that this also implies that if $\mathbf{m}\,f \in I : (x_1x_2\cdots x_k)^\infty$ for some monomial $\mathbf{m}$, then $f \in I : (x_1x_2\cdots x_k)^\infty$. This property will be used several times later on.

Geometrically, saturating an ideal J has the effect of removing components of the variety $V(J)$ that are contained entirely in a coordinate hyperplane $x_i = 0$. This is a sensible operation for the *slack variety* $V(I_P)$ because each variable represents the distance from a vertex $\mathbf{v}$ to a facet that does not contain $\mathbf{v}$.

1.4 Graphic polytopes

In [10], a second ideal associated to a polytope, T_P, is introduced in order to study projective uniqueness from an algebraic point of view. This new ideal is toric. To introduce the ideal, we first define the *non-incidence graph* of a polytope P. This is the bipartite graph $\mathbf{G}_P$ on $\text{Facets}(P) \sqcup \text{Vert}(P)$ with an edge connecting facet F with vertex $\mathbf{v}$ if and only if $\mathbf{v} \notin F$. The edges of $\mathbf{G}_P$ are thus labeled by the variables that appear in the symbolic slack matrix of P.

Example 1.4.1. Consider the polytope P from Example 1.3.2. Its non-incidence graph $\mathbf{G}_P$ is



To every collection C of oriented edges in this graph we can associate a binomial in the following way:

- let $x_1, \ldots, x_n$ be the labels (variables) of all edges of C that, according to the orientation, go from Facets(P) to Vert(P)
- let $y_1, \ldots, y_m$ be the labels (variables) of all edges of C that, according to the orientation, go from Vert(P) to Facets(P).

The binomial associated to C is $f_C := x_1 \cdots x_n - y_1 \cdots y_m$. If C is a simple cycle, we implicitly suppose its edges are oriented in order to form a directed cycle. So we can talk of the binomial f_C associated to a simple cycle, which is unique up to sign.

Definition 1.4.2. Let P be a polytope. Then

$$T_P := \langle f_C : C \text{ is a chordless cycle of } \mathbf{G}_P \rangle.$$

This is in fact the toric ideal associated to the vertex-edge incidence matrix of $\mathbf{G}_P$ (see [10]). We recall that the toric ideal associated to a $d \times k$ integer matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_k$ is $\ker \pi$ where

$$\pi : \mathbb{C}[x_1, \ldots, x_k] \longrightarrow \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$$

is the $\mathbb{C}$-algebra homomorphism such that $x_i \mapsto \mathbf{t}^{\mathbf{a}_i}$. Also T_P is generated by the binomials of all oriented cycles, a consequence of the Cycle-Splitting Lemma (Lemma 2.1.2) which will be introduced in Section 2.1.

In general there is no obvious relationship between the ideals I_P and T_P. However there is an important special case in which they are indeed related: that of 2-level polytopes.

Definition 1.4.3. A polytope P is *2-level* if for every facet F of P, the vertices of P that are not in F are all contained in a single parallel translate of $\operatorname{aff}(F)$. Equivalently, P is 2-level if $S_P(\mathbf{1})$ is a slack matrix of P. We will say that P is *morally 2-level* if $S_P(\mathbf{1})$ is a generalized slack matrix of P.

Moral 2-levelness, unlike 2-levelness, is a combinatorial property. The 3-dimensional bisimplex is an example of a polytope that is morally 2-level but not 2-level. To see that the 3-dimensional bisimplex is morally 2-level, note that its dual is the triangular prism which has 2-level realizations.

Theorem 1.4.4 ([10]). *Let P be a full-dimensional polytope.*

1. *P is morally 2-level if and only if $I_P \subseteq T_P$.*

2. *If $I_P = T_P$, then P is projectively unique.*

Definition 1.4.5. If $I_P = T_P$, we say that P has a *graphic slack ideal* and that P is a *graphic polytope*.

Theorem 1.4.4 implies that graphic polytopes are a subset of both morally 2-level and projectively unique polytopes. Since in any given dimension the number of morally 2-level polytopes must be finite (see [1, §6]), so is the number of graphic polytopes, hence this must be a much more restrictive condition than simply being projectively unique. However, all the 11 known examples of 4-dimensional projectively unique polytopes are indeed graphic.

Since graphic polytopes are morally 2-level, they have rational realizations. The Perles polytope (see [14]), which does not have rational realizations, is an example of a polytope that is projectively unique but not graphic.

1.5 Order polytopes

As mentioned above there are many known classes of 2-level polytopes for which one might be able to apply Theorem 1.4.4. We will focus on the following combinatorially appealing class of 2-level polytopes introduced by Stanley [24].

Definition 1.5.1. Let $\mathcal{P} = ([d], \preceq)$ be a poset. The *order polytope* of $\mathcal{P}$ is

$$\mathcal{O}(\mathcal{P}) := \{\mathbf{t} \in \mathbb{R}^d : 0 \le t_i \le 1 \text{ for all } i \in [d], \text{ and } t_i \le t_j \text{ if } i \preceq j\}.$$

In Chapter 3, we will apply our results in Chapter 2 to present a large family of order polytopes that are graphic and thus projectively unique. In our study of order polytopes we will use the following definitions related to posets.

Definition 1.5.2. Let $\mathcal{P} = ([d], \preceq)$ be a poset.

1. A subset of $[d]$ is a *chain* (respectively *antichain*) if its elements are pairwise comparable (respectively pairwise incomparable.)

2. A subset J of $[d]$ is called a *filter* of $\mathcal{P}$ if whenever $x \in J$ and $y \succeq x$, then $y \in J$. If $S \subseteq [d]$, the set $(S) := \{x \in [d] : x \succeq s \text{ for some } s \in S\}$ is called the *filter generated by* S.

3. The function $\chi_S^{\mathcal{P}} := [d] \to \{0, 1\}$ given by $f(x) = 1$ if $x \in S$ and $f(x) = 0$ if $x \notin S$ is called the *characteristic function* of S. When the poset we are working with is fixed, we denote $\chi_S^{\mathcal{P}}$ simply by χ_S. We can also think of the characteristic function as a vector.

4. The poset $\mathcal{P}$ is *ranked* if there is a function $\rho : [d] \to \mathbb{N}$ such that $\rho(y) = \rho(x) + 1$ for all cover relations $x \prec y$ in $\mathcal{P}$. (We do not insist that ρ be unique.) The *rank* of a finite ranked poset is defined as the maximum length of a maximal chain.

There is a bijective correspondence between filters and antichains of $\mathcal{P}$. Given a filter, take its minimal elements to obtain an antichain, and given an antichain, take the filter generated by this antichain.

Theorem 1.5.3 ([24]). *Let $\mathcal{P} = ([d], \preceq)$ be a poset. Then $\mathcal{O}(\mathcal{P}) \subseteq \mathbb{R}^d$ is a full-dimensional polytope. Its vertices are precisely the characteristic vectors χ_J where J is a filter. Thus $\mathcal{O}(\mathcal{P})$ is a 0/1-polytope. The facets are the following sets*

- $\{\mathbf{t} \in \mathcal{O}(\mathcal{P}) : t_i = 0\}$ *where i is a minimal element of $\mathcal{P}$,*
- $\{\mathbf{t} \in \mathcal{O}(\mathcal{P}) : t_i = t_j\}$ *where $i \prec j$, and*
- $\{\mathbf{t} \in \mathcal{O}(\mathcal{P}) : t_i = 1\}$ *where i is a maximal element of $\mathcal{P}$.*

In other words, vertices are given by filters (or antichains) and the facets by covers, minimal and maximal elements of $\mathcal{P}$. Each facet of $\mathcal{O}(\mathcal{P})$ is defined by an inequality of the form $t_i \geq 0$, $t_i \leq t_j$, or $t_i \leq 1$ depending if the facet comes from a minimal element, a cover, or a maximal element. The notation $F : t_i \geq 0$ will mean that the facet F comes from the minimal element i. Analogous notations will be used for the other types of facets. Using the above inequalities and the fact that order polytopes are 0/1-polytopes, we can see they have 0/1-slack matrices, that is, they are 2-level.

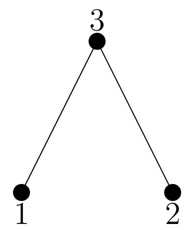
Example 1.5.4.

1. The empty poset has exactly one antichain: the empty set. Thus its order polytope is a single point.

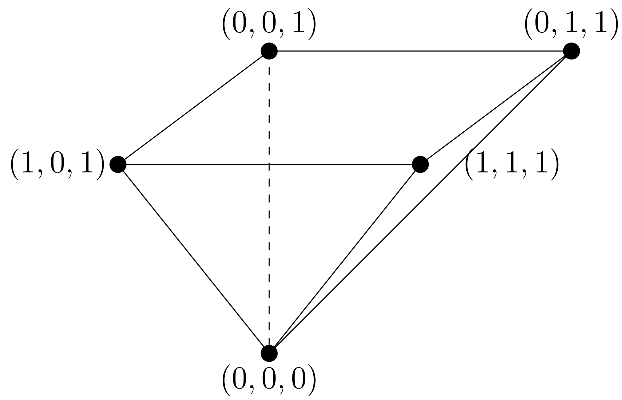
2. If $\mathcal{P}$ is the chain $1 \prec 2 \prec \ldots \prec d-1 \prec d$, then $\mathcal{O}(\mathcal{P}) = \{\mathbf{t} \in \mathbb{R}^d : 0 \leq t_1 \leq t_2 \leq \ldots \leq t_{d-1} \leq t_d \leq 1\}$; that is, $\mathcal{O}(\mathcal{P})$ is a d-simplex.

3. If $\mathcal{P}$ is an antichain with d elements, then $\mathcal{O}(\mathcal{P}) = \{\mathbf{t} \in \mathbb{R}^d : 0 \leq t_i \leq 1 \text{ for all } 1 \leq i \leq d\}$; that is, $\mathcal{O}(\mathcal{P})$ is a d-cube.

Example 1.5.5.



$\mathcal{P}$



$\mathcal{O}(\mathcal{P})$

1.6 Spectrahedral lifts

In optimization, it is useful to ask if a given convex set is the image under a linear map of a geometric object over which is more efficient to optimize a linear function than over the original convex set. The geometric objects usually considered are affine slices of closed convex cones. Among these, polyhedra (linear optimization) and spectrahedra (semidefinite optimization) are of special interest. They are obtained as affine slices of the positive orthants $\mathbb{R}^k_+$ and affine slices of the cones of $k \times k$ real positive semidefinite matrices $\mathcal{S}^k_+$, respectively.

Definition 1.6.1 ([11])**.** Let $C \in \mathbb{R}^n$ be a convex set and $K \subseteq \mathbb{R}^m$ a full-dimensional closed convex cone. We say that $Q \subseteq \mathbb{R}^m$ is a *K-lift* of C if there is an affine space $L \in \mathbb{R}^m$ and a linear transformation $\pi : \mathbb{R}^m \to \mathbb{R}^n$ such that $Q = K \cap L$ and $\pi(Q) = C$. If L intersects the interior of K, we say that Q is a *proper K-lift* of C.

Of course it is not always true that C has a K-lift. Before presenting necessary and sufficient conditions for this to happen, let us recall some concepts about convexity. The extreme points $\text{ext}(C)$ of a convex set C are by definition all the points in C that cannot be expressed as a convex combination of other two points in C, and the polar of C is the set

$$C^\circ := \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1,\ \forall \mathbf{y} \in C\}.$$

We also recall the dual of a cone K, which is defined as

$$K^* := \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0,\ \forall \mathbf{y} \in K\}.$$

The cones we are interested in, $\mathbb{R}^k_+$, $\mathcal{S}^k_+$ are self-dual. In the first case, the lifts are called *linear lifts*; in the second case, they are called *positive semidefinite lifts*.

Definition 1.6.2 ([11])**.** Let $C \in \mathbb{R}^n$ be a convex set and $K \subseteq \mathbb{R}^m$ a full-dimensional closed convex cone. The *slack operator* of C is the function $S_C : \text{ext}(C^\circ) \times \text{ext}(C) \to \mathbb{R}$ defined by $S_C(\mathbf{x}, \mathbf{y}) = 1 - \langle \mathbf{x}, \mathbf{y} \rangle$. We say that S_C is *K-factorizable* if there exist functions $A : \text{ext}(C^\circ) \to K^*$ and $B : \text{ext}(C) \to K$ such that $S_C(\mathbf{x}, \mathbf{y}) = \langle A(\mathbf{x}), B(\mathbf{y}) \rangle$ for all $(\mathbf{x}, \mathbf{y}) \in \text{ext}(C^\circ) \times \text{ext}(C)$.

Theorem 1.6.3 ([11])**.** *If C has a proper lift, then S_C is K-factorizable. Conversely, if S_C is K-factorizable, then C has a K-lift.*

Readers who want to go deeper in the topic of K-lifts are reffered to [11]. For example, the condition in Theorem 1.6.3 that the lift has to be proper can be dropped if we consider the cones $\mathbb{R}^k_+$ or $\mathcal{S}^k_+$.

Now let us translate the concepts presented above to the case we are interested in, where $C = P$ is a polytope. In this case P° is also a polytope. We also have $\text{ext}(P) = \text{Vert}(P)$, $\text{ext}(P^\circ) = \text{Vert}(P^\circ)$ and

$$P = \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1, \ \forall \mathbf{y} \in P^\circ\} = \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1, \ \forall \mathbf{y} \in \text{Vert}(P^\circ)\}.$$

For each $\mathbf{y} \in \text{Vert}(P^\circ)$, the inequality $1 - \langle \mathbf{x}, \mathbf{y} \rangle \geq 0$ gives rise to a facet of P. Since vertices of P° are in correspondence with facets of P, the matrix $[S_C(\mathbf{x}, \mathbf{y})]$ in $\mathbb{R}^{\text{Vert}(P^\circ) \times \text{Vert}(P)}$ is a slack matrix of P. Thus, Theorem 1.6.3 can be reformulated in terms of slack matrices instead of the slack operator.

Definition 1.6.4 ([11])**.** Let $M \in \mathbb{R}^{m \times n}$ be a nonnegative matrix and K a closed convex cone. A K-factorization of M is a pair of ordered sets $a_1, \ldots, a_m \in K^*$ and $b_1, \ldots, b_n \in K$ such that $M_{ij} = \langle a_i, b_j \rangle$.

Theorem 1.6.5 ([11])**.** *If a full-dimensional polytope P has a proper K-lift, then every slack matrix of P admits a K-factorization. Conversely, if some slack matrix of P has a K-factorization, then P has a K-lift.*

The process of constructing a K-lift from a K-factorization is illustrated in Section 4.2. We end this section by relating the concepts presented above with spectrahedral lifts of polytopes.

Definition 1.6.6. A *spectrahedron* is a set of the form

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

where $A_0, A_1, \ldots, A_n \in \mathbb{R}^{k \times k}$ are real symmetric matrices of the same size.

Definition 1.6.7. A *spectrahedral lift* of a polytope P is a spectrahedron S for which there is a linear transformation π such that $\pi(S) = P$.

Spectrahedra are related to affine slices of the cones $\mathcal{S}^k_+$. If $L = A_0 + \text{Span}\{A_1, \ldots, A_n\}$, where $A_1, \ldots, A_n$ are linearly independent in $\mathcal{S}^k$, then

$$\mathcal{S}^k_+ \cap L = \{A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0 : \mathbf{x} \in \mathbb{R}^k\}$$

is affinely equivalent to the spectrahedron

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}.$$

If $A_1, \ldots, A_n$ are not linearly independent, take a basis $\{B_1, \ldots, B_m\}$ of its span. For $\mathbf{x} \in \mathbb{R}^n$, we can find a unique $\mathbf{y} \in \mathbb{R}^m$ such that $A_0 + x_1 A_1 + \ldots + x_n A_n = A_0 + y_1 B_1 + \ldots + y_m B_m$, and this mapping give us a way to affinely transform the spectrahedron

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

onto the spectrahedron

$$\{\mathbf{y} \in \mathbb{R}^m : A_0 + y_1 B_1 + \ldots + y_m B_m \succeq 0\}.$$

In conclusion, from a $\mathcal{S}^k_+$-lift for some $k \in \mathbb{Z}^+$ we can obtain a spectrahedral lift and vice versa. Not strangely, sometimes spectrahedra are defined as affine slices of the cones $\mathcal{S}^k_+$, and both definitions are used interchangeably.

For more about spectrahedral lifts, we recommend the survey [29] which also touches the topics in the next section.

1.7 Psd-minimal polytopes

Sometimes, given a polytope P and a family of cones $\{K_k : k \in \mathbb{Z}^+\}$, like $\{\mathbb{R}^k_+ : k \in \mathbb{Z}^+\}$ or $\{\mathcal{S}^k_+ : k \in \mathbb{Z}^+\}$, we are interested in finding the smallest k for which there is a K_k-lift of P.

Definition 1.7.1. The *positive semidefinite rank*, or *psd-rank* for short, of a polytope P is the smallest k for which there is a $\mathcal{S}^k_+$-lift of P. Alternatively, it is the smallest k for which there are real symmetric matrices $A_0, A_1, \ldots, A_n \in \mathbb{R}^{k\times k}$ such that

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

is a spectrahedral lift of P. The psd-rank of the polytope P is denoted by $\mathrm{rank}_{\mathrm{psd}}(P)$.

It happens that $\mathrm{rank}_{\mathrm{psd}}(P)$ of a d-dimensional polytope is bounded below by $d+1$, and those for which this inequality holds are our objects of interest in this section.

Proposition 1.7.2 ([13])**.** *If P is a d-polytope, then* $\mathrm{rank}_{\mathrm{psd}}(P) \geq d+1$.

Definition 1.7.3. Let P be a d-polytope. If $\mathrm{rank}_{\mathrm{psd}}(P) = d+1$, we say that P is *psd-minimal*.

Definition 1.7.4. Let P be a d-polytope and S a spectrahedral lift of P. We say that S is a *psd-minimal lift* of P if

$$S = \{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

for some real symmetric matrices $A_0, A_1, \ldots, A_n \in \mathbb{R}^{(d+1)\times(d+1)}$.

Roughly speaking psd-minimal polytopes are those with a spetrahedral lift as small as possible, and they are the perfect candidates for translating linear programs on them into semidefinite programs on their psd-minimal lift. The study of psd-minimal polytopes began in [13], where some basic results are proved and a full characterization of 2- and 3-dimensional psd-minimal polytopes is given. In [12], the 4-dimensional polytopes that are psd-minimal are characterized. The problem of characterization in higher dimensions is open. For more about the psd-rank beyond psd-minimality, we recommend [5].

The problem of determining if a given polytope is psd-minimal can be reduced to knowing if the slack matrix of the polytope can be factored in a precise way. Since the existence of this factorization is not affected if we positively scale its rows and columns, the following definitions are useful.

Definition 1.7.5. Let S be a slack matrix of a polytope, then a *positive scaling* of S is any matrix of the form $D_1 S D_2$ where D_1 and D_2 are positive diagonal matrices (of the appropriate size). The set of positive scalings of slack matrices of P will be denoted by $\mathbf{SC}_P$.

Sometimes it is also helpful to scale the symbolic slack matrix in order to fix some variables to 1 and to relabel the remaining variables. With this, we can reduce variables and thus simplify calculations. We illustrate this by obtaining an *scaled symbolic slack matrix* for the hexagon.

Example 1.7.6. The symbolic slack matrix of the hexagon is

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array} \\ \begin{bmatrix} 0 & 0 & z_1 & z_2 & z_3 & z_4 \\ z_5 & 0 & 0 & z_6 & z_7 & z_8 \\ z_9 & z_{10} & 0 & 0 & z_{11} & z_{12} \\ z_{13} & z_{14} & z_{15} & 0 & 0 & z_{16} \\ z_{17} & z_{18} & z_{19} & z_{20} & 0 & 0 \\ 0 & z_{21} & z_{22} & z_{23} & z_{24} & 0 \end{bmatrix} \end{array},$$

which can be scale to the matrix

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & x_1 & x_2 & 1 \\ 1 & 0 & 0 & 1 & x_3 & x_4 \\ x_5 & 1 & 0 & 0 & 1 & x_6 \\ 1 & x_7 & 1 & 0 & 0 & x_8 \\ x_9 & 1 & x_{10} & 1 & 0 & 0 \\ 0 & x_{11} & x_{12} & x_{13} & 1 & 0 \end{bmatrix} \end{array}.$$

We have scaled the symbolic slack matrix by fixing to 1 the following entries:

1. entry $(F_6, \mathbf{v}_5)$ by scaling column $\mathbf{v}_5$
2. entry $(F_3, \mathbf{v}_5)$ by scaling row F_3
3. entry $(F_3, \mathbf{v}_2)$ by scaling column $\mathbf{v}_2$
4. entry $(F_5, \mathbf{v}_2)$ by scaling row F_5
5. entry $(F_5, \mathbf{v}_4)$ by scaling column $\mathbf{v}_4$
6. entry $(F_2, \mathbf{v}_4)$ by scaling row F_2
7. entry $(F_2, \mathbf{v}_1)$ by scaling column $\mathbf{v}_1$
8. entry $(F_4, \mathbf{v}_1)$ by scaling row F_4
9. entry $(F_4, \mathbf{v}_3)$ by scaling column $\mathbf{v}_3$
10. entry $(F_1, \mathbf{v}_3)$ by scaling row F_1
11. entry $(F_1, \mathbf{v}_6)$ by scaling column $\mathbf{v}_6$.

Notice that each step does not damage the already fixed 1's, and thus for every slack matrix S of a hexagon, we can find positive diagonal matrices D_1 and D_2 such that $D_1 S D_2 = S_P(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha} \in \mathbb{R}^{13}_{++}$.

We can use the observation made in the previous example to give a definition of a scaled slack matrix.

Definition 1.7.7. Let P be a polytope, and let $S_P(x_1 \dots, x_k)$ be a matrix with the same zero pattern of the symbolic slack matrix of P. If for every slack matrix S of P, we can find positive diagonal matrices D_1 and D_2 such that $D_1 S D_2 = S_P(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha} \in \mathbb{R}^{k}_{++}$, then we say that $S_P(x_1 \dots, x_k)$ is a *scaled slack matrix* of P.

Observation. As is noted in [10], we can scale the rows and columns of the symbolic slack matrix of a polytope P so that it has ones in the entries indexed by the edges in a maximal spanning forest of the graph $\mathbf{G}_P$. In some cases the rank restriction of the slack matrix may fix some of the remaining entries as is illustrated by the next example.

Example 1.7.8. The symbolic slack matrix of any quadrilateral Q is of the form

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array} \\ \begin{bmatrix} 0 & 0 & z_1 & z_2 \\ z_3 & 0 & 0 & z_4 \\ z_5 & z_6 & 0 & 0 \\ 0 & z_7 & z_8 & 0 \end{bmatrix} \end{array},$$

which can be scaled to the matrix

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & x \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Notice that there is a 1 in the entries indexed by the edges in a maximal spanning forest of $\mathbf{G}_Q$. Since the rank of a slack matrix of Q is 3, and $x - 1$ is a 4-minor of the above matrix, any slack matrix of Q scaled as above must end up with a 1 in position $(F_1, \mathbf{v}_4)$. Thus we can also consider the matrix

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

as a scaled slack matrix. This is not strange since quadrilaterals are projectively unique. The symbolic slack matrix of a quadrilateral will appear when we work with 3-dimensional polytopes with a vertex of degree 4, and the observation made in this example will be useful in Chapter 5.

We recall that the Hadamard product of two matrices of the same size, and denoted with $\odot$, is the element-wise product of the two matrices.

Theorem 1.7.9 ([13])**.** *Let P be a d-polytope, and let $S \in \mathbf{SC}_P$. Then P is psd-minimal if and only if there is a real matrix M of rank $d+1$ such that $S = M \odot M$.*

Since 2-level polytopes are those with a 0/1 slack matrix, the above result implies that 2-level polytopes are psd-minimal. The d-polytopes with $d+1$ vertices are simplices, which are 2-level and thus psd-minimal. It also happens that d-polytopes with $d+2$ vertices are psd-minimal.

Proposition 1.7.10 ([13])**.** *Every d-polytope with $d+2$ vertices or facets is psd-minimal.*

The following result will help us to find (or rule out) psd-minimal polytopes of a given dimension, when we know which ones are psd-minimal in lower dimensions.

Proposition 1.7.11 ([13])**.** *A psd-minimal polytope has psd-minimal faces.*

With the appropriate ordering of vertices and facets, if P an Q are projectively equivalent, then $\mathbf{SC}_P = \mathbf{SC}_Q$. Also if P and P^* are dual of each other, then matrices in $\mathbf{SC}_{P^*}$ correspond to the transposes of the matrices in $\mathbf{SC}_P$. We have then the following results about psd-minimality.

Proposition 1.7.12. *psd-minimality is preserved by projective equivalence.*

Proposition 1.7.13. *If P is a psd-minimal polytope, then there is a dual polytope of P that is also psd-minimal.*

Since we are thinking of duality as a combinatorial operation, the result is not very specific about which geometric dual is psd-minimal, and we do not need this specificity. Nonetheless, it can be seen that if P is psd-minimal and $\mathbf{0} \in \operatorname{int}(P)$, then $P^\circ := \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1,\ \forall \mathbf{y} \in P\}$ is also psd-minimal.

We remark that for a given dimension d and positive integer k, there are only finitely many d-polytopes with psd-rank k (see Corollary 4 in [11]). In particular, there are only finitely many psd-minimal d-polytopes for a given dimension d.

Proposition 1.7.14 ([13],[12])**.** *The psd-minimal polygons are the triangles and the quadrilaterals. The psd-minimal 3-polytopes are the simplices, bisimplices, quadrilateral pyramids, biplanar octahedra (octahedra such that there are two different planes containing 4 vertices each) and their duals.*

1.8 Matchings and matching polytopes

In this section, we recall some basic concepts and results about matchings, which have been of special interest in graph theory and combinatorial optimization.

Definition 1.8.1. A *matching* M of a graph $G = (V, E)$ is a collection of edges of G such that no two different edges in M are incident to a common vertex. If $|M| = k$, we say that M is a *k-matching*, and the matching is *perfect* if every vertex of G belongs to an edge in M. The function $\chi(M) \in \mathbb{R}^E$ defined by

$$\chi(e) = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}$$

is called the characteristic function of M.

Given a graph, we can construct a polytope that encodes its matching structure.

Definition 1.8.2. Let G be a graph, then the polytope

$$\mathrm{M}(G) := \mathrm{conv}\{\chi(M) : M \text{ is a matching of } G\}$$

in $\mathbb{R}^E$ is called the *matching polytope* of G.

Associated to this construction is the fractional matching polytope, which is a natural linear relaxation of the matching polytope.

Definition 1.8.3. Let G be a graph, then the polytope

$$\mathrm{FM}(G) := \{\mathbf{x} \in \mathbb{R}^E : \sum_{e \ni v} x_e \leq 1 \ \ \forall v \in V,\ x_e \geq 0 \ \ \forall e \in E\}$$

in $\mathbb{R}^E$ is called the *fractional matching polytope* of G.

In general the matching polytope and the fractional matching polytope are not equal. Nonetheless they are equal for bipartite graphs.

Theorem 1.8.4. *If G is a bipartite graph, then* $\mathrm{M}(G) = \mathrm{FM}(G)$.

So for bipartite graphs, we know the V-representation and a simple H-representation for the matching polytope. Another important result for matchings in bipartite graphs is Hall's Marriage Theorem.

Theorem 1.8.5 (Hall's Marriage Theorem)**.** *Let G be a bipartite graph on $A \sqcup B$. Then G has a matching that covers A if and only if for every $S \subseteq A$, the set of neighbors of S (the set of vertices that are connected with at least one vertex in S), denoted by $\Gamma(S)$, satisfies $|\Gamma(S)| \geq |S|$.*

Also, we will need the following well known result about the union of two matchings.

Lemma 1.8.6. *Let G be a bipartite graph on $A \sqcup B$, $V_1 \subseteq A$, and $V_2 \subseteq B$. If M_1 is a matching that covers V_1 and M_2 a matching that covers V_2, then $M_1 \cup M_2$ contains a matching that covers $V_1 \cup V_2$.*

We can also construct polytopes from matchings of a given size, which are less well-studied for k other than the maximum possible size.

Definition 1.8.7. Let G be a graph and $k \in \mathbb{N}$. The polytope

$$\mathrm{M}_k(G) := \mathrm{conv}\{\chi(M) : M \text{ is a } k\text{-matching of } G\}$$

in $\mathbb{R}^E$ is called the *k-matching polytope* of G, and the polytope

$$\mathrm{FM}_k(G) := \{\mathbf{x} \in \mathbb{R}^E : \sum_{e \ni v} x_e \leq 1 \ \ \forall v \in V, \ x_e \geq 0 \ \ \forall e \in E, \ \sum_{e \in E} x_e = k\}$$

is called the *fractional k-matching polytope* of G.

We will see in Chapter 6, Section 6.1 that again these two polytopes are equal for bipartite graphs. Even further, we can work with matchings up to a given size and define analogously the polytopes $M_{\leq k}(G)$ and $FM_{\leq k}(G)$ and still obtain the same equality for bipartite graphs.

We have defined the polytope $\mathrm{M}_k(G)$ for a bipartite graph $G = (V, E)$ on $A \sqcup B$ as living in $\mathbb{R}^E$, but we can also think of it as a polytope in $\mathbb{R}^{m \times n}$, where m and n are the sizes of A and B respectively, in the following way: if $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, then a matching M of G can be represented as a 0/1 matrix of size $m \times n$ with a one in those entries (i, j) such that $\{a_i, b_j\} \in M$ and zeros elsewhere. In this case $\mathrm{M}_k(G) = \mathrm{conv}\{M \in \mathbb{R}^{m \times n} : M \text{ is a } k\text{-matching of } G\}$, and $\mathrm{FM}(G)$ is the set of matrices $X \in \mathbb{R}^{m \times n}$ such that the entries in each row and column sum to at most one, $X_{ij} = 0$ when $\{a_i, b_j\} \notin E$, and all the entries of X sum to k.

Of special interest is when $G = \mathrm{K}_{n,n}$. With the matrix representation for matchings, the n-matchings (wich happen to be the perfect matchings) corresponds to permutation matrices.

Definition 1.8.8. Let $n \in \mathbb{N}$. The polytope

$$\begin{aligned} B_n := \{N \in \mathbb{R}^{n \times n} : &\sum_{j=1}^{n} x_{ij} = 1 \ \ \forall\, 1 \leq i \leq n, \\ &\sum_{i=1}^{n} x_{ij} = 1 \ \ \forall\, 1 \leq j \leq n, \\ &x_{ij} \geq 0 \ \ \forall\, 1 \leq i, j \leq n\} \end{aligned}$$

is called a Birkhoff polytope.

We can see that $B_n = \mathrm{FM}_n(K_{n,n})$, and the equality $\mathrm{M}_n(K_{n,n}) = \mathrm{FM}_n(K_{n,n})$ is the famous Birkhoff-von Neumann Theorem which says that every doubly stochastic matrix is a convex combination of permutation matrices. Also the polytopes $\mathrm{M}_k(\mathrm{K}_{m,n})$ are studied in [6] under the name of k-assignment polytopes.

Using the Birkhoff polytopes as example, we transition to the concept of normality.

Definition 1.8.9. A polytope P is normal if for all $t \in \mathbb{N}$ and every integer point (points with integer coordinates) in $tP := \{t\mathbf{x} : \mathbf{x} \in P\}$ is equal to the sum of t integer points in P.

It is known that Birkhoff polytopes are normal, and in Chapter 6, Section 6.2 we generalize that by showing that in general k-matching polytopes of bipartite graphs are normal.

For a thorough reference on matchings, see the book [17].

Chapter 2

McMullen's operations on graphicality

We recall that a combinatorial polytope P is said to be projectively unique if it has a single realization up to projective transformations. In other words, P is projectively unique if any two (embedded) polytopes with its combinatorial structure can be mapped to each other by a projective transformation (see Section 1.2). Projectively unique polytopes form a very interesting class, where the combinatorics contain all essential information and the realization space is trivial in a very strong sense. In contrast, realization spaces of polytopes can be arbitrarily complicated in general [22].

The study of projectively unique polytopes for their own sake goes back more than fifty years, with important pioneering work of Perles, Shepard and McMullen (see for instance [14, Section 4.8],[21] and [19]), but the full characterization of all three-dimensional projectively unique polytopes goes back even further, being a consequence of Steinitz's work in the early twentieth century [27]. In dimension three, a polytope is projectively unique if and only if it has at most 9 edges (simplices, square pyramids, triangular prisms and triangular bipyramids). In dimension two, it is a simple exercise to see that only triangles and quadrilaterals are projectively unique.

In higher dimensions, projective uniqueness is much more elusive. In dimension four, there is a list of 11 combinatorial classes of projectively unique polytopes conjectured to be complete by Shephard and McMullen ([19]). A weaker, more general question, posed by Perles and Shephard in [21] asks if the number of such combinatorial classes of polytopes in a fixed dimension $d \geq 4$ is even finite. This was answered negatively for $d \geq 69$ in [2], but remains open for $4 \leq d \leq 68$.

Our work merges two ideas from the literature on projective uniqueness. Very recently, the concept of *slack ideals* introduced in [12, 9, 10] presents an algebraic take on the study of realization spaces (see Section 1.3). In [10] a subclass of projectively unique polytopes was defined, the *graphic polytopes*, for which one has an algebraic certificate of projective uniqueness (see Section 1.4). On the other hand, an important early result in the study of projective uniqueness is due to [19], where certain operations on polytopes are introduced and proven to preserve projective uniqueness. Our main result is that these same operations also preserve graphicality. Note that this result neither implies nor is implied by the original McMullen result. This connection gives us an algebraic version of the McMullen's result that can be used to create large families of graphic polytopes, as well as to prove graphicality for particular polytopes of interest.

Graphic polytopes are not only a subclass of projectively unique polytopes but also a subclass of *morally 2-level* polytopes. Morally 2-level polytopes are those that have 0/1 generalized slack matrices, and in particular they include all 2-level polytopes. Such polytopes play an important role in the theory of semidefinite representations of polyhedra and have been the focus of recent interest. Moreover, they comprise a very large family that includes many interesting polytopes; see for example [1] for combinatorially relevant examples and [4] for a full enumeration in dimension up to 7.

We saw in Section 1.2 the operations on (combinatorial) polytopes introduced in [19] by McMullen that preserve projective uniqueness, as seen in Theorem 1.2.6. In this chapter we will show that these operations also preserve graphicality. Note that this neither implies nor is implied by the results of McMullen, since we have both a stronger hypothesis and a stronger conclusion. To be more precise, we will prove the following graphical version of Theorem 1.2.6.

Theorem 2.0.1. *Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ be two full-dimensional polytopes. Then:*

1. *$P \vee Q$ is graphic if and only if P and Q are graphic,*
2. *if P and Q are graphic, then so is $P \oplus_{(\mathbf{v},\mathbf{w})} Q$ for any vertices $\mathbf{v}$ of P and $\mathbf{w}$ of Q, and*
3. *if P is graphic and is not the vertex sum of two polytopes at $\mathbf{p}$, then $P_{\mathbf{p}}$ is graphic.*

This is a direct analogue of Theorem 1.2.6. Note that while the first two parts of Theorem 2.0.1 show that some operations unconditionally preserve graphicality, the same is not true for the last part. The condition of not being a vertex sum of two polytopes is not very natural in an algebraic setting. We therefore derive a necessary and sufficient algebraic condition (fully described later in the chapter) for a vertex split to be projectively unique. This condition can be easier to check than McMullen's original geometric condition.

Before setting out to prove this theorem, we will first show that duality preserves graphicality.

Proposition 2.0.2. *Let P and P^* be dual polytopes. Then P is graphic if and only if P^* is graphic.*

Proof. Since the symbolic slack matrices of P and P^* are transposes of each other, we have $I_P = I_{P^*}$. Also, the non-incidence graphs of P and P^* are the same and thus $T_P = T_{P^*}$. From these equalities, the result follows. □

Proposition 2.0.2 allows us to translate results from vertex splitting to facet wedging, which will be convenient later on in the context of order polytopes.

2.1 Auxiliary results

To show that the McMullen operations also preserve graphicality, we will analyze their effect on the slack ideals I_P and the toric ideals T_P. To do that we will use two technical auxiliary results that we present in this section.

The first of these results is simply a restatement of the usual argument used to show that slack matrices of d-dimensional polytopes have rank $d+1$ by showing that certain submatrices associated to flags of faces are triangular.

Lemma 2.1.1 (Flag Lemma [9]). *Let P be a d-polytope and*

$$\emptyset = g_{-1} \subset g_0 \subset \cdots \subset g_{d-1} \subset g_d = P$$

be a complete flag of faces of P. Let $G_0, \ldots, G_d$ be facets of P such that $g_k = G_d \cap G_{d-1} \cap \ldots G_{k+1}$ for $k = -1, 0, \ldots, d-1$ and $\mathbf{w}_0, \ldots, \mathbf{w}_d$ be vertices of P such that $\mathbf{w}_k \in g_k \setminus g_{k-1}$ for $k = 0, \ldots, d$. Then the $(d+1) \times (d+1)$ submatrix $A(\mathbf{x})$ formed from the rows of $S_P(\mathbf{x})$ indexed by $G_0, \ldots, G_d$ and the columns indexed by $\mathbf{w}_0, \ldots, \mathbf{w}_d$ is upper triangular with variables on the diagonal. In particular, the $(d+1)$-minor given by its determinant is a nonzero monomial.

Proof. For each $j = 0, \ldots, d$, we have $\mathbf{w}_j \in g_j = G_d \cap \cdots \cap G_{j+1}$, so $a_{ij} = 0$ for each $i > j$. But $\mathbf{w}_j \notin g_{j-1} = G_d \cap \cdots \cap G_{j+1} \cap G_j$, so $\mathbf{w}_j \notin G_j$ and so a_{jj} is a variable for each j. That is, $A(\mathbf{x})$ is upper triangular with variables on the diagonal. □

The second basic result we will be repeatedly using relates the cycle space of the non-incidence graph of a polytope with its slack ideal. Here, if C is a collection of oriented edges of $\mathbf{G}_P$, then $\overline{C}$ denotes the same collection of edges but with opposite orientations. Also, if C_1 and C_2 are collections of oriented edges, then we can find collections C_1', C_2' and C_0 such that $C_1 = C_1' \cup C_0$, $C_2 = C_2' \cup \overline{C_0}$, and there is no edge that is in both C_1' and C_2' but with different orientation. Using this, we define $C_1 + C_2 := C_1' \cup C_2'$.

Lemma 2.1.2 (Cycle-Splitting Lemma). *Let C_1 and C_2 be collections of oriented edges of $\mathbf{G}_P$. If f_{C_1} and f_{C_2} both belong to I_P, then $f_{C_1+C_2} \in I_P$.*

Proof. Using the notation above, let $f_{C_1'} := \mathbf{m}_1 - \mathbf{n}_1$, $f_{C_2'} := \mathbf{m}_2 - \mathbf{n}_2$, and $f_{C_0} = \mathbf{m}_3 - \mathbf{n}_3$. Thus $f_{C_1} = \mathbf{m}_1\,\mathbf{m}_3 - \mathbf{n}_1\,\mathbf{n}_3$ and $f_{C_2} := \mathbf{m}_2\,\mathbf{n}_3 - \mathbf{n}_2\,\mathbf{m}_3$. Since f_{C_1} and f_{C_2} are in I_P, so is $\mathbf{m}_2\,f_{C_1} + \mathbf{n}_1\,f_{C_2} = \mathbf{m}_1\,\mathbf{m}_2\,\mathbf{m}_3 - \mathbf{n}_1\,\mathbf{n}_2\,\mathbf{m}_3 = \mathbf{m}_3\,f_{C_1+C_2}$. Since I_P is saturated, we conclude that $f_{C_1+C_2} \in I_P$. □

We call this result the Cycle-Splitting Lemma because we are going to apply it in the case that C_1, C_2 and $C_1 + C_2$ are cycles.

2.2 The join operation

The first and simplest of the operations introduced by McMullen is that of the join. An important special case of this construction is the pyramid over a polytope P, which is the join of P with a point. The slack ideal of the pyramid is exactly the same as the slack ideal of the original polytope, so all algebraic properties, including graphicality, are preserved.

Even in the general case, the join operation is very easy to interpret in terms of slack matrices. In fact, from the description of its facets and vertices in Section 1.2 we see that the symbolic slack matrix of $P \vee Q$ is given by

$$S_{P\vee Q}(\mathbf{x},\mathbf{y}) = \begin{bmatrix} S_P(\mathbf{x}) & O \\ O & S_Q(\mathbf{y}) \end{bmatrix}$$

where $S_P(\mathbf{x})$ and $S_Q(\mathbf{y})$ are the symbolic slack matrices of P and Q, respectively. This makes the slack ideal of the join easy to describe in terms of the original slack ideals.

Lemma 2.2.1. *Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ be two full-dimensional polytopes and $I_P \subseteq \mathbb{R}[\mathbf{x}]$ and $I_Q \subseteq \mathbb{R}[\mathbf{y}]$ their slack ideals. Then*

$$I_{P\vee Q} = \langle I_P \rangle + \langle I_Q \rangle.$$

Moreover $I_{P\vee Q} \cap \mathbb{R}[\mathbf{x}] = I_P$ and $I_{P\vee Q} \cap \mathbb{R}[\mathbf{y}] = I_Q$.

Proof. We first note that by [28, Lemma 2.6], $\langle I_P \rangle + \langle I_Q \rangle$ is saturated because I_P and I_Q are saturated ideals in two polynomial rings on disjoint sets of variables. Thus, to prove $I_{P\vee Q} \subseteq \langle I_P \rangle + \langle I_Q \rangle$ it is enough to show that any $(d+e+3)$-minor of $S_{P\vee Q}$ is in $\langle I_P \rangle + \langle I_Q \rangle$. Let m be a nonzero $(d+e+3)$-minor of $S_{P\vee Q}(\mathbf{x},\mathbf{y})$. By the block structure of $S_{P\vee Q}$, $m(\mathbf{x},\mathbf{y}) = m'(\mathbf{x})m''(\mathbf{y})$ where m' is an r-minor of $S_P(\mathbf{x})$ and m'' is an s-minor of $S_P(\mathbf{y})$ for some r and s that sum to $d+e+3$. By the pigeonhole principle, either $r \geq d+2$ or $s \geq e+2$. Without loss of generality, assume the former. It is easy to see that $\text{Minors}_r(S_P(\mathbf{x})) \subseteq \text{Minors}_{d+2}(S_P(\mathbf{x}))$ by way of Laplace expansion, so $m'(\mathbf{x}) \in I_P$ and hence $m(\mathbf{x},\mathbf{y}) \in \langle I_P \rangle$, concluding the proof of the forward inclusion.

To prove the reverse inclusion it is enough to show that any $(d+2)$-minor of $S_P(\mathbf{x})$ is in $I_{P\vee Q}$. The symmetry between P and Q and the fact that $I_{P\vee Q}$ is saturated then will imply it. Let $m(\mathbf{x})$ be the $(d+2)$-minor of $S_P(\mathbf{x})$ obtained from an arbitrary $(d+2)\times(d+2)$ submatrix T. By the Flag Lemma, we know there is some $(e+1)\times(e+1)$ triangular submatrix U of $S_Q(\mathbf{y})$ with variables on the diagonal. Form a $(d+e+3)\times(d+e+3)$ submatrix of $S_{P\vee Q}(\mathbf{x},\mathbf{y})$ containing both T and U. The block structure will imply that the associated $(d+e+3)$-minor is simply $m(\mathbf{x})n(\mathbf{y})$ where $n(\mathbf{y})$ is a monomial in the variables $\mathbf{y}$. Thus $m(\mathbf{x})n(\mathbf{y}) \in I_{P\vee Q}$, and by saturation this implies $m(\mathbf{x})$ belongs to $I_{P\vee Q}$.

We are left to prove that $I_{P\vee Q} \cap \mathbb{R}[\mathbf{x}] = I_P$ since the analogous result for Q follows from the symmetry of the construction. We prove the forward inclusion since the other is clear. Let $f(\mathbf{x}) \in I_{P\vee Q} = \langle I_P \rangle + \langle I_Q \rangle$. Then there are polynomials $f_i \in I_P, g_j \in I_Q$ and $p_i, q_j \in \mathbb{C}[\mathbf{x},\mathbf{y}]$ such that

$$f(\mathbf{x}) = \sum_i p_i(\mathbf{x},\mathbf{y})f_i(\mathbf{x}) + \sum_j q_j(\mathbf{x},\mathbf{y})g_j(\mathbf{y}).$$

Evaluating the expression at some $\tilde{\mathbf{y}}$ such that $S_Q(\tilde{\mathbf{y}})$ is a true slack matrix of Q, we have that $g_j(\tilde{\mathbf{y}}) = 0$ hence

$$f(\mathbf{x}) = \sum_i p_i(\mathbf{x}, \tilde{\mathbf{y}}) f_i(\mathbf{x}) \in I_P,$$

proving the claim. □

The join is even easier to understand when applied to T_P and T_Q. Since the non-incidence graph of $P \vee Q$ is just the disjoint union of the non-incidence graphs of P and Q, we have by definition that

$$T_{P \vee Q} = \langle T_P \rangle + \langle T_Q \rangle.$$

By the same argument used above but with $\tilde{\mathbf{y}}$ being the all-ones vector, we obtain that $T_{P\vee Q} \cap \mathbb{R}[\mathbf{x}] = T_P$ and $T_{P\vee Q} \cap \mathbb{R}[\mathbf{y}] = T_Q$. From these properties it is easy to show that the join preserves graphicality.

Theorem 2.2.2. *Let $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ be two full-dimensional polytopes. Then*

1. *$I_{P\vee Q} \subseteq T_{P\vee Q}$ if and only if $I_P \subseteq T_P$ and $I_Q \subseteq T_Q$.*

2. *$T_{P\vee Q} \subseteq I_{P\vee Q}$ if and only if $T_P \subseteq I_P$ and $T_Q \subseteq I_Q$.*

3. *In particular, $P \vee Q$ is graphic if and only if P and Q are graphic.*

Proof. The first two statements follow from Lemma 2.2.1 and the properties of $T_{P\vee Q}$ seen above by intersecting with $\mathbb{R}[\mathbf{x}]$ and $\mathbb{R}[\mathbf{y}]$. The third statement follows immediately from the first two. □

Observation. Note that by Theorem 1.4.4 (1), (1) is equivalent to saying that $P \vee Q$ is morally 2-level if and only if P and Q are morally 2-level. Also, from the block structure of $S_{P\vee Q}(\mathbf{x}, \mathbf{y})$, it is clear that $P \vee Q$ is projectively unique if and only if P and Q are projectively unique (see Theorem 1.3.6.)

2.3 The vertex sum operation

We now turn to a more involved operation: the vertex sum. Before proving that this operation preserves graphicality, we will explain its effect on slack matrices. For simplicity we work with the *support* of each symbolic slack matrix, which is the matrix obtained by replacing each variable by a 1.

Let P be a d-dimensional polytope and Q an e-dimensional polytope, embedded in such a way that their affine spans intersect only in a single common vertex $\mathbf{p}$ as in Definition 1.2.5 (2). Denote by $\mathcal{F}$ and $\mathcal{G}$ the sets of facets of P and Q, respectively, that contain $\mathbf{p}$, while $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are the sets of facets that do not contain $\mathbf{p}$. Suppose that $\overline{\mathcal{F}}$ has r elements while $\overline{\mathcal{G}}$ has s. Furthermore, let $V := \mathrm{Vert}(P) \setminus \{\mathbf{p}\}$ and $W := \mathrm{Vert}(Q) \setminus \{\mathbf{p}\}$. With this notation, the supports of the slack matrices of P and Q are as follows:

$$S_P(\mathbf{1}) = \begin{array}{c} \\ \scriptstyle\mathcal{F} \\ \scriptstyle\overline{\mathcal{F}} \end{array} \begin{array}{c} \begin{array}{cc} \scriptstyle V & \scriptstyle \mathbf{p} \end{array} \\ \begin{bmatrix} A & \mathbf{0} \\ \overline{A} & \mathbf{1}_r \end{bmatrix} \end{array} \quad \text{and} \quad S_Q(\mathbf{1}) = \begin{array}{c} \\ \scriptstyle\mathcal{G} \\ \scriptstyle\overline{\mathcal{G}} \end{array} \begin{array}{c} \begin{array}{cc} \scriptstyle W & \scriptstyle \mathbf{p} \end{array} \\ \begin{bmatrix} B & \mathbf{0} \\ \overline{B} & \mathbf{1}_s \end{bmatrix} \end{array}.$$

From the description of the vertices and facets of a vertex sum in Section 1.2, we can conclude that the support of the slack matrix of $P \oplus_{\mathbf{p}} Q$ has the form

$$S_{P\oplus_{\mathbf{p}}Q}(\mathbf{1}) = \begin{array}{c} \\ \scriptstyle\mathcal{F}\oplus_{\mathbf{p}}Q \\ \scriptstyle P\oplus_{\mathbf{p}}\mathcal{G} \\ \scriptstyle\overline{\mathcal{F}}\vee\overline{\mathcal{G}} \end{array} \begin{array}{c} \begin{array}{ccc} \scriptstyle V & \scriptstyle \mathbf{p} & \scriptstyle W \end{array} \\ \begin{bmatrix} A & \mathbf{0} & O \\ O & \mathbf{0} & B \\ \overline{A}\otimes\mathbf{1}_s & \mathbf{1}_{r+s} & \mathbf{1}_r\otimes\overline{B} \end{bmatrix} \end{array},$$

where $\mathcal{F} \oplus_{\mathbf{p}} Q := \{F \oplus_{\mathbf{p}} Q : F \in \mathcal{F}\}$, $P \oplus_{\mathbf{p}} \mathcal{G} := \{P \oplus_{\mathbf{p}} G : G \in \mathcal{G}\}$ and $\overline{\mathcal{F}} \vee \overline{\mathcal{G}} := \{F \vee G : F \in \overline{\mathcal{F}}, G \in \overline{\mathcal{G}}\}$. Note that the rows indexed by the facets in $\overline{\mathcal{F}} \vee \overline{\mathcal{G}}$ are all possible concatenations of rows of $\overline{A}$ and rows of $\overline{B}$ with a 1 in the middle.

With this observation, it is easy to show that if P and Q are morally 2-level then so is $P \oplus_{\mathbf{p}} Q$. Recall that P and Q being morally 2-level just means that $S_P(\mathbf{1})$ and $S_P(\mathbf{1})$ are of rank $d+1$ and $e+1$, respectively. Looking at the submatrix of $S_{P\oplus_{\mathbf{p}}Q}(\mathbf{1})$ whose columns are indexed by V and $\mathbf{p}$, we see that the rows there are precisely those of $S_P(\mathbf{1})$ with some repetitions, so it has rank $d+1$, while the submatrix of the columns indexed by W and $\mathbf{p}$ similarly has rank $e+1$. Since the two submatrices share a column, the total rank of $S_{P\oplus_{\mathbf{p}}Q}(\mathbf{1})$ is at most $(d+1)+(e+1)-1 = d+e+1$. Since $P \oplus_{\mathbf{p}} Q$ is $(d+e)$-dimensional, the rank must in fact be exactly $d+e+1$, so the vertex sum is indeed morally 2-level. Using Theorem 1.4.4 to translate this result to an algebraic language, we obtain the following.

Lemma 2.3.1. *Let P, Q be polytopes. If $I_P \subseteq T_P$ and $I_Q \subseteq T_Q$, then $I_{P\oplus_{\mathbf{p}}Q} \subseteq T_{P\oplus_{\mathbf{p}}Q}$.*

It remains to show that the same implication holds when we reverse the inclusions. This requires more involved reasoning as we see next.

Theorem 2.3.2. *Let P, Q be polytopes. If $T_P \subseteq I_P$ and $T_Q \subseteq I_Q$, then $T_{P\oplus_{\mathbf{p}}Q} \subseteq I_{P\oplus_{\mathbf{p}}Q}$. In particular, if P and Q are graphic then $P \oplus_{\mathbf{p}} Q$ is graphic.*

Proof. The second statement follows from the first one together with Lemma 2.3.1, so it is enough to prove the first.

Suppose $T_P \subseteq I_P$ and $T_Q \subseteq I_Q$ and let C be a chordless cycle of $\mathbf{G}_{P\oplus_{\mathbf{p}}Q}$. We have to show that the binomial associated to C is in $I_{P\oplus Q}$.

Again let $V = \mathrm{Vert}(P) \setminus \{\mathbf{p}\}$ and $W = \mathrm{Vert}(Q) \setminus \{\mathbf{p}\}$. We will continue the proof by cases depending on the form of C.

Case 1: Suppose C contains only vertices from one polytope, which we can by symmetry consider to be P. In other words suppose C contains no vertex from W, and comes from the submatrix indexed by V and $\mathbf{p}$.

If C does not contain any pair of facets $F \vee G_j$, $F \vee G_k$, for $j \neq k$, then it does not use any pair of rows with the same support (recall that the nonzero rows of this submatrix are just copies of the rows of S_P), and therefore there is a submatrix S of $S_{P \oplus_{\mathbf{p}} Q}$ that indexes all the facets and vertices in C and has the same support as S_P. Let T be any $(d+2) \times (d+2)$ submatrix of S. Note that the submatrix B of S_Q has an $e \times e$ triangular submatrix M. To see this, apply the Flag Lemma to it with a flag starting at the vertex $\mathbf{p}$, and then remove the column indexed by $\mathbf{p}$ and the row indexed by the unique facet that does not contain $\mathbf{p}$. Now $\det(T) \cdot \det(M)$ is a $(d+e+2)$-minor of $S_{P \oplus_{\mathbf{p}} Q}$ by the block structure of this matrix. Since $\det(M)$ is a monomial, this proves that the saturation of the ideal I generated by the $(d+2)$-minors of S is contained in $I_{P \oplus_{\mathbf{p}} Q}$. Since I is isomorphic to I_P, it contains the binomial f_C by the graphicality of P. Thus $f_C \in I \subseteq I_{P \oplus_{\mathbf{p}} Q}$ as we needed to show.

Now suppose that C uses a pair of repeated rows; that is to say, it contains a pair of facets $F \vee G_1$, $F \vee G_2$ where F is a facet of P and G_1, G_2 are distinct facets of Q. Since they are connected to precisely the same nodes in $V \cup \{\mathbf{p}\}$, then the chordlessness of C forces it to be of the form

$$C = \mathbf{u}_1 (F \vee G_1)\, \mathbf{u}_2 (F \vee G_2)\, \mathbf{u}_1,$$

where $\mathbf{u}_1, \mathbf{u}_2 \in V \cup \{\mathbf{p}\}$, which yields the binomial $f_C = z_{11}z_{22} - z_{12}z_{21} \in T_{P \oplus_{\mathbf{p}} Q}$.

Remember that the submatrix B of S_Q has an $e \times e$ triangular submatrix M. Furthermore, by applying the Flag Lemma to P with a flag ending at F, S_P has a $(d+1) \times (d+1)$ upper triangular submatrix U where the bottom row is indexed by F and the rightmost column by $\mathbf{u}_1$. Let N be the matrix obtained from U by removing that row and column. Looking at the structure of $S_{P \oplus_{\mathbf{p}} Q}$, we then see that it has a $(d+e+2) \times (d+e+2)$ submatrix of the form

$$\begin{array}{c} \\ \\ \\ {\scriptstyle F\vee G_1} \\ {\scriptstyle F\vee G_2} \end{array} \begin{array}{c} \begin{array}{cccc} & & {\scriptstyle \mathbf{u}_1} & {\scriptstyle \mathbf{u}_2} \end{array} \\ \left[\begin{array}{cccc} M & O & \mathbf{0} & \mathbf{0} \\ * & N & * & * \\ * & \mathbf{0}^T & z_{11} & z_{12} \\ * & \mathbf{0}^T & z_{21} & z_{22} \end{array} \right] \end{array}.$$

The determinant of this submatrix equals $\det(M) \cdot \det(N) \cdot f_C$ which is a monomial times f_C, and thus $f_C \in I_{P \oplus_{\mathbf{p}} Q}$.

Case 2: Suppose now that C contains a vertex of V and a vertex of W. By analyzing the structure of the slack matrix, we see that to go from a vertex in V to a vertex in W in the graph $G_{P \oplus_{\mathbf{p}} Q}$, one must go through a facet of the form $F \vee G$. For the cycle C to pass through V and W, that must happen at least twice. Since $\mathbf{p}$ is connected to all those facets, if $\mathbf{p}$ were in C it would have a chord. Hence we know that $\mathbf{p} \notin C$ and so we can write

$$C = (F_0 \vee G_0)A_0(F_1 \vee G_1)A_1 \dots A_{2k-2}(F_{2k-1} \vee G_{2k-1})A_{2k-1}(F_0 \vee G_0)$$

where A_i is a path whose vertices are all in V (if i is even) or all in W (if i is odd) and F_i (resp. G_i) are facets of P (resp. of Q) that do not contain $\mathbf{p}$. But then there is an edge in the nonincidence graph from $\mathbf{p}$ to $F_i \vee G_i$ for each i, so the graph contains cycles $C_0, \dots, C_{2k-1}$ where

$$C_i = (F_i \vee G_i)A_i(F_{i+1} \vee G_{i+1})\,\mathbf{p}(F_i \vee G_i)$$

with indices taken modulo $2k$. In particular, C_i never contains both a vertex in V and a vertex in W, so for each i, $f_{C_i} \in I_{P \oplus_\mathbf{p} Q}$ by the previous case. Then by the Cycle-Splitting Lemma (Lemma 2.1.2), $f_C \in I_{P \oplus_\mathbf{p} Q}$. □

2.4 The vertex splitting operation

We proceed now with the operation of vertex splitting. The study of graphicality under this operation turns out to be more delicate, as it is not the case that it is unconditionally preserved.

In terms of slack matrices, the operation of vertex splitting is again quite simple. As in the previous section, denote by $\mathcal{F}$ and $\overline{\mathcal{F}}$ the sets of facets of P that, respectively, contain or do not contain $\mathbf{p}$. Furthermore, let $V := \mathrm{Vert}(P) \setminus \{\mathbf{p}\}$. If the symbolic slack matrix of P has support of the form

$$S_P(\mathbf{1}) = \begin{array}{c} \\ \scriptstyle\mathcal{F} \\ \scriptstyle\overline{\mathcal{F}} \end{array} \begin{array}{c} \begin{array}{cc} \scriptstyle V & \scriptstyle\mathbf{p} \end{array} \\ \begin{bmatrix} A & \mathbf{0} \\ \overline{A} & \mathbf{1} \end{bmatrix} \end{array}$$

then the support of the slack matrix of the vertex split at $\mathbf{p}$ is

$$S_{P_\mathbf{p}}(\mathbf{1}) = \begin{array}{c} \\ \scriptstyle\mathcal{F} \\ \scriptstyle\overline{\mathcal{F}} \\ \scriptstyle\widehat{\mathcal{F}} \end{array} \begin{array}{c} \begin{array}{ccc} \scriptstyle V & \scriptstyle\overline{\mathbf{p}} & \scriptstyle\widehat{\mathbf{p}} \end{array} \\ \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \overline{A} & \mathbf{1} & \mathbf{0} \\ \overline{A} & \mathbf{0} & \mathbf{1} \end{bmatrix} \end{array}.$$

As with vertex sums, it is not difficult to see that this preserves moral 2-levelness. We need to prove that $S_{P_\mathbf{p}}(\mathbf{1})$ has rank $d+2$ when $S_P(\mathbf{1})$ has rank $d+1$. Indeed, we get that the submatrix of $S_{P_\mathbf{p}}(\mathbf{1})$ with rows indexed by $\mathcal{F}$ and $\overline{\mathcal{F}}$ is exactly $S_P(\mathbf{1})$ with a zero column added, hence has also rank $d+1$. Adding the rows indexed by $\widehat{\mathcal{F}}$ increases the rank by just one, since they can all be attained from the corresponding rows of $\overline{\mathcal{F}}$ by adding $\begin{bmatrix} 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$ which is the difference between the first row indexed by $\widehat{\mathcal{F}}$ and the first row indexed by $\overline{\mathcal{F}}$. In algebraic terms, we have just proved the following result.

Lemma 2.4.1. *Let $P_\mathbf{p}$ be obtained from P by splitting a vertex $\mathbf{p}$. If $I_P \subseteq T_P$, then $I_{P_\mathbf{p}} \subseteq T_{P_\mathbf{p}}$.*

In order to study the other inclusions, we need to consider more closely the structure of the non-incidence graph of the vertex split. In Figure 2.1 one can see the structure of the graph of the vertex split described above.

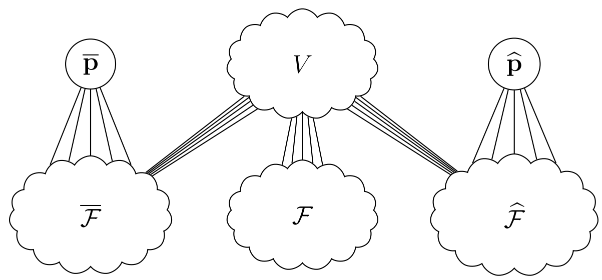


Figure 2.1: Structure of the non-incidence graph of a vertex split

We will be especially interested in two classes of cycles. The first is the class of 4-cycles of the form

$$\overline{F}\,\mathbf{v}\,\widehat{F}\,\mathbf{w}\,\overline{F}$$

where $\mathbf{v}, \mathbf{w} \in V$, and $\overline{F} \in \overline{\mathcal{F}}$ and $\widehat{F} \in \widehat{\mathcal{F}}$ are associated to the same facet F of P. We will say these are cycles *of type A*. The second class, the *cycles of type B*, are those 8-cycles of the form

$$\overline{\mathbf{p}}\overline{F}\,\mathbf{v}\,\widehat{F}\widehat{\mathbf{p}}\widehat{G}\,\mathbf{w}\,\overline{G}\overline{\mathbf{p}}$$

where, as before, $\mathbf{v}, \mathbf{w} \in V$, $\overline{F}, \overline{G} \in \overline{\mathcal{F}}$ and $\widehat{F}, \widehat{G} \in \widehat{\mathcal{F}}$, $\overline{F}$ and $\widehat{F}$ are derived from the same facet F of P and similarly with $\overline{G}$ and $\widehat{G}$. It turns out that these are the two types of cycles that must be checked in order for the inclusion of the toric ideal in the slack ideal to be maintained.

Proposition 2.4.2. *Let $P_{\mathbf{p}}$ be the vertex split of P as above. If $T_P \subseteq I_P$, then $T_{P_{\mathbf{p}}} \subseteq I_{P_{\mathbf{p}}}$ if and only if the binomials associated to all cycles of type A and B are in $I_{P_{\mathbf{p}}}$.*

Proof. We simply need to show that if all the binomials associated to cycles of type A and B are in $I_{P_{\mathbf{p}}}$, then for any oriented cycle C of $\mathbf{G}_{P_{\mathbf{p}}}$, its associated binomial f_C is in $I_{P_{\mathbf{p}}}$, as the other implication is trivial. We will prove this by induction on the number of nodes of C belonging to $\widehat{\mathcal{F}}$.

We start by considering the case in which C does not contain any elements of $\widehat{\mathcal{F}}$. In this case, C also cannot contain $\widehat{\mathbf{p}}$ since it only connects to $\widehat{\mathcal{F}}$. So the cycle is entirely contained in the submatrix obtained from $S_{P_{\mathbf{p}}}$ by removing the rows indexed by $\widehat{\mathcal{F}}$ and the column indexed by $\widehat{\mathbf{p}}$. We can identify this submatrix with S_P. Under this identification we have that f_C is in I_P, but any $(d+2)$-minor of S_P can be completed to a $(d+3)$-minor of $S_{P_{\mathbf{p}}}$ by adding one row from $\widehat{\mathcal{F}}$ and column $\widehat{\mathbf{p}}$. The block structure of such a matrix guarantees that we are only multiplying the old minor by a new variable that will be saturated out, hence $f_C \in I_P \subseteq I_{P_{\mathbf{p}}}$ and we obtain the desired result.

Suppose now that C includes an element $\widehat{F}$ in $\widehat{\mathcal{F}}$. We consider the following two cases.

If $\widehat{F}$ is not connected to $\widehat{\mathbf{p}}$ in C, then we can write

$$C = \mathbf{v}\,\widehat{F}\,\mathbf{w}\,\Gamma\,\mathbf{v}$$

where $\mathbf{v}, \mathbf{w} \in V$ and Γ is a path. Then we can write $C = C_1 + C_2$ where

$$C_1 = \mathbf{v}\,\overline{F}\,\mathbf{w}\,\Gamma\,\mathbf{v},\ C_2 = \overline{F}\,\mathbf{v}\,\widehat{F}\,\mathbf{w}\,\overline{F}.$$

Now C_1 is an oriented cycle containing one fewer element of $\widehat{\mathcal{F}}$ than C, so $f_{C_1} \in I_{P_\mathbf{p}}$ by induction. Also C_2 is a cycle of type A, so $f_{C_2} \in I_{P_\mathbf{p}}$ by assumption. Thus by the Cycle-Splitting Lemma, we conclude that $f_C \in I_{P_\mathbf{p}}$.
On the other hand, if $\widehat{F}$ is connected to $\widehat{\mathbf{p}}$ in C, then we can write

$$C = \mathbf{v}\,\widehat{F}\widehat{\mathbf{p}}\widehat{G}\,\mathbf{w}\,\Gamma\,\mathbf{v}$$

where $\mathbf{v}, \mathbf{w} \in V$, $\widehat{G} \in \widehat{\mathcal{F}}$ and Γ is a path. In this case, we can write $C = C_1 + C_2$ where

$$C_1 = \mathbf{v}\,\overline{F}\overline{\mathbf{p}}\overline{G}\,\mathbf{w}\,\Gamma\,\mathbf{v},\ C_2 = \overline{\mathbf{p}}\overline{F}\,\mathbf{v}\,\widehat{F}\widehat{\mathbf{p}}\widehat{G}\,\mathbf{w}\,\overline{G}\overline{\mathbf{p}}.$$

Again C_1 contains fewer elements of $\widehat{\mathcal{F}}$ than C, so by induction $f_{C_1} \in I_{P_\mathbf{p}}$. Also C_2 is a cycle of type B, so $f_{C_2} \in I_{P_\mathbf{p}}$ by assumption. Thus by the Cycle-Splitting Lemma, we conclude that $f_C \in I_P$.

□

Corollary 2.4.3. *Suppose $T_P \subseteq I_P$ and the binomials associated to all cycles of type A belong to $I_{P_\mathbf{p}}$. If C is an oriented cycle of $\mathbf{G}_{P_\mathbf{p}}$ that either does not contain $\widehat{\mathbf{p}}$ or does not contain $\overline{\mathbf{p}}$, then $f_C \in I_{P_\mathbf{p}}$.*

Proof. If C does not contain $\widehat{\mathbf{p}}$, then the conclusion follows as in the proof of Proposition 2.4.2 because cycles of type B are only invoked when $\widehat{\mathbf{p}}$ appears in the original cycle. If C does not contain $\overline{\mathbf{p}}$, then again apply the proof of Proposition 2.4.2 but interchanging $\widehat{\mathcal{F}}$ with $\overline{\mathcal{F}}$ and $\widehat{\mathbf{p}}$ with $\overline{\mathbf{p}}$. □

We proceed to show when even these special cycles need not be checked. This can be characterized purely in terms of the non-incidence graph of the original polytope as follows.

Lemma 2.4.4. *Let P be a polytope such that $T_P \subseteq I_P$ and $\mathbf{G}$ be the connected component of $\mathbf{G}_P$ that contains $\mathbf{p}$. Then:*

1. *If the graph obtained from $\mathbf{G}$ by removing $\mathbf{p}$ is connected, then $T_{P_\mathbf{p}} \subseteq I_{P_\mathbf{p}}$ if and only if the binomials associated to all cycles of type A are in $I_{P_\mathbf{p}}$.*

2. *If the graph obtained from $\mathbf{G}$ by removing $\mathbf{p}$ and all its neighbors is connected, then the binomials associated to all cycles of types A are in $I_{P_\mathbf{p}}$.*

Proof. We will start by proving (1). Let C be a cycle of type B of the form $\overline{\mathbf{p}}\overline{F}\,\mathbf{v}\,\widehat{F}\widehat{\mathbf{p}}\widehat{G}\,\mathbf{w}\,\overline{G}\overline{\mathbf{p}}$. Since $\mathbf{p}$ is not a cut-vertex of $\mathbf{G}$, there is a path Γ in $\mathbf{G}_P$ from $\mathbf{v}$ to $\mathbf{w}$ that does not pass through $\mathbf{p}$. Then Γ is also a path in $\mathbf{G}_{P_\mathbf{p}}$ that does not pass through $\widehat{\mathcal{F}}$ nor $\overline{\mathbf{p}}$. Let C_1 be the cycle in $\mathbf{G}_{P_\mathbf{p}}$ obtained by joining Γ to the path $\overline{G}\overline{\mathbf{p}}\overline{F}\,\mathbf{v}$ and C_2 be the cycle $\mathbf{v}\,\widehat{F}\widehat{\mathbf{p}}\widehat{G}\,\mathbf{w}\,\Gamma\,\mathbf{v}$. Now C_1 contains no elements of $\widehat{\mathcal{F}}$, so $f_{C_1} \in I_{P_\mathbf{p}}$ by the same argument as in the base case of the proof of Proposition 2.4.2. Also, C_2 does not contain $\overline{\mathbf{p}}$ so $f_{C_2} \in I_{P_\mathbf{p}}$ by Corollary 2.4.3. Since $C = C_1 + C_2$, $f_C \in I_{P_\mathbf{p}}$ by the Cycle-Splitting Lemma.

We now prove (2). Suppose we have a cycle of type A of the form $C = \overline{F}\,\mathbf{v}\,\widehat{F}\,\mathbf{w}\,\overline{F}$. By hypothesis the vertices $\mathbf{v}$ and $\mathbf{w}$ are connected in $\mathbf{G}_P$ by a path Γ passing only through V and $\mathcal{F}$. This means that $C_1 = \mathbf{v}\,\Gamma\,\mathbf{w}\,\overline{F}\,\mathbf{v}$ is a cycle that does not contain elements in $\widehat{\mathcal{F}}$, hence $f_{C_1} \in I_{P_\mathbf{p}}$, and similarly for $C_2 = \mathbf{v}\,\widehat{F}\,\mathbf{w}\,\Gamma\,\mathbf{v}$. Since $C = C_1 + C_2$, we have $f_C \in I_{P_\mathbf{p}}$ by the Cycle-Splitting Lemma.

□

This Lemma can be improved by noticing that the conditions for the second statement almost always imply the conditions for the first statement.

Proposition 2.4.5. *Let P be a polytope such that $T_P \subseteq I_P$ and let $\mathbf{G}$ be the connected component of $\mathbf{G}_P$ that contains $\mathbf{p}$. If the graph obtained from $\mathbf{G}$ by removing $\mathbf{p}$ and all its neighbors is connected, then $T_{P_\mathbf{p}} \subseteq I_{P_\mathbf{p}}$.*

Proof. Without loss of generality, we can suppose $\mathbf{G}_P = \mathbf{G}$ (that is, P is not the join of two polytopes). First note that since removing $\mathbf{p}$ and its neighbours leaves the graph connected, the only way that removing $\mathbf{p}$ could disconnect $\mathbf{G}$ is if at least one of its neighbours becomes isolated (note that two neighbours can't connect to each other since $\mathbf{G}$ is bipartite). In this case there is a facet that contains every vertex but $\mathbf{p}$, which implies that P is a pyramid with $\mathbf{p}$ as its apex. But vertex splitting the apex of a pyramid coincides with taking another pyramid over it, which preserves the inclusion $T_P \subseteq I_P$ (it is a special case of the join), so we are done with this case.

On the other hand, if removing $\mathbf{p}$ leaves a connected graph, than we have the result by combining the two parts of Lemma 2.4.4. □

It remains to see which polytopes do not verify the conditions of the previous proposition. Suppose P is a d-dimensional polytope with N vertices and $\mathbf{p}$ one of its vertices. If removing $\mathbf{p}$ and its neighbors increases the number of connected components of $\mathbf{G}_P$, then the slack matrix of P can be written as

$$S_P = \begin{array}{c} \\ \mathcal{F}_1 \\ \mathcal{F}_2 \\ \overline{\mathcal{F}} \end{array} \begin{array}{c} \begin{array}{ccc} V_1 & \mathbf{p} & V_2 \end{array} \\ \left[\begin{array}{ccc} A & \mathbf{0} & O \\ O & \mathbf{0} & B \\ \overline{A} & \mathbf{1} & \overline{B} \end{array}\right] \end{array}.$$

By Corollary 3 of [8], a matrix is the slack matrix of a polytope if and only if its rows generate the cone obtained by intersecting its row space with the nonnegative orthant. In this case we get a simple characterization of the extreme rays of $\text{Row}(S_P) \cap \mathbb{R}^N_+$ in terms of the matrices

$$S' = \begin{array}{c} \\ \mathcal{F}_1 \\ \overline{\mathcal{F}} \end{array} \begin{array}{c} \begin{array}{cc} V_1 & \mathbf{p} \end{array} \\ \left[\begin{array}{cc} A & \mathbf{0} \\ \overline{A} & \mathbf{1} \end{array} \right] \end{array} \quad \text{and } S'' = \begin{array}{c} \\ \mathcal{F}_2 \\ \overline{\mathcal{F}} \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{p} & V_2 \end{array} \\ \left[\begin{array}{cc} \mathbf{0} & B \\ \mathbf{1} & \overline{B} \end{array} \right] \end{array}.$$

Lemma 2.4.6. *Let $n := |V_1|$ and $m := |V_2|$. A vector $(p, r, q) \in \mathbb{R}^{n+1+m}$ is a generator of $\text{Row}(S_P) \cap \mathbb{R}^{n+1+m}_+$ if (p, r) and (r, q) are generators of $\text{Row}(S') \cap \mathbb{R}^{n+1}_+$ and $\text{Row}(S'') \cap \mathbb{R}^{1+m}_+$ respectively.*

Proof. Since $\dim \text{Row}(S_P) = \dim \text{Col}(S_P)$, $\text{rank}[\overline{A}\ \mathbf{1}\ \overline{B}] = 1$. Thus $\text{Row}(S_P) = (\text{Row}(A), \mathbf{0}, O) + \mathbb{R}(a, 1, b) + (O, \mathbf{0}, \text{Row}(B)) \subseteq \mathbb{R}^{n+1+m}$, where $[a\ 1\ b]$ is the first row indexed by $\overline{\mathcal{F}}$ in S_P. Note that the projection of $\text{Row}(S_P) \cap \mathbb{R}^{n+1+m}_+$ into the first $n+1$ coordinates is simply $\text{Row}(S') \cap \mathbb{R}^{n+1}_+$, while the projection into the last $1+m$ is $\text{Row}(S'') \cap \mathbb{R}^{1+m}_+$. The inverse image of an extreme ray by a linear projection is a face, and the intersection of faces is still a face. Since the intersection of the inverse images, by each of the projections, of the rays generated by (p, r) and (r, q) is the ray generated by (p, r, q), then it must be a face, hence extreme, whenever (p, r) and (r, q) are. □

This means that every generator of the cone $\text{row}(S') \cap \mathbb{R}^{n+1}_+$ appears as a row in S', and similarly for S''. Then by removing redundant and repeated rows from each of S' and S'', we obtain the slack matrices of some polytopes Q and R. Now the slack matrix of $Q \oplus_{\mathbf{p}} R$ is a submatrix of S_P whose row space equals the row space of S_P. But S_P, being the slack matrix of P, has no redundant rows. So in fact this submatrix is all of S_P. That is, $P = Q \oplus_{\mathbf{p}} R$.

This allows us to extract a purely geometrical sufficient condition for vertex splitting to preserve graphicality, and it once more matches McMullen's condition.

Theorem 2.4.7. *If P satisfies $T_P \subseteq I_P$ and is not the vertex sum of two polytopes at $\mathbf{p}$, then the vertex splitting of P at $\mathbf{p}$ verifies $T_{P_{\mathbf{p}}} \subseteq I_{P_{\mathbf{p}}}$. In particular, if P is graphic and not the vertex sum of two polytopes at $\mathbf{p}$, then $P_{\mathbf{p}}$ is graphic.*

Chapter 3

Projectively unique order polytopes

A perfect candidate to apply the methods in Chapter 2 is the family of *order polytopes*. Order polytopes were introduced in [24]. They are constructed from finite posets, and we can translate properties of the poset into properties of its order polytope. They are very interesting objects; their vertices and facets are easy to describe, and furthermore they are 0/1-polytopes and 2-level.

Order polytopes offer us a combinatorial window on the phenomena we are studying, since we can understand visually at the level of posets the operations that are being applied to high-dimensonal polytopes, which are much harder to internalize. We use the main result in the previous chapter to, in particular, prove that order polytopes from finite ranked posets with no 3-antichain are graphic, and therefore projectively unique. As a side effect, we obtain a tool to generate many low-dimensional, easy to understand, examples of graphic polytopes.

3.1 Operations on finite posets and their order polytopes

It turns out that several simple operations on posets correspond precisely to applying the operations introduced in Section 1.2 to their order polytopes. This opens the possibility of applying the results developed in the previous chapter to the question of graphicality of order polytopes, a task that we will undertake in Section 3.2.

We start by defining three simple operations that depend only on the posets involved, without any additional choices.

Definition 3.1.1. Let $\mathcal{P}$ and $\mathcal{Q}$ be posets on the disjoint sets X and Y respectively. We then define the following new posets.

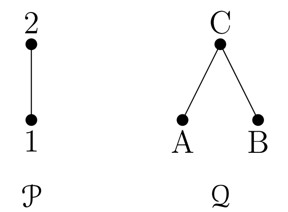
1. $\mathcal{P}_{\text{rev}}$, the *reverse* of $\mathcal{P}$, is the poset on X obtained by reversing the order of $\mathcal{P}$.

2. $\mathcal{P} \vee \mathcal{Q}$, the *join* of $\mathcal{P}$ and $\mathcal{Q}$, is the poset defined by taking all the elements and relations of $\mathcal{P}$ and $\mathcal{Q}$, and adding a new element, $*$, that is greater than each element in X and less than each element in Y.

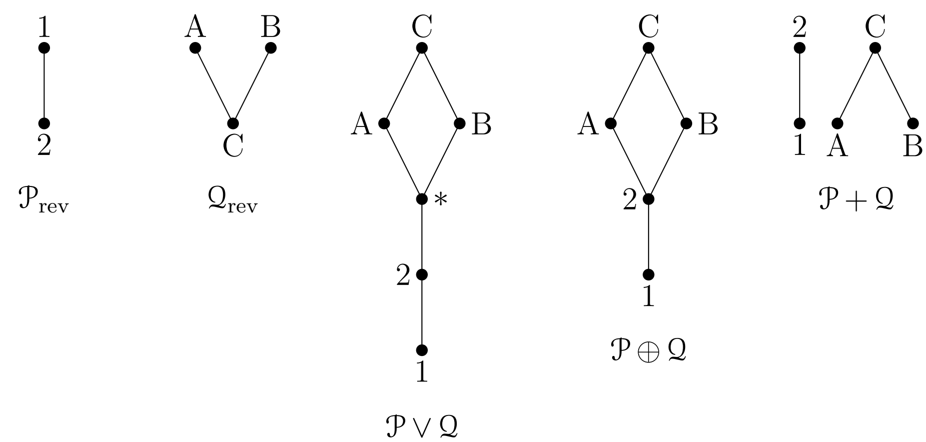
3. $\mathcal{P} \oplus \mathcal{Q}$, the *ordinal sum* of $\mathcal{P}$ and $\mathcal{Q}$, is the poset defined by taking all the elements and relations of $\mathcal{P}$ and $\mathcal{Q}$ and imposing that each element of X is less than every element of Y.

4. $\mathcal{P} + \mathcal{Q}$, the *direct sum* of $\mathcal{P}$ and $\mathcal{Q}$, is the poset defined by simply taking the union of all the elements and relations of $\mathcal{P}$ and $\mathcal{Q}$, with no additional relations.

Example 3.1.2. Consider the posets $\mathcal{P}$ and $\mathcal{Q}$ given by the following Hasse diagrams.



We illustrate the operations defined above applied to $\mathcal{P}$ and $\mathcal{Q}$.



We will now see that all these operations on posets induce simple operations on their order polytopes.

Proposition 3.1.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite posets on disjoint ground sets X and Y. Then we have the following relations between order polytopes.*

1. *$\mathcal{O}(\mathcal{P})$ is affinely equivalent to $\mathcal{O}(\mathcal{P}_{\text{rev}})$;*

2. *$\mathcal{O}(\mathcal{P} \vee \mathcal{Q})$ is combinatorially equivalent to $\mathcal{O}(\mathcal{P}) \vee \mathcal{O}(\mathcal{Q})$;*

3. *$\mathcal{O}(\mathcal{P} \oplus \mathcal{Q})$ is combinatorially equivalent to $\mathcal{O}(\mathcal{P}) \oplus_{(\mathbf{v},\mathbf{w})} \mathcal{O}(\mathcal{Q})$, where $\mathbf{v}$ is the vertex in $\mathcal{O}(\mathcal{P})$ given by the empty filter and $\mathbf{w}$ is the vertex in $\mathcal{O}(\mathcal{Q})$ given by the complete filter.*

4. *$\mathcal{O}(\mathcal{P} + \mathcal{Q}) = \mathcal{O}(\mathcal{P}) \times \mathcal{O}(\mathcal{Q})$.*

Proof. (1) The affine function $\phi : \mathbb{R}^d \to \mathbb{R}^d$ given by $\phi(\mathbf{x}) = 1 - \mathbf{x}$ is an affine isomorphism between $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{P}_{\text{rev}})$.

(2) Note that we can naturally identify the facets of $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{Q})$ with those of $\mathcal{O}(\mathcal{P} \vee \mathcal{Q})$. Facets coming from cover relations, minimal elements of $\mathcal{P}$ and maximal elements of $\mathcal{Q}$ are still present in $\mathcal{O}(\mathcal{P} \vee \mathcal{Q})$. Those coming from maximal elements of $\mathcal{P}$ can be identified with inequalities given by the cover relations of $*$, and similarly with the minimal elements of $\mathcal{Q}$.

As for the vertices, note that if any filter of $\mathcal{P} \vee \mathcal{Q}$ contains an element of X, then it also contains $*$ and every element of Y. So any filter either does not contain $*$ nor any element of X, or it contains $*$ and every element of Y. We can identify the filters of $\mathcal{P}$ with those of $\mathcal{P} \vee \mathcal{Q}$ that contain $*$ by adding $*$ and all elements of Y, while the filters of $\mathcal{Q}$ identify directly with filters of $\mathcal{P} \vee \mathcal{Q}$.

We thus have an identification between the facets of $\mathcal{O}(\mathcal{P} \vee \mathcal{Q})$ and the union of those of $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{Q})$ and similarly for vertices. It is now easy to check that under that identification, every vertex of $\mathcal{O}(\mathcal{Q})$ belongs to every facet of $\mathcal{O}(\mathcal{P})$ and vice-versa. Moreover, the identifications respect vertex-facets incidences inside each of the posets. This implies that $\mathcal{O}(\mathcal{P} \vee \mathcal{Q})$ and $\mathcal{O}(\mathcal{P}) \vee \mathcal{O}(\mathcal{Q})$ have the same vertex-facet incidences, hence are combinatorially equivalent.

(3) We will proceed as in (2). Again, note that facets coming from cover relations, minimal elements of $\mathcal{P}$ and maximal elements of $\mathcal{Q}$ are still present in $\mathcal{O}(\mathcal{P} \oplus \mathcal{Q})$. Furthermore, one can map filters of $\mathcal{P}$ to filters of $\mathcal{P} \oplus \mathcal{Q}$ by adding all elements of Y, while filters of $\mathcal{Q}$ again directly yield filters of $\mathcal{P} \oplus \mathcal{Q}$. However this will map two filters to the same: the empty filter of $\mathcal{P}$ and the complete filter Y of $\mathcal{Q}$. Moreover, facets of $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{Q})$ that do not contain these filters are precisely those that correspond to maximal elements of $\mathcal{P}$ and minimal elements of $\mathcal{Q}$, respectively, and are not present in $\mathcal{O}(\mathcal{P} \oplus \mathcal{Q})$. Instead for any pair of such facets, we have a single new facet given by the cover relation between the maximal element of $\mathcal{P}$ and the minimal element of $\mathcal{Q}$ that was introduced by $\oplus$. This gives us again a one to one identification between facets and vertices of $\mathcal{O}(\mathcal{P} \oplus \mathcal{Q})$ and those of $\mathcal{O}(\mathcal{P}) \oplus_{(\mathbf{v},\mathbf{w})} \mathcal{O}(\mathcal{Q})$, and it is once again easy to see that the incidence relations are preserved. Thus $\mathcal{O}(\mathcal{P} \oplus \mathcal{Q})$ and $\mathcal{O}(\mathcal{P}) \oplus_{(\mathbf{v},\mathbf{w})} \mathcal{O}(\mathcal{Q})$ are combinatorially equivalent.

(4) We need only to observe that a set of elements of $X \cup Y$ is a filter on $\mathcal{P} + \mathcal{Q}$ if and only if its restrictions to X and Y are filters of $\mathcal{P}$ and $\mathcal{Q}$ respectively. This means that the set of vertices of $\mathcal{O}(\mathcal{P} \vee \mathcal{Q})$ is the set of all pairs $(\mathbf{v}, \mathbf{w})$ where $\mathbf{v}$ and $\mathbf{w}$ are vertices of $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{Q})$ respectively, proving the result. □

As a particular case of (2), note that given a poset $\mathcal{P}$, the poset $\mathcal{P}^{\Delta}$ obtained from $\mathcal{P}$ by adjoining a new universal maximum is simply the join of $\mathcal{P}$ with the empty poset, which implies that $\mathcal{O}(\mathcal{P}^{\Delta})$ is simply the join of $\mathcal{O}(\mathcal{P})$ with a point, i.e., a pyramid over $\mathcal{O}(\mathcal{P})$.

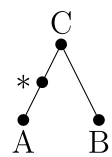
We now introduce two other operations on posets that depend on more than just their ground sets.

Definition 3.1.4. Let $\mathcal{P}$ be a poset.

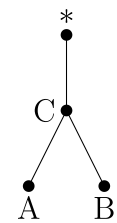
1. For $a \prec b$ in $\mathcal{P}$ we define $\mathcal{P}_{a \prec b}$ by adding a new element $*$ and replacing $a \prec b$ by $a \prec * \prec b$. We say that $\mathcal{P}_{a \prec b}$ is obtained from $\mathcal{P}$ by *splitting the cover* $a \prec b$.

2. For a a maximal (minimal) element of $\mathcal{P}$ we define $\mathcal{P}_a$ by adding a new element $*$ and the cover $a \prec *$ (respectively $* \prec a$). We say that $\mathcal{P}_a$ is obtained from $\mathcal{P}$ by *splitting the maximal (minimal) element* a.

Example 3.1.5. Let Q be the poset of Example 3.1.2. If we split the cover $A \prec C$ or the maximal element C we obtain the following posets.



$\mathcal{Q}_{A \prec C}$



$\mathcal{Q}_C$

Since covers and maximal/minimal elements determine the facets of the order polytope, it is not surprising that the operations of splitting in finite posets are related to the operation of facet wedging on polytopes.

Proposition 3.1.6. *Let $\mathcal{P}$ be a poset, let $a \prec b$ be a cover relation in $\mathcal{P}$ and c be a minimal or maximal element. Then*

1. *$\mathcal{O}(\mathcal{P}_{a \prec b})$ is the facet wedge of $\mathcal{O}(\mathcal{P})$ at $F : t_a \leq t_b$;*

2. *$\mathcal{O}(\mathcal{P}_c)$ is the facet wedge of $\mathcal{O}(\mathcal{P})$ with respect to the facet cut by $0 \leq t_c$ or $t_c \leq 1$, depending on whether c is minimal or maximal.*

Proof. We will prove only (1), since the proof of (2) is completely analogous. Note that splitting a cover relation replaces one facet given by $F : t_a \leq t_b$ by two new ones given by $\hat{F} : t_a \leq t_*$ and $\tilde{F} : t_* \leq t_b$. In terms of vertices, if b is not in a filter, then neither is $*$, while if a is in the filter then so is $*$. This implies that there is a bijection between vertices of $\mathcal{O}(\mathcal{P})$ that are in the facet given by $t_a \leq t_b$ and vertices $\mathbf{v}$ in $\mathcal{O}(\mathcal{P}_{a \prec b})$ that satisfy $v_a = v_* = v_b$. However, if a is not in the filter and b is, then adding $*$ to the filter maintains it as a filter. So each vertex $\mathbf{v}$ in $\mathcal{O}(\mathcal{P})$ with $v_a = 0$ and $v_b = 1$ corresponds to two vertices in $\mathcal{O}(\mathcal{P}_{a \prec b})$: the first vertex $\hat{\mathbf{v}}$ is obtained by adding $v_* = 0$ and the second one $\tilde{\mathbf{v}}$ is obtained by adding $v_* = 1$. If the slack matrix of $\mathcal{O}(\mathcal{P})$ is of the form below on the left, then that of $\mathcal{O}(\mathcal{P}_{a \prec b})$ will be as below on the right.

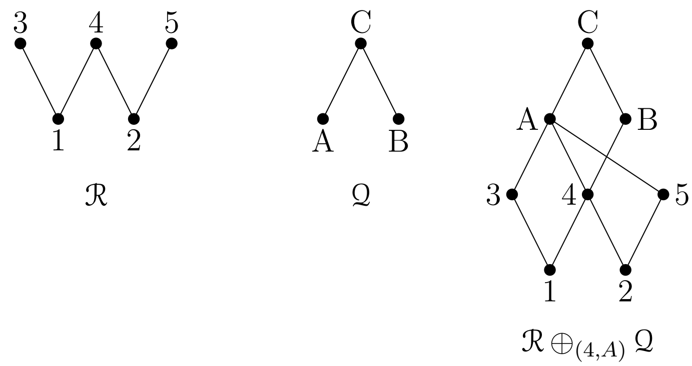
$$\begin{array}{c} \\ \\ {\scriptstyle F} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle V} & {\scriptstyle W} \end{array} \\ \left[\begin{array}{cc} A & B \\ \mathbf{1}^T & \mathbf{0}^T \end{array} \right] \end{array}, \qquad \begin{array}{c} \\ \\ {\scriptstyle \hat{F}} \\ {\scriptstyle \tilde{F}} \end{array} \begin{array}{c} \begin{array}{ccc} {\scriptstyle \hat{V}} & {\scriptstyle \tilde{V}} & {\scriptstyle W} \end{array} \\ \left[\begin{array}{ccc} A & A & B \\ \mathbf{0}^T & \mathbf{1}^T & \mathbf{0}^T \\ \mathbf{1}^T & \mathbf{0}^T & \mathbf{0}^T \end{array} \right] \end{array}.$$

The second matrix is precisely the slack matrix of the facet wedge of $\mathcal{O}(\mathcal{P})$ at F, giving us the desired result. □

The last operation we will introduce is a weaker version of the ordinal sum.

Definition 3.1.7. Let $\mathcal{P}$ and $\mathcal{Q}$ be posets on disjoint sets, a a maximal element of $\mathcal{P}$ and b a minimal element of $\mathcal{Q}$. The *the partial ordinal sum of* $\mathcal{P}$ *and* $\mathcal{Q}$ *with respect to* a *and* b, denoted by $\mathcal{P} \oplus_{(a,b)} \mathcal{Q}$ is the poset attained by taking all elements and relations of $\mathcal{P}$ and $\mathcal{Q}$ and adding the relations that a is less than all elements of $\mathcal{Q}$ while b is greater than all elements of $\mathcal{P}$.

Example 3.1.8. Consider the posets $\mathcal{R}$ and $\mathcal{Q}$ with the Hasse diagrams below on the left and center. On the right is the Hasse diagram of the partial ordinal sum of $\mathcal{R}$ and $\mathcal{Q}$ with respect to 4 and A.



Once more this operation has a simple interpretation in terms of its effect on the order polytopes.

Proposition 3.1.9. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be finite posets on disjoint sets,* a *a maximal element of* $\mathcal{P}$ *and* b *a minimal element of* $\mathcal{Q}$*. Then* $\mathcal{O}(\mathcal{P} \oplus_{(a,b)} \mathcal{Q})$ *is the facet product* $\mathcal{O}(\mathcal{P}) \otimes_F \mathcal{O}(\mathcal{Q})$ *where we identify with* F *the facets* $F_1 : t_a \leq 1$ *of* $\mathcal{O}(\mathcal{P})$ *and* $F_2 : t_b \geq 0$ *of* $\mathcal{O}(\mathcal{Q})$*.*

Proof. In terms of facets, all the facets except the ones corresponding to maximal elements of $\mathcal{P}$ and minimal elements of $\mathcal{Q}$ are preserved untouched. For the maximal elements c of $\mathcal{P}$ different from a, we can identify the facets of $\mathcal{O}(\mathcal{P})$ given by $t_c \leq 1$ with the new facets $t_c \leq t_b$ of $\mathcal{O}(\mathcal{P} \oplus_{(a,b)} \mathcal{Q})$. We can similarly identify the facets corresponding to minimal elements d of $\mathcal{Q}$ different from b with the facets $t_a \leq t_d$. We are left with facets F_1 and F_2, corresponding to the two special extremal elements, that will disappear and be replaced with a single facet $F : t_a \leq t_b$.

In terms of vertices, any filter of $\mathcal{P}$ that contains a gives rise to a filter of $\mathcal{P} \oplus_{(a,b)} \mathcal{Q}$ only by adding all elements of $\mathcal{Q}$, and those are the only filters in that poset that contain a. Similarly, any filter in $\mathcal{Q}$ that does not contain b is also a filter of $\mathcal{P} \oplus_{(a,b)} \mathcal{Q}$, and those are the only filters in that polytope that do not contain b. This means that the only vertices that we have to deal with are those filters of $\mathcal{P} \oplus_{(a,b)} \mathcal{Q}$ that do not contain a but do contain b. The restriction of any such filter to the elements of $\mathcal{P}$ and $\mathcal{Q}$ gives rise to a pair of filters in those posets. Moreover, the union of a filter of $\mathcal{P}$ not containing a and a filter of $\mathcal{Q}$ containing b is always a valid filter in $\mathcal{P} \oplus_{(a,b)} \mathcal{Q}$.

This means that a vertex in $\mathcal{O}(\mathcal{P} \oplus_{(a,b)} \mathcal{Q})$ can be identified with either a vertex in V, where V is the set of vertices in $\mathcal{O}(\mathcal{P})$ in the facet F_1, with a vertex in W, where W is the set of vertices in $\mathcal{O}(\mathcal{Q})$ in the facet F_2, or with a pair of vertices $(v, w) \in \bar{V} \times \bar{W}$ where $\bar{V}$ and $\bar{W}$ are, respectively, the vertices of $\mathcal{O}(\mathcal{P})$ not in F_1 and those of of $\mathcal{O}(\mathcal{Q})$ not in F_2. It is now easy to check that if the slack matrices of $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{Q})$ are the ones below on the left and center, with the identifications introduced above, then the slack matrix of $\mathcal{O}(\mathcal{P} \oplus_{(a,b)} \mathcal{Q})$ is the one below on the right.

$$\begin{array}{c} \\ \mathcal{F} \\ F_1 \end{array}\!\!\begin{array}{c} \begin{matrix} V & \bar{V} \end{matrix} \\ \begin{bmatrix} A & B \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \mathcal{F}' \\ F_2 \end{array}\!\!\begin{array}{c} \begin{matrix} W & \bar{W} \end{matrix} \\ \begin{bmatrix} C & D \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \mathcal{F} \\ \mathcal{F}' \\ F \end{array}\!\!\begin{array}{c} \begin{matrix} V & W & \bar{V}\times\bar{W} \end{matrix} \\ \begin{bmatrix} A & O & B \otimes \mathbf{1}^T \\ O & C & \mathbf{1}^T \otimes D \\ \mathbf{0}^T & \mathbf{0}^T & \mathbf{1}^T \end{bmatrix} \end{array}$$

The last matrix is precisely the slack matrix of $\mathcal{O}(\mathcal{P}) \otimes_F \mathcal{O}(\mathcal{Q})$, concluding the proof. □

3.2 Graphicality of order polytopes

In this section, we put together the work developed in the previous chapter and in the previous section to derive sufficient conditions for graphicality, and consequently projective uniqueness, to arise in order polytopes. In particular, we will prove that every finite ranked poset with no 3-antichain has a graphic order polytope.

We begin by stating several graphicality results that follow immediately from the descriptions of order polytopes in Section 3.1 along with the general results on graphicality in Chapter 2. The operations denoted in this statement are the ones introduced in Definitions 3.1.1 and 3.1.7.

Proposition 3.2.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be finite posets on disjoint ground sets. Then if $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{Q})$ are projectively unique (respectively graphic), so are $\mathcal{O}(\mathcal{P}_{\text{rev}})$, $\mathcal{O}(\mathcal{P} \vee \mathcal{Q})$, $\mathcal{O}(\mathcal{P} \oplus \mathcal{Q})$ and $\mathcal{O}(\mathcal{P} \oplus_{(a,b)} \mathcal{Q})$.*

Proof. This is immediate from Propositions 3.1.3 and 3.1.9 that describe the effect of the operations in the order polytopes and Theorems 1.2.6 and 2.0.1 that show that they all preserve projective uniqueness and graphicality. □

Another operation that preserves graphicality is the splitting of extremal elements, but that is a little more delicate to show.

Proposition 3.2.2. *Let $\mathcal{P}$ be a poset such that $\mathcal{O}(\mathcal{P})$ is graphic, and c one of its extremal elements. Then $\mathcal{O}(\mathcal{P}_c)$ is graphic.*

Proof. From Proposition 3.1.6 we know that the operation of splitting c correspond to facet wedging in the order polytope with respect to the facet F given by the extremal element c. Moreover, Proposition 2.4.5 gives us a sufficient condition for such operation to preserve graphicality: that removing the facet F and all of its neighbors from the non-incidence graph of $\mathcal{O}(\mathcal{P})$ does not create any new connected components.

We may assume c is a maximal element since reversing the poset preserves graphicality. Take any pair of facets $\bar{F}$ and $\tilde{F}$. There must be a path Γ between them in the original non-incidence graph, say $\bar{F} w_0 F_1 w_1 \ldots F_t w_t \tilde{F}$. Recall that each w_i corresponds to a filter of the poset and that adding a maximal element to a filter preserves the filter property, so let $w_i' = w_i \cup \{c\}$. None of the w_i' is a neighbor of F.

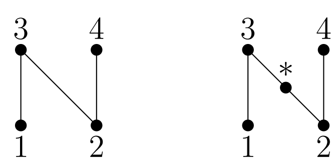
Now form a new sequence Γ' by starting with Γ and replacing w_i by w_i' for each i. If $w_i' \neq w_i$, then the only facet that neighbors w_i but not w_i' is F, so the only potential problem is that F might belong to Γ, in which case Γ' lacks one edge to be a path. In this case, let F' be the facet associated to a cover $d \lessdot c$ (or the facet induced by c being a minimal element if c is both maximal and minimal) and modify Γ' by replacing F by F'. Now if w_i neighbors F, then it corresponds to a filter that does not contain c, hence also does not contain d. Then w_i' corresponds to a filter that does contain c but does not contain d, so w_i' neighbors F'. Thus Γ' is now a path from $\bar{F}$ to $\tilde{F}$ that avoides F and all of its neighbors, so the condition in Proposition 2.4.5 is satisfied. □

Two other operations were defined in Section 3.1 the direct sum of posets and cover splitting. It is not hard to see that these operations do not universally preserve graphicality.

Example 3.2.3. If we consider $\mathcal{P}$ and $\mathcal{Q}$ to be posets on one and two elements and no relations as represented below, then their order polytopes are a segment and a square, both of which are graphic. However $\mathcal{O}(\mathcal{P}+\mathcal{Q})$ is a cube, which is not projectively unique and hence not graphic

$\mathcal{P}$ $\mathcal{Q}$ $\mathcal{P}+\mathcal{Q}$

Similarly, if $\mathcal{R}$ is the poset shown below, it is not hard to see that its order polytope is graphic, as $\mathcal{P}$ is a partial ordinal sum of two posets with two elements and no relations. However, one can computationally check that $\mathcal{R}_{2\lessdot 3}$ is not projectively unique or graphic.



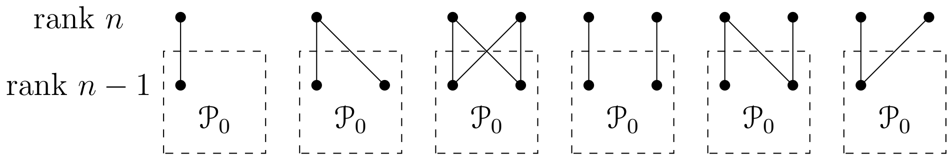
$\mathcal{R}$ $\mathcal{R}_{2\lessdot 3}$

In both of these cases, non-graphicality seems related to the antichain of size three. If $\mathcal{P}$ is a poset with an antichain of size three, then $\mathcal{O}(\mathcal{P})$ always has a face which is a 3-cube. This face is not projectively unique. This does not imply that $\mathcal{O}(\mathcal{P})$ itself cannot be projectively unique (see the discussion of non-prescribable faces in [9]) but it strongly suggests that it may not be. If we rule these antichains out, we can show graphicality for ranked posets.

Theorem 3.2.4. *Let $\mathcal{P}$ be a finite ranked poset with no 3-antichain. Then $\mathcal{O}(\mathcal{P})$ is graphic, and therefore projectively unique.*

Proof. We prove this by induction on the rank of $\mathcal{P}$. If $\mathcal{P}$ has rank 0, then its order polytope is a segment or a square, both of which are graphic. Also note that if there are no 3-antichains, then the only possibility for the Hasse diagram to be disconnected is that $\mathcal{P}$ is the direct sum of two chains. In this case, $\mathcal{O}(\mathcal{P})$ is graphic, as it can be attained from successively splitting maximal elements starting with the poset of two unrelated elements. Thus our strategy will be to assume that this happens for all finite ranked posets of rank $n-1$ and prove it for finite ranked posets of rank n such that the Hasse diagram is connected.

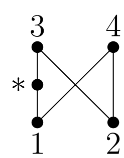
Note that elements of top rank are maximal and elements of constant rank form an antichain. Let $\mathcal{P}_0$ be the poset obtained from $\mathcal{P}$ by removing its top ranked elements, which we can suppose without loss of generality have rank n. The Hasse diagram of $\mathcal{P}$ is constructed from that of $\mathcal{P}_0$ by connecting the rank n elements to the rank $n-1$ elements in one of the following ways.



Note now that the first and fourth cases are obtained by splitting maximal elements of $\mathcal{P}_0$. The second, third and sixth cases are ordinal sums of $\mathcal{P}_0$ with the posets of a single element (in the second) and two unrelated elements (in the third and sixth). Note that, in these three cases, the rank $n-1$ elements drawn must be the full set of maximal elements of $\mathcal{P}_0$, as otherwise there would be a 3-antichain in $\mathcal{P}$. Finally, the fifth case is a partial ordinal sum of $\mathcal{P}_0$ with the poset of two unrelated elements. Since we have seen that all these operations preserve graphicality, the result follows. □

Note that this theorem is not exhaustive of all graphic order polytopes, since the ranked condition is not necessary.

Example 3.2.5. Let $\mathcal{P}$ be the following poset.

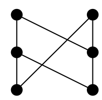


This is not a ranked poset, but it is easy to check computationally that it is graphic. In fact, it can be attained from a ranked poset with no 3-antichains by splitting a cover, and one could show more generally that cover splitting preserves graphicality under mild assumptions (essentially that it does not create a 3-antichain).

To conclude this discussion we present two conjectures on the necessary and sufficient conditions for order polytopes to be graphic and projectively unique. First we conjecture that the ranked condition can simply be dropped from Theorem 3.2.4.

Conjecture 3.2.6. *Let $\mathcal{P}$ be a finite poset (not necessarily ranked) with no 3-antichain. Then $\mathcal{O}(\mathcal{P})$ is graphic.*

Example 3.2.7. Dealing with cover-splitting would not be sufficient to prove Conjecture 3.2.6. The following unranked poset cannot be obtained by cover-splitting or any of the previously considered operations from another poset with fewer elements.



However, with the help of Antonio Macchia and Amy Wiebe and their Macaulay2 package for slack ideals [18], we were able to verify that the order polytope of this poset is at least projectively unique.

Secondly, we believe that having no 3-antichain is actually a necessary condition even for projective uniqueness.

Conjecture 3.2.8. *Let $\mathcal{P}$ be a finite poset. If $\mathcal{O}(\mathcal{P})$ is projectively unique, then $\mathcal{P}$ has no antichain of size 3.*

Note that these two conjectures together would in particular imply that projectively unique order polytopes are all graphic.

Chapter 4

Complex psd-minimal polygons

In this chapter, we begin the study of complex psd-minimal polytopes. In Section 1.6, we learn about K-lifts of polytopes where K is a closed convex cone. We remarked what happens when K is $\mathbb{R}^k_+$ and particularly when it is $\mathcal{S}^k_+$. But now we consider another interesting case, when K is the cone of $k \times k$ positive semidefinite Hermitian matrices $\mathcal{H}^k_+$. This is a closed convex cone in the real space $\mathcal{H}^k$ of $k \times k$ Hermitian matrices which, as in the two previous cases, happens to be self-dual.

There are several good reasons to be interested in complex positive semidefinite lifts, and particularly on minimal ones. On the one hand, we will see that a polytope with a complex positive semidefinite lift of size k has a real positive semidefinite lift of size at most $2k$. This means that if we can find an easy characterization for polytopes with small complex positive semidefinite lifts, we can get a large class of polytopes with small (although slightly larger) real positive semidefinite lifts. On the other hand, as seen in Theorem 1.6.3, complex positive semidefinite lifts are related with complex positive semidefinite factorizations of matrices. Going from factorizations of matrices using real positive semidefinite to those using complex positive semidefinite factorizations is a very natural step, and the existance of small such factorizations, which is equivalent to the existence of small complex postive semidenifite lifts, plays a relevant role in quantum information theory (see for example [16] and [26]).

In Section 1.6, we see the relation between spectrahedra and affine slices of the cones $\mathcal{S}^k_+$ which makes us consider in this new case objects of the form

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

where $A_0, A_1, \ldots, A_n \in \mathbb{C}^{k\times k}$ are Hermitian matrices of the same size. It happens that these objects are still spectrahedra. That is, we can replace the Hermitian matrices with real symmetric matrices. To see this, note that if we have a $k \times k$ Hermitian matrix A and replace each of its entries $a + bi$ with the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

we obtain a $2k \times 2k$ real symmetric matrix $\hat{A}$. Indeed the mapping $A \mapsto \hat{A}$ defines an isomorphism between the real vector space $\mathcal{H}^k$ of Hermitian matrices of size $k \times k$ and the real vector space $\mathcal{S}^{2k}$ of real symmetric matrices of size $2k \times 2k$, and we have

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\} = \{\mathbf{x} \in \mathbb{R}^n : \widehat{A_0} + x_1 \widehat{A_1} + \ldots + x_n \widehat{A_n} \succeq 0\}.$$

Many of the definitions and results we presented in Section 1.6, about (real) psd-minimality can be transformed easily into the complex world with minor changes and practically the same proofs.

Definition 4.0.1. The *complex positive semidefinite rank*, or *complex psd-rank* for short, of a polytope P is the smallest k for which there is a $\mathcal{H}^k_+$-lift of P. Alternatively, it is the smallest k for which there are real Hermitian matrices $A_0, A_1, \ldots, A_n \in \mathbb{R}^{k \times k}$ such that

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

is a spectrahedral lift of P. The complex psd-rank of the polytope P is denoted by $\mathrm{rank}^{\mathbb{C}}_{\mathrm{psd}}(P)$.

Proposition 4.0.2 ([7])**.** *If P is a d-polytope, then* $\mathrm{rank}_{\mathrm{psd}}(P) \geq \mathrm{rank}^{\mathbb{C}}_{\mathrm{psd}}(P) \geq d + 1$.

Motivated by the proposition above, we say that a d-polytope P is *complex psd-minimal* if $\mathrm{rank}^{\mathbb{C}}_{\mathrm{psd}}(P) = d + 1$. If S is a spectrahedral lift of P such that

$$S = \{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

for some Hermitian matrices $A_0, A_1, \ldots, A_n \in \mathbb{C}^{(d+1)\times(d+1)}$, we say that S is a *complex psd-minimal lift* of P.

Again, we have a nice way to determine if a polytope is complex psd-minimal.

Theorem 4.0.3 ([7])**.** *Let P be a d-polytope, and let $S \in \mathbf{SC}_P$. Then P is complex psd-minimal if and only if there is a complex matrix M of rank $d + 1$ such that $S = M \odot \overline{M}$.*

The above theorem can be paraphrased in the following way.

Theorem 4.0.4 ([7])**.** *Let P be a d-polytope, and let $S_P(x_1, \ldots, x_k)$ be a scaled symbolic slack matrix. If $S_P(\boldsymbol{\alpha}) \in \mathbf{SC}_P$, then P is complex psd-minimal if and only if there is $\boldsymbol{\zeta} \in (\mathbb{C}^*)^k$ such that $S_P(\boldsymbol{\alpha}) = S_P(\boldsymbol{\zeta}) \odot \overline{S_P(\boldsymbol{\zeta})}$ and* $\mathrm{rank}\, S_P(\boldsymbol{\zeta}) = d + 1$.

The proof of the following proposition is virtually the same as in the real case.

Proposition 4.0.5. *If P is a complex psd-minimal polytope, then*

1. *it has complex psd-minimal faces.*
2. *any projectively equivalent polytope is also complex psd-minimal.*
3. *there is a dual polytope that is also complex psd-minimal.*

As in the real case, for a given dimension d, there are only finitely many complex psd-minimal d-polytopes. This is because any polytope with complex psd-rank $d+1$ has a real psd-rank of at most $2(d+1)$, and there are only finitely many d-polytopes with this property as we remarked in Section 1.7.

It can be said that the study of complex psd-minimal polytopes began in [7] where the authors proved that the regular hexagon is complex psd-minimal but pentagons are not. It is important to remark that (real) psd-minimal polytopes are also complex psd-minimal, and the regular hexagon is the first example of a polytope that is complex psd-minimal but not (real) psd-minimal.

Until now, we did not even know which are the hexagons that work, nor what happens with polygons with more than six vertices. In this chapter we will give the complete characterization of the polygons that are complex psd-minimal. Namely, we will prove that the complex psd-minimal polygons are precisely the triangles, the quadrilaterals, and the Pappus hexagons, which we will define in the next section. This will give closure to complex psd-minimality in 2 dimensions. In the next chapter, we will initiate the study of complex psd-minimal 3-polytopes.

Recall that if P is a d-polytope, then $\text{Minors}_{d+2}(S_P(\mathbf{x})) := \langle (d+2) - \text{minors of } S_P(\mathbf{x})\rangle$, and the saturation of this ideal with respect to all variables is the slack ideal I_P. We present now the main tool used to prove that a polytope is not complex psd-minimal.

Lemma 4.0.6 ([7], Trinomial Obstruction). *Let P be a d-polytope, and let $S_P(\mathbf{x}) = S_P(x_1, \ldots, x_k)$ be a scaled symbolic slack matrix of P. Suppose there is a trinomial of the form $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} + \mathbf{x}^{\mathbf{c}}$, $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{N}^k$, in $\text{Minors}_{d+2}(S_P(\mathbf{x}))$. If there are $\boldsymbol{\alpha}, \boldsymbol{\zeta} \in \mathbb{C}^k$ such that $S_P(\boldsymbol{\alpha}) = S_P(\boldsymbol{\zeta}) \odot \overline{S_P(\boldsymbol{\zeta})}$ with $\operatorname{rank} S_P(\boldsymbol{\alpha}) = \operatorname{rank} S_P(\boldsymbol{\zeta}) = d+1$, then $\operatorname{Re}(\boldsymbol{\zeta}^{\mathbf{a}}\overline{\boldsymbol{\zeta}^{\mathbf{c}}}) = 0$.*

Proof. Since $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} + \mathbf{x}^{\mathbf{c}} \in \text{Minors}_{d+2}(S_P(\mathbf{x}))$ and $\operatorname{rank} S_P(\boldsymbol{\alpha}) = \operatorname{rank} S_P(\boldsymbol{\zeta}) = d+1$, we have $\boldsymbol{\alpha}^{\mathbf{a}} - \boldsymbol{\alpha}^{\mathbf{b}} + \boldsymbol{\alpha}^{\mathbf{c}} = 0$ and $\boldsymbol{\zeta}^{\mathbf{a}} - \boldsymbol{\zeta}^{\mathbf{b}} + \boldsymbol{\zeta}^{\mathbf{c}} = 0$. So $\boldsymbol{\alpha}^{\mathbf{b}} = \boldsymbol{\alpha}^{\mathbf{a}} + \boldsymbol{\alpha}^{\mathbf{c}}$ and $\boldsymbol{\zeta}^{\mathbf{b}} = \boldsymbol{\zeta}^{\mathbf{a}} + \boldsymbol{\zeta}^{\mathbf{c}}$. Thus

$$\begin{aligned}\boldsymbol{\alpha}^{\mathbf{a}} + \boldsymbol{\alpha}^{\mathbf{c}} &= \boldsymbol{\alpha}^{\mathbf{b}} \\ &= \boldsymbol{\zeta}^{\mathbf{b}}\overline{\boldsymbol{\zeta}^{\mathbf{b}}} \\ &= (\boldsymbol{\zeta}^{\mathbf{a}} + \boldsymbol{\zeta}^{\mathbf{c}})\left(\overline{\boldsymbol{\zeta}^{\mathbf{a}}} + \overline{\boldsymbol{\zeta}^{\mathbf{c}}}\right) \\ &= \boldsymbol{\alpha}^{\mathbf{a}} + 2\operatorname{Re}(\boldsymbol{\zeta}^{\mathbf{a}}\overline{\boldsymbol{\zeta}^{\mathbf{c}}}) + \boldsymbol{\alpha}^{\mathbf{c}}.\end{aligned}$$

It follows that $\operatorname{Re}(\boldsymbol{\zeta}^{\mathbf{a}}\overline{\boldsymbol{\zeta}^{\mathbf{c}}}) = 0$. □

Observation. In the previous lemma, we can replace $\text{Minors}_{d+2}(S_P(\mathbf{x}))$ by the bigger ideal I_P, but we do not need it for the results presented here.

4.1 Characterization of complex psd-minimal polygons

The purpose of this section is to characterize all the complex psd-minimal polygons. More precisely, we will prove that the complex psd-minimal polygons are precisely the triangles, the quadrilaterals, and the Pappus hexagons, which we will define now.

Definition 4.1.1 (Concurrent Lines)**.** Given three different lines in $\mathbb{R}^d$, we say that they are *concurrent* if the three of them intersect in a single point or if they are parallel (intersect at infinity).

Definition 4.1.2 (Pappus Hexagon)**.** A hexagon with consecutive vertices $\mathbf{v}_1, \ldots, \mathbf{v}_6$ is called a *Pappus' hexagon* if the following conditions hold:

- The lines $\overleftrightarrow{\mathbf{v}_1\mathbf{v}_2}$, $\overleftrightarrow{\mathbf{v}_3\mathbf{v}_6}$ and $\overleftrightarrow{\mathbf{v}_4\mathbf{v}_5}$ are concurrent
- The lines $\overleftrightarrow{\mathbf{v}_2\mathbf{v}_3}$, $\overleftrightarrow{\mathbf{v}_1\mathbf{v}_4}$ and $\overleftrightarrow{\mathbf{v}_5\mathbf{v}_6}$ are concurrent
- The lines $\overleftrightarrow{\mathbf{v}_1\mathbf{v}_6}$, $\overleftrightarrow{\mathbf{v}_2\mathbf{v}_5}$ and $\overleftrightarrow{\mathbf{v}_3\mathbf{v}_4}$ are concurrent.

We called them this way, since these are the type of hexagons that appear in the dual version of the Pappus' Theorem in projective geometry, which says that if a hexagon satisfies two of the three conditions above, then it also satisfies the other one. Compare the definition with the related notion of Desarguian hexagon, which plays a role in linear extension complexity [20]. For examples of Pappus hexagons, see Figure 4.1.

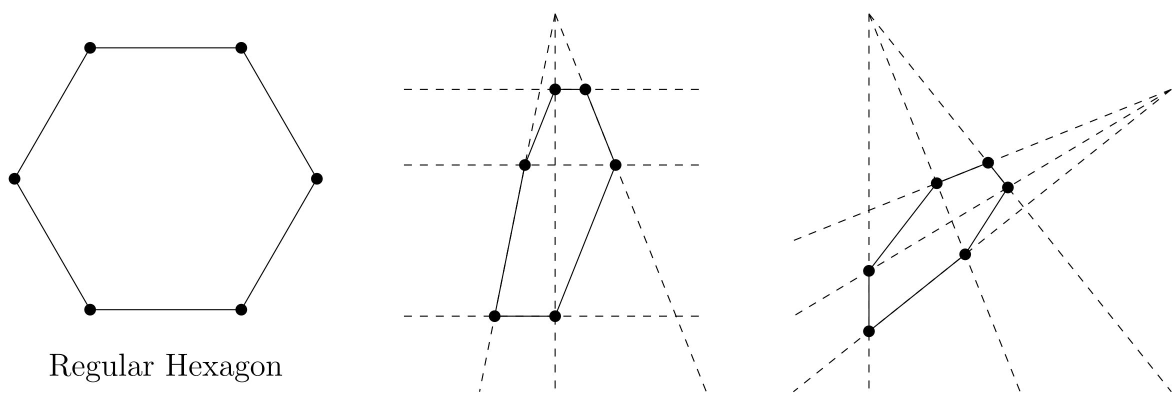


Figure 4.1: Examples of Pappus hexagons

As was mentioned before, from [7] we know that pentagons are not complex psd-minimal. We present now this proof for completeness in our treatment of polygons.

Lemma 4.1.3 ([7])**.** *Pentagons are not complex psd-minimal.*

Proof. The symbolic slack ideal of a pentagon P is

$$\begin{bmatrix} 0 & 0 & z_1 & z_2 & z_3 \\ z_4 & 0 & 0 & z_5 & z_6 \\ z_7 & z_8 & 0 & 0 & z_9 \\ z_{10} & z_{11} & z_{12} & 0 & 0 \\ 0 & z_{13} & z_{14} & z_{15} & 0 \end{bmatrix},$$

which can be scaled to

$$S_P(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 1 & x_1 & 1 \\ 1 & 0 & 0 & 1 & x_2 \\ x_3 & 1 & 0 & 0 & 1 \\ 1 & x_4 & 1 & 0 & 0 \\ 0 & 1 & x_5 & x_6 & 0 \end{bmatrix}.$$

Suppose P is complex psd-minimal. Then there are $\boldsymbol{\alpha} \in \mathbb{R}^6_{++}$ and $\boldsymbol{\zeta} \in (\mathbb{C}^*)^6$ such that $S_P(\boldsymbol{\alpha})$ is a slack matrix of P, $\operatorname{rank} S_P(\boldsymbol{\zeta}) = 3$, and $S_P(\boldsymbol{\alpha}) = S_P(\boldsymbol{\zeta}) \odot \overline{S_P(\boldsymbol{\zeta})}$. Since the polynomials $x_1 - x_3x_4 + 1$, $x_2 - x_4x_5 + 1$ and $x_4 - x_1x_2 + 1$ are 4-minors of $S_P(\mathbf{x})$, by Lemma 4.0.6, we have $\operatorname{Re}(\zeta_1) = \operatorname{Re}(\zeta_2) = \operatorname{Re}(\zeta_4) = 0$. That is, $\zeta_1, \zeta_2, \zeta_4 \in i\mathbb{R}$.

On the other hand, since $x_4 - x_1x_2 + 1$ is a 4-minors of $S_P(\mathbf{x})$ and $\operatorname{rank} S_P(\zeta) = 3$, then $\zeta_4 = \zeta_1\zeta_2 - 1$. Since $\zeta_1, \zeta_2 \in i\mathbb{R}$, $\zeta_4 = \zeta_1\zeta_2 - 1 \in \mathbb{R}$. Thus $\zeta_4 = 0$, a contradiction. □

We already know that the triangles and the quadrilaterals are (real) psd-minimal, so they are also complex psd-minimal. We also know that pentagons are not complex psd-minimal. So the plan is the following: first, we will rule out the n-gons with $n \geq 7$; next, we will prove that if a hexagon is complex psd-minimal, then it is a Pappus' hexagon; finally, we will find a Hadamard factorization as in Theorem 4.0.3 which certifies that Pappus hexagons are complex psd-minimal.

We will now introduce some notation we will use throughout this section. Let P be a n-gon with $n \geq 6$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_6$ be six consecutive vertices of P, $F_1 := [\mathbf{v}_1, \mathbf{v}_2], \ldots, F_5 := [\mathbf{v}_5, \mathbf{v}_6]$ and let F_6 be the other facet containing $\mathbf{v}_6$. Then the symbolic slack matrix of P has a submatrix of the form

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array} \\ \begin{bmatrix} 0 & 0 & z_1 & z_2 & z_3 & z_4 \\ z_5 & 0 & 0 & z_6 & z_7 & z_8 \\ z_9 & z_{10} & 0 & 0 & z_{11} & z_{12} \\ z_{13} & z_{14} & z_{15} & 0 & 0 & z_{16} \\ z_{17} & z_{18} & z_{19} & z_{20} & 0 & 0 \\ \boxed{z_{25}} & z_{21} & z_{22} & z_{23} & z_{24} & 0 \end{bmatrix} \end{array},$$

where $z_1 \ldots, z_{24}$ are variables, and z_{25} is a variable if $n > 7$ and 0 if $n = 6$.

We scale the symbolic slack matrix using the same scaling as in Example 1.7.6 in Chapter 1, Section 1.7. Thus we have a scaled symbolic slack matrix $S_P(x_1, \ldots, x_k)$ that has a submatrix that looks like

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & x_1 & x_2 & 1 \\ 1 & 0 & 0 & 1 & x_3 & x_4 \\ x_5 & 1 & 0 & 0 & 1 & x_6 \\ 1 & x_7 & 1 & 0 & 0 & x_8 \\ x_9 & 1 & x_{10} & 1 & 0 & 0 \\ \boxed{x_{14}} & x_{11} & x_{12} & x_{13} & 1 & 0 \end{bmatrix} \end{array},$$

where $x_1 \dots, x_{13}$ are variables, and x_{14} is a variable if $n > 7$ and 0 if $n = 6$.

The scaling used will be used again in this and the following section, and we will refer to it as **our scaling**. Selecting a scaling simplifies calculations, and the one we use gives us several trinomial 4-minors which are helpful due to Lemma 4.0.6. Now, the following polynomials belong to the ideal J generated by the 4-minors of the above submatrix (and thus of the scaled symbolic matrix):

- $x_1 - x_5x_7 + 1$
- $x_7 - x_1x_3 + x_2$
- $x_3 - x_7x_{10} + 1$
- $x_7x_9 - 1 + x_2$
- $x_3 - x_2x_4 + x_8$
- $x_8x_{10} - x_3x_6 + x_4$
- $x_8x_{14} - x_8x_9x_{11} + 1$
- $x_8x_{10}x_{11} - x_8x_{12} + x_4$

If P is a complex psd-minimal n-gon with $n \geq 6$, then we can find $\boldsymbol{\alpha} \in \mathbb{R}^k_{++}$ and $\boldsymbol{\zeta} \in (\mathbb{C}^*)^k$ such that $S_P(\boldsymbol{\alpha}) = S_P(\boldsymbol{\zeta}) \odot \overline{S_P(\boldsymbol{\zeta})}$ with rank $S_P(\boldsymbol{\alpha}) = \text{rank}\, S_P(\boldsymbol{\zeta}) = 3$, and we have the following result.

Proposition 4.1.4. *If P is a complex psd-minimal n-gon with $n \geq 6$ and $\boldsymbol{\alpha}$ and $\boldsymbol{\zeta}$ are as above, then*

1. *P is a hexagon.*
2. *$\zeta_3 \in i\mathbb{R}$, $\zeta_8 \in \mathbb{R}$ and $\alpha_9 = \zeta_9 = 1$.*

Proof. Using Lemma 4.0.6 with the above polyonomials, we obtain:

- $\text{Re}(\zeta_1) = 0$
- $\text{Re}(\zeta_7\overline{\zeta_2}) = 0$
- $\text{Re}(\zeta_3) = 0$
- $\text{Re}(\zeta_7\zeta_9\overline{\zeta_2}) = 0$
- $\text{Re}(\zeta_3\overline{\zeta_8}) = 0$
- $\text{Re}(\zeta_8\zeta_{10}\overline{\zeta_4}) = 0$
- $\text{Re}(\zeta_8\zeta_{14}) = 0$
- $\text{Re}(\zeta_8\zeta_{10}\zeta_{11}\overline{\zeta_4}) = 0.$

These results can be used to prove that $\zeta_1 \in i\mathbb{R}$, $\zeta_3 \in i\mathbb{R}$, $\zeta_8 \in \mathbb{R}$, $\zeta_{14} \in i\mathbb{R}$, $\zeta_9 \in \mathbb{R}$, and $\zeta_{11} \in \mathbb{R}$.

Since the polynomial $x_8x_{14} - x_8x_9x_{11} + 1$ is in J and rank $S_P(\boldsymbol{\zeta}) = 3$, we have $\zeta_8\zeta_{14} - \zeta_8\zeta_9\zeta_{11} + 1 = 0$ and thus

$$\zeta_{14} = \frac{\zeta_8\zeta_9\zeta_{11} - 1}{\zeta_8} \in \mathbb{R}.$$

Since $\zeta_{14} \in i\mathbb{R}$ and $\zeta_{14} \in \mathbb{R}$, $\zeta_{14} = 0$. Therefore P has to be a hexagon, proving (1).

Since $x_7 - x_1x_3 + x_2$ and $x_7x_9 - 1 + x_2$ belong to J, $\zeta_7 - \zeta_1\zeta_3 + \zeta_2 = 0$ and $\zeta_7\zeta_9 - 1 + \zeta_2 = 0$. Thus $\zeta_1\zeta_3 - \zeta_7 = 1 - \zeta_7\zeta_9$, from which we obtain $\zeta_7(\zeta_9 - 1) = 1 - \zeta_1\zeta_3$. After taking imaginary part at both sides, we get $(\zeta_9 - 1)\,\text{Im}(\zeta_7) = 0$.

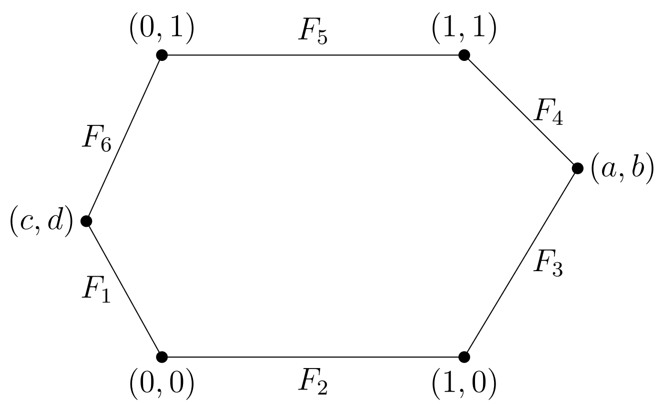
Suppose $\mathrm{Im}(\zeta_7) = 0$, that is $\zeta_7 \in \mathbb{R}$. Then $\zeta_7 - \zeta_1\zeta_3 + \zeta_2 = 0$ implies that $\zeta_2 \in \mathbb{R}$. Now we saw that $\mathrm{Re}(\zeta_7\overline{\zeta_2}) = 0$, so we also have $\zeta_2 \in i\mathbb{R}$. Thus $\zeta_2 = 0$, a contradiction. So $\zeta_9 = 1$, and thus $\alpha_9 = 1$, concluding the proof of (2). $\square$

Proposition 4.1.4 tell us that any n-gon with $n > 6$ cannot be complex psd-minimal. Now we want to prove that the complex psd-minimal hexagons are precisely the Pappus hexagons.

Proposition 4.1.5. *If P is a complex psd-minimal hexagon, then it has to be a Pappus hexagon.*

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_6$ be the consecutive vertices of P. We only prove that the lines $\overleftrightarrow{\mathbf{v}_1\mathbf{v}_2}$, $\overleftrightarrow{\mathbf{v}_3\mathbf{v}_6}$ and $\overleftrightarrow{\mathbf{v}_4\mathbf{v}_5}$ are concurrent, since the proofs of the other conditions are analogous.

We can always find a projective transformation that sends $\mathbf{v}_1$ to (0,0), $\mathbf{v}_2$ to (1,0), $\mathbf{v}_4$ to (1,1), and $\mathbf{v}_5$ to (0,1). Thus P can be projectively transformed to a hexagon of the form



where $a > 1$, $c < 0$ and $0 < b, d < 1$. Since complex psd-minimality is preserved by projective equivalence, this new hexagon is still complex-psd minimal.

We use that

- $F_1 : dx - cy \geq 0$
- $F_2 : y \geq 0$
- $F_3 : (a-1)y - b(x-1) \geq 0$
- $F_4 : (b-1)(x-1) - (y-1)(a-1) \geq 0$
- $F_5 : 1 - y \geq 0$
- $F_6 : c(y-1) - (d-1)x \geq 0$

to obtain the slack matrix

$$\begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{array} \begin{array}{cccccc} (c,d) & (0,0) & (1,0) & (a,b) & (1,1) & (0,1) \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & d & ad-bc & d-c & -c \\ d & 0 & 0 & b & 1 & 1 \\ (a-1)d-b(c-1) & b & 0 & 0 & a-1 & a+b-1 \\ (b-1)(c-1)-(d-1)(a-1) & a-b & a-1 & 0 & 0 & 1-b \\ 1-d & 1 & 1 & 1-b & 0 & 0 \\ 0 & -c & 1-c-d & c(b-1)-(d-1)a & 1-d & 0 \end{bmatrix}.$$

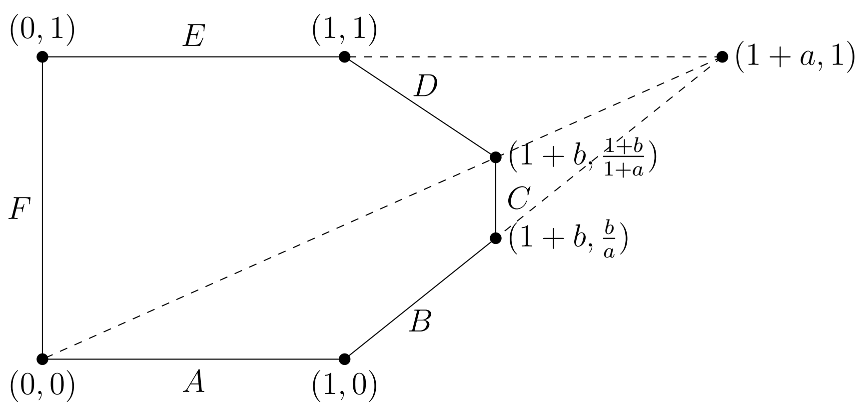
We scale it with our scaling, and this give us a new matrix whose $(F_5, (c,d))$-entry is now $\frac{b(d-1)}{(b-1)d}$. According to Proposition 4.1.4 ($\alpha_9 = 1$), we have $\frac{b(d-1)}{(b-1)d} = 1$, thus $b = d$. This implies that $\overleftrightarrow{(0,0)(1,0)}$, $\overleftrightarrow{(c,d)(a,b)}$ and $\overleftrightarrow{(0,1)(1,1)}$ are concurrent. Since concurrency is preserved by projective transformations, $\overleftrightarrow{\mathbf{v}_1\mathbf{v}_2}$, $\overleftrightarrow{\mathbf{v}_3\mathbf{v}_6}$ and $\overleftrightarrow{\mathbf{v}_4\mathbf{v}_5}$ are also concurrent. □

Lemma 4.1.6. *Let P be a Pappus hexagon. Take any slack matrix of P and scale it using our scaling, then the resulting matrix is of the form*

$$S_{a,b} := \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{a-b}{a(1+b)} & 1 \\ 1 & 0 & 0 & 1 & a & \frac{a(1+a)(1+b)}{a-b} \\ \frac{1+b}{b} & 1 & 0 & 0 & 1 & \frac{a(1+b)^2}{b(a-b)} \\ 1 & \frac{(1+a)b}{a(1+b)} & 1 & 0 & 0 & 1 \\ 1 & 1 & \frac{a(1+b)}{b} & 1 & 0 & 0 \\ 0 & 1 & \frac{a^2(1+b)^2}{b(a-b)} & \frac{(1+a)(1+b)}{a-b} & 1 & 0 \end{bmatrix}$$

for some $a > b > 0$.

Proof. Let P be a Pappus hexagon with consecutive vertices $\mathbf{v}_1, \ldots, \mathbf{v}_6$. Without loss of generality, we can suppose that P looks like



where $a > b > 0$. This is because we can send through a projective transformation $\mathbf{v}_1$ to (0,0), $\mathbf{v}_2$ to (1,0), $\mathbf{v}_5$ to (1,1) and $\mathbf{v}_6$ to (0,1), and use that projective transformations preserve concurrency.

We use that

- $A : y \geq 0$
- $B : ay - x + 1 \geq 0$
- $C : 1 + b - x \geq 0$
- $D : (1+a)b(1-y) - (a-b)(x-1) \geq 0$
- $E : 1 - y \geq 0$
- $F : x \geq 0$

to obtain the slack matrix of P

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} (0,0) & (1,0) & (1+b,\frac{b}{a}) & (1+b,\frac{1+b}{1+a}) & (1,1) & (0,1) \end{array} \\ \begin{bmatrix} 0 & 0 & \frac{b}{a} & \frac{1+b}{1+a} & 1 & 1 \\ 1 & 0 & 0 & \frac{a-b}{1+a} & a & 1+a \\ 1+b & b & 0 & 0 & b & 1+b \\ a(1+b) & (1+a)b & \frac{b(a-b)}{a} & 0 & 0 & a-b \\ 1 & 1 & \frac{a-b}{a} & \frac{a-b}{1+a} & 0 & 0 \\ 0 & 1 & 1+b & 1+b & 1 & 0 \end{bmatrix} \end{array}.$$

If we scaled this matrix with our scaling, we obtain the intended matrix. □

Let $S_{a,b}$ be as above, and

$$M_{\xi_1,\xi_2} := \begin{bmatrix} 0 & 0 & 1 & \frac{1}{\xi_1} & \xi_2 & 1 \\ 1 & 0 & 0 & 1 & \xi_1 & \frac{1+\xi_1}{\xi_2} \\ \frac{1+\xi_1}{\xi_1(1-\xi_2)} & 1 & 0 & 0 & 1 & \frac{1+\xi_1}{\xi_1\xi_2(1-\xi_2)} \\ 1 & 1-\xi_2 & 1 & 0 & 0 & 1 \\ 1 & 1 & \frac{1+\xi_1}{1-\xi_2} & 1 & 0 & 0 \\ 0 & 1 & \frac{1+\xi_1}{\xi_2(1-\xi_2)} & \frac{1+\xi_1}{\xi_1\xi_2} & 1 & 0 \end{bmatrix}$$

where $\xi_1 = \pm\sqrt{a}i$ and $\xi_2 = \frac{a-b}{a(1+b)} \pm \frac{\sqrt{(1+a)b(a-b)}}{a(1+b)}i$ (each of the four possibilities work). Then it can be seen that M_{ξ_1,ξ_2} has rank ≥ 3 by Lemma 1.3.3 since it has the same zero pattern of the slack matrices of hexagons. Its rank is exactly 3 since all of its 4-minors vanish. We also have $S_{a,b} = M_{\xi_1,\xi_2} \odot \overline{M_{\xi_1,\xi_2}}$. These observations combined with Lemma 4.1.6 and Theorem 4.0.3 give us the following result.

Proposition 4.1.7. *If P is a Pappus hexagon, then P is complex psd-minimal.*

Thus a hexagon is complex psd-minimal if and only if it is a Pappus hexagon. We also know that triangles and quadrilaterals are complex psd-minimal, but n-gons with $n = 5$ or $n > 6$ are not. Thus we can conclude the main result of this section.

Theorem 4.1.8 (Characterization of complex psd-minimal polygons)**.** *The complex psd-minimal polygons are precisely the triangles, the quadrilaterals, and the Pappus hexagons.*

We end up this section, complementing the relation between $S_{a,b}$ and M_{ξ_1,ξ_2}, explaining where this last matrix comes from, and that is going to be useful in Chapter 5.

Lemma 4.1.9. *Let*

$$S(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 1 & x_1 & x_2 & 1 \\ 1 & 0 & 0 & 1 & x_3 & x_4 \\ x_5 & 1 & 0 & 0 & 1 & x_6 \\ 1 & x_7 & 1 & 0 & 0 & x_8 \\ x_9 & 1 & x_{10} & 1 & 0 & 0 \\ 0 & x_{11} & x_{12} & x_{13} & 1 & 0 \end{bmatrix}$$

and $\boldsymbol{\alpha} \in \mathbb{R}^{13}_{++}$ *such that* $\operatorname{rank} S(\boldsymbol{\alpha}) = 3$. *Then*

$$S(\boldsymbol{\alpha}) = S(\boldsymbol{\zeta}) \odot \overline{S(\boldsymbol{\zeta})} \text{ for some } \boldsymbol{\zeta} \in (\mathbb{C}^*)^{13} \text{ with } \operatorname{rank} S(\boldsymbol{\zeta}) = 3 \qquad (*)$$

if and only if

$$S(\boldsymbol{\alpha}) = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{a} & \frac{a-b}{a(1+b)} & 1 \\ 1 & 0 & 0 & 1 & a & \frac{a(1+a)(1+b)}{a-b} \\ \frac{1+b}{b} & 1 & 0 & 0 & 1 & \frac{a(1+b)^2}{b(a-b)} \\ 1 & \frac{(1+a)b}{a(1+b)} & 1 & 0 & 0 & 1 \\ 1 & 1 & \frac{a(1+b)}{b} & 1 & 0 & 0 \\ 0 & 1 & \frac{a^2(1+b)^2}{b(a-b)} & \frac{(1+a)(1+b)}{a-b} & 1 & 0 \end{bmatrix}$$

for some $a > b > 0$. *In this case* $S(\boldsymbol{\zeta})$ *has to be of the form*

$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{\xi_1} & \xi_2 & 1 \\ 1 & 0 & 0 & 1 & \xi_1 & \frac{1+\xi_1}{\xi_2} \\ \frac{1+\xi_1}{\xi_1(1-\xi_2)} & 1 & 0 & 0 & 1 & \frac{1+\xi_1}{\xi_1\xi_2(1-\xi_2)} \\ 1 & 1-\xi_2 & 1 & 0 & 0 & 1 \\ 1 & 1 & \frac{1+\xi_1}{1-\xi_2} & 1 & 0 & 0 \\ 0 & 1 & \frac{1+\xi_1}{\xi_2(1-\xi_2)} & \frac{1+\xi_1}{\xi_1\xi_2} & 1 & 0 \end{bmatrix}$$

where $\xi_1 = \pm\sqrt{a}i$ *and* $\xi_2 = \frac{a-b}{a(1+b)} \pm \frac{\sqrt{(1+a)b(a-b)}}{a(1+b)}i$, *and each of the four possibilities satisfies* $(*)$.

Proof. We already proved the if and only if part. So we only need to prove that $S(\boldsymbol{\zeta})$ has to be of the given form.

By Proposition 4.1.4 ($\zeta_9 = 1$), we know that $S(\boldsymbol{\zeta})$ has to be of the form

$$\begin{bmatrix} 0 & 0 & 1 & \zeta_1 & \zeta_2 & 1 \\ 1 & 0 & 0 & 1 & \zeta_3 & \zeta_4 \\ \zeta_5 & 1 & 0 & 0 & 1 & \zeta_6 \\ 1 & \zeta_7 & 1 & 0 & 0 & \zeta_8 \\ 1 & 1 & \zeta_{10} & 1 & 0 & 0 \\ 0 & \zeta_{11} & \zeta_{12} & \zeta_{13} & 1 & 0 \end{bmatrix}$$

where the ζ_i's are nonzero complex numbers. After replacing each ζ_i with the variable x_i, let J' be the ideal generated by the 4-minors.

Now the following polynomials belong to the ideal J':

- $x_1x_3 - 1$
- $x_4x_2 - x_3 - x_8$
- $x_5(1 - x_2)x_3 - 1 - x_3$
- $x_6x_2(x_2 - 1)x_3 - (x_2 - 1)x_3 + (x_2x_3 + 1)x_8$

- $x_7 - 1 + x_2$
- $x_{10}(1 - x_2) - 1 - x_3$
- $x_{11}x_8 - 1$
- $x_{12}x_2(x_2 - 1)x_8 - (x_2 - 1)x_8 + x_2 + x_3$
- $x_{13}x_2(x_2 - 1)x_3 - (x_2 - 1)x_3 + (x_2x_3 + 1)x_8$.

We can use these polynomials to express the ζ_i's in terms of ζ_2, ζ_3 and ζ_8. Thus

$$S(\boldsymbol{\zeta}) = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{\zeta_3} & \zeta_2 & 1 \\ 1 & 0 & 0 & 1 & \zeta_3 & \frac{\zeta_3+\zeta_8}{\zeta_2} \\ \frac{1+\zeta_3}{(1-\zeta_2)\zeta_3} & 1 & 0 & 0 & 1 & \frac{(\zeta_2-1)\zeta_3-(\zeta_2\zeta_3+1)\zeta_8}{\zeta_2(\zeta_2-1)\zeta_3} \\ 1 & 1-\zeta_2 & 1 & 0 & 0 & \zeta_8 \\ 1 & 1 & \frac{1+\zeta_3}{1-\zeta_2} & 1 & 0 & 0 \\ 0 & \frac{1}{\zeta_8} & \frac{(\zeta_2-1)\zeta_8-\zeta_2-\zeta_3}{\zeta_2(\zeta_2-1)\zeta_8} & \frac{\zeta_3+\zeta_8}{\zeta_2\zeta_3\zeta_8} & 1 & 0 \end{bmatrix}.$$

By Proposition 4.1.4, $\zeta_3 \in i\mathbb{R}$. Since $\zeta_3\overline{\zeta_3} = a$, it follows that $\zeta_3 = \pm\sqrt{a}i$. We also have $(1-\zeta_2)(1-\overline{\zeta_2}) = \zeta_7\overline{\zeta_7} = \frac{(1+a)b}{a(1+b)}$; simplifying this expression lead us that $\operatorname{Re}(\zeta_2) = \frac{a-b}{a(1+b)}$. Since $\zeta_2\overline{\zeta_2} = \frac{a-b}{a(1+b)}$, then $\operatorname{Im}(\zeta_2) = \pm\frac{\sqrt{(1+a)b(a-b)}}{a(1+b)}$.

By Proposition 4.1.4, $\zeta_8 \in \mathbb{R}$. Since $\zeta_8\overline{\zeta_8} = 1$, then $\zeta_8 = \pm 1$. Suppose that $\zeta_8 = -1$, and let us arrive to a contradiction. In this scenario, $\zeta_6 = \frac{2\zeta_2\zeta_3-\zeta_3+1}{\zeta_2(\zeta_2-1)\zeta_3}$ and $\zeta_{12} = \frac{1-2\zeta_2-\zeta_3}{\zeta_2(1-\zeta_2)}$.

Since $\zeta_6\overline{\zeta_6} = \frac{a(1+b)^2}{b(a-b)}$, then $(2\zeta_2\zeta_3 - \zeta_3 + 1)(2\overline{\zeta_2\zeta_3} - \overline{\zeta_3} + 1) = 1 + a$. This simplifies to $4\operatorname{Re}(\zeta_2\zeta_3) = 1$.

Since $\zeta_{12}\overline{\zeta_{12}} = \frac{(1+a)(1+b)}{a-b}$, then $(1 - 2\zeta_2 - \zeta_3)(1 - 2\overline{\zeta_2} - \overline{\zeta_3}) = 1 + a$. This simplifies to $4\operatorname{Re}(\zeta_2\overline{\zeta_3}) = 1$ which implies that $4\operatorname{Re}(\zeta_2\zeta_3) = -1$ since $\zeta_3 \in i\mathbb{R}$, thus arriving at a contradiction.

Then $\zeta_8 = 1$, and this concludes the proof. □

4.2 An explicit complex psd-minimal lift for the regular hexagon

In this section we will explain how to construct a complex psd-minimal lift of a complex psd-minimal polytope P of dimension d, when we know a Hadamard factorization as in Theorem 4.0.3, separating the process in intermediate steps and using the regular hexagon as an example. Although we already know that the regular hexagon is complex psd-minimal from [7], this is the first time that a complex psd-minimal lift is presented explicitly. The procedure used here is the one that is used in the proof of Theorem 1.6.3 in [11].

Consider the regular hexagon

$$H := \operatorname{conv}\left\{\left(\cos\frac{2j\theta}{6}, \sin\frac{2j\theta}{6}\right) : j = 0, 1, 2, 3, 4, 5\right\}.$$

We need to find $n \in \mathbb{Z}^+$ and Hermitian matrices $A_1 \ldots, A_n$ of size 3×3 such that the spectahedron

$$\{\mathbf{x} \in \mathbb{R}^n : A_0 + x_1 A_1 + \ldots + x_n A_n \succeq 0\}$$

is a spectrahedral lift of H. We construct the lift in such a way that the regular hexagon is precisely the projection onto the first two conponents of the spectrahedron.

1. Find a slack matrix S of P.

Let $\mathbf{v}_j := \left(\cos \frac{2j\theta}{6}, \sin \frac{2j\theta}{6}\right)$ and using the facet descriptions

- $F_1 : 1 - x - \frac{y}{\sqrt{3}} \geq 0$
- $F_2 : 1 - \frac{2y}{\sqrt{3}} \geq 0$
- $F_3 : 1 + x - \frac{y}{\sqrt{3}} \geq 0$
- $F_4 : 1 + x + \frac{y}{\sqrt{3}} \geq 0$
- $F_5 : 1 + \frac{2y}{\sqrt{3}} \geq 0$
- $F_6 : 1 - x + \frac{y}{\sqrt{3}} \geq 0$

we obtain the nice slack matrix

$$S_0 = \begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}.$$

2. Find a matrix M of rank $d+1$ such that $S = M \odot \overline{M}$.

A factorization of S_0 as in Theorem 4.0.3 was already presented in [7]. We illustrate here a systematic procedure to find one, using Lemma 4.1.9, that works not only for the regular hexagon but for all Pappus' hexagons. Using our special scaling, we obtain the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & \frac{1}{2} & 1 \\ 1 & 0 & 0 & 1 & 1 & 4 \\ 4 & 1 & 0 & 0 & 1 & 8 \\ 1 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 8 & 4 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

which, according to Lemma 4.1.9, it can be written as

$$\begin{bmatrix} 0 & 0 & 1 & -i & \frac{1}{2}+\frac{1}{2}i & 1 \\ 1 & 0 & 0 & 1 & i & 2 \\ 2 & 1 & 0 & 0 & 1 & 2-2i \\ 1 & \frac{1}{2}-\frac{1}{2}i & 1 & 0 & 0 & 1 \\ 1 & 1 & 2i & 1 & 0 & 0 \\ 0 & 1 & 2+2i & -2i & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & i & \frac{1}{2}-\frac{1}{2}i & 1 \\ 1 & 0 & 0 & 1 & -i & 2 \\ 2 & 1 & 0 & 0 & 1 & 2+2i \\ 1 & \frac{1}{2}+\frac{1}{2}i & 1 & 0 & 0 & 1 \\ 1 & 1 & -2i & 1 & 0 & 0 \\ 0 & 1 & 2-2i & 2i & 1 & 0 \end{bmatrix}$$

where these two matrices have rank 3.

In general for matrices, if $S = DS_0D'$, where $D = \operatorname{diag}(d_1, \ldots, d_m)$ and $D' = \operatorname{diag}(d'_1, \ldots, d'_n)$ are positive diagonal matrices, and $S = M \odot \overline{M}$ ($M = [m_{ij}]$), then $S_0 = M_0 \odot \overline{M_0}$ where $M_0 := [\frac{m_{ij}}{d_i d_{j'}}]$ (since M_0 is a positive scaling of M, they have the same rank). Using this, we obtain that $S_0 = M_0 \odot \overline{M_0}$, where

$$M_0 := \begin{bmatrix} 0 & 0 & 1 & -\sqrt{2}i & 1+i & 1 \\ 1 & 0 & 0 & 1 & \sqrt{2}i & \sqrt{2} \\ \sqrt{2} & 1 & 0 & 0 & 1 & 1-i \\ \sqrt{2} & 1-i & 1 & 0 & 0 & 1 \\ 1 & \sqrt{2} & \sqrt{2}i & 1 & 0 & 0 \\ 0 & 1 & 1+i & -\sqrt{2}i & 1 & 0 \end{bmatrix}$$

has rank 3.

3. Suppose P has vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and facets $F_i : h_i(\mathbf{t}) \geq 0$, $1 \leq i \leq m$, so that $S = [h_i(\mathbf{v}_j)]_{m \times n}$. By Theorem 1.6.5, there are matrices $A_1, \ldots, A_m, B_1, \ldots, B_n \in \mathcal{H}_+^{d+1}$ such that $h_i(\mathbf{v}_j) = \langle A_i, B_j \rangle$, where $\langle A, B \rangle := \operatorname{Trace}(AB)$ is the standard inner product in $\mathcal{H}^{d+1}$. Find them in the following way: find a a rank factorization of M, that is, find matrices $A \in \mathbb{C}^{m \times (d+1)}$ and $B \in \mathbb{C}^{(d+1) \times n}$ such that $M = AB$. Define $A_i := (\operatorname{row}_i A)^*(\operatorname{row}_i A)$ for $1 \leq i \leq m$, and $B_j := (\operatorname{col}_j B)(\operatorname{col}_j B)^*$ for $1 \leq j \leq n$. It is easy to see that they satisfy the required properties.

In the case of the regular hexagon, the rank factorization

$$M_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \sqrt{2} & 1 & 0 \\ \sqrt{2} & 1-i & 0 \\ 1 & \sqrt{2} & \sqrt{2}i \\ 0 & 1 & 1+i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & \sqrt{2}i & \sqrt{2} \\ 0 & 1 & 0 & -\sqrt{2} & 1-2i & -1-i \\ 0 & 0 & 1 & -\sqrt{2}i & 1+i & 1 \end{bmatrix}$$

give us

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 2 & \sqrt{2}-\sqrt{2}i & \sqrt{2} \\ \sqrt{2}+\sqrt{2}i & 2 & 1+i \\ \sqrt{2} & 1-i & 1 \end{bmatrix}, A_5 = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{2}i \\ \sqrt{2} & 2 & 2i \\ -\sqrt{2}i & -2i & 2 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1+i \\ 0 & 1-i & 2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & -\sqrt{2} & \sqrt{2}i \\ -\sqrt{2} & 2 & -2i \\ -\sqrt{2}i & 2i & 2 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 2 & -2\sqrt{2}+\sqrt{2}i & \sqrt{2}+\sqrt{2}i \\ -2\sqrt{2}-\sqrt{2}i & 5 & -1-3i \\ \sqrt{2}-\sqrt{2}i & -1+3i & 2 \end{bmatrix}, B_6 = \begin{bmatrix} 2 & -\sqrt{2}+\sqrt{2}i & \sqrt{2} \\ -\sqrt{2}-\sqrt{2}i & 2 & -1-i \\ \sqrt{2} & -1+i & 1 \end{bmatrix}.$$

4. Consider $\Gamma := \{Z \in \mathcal{H}_+^{d+1} : \text{there is } \mathbf{t} \in \mathbb{R}^d \text{ such that } h_i(\mathbf{t}) = \langle A_i, Z\rangle \text{ for all } 0 \leq i \leq m\}$, which is an affine slice of $\mathcal{H}_+^{d+1}$. If $Z \in \Gamma$, then there is a unique $\mathbf{t} \in \mathbb{R}^d$ such that $h_i(\mathbf{t}) = \langle A_i, Z\rangle$ for all $0 \leq i \leq m$, and this $\mathbf{t}$ has to be in P. Finally, the function $Z \in \Gamma \mapsto \mathbf{t} \in P$ can be extended to a linear transformation $\pi : \mathcal{H}^r \to \mathbb{R}^d$ such that $P = \pi(\Gamma)$. Thus, Γ is a $\mathcal{H}_+^{d+1}$-lift of P.

Consider the set Γ corresponding to the case of the regular hexagon. If

$$Z = \begin{bmatrix} z_1 & z_2 & z_3 \\ \overline{z_2} & z_4 & z_5 \\ \overline{z_3} & \overline{z_5} & z_6 \end{bmatrix} \in \mathcal{H}_+^3$$

($z_1, z_4, z_6 \in \mathbb{R}$), then the equations $h_i(x,y) = \langle A_i, Z\rangle$, $1 \leq i \leq 6$ become:

1. $1 - x - \frac{y}{\sqrt{3}} = z_6$
2. $1 - \frac{2y}{\sqrt{3}} = z_1$
3. $1 + x - \frac{y}{\sqrt{3}} = 2z_1 + 2\sqrt{2}\,\mathrm{Re}(z_2) + z_4$
4. $1 + x + \frac{y}{\sqrt{3}} = 2z_1 + 2\sqrt{2}\,\mathrm{Re}(z_2) + 2\sqrt{2}\,\mathrm{Im}(z_2) + 2\sqrt{2}\,\mathrm{Re}(z_3) + 2z_4 + 2\,\mathrm{Re}(z_5) - 2\,\mathrm{Im}(z_5) + z_6$
5. $1 + \frac{2y}{\sqrt{3}} = z_1 + 2\sqrt{2}\,\mathrm{Re}(z_2) - 2\sqrt{2}\,\mathrm{Im}(z_3) + 2z_4 - 4\,\mathrm{Im}(z_5) + 2z_6$
6. $1 - x + \frac{y}{\sqrt{3}} = z_4 + 2\,\mathrm{Re}(z_5) - 2\,\mathrm{Im}(z_5) + 2z_6.$

That is, Γ is the set of all matrices

$$Z = \begin{bmatrix} z_1 & z_2 & z_3 \\ \overline{z_2} & z_4 & z_5 \\ \overline{z_3} & \overline{z_5} & z_6 \end{bmatrix} \in \mathcal{H}_+^3$$

that satisfies (1) to (6) for some $(x,y) \in \mathbb{R}^2$. But notice that from (2), $y = \frac{\sqrt{3}}{2}(1 - z_1)$, and from (1), $x = 1 - \frac{y}{\sqrt{3}} - z_6 = \frac{1}{2} + \frac{z_1}{2} - z_6$. Using this to simplify (3) to (6), we obtain

3'. $\mathrm{Re}(z_2) = \frac{1 - z_1 - z_4 - z_6}{2\sqrt{2}}$

4'. $\mathrm{Im}(z_2) = -\mathrm{Re}(z_3)$

5'. $\mathrm{Im}(z_3) = \frac{1}{2\sqrt{2}} - \frac{z_1}{2\sqrt{2}} - \frac{z_4}{2\sqrt{2}} - \frac{z_6}{2\sqrt{2}} - \frac{\mathrm{Re}(z_5)}{\sqrt{2}}$

6'. $\mathrm{Im}(z_5) = \mathrm{Re}(z_5) - \frac{1}{2} + \frac{z_1}{2} + \frac{z_4}{2} + \frac{z_6}{2}.$

Thus Γ is the set of all

$$Z = \begin{bmatrix} z_1 & z_2 & z_3 \\ \overline{z_2} & z_4 & z_5 \\ \overline{z_3} & \overline{z_5} & z_6 \end{bmatrix} \in \mathcal{H}_+^3$$

that satisfy (3') to (6'), and we can use the transformations $x = \frac{1}{2} + \frac{z_1}{2} - z_6$, $y = \frac{\sqrt{3}}{2}(1 - z_1)$ to send Γ onto P.

We can make the things simpler with the change of coordinates

- $x_1 = \frac{1}{2} + \frac{z_1}{2} - z_6$
- $x_2 = \frac{\sqrt{3}}{2}(1 - z_1)$
- $x_3 = z_4$
- $x_4 = \mathrm{Re}(z_3)$
- $x_5 = \mathrm{Re}(z_5)$.

From here we obtain that

- $z_1 = 1 - \frac{2}{\sqrt{3}}x_2$
- $z_2 = \frac{1}{2\sqrt{2}}(-1 + x_1 + \sqrt{3}x_2 - x_3) - x_4 i$
- $z_3 = x_4 + \frac{1}{2\sqrt{2}}(-1 + x_1 + \sqrt{3}x_2 - x_3 - 2x_5)i$
- $z_4 = x_3$
- $z_5 = x_5 + \frac{1}{2}(1 - x_1 - \sqrt{3}x_2 + x_3 + 2x_5)i$
- $z_6 = 1 - x_1 - \frac{x_2}{\sqrt{3}}$,

and we can conclude that

$$\Gamma = \left\{ \begin{bmatrix} 1 & -\frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{i}{2} \\ \frac{i}{2\sqrt{2}} & -\frac{i}{2} & 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 & \frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 & -\frac{i}{2} \\ -\frac{i}{2\sqrt{2}} & \frac{i}{2} & -1 \end{bmatrix} + x_2 \begin{bmatrix} -\frac{2}{\sqrt{3}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}i}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} & 0 & -\frac{\sqrt{3}i}{2} \\ -\frac{\sqrt{3}i}{2\sqrt{2}} & \frac{\sqrt{3}i}{2} & -\frac{1}{\sqrt{3}} \end{bmatrix} \right.$$
$$\left. x_3 \begin{bmatrix} 0 & -\frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 1 & \frac{i}{2} \\ \frac{i}{2\sqrt{2}} & -\frac{i}{2} & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & -i & 1 \\ i & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 1+i \\ \frac{i}{\sqrt{2}} & 1-i & 0 \end{bmatrix} \succeq 0 : \mathbf{x} \in \mathbb{R}^5 \right\}.$$

The matrices that accompany the x_k's are linearly independent which give us the next result.

Proposition 4.2.1. *Let*

$$S := \left\{ \mathbf{x} \in \mathbb{R}^5 : \begin{bmatrix} 1 & -\frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{i}{2} \\ \frac{i}{2\sqrt{2}} & -\frac{i}{2} & 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 & \frac{1}{2\sqrt{2}} & \frac{i}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 & -\frac{i}{2} \\ -\frac{i}{2\sqrt{2}} & \frac{i}{2} & -1 \end{bmatrix} + x_2 \begin{bmatrix} -\frac{2}{\sqrt{3}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}i}{2\sqrt{2}} \\ \frac{\sqrt{3}}{2\sqrt{2}} & 0 & -\frac{\sqrt{3}i}{2} \\ -\frac{\sqrt{3}i}{2\sqrt{2}} & \frac{\sqrt{3}i}{2} & -\frac{1}{\sqrt{3}} \end{bmatrix} \right.$$
$$\left. x_3 \begin{bmatrix} 0 & -\frac{1}{2\sqrt{2}} & -\frac{i}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 1 & \frac{i}{2} \\ \frac{i}{2\sqrt{2}} & -\frac{i}{2} & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & -i & 1 \\ i & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 1+i \\ \frac{i}{\sqrt{2}} & 1-i & 0 \end{bmatrix} \succeq 0 \right\}.$$

We have $H = \pi_{x_1,x_2}(S)$ where $\pi_{x_1,x_2} : \mathbb{R}^5 \to \mathbb{R}^2$ is the projection into the first two components.

Chapter 5

Complex psd-minimal 3-polytopes

In this chapter, we begin the study of complex psd-minimal 3-polytopes. The ideal scenario will be to obtain a characterization of those 3-polytopes that are complex psd-minimal in the same way we obtain one for the 2-dimensional case. There, the Pappus hexagons are the only new psd-minimal polygons that appear when we pass from the real to the complex case. Surprisingly, there are more combinatorial classes of 3-polytopes with a complex psd-minimal realization than we originally thought. Although a complete characterization of complex psd-minimal polytopes is not given, we present some important advances.

We want to identify combinatorial classes for which there is a complex psd-minimal 3-polytope with that combinatorial structure, as we did with hexagons in the 2-dimensional case. It is important to mention that if a polytope is (complex) psd-minimal, then it is not necessarily true that every combinatorially equivalent polytope is also (complex) psd-minimal. The natural way to mod out polytopes is through projective equivalence since this preserve (complex) psd-minimality. Nonetheless by identifying combinatorial classes for which there is no (complex) psd-minimal polytope, we can try to identify *combinatorial obstructions* for (complex) psd-minimality. These are combinatorial properties that automatically rule out the possibility of a polytope to be (complex) psd-minimal.

After a combinatorial class of a 3-polytope with a complex psd-minimal realization is identified, we can work within this combinatorial class and try to parametrize the family of the complex psd-minimal ones. This methodology is used in the characterization of real psd-minimal polytopes in dimensions 2, 3 and 4 in [13] and [12]. Nonetheless we will not try to parametrize these families in our work.

Since we will work with combinatorial classes, we will make use of the graph of a polytope. This is the graph whose vertices and edges are respectively the vertices and edges of the polytope. For 3-polytopes, these graphs are called polyhedral graphs, and they are precisely the 3-connected planar graphs by Steinitz's Theorem. In any planar embedding of a polyhedral graph, each region corresponds to a facet of the polytope.

One way to prove that a given combinatorial class has a complex psd-minimal realization is to produce a generalized slack matrix (see Definition 1.3.4 and Theorem 1.3.5) that can be factored as in Theorem 4.0.3. When using such a matrix, we present the polyhedral graph with labels so that the reader can see which ordering on the facets and vertices is being used.

The first combinatorial obstruction we will use comes from the fact that a complex psd-minimal polytope has complex psd-minimal faces.

Proposition 5.0.1. *A complex psd-minimal 3-polytope can only have triangular, quadrilateral and hexagonal facets, and vertices of degree 3, 4 and 6.*

Proof. The first part is consequence of Proposition 4.0.5 and Theorem 4.1.8. The second part is the dual statement. □

5.1 Complex psd-minimal 3-polytopes with a vertex of degree 6

In this section, we will identify all the combinatorial classes in which there is a complex psd-minimal 3-polytope with a vertex of degree 6. More precisely we will prove first that if a complex psd-minimal 3-polytope has a vertex of degree 6, then it must have exactly 7 vertices. After that, we will prove that if a complex psd-minimal 3-polytope has 7 vertices, one of degree 6, then there exists a combinatorially equivalent polytope that is complex psd-minimal. The previous results can be dualized to obtain similar results for complex psd-minimal 3-polytopes with a hexagonal facet.

Lemma 5.1.1. *Let P be a 3-polytope, and let S be a scaled symbolic slack matrix of P. Suppose P has a vertex $\mathbf{v}$ of degree n with neighbors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, where $\mathbf{v}_i, \mathbf{v}, \mathbf{v}_{i+1}$ belong mod n to the facet F_i, $1 \leq i \leq n$. Also suppose $\mathbf{v}'_1, \ldots, \mathbf{v}'_m$ are vertices of P which are not neighbors of $\mathbf{v}$. If $\operatorname{rank} S(\boldsymbol{\zeta}) = 4$ for some vector $\boldsymbol{\zeta}$ with nonzero complex entries, then the submatrix of $S(\boldsymbol{\zeta})$ indexed by the rows $F_1, \ldots, F_n$ and columns $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}'_1, \ldots, \mathbf{v}'_m$ has rank 3.*

Proof. Let G be the symbolic slack matrix of the n-polygon, and let F be a facet not containing $\mathbf{v}$. Then $S(\boldsymbol{\zeta})$ contains a submatrix of the form

$$S_0 = \begin{array}{c} \\ F_1 \\ \vdots \\ F_n \\ F \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{v} & \mathbf{v}_1,\ldots,\mathbf{v}_n & \mathbf{v}'_1,\ldots,\mathbf{v}'_m \end{array} \\ \left[\begin{array}{ccc} 0 & & \\ \vdots & G(\boldsymbol{\zeta}') & * \\ 0 & & \\ c & * & * \end{array}\right] \end{array}$$

for some nonzero complex number c and some vector $\boldsymbol{\zeta}'$ with nonzero complex entries.

If there are r-linearly independent rows in the first n rows of S_0, then by its zero pattern, we have that S_0 has rank equal to $r+1$. Since $\operatorname{rank} S(\boldsymbol{\zeta}) = 4$, then r is at most 3. On the other hand, $G(\boldsymbol{\zeta}')$ has rank at least 3 by Lemma 1.3.3. Thus $r = 3$, and the result follows. □

Lemma 5.1.2. *Let*

$$S(\mathbf{x}) := \begin{bmatrix} 0 & 0 & 1 & x_1 & x_2 & 1 & 1 \\ 1 & 0 & 0 & 1 & x_3 & x_4 & x_{14} \\ x_5 & 1 & 0 & 0 & 1 & x_6 & x_{15} \\ 1 & x_7 & 1 & 0 & 0 & x_8 & x_{16} \\ x_9 & 1 & x_{10} & 1 & 0 & 0 & x_{17} \\ 0 & x_{11} & x_{12} & x_{13} & 1 & 0 & x_{18} \end{bmatrix}.$$

Then there is no vector $\boldsymbol{\alpha}$ of positive entries, and no vector $\boldsymbol{\zeta}$ of nonzero complex entries such that $S(\boldsymbol{\alpha}) = S(\boldsymbol{\zeta}) \odot \overline{S(\boldsymbol{\zeta})}$ and $\operatorname{rank} S(\boldsymbol{\alpha}) = \operatorname{rank} S(\boldsymbol{\zeta}) = 3$.

Proof. Suppose there is a vector $\boldsymbol{\alpha}$ of positive entries, and a vector $\boldsymbol{\zeta}$ of nonzero complex entries such that $S_P(\boldsymbol{\alpha}) = S_P(\boldsymbol{\zeta}) \odot \overline{S_P(\boldsymbol{\zeta})}$ and $\operatorname{rank} S_P(\boldsymbol{\alpha}) = \operatorname{rank} S_P(\boldsymbol{\zeta}) = 3$. If we removed the last column of $S(\mathbf{x})$, we obtain a scaled symbolic slack matrix of the hexagon which will be denoted by S_H. By Lemma 1.3.3, $\operatorname{rank} S_H(\alpha_1, \ldots, \alpha_{13}) \geq 3$; on the other side $\operatorname{rank} S_H(\alpha_1, \ldots, \alpha_{13}) \leq \operatorname{rank} S(\boldsymbol{\alpha}) = 3$. Thus $\operatorname{rank} S_H(\alpha_1, \ldots, \alpha_{13}) = 3$, and therefore $\alpha_1, \ldots, \alpha_{13}$ and $\zeta_1, \ldots, \zeta_{13}$ are as in Corollary 4.1.9, where $a = A^2, b = B^2$, $\xi_1 = Ai$ and $\xi_2 = \frac{A^2-B^2}{A^2(1+B^2)} + \frac{B\sqrt{(1+A^2)(A^2-B^2)}}{A^2(1+B^2)}i$ for some real numbers $A, B \neq 0$.

Since α_8, α_9, α_{11}, ζ_8, ζ_9 and ζ_{11} are all equal to 1, we consider the matrix

$$\begin{bmatrix} 0 & 0 & 1 & x_1 & x_2 & 1 & 1 \\ 1 & 0 & 0 & 1 & x_3 & x_4 & x_{14} \\ x_5 & 1 & 0 & 0 & 1 & x_6 & x_{15} \\ 1 & x_7 & 1 & 0 & 0 & 1 & x_{16} \\ 1 & 1 & x_{10} & 1 & 0 & 0 & x_{17} \\ 0 & 1 & x_{12} & x_{13} & 1 & 0 & x_{18} \end{bmatrix}.$$

Using computational algebra software, it can be checked that the following polynomials belong to the ideal generated by the 4-minors of this matrix:

- $x_{15} - x_5x_{14} - x_{18} + x_{12}$
- $x_2x_{16} - x_2x_{14} - x_2x_7x_{18} + x_3 + x_7$
- $x_{17} - x_{14} - x_{18} + x_4$.

This give us the relations:

- $\alpha_{15} = \alpha_5\alpha_{14} + \alpha_{18} - \alpha_{12}$
- $\alpha_{16} = \alpha_{14} + \alpha_7\alpha_{18} - \frac{\alpha_3}{\alpha_2} - \frac{\alpha_7}{\alpha_2}$
- $\alpha_{17} = \alpha_{14} + \alpha_{18} - \alpha_4$.
- $\zeta_{15} = \zeta_5\zeta_{14} + \zeta_{18} - \zeta_{12}$
- $\zeta_{16} = \zeta_{14} + \zeta_7\zeta_{18} - \frac{\zeta_3}{\zeta_2} - \frac{\zeta_7}{\zeta_2}$
- $\zeta_{17} = \zeta_{14} + \zeta_{18} - \zeta_4$.

Let $\operatorname{Re}(\zeta_{14}) = a_1$, $\operatorname{Im}(\zeta_{14}) = b_1$, $\operatorname{Re}(\zeta_{18}) = a_2$ and $\operatorname{Im}(\zeta_{18}) = b_2$. Using that $\alpha_{15} = \zeta_{15}\overline{\zeta_{15}}$, $\alpha_{16} = \zeta_{16}\overline{\zeta_{16}}$ and $\alpha_{17} = \zeta_{17}\overline{\zeta_{17}}$, we obtain after some simplification:

- $A^5(1+B^2)^2\sqrt{1+A^2} + (\sqrt{A^2-B^2}(A^2-B^2)B + AB^2(A^2-B^2)\sqrt{1+A^2})(a_1a_2+b_1b_2) - (\sqrt{A^2-B^2}AB(A^2-B^2) - B^2(A^2-B^2)\sqrt{1+A^2})(b_1a_2-a_1b_2) - A^2B(1+B^2)\sqrt{A^2-B^2}(1+A^2)a_1 - A^2(1+B^2)(A^2-B^2)\sqrt{1+A^2}b_1 - (-A^2B\sqrt{A^2-B^2}(A^2-B^2) + A^2B^3\sqrt{A^2-B^2}(1+A^2) + 2AB^2(A^2-B^2)\sqrt{1+A^2})a_2 - (AB\sqrt{A^2-B^2}(A^2-B^2) - AB^3\sqrt{A^2-B^2}(1+A^2) + 2A^2B^2(A^2-B^2)\sqrt{1+A^2})b_2 = 0$

- $A^6(1+B^2)^2 - A^3B(1+B^2)\sqrt{1+A^2}\sqrt{A^2-B^2} + A^2B^2(1+A^2)(1+B^2) + (1+A^2)B^2(A^2-B^2)(a_1a_2+b_1b_2) - B\sqrt{1+A^2}\sqrt{A^2-B^2}(A^2-B^2)(b_1a_2-a_1b_2) - A^3B(1+B^2)\sqrt{1+A^2}\sqrt{A^2-B^2}\, a_1 - (A^3(1+B^2)(A^2-B^2) - A^2B(1+B^2)\sqrt{1+A^2}\sqrt{A^2-B^2})b_1 - (AB^3(1+A^2)\sqrt{1+A^2}\sqrt{A^2-B^2} - AB\sqrt{1+A^2}\sqrt{A^2-B^2}(A^2-B^2) + (1+A^2)B^2(A^2-B^2))a_2 + (-AB^2(1+A^2)(A^2-B^2) + (1+A^2)B^3\sqrt{1+A^2}\sqrt{A^2-B^2} - AB^2(1+A^2)(A^2-B^2))b_2 = 0$

- $A^2(1+A^2)(1+B^2) + (a_1a_2+b_1b_2)(A^2+B^2) - ((A^2-B^2) + AB\sqrt{A^2-B^2}\sqrt{1+A^2})(a_1+a_2) - (A(A^2-B^2) - \sqrt{A^2-B^2}\sqrt{1+A^2}B)(b_1+b_2) = 0.$

If we define $u := \sqrt{1+A^2}$ and $\nu := \sqrt{A^2-B^2}$, the above expressions simplify as follows:

- $A^5(1+B^2)^2u + (\nu^3B + AB^2\nu^2u)(a_1a_2+b_1b_2) - (\nu^3AB - B^2\nu^2u)(b_1a_2-a_1b_2) - A^2B(1+B^2)\nu u^2a_1 - A^2(1+B^2)\nu^2ub_1 - (-A^2B\nu^3 + A^2B^3\nu u^2 + 2AB^2\nu^2u)a_2 - (AB\nu^3 - AB^3\nu u^2 + 2A^2B^2\nu^2u)b_2 = 0$

- $A^6(1+B^2)^2 - A^3B(1+B^2)u\nu + A^2B^2u^2(1+B^2) + u^2B^2\nu^2(a_1a_2+b_1b_2) - Bu\nu^3(b_1a_2-a_1b_2) - A^3B(1+B^2)u\nu a_1 - (A^3(1+B^2)\nu^2 - A^2B(1+B^2)u\nu)b_1 - (AB^3u^3\nu - ABu\nu^3 + u^2B^2\nu^2)a_2 + (-AB^2u^2\nu^2 + u^3B^3\nu - AB^2u^2\nu^2)b_2 = 0$

- $A^2u^2(1+B^2) + (a_1a_2+b_1b_2)\nu^2 - (\nu^2 + AB\nu u)(a_1+a_2) - (A\nu^2 - \nu uB)(b_1+b_2) = 0.$

Consider the ideal I in $\mathbb{C}[A, B, a_1, b_1, a_2, b_2, u, \nu]$ generated by the above three polynomials. It can be checked that $\nu^4 \in I$. This implies that $A^2 = B^2$ which is a contradiction since $a > b > 0$. □

Corollary 5.1.3. *A complex psd-minimal 3-polytope with a vertex of degree 6 has exactly 7 vertices.*

Proof. Let P be a complex psd-minimal 3-polytope with a vertex $\mathbf{v}$ of degree 6, and let $\mathbf{v}_1, \ldots, \mathbf{v}_6$ be its neighbors. Let F_i be the facet containing $\mathbf{v}$, $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ mod 6. Suppose there is a vertex $\mathbf{v}'$ not adjacent to $\mathbf{v}$, which we can suppose without loss of generality that it does not belong to F_1.

We can scale the symbolic slack matrix $\hat{S}$ of P to obtain a matrix which has a submatrix of the form

$$S = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 & \mathbf{v}' \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & x_1 & x_2 & 1 & 1 \\ 1 & 0 & 0 & 1 & x_3 & x_4 & x_{14} \\ x_5 & 1 & 0 & 0 & 1 & x_6 & x_{15} \\ 1 & x_7 & 1 & 0 & 0 & x_8 & x_{16} \\ x_9 & 1 & x_{10} & 1 & 0 & 0 & x_{17} \\ 0 & x_{11} & x_{12} & x_{13} & 1 & 0 & x_{18} \end{bmatrix} \end{array}.$$

If P is complex psd-minimal, then by Lemma 4.0.4, there is a vector $\hat{\boldsymbol{\alpha}}$ of positive entries and a vector $\hat{\boldsymbol{\zeta}}$ of nonzero complex numbers such that $\hat{S}(\hat{\boldsymbol{\alpha}}) = \hat{S}(\hat{\boldsymbol{\zeta}}) \odot \overline{\hat{S}(\hat{\boldsymbol{\zeta}})}$ with $\operatorname{rank} \hat{S}(\hat{\boldsymbol{\alpha}}) = \operatorname{rank} \hat{S}(\hat{\boldsymbol{\zeta}}) =$

4. Thus, there is a vector $\boldsymbol{\alpha}$ of positive entries and a vector $\boldsymbol{\zeta}$ of nonzero complex numbers such that $S(\boldsymbol{\alpha}) = S(\boldsymbol{\zeta}) \odot \overline{S(\boldsymbol{\zeta})}$, and by Lemma 5.1.1, $\operatorname{rank} S(\boldsymbol{\alpha}) = \operatorname{rank} S(\boldsymbol{\zeta}) = 3$. This contradicts Lemma 5.1.2, so P cannot be complex psd-minimal. □

In Figure 5.1 there is a list of all polyhedral graphs with 7 vertices one of them of degree 6; we have used [30] as reference for a complete list of heptahedral graphs. We will see that all the combinatorial classes in Figure 5.1 have at least one complex psd-minimal realization. The same is true for the ones in Figure 5.2 since they are the duals of the ones in Figure 5.1, which corresponds to polyhedral graphs with 7 facets one of them hexagonal.

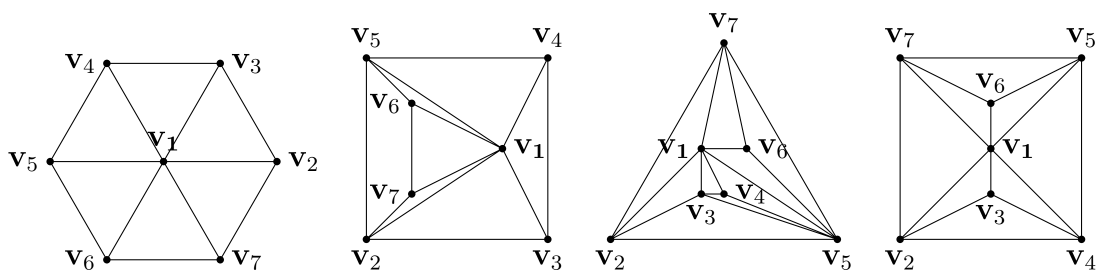


Figure 5.1: Polyhedral graphs with a vertex of degree 6 and a complex psd-minimal realization

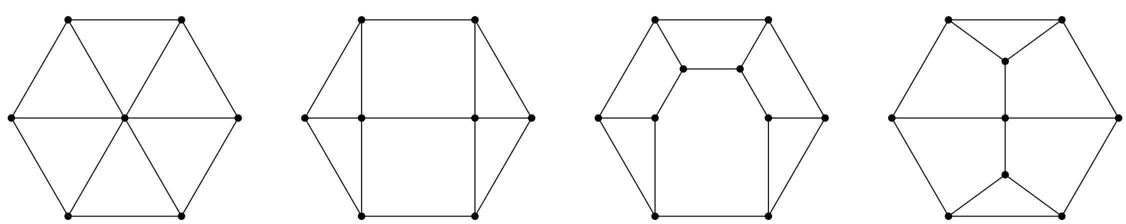


Figure 5.2: Polyhedral graphs with a hexagonal facet and a complex psd-minimal realization

Proposition 5.1.4. *If a 3-polytope has 7 vertices one of them of degree 6 (or 7 facets one of them hexagonal), then there is a combinatorially equivalent polytope that is complex psd-minimal.*

Proof. For each of the four combinatorial classes in Figure 5.1, we need to find a generalized slack matrix S and a complex matrix M of rank 4 such that $S = M \odot \overline{M}$.

$$1.\ S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & 4 & 1 & 0 & 0 & 1 & 8 \\ 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ M = \begin{bmatrix} 0 & 0 & 0 & 1 & -i & \frac{1}{2}+\frac{1}{2}i & 1 \\ 0 & 1 & 0 & 0 & 1 & i & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 2-2i \\ 0 & 1 & \frac{1}{2}-\frac{1}{2}i & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2i & 1 & 0 & 0 \\ 0 & 0 & 1 & 2+2i & -2i & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. $S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & 4 & 1 & 0 & 0 & 1 & 8 \\ 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 & 4 & 1 & 0 \\ 1 & 0 & 1 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \; M = \begin{bmatrix} 0 & 0 & 0 & 1 & -i & \frac{1}{2}+\frac{1}{2}i & 1 \\ 0 & 1 & 0 & 0 & 1 & i & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 2-2i \\ 0 & 1 & \frac{1}{2}-\frac{1}{2}i & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2i & 1 & 0 & 0 \\ 0 & 0 & 1 & 2+2i & -2i & 1 & 0 \\ 1 & 0 & -i & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & -2i \end{bmatrix}.$

3. $S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & 4 & 1 & 0 & 0 & 1 & 8 \\ 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 & 4 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & \frac{1}{2} & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 4 \\ 1 & 1 & 1 & 4 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{4} & 2 & 0 & \frac{1}{4} & 0 \end{bmatrix}, \; M = \begin{bmatrix} 0 & 0 & 0 & 1 & -i & \frac{1}{2}+\frac{1}{2}i & 1 \\ 0 & 1 & 0 & 0 & 1 & i & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 2-2i \\ 0 & 1 & \frac{1}{2}-\frac{1}{2}i & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2i & 1 & 0 & 0 \\ 0 & 0 & 1 & 2+2i & -2i & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & \frac{1}{2}+\frac{1}{2}i & 1 \\ 1 & -i & 0 & 0 & 0 & 1 & -2i \\ 1 & -i & -i & 2 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 1+i & 0 & \frac{1}{2} & 0 \end{bmatrix}.$

4. $S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 4 \\ 0 & 4 & 1 & 0 & 0 & 1 & 8 \\ 0 & 1 & \frac{1}{2} & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 & 4 & 4 & 8 \\ 1 & 4 & 4 & 8 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \; M = \begin{bmatrix} 0 & 0 & 0 & 1 & -i & \frac{1}{2}+\frac{1}{2}i & 1 \\ 0 & 1 & 0 & 0 & 1 & i & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 2-2i \\ 0 & 1 & \frac{1}{2}-\frac{1}{2}i & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2i & 1 & 0 & 0 \\ 0 & 0 & 1 & 2+2i & -2i & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2i & 2+2i \\ 1 & -2i & -2i & 2-2i & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$

□

5.2 Complex psd-minimal 3-polytopes with six vertices

Recall from Chapter 1, Section 1.7 that any d-polytope with $d+1$ or $d+2$ vertices is psd-minimal. In the 3-dimensional case, this tell us that simplices, bisimplices and quadrilateral pyramids are psd-minimal. If we want to find new complex psd-minimal 3-polytopes we need to start with those which have six vertices. In this section we find all the combinatorial classes of 3-polytopes with six vertices or six facets (by duality) which have a complex psd-minimal realization.

We use [31] as reference for a complete list of hexahedral graphs. By Proposition 5.0.1, we only need to consider the ones in Figure 5.3 and their duals in Figure 5.4. We will see that all of them have complex psd-minimal realizations.

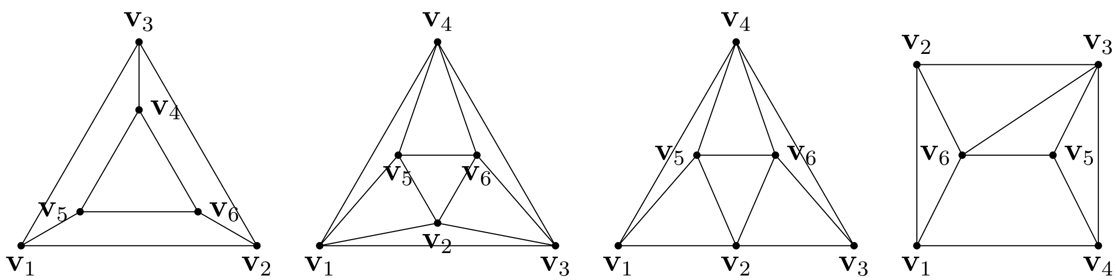


Figure 5.3: Polyhedral graphs with six vertices and a complex psd-minimal realization

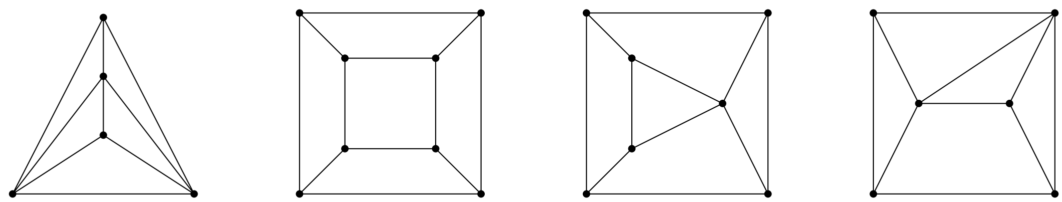


Figure 5.4: Polyhedral graphs with six facets and a complex psd-minimal realization

Proposition 5.2.1. *Figure 5.3 (5.4) gives a complete list of combinatorial classes of 3-polytopes with six vertices (facets) and a complex psd-minimal realization.*

Proof. We already notice that the hexahedral graphs in Figure 5.3 are our candidates for combinatorial classes of 3-polytopes with six vertices and a complex psd-minimal realization. Let us see that all of them works.

1. The first one corresponds to a triangular prism which has five facets, thus by Proposition 1.7.10, all its realizations are complex psd-minimal.

2. The second one corresponds to an octahedron which we know has real psd-minimal realizations by Proposition 1.7.14.

For the rest we need to find a generalized slack matrix S and a complex matrix M of rank 4 such that $S = M \odot \overline{M}$.

3. $$S = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1-i & 0 & 0 \\ i & 0 & 0 & 1 & 1 & 0 \\ i & 1 & 0 & 0 & 1 & 0 \\ 1 & 1-i & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -i & 1 \end{bmatrix}.$$

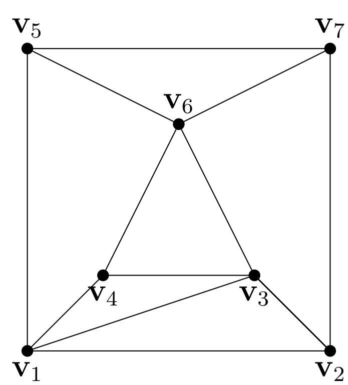
4. $S = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\ 1 & 0 & 0 & 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2}+\frac{1}{2}i & 1 \\ 1 & 0 & 0 & 1 & \frac{1}{2}-\frac{1}{2}i & 0 \\ 1+i & 1 & 0 & i & 0 & 0 \end{bmatrix}$.

□

5.3 Complex psd-minimal 3-polytopes with seven vertices

In this section, we will determine all the combinatorial classes of 3-polytopes with seven vertices (facets) that have at least one complex psd-minimal realization. We use [30] as reference for a complete list of heptahedral graphs. We already know which ones work if there is a vertex of degree six (Figure 5.1). So we only need to consider the ones with triangular and quadrilateral facets and vertices of degree 3 and 4. We will rule out first the one that does not work. The remaining ones appear in Figure 5.5 (their duals appear in Figure 5.6), and we will see that all of them have complex psd-minimal realizations.

Lemma 5.3.1. *If P is a 3-polytope with graph*



(which is self-dual), then P is not complex psd-minimal.

Proof. A scaled slack matrix of P is

$$S_P(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & x_1 & 1 \\ 0 & 1 & 0 & 0 & 1 & x_2 & x_3 \\ 0 & 1 & 1 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & x_5 & 0 & x_6 \\ 1 & 0 & 0 & x_7 & x_8 & 0 & 0 \\ 1 & x_9 & x_{10} & x_{11} & 0 & 0 & 0 \end{bmatrix}$$

(there is an additional 1 in position (1,5) since the slack matrix of a quadrilateral appears from row 1 to 4 and from column 2 to 5; see the observation after Definition 1.7.7). Let $S_P(\boldsymbol{\alpha}) \in \mathbf{SC}_P$

and suppose P is complex psd-minimal. By Theorem 4.0.4, there is $\boldsymbol{\zeta} \in (\mathbb{C}^*)^{11}$ such that $S_P(\boldsymbol{\alpha}) = S_P(\boldsymbol{\zeta}) \odot \overline{S_P(\boldsymbol{\zeta})}$ and $\operatorname{rank} S_P(\boldsymbol{\zeta}) = d+1$.

The following trinomials belong to the ideal J generated by the 5-minors of the above matrix: $x_2 - x_1 + 1$, $x_4 - x_3 + 1$ and $x_1x_4 - x_1x_6 + 1$. Applying Lemma 4.0.6 to these polynomials, we obtain $\mathrm{Re}(\zeta_2) = \mathrm{Re}(\zeta_4) = \mathrm{Re}(\zeta_1\zeta_4) = 0$, and thus $\mathrm{Im}(\zeta_1) = 0$. Since $x_2 - x_1 + 1 \in J$, $\zeta_1 = \zeta_2 + 1$. It follows that $\mathrm{Re}(\zeta_1) = 1$, so $\zeta_1 = 1$. This implies that $\zeta_2 = 0$ which is a contradiction. $\square$

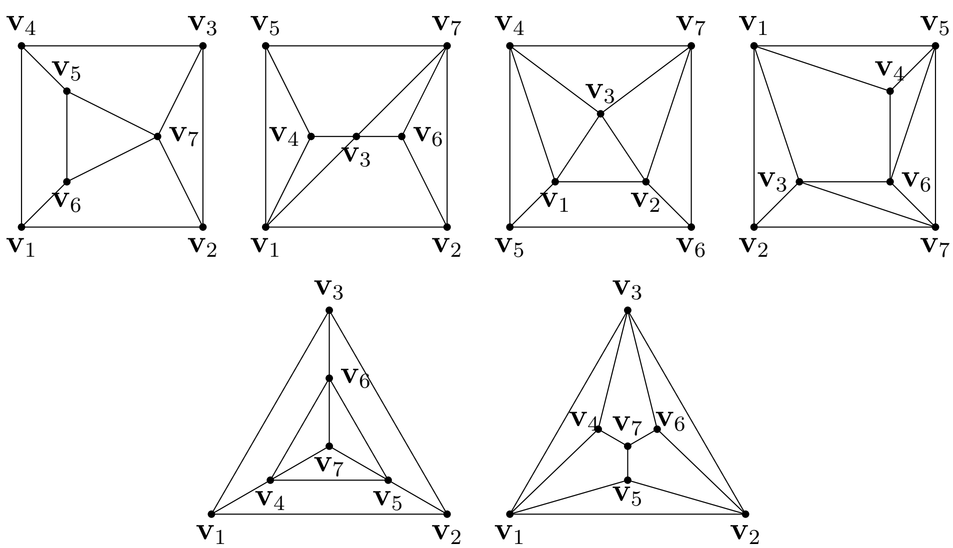


Figure 5.5: Polyhedral graphs with seven vertices none with degree 6 and a complex psd-minimal realization

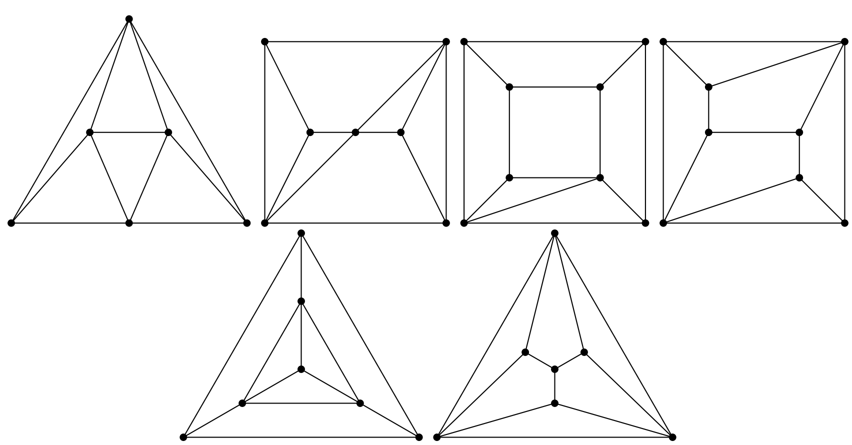


Figure 5.6: Polyhedral graphs with seven facets none hexagonal and a complex psd-minimal realization

Proposition 5.3.2. *Figure 5.1 with Figure 5.5 (5.2 with 5.6) gives a complete list of combinatorial classes of 3-polytopes with seven vertices (facets) and a complex psd-minimal realization.*

Proof. We already notice that the heptahedral graphs in Figure 5.1 with those in Figure 5.5 are our candidates for combinatorial classes of 3-polytopes with seven vertices and a complex psd-minimal realization. We saw the ones in Figure 5.1 work. Let us see that the ones in Figure 5.5 also work.

1. Notice that this polyhedral graph has six facets. From Proposition 5.2.1, we already know it has a complex psd-minimal realization.

For the rest we need to find a generalized slack matrix S and a complex matrix M of rank 4 such that $S = M \odot \overline{M}$.

2. $$S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{5}{2} & 1 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1+i \\ 0 & 1 & 1 & 0 & 0 & 1+i & i \\ 0 & 0 & 1 & 1 & 0 & i & 0 \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{2}-\frac{1}{2}i & 0 \\ 1 & \frac{1}{2}+\frac{1}{2}i & 0 & 1 & \frac{1}{2}+\frac{1}{2}i & 0 & 0 \\ 1 & 0 & \frac{1}{2}-\frac{1}{2}i & \frac{3}{2}+\frac{1}{2}i & i & 0 & 0 \end{bmatrix}.$$

3. $$S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1+i & 1 \\ 0 & 1 & 1 & 0 & 0 & i & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & i & 0 & 0 \\ 1 & 0 & 0 & 1 & 1+i & 1 & 0 \\ 1 & 1 & 0 & 0 & 1+i & 1+i & 0 \\ 1 & 1 & 1+i & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4. $$S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & i & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1-i \\ 0 & 1 & 1 & 0 & 0 & 1 & -i \\ 0 & 0 & 1 & 1 & 0 & 1+i & 0 \\ 1 & 0 & 0 & 1 & \frac{1}{2}-\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2}+\frac{1}{2}i & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2}+\frac{1}{2}i & \frac{1}{2}i & \frac{1}{2}i & 0 & 0 & 0 \\ 1 & i & -\frac{1}{2}+\frac{1}{2}i & 0 & 0 & 0 & \frac{1}{2}+\frac{1}{2}i \end{bmatrix}.$$

5. $$S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & \frac{1}{4} & \frac{13}{4} \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 0 & i & 1 & -\frac{1}{2}i & 1+\frac{3}{2}i \\ 0 & 1 & 0 & 0 & 1 & 0 & i \\ 0 & 0 & 2i & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & i & 0 & 0 & i \\ i & 1+i & i & 0 & 1 & 0 & 0 \\ 1 & 1 & 1+2i & 0 & 0 & 1 & 0 \\ 1-i & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$6.\ S = \begin{bmatrix} 0 & 0 & 0 & 1 & \frac{1}{2} & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 \end{bmatrix},\ M = \begin{bmatrix} 0 & 0 & 0 & 1 & -\frac{1}{2}-\frac{1}{2}i & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1-i & 1 \\ 0 & 0 & i & i & 0 & 1+i & 1 \\ 1 & 0 & 0 & 1+i & 1-i & 0 & 1 \\ 1 & i & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1-i & 0 \\ 1 & 0 & 1 & 1+i & 0 & 0 & 0 \end{bmatrix}.$$

□

In addition to the work presented in this thesis, for combinatorial classes of 3-polytopes with 8 vertices, we have obtained examples with and without a complex psd-minimal realization. Our purpose is to find new combinatorial obstructions that can help us to characterize the complex psd-minimal 3-polytopes or at least give us a nice upper bound on the number of vertices/facets a complex psd-minimal 3-polytope can have.

Chapter 6

k-matching polytopes of bipartite graphs

Matchings are important in combinatorial optimization not only for their applications, but because of their intriguing status from the viewpoint of computational complexity. There is a polynomial-time algorithm to find a perfect matching if one exists, or show that one does not exist. This is the method of augmenting paths, which is one of the first non-trivial polynomial algorithms that we encounter in combinatorial optimization.

Furthermore, although decision problems such as "does G have a perfect matching?", or "does G have a k-matching?" are in P, the counting problem "how many perfect matchings does G have?" is #P-complete, even for bipartite graphs. That means that counting matchings is just as hard as counting 3-colorings or large cliques or other objects for which the decision problem is NP-complete. There is also literature about lifts of matching polytopes such as [15] and [23], and even an entire book [17] dedicated to matchings in general.

In this last chapter, we study k-matching polytopes of bipartite graphs. The polytopes of all matchings and of maximal matchings are well-studied but much less so when we only consider matchings of a given size. Our interest in k-matching polytopes originated from the fact that the slack ideal of a d-polytope P is constructed from the $(d+2)$-minors of the symbolic slack matrix, and the monomials that appear in this minors correspond to $(d+2)$-matchings of the non-incidence graph $\mathbf{G}_P$.

We will prove two important results about k-matching polytopes. First, we will prove that the k-matching polytope is equal to the fractional k-matching polytope for bipartite graphs. After that we will prove that k-matching polytopes of bipartite graphs are normal, that is, every integer point in its t-dilate is the sum of t integers points of the original polytope. This result was known for Birkhoff polytopes $B_n = M_n(\mathrm{K}_{n,n})$ and we show it remains true in this more general setting.

6.1 The H-representation of the k-matching polytope of a bipartite graph

In this section we extend a standard result on matchings in bipartite graphs, $\mathrm{M}(G) = \mathrm{FM}(G)$, to k-matchings. We will prove that $\mathrm{M}_k(G) = \mathrm{FM}_k(G)$ if G is a bipartite graph. This result was known and proved in [6] for the special case of k-assignment polytopes, that is, when $G = K_{m,n}$.

Theorem 6.1.1. *Let $G = (V, E)$ be a bipartite graph and $k \in \mathbb{N}$. Then $\mathrm{M}_k(G) = \mathrm{FM}_k(G)$.*

Proof. Clearly $\mathrm{M}_k(G) \subseteq \mathrm{FM}_k(G)$ and $\mathrm{FM}_k(G) \cap \mathbb{Z}^E \subseteq \mathrm{M}_k(G)$. So it is enough to prove that every vertex of $\mathrm{FM}_k(G)$ has integer coordinates. We do so by showing that if $\mathbf{x}$ is not an integer point of $\mathrm{FM}_k(G)$, then it can be written as $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$ where $\mathbf{x}', \mathbf{x}'' \in \mathrm{FM}_k(G) \setminus \{\mathbf{x}\}$, and thus it cannot be a vertex of $\mathrm{FM}_k(G)$.

Let $\mathbf{x}$ be a point of $\mathrm{FM}_k(G)$ that is not integer, and consider the subgraph $H := (V, F)$ where $F := \{e \in E : x_e \notin \{0, 1\}\}$. We divide the proof by cases, depending if H does or does not have certain properties, and some of the cases overlap.

Case 1: H has a cycle.

If H has a cycle, then it has to be an even cycle since G is bipartite. Let $e_1, \dots, e_m$ be the edges that appear in the cycle in that precise order. Define $\varepsilon := \min\{x_{e_1}, \dots, x_{e_m}, 1 - x_{e_1}, \dots, 1 - x_{e_m}\}$. Let $\mathbf{x}' = (x'_e)_{e \in E}$ and $\mathbf{x}'' = (x''_e)_{e \in E}$ where

$$x'_e = \begin{cases} x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd} \\ x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even} \\ x_e & \text{otherwise,} \end{cases} \tag{$*$}$$

and

$$x''_e = \begin{cases} x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd} \\ x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even .} \\ x_e & \text{otherwise.} \end{cases} \tag{$**$}$$

Then $\mathbf{x}', \mathbf{x}'' \in \mathrm{FM}_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$.

Case 2: H has no cycles.

Let P be a maximal path of H with consecutive vertices $v_1, v_2, \dots, v_m, v_{m+1}$. Then $x_e = 0$ if e is an edge of G incident to v_1 different from $\{v_1, v_2\}$. Indeed, it cannot be equal to 1 due to the inequality $\sum_{e \ni v_1} x_e \leq 1$, and it cannot be strictly between 0 and 1 since otherwise the path could be extended. A similar analysis can be done for v_{m+1}. Thus $\sum_{e \ni v_1} x_e = x_{\{v_1, v_2\}} < 1$ and $\sum_{e \ni v_{m+1}} x_e = x_{\{v_m, v_{m+1}\}} < 1$.

Furthermore, if P is of even length with consecutive edges $e_1, \dots, e_m$, take $\varepsilon := \min\{x_{e_1}, \dots, x_{e_m}, 1-$

$x_{e_1}, \dots, 1 - x_{e_m}\}$. Define $\mathbf{x}'$ and $\mathbf{x}''$ as in $(*)$ and $(**)$. Then $\mathbf{x}', \mathbf{x}'' \in \mathrm{FM}_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$. In conclusion, if we can find a maximal path of even length in H we are done.

Subcase 2.1: H has no cycles, but it is connected.

If P is of even length, we are done, so suppose P has odd length. We analize the cases $H = P$ and $H \neq P$ separately.

Suppose $H = P$. If $\sum_{e \ni v} x_e = 1$, for all interior vertex v of H, then

$$\begin{aligned} k &= \sum_{e \in E} x_e \\ &= \sum_{e \in H} x_e + \text{ some integer} \\ &= (x_{e_1} + x_{e_2}) + (x_{e_3} + x_{e_4}) + \cdots + (x_{e_{m-2}} + x_{e_{m-1}}) + x_{e_m} + \text{ some integer} \\ &= 1 + 1 + \cdots + 1 + x_{e_m} + \text{ some integer} \\ &= x_{e_m} + \text{ some integer} \end{aligned}$$

which is a contradiction since x_{e_m} is not an integer. So there is an interior vertex v of H such that $\sum_{e \ni v} x_e < 1$. This vertex v divides H into two subpaths and one of them has to be of even length. Let's called this subpath P_0, and suppose without loss of generality that its consecutive edges are $e_1, \dots, e_n$ $(n < m)$. Take $\varepsilon := \min\{x_{e_1}, \dots, x_{e_n}, 1 - x_{e_1}, \dots, 1 - x_{e_{n-1}}, 1 - x_{e_n} - x_{e_{n+1}}\}$. Now define $\mathbf{x}'$ and $\mathbf{x}''$ as in $(*)$ and $(**)$ with m replaced by n. Then $\mathbf{x}', \mathbf{x}'' \in \mathrm{FM}_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$.

Now suppose $H \neq P$, then there is an edge e_1' that goes from an interior vertex v_i of P (it cannot be an end-vertex by the maximality of P) to a vertex of H not belonging to P (since there is no cycles in H). Extend the path $e_1, \dots, e_{i-1}, e_1'$ as much as posible to the maximal path $e_1, \dots, e_{i-1}, e_1', \dots, e_j'$. Then one of the maximal paths $e_1, \dots, e_{i-1}, e_1', \dots, e_j'$ or $e_m, \dots, e_i, e_1', \dots, e_j'$ is of even length, and we are done.

Subcase 2.2: H has no cycles and is not connected.

Finally, if H is disconnected, then it has two maximal paths P and P' with no vertex in common. If one of them is of even length, we are done, so suppose both of them are of odd length. Let $e_1, \dots, e_m$ be the consecutive edges of P and $e_1', \dots, e_t'$ the consecutive edges of P'. Define $\varepsilon := \min\{x_{e_1}, \dots, x_{e_m}, x_{e_1'}, \dots, x_{e_t'}, 1 - x_{e_1}, \dots, 1 - x_{e_m}, 1 - x_{e_1'}, \dots, 1 - x_{e_t'}\}$. Let $\mathbf{x}' = (x_e')_{e \in E}$ and $\mathbf{x}'' = (x_e'')_{e \in E}$ where

$$x_e' = \begin{cases} x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd} \\ x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even} \\ x_e - \varepsilon & \text{if } e = e_i' \text{ for some } i \in [t] \text{ odd} \\ x_e + \varepsilon & \text{if } e = e_i' \text{ for some } i \in [t] \text{ even} \\ x_e & \text{otherwise} \end{cases}$$

and

$$x''_e = \begin{cases} x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd} \\ x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even} \\ x_e + \varepsilon & \text{if } e = e'_i \text{ for some } i \in [t] \text{ odd} \\ x_e - \varepsilon & \text{if } e = e'_i \text{ for some } i \in [t] \text{ even} \\ x_e & \text{otherwise} \end{cases}.$$

Then $\mathbf{x}', \mathbf{x}'' \in \mathrm{FM}_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$.

□

Observation. The same proof also works to show that $\mathrm{M}_{\leq k}(G) = \mathrm{FM}_{\leq k}(G)$ for bipartite graphs. A similar result holds for $\geq$ instead of $\leq$.

6.2 The normality of k-matching polytopes of bipartite graphs

The main objective in this section is to prove that k-matching polytopes of bipartite graph are normal. As explained in Chapter 1, Section 1.8, we will think of a matching in a bipartite graph not only as a set of edges but also as a matrix. Our starting point is the following result.

Lemma 6.2.1. *If G is a bipartite graph such that $\mathrm{M}_k(G)$ is normal, then so is $\mathrm{M}_k(H)$ for every subgraph H of G.*

Proof. By adding, if necessary, vertices with no incident edges, we can suppose without loss of generality that H and G has the same vertices. If N is an integer point of $t\,\mathrm{M}_k(H)$, N is also an integer point of $t\,\mathrm{M}_k(G)$. Since $\mathrm{M}_k(G)$ is normal, $N = M_1 + \cdots + M_t$ where each M_i is a k-matching of G. Since N and the M_i's have nonnegative entries, every time that N has a zero in the (i, j)-entry so do the M_i's. Thus the M_i's are k-matchings of H, and therefore $\mathrm{M}_k(H)$ is also normal. □

Since every bipartite graph is a subgraph of the complete bipartite graph $\mathrm{K}_{n,n}$ for some $n \in \mathbb{Z}^+$, it is enough to prove our main result for the polytopes $\mathrm{M}_k(\mathrm{K}_{n,n})$, and that is what we are going to do.

Definition 6.2.2. Let G be a bipartite graph with bipartite sets $A = \{a_1, \cdots, a_m\}$ and $B = \{b_1, \cdots, b_n\}$. If $N \in t\,\mathrm{M}_k(G)$ $(t \in \mathbb{Z}^+)$ is an integer point, then the graph H induced by N is the graph with the same vertices of G and such that $\{a_i, b_j\}$ is an edge of H if and only if the entry (i, j) of N is nonzero.

For completeness we present a proof that Birkhoff polytopes $B_n = \mathrm{M}_n(\mathrm{K}_{n,n})$ are normal (Theorem 6.2.4). This is a known result in matching theory; nonetheless it is difficult to find a reference for its proof. There is nothing innovative in the proof presented here, which we believe is the natural one obtained from Hall's Marriage Theorem. We begin by extracting a lemma which we will use later on.

Lemma 6.2.3. *Let $N \in tB_n$ be a an integer point. Then the graph induced by N has a perfect matching M, and thus $N - M \in (t-1)B_n$.*

Proof. Let H be the graph induced by N, and suppose it has no perfect matching. By Hall's Marriage Theorem, there is a subset S of A such that $|\Gamma(S)| < |S|$, where $\Gamma(S)$ is the set of vertices in B that are connected in H by an edge to at least one element in S. Let $k = |S|$ and, without loss of generality, assume $S = \{a_1, \dots, a_k\}$ and $B \setminus \Gamma(S) = \{b_1, \dots, b_m\}$ where $m = n - |\Gamma(S)|$. From $|\Gamma(S)| < |S|$, it follows that $m = n - |\Gamma(S)| > n - |S|$, so $m \geq n - k + 1$. If $b_j \notin \Gamma(S)$, then the entries $N_{1j}, \dots, N_{kj}$ of N are zero. Thus in the upper-left corner of N, there is a submatrix of zeros of size $k \times (n - k + 1)$.

Since $N \in tB_n$, the entries of N in each row and column have to sum up to t. This means that the entries of N in the upper-right $k \times (k-1)$-submatrix of N must sum up to kt, but on the other hand, the entries in the right $n \times (k-1)$ submatrix of N must sum up to $(k-1)t$. This is a contradiction. □

Theorem 6.2.4. *The Birkhoff polytope B_n is normal.*

Proof. We prove by induction on t that every integer point in tB_n is the sum of t integer points in B_n. The result clearly holds for $t = 1$. So suppose it is true for $t - 1$, and we will prove that it also holds for t.

Let N be an integer point of tB_n. By Lemma 6.2.3, the graph induced by N has a perfect matching M, and thus $N - M \in (t-1)B_n$ which by induction can be written as the sum of $t - 1$ integer points of B_n. Notice that M is an integer point of B_n, so N is the sum of t integer points of B_n. □

The plan now is to generalize both Lemma 6.2.3 and Theorem 6.2.4 for the polytopes $\mathrm{M}_k(\mathrm{K}_{n,n})$. For that, we need the next auxiliary result.

Lemma 6.2.5. *Let G be a bipartite graph on $A \sqcup B$. Given $A' \subseteq A$ with $|A'| = r$, $B' \subseteq B$ with $|B'| = c$, and a nonnegative integer $k \leq r + c$, suppose:*

- *there is a k-matching M_1 of G covering A'*
- *there is a k-matching M_2 of G covering B'*
- *there is a $(r + c - k)$-matching M_3 of the subgraph induced by $A' \sqcup B'$.*

Then $M_1 \cup M_2 \cup M_3$ contains a k-matching of G covering $A' \sqcup B'$.

Proof. We prove this by induction on $r + c - k$. If $r + c - k = 0$, we use Lemma 1.8.6: if V_1 is the set of all vertices in A that are covered by M_1, and V_2 is the set of all vertices in B that are covered by M_2, then $M_1 \cup M_2$ contains a matching M that covers $V_1 \cup V_2$. This matching M contains at least k edges since $|V_1| = |M_1| = k$, and it uses at most k edges to cover $A' \sqcup B'$ since $k = r + c$. So we can remove edges from M to obtain a k-matching covering $A' \sqcup B'$. Suppose now that $r+c-k>0$ and that the result holds for smaller values.

Case 1: M_1 or M_2 contains an edge from A' to B'. Without loss of generality we assume the edge is in M_1. If M_1 contains the edge $e=\{a,b\}$ where $a\in A'$ and $b\in B'$, define $M_1' := M_1\setminus\{e\}$, $M_2' := M_2\setminus\{e'\}$ where e' is the edge of M_2 incident to b, and M_3' as the set obtained from M_3 by removing from this all the edges incident to a and b; in case that there is no edge in M_3 incident to a or b, remove any arbitrary edge of M_3. Thus M_3' has one or two element less than M_3.

If $|M_3\setminus M_3'|=1$, we have

- a matching M_1' of size $k-1$ covering $A'\setminus\{a\}$
- a matching M_2' of size $k-1$ covering $B'\setminus\{b\}$
- a matching M_3' of size $r+c-k-1=(r-1)+(c-1)-(k-1)$ in the subgraph induced by $(A'\setminus\{a\})\sqcup(B'\setminus\{b\})$,

so by the induction hypothesis $M_1'\cup M_2'\cup M_3'$ contains a matching of size $k-1$ covering $(A'\setminus\{a\})\sqcup(B'\setminus\{b\})$. Add e to this matching, and the result follows.

If $|M_3\setminus M_3'|=2$, we have two edges in M_3 of the form $e_1=\{a,b'\}$ and $e_2=\{a',b\}$ with $a\neq a'\in A'$ and $b\neq b'\in B$. Remove from M_1' the edge that is incident to a' and from M_2' the edge that is incident to b'. Call this new sets M_1'' and M_2''. We have then

- a matching M_1'' of size $k-2$ covering $A'\setminus\{a,a'\}$
- a matching M_2'' of size $k-2$ covering $B'\setminus\{b,b'\}$
- a matching M_3' of size $r+c-k-2=(r-2)+(c-2)-(k-2)$ in the subgraph induced by $(A'\setminus\{a,a'\})\sqcup(B'\setminus\{b,b'\})$,

so by the induction hypothesis $M_1''\cup M_2''\cup M_3'$ contains a matching of size $k-2$ covering $(A'\setminus\{a,a'\})\sqcup(B'\setminus\{b,b'\})$. Add e_1 and e_2 to this matching, and the result follows.

Case 2: Neither M_1 nor M_2 has an edge from A' to B'. Let A'' be the vertices in A' that are covered by M_3, and define B'' similarly. Let $M_1' := \{e\in M_1 : e \text{ is incident to a vertex in } A'\setminus A''\}$, and define M_2' similarly. Notice that $|M_1'|=|A'|-|A''|=r-(r+c-k)=k-c$, and similarly, $|M_2'|=k-r$. Then $M_1'\sqcup M_2'\sqcup M_3$ is a matching of G of size $|M_1'|+|M_2'|+|M_3|=(k-c)+(k-r)+(r+c-k)=k$ covering $A'\sqcup B'$ (no induction is required in this case). $\square$

To make it easier to understand the following proves, let us consider the elements in $\mathrm{M}_k(\mathrm{K}_{n,n})$ and

$t\,\mathrm{M}_k(\mathrm{K}_{n,n})$ as matrices. By Theorem 6.1.1,

$$\begin{aligned}\mathrm{M}_k(\mathrm{K}_{n,n}) := \{X \in \mathbb{R}^{n\times n} : &\sum_{j=1}^{n} x_{ij} \leq 1 \;\; \forall i \in [n],\; \sum_{i=1}^{n} x_{ij} \leq 1 \;\; \forall j \in [n],\\ &x_{ij} \geq 0 \;\; \forall i,j \in [n],\\ &\sum_{i,j} x_{ij} = k\}.\end{aligned}$$

Since $X \in t\,\mathrm{M}_k(\mathrm{K}_{n,n})$ if and only if $\frac{1}{t}X \in \mathrm{M}_k(\mathrm{K}_{n,n})$, we have

$$\begin{aligned}t\,\mathrm{M}_k(\mathrm{K}_{n,n}) := \{X \in \mathbb{R}^{n\times n} : &\sum_{j=1}^{n} x_{ij} \leq t \;\; \forall i \in [n],\; \sum_{i=1}^{n} x_{ij} \leq t \;\; \forall j \in [n],\\ &x_{ij} \geq 0 \;\; \forall i,j \in [n],\\ &\sum_{i,j} x_{ij} = tk\}.\end{aligned}$$

Lemma 6.2.6. *Let $N \in t\,\mathrm{M}_k(\mathrm{K}_{n,n})$ be an integer point. Then the graph induced by N has a k-matching M such that $N - M \in (t-1)\,\mathrm{M}_k(\mathrm{K}_{n,n})$.*

Proof. The case $k = n$ is Lemma 6.2.3. We prove this result by induction on k. The case $k = 0$ is trivial, so suppose $0 < k < n$, and that the result is true for smaller values.

Since $k < n$, there is a row i such that the sum of its entries is less than t, since otherwise the sum of all the entries in N would be tn. Similarly there is a column j such that the sum of its entries is less than t. We add 1 to the entry (i, j) to obtain a new matrix N_1. We now repeat this process to N_1 to obtain a new matrix N_2. Repeating this process $t(n-k)$ times, we obtain a matrix $N_{t(n-k)} \in B_n$. By the normality of B_n, $N_{t(n-k)} = M_1^* + \cdots + M_t^*$ where each M_i^* is a perfect matching of $K_{n,n}$.

Think of the 1's we added to N to obtain $N_{t(n-k)}$ as colored with red, and think of each original entry of N as the sum of black 1's. Since $N_{t(n-k)} = M_1^* + \cdots + M_t^*$, each 1 in $N_{t(n-k)}$ (black or red) has to appear in one of the M_i^*'s. The total number of black 1's that appear in all the M_i^*'s is tk, so the average number of black 1's in each perfect matching is $tk/t = k$. This implies that among the M_i^*'s there is at least one, which we denote by M^*, with at least k black 1's. This implies that there is a k-matching M of H, where H is the graph induced by N. The problem is that $N - M$ does not necessarily belong to $(t-1)\,\mathrm{M}_k(\mathrm{K}_{n,n})$ since may have rows or columns that sum up to t, so further analysis has to be done.

Since we never add red 1's in those rows and columns whose entries sum up to t, we have a black 1 in each of these rows and columns in M^*. Let r and c be the number of rows and columns of N, respectively, whose entries sum up to t. If $r + c \leq k$, then we can always take k black 1's from M^*, to form a k-matching M of H, in such a way that we take all 1's in those rows and columns whose entries sum up to t. This implies that $N - M \in (t-1)\,\mathrm{M}_k(\mathrm{K}_{n,n})$.

Nevertheless, since $r, c \leq k$ (otherwise the sum of the entries of N would be greater than tk), we can always find a k-matching M_1 that covers

$$A' := \{\text{vertices in } A \text{ whose respective rows sum up to } t\}$$

and a k-matching M_2 that covers

$$B' := \{\text{vertices in } B \text{ whose respective columns sum up to } t\}.$$

Let's consider now the case $r + c > k$. Without loss of generality suppose that the first r rows and the first c columns of N are the ones whose entries sum up to t. Then N can be written in blocks as

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where A_1 is a block of size $r \times c$. If a_i denotes the sum of the entries in A_i, then $a_1 + a_2 + a_3 \leq tk$. Thus $tr + tc = (a_1 + a_2) + (a_1 + a_3) \leq a_1 + tk$, and therefore $t(r + c - k) \leq a_1$.

If $r = c = k$, A_2, A_3, and A_4 would be blocks with only zeros, and $A_1 \in tB_k$. By Lemma 6.2.3, the graph induced by A_1 has a perfect matching P, and in this case the k-matching

$$M = \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

satisfies that $N - M \in (t-1)\,\mathrm{M}_k(\mathrm{K}_{n,n})$.

So suppose now that r or c is not equal to k, and thus $0 < r + c - k < k$.

We reduce some positive entries of A_1 to obtain a matrix C such that its entries sum up to $t(r + c - k)$, which we can do since $t(r + c - k) \leq a_1$. Then

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in t\,\mathrm{M}_{r+c-k}(\mathrm{K}_{n,n}).$$

Since $0 < r + c - k < k$, by induction, the graph induced by

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

has a matching of size $r + c - k$ of the form

$$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

From the way C is obtained, we have that the induced graph of

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

is a subgraph of the induced graph H of N. Thus

$$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

is also a matching of H of size $r + c - k$. That is, we have find a matching M_3 of size $r + c - k$ in the subgraph induced by $A' \sqcup B'$. We use Lemma 6.2.5 to conclude that there is a matching M of H of size k covering $A' \sqcup B'$, which implies that $N - M \in (t-1)\,\mathrm{M}_k(\mathrm{K}_{n,n})$. □

Proposition 6.2.7. *The polytope* $\mathrm{M}_k(\mathrm{K}_{n,n})$ *is normal.*

Proof. We need to prove that if $N \in t\,\mathrm{M}_k(\mathrm{K}_{n,n})$, then it can be written as the sum of t integer points of $\mathrm{M}_k(\mathrm{K}_{n,n})$, that is, as the sum of t k-matchings of $\mathrm{K}_{n,n}$. We prove this by induction on t.

The case $t = 1$ is clear. Suppose the result is true for $t - 1$, and let us prove it for t. By Lemma 6.2.6, there is a k-matching of the induced graph of N, which is also a k-matching of $\mathrm{K}_{n,n}$, such that $N - M \in (t-1)\,\mathrm{M}_k(\mathrm{K}_{n,n})$. By the induction hypothesis, $N - M$ is the sum of $t - 1$ k-matchings of $\mathrm{K}_{n,n}$. Thus N is the sum of t k-matchings of $\mathrm{K}_{n,n}$. □

Theorem 6.2.8. *Let G be a bipartite graph and $k \in \mathbb{N}$. Then the polytope* $\mathrm{M}_k(G)$ *is normal.*

Proof. As we remarked before, every bipartite graph can be seen as a subgraph of $\mathrm{K}_{n,n}$ for some $n \in \mathbb{Z}^+$. Thus, our result follows from Lemma 6.2.1 and Proposition 6.2.7. □

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