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THE MAXIMAL TORI THEOREM AND THE MACKEY MACHINERY

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Content

Introduction

The objective of this project is to document two of the results used in the study of representations of compact Lie groups, these are: The Maximal Tori Theorem and Mackey's Normal Subgroup Analysis or machinery. The first of these results establishes that given a compact Lie group G, every element of G is contained in a maximal torus and that the maximal tori are conjugated to each other. Given T a maximal torus with normalizer N, the quotient group W = N / T is known as the Weyl group. A consequence of the maximal tori theorem is that the invariant functions under conjugation of G are in correspondence with the torus functions invariant under the action of W. This allows us to understand the characters of the representations of G, and, consequently, gives us information about the representations themselves. On the other hand, the second result gives us a way to calculate explicitly the decomposition of a representation of a compact Lie group as a direct sum of irreducible representations. This is accomplished by relating the representations of the group to those of a closed normal subgroup.

Chapter 1

Maximal tori of a compact Lie group

1.1 Definitions and preliminaries

The main purpose of this section is to give an exposition of some of the tools of Lie theory used in the study of Maximal tori. We will also explore some of the central definitions that will be used along this chapter.

Definition 1.1.1. *A Lie group G is a smooth real manifold of finite dimension with a group structure such that the maps*

$$\begin{aligned} G \times G &\longrightarrow G \qquad & (x, y) &\longrightarrow xy \\ G &\longrightarrow G \qquad & x &\longrightarrow x^{-1} \end{aligned}$$

are smooth.

The following result will give us an easy way of identifying Lie subgroups

Theorem 1.1.1. *(Cartan's closed subgroup theorem) Let G be a Lie group. Let H be a subgroup of G that is closed then H is an embedded Lie subgroup*

The proof of this theorem can be found in [Lee13] page 523. The following is an important algebraic structure used in the study of Lie groups.

Definition 1.1.2. *A Lie algebra $\mathfrak{g}$ is a vector space over a field F together with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called the Lie bracket, that satisfies the following conditions:*

- *Bilinearity: for all* $x, y, z \in \mathfrak{g}$ *and for all* $a, b \in F$

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

- *Anticommutativity: For all* $x, y \in \mathfrak{g}$

$$[x, y] = -[y, x]$$

- *Jacobi identity: For all* $x, y, z \in \mathfrak{g}$

$$[x, [y, z]] + [y, [z, x]] + [z[x, y]] = 0$$

Given a lie group G there is a way to associate a lie algebra to G as follows. Given a vector field $X : G \longrightarrow TG$ we say X is left invariant if for all $g, h \in G$ we have:

$$(dl_g)_h(X_h) = X_{gh}$$

Where $l_g : G \longrightarrow G$ its given by $x \longrightarrow gx$ and $(dl_g)_h$ is just the differential of l_g at h. Let Lie(G) be the set of all left invariant vector fields of G. There is a fact from differential geometry that Lie(G) is a Lie algebra with the bracket of vector fields given by $[X, Y] = XY - YX$ and that Lie(G) is isomorphic to T_eG where the isomorphism is given by $ev : Lie(G) \longrightarrow T_eG : V \longrightarrow V_e$.

The most important connection between a Lie group and its Lie algebra is called the exponential map which is defined as follows:

Definition 1.1.3. *Let G be a Lie group with Lie algebra* $\mathfrak{g}$*, the exponential map* $exp : \mathfrak{g} \longrightarrow G$ *is given by* $exp(X) = \gamma(1)$ *where* γ *is the unique integral curve starting at the identity or in other words the unique one-parameter subgroup generated by the field X*

Remark. *In the previous definition we used the well known fact that left invariant vector fields are complete. The proof can be found in [Lee13] theorem 9.18*

The exponential map has the following important properties whose proof can be found in [Lee13] proposition 20.8

Theorem 1.1.2.

- *Let G, H be Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\varphi : G \longrightarrow H$ be a homomorphism of Lee groups. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi_*} & \mathfrak{h} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow{\varphi} & H \end{array}$$

- $D_1(exp) : T_0\mathfrak{g} \cong \mathfrak{g} \longrightarrow \mathfrak{g}$ *is the identity map*

An important application of the previous theorem is the related to the adjoint representation. recall that G acts in itself by conjugation, therefore for every $g \in G$ we have the smooth function

$$Ad_g : G \longrightarrow G : h \longrightarrow ghg^{-1}$$

Derivating we get the linear map $D_1(Ad_g) : \mathfrak{g} \longrightarrow \mathfrak{g}$. We have that $Ad_g \circ Ad_h = Ad_{gh}$ therefore by the chain rule we obtain that $D_1(Ad_g) \circ D_1(Ad_h) = D_1(Ad_{gh})$ and thus we get the map

$$Ad : G \longrightarrow Gl(\mathfrak{g}) : g \longrightarrow D_1(Ad_g)$$

That is an homomorphism of lie groups called the adjoint representation of G. finally By a direct application of theorem 1.1.2 in our new constructions we obtain the following corollary

Corollary 1.1.1. *The following diagrams are commutative*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{D_1(Ad_g)} & \mathfrak{g} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow{Ad_g} & G \end{array}$$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{D_1(Ad)} & End(\mathfrak{g}) \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow{Ad} & GL(\mathfrak{g}) \end{array}$$

Now we are ready to introduce the definition of a Torus

Definition 1.1.4. *A Lie subgroup $T \subset G$ is said to be a maximal torus if it is compact, connected, abelian and there is no other torus T' such that $T \subset T' \subset G$*

If a T and T' are two torus and $T \subsetneq T'$ then $dim(T) < dim(T')$, since tori are compact and connected. Therefore if there where no maximal tori then there would be an infinite increasing sequence of tori $T_0 \subsetneq T_1 \subsetneq T_2 \dots$ with $dim(T_n) < dim(T_{n+1})$ implying the G is infinite dimensional therefore maximal tori must exist. Our next goal is to show that a torus defined in the way that we defined it matches with the usual definition of a torus i.e $T \cong \mathbb{R}^k/\mathbb{Z}^k$ for some $k \in \mathbb{N}$. But before we need to prove three useful lemmas.

Lemma 1.1.1. *Let G be a connected Lie group and let U be neighborhood of e. Then U generates G i.e:*

$$G = \{g_1 g_2 g_3 \dots g_n : g_i \in U, n \in \mathbb{N}\}$$

Proof. Let $V = U \cap U^{-1}$ with U^{-1} the set of inverses of elements in U. We will prove that V generates G and U will generate G since $V \subset U$. Let $S = \{g_1 g_2 g_3 \dots g_n : g_i \in V, n \in \mathbb{N}\}$ to check that $G = S$ is enough to show that S is a non empty clopen set since G is connected. $V \neq \emptyset$ because $e \in V$ therefore $S \neq \emptyset$. V is open because is the preimage of U under the continuous map $x \to x^{-1}$ and gV is also open because is the preimage of V under the continuous map $x \to g^{-1}x$. S is open since for every $g \in S$:

$$g \in gV \subset S$$

Finally S is closed since if $g \notin S$ we have:

$$g \in gV \subset G - S$$

hence S is closed and by connectedness of G we have $S = G$ □

Lemma 1.1.2. *Let G a compact lie group with lie algebra $\mathfrak{g}$*

1. *The function $exp : \mathfrak{g} \longrightarrow G$ is a local difeomorphism near to 0.*

2. *if G is connected $exp(\mathfrak{g})$ generates G*

Proof.

1. By theorem 1.1.2 $D(exp) = Id_{\mathfrak{g}}$ then the result follows from the inverse function theorem

2. It follows from Lemma 1.1.4 and 1

□

Lemma 1.1.3. *Let G be a compact Lie group with with $\mathfrak{g} = Lie(G)$. If $X, Y \in \mathfrak{g}$ with $[X, Y] = 0$ then e^X and e^Y commute. Additionally $\mathfrak{g}$ is a abelian if and only if G^0 is abelian and in this case $e^X e^Y = e^{X+Y}$*

Proof. Let $[X, Y] = 0$ then using corollary 1.4.1 we have that

$$\begin{aligned} Ad_{exp(X)}(exp(Y)) &= exp(D_1(Ad_{exp(X)})(Y)) \\ &= exp(Ad(exp(X)(Y)) \\ &= exp(exp(D_1(Ad)(X))(Y)) \\ &= exp(Y) \end{aligned}$$

Where in the last step we used that since $D_1(Ad)(X)Y = [X, Y] = 0$ then

$$exp(D_0(Ad)(X))(Y) = exp([X, Y])(Y) = \sum_{k=0}^{\infty} \frac{[X, Y]^k}{k!} Y = Y$$

Hence e^X and e^Y commute and therefore if $\mathfrak{g}$ is abelian, the group generated by $exp(\mathfrak{g})$ is abelian and by the lemma 1.1.5 G^0 is abelian. If G^0 is abelian then $Ad|_{G^0} = Id_{G^0}$ and thus $D_0(Ad_{G^0}) = 0$. Since $D_0(Ad_{G^0})$ is the commutator in $\mathfrak{g}$, $\mathfrak{g}$ is abelian. Finally using the Cambpbell, Baker, Hausdorff formula we get.

$$e^X e^Y = exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [Y, X]] \dots) = e^{X+Y}$$

□

Now we are ready to proof the following theorem

Theorem 1.1.3. *If T is a Torus with $dim(T) = k$ then $T \cong \mathbb{R}^k/\mathbb{Z}^k$*

Proof. Let $\mathfrak{t} = Lie(T)$. Since $T = T^0$ is abelian the exponential map is an homeomorphism. It is surjective because using lemma 1.1.5 we have that given an $x \in T$ x can be written as $exp(g_1)exp(g_2)\dots exp(g_n)$ and by lemma 1.1.6 this is equal to $exp(g_1 + g_2 \cdots + g_n)$. Therefore we get that $T \cong \mathfrak{t}/ker(exp)$. On the other side $\mathfrak{t} \cong \mathbb{R}^k$ and $ker(exp)$ is discrete since by lemma 1.1.5 it is a local diffeomorphism near to 0 and thus $ker(exp) \cong \mathbb{Z}^r$ for some r. finally we have that $T = \mathbb{R}^k/\mathbb{Z}^r = T \times \mathbb{R}^{k-r}$ but by compactness of T k must be equal to r and thus $T \cong \mathbb{R}^k/\mathbb{Z}^k$ □

Now we are going to focus our attention on elements of T called the generators and our main tool to do that is called the Kronecker theorem.

Definition 1.1.5. *An element $t \in T \cong \mathbb{R}^k/\mathbb{Z}^k$ is a generator if the generated subgroup $\langle t \rangle$ is dense in T*

Theorem 1.1.4 (Kronecker)**.** *Let $(t_1, t_2, t_3 \dots t_k) \in \mathbb{R}^k$ and t be the its image under $exp : \mathbb{R}^k \longrightarrow T \cong \mathbb{R}^k/\mathbb{Z}^k$. t is a generator if and only if $1, t_1, t_2 \dots t_k$ are linearly independent over $\mathbb{Q}$*

Proof. By theorem 1.1.2 and the discussion on the proof of the previous theorem we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{i} & \mathbb{R}^k & \xrightarrow{exp} & T & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \varphi_*|_{\mathbb{Z}^k}} & & \Big\downarrow{\scriptstyle \varphi_*} & & \Big\downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i} & \mathbb{R} & \xrightarrow{exp} & S^1 & \longrightarrow & 0 \end{array}$$

Where $\varphi : T \longrightarrow S^1$ is an arbitrary homomorphism. Note that $\varphi_*(t_1, t_2 \dots t_k) = \alpha_1 t_1 + \alpha_2 t_2 \dots + \alpha_k t_k$. Now lets show that the following are equivalent:

1. $1, t_1, t_2 \dots t_k$ are linearly dependent over $\mathbb{Q}$
2. $\sum_{i=1}^{k} q_i r_i \in \mathbb{Q}$ for a k-tuple $0 \neq (q_1, q_2, \dots q_k) \in \mathbb{Q}^k$
3. $\sum_{i=1}^{k} \alpha_i r_i \in \mathbb{Z}$ for a k-tuple $0 \neq (\mathbb{Z}_1, \mathbb{Z}_2, \dots \mathbb{Z}_k) \in \mathbb{Z}^k$
4. t mod $\mathbb{Z}^k$ is in the kernell of a non trivial homomorphism
5. t mod $\mathbb{Z}^k$ is not a generator of T

$1 \iff 2$: follows straightforward from the definition. $2 \implies 3$ follows by multiplying by the least common multiple of the denominators of the $q_i's$ and $3 \implies 2$ is trivial.

$3 \implies 4$: suppose $\alpha_1 t_1 + \alpha_2 t_2 \dots + \alpha_k t_k \in \mathbb{Z}$ Let $\varphi : T \longrightarrow S^1$ be a non trivial homomorphism of Lie groups. Note that $exp : \mathbb{R}^k \longrightarrow T$ is $t \longrightarrow t \, mod \, \mathbb{Z}^k$ therefore:

$$\begin{aligned} \varphi(t \, mod \, \mathbb{Z}^k) &= \varphi(exp(\mathbb{Z}^k)) \\ &= exp(\varphi_*(t)) \\ &= exp(\alpha_1 t_1 + \alpha_2 t_2 \dots + \alpha_k t_k) \\ &= 0 \end{aligned}$$

$4 \implies 3$: suppose that $t\, mod\, \mathbb{Z}^k \in ker(\varphi)$ with φ a non trivial homomorphism. We have that:

$$\begin{aligned} \varphi(t\, mod\, \mathbb{Z}^k) &= \varphi(exp(\mathbb{Z}^k)) \\ &= exp(\varphi_*(t)) \\ &= exp(\alpha_1 t_1 + \alpha_2 t_2 \cdots + \alpha_k t_k) \\ &= 0 \end{aligned}$$

Thus we have $\alpha_1 t_1 + \alpha_2 t_2 \cdots + \alpha_k t_k \in \mathbb{Z}^k$

$4 \implies 5$: Suppose that $t mod \mathbb{Z}^k \in ker(\varphi)$ with φ a non trivial homomorphism, then φ is not all T and is a closed subspace. By Cartan's theorem is a Lie subgroup and since is proper $t\, mod\, \mathbb{Z}^k$ does not generates T

$5 \implies 4$: If $t\, mod\, \mathbb{Z}^k$ is not a generator of T it is contains in a proper subgroup H. The quotient space T/H is a non trivial abelian connected compact Lie group and hence a torus. Therefore $T/H \cong \mathbb{R}^r/\mathbb{Z}^r$ for some $r > 0$ and t is in the kernel of

$$T \longrightarrow T/H \cong \mathbb{R}^k/\mathbb{Z}^k \cong S^1 \times S^1 \times S^1 \dots S^1 \xrightarrow{\text{proj}} S^1$$

□

Corollary 1.1.2. *The set of generators of a torus T is dense in T*

Proof. Let A be the set of k-tuples $(t_1, t_2 \dots t_k) \in \mathbb{R}^k$ such that $1, t_1, t_2 \dots t_k$ are linearly independent over $\mathbb{Q}$. The set A is co-countable because if $1, t_2, t_3 \dots t_i$ are linearly independent, the linear independence of $1, t_2, \dots t_{i+1}$ excludes only countable many elements which are precisely the rational multiples of t_{i+1}. Since any co-countable set in $\mathbb{R}^k$ is dense, A is dense in $\mathbb{R}^k$. By the Kronecker theorem the set $exp(A)$ is the set of generators of T and finally by continuity and surjectivity of $exp : \mathbb{R}^k \longrightarrow T$ we have:

$$T = exp(\mathbb{R}^k) = exp(\overline{A}) \subseteq \overline{exp(A)} \subseteq T$$

□

Now let's introduce an important tool called the Weyl group

Definition 1.1.6. *Let T be a maximal torus in G, recall that the normalizer of T in G denoted by $N(T)$ is the subgroup*

$$N(T) = \{g \in G : gTg^{-1} = T\}$$

then the Weyl group of G with respect to T is $W(T) = N(T)/T$

Since maximal tori are not unique, we must proof that the previous definition doesn't depend on the choice of T. In section 3 of this chapter we will prove that conjugate tori are conjugate, therefore is enough to show that maximal tori give rise to isomorphic Weyl groups.

Lemma 1.1.4. *If T and T' are conjugate then $W(T) \cong W(T')$*

Proof. Since T and T' are conjugated the there is a $g \in G$ such that $T' = gTg^{-1}$. Let $\varphi : N(T') \longrightarrow N(T)/T$ be the function given by:

$$\varphi(x) = g^{-1}xgT$$

We will show that φ is a well defined surjective isomorphism with $Ker(\varphi) = T'$ and the result will follow from the first homomorphism theorem. To check it is well defined we have to check that $g^{-1}xg \in N(T)$ for all $x \in N(T')$.

$$g^{-1}xgT(g^{-1}xg)^{-1} = g^{-1}xgTg^{-1}x^{-1}g = g^{-1}xT'x^{-1}g = g^{-1}T'g^{-1} = T$$

Now let yT be an element of $N(T)/T$. $gyg^{-1} \in (T')$ because:

$$gyg^{-1}T'(gyg^{-1})^{-1} = gyg^{-1}T'gy^{-1}g^{-1} = gyTy^{-1}g^{-1} = gTg^{-1} = T'$$

and $\varphi(gyg^{-1}) = y$ therefore φ is surjective. Finally we have to check that $Ker(\varphi) = T'$. let $x \in Ker(\varphi)$ then $g^{-1}xgT = eT$ which means that $g^{-1}xg \in T$ hence $x \in T'$. now let $x \in T'$ then:

$$\varphi(x) = g^{-1}xgT = eT$$

since $g^{-1}xgT$. Therefore $Ker(\varphi) = T'$ and by the first isomorphism theorem we have $N(T')/T' \cong N(T)/T$ □

The normalizer is a compact space since it can be viewed as the preimage of T under the automorphism $f : G \longrightarrow G : g \longrightarrow gtg^{-1}$ where t is a generator of T that exist by corollary 1.1.2. Therefore the Weyl group is compact and thus if we proof that it is discrete follows that is finite. Before doing that, is useful to use an idea similar to the one used in the proof of Kronecker's theorem to understand the torus automorphisms through the following lemma

Lemma 1.1.5. *Let $\varphi \in Aut(T)$ using theorem 1.1.2 we get thee following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{i} & \mathbb{R}^k & \xrightarrow{exp} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_*|_{\mathbb{Z}^k}} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{i} & \mathbb{R}^k & \xrightarrow{exp} & T & \longrightarrow & 0 \end{array}$$

The map $\Phi : Aut(G) \longrightarrow Aut(\mathbb{Z}^k) : \varphi \longrightarrow \varphi_*|_{\mathbb{Z}^k}$ is an isomorphism

Proof. To see that Φ is an homomorphism let $\varphi, \psi \in Aut(T)$ then using the chain rule we get:

$$\Phi(\varphi \circ \psi) = (\varphi \circ \psi)_*|_{\mathbb{Z}^k} = (\varphi_* \circ \psi_*)|_{\mathbb{Z}^k} = \varphi_*|_{\mathbb{Z}^k} \circ \psi_*|_{\mathbb{Z}^k}$$

Injectivity: if $\varphi_*|_{\mathbb{Z}^k} = Id_{\mathbb{Z}^k}$ then $\varphi_* = Id_{\mathbb{R}^k}$ since $\forall (t_1, t_2 \dots t_k)$ we have :

$$\begin{aligned} \varphi_*(t_1, t_2 \dots t_k) &= t_1\varphi_*(e_1) + t_2\varphi_*(e_2) \dots + t_k\varphi(e_k) \\ &= t_1e_1 + t_2e_2 + \dots t_ke_k \\ &= (t_1, t_2, \dots t_k) \end{aligned}$$

Where e_i is the vector with a 1 in its i component and 0 in the rest of components.

Surjectivity: Let $F \in Aut(\mathbb{Z}^k)$ and define $F' : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ by

$$F'(t_1e_1 + t_2e_2 \dots + t_ke_k) = t_1F(e_1) + t_2 \dots + t_kF(e_k)$$

Let $\varphi : T \longrightarrow T$ be the function given by $\varphi(t \, mod \, \mathbb{Z}^k) = exp(F'(t_1, \dots t_k))$. We want to check $\varphi \in Aut(T)$. To see that φ is well defined suppose that $t \, mod \, \mathbb{Z}^k = (t_1, t_2 \dots t_k) \, mod \, \mathbb{Z}^k = (l_1, l_2 \dots l_k) \, mod \, \mathbb{Z}^k = l \, mod \, \mathbb{Z}^k$ then $t_i = l_i + m_i$ for some $m_i \in \mathbb{Z}$ therefore:

$$\begin{aligned} \varphi(t \, mod \, \mathbb{Z}^k) &= exp(F'(t_1, t_2 \dots t_k)) \\ &= exp(t_1F(e_1) + t_2F(e_2) \dots t_kF(e_k))) \\ &= exp(l_1F(e_1) + l_2F(e_2) \dots l_kF(e_k) + m_1F(e_1) \dots + l_kF(e_k)) \\ &= exp(l_1F(e_1) + l_2F(e_2) \dots l_kF(e_k)) \\ &= \varphi(l \, mod \, \mathbb{Z}^k) \end{aligned}$$

φ is surjective because is the composition of two surjective maps. If $\varphi(t \, mod \, \mathbb{Z}^k) = 0 \, mod \mathbb{Z}^k \implies exp(F'(t)) = 0 \implies F'(t) \in \mathbb{Z}^k \implies t \in \mathbb{Z}^k \implies t \, mod \, \mathbb{Z}^k = 0$. Hence φ is injective and $\varphi \in Aut(G)$. Finally Φ is surjective because $\Phi(\varphi) = \varphi_*|_{\mathbb{Z}^k} = (exp_* \circ F'_*)|_{\mathbb{Z}^k} = F'_*|_{\mathbb{Z}^k} = F'|_{\mathbb{Z}^k} = F$ where we used that the derivative of a linear map its equal to itself and theorem 1.1.2 □

The normalizer of a torus T acts on the torus by conjugation. This action can be viewed as the map:

$$N \times T \longrightarrow T : (n, t) \longrightarrow ntn^{-1}$$

This action induces an action of the Weyl group over the torus given by:

$$W(T) \times T \longrightarrow T : (nT, t) \longrightarrow nTn^{-1}$$

This action is well defined because if $nT = n'T$ then there is a $t' \in T$ such that $n = n't'$ therefore:

$$ntn^{-1} = n't'tt'^{-1}n'^{-1} = n'tn'$$

Using this idea and the previous lemma we are ready to show our last result of this section

Theorem 1.1.5. *Let T be a maximal torus then $W(T)$ is finite*

Proof. We already proved observed that the Weyl group is compact and therefore is enough to show that is discrete. First of all, notice that the action of N over T can be viewed as the following continuous map:

$$f : N \longrightarrow Aut(T) \cong GL(k, \mathbb{Z}) : n \longrightarrow Ad_{n_*}|_{\mathbb{Z}^k}$$

First of all notice that $Id \in f(N_0)$, but $f(N_0)$ is a connected subspace of the discrete space $Gl(k, \mathbb{Z})$ hence $Im(N_0) = \{Id\}$ and N_0 acts trivially on T. Let $\alpha : \mathbb{R} \longrightarrow N_0$ be a one parameter group i.e an homeomorphism $(\mathbb{R}, +) \longrightarrow (N_0, \cdot)$. Since $\alpha(\mathbb{R}) \subset N_0$ and N_0 acts trivially by conjugation over T, $\alpha(\mathbb{R})$ acts trivially over T. Hence $\alpha(\mathbb{R}) \cdot T$ is connected in N_0. Thus $\alpha(\mathbb{R}) \cdot T$ is a connected abelian group i.e a torus and since it contains T and T is maximal by hypothesis we have $\alpha(\mathbb{R}) \cdot T = T$ which means $\alpha(\mathbb{R}) \subseteq T$. N_0 is closed because is a connected component. Thus, by the Cartan's closed subgroup theorem is a Lie group. Also $\alpha(\mathbb{R})$ covers an empty neighborhood of the identity on N_0, therefore $\alpha(\mathbb{R})$ generates N_0. in consequence $T \subseteq N_0 \subseteq T$.

By the previous discussion $W(T) = N/T = N/N_0$. Now we want to show that is discrete. Let $nN_0 \in W(T)$ and $q : N \longrightarrow W(T)$ be the canonical projection on the quotient. $\{nN_0\}$ is open in W(T) iff $q^{-1}(\{nN_0\})$ is open on N.

$$\begin{aligned} q^{-1}(\{nN_0\}) &= \{g \in N(T) : gN_0 = nN_0\} \\ &= \{g \in N(T) : \exists x \in N_0 \;\; g = nx\} \\ &= L_n^{-1}(N_0) \end{aligned}$$

Where $L_n : N \longrightarrow N$ is the continuous map $x \longrightarrow nx$. N_0 is open in N being a connected component and thus $q^{-1}\{nN_0\}$ is open which implies the Weyl group is discrete, hence W(T) is finite

□

1.2 Degree of a smooth mapping

Let M and N be two oriented, closed and smooth manifolds of dimension n and let $f : M \longrightarrow N$ be an smooth map. the degree of f its defined as follows.

$$deg(f) = \int_M f^*(\omega)$$

where ω is an n form of N with integral 1. in order to prove that $deg(f)$ doesn't depend on the choice of ω we need the following result.

Theorem 1.2.1. *(Poincaré Duality Theorem) Let N be an oriented closed manifold (Compact without boundary) of dimension n. Define a map $D : \Omega^k(N) \longrightarrow \Omega^{n-k}(N)^*$ given by:*

$$D(\omega)(\eta) = \int_N \omega \wedge \eta$$

then the induced linear map $D : H^k_{dR}(N) \longrightarrow H^{n-k}_{dR}(N)*$ is an isomorphism for each $k \leq n$.

Whose proof can be found in [BT11] page 64.

Proposition 1.2.1. *deg(f) is well defined*

Proof. Let ω and ω' be two volume forms of N with $1 = \int_N \omega = \int_N \omega'$ by the Poincaré Duality Theorem we have the isomorphism

$$D : H^n_{dR}(N) \longrightarrow H^0_{dR}(N)^*$$

given by

$$D([\eta])([f]) = \int_N (f\eta)$$

Since N is connected we have that $H^0_{dR}(N) = \{[f] : df = 0\} = \{[f] : \text{f is constant}\}$ therefore we have that for every $[f] \in H^0_{dR}(N)$

$$D([\omega])([f]) = \int_N f\omega = f\int_n \omega = f\int_n \omega' = \int_N f\omega' = D([\omega'])(f)$$

Hence $D([\omega]) = D([\omega'])$ but since D is a isomorphism we have that $[\omega] = [\omega']$ therefore there is an n-1 η form of N such that $\omega - \omega' = d\eta$. Finally using the fact that the exterior derivative and the pullback of a function commutes together with the stokes theorem we have that:

$$\begin{aligned}
\int_N f^*(\omega) &= \int_N f^*(\omega' + d\eta) \\
&= \int_N f^*(\omega') + f^*(d\eta) \\
&= \int_N f^*(\omega') + \int_N df^*(\eta) \\
&= \int_N f^*(\omega')
\end{aligned}$$

□

Proposition 1.2.2. *If $f : M \longrightarrow N$ is a smooth function between closed connected manifolds that is not surjective then deg(f)=0*

Proof. Since M is compact, $f(M)$ is compact, therefore $f(M)$ is closed since N is a Hausdorff space. f is not surjective and therefore we can take $x \in N - f(M)$ and a neighborhood V of x such that $f(M) \cap V = \emptyset$ and that is diffeomorphic to an open set of $\mathbb{R}^n$. now let B be a closed neighborhood of x contained in V. There is a bump function g for B with $supp(g) \subset V$. Let ω be a smooth n form of N such that $\int_N \omega = 1$ and define the form $\eta = \frac{g}{\lambda}\omega$ where $\lambda = \int_N g\omega$. Then, η is an n form of volume that is 0 outside of V with integral 1. Now we choose $(x_1, x_2, \dots x_n)$ to be coordinates for V in which we can write $\eta = \frac{g}{\lambda} dx_1 \wedge dx_2 \wedge dx_3 \wedge \dots dx_n$ therefore $f^*(\eta) = \frac{g \circ f}{\lambda} d(x_1 \circ f) \wedge d(x_2 \circ f) \wedge \dots d(x_n \circ f) = 0$ because $g \circ f = 0$, finally we have:

$$deg(f) = \int_M f^*(\eta) = 0$$

Before giving the main result in this section we need the following definition:

□

Definition 1.2.1. *Let $f : M \longrightarrow N$ be a smooth map between compact, connected, and orientable manifolds of dimension n. Let ω be a volume form of N with volume 1 and ω' be a volume form of M with volume 1. The determinant of f, $det(f) : M \longrightarrow \mathbb{R}$ is determined by the equation:*

$$f^*(\omega) = det(f)\omega'$$

Theorem 1.2.2. *Let M, N be compact connected orientable manifolds of dimension n. If $q \in N$ is a regular value of f with $f^{-1}(q) = \{p_1, \dots p_l\}$ then:*

$$deg(f) = \sum_{p \in f^{-1}(q)} sgn(det(f)(p))$$

Proof. Let Ω_c^n the set of n-forms with compact support. Recall that if $\varphi : M \longrightarrow N$ is a diffeomorphism then for all $\alpha \in \Omega_c^n$

$$\int_M \alpha = \int_N \xi \varphi^* \alpha$$

Where $\xi = \pm 1$ depending if φ preserves or reverses the orientation or not. Now, lets show that the degree of a map is invariant under homotopy classes. Let $f : M \times [0,1] \longrightarrow N$ be an smooth homotopy. If $\alpha \in \Omega^n(N)$ then $d\alpha \in \Omega^{n+1}(N) = \{0\}$. Therefore:

$$\begin{aligned} 0 = \int_{M \times [0,1]} f^* d\alpha &= \int_{M \times [0,1]} df^* \alpha \\ = \int_{\partial M \times [0,1]} f^* \alpha &= \int_M f_1^* \alpha + \int_{-M} f_0^* \alpha \\ = \int_M f_1^* \alpha - \int_M f_0^* \alpha & \end{aligned}$$

By the inverse function theorem f is a local diffeomorphism around each p_i, Since q is a regular value of f with $f^{-1}(q) = \{p_1 \dots p_l\}$. Therefore, there are neighborhoods U_i of each p_i and a neighborhood U of q such that $f|_{U_i} : U_i \longrightarrow U$ is a diffeomorphism and $f^{-1}(U) = \bigsqcup U_i$. If $supp(\alpha) \subseteq U$ the theorem follows from the fact mentioned at the beginning of the proof. Thus, if we could find a diffeomorphism $\varphi : N \longrightarrow N$ homotopically equivalent to the identity such that $supp(\alpha) \subset \varphi(U)$ which implies $supp(\varphi^*\alpha) \subset U$ we would have that:

$$\begin{aligned} deg(f) &= \int_M f^*(\alpha) = \int_M f^*(\varphi^*(\alpha)) \\ &= \int_{\bigsqcup U_i} f^*(\varphi^*(\alpha)) = \sum_i \int_{U_i} f^*(\varphi^*(\alpha)) \\ &= \sum_i sgn(det(f)(p_i)) \end{aligned}$$

We will prove that $\{\varphi(U) : \varphi : N \longrightarrow N, \varphi \text{ a diffeomorphism}\}$ covers N and the theorem will follow by taking a partition of unity $\{\psi_j : j \in J\}$ since it is true for each term of the decomposition $\alpha = \sum_{j \in J} \psi_j \alpha$. Let $x \in N$, if x and q are in a compact set completely contained in a domain of a chart, we can construct the diffeomorphism by taking the integral curve of a vector field vanishing outside the domain of the chart. If q and x are not contained in a set with this characteristics we can connect them by elements $x_1, x_2 \dots x_n$ satisfying that x_{j-1}, x_j are contained in a compact set contained in the domain of some chart.

□

1.3 Maximal tori theorem

First of all we need to set the basis of integration on Lie groups. Specifically, we need to exhibit volume forms for a Lie group G and the quotient G/H where H is a closed subgroup.

Definition 1.3.1. *A differential form ω of a Lie group G is called left invariant if for all g in G we have that $L_g^*\omega = \omega$ where L_g is the function $G \longrightarrow G : h \longrightarrow gh$*

Theorem 1.3.1. *In a Lie group G there is a volume form that is left invariant and unique up to a multiplication by a constant*

Proof. Let $\{e_1^*, e_2^* \dots e_n^*\}$ be a basis of $T_1^*G \cong \mathfrak{g}$ and let $\omega_e = e_1^* \wedge e_2^* \cdots \wedge e_n^*$. Now define the differential form $\omega : G \longrightarrow \bigwedge^n T_e^*G : g \longrightarrow \omega_g := L_{g^{-1}}^*\omega_e$. ω is left invariant since:

$$\begin{aligned}(L_g^*\omega)_h &= L_g^*\omega_{gh}\\ &= L_g^*L_{h^{-1}g^{-1}}^*\omega_e\\ &= (L_{h^{-1}g^{-1}} \circ L_g)^*\omega_e\\ &= L_{h^{-1}}^*\omega_e\\ &= \omega_h\end{aligned}$$

Now we need to show that it is a volume form. Suppose by contradiction that there is a $g \in G$ such that $\omega_g = 0$ then $L_{g^{-1}}^*\omega_e$. Let $v_1, v_2 \dots v_n$ be linearly independent vectors on T_gG. Since L_g is a diffeomorphism, L_{g_*} is an isomorphism and hence $L_{g_*^{-1}}(v_1), L_{g_*^{-1}}(v_2), \dots L_{g_*^{-1}}(v_n)$ are linearly independent on T_eG. We have that $0 = \omega_e(L_{g_*^{-1}}(v_1), L_{g_*^{-1}}(v_2), \dots L_{g_*^{-1}}(v_n))$. Let $T : T_eG \longrightarrow T_eG$ be the linear transformation given by $T(e_i) = L_{g_*^{-1}}(v_i)$. By properties of alternating maps, we get that

$$\begin{aligned}0 &= \omega_e(L_{g_*^{-1}}(v_1), L_{g_*^{-1}}(v_2), \dots L_{g_*^{-1}}(v_n))\\ &= \omega_e(T(e_1), T(e_2), \dots T(e_n))\\ &= det(T)\omega_e(e_1, e_2 \dots e_n)\\ &= det(T)\end{aligned}$$

Therefore $det(T) = 0$ and $L_{g_*^{-1}}(v_1), L_{g_*^{-1}}(v_2), \dots L_{g_*^{-1}}(v_n)$ are linearly dependent which is a contradiction.
Finally, to proof uniqueness suppose there is another left invariant volume form ω'. Since $\bigwedge^n T_e^*G$ has dimension 1, there is a constant C such that:

$$\omega'_e = C\omega_e$$

Hence, for all $g \in G$

$$\omega'_g = L_{g^{-1}}^*\omega'_e = L_{g^{-1}}^*C\omega_e = CL_{g^{-1}}^*\omega_e = C\omega_g$$

□

If G is a lie group with a closed subgroup H, H acts smoothly, freely and properly over G by left multiplication. Therefore, by the quotient manifold theorem, whose proof can be found in [Lee13] theorem 21.12, G/H is a manifold of dimension $dim(G) - dim(H)$ and $\pi : G \longrightarrow G/H$ is an smooth submersion. Therefore, it makes sense to study its orientability

Theorem 1.3.2. *If H is a closed connected Lie subgroup of a compact lie group G then G/H is orientable.*

Proof. Let $n = dim(G)$ and $k = dim(H)$. Just as in the proof of theorem 1.3.1 we can build a non zero element of $\bigwedge^{n-k} T^*_{eH}G/H$ that we will denote by τ_e. Now define the volume form $\tau : G/H \longrightarrow \bigwedge^{n-k} T^*_{eH}G/H : gH \longrightarrow L^*_{g^{-1}}(\tau_e)$. Where $L_g : G/H \longrightarrow G/H$ is the function of left multiplication by g over G/H. The proof of left invariance, uniqueness and that is a volume form is equal to the proof of theorem 1.3.1 so we only need to proof that is well defined. Suppose $gh = g'H$ then there is an $h \in H$ such that $g' = gh$. We want to check that

$$\begin{aligned} L^*_{g^{-1}}\tau_e = L^*_{g'^{-1}}\tau_e &= L^*_{(gh)^{-1}}\tau_e \\ &= L^*_{h^{-1}g^{-1}}\tau_e = L^*_{g'^{-1}}L^*_{h^{-1}}\tau_e \end{aligned}$$

Therefore enough to proof that for all h in H $L^*_{h^{-1}}\tau_e = \tau_e$. $L^*_{h^{-1}}\tau_e \in \bigwedge^{n-k} T^*_{eH}G/H$ and since $\bigwedge^{n-k} T^*_{eH}G/H$ is a one dimensional space we can write $L^*_{h^{-1}}\tau_e = \lambda(h)\tau_e$ where $\lambda(h) \in \mathbb{R}$. The map $\lambda : H \longrightarrow \mathbb{R} - \{0\}$ is continuous and is group homomorphism since if $h_1, h_2 \in H$:

$$\begin{aligned} \lambda(h_1h_2)\tau_e = L^*_{(h_1h_2)^{-1}}\tau_e &= L^*_{h_1^{-1}}L^*_{h_2^{-1}}\tau_e \\ &= L^*_{h_1}\lambda(h_2)\tau_e = \lambda(h_1)\lambda(h_2)\tau_e \end{aligned}$$

H is compact because is a closed subset in a compact space and is connected by hypothesis. Thus, $\lambda(H) = [a, b]$ and since $\lambda(e) = 1$, $[a, b] \subset \mathbb{R}^{>0}$. Suppose there is a $h \in H$ such that $\lambda(h) \neq 1$ then $\lambda^n(h) = \lambda(h^n) \in [a, b]$ for all $n \in \mathbb{N}$ which is a contradiction. Therefore $\lambda(h) = 1$ for all $h \in H$ and $L^*_{h^{-1}}\tau_e = \tau_e$ as we wanted. □

It is known that that if G is a compact connected Lie group with Lie algebra $\mathfrak{g}$ there is a symmetric bilinear form $\langle \cdot\, , \cdot \rangle$ on $\mathfrak{g}$ such that for all $g \in G$ and $X, Y \in \mathfrak{g}$

$$\langle Ad(g)X, Ad(g)Y \rangle = \langle X, Y \rangle$$

If T is a maximal torus of a lie group G satisfying the above condition, the Lie algebra $\mathfrak{t}$ of T is a Lie subalgebra of $\mathfrak{g}$. Thus $\mathfrak{g}$ can be decomposed as the direct sum of $\mathfrak{g}$ and its orthogonal complement with respect to $\langle \cdot\,,\cdot\rangle$ namely $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^{\perp}$.

Proposition 1.3.1. *$Ad|_T$ acts trivially in any element of $\mathfrak{t}$ and non trivially in any non zero vector of $\mathfrak{t}^{\perp}$. Moreover for all $t \in \mathfrak{t}$ $Ad(t) : \mathfrak{g} \longrightarrow \mathfrak{g}$ preserves $\mathfrak{t} \oplus \mathfrak{t}^{\perp}$*

Proof. Let $X \in \mathfrak{t}$ and $\gamma : \mathbb{R} \longrightarrow G$ the integral curve of V that pass through the identity, then:

$$\begin{aligned} Ad(t)X = D_1 Ad_t x &= \frac{d}{ds}\Big|_{s=0}(Ad_t \circ \gamma)(s) \\ &= \frac{d}{ds}\Big|_{s=0} t\gamma(s)t^{-1} = \frac{d}{ds}\Big|_{s=0}\gamma(s) = X \end{aligned}$$

Now suppose $X \in \mathfrak{t}^{\perp}$ and $Ad(t)X = X$. Let γ be the integral curve of X passing through the identity and define $\beta : \mathbb{R} \longrightarrow G$ as $s \longrightarrow t\gamma(s)t^{-1}$ then:

$$\begin{aligned} X = Ad(t)X = D_1 Ad_t &= \frac{d}{ds}\Big|_{s=0}(Ad_t \circ \gamma)(s) \\ &= \frac{d}{ds}\Big|_{s=0} t\gamma(s)t^{-1} = \frac{d}{ds}\Big|_{s=0}\beta(s) \end{aligned}$$

Hence β is a maximal integral curve of X and by the uniqueness of integral curves $t\gamma = \gamma t$. Then $\gamma(\mathbb{R})$ commutes with T and $\gamma(\mathbb{R}) \cdot T$ is an abelian connected group containing T but by maximality of T we have that $\gamma(\mathbb{R})T = T$. Thus, $\gamma(\mathbb{R}) \subset T$ and $X \in \mathfrak{t} \cap \mathfrak{t}^{\perp} \implies X = 0$. Finally to show that $Ad(t)$ preserves $\mathfrak{t} \oplus \mathfrak{t}^{\perp}$ let $X \in \mathfrak{t}$ and $Y \in \mathfrak{t}^{\perp}$ then:

$$\langle X, Ad(t)Y\rangle = \langle Ad(t^{-1})X, Ad(t^{-1})Ad(t)Y\rangle = \langle X, Y\rangle = 0$$

Thus, $Ad(t)Y \in \mathfrak{t}^{\perp}$ □

The canonical projection $\pi : G \longrightarrow G/T$ induces the map $\pi_* : \mathfrak{t} \oplus \mathfrak{t}^{\perp} \longrightarrow T_{eT}G/T$ whose restriction give rise to an isomorphism $\mathfrak{t}^{\perp} \cong T_{eT}G/T$. We will denote by $Ad_{G/T} : T \longrightarrow Aut(Lie(G/T))$ the induced action of T over G/T. Let dg, dt, dgT be the left invariant volume forms over $G, T, G/T$ respectively, we can assume they have volume 1. If $n = dim(G)$ and $k = dim(T)$ we have that $\pi^*(dgT) \in \Omega^{n-k}(G)$. Now let $Pr_2 : \mathfrak{g} \cong Lie(G/T) \oplus \mathfrak{t} \longrightarrow \mathfrak{t}$ be the projection on $\mathfrak{t}$. Since the form $dt_e \bigwedge^k \mathfrak{t}$ it gives rise to the form $Pr_2^*(dt_e) \in \bigwedge^k \mathfrak{g}$ which determines a unique left invariant volume form that we will denote by $d\tau$ and that satisfies $d\tau|_T = dt$. $\pi^* dgT \wedge d\tau$ is an invariant volume form of G therefore $\pi^* dgT \wedge d\tau = cdg$ and we can choose the signs to ensure $c > 0$. our next goal is to proof that $c = 1$. in order to do that we need the following theorem:

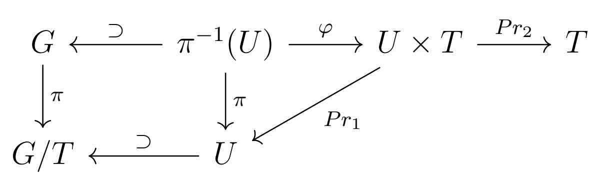
Theorem 1.3.3. *Let G be a compact Lie group with a closed subgroup H and dgH be the left invariant volume form with volume 1 of G/H then for all $f : G \longrightarrow \mathbb{R}$ continuous*

$$\int_G f(g)dg = \int_{G/H} \left(\int_H f(gh)dh \right) dgH$$

Let (U, φ) be a chart of the T-principal bundle $\pi : G \longrightarrow G/T$. If $f : G \longrightarrow \mathbb{R}$ is a non negative function different from zero with support contained in $\pi^{-1}(U)$:

$$0 \neq \int_G f dg = \int_{G/T} \left(\int_T f(gt)dt \right) dgT = \int_U \left(\int_T f(gt)dt \right) dgT$$

Now we are going to make a change of coordinates following the next diagram



Which is commutative since φ is a chart of the T principal bundle. We get,

$$\begin{aligned}
\int_U \left(\int_T f(gt)dt \right) dgT &= \int_U \left(\int_T f\varphi^{-1}(u,t) Pr_2^* dt \right) Pr_1^* dgT \\
&= \int_{U \times T} f\varphi^{-1}(u,t) Pr_1^* dgT \wedge Pr_2^* dt \\
&= \int_{U \times T} f\varphi^{-1} (\varphi^{-1})^* \pi^* dgT \wedge d\tau \\
&= c \int_G f dg
\end{aligned}$$

Hence $c = 1$. We have the volume form of G $dg = \pi^* d(gT) \wedge d\tau$ and the volume form of $G/T \times T$ $\alpha \coloneqq pr_1^* d(gT) \wedge pr_2^* dt$. Recall from the previous section that the determinant of a function $\psi : G/T \times T \longrightarrow G$ is determined by the equation:

$$\psi^* dg = det(\psi)\alpha$$

Proposition 1.3.2. *Let* $q : G/T \times T \longrightarrow G$ *be the conjugation map* $(gT, t) \longrightarrow gtg^{-1}$. *Then:*

$$det(q)(gT, t) = det(Ad_{G/T}(t^{-1}) - I_{G/T})$$

Proof. To compute the determinant of q at point (gT, t) we will use the fact that the volume forms dg and $d(gT)$ are left invariant under the action of G and that the volume form dt is invariant under the action of T in order to make some suitable left transformations to compute the determinant of a map that sends (eT, e) to e. Specifically, we have the composition:

$$G \times T \xrightarrow{(L_g, L_t)} G \times T \xrightarrow{\tilde{q}} G \xrightarrow{L_{gt^{-1}g^{-1}}} G$$

which is the map

$$(x, y) \longrightarrow (gx, ty) \longrightarrow (gx)(ty)(gx)^{-1} \longrightarrow gt^{-1}xtyx^{-1}g^{-1} = Ad_g(Ad_{t^{-1}}(x)yx^{-1})$$

This map sends (eT, e) to e thus the determinant we are looking for is the determinant of the differential of this composition restricted to $Lie(G/T) \oplus \mathfrak{t} \subseteq \mathfrak{g} \oplus \mathfrak{t} = Lie(G \times T)$. $Ad(g) = D_e Ad_g$ has determinant 1 since is an orthogonal operator. Now using that fact that the differential of the product map $m : G \times G \longrightarrow G : (g, h) \longrightarrow gh$ is the sum and that the differential of the inverse map $i : G \longrightarrow G : g \longrightarrow g^{-1}$ is taking the additive inverse we get that the differential of q at (gT, t) is the differential of the endomorphism:

$$(X, Y) \longrightarrow Ad_{G/T}(t^{-1})(X) + Y - X$$

Which goes from $Lie(G/T) \oplus \mathfrak{t}$ to itself. Its matrix form is:

$$\left(\begin{array}{c|c} Ad_{G/T}(t^{-1}) - I_{G/T} & 0 \\ \hline 0 & I_T \end{array}\right)$$

And thus its determinat is $det(Ad_{G/T}(t^{-1}) - I_{G/T})$

□

The previous proposition and the next one are part of the proof of a the main lemma which is the final step in the proof of the torus theorem.

Proposition 1.3.3. *Let* $t \in T$ *be a generator then:*

1. $|q^{-1}(t)| = |W(T)|$

2. $det(q) > 0$ on the points $q^{-1}(t)$

Proof.

1. Let N(T) be the normalizer of T. We have that:

$$q(sT, s) = t \iff gsg^{-1} = t \iff s = g^{-1}tg$$
$$\iff g^{-1}tg \in T \iff g \in N(T)$$

   Hence $q^{-1}(t) = \{(gT, g^{-1}tg) : g \in N(T)\}$. Given $g_1, g_2 \in N(T)$ such that $g_2 \in g_1T$ there is an $s \in T$ such that $g_2 = sg_1$ therefore

$$g_2^{-1}tg_2 = g_1^{-1}s^{-1}tsg_1 = g_1^{-1}tg_1$$

   On the other side if $g_1T \neq g_2T$ then $(g_1T, g_1^{-1}tg_1) \neq (g_2T, g_2^{-1}tg_2)$. Thus, $|q^{-1}(t)| = |W|$

2. By the previous proposition the determinant we want to compute is the determinant of $Ad_{G/T}(t^{-1}) - I_{G/T}$. Thus if we show that $Ad_{G/T}(t^{-1}) - I_{G/T}$ doesn't have real eigenvalues, the eigenvalues will come in complex conjugate pairs proving that $det(q) > 0$.

   Suppose $Ad_{G/T}(t^{-1}) - I_{G/T}$ has a real eigenvalue, then $Ad_{G/T}(t^{-1})$ also has a real eigenvalue. This eigenvalue is equal to ± 1 since $Ad_{G/T}(t^{-1})$ is orthogonal. If it is equal to -1 then 1 is an eigenvalue of $Ad_{G/T}(t^{-2})$. Since t^{-2} is also a generator of T is enough to show that $Ad_{G/T}(t)$ has not 1 as an eigenvalue for every generator $t \in T$.

   Now suppose there is an $X \in Lie(G/T)$ such that $Ad_{G/T}(t)(X) = X$ then X is fixed by all the elements of T since t is a generator. By corollary 1.1.1 the one parameter subgroup $H = \{exp(sX) : s \in \mathbb{R}\}$ is fixed by the conjugation action of T. Thus, HT is an abelian connected group containing T. Therefore $HT = T$ and $H \subseteq T$, hence $X \in Lie(G/T) \cap \mathfrak{t} = \{0\}$

□

The previous two propositions together with theorem 1.2.2 have as an immediate consequence the main lemma used in the proof of the torus theorem.

Lemma 1.3.1. *Let G be a connected compact Lie group and T be a maximal torus of G. The map*

$$\begin{aligned} q :& G/T \times T \longrightarrow G \\ &(gT, t) \longrightarrow gtg^{-1} \end{aligned}$$

Has $deg(q) = |W(T)|$ *in particular* q *is surjective.*

Finally, we can proof the main result of this section.

Theorem 1.3.4 (Maximal tori Theorem)**.** *Every two maximal tori in a compact connected Lie group G are conjugated and every element of the group is contained in a maximal torus*

Proof. Let T and T' be maximal two maximal tori of G and t' be a generator of T'. Since q is surjective there is a $g \in G$ such that $t' \in gTg^{-1}$. Therefore, $T' \subseteq gTg^{-1}$ and by maximality of T' we have $T' = gTg^{-1}$. Finally also by the surjectivity of q every element of G is contained in a the conjugation of a torus □

Chapter 2

Mackey machinery for compact Lie groups

2.1 The Fell topology

In this section we will introduce the topology on the space of irreducible representations. Also, we are going to state a result that will simplify our analysis of representations in the future. The strategy is to first define the Fell topology on irreducible representations of C^* algebras. After that we are going to assign a C^* algebra to a lie group and translate the topology in the space of representations of the algebra to the space of irreducible representations of the group.

Definition 2.1.1. *A C^* algebra $\mathcal{A}$ is a Banach algebra over $\mathbb{C}$ with a map $\mathcal{A} \longrightarrow \mathcal{A} : x \longrightarrow x^*$ such that for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:*

- $x^{**} = x$
- $(x + y)^* = x^* + y^*$
- $(xy)^* = y^* x^*$
- $(\lambda x)^* = \overline{\lambda} x^*$
- $||x^* x|| = ||x||^2$

The last condition is called C^* condition and an Banach algebra with all the previous conditions except by the C^* condition is called a $*-$algebra

Definition 2.1.2. *Let $\mathcal{A}$ be a C^* algebra. An ideal I of $\mathcal{A}$ is called primitive if $I = ker(\pi)$ with π an irreducible representation of $\mathcal{A}$. The set of primitive ideals will be denoted $Prim(\mathcal{A})$*

Definition 2.1.3. *The hull-kernell is this topology where the closures are of the following form:*

$$\overline{\mathcal{J}} = \{I \in Prim(\mathcal{A}) : I \supseteq \bigcap_{J \in \mathcal{J}} J\} \tag{2.1}$$

where $\mathcal{J} \subseteq Prim(\mathcal{A})$

Let $\widehat{\mathcal{A}}$ be the set of equivalence classes of irreducible representations of $\mathcal{A}$ and let $k : \widehat{\mathcal{A}} \longrightarrow Prim(\mathcal{A})$ be the canonical map $\pi \longrightarrow ker(\pi)$

Definition 2.1.4. *The Fell topology over $\widehat{\mathcal{A}}$ is the initial topology of the map $k : \pi \longrightarrow ker(\pi)$ i.e U is an open set in $\widehat{\mathcal{A}}$ if and only if $U = k^{-1}(V)$ with V open in $Prim(\mathcal{A})$*

Given a Lie group G the space $L^1(G)$ withe the norm $||\cdot||_1$ together with convolution and involution given by $f^*(g) = \overline{f(g^{-1})}$ is a $*$ algebra but it does not satisfy the C^* condition. Therefore we need to introduce a norm for $L^1(G)$

Definition 2.1.5. *Let $||\cdot||_*$ be the norm on $L^1(G)$ given by:*

$$||f||_* = \sup\{||\pi(f)|| : \pi \in Irr(G)\}$$

With $Irr(G)$ the set of equivalence classes of unitary representation of G

With this norm $L^1(G)$ satisfies all the properties of being a C^* algebra except completeness.

Definition 2.1.6. *The C^* algebra of a Lie group G denoted by $C^*(G)$ is the competition of $(L^1(G), ||\cdot||_*)$*

it is known that there is a bijection between equivalence classes of unitary representations of the Lie group G and representations of $C^*(G)$

Definition 2.1.7. *The Fell topology over the set of unitary representations of G $\widehat{G}$ is the one that makes the bijection $\widehat{G} \longrightarrow \widehat{C^*(G)}$ a homeomorphism*

Finally, we will also use the following result that can be found in the proposition 1.70 of [KT12]

Proposition 2.1.1. *If G is compact, then $\widehat{G}$ is discrete*

2.2 Central extensions and representations

From now on G will denote a compact Lie group and A will denote a closed normal subgroup of G. We have the following extension:

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

Where we define Q to be G/N. Given an irreducible representation of A $\rho : A \longrightarrow U(V_\rho)$ and an element $g \in G$ we define the irreducible representation of A $g \cdot \rho$ by:

$$(g \cdot \rho)(a) = \rho(g^{-1}ag)$$

It is well defined because A is normal in G. These definitions gives us a left action of G over the set of equivalence classes of irreducible unitary representations of A that we will denote by $Irr(A)$. Let $G_{[\rho]}$ be the stabilizer or isotropy group of the representation ρ. For each $g \in G_{[\rho]}$ there is a unitary matrix U_g such that for all $a \in A$

$$\rho(g^{-1}ag) = U_g^{-1}\rho(a)U_g$$

If $[\rho] \in Irr(A)$ and $a \in A$ then $a \cdot \rho(x) = \rho(a^{-1}xa) = \rho^{-1}(a)\rho(x)\rho(a)$. Therefore $A \subseteq G_{[\rho]}$ and we have the following extension:

$$1 \longrightarrow A \longrightarrow G_{[\rho]} \longrightarrow Q_{[\rho]} \to 1$$

Where we define $Q_{[\rho]}$ as $G_{[\rho]}/A$. If A is central in G, $G_{[\rho]} = G$ and $Q_{[\rho]}$

Lemma 2.2.1. *There is a unique homomorphism $\gamma : G_{[\rho]} \longrightarrow PU(V_\rho)$ such that the following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & G_{[\rho]} \\ \rho\big\downarrow & & \big\downarrow\gamma \\ U(V_\rho) & \xrightarrow{\;p\;} & UP(V_\rho) \end{array}$$

Proof. Let $\gamma(g) := [U_g]$.It is well defined since if U'_g is another isomorphism of irreducible representations it is a multiple of U_g by the shur's lemma therefore $[U_g] = [U'_g]$. similarly γ is an homomorphism because if $g, h \in G_g$, U_{gh} and U_gU_h are multiples by shur's lemma. The diagram is commutative since $\forall a \in A$

$$p(\rho(a)) = [\rho(a)] = [i(a)] = \gamma(i(a))$$

□

Let $p : U(V_\rho) \longrightarrow PU(V_\rho) \cong \mathrm{Inn}(V_\rho)$ be the map that sends u to conjugation by u. We have the Canonical central extension:

$$1 \to S^1 \xrightarrow{i} U(V_\rho) \xrightarrow{p} PU(V_\rho) \to 1$$

Now let $\widetilde{G}_{[\rho]}$ be the fiber product $G_{[\rho]} \times_{PU(V_\rho)} U(V_\rho) = \{(g,u) \in G_{[\rho]} \times U(V_\rho) : \gamma(g) = p(u)\}$. $\widetilde{G}_{[\rho]}$ is the pullback of γ and p, hence by the universal property we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{G}_{[\rho]} & \xrightarrow{\pi_1} & G_{[\rho]} \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \gamma} \\ U(V_\rho) & \xrightarrow{p} & PU(V_\rho) \end{array}$$

With π_1 and π_2 the projections on the first and second component respectively. We have that $1 \times S^1 \subseteq \widetilde{G}_{[\rho]}$ since $\gamma(1) = 1 = p(S^1)$. The inclusion denoted by ι gives us the following sequence:

$$1 \to S^1 \xrightarrow{\iota} \widetilde{G}_{[\rho]} \xrightarrow{\pi_1} G_{[\rho]} \to 1$$

Which is exact because $ker(\pi_1) = \{(g,u) \in \widetilde{G}_{[\rho]} : g = 1\} = 1 \times S^1 = \iota(S^1)$. S^1 is central because $Z(U(V_{[\rho]})) = S^1$ and $1 \in Z(G_{[\rho]})$. Therefore, $1 \times S^1 \subseteq Z(\tilde{G}_{[\rho]})$. Thus, it is a central extension and we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^1 & \xrightarrow{i} & \widetilde{G}_{[\rho]} & \xrightarrow{\pi_1} & G_{[\rho]} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \gamma} & & \\ 1 & \longrightarrow & S^1 & \xrightarrow{i} & U(V_\rho) & \xrightarrow{p} & PU(V_\rho) & \longrightarrow & 1 \end{array}$$

Lets consider the homomorphism $i \times \rho : A \longrightarrow G_{[\rho]} \times U(V_\rho)$. Its image is contained in $\widetilde{G}_{[\rho]}$ because of the diagram of lemma 2.2.1.

Lemma 2.2.2. $(i \times \rho)(A) \triangleleft \widetilde{G}_{[\rho]}$

Proof. Let $(g,u) \in \widetilde{G}_{[\rho]}$ and $a \in A$. Since $\gamma(g) = p(u)$ we have that

$$\rho(g^{-1}ag) = u^{-1}\rho(a)U$$

Hence we get that:

$$\begin{aligned} (g,u)^{-1}(i \times \rho)(a)(g,u) &= (g^{-1}ag, u^{-1}\rho(a)u) \\ &= (i(g^{-1}ag, \rho(g^{-1}a)g) \in (i \times \rho)(A) \end{aligned}$$

□

Thus $G_{[\rho]}/(i\times\rho)(A)$ is a lie group and we have the following exact sequence:

$$1 \to A \xrightarrow{i\times\rho} \widetilde{G}_{[\rho]} \xrightarrow{q} \widetilde{Q}_{[\rho]} \to 1$$

Where $\widetilde{Q}_{[\rho]} := \widetilde{G}_{[\rho]}/(i\times\rho)(A)$ and q is the canonical projection. Now the inclusion $\iota : 1\times S^1 \longrightarrow \widetilde{G}_{[\rho]}$, induces the map $\tilde{\iota} : 1\times S^1 \to \widetilde{Q}_{[\rho]}$, given by $\tilde{\iota} = q\circ\iota$. Therefore, we have the following sequence

$$1 \to S^1 \xrightarrow{\tilde{\iota}} \widetilde{Q}_{[\rho]} \to Q \to 1$$

Our next goal is to show it is a central extension. First notice that S^1 is central because we already showed that S^1 is a subset of $Z(G_{[\rho]})$ and since $Z(\widetilde{G}_{[\rho]})/(i\times\rho)(A) \subseteq Z(\widetilde{G}_{[\rho]}/(i\times\rho)(A))$ it follows that $S^1 \subseteq Z(\widetilde{G}_{[\rho]})$. To show that the sequence is exact we need the following lemma.

Lemma 2.2.3. $\tilde{\iota}(S^1) \cong \frac{(1\times S^1)(i\times\rho)(A)}{(i\times\rho)(A)}$

Proof. Since $\tilde{\iota} = q\circ\iota$, we have $\tilde{\iota}(S^1) = \iota(S^1)/(i\times\rho)(A)$. Thus, we have to prove that $\iota(S^1) \cong (1\times S^1)(i\times\rho)(A)$ But $\iota(S^1) = q^{-1}(\widetilde{\iota}(S^1))$ Hence this is equivalent to $q^{-1}(\widetilde{\iota}(S^1)) \cong (1\times S^1)(i\times\rho)(A)$. First let $g\in q^{-1}(\widetilde{\iota}(S^1))$, then $q(g) = \widetilde{\iota}(h)$ for some $h\in S^1$. Therefore: $q(g) = q\circ\iota(h) \implies g^{-1}\iota(h) \in (i\times\rho)(A)$ and there is a $n\in(i\times\rho)(A)$ such that $g^{-1}\iota(h) = n$ then $g = i(h)n^{-1} \in (1\times S^1)(i\times\rho)(A)$ and $q^{-1}(\widetilde{\iota}(S^1)) \subseteq (1\times S^1)(i\times\rho)(A)$. Now if $g\in(1\times S^1)(i\times\rho)(A)$ then $g = ab$ with $a\in\iota(A)$ and $b\in(i\times\rho)(A)$. Let $a'\in S^1$ such that $a = \iota(a')$, then:

$$q(g) = q(ab) = q(a)q(b) = q(\iota(a')) = \tilde{\iota}(a') \in \tilde{\iota}(S^1)$$

Therefore $q^{-1}(\widetilde{\iota}(S^1)) \cong (1\times S^1)(i\times\rho)(A)$ as we wanted ☐

Now to show that the previous sequence is exact notice that by the third theorem of isomorphism:

$$\frac{\frac{\widetilde{G}_{[\rho]}}{(i\times\rho)(A)}}{\frac{(1\times S^1)(i\times\rho)(A)}{(i\times\rho)(A)}} \cong \frac{\widetilde{G}_{[\rho]}}{(1\times S^1)(i\times\rho)(A)} \cong \frac{\frac{\widetilde{G}_{[\rho]}}{1\times S^1}}{\frac{(1\times S^1)(i\times\rho)(A)}{1\times S^1}} \cong \frac{G_{[\rho]}}{i(A)} \cong Q_{[\rho]}$$

summarizing, there is the commutative diagram of extensions of Lie groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A & \xrightarrow{i} & G_{[\rho]} & \longrightarrow & Q_{[\rho]} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\iota} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & A & \xrightarrow{i\times\rho} & \widetilde{G}_{[\rho]} & \longrightarrow & \widetilde{Q}_{[\rho]} & \longrightarrow & 1
\end{array}$$

And the commutative diagram of central extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S^1 & \xrightarrow{i} & \widetilde{G}_{[\rho]} & \xrightarrow{\pi_1} & G_{[\rho]} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \pi_2} & & \downarrow & & \\
1 & \longrightarrow & S^1 & \xrightarrow{\iota} & \widetilde{Q}_{[\rho]} & \xrightarrow{p} & Q_{[\rho]} & \longrightarrow & 1
\end{array}$$

Proposition 2.2.1. *The irreducible representation ρ can be extended to $G_{[\rho]}$ if and only if the central extension*

$$1 \to S^1 \xrightarrow{\widetilde{\iota}} \widetilde{Q}_{[\rho]} \to Q_{[\rho]} \to 1$$

is trivial.

Proof. If the extension is trivial, then the extension:

$$1 \to S^1 \to \widetilde{G}_{[\rho]} \xrightarrow{\pi_1} G_{[\rho]} \to 1$$

Therefore there is a lie homomorphism of Lie groups $\sigma : G_{[\rho]} \longrightarrow \widetilde{G}_{[\rho]}$ that is a right inverse of π_1. Recall that $\pi_2 : \widetilde{G}_{[\rho]} \longrightarrow U(V_\rho)$ is the projection on the second component. Then, the map $\pi_2 \circ \sigma : G_{[\rho]} \longrightarrow U(V_\rho)$ is a representation of $G_{[\rho]}$ and for $a \in A$

$$\pi_2(\sigma(a)) = \rho(a)$$

On the other hand if $\widetilde{\rho} : G[\rho] \longrightarrow U(V_\rho)$ is an extension of ρ, the map $p \circ \widetilde{\rho}$ makes the following diagram commute:

$$\begin{array}{ccc}
\widetilde{G}_{[\rho]} & \xrightarrow{\pi_1} & G_{[\rho]} \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle p \circ \widehat{\rho}} \\
U(V_\rho) & \xrightarrow{p} & PU(V_\rho)
\end{array}$$

Thus, $\gamma = p \circ \widetilde{\rho}$. Therefore we can write the fibered product $\widetilde{G}_{[\rho]}$ as $\{(g, u) \in G_{[\rho]} \times U(V_\rho) : p \circ \widetilde{\rho}(g) = p(u)\}$ therefore we can define the Lie group homomorphism $\sigma : G_{[\rho]} \longrightarrow \widetilde{G}_{[\rho]} : g \longrightarrow (g, \widetilde{\rho}(g))$. σ is a right inverse of π_1. Thus, the sequence

$$1 \to S^1 \to \widetilde{G}_{[\rho]} \to G_{[\rho]} \to 1$$

Is trivial and in consequence our initial sequence is trivial. □

2.3 Semidirect products

Now we will suppose that G is a semidirect product $G = A \rtimes Q$ where A is a normal subgroup of G. For a group G let $\widehat{G}$ be the set of equivalence classes of one dimensional unitary representations of G. As mentioned before the group G acts on the set $\widehat{A}$ as follows. For a $g \in G$ and a $\chi \in \widehat{A}$

$$g \cdot \chi(a) \coloneqq \chi(g^{-1}ag)$$

The restriction map $\Gamma : \widehat{G} \longrightarrow \widehat{A}$ that takes a $\gamma \in G$ and sends it to $\gamma|_A$ has its image contained on the ser of fixed points of the previous action $\widehat{A}^G$ because if $\gamma \in \widehat{G}$ then for all $a \in A$

$$g \cdot \gamma|_A(a) \coloneqq \gamma(gag^{-1}) = \gamma(g)\gamma(a)\gamma(a)^{-1} = \gamma(a)$$

Proposition 2.3.1. *The restriction map*

$$\Gamma : \widehat{G} \longrightarrow \widehat{A}^Q \times \widehat{Q}$$

that takes $\gamma \in \widehat{G}$ *and sends it to* $(\gamma|_A, \gamma|_Q)$ *is an isomorphism.*

Proof. We will show this by building the inverse of Γ. Let $\alpha \in \widehat{A}$ and $\beta \in \widehat{Q}$ and define the product $\alpha \rtimes \beta$ as follows. For a $g \in G$ there is a unique pair $(a, q) \in A \times Q$ such that $g = aq$, let $\alpha \rtimes \beta(g) = \alpha(a) \rtimes \beta(q)$. Now we want to see that $\alpha \rtimes \beta \in \widehat{G}$, for all $g_1, g_2 \in G$ with $g_1 = h_1q_1$ $g_2 = h_2q_2$ we have

$$\begin{aligned}
\alpha \rtimes \beta(g_1g_2) &= \alpha \rtimes \beta(h_1q_1h_2q_2) \\
&= \alpha \rtimes \beta(h_1q_1h_2q_2q_1^{-1}q_1q_2) \\
&= \alpha(h_1q_1h_2q_1^{-1})\beta(q_1q_2) \\
&= \alpha(h_1)\alpha(q_1)\alpha(h_2)\alpha(q_1)^{-1}\beta(q_1)\beta(q_2) \\
&= \alpha(h_1)\beta(q_1)\alpha(h_2)\beta(h_2) \\
&= \alpha \rtimes \beta(h_1, q_1)\alpha \rtimes \beta(h_2, q_2)
\end{aligned}$$

The map $(\alpha, \beta) \longrightarrow \alpha \rtimes \beta$ is the inverse of Γ hence it is bijective. □

Proposition 2.3.2. *Let* $G = A \rtimes Q$ *with* A *an abelian normal subgroup of G. For all irreducible representations* ρ *of* A *the group extension*

$$1 \longrightarrow S^1 \longrightarrow \widetilde{Q}_{[\rho]} \longrightarrow Q_{[\rho]} \longrightarrow 1$$

Is trivial.

Proof. If A is abelian all its irreducible representations are in $\widehat{A}$. Given a $\rho \in \widehat{A}$ for a $g \in G_{[\rho]}$

$$\rho(g^{-1}ag) = U_g^{-1}\rho(a)U_g = \rho(a)$$

Where the last equality follows because $U_g, \rho(a), U_g^{-1} \in S^1$ Hence the commute. Therefore every representation is invariant under the action of $G_{[\rho]}$. Now lets see that $G_{[\rho]} = A \rtimes Q_[\rho]$. Given $(\widetilde{a}, q) \in G_{[\rho]}$ we have that:

$$\rho(a) = \rho((\widetilde{a}q)^{-1}a(\widetilde{a}q)) = \rho(q^{-1}\widetilde{a}^{-1}a\widetilde{a}q) = \rho(q^{-1}aq)$$

Thus, $q \in Q_{[\rho]}$ and $(\widetilde{a}, q) \in A \rtimes Q_{[\rho]}$. If $(\widetilde{a}, q) \in A \rtimes Q_{[\rho]}$ Then:

$$\begin{aligned}
\widetilde{a}q\rho(a) &= \rho((\widetilde{a}q)^{-1}a(\widetilde{a}q)) \\
&= \rho(q^{-1}\widetilde{a}^{-1}a\widetilde{a}q) \\
&= \rho(q^{-1}aq) \\
&= \rho(a)
\end{aligned}$$

Therefore, $Q_{[\rho]} \rtimes A = G_{[\rho]}$. Thus, by taking the trivial representation of $Q_{[\rho]}$ we can extend ρ to G_ρ and by the proposition 2.2.1 the required central extension is trivial. □

2.4 Vector bundle decomposition

Lets start with a few definitions.

Definition 2.4.1. *A G-space X is a topological space with continuous action of a topological group G over X.*

Definition 2.4.2. *A G-vector bundle over a G-space X is a G space E together with an equivariant map $p : E \longrightarrow X$ such that:*

- $p : E \longrightarrow X$ *is a complex vector bundle over X*
- *For all $g \in G$ and $x \in X$ the action $E_x \longrightarrow E_{gx}$ is an homomorphism of vector spaces.*

Let E be a G-vector bundle over a G-space X with A a normal closed subgroup of a compact group G. If we take the action of A to be trivial en X then each fiber is a representation of A. By compactness of A we can use the Weyl unitarian trick and ensure they are unitary representations.

Definition 2.4.3. *In the previous context if $\rho : A \longrightarrow U(V_\rho)$ is an irreducible representation of A, a $G_{[\rho]}$-vector bundle $p : E \longrightarrow X$ is $(G_{[\rho]}, \rho)$ isotypical if each fiber E_x is a representation of A isomorphic to a multiple of ρ*

By the proposition 2.2 of [Seg68] The restriction $E|_A$ can be decomposed into the isotypical parts:

$$E|_A \cong \bigoplus_{[\rho]\in Irr(A)} \rho \otimes Hom_A(\rho, E)$$

Where ρ is the trivial A bundle $X \times V_\rho \longrightarrow X$ and $Hom_A(E_1.E_2)$ is the vector bundle where with fiber of x equal to the equivariant maps from $(E_1)_x$ to $(E_2)_x$ with the trivial action. Define $E_\rho \coloneqq \rho \otimes Hom_A(\rho, E)$ it is called the isotypical of E.

Remark. *Other way to say that E_x is a representation of A isomorphic to a multiple of ρ is that the map*

$$\begin{aligned} \beta : E_\rho &\longrightarrow E \\ v \otimes f &\longrightarrow f(v) \end{aligned}$$

is an isomorphism of A-vector bundles.

Now we are ready to proof the first theorem of the Mackey machinery

Theorem 2.4.1. *Given a G-space X with A acting trivially in X there is a one to one correspondence between the isomorphism classes of $G_{[\rho]}$ vector bundles over X that are $(G_{[\rho]}, \rho)$ isotypical and classes of isomorphisms of $\widetilde{Q}_{[\rho]}$ vector bundles over X in which S^1 acts by multiplication by inverses*

Proof. The proof is divided in 6 steps. On the first two steps we will define the correspondence and then we will define an inverse to that map. Let $p : E \longrightarrow X$ a vector bundle $(G_{[\rho]}, \rho)$ isotypical. $Hom_A(\rho, E)$ is a complex vector bundle.

1. The firs step is to give $Hom_A(\rho, E)$ the structure of a $\widetilde{Q}_{[\rho]}$ bundle by defining an action of $\widetilde{Q}_{[\rho]}$ over $Hom_A(\rho, E)$ and over X. Given a $\varphi \in Hom_A(\rho, E)$ and a $\widetilde{g} \in \widetilde{G}_{[\rho]}$

$$(\widetilde{g} \cdot \varphi)(v) = \pi_1(\widetilde{g})\varphi(\pi_2(\widetilde{g})^{-1}v)$$

Let $\widetilde{a} \in (i \times \rho)(A)$ then:

$$\begin{aligned}(\widetilde{a} \cdot \varphi)(v) &= \pi_1(\widetilde{a})\varphi(\pi_2(\widetilde{a})^{-1}v) \\ &= a\varphi(\rho(a)^{-1}v) \\ &= \varphi(\rho(a)\rho(a)^{-1}v) \\ &= \varphi(v)\end{aligned}$$

Therefore, the action of $(i \times \rho)(A)$ is trivial and hence there is an induced action of $\widetilde{Q}_{[\rho]}$ over $Hom_A(\rho, E)$. Finally G acts over X, thus, there is an induced action of $G_{[\rho]}$ over X. Since the action of A over X is trivial there is induced action of $Q_{[\rho]}$ thus, the homomorphism discused on section 2.2 $\widetilde{Q}_{[\rho]} \longrightarrow Q_{[\rho]}$ can be used to give an action of $\widetilde{Q}_{[\rho]}$ over X.

2. In this step we will show that $S^1 = ker(\pi_1)$ acts by multiplication by inverses. Let $\lambda \in S^1$.

$$\begin{aligned}(\lambda \cdot \varphi)(v) &= \varphi(\pi_2(\lambda)^{-1}) \\ &= \varphi(\lambda^{-1}v) \\ &= \lambda^{-1}\varphi(v)\end{aligned}$$

Where the last equality follows by linearity. Define the transformation:

$$[E] \longrightarrow [Hom_A(\rho, E)]$$

To show that this is a bijective correspondence between between classes of of $\widetilde{Q}_{[\rho]}$ vector bundles over X in which S^1 acts by multiplication by inverses and classes of $G_{[\rho]}$ vector bundles over X that are $(G_{[\rho]}, \rho)$ isotypical we will build the inverse of the map.

3. Let $p : F \longrightarrow X$ be a $\widetilde{Q}_{[\rho]}$ vector bundle in which S^1 acts by multiplication by inverses. In this step we will give to the vector bundle $\rho \otimes F$ an action of $G_{[\rho]}$. For a $\widetilde{g} \in \widetilde{G}_{[\rho]}$ and a simple tensor $v \otimes f \in \rho \otimes F$

$$\widetilde{g} \cdot (v \otimes f) = (\pi_2(\widetilde{g})v) \otimes (\widetilde{g}(i \times \rho)(A) \cdot f)$$

Where $\widetilde{g}(i \times \rho)(A)$ The coset of $\widetilde{g}$ on $\widetilde{Q}_{[\rho]}$. This definitions gives us an action of $\widetilde{G}_{[\rho]}$.If $\lambda \in S^1 = ker(\pi_1)$

$$\begin{aligned} \lambda \cdot (v \otimes f) &= (\pi_2(\lambda)v) \otimes (\lambda(i \times \rho)(A) \cdot f) \\ &= (\lambda v \otimes \lambda^{-1} f) \\ &= v \otimes f \end{aligned}$$

Therefore S^1 acts trivially and there is an induced action of $G_{[\rho]} \cong \widetilde{G}_{[\rho]}/S^1$

4. The next step is to show that $\rho \otimes F$ is a vector bundle $(G_{[\rho]}, \rho)$ isotypical. Given a $a \in A$ let $\widetilde{a} \in (i \times \rho)(A)$

$$\begin{aligned} a \cdot (v \otimes f) &= \widetilde{a}(v \otimes f) \\ &= (\pi_2(\widetilde{a})v) \otimes (\widetilde{a}(i \times \rho)(A) \cdot f) \\ &= (\rho(a)v) \otimes f \end{aligned}$$

And since being a vector bundle E is $(G_{[\rho]}, \rho)$ isotyppical if and only if the fibers $E_x \cong \rho^m$ for some $m \in \mathbb{N}$ then $\rho \otimes F$ is (G_ρ, ρ) isotypical. Thus we can define the map

$$[F] \longrightarrow [\rho \otimes F]$$

The following two steps shows that this map is the inverse of the map defined before.

5. First we will prove that $\rho \otimes Hom_A(\rho, E) \cong E$ as $(G_{[\rho]}, \rho)$ isotypical vector bundles. The evaluation map

$$\begin{aligned} ev : &\rho \otimes Hom_A(\rho, E) \longrightarrow E \\ &v \otimes \varphi \longrightarrow \varphi(v) \end{aligned}$$

Is an isomorphism of vector bundles since we already proof that $\rho \otimes Hom_A(\rho, E)$ is $(G_{[\rho]}, \rho)$ isotypical. Therefore the it only remains to follow that it is $G_{[\rho]}$ equivariant. Let $g \in G_{[\rho]}$ and $\widetilde{g} \in \widetilde{G}_{[\rho]}$ with $\pi_1(\widetilde{g}) = g$

$$
\begin{aligned}
ev(g \cdot (v \otimes \varphi)) &= ev(\widetilde{g}(v \otimes \varphi)) \\
&= ev(\pi_2(\widetilde{g})v \otimes (\widetilde{g} \cdot \varphi)) \\
&= (\widetilde{g} \cdot \varphi)(\pi_2(\widetilde{g})v) \\
&= g\varphi(\pi_2(\widetilde{g})^{-1}\pi_2(\widetilde{g})v) \\
&= g\varphi(v) \\
&= g \cdot ev(v \otimes \varphi)
\end{aligned}
$$

Thus, $\rho \otimes Hom_A(\rho, E) \cong E$ as $(G_{[\rho]}, \rho)$ isotypical vector bundles.

6. Finally we need to proof that $F \cong Hom_A(\rho, \rho \otimes F)$ as $\widetilde{Q}_{[\rho]}$ vector bundles in which S^1 acts by multiplication by inverses. The map

$$
\begin{aligned}
F &\longrightarrow Hom_A(\rho, \rho \otimes F) \\
f &\longrightarrow (v \longrightarrow v \otimes f)
\end{aligned}
$$

Is the isomorphism of $\widetilde{Q}_{[\rho]}$ vector bundles we are looking for

□

2.5 Induction

Given $H \subseteq G$ a subgroup of finite index of the compact Lie group G and $E \xrightarrow{\pi} X$ a H-vector bundle with X a compact space. We can choose an element of each left coset of H in G. By doing that we obtain a subset R of G called system of representatives of G/H. Each $g \in G$ can be written uniquely as $g = rs$ with $r \in R = \{r_1 \dots r_n\}$ and $s \in H$, thus $G = \bigsqcup_{i=1}^{n} r_i H$ where we can choose r_1 to be the identity of G. Consider the vector bundle $F = \bigoplus_{i=1}^{n} (r_i^{-1})^* E$ where the space $(r_i^{-1})^* E$ is the space that makes the following diagram commute

$$
\begin{array}{ccc}
(r_i^{-1})^* E & \longrightarrow & E \\
\downarrow & & \downarrow \\
X & \xrightarrow{r_i^{-1}} & X
\end{array}
$$

Or in other word the pullback or fibered product $E \times_X X$. We will call the projection of F over X $\pi_F : F \longrightarrow X$. We can define an action of G over F as follows. Let $f \in F$ the f is of the form

$$f = (f_{r_1}, f_{r_2} \dots f_{r_n})$$

With $f_{r_i} \in (r_i^{-1}) * E$. If $\pi_F : F \longrightarrow X$ then $f_{r_i} = (x, e)$ where $e \in E_{r_i^{-1}x}$. As mentioned before every $g \in G$ is in a coset of some r_j therefore gr_i can be writen as

$$gr_i = r_j s$$

For some $s \in H$. For a $g \in G$ let

$$g \cdot f_{r_i} = (gx, sc) \in (r_j^{-1})^* E$$

The vector bundle F with this action is denoted $\mathrm{Ind}_H^G(E)$. The opposite process of obtaining H-vector bundles from G-vector bundles by restriction is denoted $\mathrm{Res}_H^G(E)$ and is related to $\mathrm{Ind}_H^G(G)$ thrugh the following theorem

Theorem 2.5.1 (Frobenius reciprocity)**.** *For all G-vector bundle F over X*

$$Hom_G(Ind_H^G(E), F) \cong Hom_H(E, Res_H^G(F))$$

The proof can be found on [CRV19]. With this tool we are ready to prove the second theorem of the Mackey machinery.

Theorem 2.5.2. *Let G be a compact Lie group with a closed normal subgroup A and X be a compact G-space with A acting trivially on X. If $p : E \longrightarrow X$ is a finite dimensional G-vector bundle then we have the ismorphism of G-vector bundles*

$$\bigoplus_{[\rho] \in G/Irr(A)} Ind_{G_{[\rho]}}^G(\rho \otimes Hom_A(\rho, E)) \longrightarrow E$$

Where $G/Irr(A)$ are representatives of the orbits of the G-action on the set of isomorphisms classes of representations of A.

Proof. Each $(r_i^{-1}) * E$ has a natural structure of $r_i H r_i^{-1}$ vector bundle because if $g \in r_i H r_i^{-1}$, $g = r_i h r_i^{-1}$, therefore:

$$g \cdot (x, e) = (gx, he)$$

Thus, $g \cdot e(\in r_i^{-1})^* E$. Since E is of finite dimension and by compactness there is a finite number of classes $[\rho]$ such that

$$Hom_A(\rho, E) \neq 0$$

Furthermore if $Hom_A(\rho, E) \neq 0$ then $Hom_A(g \cdot \rho, E) \neq 0$ for all g in G. If $\varphi : (V, \rho) \longrightarrow E$ is a non trivial equivariant homomorphism then

$$\begin{aligned} \widetilde{\varphi} : (V, g{\cdot}\rho) &\longrightarrow E \\ v &\longrightarrow g\varphi(v) \end{aligned}$$

is a non trivial element in $Hom_A(g{\cdot}\rho, E)$. Thus all the elements of its orbit appears in E. Since $G/G_{[\rho]} \cong G \cdot [\rho]$ the index $[G : G_{[\rho]}]$ is finite and we can apply the previous construction of the induced vector bundle. Let $R = \{r_1, r_2 \ldots r_n\}$ be a system of representatives of $G/G_{[\rho]}$. -now lets build the following isomorphism between $G_{[r_i\rho]}$ vector bundles.

$$(r_i^{-1})E_\rho \cong E_{r_i \cdot \rho}$$

Let $\xi = (x, \psi) \in (r_i^{-1})^* E_\rho$ with $\psi \in Hom_A(\rho, E_{r_i^{-1}x})$ tehe isomorphism is given by

$$\begin{aligned} \widetilde{\psi} : (V, r_i \cdot \rho) &\longrightarrow E_x \\ v &\longrightarrow r_i\psi(v) \end{aligned}$$

It is A-equivariant because for every $a \in A$

$$\begin{aligned} \widetilde{\psi}(a \cdot v) &= \widetilde{\psi}(r_i^{-1} a r_i v) \\ &= r_i\psi(r_i^{-1} a r_i v) \\ &= a r_i \psi(v) \\ &= a\widetilde{\psi}(v) \end{aligned}$$

Finally the isomorphism is given by the following map

$$\begin{aligned} \mathrm{Ind}_{G^G_{[\rho]}}(E_\rho) &\longrightarrow E \\ (x, v \otimes \psi) \in (r_i^{-1})^*(E_\rho) &\longrightarrow r_i\psi(v) \end{aligned}$$

□

2.6 Example

Let $G = D_8$ with $A = \mathbb{Z}^4$. $D_8 = \langle a, b | a^4 = b^2 = e, bab^{-1} = a^{-1}\rangle \cong \mathbb{Z}^4 \rtimes \mathbb{Z}^2$. Since D_8 is a semidirect product by proposition 2.3.1 $\widehat{D} = \widehat{\mathbb{Z}}_4^{\mathbb{Z}_2} \times \widehat{\mathbb{Z}}_2$. We know that $\widehat{\mathbb{Z}}_2 = \{1, sign\}$ and that $\widehat{\mathbb{Z}}_4$. $\mathbb{Z}_2$ acts in $\widehat{\mathbb{Z}_4}$, lets compute the orbits of the action.

$$
\begin{aligned}
b \cdot 1(a) &= 1(bab^{-1}) = 1(a^{-1}) = 1 = 1(a)\\
b \cdot \rho(a) &= \rho(bab^{-1}) = \rho(a^{-1}) = -i = \rho^2(a)\\
b \cdot \rho^2(a) &= \rho^2(bab^{-1}) = \rho^2(a^{-1}) = -1 = \rho^2(a)\\
b \cdot \rho^3(a) &= \rho^3(bab^{-1}) = \rho^3(a^{-1}) = i = \rho(a)
\end{aligned}
$$

Thus, the orbits of the action are given by

$$\mathbb{Z}_2/\widehat{\mathbb{Z}}_4 = \{\{1\}, \{\rho^2\}, \{\rho, \rho^3\}\}$$

And the fixe points are $\widehat{\mathbb{Z}}_4^{\mathbb{Z}_2} = \{1, \rho^2\}$ And the four elements of D_8 are

$$1 \rtimes 1, 1 \rtimes sign, \rho^2 \rtimes 1, \rho^2 \rtimes sign$$

Thus, D_8 must have a two dimensional representation. D_8 acts in $\widehat{\mathbb{Z}}_4$ Lets compute the orbits of this action. In general for $0 \le m \le 4$ and $1 \le k \le 3$ we have:

$$
\begin{aligned}
a^m \rho^k(a) &= \rho^k(a^m a a^{-m}) = \rho^k(a)\\
a^m b \rho^k(a) &= \rho^k(a) = \rho^k(a^m bab^{-1}a^{-m}) = \rho^k(a^{-1})
\end{aligned}
$$

Therefore the orbits of this action are the same orbits of the previous action

$$D_8/\widehat{\mathbb{Z}}_4 = \{\{1\}, \{\rho^2\}, \{\rho, \rho^3\}\}$$

Thus, $G_{[1]} = G_{[\rho^2]} = D_8$, $Q_{[\rho]} = Q_{[1]} = D_8/\mathbb{Z}_4 \cong \mathbb{Z}_2$, $G_{[\rho]} = \mathbb{Z}_4$, $Q_{[\rho]} = 1$ and by proposition 2.3.2 all the central extensions

$$1 \longrightarrow S^1 \longrightarrow \widetilde{Q}_{[\varphi]} \longrightarrow Q_{[\varphi]} \longrightarrow 1$$

Are trivial for all $\varphi \in \widehat{\mathbb{Z}}_4$. The two dimensional irreducible representation of D_8 is an induction of an irreducible representation of ρ. In this case $G_{[\rho]} = A$ thus, we can take the induction $\mathrm{Ind}_{\mathbb{Z}_4}^{D_8}(\rho)$ as the two dimensional representation of D_8, and we have,

$$R(D_8) \cong R(\mathbb{Z}_2) \oplus R(e) \oplus R(\mathbb{Z}_2)$$

Which explicitly is

$$R(D_8) = \mathbb{Z}\langle 1 \rtimes 1, 1 \rtimes sign\rangle \oplus \mathbb{Z}\langle \mathrm{Ind}_{\mathbb{Z}_4}^{D_8}(\rho)\rangle \oplus \mathbb{Z}\langle \rho^2 \rtimes 1, \rho^2 \rtimes sign\rangle$$

Bibliography

[ABV20] Andrés Angel, Edward Becerra, and Mario Velásquez. Proper actions and decompositions in equivariant k-theory, 2020.

[BD] T. Bröcker and T.tom Dieck. *Representations of Compact Lie Groups.* Springer-Verlag Berlin Heidelberg year = 2003.

[BT11] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology.* Springer, 2011.

[CRV19] Germán Combariza, Juan Rodríguez, and Mario Velásquez. Induced character in equivariant k-theory and wreath products, 2019.

[Dor94] Robert S. Doran. *C*-algebras: 1943-1993: a fifty year celebration: AMS special session commemorating the first fifty years of c*-algebra theory, January 13-14, 1993, San Antonio, Texas.* American Mathematical Society, 1994.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, 2002.

[KT12] Eberhard Kaniuth and Keith F. Taylor. *Induced Representations of Locally Compact Groups.* Cambridge Tracts in Mathematics. Cambridge University Press, 2012.

[Lee13] John M. Lee. *Introduction to smooth manifolds.* Springer, 2 edition, 2013.

[Mur13] Takumi Murayama. Maximal tori and the weyl integration formula, apr 2013.

[Per11] Marco Perez. *THE MAXIMAL TORUS OF A COMPACT LIE GROUP.* Universite du Quebec Montreal Departement de mathematiques, 2011.

[Seg68] Graeme Segal. Equivariant k-theory. *Publications Mathématiques de l'IHÉS*, 34:129–151, 1968.