The Gauss-Manin connection for the Legendre family of elliptic curves

Daniel R. Barrero R.

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Introduction

In the nineteenth century, Lazarus Fuchs noticed that the periods of differentials of the second kind on an elliptic curve satisfy a linear differential equation with regular singular points. This is a differential equation of order two with three such singularities, which implies it is a particular case of the *hypergeometric differential equation* previously studied by Gauss.

Decades later, in the second half of the twentieth century, Manin formulated a way to differentiate cohomology classes of differentials of the second kind with respect to a parameter that indexes a given family of curves of genus g. Shortly afterward, Grothendieck commented that Manin's differentiation was probably the same as a flat connection on the vector bundle determined by the cohomology sheaves of relative differential forms $\Omega^{\bullet}_{X/S}$ given by a smooth proper map $\pi : X \to S$.

In this work, we look at the Legendre family of elliptic curves from both perspectives: the "classical point of view", where the main tool is calculus and should resemble the way in which people like Fuchs and Weierstrass thought about elliptic curves and their periods; and a more "modern point of view", that which considers the Legendre family as given by a fiber bundle $\pi : X \to S$ and instead of integrating differential forms, the objects of interest are cohomology sheaves and associated vector bundles with a flat connection. This perspective should resemble the way in which people like Manin, Grothendieck, and others thought about periods of elliptic curves and differential equations.

Chapter 1

The classical point of view

Our main reference for this chapter is the first section of the book [3], by Carlson and collaborators. *The Legendre family of elliptic curves* consists of the complex projective curves X_s given by

$$y^2 = x(x-1)(x-s)$$

for $s \in \mathbf{C} - \{0, 1\}$. Since they are smooth complex projective varieties, they can be viewed as either algebraic varieties, topological/differentiable manifolds, or complex manifolds (as which they are 1-dimensional and are therefore called Riemann surfaces). After computation of invariants such as genus or Euler characteristic, it is settled that $X_s \simeq X_t$ for any $s, t \in \mathbf{C} - \{0, 1\}$ as topological or differentiable manifolds. However, we shall see that the complex manifold structure varies with s. In fact, for all $s \in \mathbf{C} - \{0, 1\}$ there exists $\epsilon > 0$ such that for all $s' \neq s'' \in \{t \in \mathbf{C} - \{0, 1\} : \ |t - s| < \epsilon\}$, $X_{s'} \not\simeq X_{s''}$ as complex manifolds.

1.1 The global differential and the complex structure

The curve X_s has a unique (up to scalar multiples) global differential 1-form, which is denoted by

$$\omega_s = \frac{dx}{y}$$

and is given in local holomorphic coordinates by

$$\omega_s = \frac{dx}{\sqrt{x(x-1)(x-s)}}.$$

Integration of ω_s along homology cycles allows one to recover the complex manifold structure of X_s. First, let us consider the ordered homology basis $\{\delta, \gamma\}$

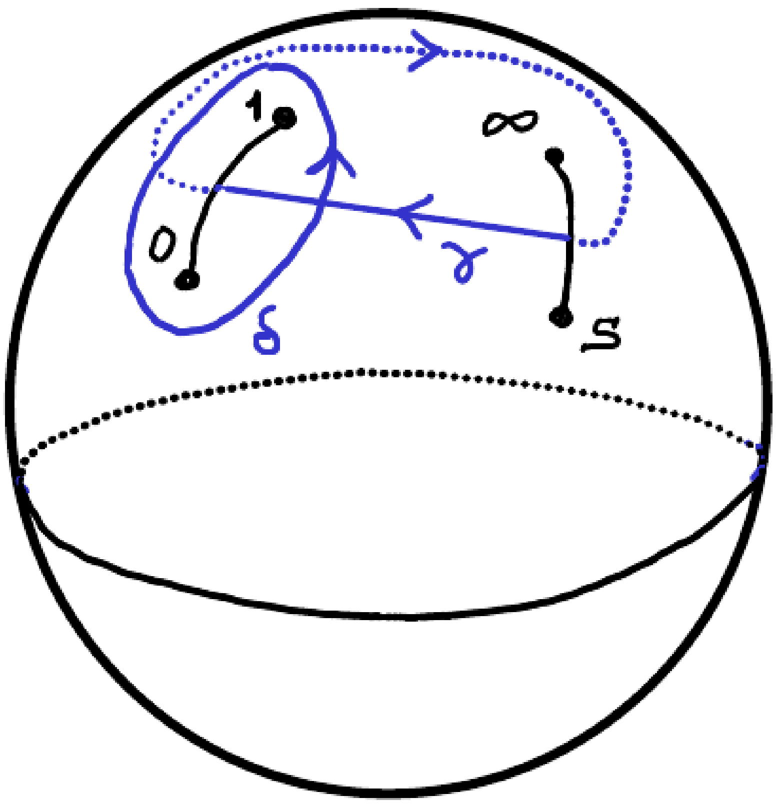

Figure 1.1: homology basis on the cut Riemann sphere

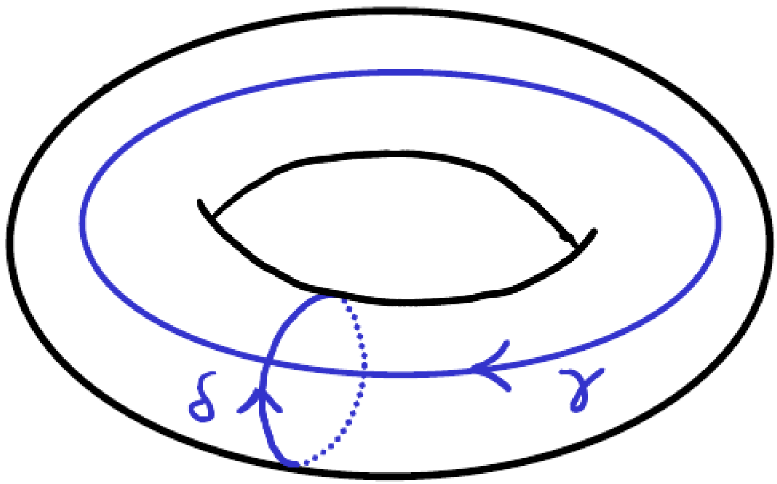

Figure 1.2: homology basis on X_s

illustrated in figures 1.1 and 1.2. These figures depict the fact that X_s is obtained by glueing together two copies of the Riemann sphere with branch cuts. The sphere with cuts represents a maximal domain where the function $y = \sqrt{x(x-1)(x-s)}$ is single-valued, which gives meaning to X_s being the Riemann surface of the global analytic function $y = \sqrt{x(x-1)(x-s)}$. The dotted line in the path of γ in figure 1.1 denotes it runs through the other sheet of X_s, as becomes clear in figure 1.2. The reader can now notice that the intersection product gives $\delta.\gamma = 1$, so that the intersection matrix of the basis $\{\delta, \gamma\}$ is

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We now define the *periods* of X_s as the integrals of ω_s along γ and δ:

$$\zeta_1 = \int_\delta \omega_s, \ \zeta_2 = \int_\gamma \omega_s.$$

Then, (for example as proved in [9]) the complex structure on X_s is given by

$$X_s \simeq \mathbf{C}/\Lambda_s$$

where Λ_s is the lattice $\mathbf{Z}\zeta_1 \oplus \mathbf{Z}\zeta_2$.

1.2 Invariants of framed elliptic curves

Although we recovered the complex structure on X_s from its defining equation, this doesn't explain how such structures vary with the parameter s.

Proposition 1. *If $f : X_s \to X_t$ is an isomorphism of complex manifolds, then $f^*\omega_t = \lambda\omega_s$ for some nonzero complex number λ.*

Proof. Complex differentiability of the map f means that in local coordinates, $Df(\partial/\partial\bar{z}) = 0$, which dually implies that $f^*(d\bar{z}) = 0$. If in addition f is biholomorphic, then $f^*(dz)$ is never zero, and therefore $f^*\alpha$ is holomorphic for any holomorphic form α defined on an open set $U \subset X_t$. Then $f^*\omega_t$ is holomorphic, and since the only global holomorphic forms on X_s are the scalar multiples of ω_s, it follows that $f^*\omega_t = \lambda\omega_s$ for some $\lambda \in \mathbf{C}$. □

Definition 1.

1. A *framed elliptic curve* is a triple (X_s, δ, γ) where X_s belongs to the Legendre family and (δ, γ) is an integral homology basis with intersection $\delta.\gamma = 1$.
2. The number $\tau(s) := \int_\gamma \omega_s / \int_\delta \omega_s \neq 0$ is called the *period ratio* of the framed elliptic curve (X_s, δ, γ).

Corollary. *Let (X_s, δ, γ), (X_t, δ', γ') be framed elliptic curves. If $f : X_s \to X_t$ is an isomorphism of complex manifolds such that $\delta' = f_*\delta$, $\gamma' = f_*\gamma$, then $\tau(t) = \tau(s)$.*

The usefulness of this corollary is that now if we can detect changes in τ, we can detect variations in the complex structure of these tori. It will turn out that for each $s \in \mathbf{C} - \{0, 1\}$, there exists a small disk $\Delta \ni s$ such that $X_t \not\cong X_s$ for all $t \in \Delta - \{s\}$; thanks to the following:

Theorem 1. *τ is holomorphic and $\tau'(s) \neq 0$ for all $s \in \mathbf{C} - \{0, 1\}$.*

So far the quantity $\tau(s)$ has been defined as a number associated to a framed elliptic curve, completely dependent on the choice of homology basis, and not as a function. We therefore turn to show that locally, $\tau(s)$ can be defined as an analytic function, and at the end of the section we will show that it's derivative is never zero.

1.3 The Picard-Fuchs differential equation

It turns out that a reasonable way to define τ is as the ratio of two linearly independent solutions to a classical differential equation, called the Picard-Fuchs equation. The equation is

$$s(s-1)\zeta'' + (2s-1)\zeta' + \frac{1}{4}\zeta = 0. \tag{1.1}$$

Power series solutions exist on any open disk in $\mathbf{C} - \{0, 1\}$ and are traditionally called hypergeometric functions.

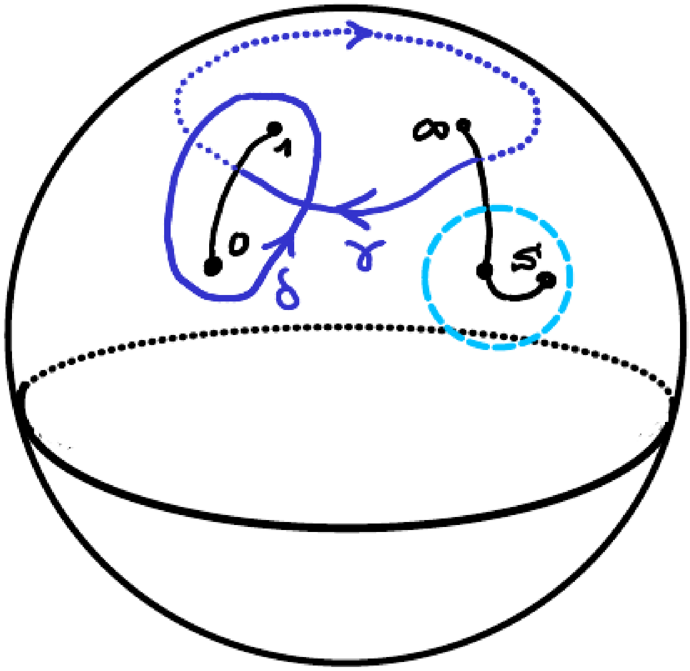

Figure 1.3: Fixed integration domains for computing periods.

1.3.1 Periods of curves in the Legendre family are solutions

Consider the line integrals

$$A(s) = \int_\delta \frac{dx}{\sqrt{x(x-1)(x-s)}},\ B(s) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-s)}}. \qquad (1.2)$$

They can be seen as functions of s by keeping the integration domains $\delta, \gamma \subset \mathbf{C}$ constant, and if we restrict to $s \in \Delta$ where Δ is a small enough open disk which does not intersect the paths, then they are actually analytic functions on Δ. This is because the function $1/\sqrt{x(x-1)(x-s)}$ for x constant, is analytic on the Riemann sphere with a cut from x to ∞. This can be seen more readily in figure 1.3.

We must pay attention to the fact that γ, when thought of as a path in $\mathbf{C}$, passes through a branch cut. This is actually not a problem, because we use the dotted line to denote the fact that γ passes to the other sheet after meeting the branch cut. A look at figure 1.4 helps to clarify this.

Now the integral along γ can be made sense of:

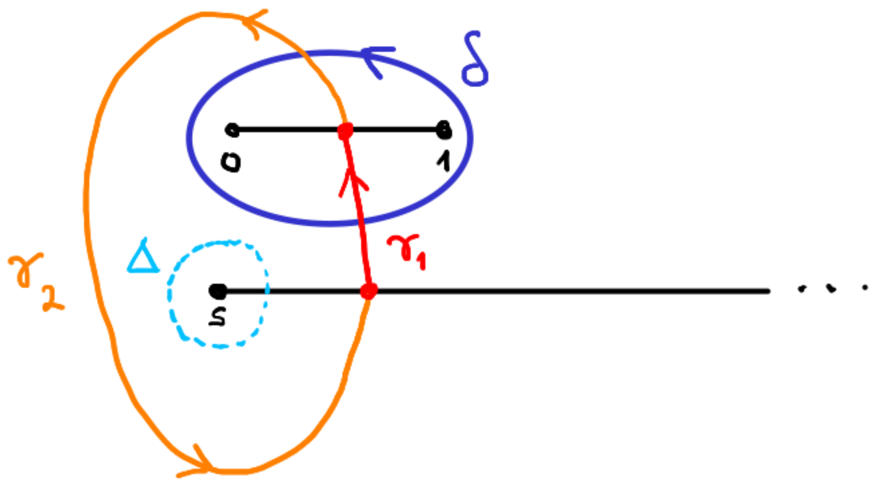

Figure 1.4: The cycles δ and γ in the complex plane with cuts, where γ is the concatenation of γ_1 and γ_2.

$$B(s) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-s)}} := \int_{\gamma_1} \frac{dx}{\sqrt{x(x-1)(x-s)}} + \int_{\gamma_2} \frac{dx}{-\sqrt{x(x-1)(x-s)}},$$

where γ_1, γ_2 are the paths depicted in figure 1.4. The only thing that remains to be shown is that these line integrals correspond with integrals of the forms ω_s along homology bases for varying s. For this we again turn to figures 1.3 and 1.4. If we delete the disk Δ from figure 1.4, we see that $\delta, \gamma_1, \gamma_2$ are defined on a domain where $y = \sqrt{x(x-1)(x-s)}$ is single-valued for all $s \in \Delta$. Therefore, given $s \in \Delta$, we define the cycles $\delta_s, \gamma_s \subset X_s$ as follows:

$$\delta_s(t) = (\delta(t), \sqrt{\delta(t)(\delta(t)-1)(\delta(t)-s)}),\ t \in [0,1]$$

$$\gamma_s(t) = \begin{cases} (\gamma(t), \sqrt{\gamma_1(2t)(\gamma_1(2t)-1)(\gamma_1(2t)-s)}),\ t \in [0,1/2] \\ (\gamma(t), \sqrt{\gamma_2(2t-1)(\gamma_2(2t-1)-1)(\gamma_2(2t-1)-s)}),\ t \in [1/2,1]. \end{cases}$$

We have thus made explicit the correspondence between two analytic functions on the domain Δ and periods of curves in the Legendre family restricted to Δ. Let us turn to showing that they are solutions of the Picard-Fuchs differential equation.

The function

$$f(x) = \frac{-1}{2}\sqrt{\frac{x(x-1)}{(x-s)^3}}$$

is a rational function on X_s such that

$$df = s(s-1)\omega_s'' + (2s-1)\omega_s' + \frac{1}{4}\omega_s$$

where

$$\omega_s' = \frac{dx}{2\sqrt{x(x-1)(x-s)^3}}$$
$$\omega_s'' = \frac{3dx}{4\sqrt{x(x-1)(x-s)^5}}.$$

That is, the forms ω_s', ω_s'' are obtained from ω_s by differentiating the coefficient functions with respect to s. Then for $\xi = \delta, \gamma$ we have

$$0 = s(s-1)\int_\xi \omega_s'' + (2s-1)\int_\xi \omega_s' + \frac{1}{4}\int_\xi \omega_s.$$

We see that the period functions $A(s)$ and $B(s)$, as defined in (1.2), are solutions to (1.1).

Period functions span the space of solutions to the Picard-Fuchs differential equation. For the fixed homology basis $\{\delta, \gamma\}$ consider the period functions

$$A(s) = \int_\delta \omega_s,$$
$$B(s) = \int_\gamma \omega_s.$$

defined on a small open disk $\Delta \subset S$.

The fact that $A(s), B(s)$ are linearly independent solutions to the PF-DE is an immediate consequence of the fact that the function $\tau(s) = A(s)/B(s)$ is non-constant. One way to establish it is by showing that $\tau'(s) \neq 0$ throughout δ. This is Theorem 1 from the previous section, which we now prove.

Proof of Theorem 1 We have $\tau(s) = A(s)/B(s)$, so that $\tau'(s) = \frac{A'(s)B(s)-A(s)B'(s)}{B(s)^2}$ and thus $\tau'(s) \neq 0$ if and only if $A'(s)B(s) - A(s)B'(s) \neq 0$. Notice that in cohomology, $[\omega_s] = A(s)\delta^* + B(s)\gamma^*$ and $[\omega_s'] = A'(s)\delta^* + B'(s)\gamma^*$. It follows that $\tau'(s) \neq 0$ if and only if $[\omega_s] \cup [\omega_s'] \neq 0$. In order to prove it, let us start by taking a closer look at the form ω_s'.

It is the meromorphic form

$$\omega'_s = \frac{dx}{2\sqrt{x(x-1)(x-s)^3}}, \tag{1.3}$$

and it turns out that this form has a pole of order 2 at $p = (s, 0)$: Notice that at this point y is a local coordinate, since p is a branching point. In order to express x locally as a function of y, we look at the defining equation for X_s^{af}, the affine part of X_s, namely the zero level set of the function $F(x,y) = y^2 - x(x-1)(x-s)$. If $\Delta \subset \mathbf{C}$ is a sufficiently small disk such that there exist holomorphic functions $x, y : \Delta \to \mathbf{C}$ such that $(x(z), y(z)) \in X_s^{af}$ defines a local parametrization of X_s^{af}, then $\frac{d}{dz}F(x(z), y(z)) = 0$ gives

$$\frac{\partial F}{\partial x}\frac{dx}{dz} = -\frac{\partial F}{\partial y}\frac{dy}{dz}$$

so that

$$\frac{dx}{dy} = \frac{2y}{x(x-1) + x(x-s) + (x-1)(x-s)} \tag{1.4}$$

defines a differential equation for x as a function of y. The Taylor expansion for $x(y)$ at $y = 0, x(0) = s$ is such that $x'(0) = 0$ and $\frac{x''(0)}{2} = \frac{1}{s(s-1)}$. This can be read from the equation (1.4) by taking the right-hand side to be $x'(y)$. One sees $x'(0) = 0$ by setting $y = 0$ in this expression, and then $\frac{x''(0)}{2} = \frac{1}{s(s-1)}$ is obtained by differentiating the right hand side of (1.4) with respect to y and then evaluating at $y = 0$. Then the Taylor expansion of $x(y)$ yields $y^2 = u(y)s(s-1)(x-s)$, where $u(y)$ is a holomorphic function such that $u(0) = 1$. We have

$$x = s + \frac{y^2}{s(s-1)} + \text{higher-order terms}$$

and therefore near p,

$$\omega_s = \frac{dx}{y} \sim \frac{2dy}{s(s-1)}.$$

Using the previous facts together with (1.3) gives

$$\omega'_s = \frac{dx}{2y(x-s)} \sim \frac{dy}{s(s-1)(x-s)} \sim \frac{dy}{y^2} + \text{regular form.} \tag{1.5}$$

Consider the exact sequence in cohomology of the pair $X_s, X_s - \{p\}$:

$$0 \longrightarrow H^1(X_s) \longrightarrow H^1(X_s - \{p\}) \xrightarrow{\delta} H^2(X_s, X_s - \{p\}).$$

Thanks to Stokes' theorem and the fact that $H^2(X_s, X_s - \{p\}) \simeq \mathbf{C}$, the coboundary map δ in the diagram can be identified with calculating the residue at p. Thanks to (1.5), we see that $res_p(\omega'_s) = 0$ and therefore ω'_s defines a class in $H^1(X_s)$. We now turn to showing that

$$\int_{X_s} [\omega_s] \cup [\omega'_s] = \frac{-4\pi i}{s(s-1)},$$

which will allow us to conclude that $\tau'(s) \neq 0$. First, notice that (1.5) implies that $\omega'_s + d(1/y)$ doesn't have a pole at p. Let U be a coordinate neighborhood of p such that $y(U) = \Delta \subset \mathbf{C}$, where Δ is the open disk of radius $\epsilon > 0$ centered at 0. Let $\rho : \mathbf{C} \to \mathbf{R}$ be a function of $|z|$ alone such that $\rho(z) = 1$ for $|z| \leq \epsilon/4$, $\rho(z) = 0$ for $|z| \geq \epsilon/2$, and ρ decreases monotonically for $\epsilon/4 < |z| < \epsilon/2$. Now consider the form

$$\theta := \omega'_s + d(\rho(y)/y).$$

This form is in the same cohomology class as ω'_s since their difference is exact. Therefore, $[\omega_s] \cup [\omega'_s]$ is represented by $\omega_s \wedge \theta$. Let $U' \subset U$ be the image of $\{|z| < \epsilon/4\}$ in our working coordinate system. Since outside U' both ω_s and ω'_s are holomorphic, and on U' both ω_s and θ are holomorphic, then

$$\int_{X_s} \omega_s \wedge \theta = \int_{X_s - U'} \omega_s \wedge \theta + \int_{U'} \omega_s \wedge \theta = \int_{X_s - U'} \omega_s \wedge \theta = \int_{X_s - U'} \omega_s \wedge d(\rho/y).$$

Now $\omega_s \wedge d(\rho/y) = d(-\rho\omega_s/y)$, so Stokes' theorem gives

$$\int_{X_s - U'} \omega_s \wedge d(\rho/y) = \int_{|y|=\epsilon/4} -\rho\omega_s/y = \int_{|y|=\epsilon/4} -\omega_s/y = \frac{-4\pi i}{s(s-1)}$$

where the last equality follows from $\omega_s \sim \frac{2dy}{s(s-1)}$ and applying the Cauchy integral formula. QED.

Corollary. *For all $s \in \mathbf{C} - \{0,1\}$ there exists $\epsilon > 0$ such that for all $s' \neq s'' \in \{t \in \mathbf{C} - \{0,1\} : |t - s| < \epsilon\}$, $X_{s'} \not\simeq X_{s''}$ as complex manifolds.*

One could say that $\tau'(s) \neq 0$ because $[\omega_s], [\omega'_s]$ form a basis for the vector space that is the cohomology of each fiber. The form ω' was obtained by differentiating ω with respect to the parameter s, which is as of yet unmotivated. In the next chapter we will work in a context where differentiating a cohomology class with respect to the parameter of the base is the result of applying a connection to the cohomology class, namely the *Gauss-Manin connection.*

1.4 Analytic continuation

After defining a period function $\zeta(s)$ on an open disk $\Delta \subset S$ centered at s_0 as above, analytic continuation of ζ may be done by regarding it as a solution to

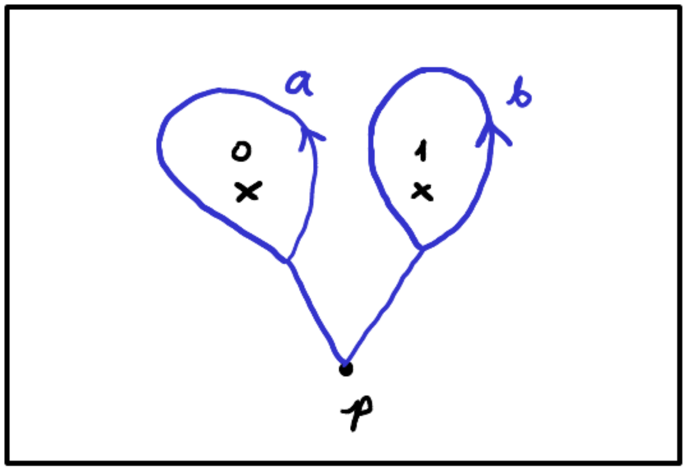

Figure 1.5: $\mathbf{C} - \{0, 1\}$ is homotopy-equivalent to the wedge of two circles.

equation 1.1 in the following way: given a piecewise smooth path $\alpha \subset S$ joining points s_0 and s_1, cover α by open disks $\Delta_j \subset S$ of the same radius centered at $\alpha(t_j)$ for a partition $0 = t_0 < t_1 < \ldots < t_n = 1$, such that $\alpha(t_{j+1}) \in \Delta_j$. Then, sequentially solve the Cauchy problem on the disk Δ_{j+1} with initial condition $\zeta_j(\alpha(t_{j+1}))$, where ζ_j is the solution on Δ_j.

However, from such a definition of analytic continuation, it is not easy to track down how the succesive solutions are given by periods of the fibers X_s. The following arguments should be accesible to the reader with basic knowledge of analytic continuation and linear homogeneous ODE over $\mathbf{C}$. We recommend the references [1],[8].

The Picard-Fuchs equation (1.1) has regular singularities at $0, 1, \infty$. Therefore, if α is a simple closed curve that encircles either 0 or 1[1], then analytic continuation of a local solution along α may produce a different solution. We recently established that the period functions $A(s), B(s)$ are linearly independent solutions of equation (1.1), and since this is a linear homogeneous equation of order 2, the functions $A(s)$ and $B(s)$ define a basis for the set of solutions to (1.1) as a $\mathbf{C}-$vector space. If we denote this space of solutions by Sol, it follows from elementary arguments that analytic continuation of A and B along α defines a linear isomorphism $\hat{\alpha} : Sol \to Sol$. This linear isomorphism is our definition of *monodromy along* α.

Since analytic continuation of a germ along a null-homotopic loop is identity,

[1]We restrict our attention to piecewise c^1 curves, and by *encircles a point* we mean that the point belongs to the bounded component of the complement of the curve.

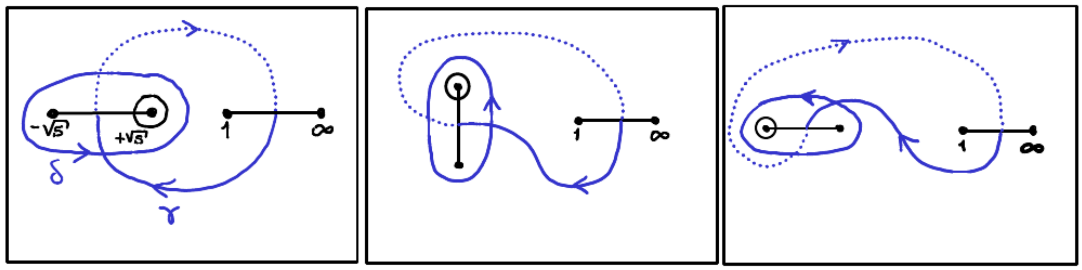

Figure 1.6: Effect of the flow of $\tilde{\eta}$ on homology of the fibers over $|s| = \epsilon$.

knowledge of monodromy along any loop is equivalent to knowledge of monodromy along generators of $\pi_1(\mathbf{C} - \{0,1\})$. This group is free on two generators since $\mathbf{C} - \{0,1\}$ is homotopic to the wedge of two circles (see figure 1.5). We proceed to compute monodromy along loops encircling 0 and 1 like those pictured in figure 1.5; we will do so by observing how homology changes under the flow of a vector field and computing intersection numbers.

1.4.1 Monodromy around 0

The family of elliptic curves given by

$$M_s = \{(x, y)|\ y^2 = (x^2 - s)(x - 1)\},\ s \in \mathbf{C} - \{0, 1\}$$

is convenient to illustrate what monodromy around 0 should look like. Consider the circle $|s| = \epsilon$. The generator for $\pi_1(\mathbf{C} - \{0,1\})$ pictured in figure 1.5 is homotopic to the loop defined by joining a path from the basepoint to the circle $|s| = \epsilon$, so we can restrict ourselves to analytic continuation along such a circle. The vector field $\partial/\partial\theta$ on the punctured disk $\{0 < |s| < 2\epsilon\}$ has $|s| = \epsilon$ as an integral curve, and can be extended to a vector field η on $\mathbf{C} - \{0, 1\}$. This vector field lifts to a vector field $\tilde{\eta}$ on the manifold $M = \{(x, y, s)|\ y^2 = (x^2 - s)(x - 1),\ s \in \mathbf{C} - \{0, 1\}\}$, such that its flow defines diffeomorphisms between the fibers of $|s| = \epsilon$. Let s_0 be an arbitrary point in this circle and let $\tilde{\eta}_{2\pi} : M_{s_0} \to M_{s_0}$ be the flow diffeomorphism that maps the fiber at s_0 to itself after a full turn of the circle. It defines a map T on homology which only depends on the homotopy class of $\tilde{\eta}_{2\pi}$.

Notice that a natural way to define charts on the curves M_s is by cutting the Riemann sphere as illustrated in the left panel of figure 1.6, with one cut being the line segment joining the square roots of s and the other cut being a ray from 1 to ∞. Then, we say the flow of η *rotates the cut from* $-\sqrt{s}$ *to* $\sqrt{s}$, and in that way it defines a family of flow diffeomorphisms between the fibers of $|s| = \epsilon$ since the map extends uniquely by continuity to the curves after defining it on their sheets. The way it acts on homology is illustrated in the sequence of

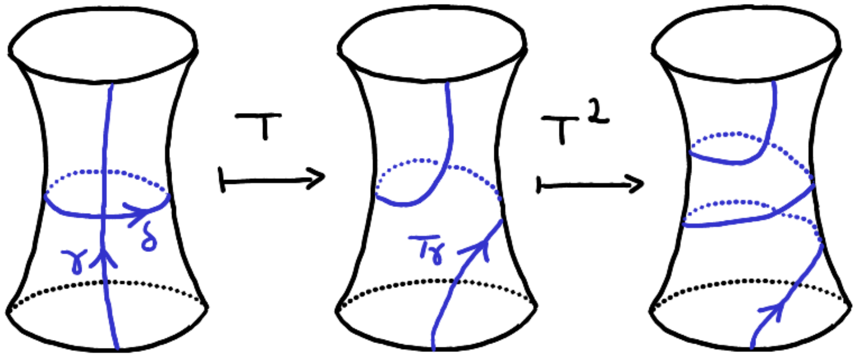

Figure 1.7: The homology transformation for the Legendre family is T^2.

pictures in figure 1.6.

If we denote by $T : H_1(M_{s_0}, \mathbf{Z}) \to H_1(M_{s_0}, \mathbf{Z})$ the map on homology, we can read the intersection numbers of $T\gamma$ and $T\delta$ with δ and γ from figure 1.6. Clearly, $T\delta = \delta$ and $T\gamma.\delta = \gamma.\delta = -1$. To obtain $T\gamma.\gamma = 1$, we superimpose the left and right pictures of figure 1.6. Then, the matrix representation of T in the basis $\{\delta, \gamma\}$ is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The family M is such that when s rotates by 2π around 0, the branching points are the same but the cut joining the square roots of s has rotated only by π. Since for our family X the cut from 0 to s rotates by 2π when s travels a full turn on the circle $|s| = \epsilon$, the map on homology given by $\tilde{\eta}_{2\pi}$ is $P := T^2$ (see figure 1.7).

$$P = T^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

1.4.2 Monodromy around 1

To compute monodromy for $s \to 1$, the flow-diffeomorphism arguments are the same as for $s \to 0$. However, it is convenient to choose a different pair of branch cuts while keeping the same homology basis, essentially because for our choice of homology basis the cycle γ encircles 1 and s. The reader may want to convince herself that choosing new branch cuts while keeping the homology cycles fixed is a reasonable thing to do.

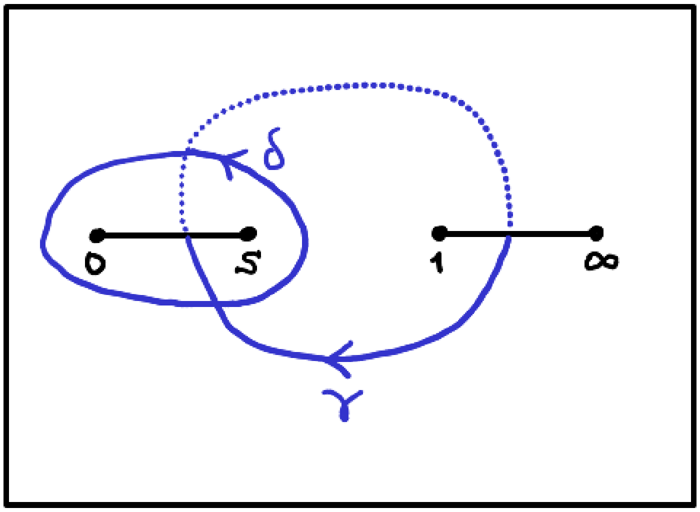

Figure 1.8: Standard homology basis with branch cuts used to compute monodromy around 0.

Figure 1.8 depicts our working homology basis together with the cuts used to compute monodromy around 0, while figure 1.9 shows the cuts best adapted for computing monodromy around 1 in the left panel. The right panel of figure 1.9 shows the change in homology when the segment joining 1 and s has turned a half-circle. Computing intersection numbers the same way as before shows that the matrix T' associated to a half-turn is

$$T' = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Then, the matrix representing monodromy around 1 for the Legendre family is

$$Q := T'^2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

1.4.3 Monodromy and analytic continuation

The reader may now reasonably wonder why knowledge of how homology of X_s changes as s winds around 0 and 1 implies knowledge of analytic continuation of the period functions $A(s), B(s)$. If s lies in a small circle that winds around 0, the homology cycles determined by flow of a vector field are homologous to δ_s and γ_s for points in the circle that are close enough to s, when thought of as homology cycles on X_s. This same reasoning is applicable for s in a small circle that winds around 1. Combining this idea with those discussed in section 1.3

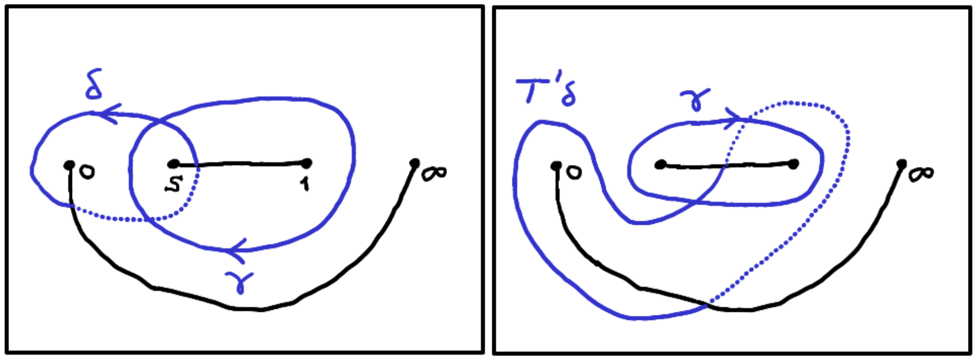

Figure 1.9: Monodromy for $s \to 1$.

related to the pictures 1.4 and 1.3, we conclude that the flow diffeomorphisms between the fibers do define analytic continuations of the period functions. In other words, if $\alpha : [0,1] \to \mathbf{C} - \{0,1\}$ is a loop that acts on homology by a matrix[2] $\mathcal{W}$ and ξ is a homology cycle on the curve X_{s_0}, then for s near s_0

$$\zeta(s) := \int_{\xi} \omega_s$$

implies

$$(\alpha \cdot \zeta)(s) = \int_{\mathcal{W}\xi} \omega_s,$$

where $\alpha \cdot \zeta$ denotes the analytic continuation of ζ along α.

Remark. Notice that the monodromy matrices P and Q define a complex representation of the fundamental group of $\mathbf{C} - \{0,1\}$. This representation is faithful, and for a proof we refer the reader to [3]. Also relevant to this discussion is the fact that the category of complex representations of the fundamental group of a space Z is equivalent to the category of locally constant sheaves of complex vector spaces on Z, a proof can be found in [4]. In the next chapter we will prove that cohomology of the curves in the Legendre family defines a sheaf locally isomorphic to $\mathbf{C}^2_{\mathbf{C}-\{0,1\}}$, which is precisely the locally constant sheaf given by the monodromy representation of the fundamental group of $\mathbf{C} - \{0,1\}$.

[2]The reader can convince himself that $\mathcal{W}$ is a word with letters P, P^{-1}, Q, Q^{-1}.

Chapter 2

The point of view of Manin and Grothendieck

This chapter can be thought of as an elaboration on a footnote at the end of Grothendieck's 1966 paper *On the de Rham cohomology of algebraic varieties* [5]. In chapter 1 we obtained the form ω_s' out of ω_s by way of differentiating its coefficient function with respect to the variable s, and then saw that they defined independent cohomology classes. Let us see that we can endow cohomology of the fibers with the structure of a vector bundle with a flat connection, such that the connection coincides with differentiation with respect to s. We start by giving a new context for the Legendre family of elliptic curves.

Define $S = \mathbf{C} - \{0,1\}$ and let X be the closure in $\mathbf{P}^2 \times S$ of the space $X^{af} := \{(x,y,s) \in \mathbf{C}^2 \times S |\ y^2 = x(x-1)(x-s)\}$. Let $\pi : X \to S$ be the restriction to X of the projection $(x,y,s) \mapsto s$.This map is a proper submersion and its fibers are the curves of the Legendre family, namely $\pi^{-1}(s) = X_s$. An important consequence of this is that in the smooth category, $\pi^{-1}(\Delta) \simeq \Delta \times T$ where $\Delta \subset S$ is a sufficiently small open disk and $T = S^1 \times S^1$. This actually was implicit in the previous chapter, for instance when we provided a homology basis for all curves above a small disk Δ by way of paths $\delta, \gamma_1, \gamma_2 : [0,1] \to \mathbf{C}$. From such data one can use the fact that $T \simeq \mathbf{R}^2/\mathbf{Z}^2$ and $X_s \simeq \mathbf{C}/A(s)\mathbf{Z} \oplus B(s)\mathbf{Z}$ to trivialize $\pi^{-1}(\Delta)$, where $A(s), B(s)$ are the periods of X_s.

Notice that the previous paragraph implies that s is a global coordinate function on X, which is acutually enough to show that the canonical map $\pi^\flat : \pi^{-1}\mathcal{O}_S \to \mathcal{O}_X$ is injective[1]. With this in mind, it is perfectly reasonable to think of differential forms on X which are *independent of* s, namely those whose expressions in local coordinates do not contain ds nor coefficient functions dependent on s. The formal picture is the following:

[1] The choice of notation is by analogy with Hartshorne's book, where the adjoint map $\mathcal{O}_S \to \pi_*\mathcal{O}_X$ is called $\pi^\sharp$.

Definition 2. 1. Pull-back of differential 1-forms along π defines a map $\delta\pi : \pi^*\Omega_S \to \Omega_X$. The *sheaf of relative 1-forms* $\Omega_{X/S}$ is the cokernel of $\delta\pi$ in the category of sheaves.

2. The *relative de Rham complex* $\Omega^\bullet_{X/S}$ is the complex

$$\mathcal{O}_X \xrightarrow{d'} \Omega_{X/S}$$

with the differential d' being the composition of $d : \mathcal{O}_X \to \Omega_X$ and the quotient map $\Omega_X \to \Omega_{X/S}$.

Let us now state the purpose of this chapter. Given a connected manifold Z, we define a *vector bundle with a flat connection on* Z to be any sheaf F isomorphic to $\mathcal{O}_Z \otimes_{\mathbf{C}_Z} \ell$, where ℓ is a *local system of rank* r, namely a sheaf locally isomorphic to $\mathbf{C}_Z^r$. Then there exists a canonical action of the tangent sheaf Θ_Z on F given by differentiation on the $\mathcal{O}_Z$ factor, this is what we refer to as the *flat connection*. Our purpose in this chapter is to show that by way of the direct image functor, $\Omega^\bullet_{X/S}$ defines a vector bundle with a flat connection on S. This will be our definition of the *Gauss-Manin connection*.

To achieve this purpose, it suffices to show that there exist a local system ℓ and a functor Φ related to direct image such that $\Phi(\Omega^\bullet_{X/S}) = \mathcal{O}_S \otimes \ell$. However, we are also interested in relating this to the previous chapter, so let us start by discussing a local system ℓ that we have encountered before.

Proposition 2. *$R^q\pi_*\mathbf{C}_X$ is a local system on S for each q and its stalk at s is $H^q(X_s, \mathbf{C})$.*

Proof. We start by proving the second statement. Let $\mathbf{C}_X \tilde{\to} I^\bullet$ be a resolution of the constant sheaf by objects acylclic for the direct image functor. Then

$$(R^q\pi_*\mathbf{C}_X)_s = H^q(\pi_*I^\bullet)_s = \varinjlim_{U\ni s} H^q(\pi_*I^\bullet)(U).$$

Since the direct limit functor is exact, it commutes with cohomology. Therefore, we have:

$$\varinjlim_{U\ni s} H^q(\pi_*I^\bullet)(U) = \varinjlim_{U\ni s} H^q(I^\bullet(\pi^{-1}U)) = H^q(\varinjlim_{U\ni s} I^\bullet(\pi^{-1}U)).$$

Since the directed system consisting of open disks Δ centered at s is cofinal with respect to the system of all neighborhoods of s, and $X \to S$ is locally trivial with fiber T, it follows that

$$H^q(\varinjlim_{U\ni s} I^\bullet(\pi^{-1}U)) = H^q(\varinjlim_{\Delta\ni s} I^\bullet(\pi^{-1}\Delta)) = H^q(\Delta\times T, \mathbf{C}) = H^q(X_s, \mathbf{C}).$$

That $R^q\pi_*\mathbf{C}_X$ is a local system follows from it being the sheafification of the presheaf $U \mapsto H^q(\pi^{-1}U, \mathbf{C}_X|_{\pi^{-1}U})$.

□

We are ready to state our result of interest:

Theorem 2.

$$R\pi_*(\Omega^\bullet_{X/S}) = \mathcal{O}_S \otimes_{\mathbf{C}_S} R\pi_*\mathbf{C}_X.$$

The next section is devoted to proving the theorem in steps, and the reader is likely to notice soon enough that it is nothing more than a particular case of the *projection formula* [6],[7].

2.1 Proof of Theorem 2

Since we are applying the derived functor $R\pi_*$, we may start by replacing $\Omega^\bullet_{X/S}$ with an equivalent object.

Proposition 3.

$$R\pi_*(\Omega^\bullet_{X/S}) = R\pi_*(\pi^{-1}\mathcal{O}_S).$$

Proof. We prove that $d(\mathcal{O}_X)\cap\delta\pi(\pi^*\Omega_S)$ is the image of the composition $d\circ\pi^\flat : \pi^{-1}\mathcal{O}_S \to \Omega_X$. First, given $f \in \pi^{-1}\mathcal{O}_S$, $\pi^\flat(f) = g\circ\pi$ for $g \in \mathcal{O}_S$. Then $d(\pi^\flat(f)) = \delta\pi(dg)$. Next, let $f \in \mathcal{O}_X$ be such that $df \in \delta\pi(\pi^*\Omega_S)$. We use the fact that s is a global coordinate to express f locally as $f(x,s)$, where x is some other holomorphic coordinate. Then $df = f_x dx + f_s ds = f_s ds$. Then f is independent of x and therefore $f \in \pi^\flat(\pi^{-1}\mathcal{O}_S)$. □

The following simple fact will save us the need to repeatedly specify the coefficient ring for the tensor product:

Proposition 4.

$$\pi_*\mathbf{C}_X = \mathbf{C}_S.$$

Proof. The constant sheaf $\mathbf{C}_S$ is determined by the property that if $U \subset S$ is a connected open set, then $\mathbf{C}_S(U) = \mathbf{C}$. To prove the desired equality of sheaves, we prove that the map π is such that $\pi^{-1}U \subset X$ is connected for $U \subset S$ a connected open set.

Let $U \subset S$ be a connected open set, and consider two arbitrary points $a, b \in \pi^{-1}U$. Take a path $\gamma \subset U$ joining πa and πb, and let $\Delta_1, \ldots, \Delta_n$ be open disks centered at points of γ such that $\pi^{-1}\Delta_i \simeq \Delta_i \times T$ and $\gamma \subset \cup_i \Delta_i$. Then standard arguments give a lifting of γ to a path $\tilde{\gamma}$ joining a and b. □

Proposition 5. *1. Fine sheaves are acyclic for the direct image functor.*

2. The constant sheaf $\mathbf{C}_X$ admits a fine resolution $(I^\bullet, d)$.

3. If M is any sheaf on X and I is a fine sheaf on X, then $M \otimes_{\mathbf{C}_X} I$ is a fine sheaf[2].

4. In the category of sheaves of complex vector spaces, if M is any sheaf, then the functor $M \otimes_{\mathbf{C}} (\bullet)$ is exact.

Proof. 1. A proof can be found in [2].

2. Let $\Omega^\bullet_{X,\mathbf{C},\infty}$ denote the complex of smooth differential forms with complex values on X. Namely, $\alpha \in \Omega^p_{X,\mathbf{C},\infty}$ is given in local coordinates by $\alpha|_V = \sum_I (u_I + iv_I) dx_{i_1} \wedge ... \wedge dx_{i_p}$ where I is the multi-index $i_1 < ... < i_p$, x_j are smooth coordinate functions on $V \subset X$, and u_I, v_I are smooth functions on $V \subset X$. It is a complex of fine sheaves since separating into real and imaginary parts gives $\Omega^\bullet_{X,\mathbf{C},\infty} \simeq \Omega^\bullet_{X,\infty} \oplus \Omega^\bullet_{X,\infty}$, where $\Omega^\bullet_{X,\infty}$ is the usual smooth de Rham complex on X. Poincaré lemma on the real and imaginary parts implies the complex is exact at each term, and if $f = u + iv$ is a complex-valued smooth function, $df = 0$ if and only if $du = 0$ and $dv = 0$, which is to say f is constant.

3. Let I be a fine sheaf on X and let M be any sheaf on X. If $\{\phi_\alpha\} : I \to I$ is a collection of endomorphisms of I with locally finite support such that $\sum_\alpha \phi_\alpha = id_I$, then $\{id \otimes \phi_\alpha\} : M \otimes I \to M \otimes I$ is a collection of endomorphisms of $M \otimes I$ with the same properties.

4. Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of sheaves of complex vector spaces on X. Then

$$0 \to M \otimes_{\mathbf{C}} A' \to M \otimes_{\mathbf{C}} A \to M \otimes_{\mathbf{C}} A'' \to 0$$

is exact if and only if for all $x \in X$ the sequence of stalks

$$0 \to (M \otimes_{\mathbf{C}} A')_x \to (M \otimes_{\mathbf{C}} A)_x \to (M \otimes_{\mathbf{C}} A'')_x \to 0$$

is exact. This is true since taking the stalk commutes with the tensor product and since all vector spaces are flat $\mathbf{C}-$modules.

□

In order to compute $R\pi_*\pi^{-1}\mathcal{O}_S$ we need an appropriate resolution of $\pi^{-1}\mathcal{O}_S$. This is the content of the next proposition.

Proposition 6. *If $(I^\bullet, d)$ is a fine resolution of $\mathbf{C}_X$, then $(\pi^{-1}\mathcal{O}_S \otimes I^\bullet, 1 \otimes d)$ is a fine resolution of $\pi^{-1}\mathcal{O}_S$.*

[2] We restrict to sheaves of complex vector spaces.

Proof. Thanks to proposition 5, $\ker(\pi^{-1}\mathcal{O}_S \otimes I^0 \to \pi^{-1}\mathcal{O}_S \otimes I^1) = \pi^{-1}\mathcal{O}_S$. Also, each term of the complex $\pi^{-1}\mathcal{O}_S \otimes I^\bullet$ is a fine sheaf. To complete the proof it suffices to prove exactness. Since $(\pi^{-1}\mathcal{O}_S \otimes I^\bullet, 1 \otimes d)$ is a complex of sheaves, it's enough to prove exactness at the level of stalks. Namely, for $n \geq 1$ consider the sequence of stalks

$$\pi^{-1}\mathcal{O}_{S,x} \otimes I^{n-1}{}_{,x} \xrightarrow{1\otimes d} \pi^{-1}\mathcal{O}_{S,x} \otimes I^{n}{}_{,x} \xrightarrow{1\otimes d} \pi^{-1}\mathcal{O}_{S,x} \otimes I^{n+1}{}_{,x}\,.$$

Let $\sum_i f_i \otimes y_i \in \pi^{-1}\mathcal{O}_{S,x} \otimes I^n{}_{,x}$ be such that $\sum_i f_i \otimes dy_i = 0$. Since $\pi^{-1}\mathcal{O}_{S,x} \otimes I^n{}_{,x}$ is a tensor product of vector spaces, the f_i may be taken to be linearly independent. Then, $\sum_i f_i \otimes dy_i = 0$ implies $dy_i = 0$ for all i. Then, exactness of

$$I^{n-1}{}_{,x} \xrightarrow{d} I^n{}_{,x} \xrightarrow{d} I^{n+1}{}_{,x}$$

yields $y_i = dy'_i$ for some $y'_i \in I^{n-1}{}_{,x}$. Then $\sum_i f_i \otimes y_i = 1 \otimes d(\sum_i f_i \otimes y'_i)$, thereby proving exactness. □

We have arrived at the final step of the proof.

Proposition 7. $R\pi_*(\pi^{-1}\mathcal{O}_S) = \mathcal{O}_S \otimes_{\mathbf{C}_S} R\pi_*\mathbf{C}_X$.

Before proving it we introduce some preliminary considerations. Using the fine resolution $\mathbf{C}_X \xrightarrow{\sim} I^\bullet$ and our previous results, we have $R\pi_*\pi^{-1}\mathcal{O}_S = \pi_*(\pi^{-1}\mathcal{O}_S \otimes_{\mathbf{C}_X} I^\bullet)$ and $\mathcal{O}_S \otimes_{\mathbf{C}_S} R\pi_*\mathbf{C}_X = \mathcal{O}_S \otimes_{\mathbf{C}_S} \pi_* I^\bullet$.

Considering the n-th terms of the respective complexes, we want to define a map $(\mathcal{O}_S \otimes_{\mathbf{C}_S} \pi_* I^n)(U) \to \pi_*(\pi^{-1}\mathcal{O}_S \otimes_{\mathbf{C}_X} I^n)(U)$.

Let $I := I^n$. There are natural maps of presheaves

$$\begin{aligned} sh \otimes \mathrm{id} : \pi^{-1}\mathcal{O}_S^{pre} \otimes^{pre} I &\to \pi^{-1}\mathcal{O}_S \otimes^{pre} I \\ sh : \pi^{-1}\mathcal{O}_S \otimes^{pre} I &\to \pi^{-1}\mathcal{O}_S \otimes I \end{aligned}$$

for which functoriality of π_* in the category of presheaves gives maps

$$\begin{aligned} \pi_*(sh \otimes \mathrm{id}) : \pi_*(\pi^{-1}\mathcal{O}_S^{pre} \otimes^{pre} I) &\to \pi_*(\pi^{-1}\mathcal{O}_S \otimes^{pre} I) \\ \pi_* sh : \pi_*(\pi^{-1}\mathcal{O}_S \otimes^{pre} I) &\to \pi_*(\pi^{-1}\mathcal{O}_S \otimes I), \end{aligned}$$

where sh denotes the appropriate sheafification. Moreover, it can be easily checked that there is an equality of presheaves $\mathcal{O}_S \otimes^{pre} \pi_* I = \pi_*(\pi^{-1}\mathcal{O}_S^{pre} \otimes^{pre} I)$. Therefore, if we prove that the composition of the chain of natural maps of presheaves

$$\mathcal{O}_S \otimes^{pre} \pi_* I \to \pi_*(\pi^{-1}\mathcal{O}_S^{pre} \otimes^{pre} I) \to \pi_*(\pi^{-1}\mathcal{O}_S \otimes^{pre} I) \to \pi_*(\pi^{-1}\mathcal{O}_S \otimes I)$$

is an isomorphism on stalks, we prove that the sheaves $\mathcal{O}_S \otimes \pi_* I$ and $\pi_*(\pi^{-1}\mathcal{O}_S \otimes I)$ are canonically isomorphic, with the natural map having the direction $\mathcal{O}_S \otimes \pi_* I \to \pi_*(\pi^{-1}\mathcal{O}_S \otimes I)$.

Remark. We have defined a map of complexes since the map $\mathcal{O}_S \otimes^{pre} \pi_* I \to \pi_*(\pi^{-1}\mathcal{O}_S^{pre} \otimes^{pre} I)$ is identity and for the remaining maps, commutativity with the differentials is given by functoriality.

The proof that the natural map $\mathcal{O}_S \otimes \pi_* I \to \pi_*(\pi^{-1}\mathcal{O}_S \otimes I)$ is an isomorphism will also be done in steps.

Lemma 1. *π is an open map.*

Proof. It follows from the fact that as a topological space, X admits a basis consisting of open sets homeomorphic to $U \times W$ where U is an open subset of S and W is an open subset of T. □

Corollary. $\varinjlim_{U \supset \pi V} G(U) = G(\pi V)$ *for any sheaf G on S.*

Lemma 2. *Let F be a presheaf on X. Define $F, \pi^{-1}s := \varinjlim_{V \supset \pi^{-1}s} F(V)$. Then $\pi_* F, s = F, \pi^{-1}s$.*

Proof. Since for all open sets $S \supset U \ni s$, $\pi^{-1}U \supset \pi^{-1}s$ holds, then there is a natural map $\pi_* F, s \to F, \pi^{-1}s$. Let us see that the open sets $\pi^{-1}U$, $U \subset S$ are cofinal in the directed system associated to $F, \pi^{-1}s$. It suffices to show that for all open sets $V \subset X$ such that $\pi^{-1}s \subset V$, there exists an open set $U \subset S$ such that $\pi^{-1}s \subset \pi^{-1}U \subset V$. Recall that fibers are compact since for all $s \in S$, $\pi^{-1}s \simeq T$ where $T = S^1 \times S^1$. Given $x \in \pi^{-1}s$, choose $x \in V_x \subset V$ such that $V_x \simeq \Delta_x \times W_x$, where $\Delta_x \subset S$ is a small open disk centered at s. In this manner we cover $\pi^{-1}s$ by open sets contained in V. By compactness of $\pi^{-1}s$ we may assume that the cover $\{\Delta_x \times W_x\}_{x \in \pi^{-1}s}$ is finite, say it consists of $\Delta_1 \times W_1, \dots, \Delta_n \times W_n$. Assuming $\Delta_1 =: U$ is the disk of smallest radius, it follows that $\pi^{-1}s \subset \pi^{-1}U \subset V$. □

Lemma 3. [3] $\pi^{pre\,-1}G = \pi^{-1}G$ *for any sheaf G on S.*

Proof. Let us start by proving that for open sets $V, V' \subset X$, $\pi(V \cap V') = \pi V \cap \pi V'$. Elementary set theory tells us that it suffices to prove $\pi V \cap \pi V' \subset \pi(V \cap V')$. If $s \in \pi V \cap \pi V'$, there exist $x \in V, x' \in V'$ such that $\pi x = \pi x' = s$. Let $\Delta \subset S$ be an open disk centered at s such that there exists a homeomorphism $\varphi : \pi^{-1}\Delta \to \Delta \times T$ such that $pr_1 \circ \varphi = \pi$. Let $\tilde{W} := V \cap \pi^{-1}\Delta$, $\tilde{W}' := V' \cap \pi^{-1}\Delta$. Then there exist open disks $\Delta_1, \Delta_2 \subset \Delta$ centered at s and open

[3] $\pi^{pre\,-1}G$ denotes the presehaf on X whose sections are $\pi^{pre\,-1}G(V) = \varinjlim_{U \supset \pi V} G(U)$. The sheaf $\pi^{-1}G$ is defined to be the sheafification of this presheaf.

sets $W, W' \subset T$ such that $\varphi x \in \Delta_1 \times W \subset \varphi\tilde{W}$ and $\varphi x' \in \Delta_2 \times W' \subset \varphi\tilde{W}'$. Then $s \in \Delta_1 \cap \Delta_2 = pr_1(\Delta_1 \cap \Delta_2 \times W \cap W') = pr_1\varphi\varphi^{-1}(\Delta_1 \cap \Delta_2 \times W \cap W') = \pi\varphi^{-1}(\Delta_1 \cap \Delta_2 \times W \cap W') \subset \pi(\tilde{W} \cap \tilde{W}') \subset \pi(V \cap V')$. Let us now check that $\pi^{pre\,-1}G$ satisfies the sheaf axioms. Let $V = \cup V_i$ and suppose $g \in \pi^{pre\,-1}G(V)$ is such that $\forall i : g|V_i = 0$. Since π is an open map, $\pi^{pre\,-1}G(V) = G(\pi V)$ and $g \in G(\pi V)$ is such that $g|\pi V_i = 0$. Since $\cup\pi V_i = \pi V$, g restricts to 0 on an open cover of πV, so that G being a sheaf implies $g = 0$. Now suppose we are given a collection $g_i \in \pi^{pre\,-1}G(V_i)$ such that $g_i = g_j$ on $V_i \cap V_j$. This is the same as a collection $g_i \in G(\pi V_i)$ such that $g_i \cap g_j$ on $\pi(V_i \cap V_j)$, but earlier we proved $\pi(V_i \cap V_j) = \pi V_i \cap \pi V_j$, so πV_i being an open cover of πV and G being a sheaf imply there exists $g \in G(\pi V) = \pi^{pre\,-1}G(V)$ such that $g|V_i = g_i$. □

Corollary. *the map $\pi_*(\pi^{-1}\mathcal{O}_S^{pre} \otimes^{pre} I) \to \pi_*(\pi^{-1}\mathcal{O}_S \otimes^{pre} I)$ is identity, hence an isomorphism on stalks.*

Lemma 4. $\varinjlim_{V \supset \pi^{-1}s}(\pi^{-1}\mathcal{O}_S \otimes I)(V) = \varinjlim_{V \supset \pi^{-1}s} \pi^{-1}\mathcal{O}_S(V) \otimes I(V)$.

Proof. Let $\alpha \in (\pi^{-1}\mathcal{O}_S \otimes I)(V)$ be a representative of an element in $(\pi^{-1}\mathcal{O}_S \otimes I), \pi^{-1}s$. By topological arguments similar to the ones we have just used, we may assume $V = \pi^{-1}U$ for U an open neighborhood of s, and that there exist $\alpha_i \in \pi^{-1}\mathcal{O}_S(V_i) \otimes I(V_i), i = 1, \dots, n$ such that $\cup_{i=1}^n V_i = V$, $\alpha|V_i = \alpha_i$ and $\pi V_i = U$ for $i = 1, \dots, n$. Consider the tensors $\alpha_i = \sum_{t=1}^N f_t \otimes y_t \in \pi^{-1}\mathcal{O}_S(V_i) \otimes I(V_i)$ and $\alpha_j = \sum_{t=1}^M f'_t \otimes z_t \in \pi^{-1}\mathcal{O}_S(V_j) \otimes I(V_j)$. Using the fact that $\pi V_i = \pi V_j = U$ and taking a basis of span$(f_1, \dots, f_N, f'_1, \dots, f'_M) \subset \mathcal{O}_S(U)$, we may assume $\alpha_i = \sum_{t=1}^N f_t \otimes y_t$ and $\alpha_j = \sum_{t=1}^N f_t \otimes z_t$ where $f_1, \dots, f_N$ are linearly independent vectors. Let $W = V_i \cap V_j$. Then $\alpha_i|W = \alpha_j|W$ yields $0 = \sum_{t=1}^N f_t \otimes (y_t|W - z_t|W)$, so $y_t = z_t$ on $V_i \cap V_j$. It follows that $\alpha = \sum_{t=1}^N f_t \otimes y_t \in \pi^{-1}\mathcal{O}_S(V) \otimes I(V)$. □

Corollary. *the map $\pi_*(\pi^{-1}\mathcal{O}_S \otimes^{pre} I) \to \pi_*(\pi^{-1}\mathcal{O}_S \otimes I)$ is an isomorphism on stalks.*

2.2 The holomorphic forms $\omega_s \in \Omega_{X_s}$

Since $R\pi_*(\Omega^\bullet_{X/S}) \simeq \mathcal{O}_S \otimes R\pi_*\mathbf{C}_X$ and $R^1\pi_*\mathbf{C}_X$ is locally isomorphic to $\mathbf{C}_S^2$, it follows that $R^1\pi_*(\Omega^\bullet_{X/S})$ is locally isomorphic to $\mathcal{O}_S \oplus \mathcal{O}_S$. If $U \subset S$ is an open set such that we can fix a homology basis δ, γ for the fibers $X_s,\ s \in U$, then taking the dual cohomology basis gives $R^1\pi_*\mathbf{C}_X|_U \simeq \mathbf{C}_U^2$ and therefore $R^1\pi_*(\Omega^\bullet_{X/S})|_U \simeq \mathcal{O}_S|_U \oplus \mathcal{O}_S|_U$.

Integration over δ, γ of the holomorphic forms ω_s for $s \in U$ gives the period functions $A(s), B(s)$ on U from the previous chapter. Then we may identify the cohomology classes $[\omega_s]$ with the values of the vector-valued function $(A(s), B(s))$ on U. That is, we identify $[\omega_s]$ for varying s with a section of $\mathcal{O}_S|_U \oplus \mathcal{O}_S|_U$. The Gauss-Manin connection, which on U is given by differentiation with respect to s, gives $(A'(s), B'(s))$ when applied to the vector-valued function that we have identified with $[\omega_s]$. But then recalling from the previous chapter we see that this is the same as the cohomology classes $[\omega_s']$, and it is in this sense that we can say that the form ω_s' is the result of applying the Gauss-Manin connection to the form ω_s.

Although the previous paragraph was a local discussion, it is clear that the cohomology classes $[\omega_s]$ and $[\omega_s']$ define two independent global sections of $R^1\pi_*(\Omega^\bullet_{X/S})$, and the vector-valued functions used to represent them locally are given by a choice of homology basis on an open set where the family is trivial.

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