Applications of Kronecker states in quantum information theory

Walther Leonardo González Olaya
201627469

Advisor: Alonso Botero Mejía PhD.

October 7, 2022

1 Introduction

Shannon developed classical information theory in 1948, quantifying information and giving a mathematical foundation to any process within it, such as transmission, compression, codification, etc. Years later, the question of how to transmit quantum information arose naturally from quantum physics; as a result, Quantum Information Theory based on Shannon ideas was developed. This new focus on information theory gave rise to new problems concerning to the successful transmission of quantum information[1]; but also allowed the introduction of new advantageous protocols, exhibiting features not achievable in the classic theory. In quantum information theory, maximally-entangled states (MES) are key to implement many well-known protocols, such as quantum teleportation, quantum error correction and quantum key distribution.

The representation theory of the symmetric group provides a mechanism that allows one to generate a wide class of maximally-entangled multipartite states. Such states, which we call Kronecker states [2], belong to the invariant subspace of products of irreducible representations of S_n. Reduced density matrices of such states in each individual subspace are completely mixed, proving that Kronecker states are MES [3].

The purpose of this work is to better understand Kronecker states and their applications in quantum information theory. In order to reveal properties of Kronecker states from different focuses, we will use them as special maximally-entangled states in some problems of quantum information theory. In particular, we purpose to implement them in four cases:

- Superadditivity of classical channel capacity in quantum channels
- Quantum error correction
- Quantum secret sharing
- Entanglement concentration

In this document we will describe the basis of each case, commenting on the possible importance of Kronecker states in each one.

2 Representation Theory of the Symmetric Group

In this section we will present the basics of the representation theory of the symmetric group [4], preparing the basis for the introduction of Kronecker states and their significance in quantum information theory.

2.1 The Symmetric Group S_n

S_n is the group of permutations of a set of n elements. Each element π of the group can be expressed as

$$\pi = [\pi(1)\pi(2)\dots\pi(n)], \tag{1}$$

where $\pi(i)$ represents the position of the element of the set which will take the position of the element i, e.g., given a set $\boldsymbol{X} = \{A, B, C, D\}$, and the permutation $\pi = [3241]$, the resultant set is $\pi\boldsymbol{X} = \{C, B, D, A\}$. In this document the cycle notation will be used to label permutations.

Definition Let $x_1, \dots, x_r$ be elements of a set of $n \geq r$ elements. The permutation which maps the elements $x_1 \to x_2, x_2 \to x_3, \dots, x_{r-1} \to x_r$ and fixes the $n-r$ remaining elements is the cycle $(x_1, \dots, x_r)$ of order r.

It is always possible to write any permutation as the product of disjoint cycles, and the decreasing orders of such cycles determine the cycle structure ρ of the permutation, e.g., the permutation $\pi = [643215]$ written in cycle notation is $\pi = (165)(24)$, with cycle structure is $\rho = (32)$.

Definition For any group G, the elements $g_1, g_2 \in G$ are conjugate if there is an element $g \in G$ such that

$$g_1 = g \cdot g_2 \cdot g^{-1} \tag{2}$$

Any group can be divided in conjugacy classes that are sets of conjugated elements. That is, the conjugacy class of g is:

$$K(g) = \{\hat{g}g\hat{g}^{-1} | \hat{g} \in G\} \tag{3}$$

For the group S_n, let $\tau = (\tau_1, \tau_2, \dots, \tau_r)$ be an $r-$cycle; then, the result of a conjugation with some permutation π is

$$\pi\tau\pi^{-1} = (\pi(\tau_1)\pi(\tau_2)\dots\pi(\tau_r)), \tag{4}$$

which is just a relabeling of the elements in τ, so in general the conjugation operation doesn't change the cycle structure of the elements in S_n. In other words, the conjugacy classes for S_n are defined for the different cycle structures. For some n, the conjugacy classes are labeled by the possible *partitions* of n, which represent the cycle structure.

Definition A partition is a sequence of positive integers ordered in non increasing order $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, with a weight $n = \sum_i \lambda_i$. If m_i represents the times that the element i appears in λ, it is possible to rewrite $\lambda = (n^1 \ldots r^{(m_r)} \ldots 2^{(m_2)} 1^{(m_1)})$,e.g., for $n = 6$ there are 11 possible partitions, $(6), (51), (42), (41^2), (3^2), (321), (31^2), (2^3)$, $(2^2, 1^2)$,$(2, 1^4)$,(1^6). So, the elements of S_6 are divided in this 11 conjugacy classes.

2.2 Young Tableaux

Each partition can be represented as a Young diagram with empty boxes in a way that λ_i is the number of boxes in the row i. Each Young diagram can be filled in different ways, the one most important for us is the standard Young tableau (SYT) in which boxes are filled with the numbers from 1 to n in a way that they are increasing from left to right in every row and increasing downwards in every column.

For example, one partition of weight $n = 5$ is $\lambda = (3, 2)$. This partition is represented by the diagram

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array},$$

whose SYT's are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \tag{5}$$

The number f^λ of SYT's associated with a partition can be calculated from the hook rule,

$$f^\lambda = \frac{n!}{\prod_{i,j} v_{i,j}}, \tag{6}$$

here $v_{i,j}$ is the number of boxes to the right and below plus one of the box (i, j) in the Young diagram. For the example studied, this is

$$\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \quad , \quad f^\lambda = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1} = 5. \tag{7}$$

2.3 Representations of S_n

It turns out that S_n has as many irreducible representations (irreps) as partitions of weight n. The dimension of each irrep is given for the number of SYT in the correspondent partition. For this reason, it is convenient to label the irreps of S_n with the

correspondent partitions λ_i of n.
The basis for the elements of S_n in the irrep $[\lambda]$ are given by the f^λ SYT's of λ, usually labeled with the correspondent Yamanouchi symbol(a list with n elements, where the $i-th$ element represents the row where the number i is in the SYT). The Yamanouchi symbols for the SYT's of $\lambda = (3,2)$ presented in Equation 5 are respectively

$$\{1,1,1,2,2\}, \{1,1,2,1,2\}, \{1,1,2,2,1\}, \{1,2,1,1,2\}, \{1,2,1,2,1\}. \tag{8}$$

The matrix representing the element π in the irrep $[\lambda]$ is $S_\lambda(\pi)$ with dimensions $f^\lambda \times f^\lambda$. Defining the axial distance in the Yamanouchi basis M

$$\rho_M(n_1, n_2) = (x_2 - x_1) + (y_1 - y_2), \tag{9}$$

where x_i, y_i are the row and column where the number n_i is in the SYT correspondent to M, and assigning the bases e_M to each Yamanouchi symbol, we can compute the rows of the matrix representation of the transposition $T_j = (j, j+1)$ in the irrep λ as

$$S_\lambda(T_j)e_M = \frac{1}{\rho_M(j+1,j)}e_M + \sqrt{1 - \frac{1}{\rho_M(j+1,j)^2}}e_{T_jM}; \tag{10}$$

if T_jM (the transposition applied to the Yamanouchi symbol) is an existing Yamanouchi symbol M_2, then $e_{T_jM} = e_{M_2}$, otherwise, $e_{T_jM} = 0$. For example, in S_4 for the irrep $\lambda = (31)$ there are three valid Yamanouchi symbols

$$M_1 = \{1,1,1,2\}, M_2 = \{1,1,2,1\}, M_3 = \{1,2,1,1\}, \tag{11}$$

which will be assigned to a three dimensional orthogonal basis

$$M_1 \to e_{M_1} = (1,0,0), M_2 \to e_{M_2} = (0,1,0), M_3 \to e_{M_3} = (0,0,1). \tag{12}$$

In this way, the transposition T_1 is

$$\begin{gathered}
S_{(31)}(T_1)e_{\{1,1,1,2\}} = 1 * e_{\{1,1,1,2\}} + 0 * e_{\{1,1,1,2\}} = (1,0,0) \\
S_{(31)}(T_1)e_{\{1,1,2,1\}} = 1 * e_{\{1,1,2,1\}} + 0 * e_{\{1,1,2,1\}} = (0,1,0) \\
S_{(31)}(T_1)e_{\{1,2,1,1\}} = -1 * e_{\{1,2,1,1\}} + 0 * e_{\{2,1,1,1\}} = (0,0,-1) \\
S_{(31)}(T_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{gathered} \tag{13}$$

In the same way T_2 and T_3 can be computed, and the other permutations of S_4 can be calculated as the product of these transpositions.

The trace of $S_\lambda(\pi)$ in this context is known as the character $\chi^\lambda(\pi)$ and it has some interesting properties. Because the trace is invariant to similarity transformations

$$\chi^\lambda(\pi) = tr(S_\lambda(\pi)) = tr(S_\lambda(\hat{\pi})S_\lambda(\pi)S_\lambda(\hat{\pi}^{-1})) \quad \forall \hat{\pi} \in S_n, \tag{14}$$

it implies that the character only depends on the conjugacy class of π, which is determined by the cycle structure ρ of π, that is

$$\chi^\lambda(\pi) = \chi^\lambda(\rho) \quad , \forall \pi \in \rho. \tag{15}$$

From the properties of the trace, we also have that

$$\chi^\lambda(\pi \oplus \hat{\pi}) = \chi^\lambda(\pi) + \chi^\lambda(\hat{\pi}), \quad \chi^\lambda(\pi \otimes \hat{\pi}) = \chi^\lambda(\pi) \cdot \chi^\lambda(\hat{\pi}). \tag{16}$$

In addition, the characters of different representations are orthogonal

$$\sum_{\pi \in S_n} \chi^\alpha(\pi)\chi^\beta(\pi) = n!\delta_{\alpha\beta}. \tag{17}$$

With this, we complete the tools needed to introduce Kronecker states in the next section.

3 Maximally-entangled states and Kronecker states

3.1 Maximally-entangled states

In quantum mechanics, two particles belonging to Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ respectively, conform a system that belongs to $\mathcal{H}_A \otimes \mathcal{H}_B$. If the total state can be written as

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B \tag{18}$$

with $|\psi\rangle_A \in \mathcal{H}_A, |\psi\rangle_B \in \mathcal{H}_B$, the state is a separable state. If such separation is impossible, the state is an entangled state, which has correlations that are not present in the classical case. In particular, a bipartite entangled state is maximally entangled if the partial trace in each subsystem is a multiple of the identity. An example of this is the two qubit state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right). \tag{19}$$

There is no possible combination of $|\psi\rangle_A, |\psi\rangle_B$ with $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$, so $|\psi\rangle_{AB}$ is an entangled stated. Because partial traces are

$$\rho_A = tr_B(|\psi\rangle_{AB} \langle\psi|_{AB}) = \frac{1}{2}\mathbb{1}_{2\times 2} = \rho_B \tag{20}$$

in fact $|\psi\rangle_{AB}$ is a maximally-entangled state. This fact has special characteristics in measurements. If we measure the first qubit in the basis $\{|0\rangle, |1\rangle\}$, we will obtain each result with probability of 1/2; if the result is $|0\rangle$, the resulting total state is $|\psi\rangle = |00\rangle$; otherwise, it is $|\psi\rangle = |11\rangle$. In each case a posterior measurement in the second qubit is fully determined by the result of the first measurement.

The same ideas can be extended to multipartite systems, where N particles in Hilbert spaces $\mathcal{H}_1, \mathcal{H}_1, \ldots \mathcal{H}_N$ compose a system in $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$. If the total state can be written as

$$|\psi\rangle = \bigotimes_{i=1}^{N} |\psi\rangle_i \tag{21}$$

with $|\psi\rangle_i \in \mathcal{H}_i$, the state is separable, otherwise, it is an entangled state. If reduced density matrices in all one-party subsystems are proportional to the identity, the total state is a maximally-entangled state [3].

3.2 Kronecker states

In the symmetric group S_n, the tensor product of two irreps can be decomposed as a direct sum of other irreps with a certain multiplicity.

$$[\alpha] \otimes [\beta] = \bigoplus_{\lambda} g_{\alpha\beta\lambda}[\lambda] \tag{22}$$

with $[\alpha], [\beta], [\lambda]$ irreps of S_n, and $g_{\alpha\beta\lambda}$ the multiplicity of the representation $[\lambda]$ in $[\alpha] \otimes [\beta]$, which is called Kronecker coefficient. Such decomposition is the same for any element π of S_n. Tracing both sides for any π and applying the properties of characters, we can obtain an explicit formula for Kronecker coefficients

$$\begin{gathered}
\chi^{\alpha}(\pi)\chi^{\beta}(\pi) = \sum_{\lambda} g_{\alpha\beta\lambda}\chi^{\lambda}(\pi) \\
\sum_{\pi \in S_n} \chi^{\alpha}(\pi)\chi^{\beta}(\pi)\chi^{\gamma}(\pi) = \sum_{\lambda} g_{\alpha\beta\lambda} \sum_{\pi \in S_n} \chi^{\lambda}(\pi)\chi^{\gamma}(\pi) \\
\frac{1}{n!}\sum_{\pi \in S_n} \chi^{\alpha}(\pi)\chi^{\beta}(\pi)\chi^{\gamma}(\pi) = g_{\alpha\beta\gamma}.
\end{gathered} \tag{23}$$

There is another interpretation for the Kronecker coefficient that is the most important one for us. The dimension of the invariant subspace in a product of three irreps is given by the Kronecker coefficient [5]. The projector of the product of irreps to the trivial representation of S_n represented by (n) is

$$\Pi_{(n)}([\alpha] \otimes [\beta] \otimes [\gamma]) = \frac{1}{n!}\sum_{\pi} \chi^{(n)}(\pi) S_{\alpha}(\pi) \otimes S_{\beta}(\pi) \otimes S_{\gamma}(\pi), \tag{24}$$

as $\chi^{(n)}(\pi) = 1$ for each element of S_n, the dimension of the invariant subspace is

$$dim([\alpha]\otimes[\beta]\otimes[\gamma])^{S_n} = tr(\Pi_{(n)}([\alpha]\otimes[\beta]\otimes[\gamma])) = \frac{1}{n!}\sum \chi^{\alpha}(\pi)\chi^{\beta}(\pi)\chi^{\gamma}(\pi) = g_{\alpha\beta\gamma}. \tag{25}$$

The same idea can be generalized when we study the product of more than three irreps; nevertheless, for simplicity, we will only describe here the product of three irreps.

We will focus our study on the elements $|K\rangle$ of this invariant subspace, in particular for $g_{\alpha\beta\gamma} > 2$. By this definition, the states $|K\rangle$ are invariant to any element $S_\alpha(\pi) \otimes S_\beta(\pi) \otimes S_\gamma(\pi)$. We call such states *Kronecker states*, and their invariant properties are the keystone of our research. First of all, let's state mathematically the properties of such states.

- Invariant, as it was discussed before,

$$S_\alpha(\pi) \otimes S_\beta(\pi) \otimes S_\beta(\pi) |K\rangle = |K\rangle \quad , \quad \forall \pi \in S_n, \forall |K\rangle \in ([\alpha] \otimes [\beta] \otimes [\gamma])^{S_n}. \tag{26}$$

- The reduced density matrix of $|K\rangle\langle K|$ in any one-party subsystem $([\alpha], [\beta], [\gamma])$ the result is a multiple of the identity. Let's call ρ_λ the partial trace of $\rho = |K\rangle\langle K|$ over all the subsystems except λ. Due to the first property, ρ_λ is invariant to elements of $[\lambda]$

$$\rho_\lambda = S_\lambda(\pi)\rho_\lambda S_\lambda(\pi)^\dagger \quad , \quad \forall \pi \in S_n. \tag{27}$$

  As λ is an irreducible representation, by Schur's lemma [4] we have that

$$\rho_\lambda \propto \mathbb{1}_\lambda. \tag{28}$$

These two properties have deep meaning in quantum mechanics due to the fact that if all one-party reduced density matrices of a state are completely mixed ($\rho_\lambda \propto \mathbb{1}$), the total density matrix represents a maximally-entangled state, as it was discussed in subsection 3.1.

One effective way to obtain Kronecker states is to take the projection of any state in $[\alpha] \otimes [\beta] \otimes [\gamma]$ to the invariant subspace,

$$|K\rangle = \Pi_{(n)}([\alpha] \otimes [\beta] \otimes [\gamma]) |v\rangle \quad , \quad \forall |v\rangle \in [\alpha] \otimes [\beta] \otimes [\gamma]. \tag{29}$$

For example, in S_3, one triplet of irreps with $g_{\alpha\beta\gamma}$ is $\alpha = \beta = \gamma = (21)$, each one of dimension 2. If we label the basis

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \to |0\rangle , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array} \to |1\rangle , \tag{30}$$

the projector to the invariant subspace can be written as

$$\Pi_{(n)} = \frac{1}{4}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)(\langle 000| - \langle 011| - \langle 101| - \langle 110|). \tag{31}$$

If we project any state in the invariant subspace, lets say $|000\rangle$ we obtain the normalized Kronecker state

$$|K\rangle = \Pi_{(n)} |000\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle), \tag{32}$$

which is the only Kronecker state in this invariant subspace (because $g_{\alpha\beta\gamma} = 1$ it is a one-dimensional space), and their reduced density matrices are

$$\rho_\alpha = tr_{\beta\gamma}(|K\rangle\langle K|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbb{1}_{2\times 2} = \rho_\beta = \rho_\gamma. \tag{33}$$

In the next chapter, we will summarize some problems in quantum information theory, explaining why these properties of Kronecker states could help us to understand and gain information about them.

4 Possible applications of Kronecker states in quantum information theory

4.1 Superadditivity of classical capacity in quantum channels

4.1.1 Classical capacity of a quantum channel

A quantum channel $\mathcal{C}$ describes the most general type of interaction of a quantum state system with an environment E. Any interaction is described with some unitary evolution U, and then the posterior quantum state is separated from the environment, that is

$$\rho \overset{\mathcal{C}}{\rightarrow} tr_E(U(\rho \otimes \omega)U^\dagger). \tag{34}$$

The same action can be described in terms of Kraus operators M_i [6]

$$\sum_{i=1}^{D} M_i M_i^\dagger = \mathbb{1}, \tag{35}$$

for some $D \geq 1$, defining the action on the state as

$$\mathcal{C}(\rho) = \sum_{i}^{D} M_i \rho M_i^\dagger. \tag{36}$$

In this representation, each channel is defined by the D matrices which compose it.

A useful quantity in quantum channels is the Von Neumann entropy

$$H(\rho) = -tr(\rho log \rho) = -\sum_i \lambda_i log \lambda_i \tag{37}$$

where $\vec{\lambda}$ is the set of eigenvalues of ρ.

In general, we are interested in the performance of the channel when we have a set of pure inputs ρ_i with probabilities π_i. Let $\hat{\rho}_i = \mathcal{C}(\rho_i)$ be the output of the channel when the input is ρ_i. Holevo's bound [7] tells us that the accessible information, the maximum amount of information that can be recovered in the output of the channel from all possible encoding and decoding protocols, in this channel has to be bounded by

$$I_{acc} \leq \chi(\mathcal{C}, \vec{\pi}, \{\rho_i\}) = H(\sum_i \pi_i \hat{\rho}_i) - \sum_i \pi_i H(\hat{\rho}_i). \tag{38}$$

The classical channel capacity, or Holevo capacity is then defined as the maximum information that can be achieved with the channel with any possible combination of π_i, ρ_i, that is

$$C(\mathcal{C}) = \sup_{\vec{\pi}, \{\rho_i\}} \chi(\mathcal{C}, \vec{\pi}, \{\rho_i\}). \tag{39}$$

4.1.2 Superadditivity of communication capacity using entangled inputs

For a long time, it was believed that the Holevo capacity of composed quantum channels was additive [8], meaning that if we have two channels $\mathcal{C}_1, \mathcal{C}_2$, then

$$C(\mathcal{C}_1 \otimes \mathcal{C}_2) = C(\mathcal{C}_1) + C(\mathcal{C}_2), \tag{40}$$

where $\mathcal{C}_1 \otimes \mathcal{C}_2$ is the composed channel. In [9], Peter Shor discussed the equivalence between the additivity of Holevo capacity and additivity of minimum output Von Neumann entropy,

$$min(H(\mathcal{C}_1 \otimes \mathcal{C}_2)) = min(H(\mathcal{C}_1)) + min(H(\mathcal{C}_2)), \tag{41}$$

where $min(H(\mathcal{C}))$ is the minimum value of the Von Neumann entropy of all possible inputs in the channel $\mathcal{C}$; concluding that subadditivity in the minimum Von Neumman output entropy means superadditivity in the Holevo capacity. In fact, in 2009, Hastings [10] proved that if a quantum channel $\mathcal{E}$ is built using D random unitary matrices $\mathbf{U}$, and its conjugate $\bar{\mathcal{E}}$ (built with the conjugated matrices $\bar{\mathbf{U}}$) are used to shape a composed channel $\mathcal{C} = \mathcal{E} \otimes \bar{\mathcal{E}}$, it is possible to achieve superadditivity of Holevo capacity with maximally-entangled states as inputs,

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{\alpha=1}^{N} |\alpha\rangle \otimes |\alpha\rangle \,. \tag{42}$$

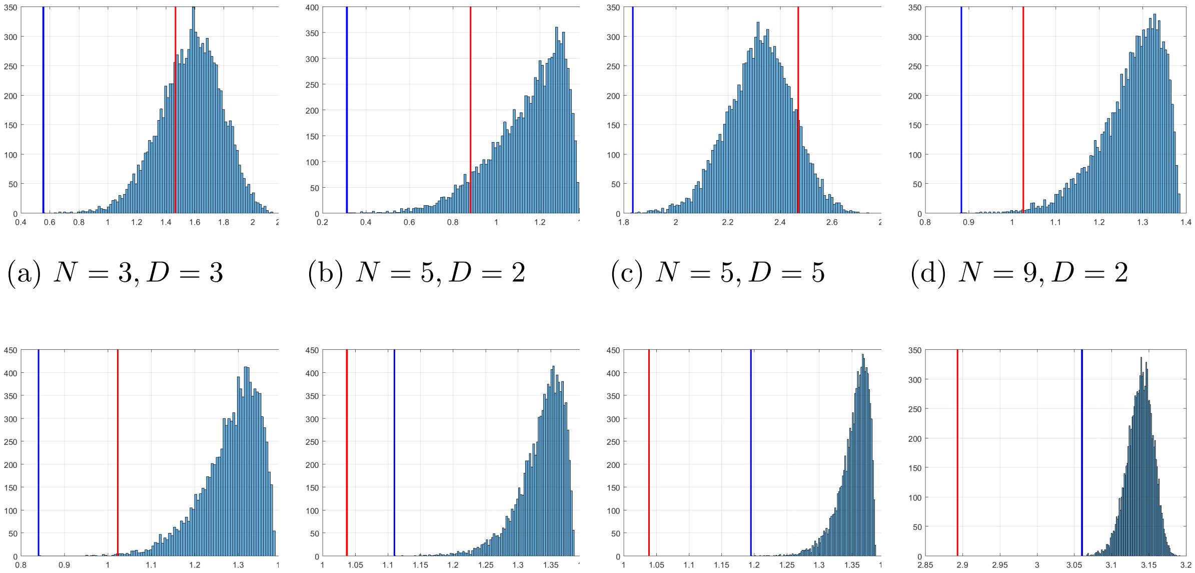
The main reason for this behaviour is that these states are invariant to some elements in the channel,

$$U_i \otimes \bar{U}_i \, |\psi\rangle = |\psi\rangle \,, \tag{43}$$

allowing a lower minimum Von Neumann output entropy when $1 < D << N$ for the composed channel than the sum of its parts

$$min(H(\mathcal{C})) \leq min(H(\mathcal{E})) + min(H(\bar{\mathcal{E}})). \tag{44}$$

This violation implies that superadditivity of Holevo capacity can be achieved. With the purpose of obtaining numerical evidence of this effect, we simulated these channels, measuring the output Von Neumann entropy of random states and MES in the form of Equation 42. It can be seen that the Von Neumann entropy of maximally-entangled states presents a gap with the Von Neumann entropy for random states for big values of N.



(a) $N = 3, D = 3$ (b) $N = 5, D = 2$ (c) $N = 5, D = 5$ (d) $N = 9, D = 2$

(e) $N = 11, D = 2$ (f) $N = 20, D = 2$ (g) $N = 30, D = 2$ (h) $N = 50, D = 5$

Figure 1: Output entropy histograms for input random states (blue) with its respective minimum (blue line) and the output entropy of the N-maximally-entangled state (red line).

This gap becomes bigger as the relation $1 < D << N$ is more strict. Even though these results are not a proof of superadditivity, they give us an idea of how for this case, MES are special in the composed channel.

4.1.3 Kronecker states in quantum channels

Following the ideas of Hastings, we want to build a channel composite of three single channels, let's call them $\mathcal{C}_\alpha, \mathcal{C}_\beta, \mathcal{C}_\gamma$, where their Krauss operators are the representations of the same D permutations in the irreps $[\alpha], [\beta], [\gamma]$ each one with probability $\frac{1}{D}$, such that the output of each channel when its input is the state ρ is

$$\mathcal{C}_\lambda(\rho) = \frac{1}{D}\sum_{i=1}^{D} S_\lambda(\pi_i)\rho S_\lambda(\pi_i)^\dagger. \tag{45}$$

The composite channel $\mathcal{C}_{\alpha\beta\gamma}$ will have an action defined by

$$\mathcal{C}_{\alpha\beta\gamma}(\rho) = \mathcal{C}_\alpha \otimes \mathcal{C}_\beta \otimes \mathcal{C}_\gamma(\rho) = \frac{1}{D^3}\sum_{i,j,k=1}^{D} S_\alpha(\pi_i)\otimes S_\beta(\pi_j)\otimes S_\gamma(\pi_k)\rho S_\alpha(\pi_i)^\dagger \otimes S_\beta(\pi_j)^\dagger \otimes S_\gamma(\pi_k)^\dagger. \tag{46}$$

Note that all Kronecker states are invariant to D elements of the channel

$$S_\alpha(\pi_i)\otimes S_\beta(\pi_i)\otimes S_\gamma(\pi_i)\left|K\right\rangle\left\langle K\right| S_\alpha(\pi_i)^\dagger \otimes S_\beta(\pi_i)^\dagger \otimes S_\gamma(\pi_i)^\dagger = \left|K\right\rangle\left\langle K\right|. \tag{47}$$

We wish to explore whether this property may imply superadditivity of the Holevo capacity for triple channels $\mathcal{C}_{\alpha\beta\gamma}$, but remains additive for channels composed of two irreps ,e.g., $\mathcal{C}_{\alpha\beta} = \mathcal{C}_\alpha \otimes \mathcal{C}_\beta$.

$$\mathcal{C}_{\alpha\beta}(\rho) = \mathcal{C}_\alpha \otimes \mathcal{C}_\beta = \frac{1}{D^2}\sum_{i,j=1}^{D} S_\alpha(\pi_i)\otimes S_\beta(\pi_j)\rho S_\alpha(\pi_i)^\dagger \otimes S_\beta(\pi_j)^\dagger, \tag{48}$$

because for these channels the invariant property disappears.
We will call $\mathcal{C}_{\alpha\beta\gamma}$ a Borromean channel, in reference to Borromean rings [11], a composition of three linked rings where taking any ring out, causes the connection between them to disappear. We have found an interesting feature in these channels: it is possible to build a non-trivial channel where the output entropy of all Kronecker states is the same. We expect that this fact will help us to determine a direct proof of superadditivity in Borromean channels.

4.2 Quantum Secret Sharing (QSS)

4.2.1 Classical secret sharing

Before introducing the QSS scheme, we will explain the basic aspects in the classical case. In classical secret sharing [12][13], a dealer $\mathcal{D}$ distributes a secret $\mathcal{S}$ among a set of players $\mathcal{P}$, such that some groups of players can reconstruct the secret. The access structure Γ is a list of groups in $\mathcal{P}$ called authorized sets, which are able to reconstruct the secret. The access structure has to be monotone, that is

$$(G \in \Gamma) \wedge (G \subseteq G') \to G' \in \Gamma. \tag{49}$$

The sets that cannot reconstruct the secret are called unauthorized sets and shape the adversary structure $\mathcal{A}$.

A secret sharing structure Γ is called perfect if each set in Γ can recover completely the secret and each set in $\mathcal{A}$ can get no information. There is a special class of perfect secret sharing schemes called (t, n) *threshold scheme*, with $1 \leq t \leq n$ and n players, such that its access structure is given by

$$\Gamma = \{G \subseteq \mathcal{P} : |G| \geq t\}. \tag{50}$$

This means that each subset of at least t players can reconstruct the secret, and any subset of less than t players has no information about the secret.

4.2.2 (t,n) Quantum threshold scheme

For QSS [14] the secret $\mathcal{S}$ is a quantum state $|\psi\rangle$ which will be codified in n qubits that will be distributed in some way to n players. The condition for the reconstruction of the secret is it has to be achieved by local operations with only the qubits of any group in the access structure; this change adds a limitation to the access structure for the QSS due to the no-cloning theorem [15]. There are no QSS access structures containing two disjoint sets; otherwise, it would be possible for those sets to reconstruct the secret independently and achieve two copies of the same state $|\psi\rangle$. As an example lets review the $(2, 3)$ quantum threshold scheme [16], indicating the possible application of Kronecker states in this kind of algorithm. For this case we have a qutrit $|\psi\rangle$ which represents the secret $\mathcal{S}$ in the scheme:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle . \tag{51}$$

The goal is to distribute this secret between three ($n = 3$) players in a way that each set of two ($t = 2$) is able to reconstruct the secret and no individual player has information about the secret. This is achieved with the mapping

$$\begin{aligned} V_{2,3}(\alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle) = |\psi'\rangle = \ & \tfrac{1}{\sqrt{3}} [\alpha(|000\rangle + |111\rangle + |222\rangle) \\ & +\beta(|012\rangle + |120\rangle + |201\rangle) \\ & +\gamma(|021\rangle + |102\rangle + |210\rangle)] , \end{aligned} \tag{52}$$

which allows one to send each of three particles to each player. Note that when computing the reduced density matrix in any one-party subsystem, no one has access to any information. For example in the first qutrit,

$$\begin{aligned} \rho_1 = tr_{23}(|\psi'\rangle \langle\psi'|) &= \frac{1}{3} \sum_{i=0}^{2} \langle i| \sum_{j=0}^{2} (\langle j| |\psi'\rangle \langle\psi'| |j\rangle) |i\rangle \\ &= \frac{1}{3}(\alpha^2 + \beta^2 + \gamma^2)(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|) = \frac{1}{3}\mathbb{1}_{3\times 3}, \end{aligned} \tag{53}$$

player one has access to no information, because its reduced density matrix is a multiple of the identity. Here, it is important to remember that for Kronecker states the reduced density matrix in any subsystem of one part is a multiple of the identity, so they also fill the requirement of no information for the adversary structure in this scheme. Continuing with the scheme, we still have to recover the secret using the information of two players. This can be achieved if we have access to the information of two players; lets say that the information of players one and two is available, that is, we can perform local operations in the first two qutrits without affecting the third one. The operation which allows us to recover the secret is

$$\mathcal{R}\left|x,y,z\right\rangle \rightarrow \left|x+y+x,x+y,z\right\rangle \quad mod(3). \tag{54}$$

Applying this transformation over $\left|\psi'\right\rangle$ we obtain

$$\begin{aligned}\mathcal{R}\left|\psi'\right\rangle = \; & \tfrac{1}{\sqrt{3}}\left[\alpha(\left|000\right\rangle + \left|021\right\rangle + \left|012\right\rangle)\right. \\ & +\beta(\left|112\right\rangle + \left|100\right\rangle + \left|121\right\rangle) \\ & \left. +\gamma(\left|221\right\rangle + \left|212\right\rangle \left|200\right\rangle\right],\end{aligned} \tag{55}$$

This state is separable

$$\mathcal{R}\left|\psi'\right\rangle = \frac{1}{\sqrt{3}}\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle + \gamma\left|2\right\rangle\right)\left(\left|00\right\rangle + \left|12\right\rangle + \left|21\right\rangle\right). \tag{56}$$

This means that player one in this case has recovered the secret using the information of player two only, which completes the scheme. We want to implement Kronecker states as codification states in schemes like this, looking for a method that allows us to recover the secret and the implications that it will have.

4.3 Quantum error correction (QEC)

Classical error correction codes are based mainly in the possibility of copying (repetition codes) or measuring (Hamming codes) [17] a set of bits; in the quantum case both approaches are not possible, because first, the no cloning theorem does not allow copying quantum states, and second, measurements over states change the state itself which is not desired in quantum information protocols. In this section we will summarize how to avoid these problems in order to create a correct quantum error correcting code.

4.3.1 3 Qubit Code (3QC)

The example of the 3QC is ideal to understand the basis for QEC [18] . In this case, we want to send a qubit through a noisy quantum channel and in the output recover the initial qubit. For this purpose the computational bases of the qubit is extended using the following mapping.

$$\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle \rightarrow \left|\psi'\right\rangle = \alpha\left|000\right\rangle + \beta\left|111\right\rangle. \tag{57}$$

This mapping can be achieved using the quantum circuit represented in Figure 2, where • represents the control qubit and $\bigoplus$ the target of a controlled not operation (CNOT)

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{58}$$

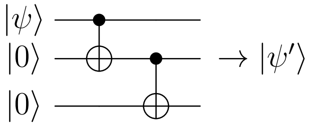


Figure 2: Quantum circuit to extend the computational bases of $|\psi\rangle$

Note how the initial state is not cloned acording to the no-cloning theorem; instead of that, the bases of the state have been extended. Because no measurement of the state has been done, its information α, β remains in the new state of three qubits (q_1, q_2, q_3). The extension of the bases allows us to fix an error when one flip $|0\rangle \rightarrow |1\rangle$ or vice versa is induced by the channel. Similar to the classical case, the codewords (bases) have a Hamming distance $d = 3$ and only $t = 1$ errors can be corrected. Now it is necessary to know which qubit was flipped by the channel; for this, two extra qubits (ancilla) are used for a parity check, permitting us to determine which qubit was flipped.

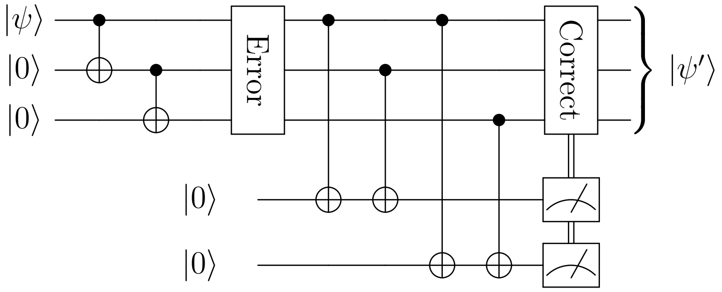


Figure 3: Detection and correction scheme of errors in 3QC

The circuit represented in Figure 3 shows how the error occurs after the bases are extended, then the ancilla qubits (q_4, q_5) are coupled to the system to check the parity of the first three qubits,

$$q_4 = mod_2(q_1 + q_2) \quad , \quad q_5 = mod_2(q_1 + q_3). \tag{59}$$

The CNOT gates in the circuit change the values for the ancilla qubits according to Equation 59. Measuring the ancilla qubits determines exactly which qubit in the system was flipped by the channel. In the next table, the possible flips with their correspondent measures of ancilla qubits are described.

Flip	$\|\psi\rangle_E$	$\|q_4q_5\rangle$
No Flip	$\alpha\|000\rangle+\beta\|111\rangle$	$\|00\rangle$
Flip qubit 1	$\alpha\|100\rangle+\beta\|011\rangle$	$\|10\rangle$
Flip qubit 2	$\alpha\|010\rangle+\beta\|101\rangle$	$\|11\rangle$
Flip qubit 3	$\alpha\|001\rangle+\beta\|110\rangle$	$\|01\rangle$

Each flip (or No flip) results in a different measurement of the ancilla qubits, so with this scheme we can determine which qubit in the system was flipped and correct it with a flip in the respective qubit, recovering the state without errors.

4.3.2 Errors in quantum circuits

Until now we have seen how the 3QC allows us to correct one flip induced by the channel; nevertheless, a bit flip is not the only error that can be induced by the channel. Generally, a state can be modified by the channel with any unitary transformation $\mathcal{U}$, which acts over the state and the environment $|E\rangle$

$$\begin{array}{rlcl} \mathcal{U}: & |0\rangle\otimes|E\rangle & \rightarrow & |0\rangle\otimes|E_{00}\rangle+|1\rangle\otimes|E_{01}\rangle \\ & |1\rangle\otimes|E\rangle & \rightarrow & |0\rangle\otimes|E_{10}\rangle+|1\rangle\otimes|E_{11}\rangle \end{array} \tag{60}$$

In this way, any interaction with the qubit $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$ results in:

$$\begin{array}{rcl} \mathcal{U}(|\psi\rangle\otimes|E\rangle) & = & \alpha(|0\rangle\otimes|E_{00}\rangle+|1\rangle\otimes|E_{01}\rangle)+\beta(|0\rangle\otimes|E_{10}\rangle+|1\rangle\otimes|E_{11}\rangle) \\ & = & (\alpha|0\rangle+\beta|1\rangle)\otimes\frac{1}{2}(|E_{00}\rangle+|E_{11}\rangle) \\ & & +(\alpha|0\rangle-\beta|1\rangle)\otimes\frac{1}{2}(|E_{00}\rangle-|E_{11}\rangle) \\ & & +(\alpha|1\rangle+\beta|0\rangle)\otimes\frac{1}{2}(|E_{01}\rangle+|E_{10}\rangle) \\ & & +(\alpha|1\rangle-\beta|0\rangle)\otimes\frac{1}{2}(|E_{01}\rangle-|E_{10}\rangle). \end{array} \tag{61}$$

As seen above, in any given circumstance, with the right basis choice, the action over the qubit can be collapsed to one of four possible results, making a measurement over the environment. These possible results correspond to the action of Pauli matrices over the initial qubit,

$$\mathbb{1}_{2\times2}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{62}$$

Where σ_x is the bit flip, making the exchange $|0\rangle\leftrightarrow|1\rangle$. σ_z let $|0\rangle$ unchanged and maps $|1\rangle\rightarrow-|1\rangle$, this transformation is called “phase flip”. Additionally $-i\sigma_y$ is a composition of bit flip and phase flip because

$$\sigma_y=i\sigma_x\sigma_z. \tag{63}$$

Finally, the identity matrix is the absence of errors. So, a complete quantum error correction code has to provide a mechanism to measure and differentiate the environment state for each possible error (or absence of it) over the qubit and then correct such error applying the correspondent Pauli matrix.

4.3.3 Stabilizers and the 5 qubit code

One powerful mechanism commonly used in QEC to differentiate the errors using measurements over the environment is the stabilizer group [19]. The stabilizer group $\mathcal{G}$ in the $N-$qubit system is defined as

$$\mathcal{G} = \{K^i | K^i \left|\psi\right\rangle = \left|\psi\right\rangle, [K^i, K^j] = 0, \forall (i,j)\}, \tag{64}$$

where $\left|\psi\right\rangle$ are the codewords of the QEC code, and K^i are generalized Pauli matrices for $N-$qubits ($N-$tensor products of two dimensional Pauli matrices). For example, for a 3 qubit system, the generators of the stabilizer group are

$$\begin{aligned} K^1 &= \sigma_x \otimes \sigma_x \otimes \sigma_x = XXX, \\ K^2 &= \sigma_z \otimes \sigma_z \otimes \mathbb{1}_{2\times 2} = ZZI, \\ K^3 &= \mathbb{1}_{2\times 2} \otimes \sigma_z \otimes \sigma_z = IZZ, \end{aligned} \tag{65}$$

where right hand expressions are the common notation for stabilizers. To find the codewords stabilized by these elements up to normalization we calculate

$$\left|0\right\rangle_L = \prod_{i=1}^{3} (I^{\otimes 3} + K^i) \left|000\right\rangle, \left|1\right\rangle_L = \prod_{i=1}^{3} (I^{\otimes 3} + K^i) \left|111\right\rangle. \tag{66}$$

However, for this case, the codewords are the same,

$$\left|0\right\rangle_L = \left|1\right\rangle_L = \frac{\left|000\right\rangle + \left|111\right\rangle}{\sqrt{2}}, \tag{67}$$

making it impossible to use this set of stabilizers as a QEC code. Nevertheless, it is remarkable that this state, the GHZ state, is a Kronecker state [2].

The smallest stabilizer code that allows one to correct one error in one qubit (two different codewords) is the 5 qubit code defined by the generators

$$\begin{aligned} K^1 &= XZZXI, \\ K^2 &= IXZZX, \\ K^3 &= XIXZZ, \\ K^4 &= ZXIXZ, \end{aligned} \tag{68}$$

whose codewords are

$$\begin{aligned} \left|0\right\rangle_L &= \tfrac{1}{4} (\left|0000\right\rangle + \left|10010\right\rangle + \left|01001\right\rangle + \left|10100\right\rangle \\ &\quad + \left|01010\right\rangle - \left|11011\right\rangle - \left|00110\right\rangle - \left|11000\right\rangle \\ &\quad - \left|11101\right\rangle - \left|00011\right\rangle - \left|11110\right\rangle - \left|01111\right\rangle \\ &\quad - \left|10001\right\rangle - \left|01100\right\rangle - \left|10111\right\rangle + \left|00101\right\rangle), \\ \\ \left|1\right\rangle_L &= XXXXX \left|0\right\rangle_L. \end{aligned} \tag{69}$$

Any single error in one qubit of the codewords can be expressed as a generalized Pauli matrix, e.g., a bit flip in the second qubit is $IXIII$; as the errors E_i belong to the generalized Pauli group, they can only commute or anticommute with the elements in the stabilizer. The structure of this code is designed in a way that each possible error commutes and anticommutes with a different set of generators of the stabilizer group; each combination is known as syndrome and can be measured, determining the error in the state, which can be corrected.

Note that in the reduced density matrix of codewords in Equation 69,for each component in both codewords are

$$\rho_i = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbb{1}_{2\times 2}; \tag{70}$$

we have reasons to think that these states are also cases of Kronecker states, and if so, we want to study a generalization of codes like this to higher dimensions using Kronecker states where the correspondent irreps in the symmetric group are the stabilizer group.

4.4 Entanglement Concentration

As discussed before, maximally-entangled states are important resources in quantum information. Because of this, the question of how to obtain MES has been broadly studied. The method proposed by Bennet et. al [20] consists of taking many copies of partially entangled states and subtracting a smaller number of perfect singlets (MES) from them. Lets consider a bipartite state in the Schmidt basis

$$|\psi\rangle = \sqrt{p_0}\,|00\rangle + \sqrt{p_1}\,|11\rangle\,. \tag{71}$$

If we take n copies of such state, the total state can be represented as

$$|\psi\rangle^{\otimes n} = \sum_{s\in\{0,1\}^n} p_0^{(n-n_1(s))/2} p_1^{n_1(s)/2}\,|s,s\rangle \tag{72}$$

where s are the possible sequences of $0's$ and $1's$ of length n, and $n_1(s)$ is the number of $1's$ in the sequence. If we separate the sequences with the same number of $1's$, i.e., with the same weight $w(s)$, we can write

$$|\psi\rangle^{\otimes n} = \sum_{n_1=0}^{n} \sqrt{\left(\begin{array}{c} n \\ n_1 \end{array}\right)} p_0^{(n-n_1)/2} p_1^{n_1/2} \sum_{w(s)=n_1} \frac{|s,s\rangle}{\sqrt{\left(\begin{array}{c} n \\ n_1 \end{array}\right)}}. \tag{73}$$

If a measurement of the weight is done in the total state, the result will be $n_1 = k$ with probability

$$P(n_1 = k) = \left(\begin{array}{c} n \\ k \end{array}\right) p_0^{n-k} p_1^k, \tag{74}$$

collapsing the total state to

$$|\psi\rangle^{\otimes n} \rightarrow \sum_{w(s)=k} \frac{|s,s\rangle}{\sqrt{\left(\begin{array}{c} n \\ k \end{array}\right)}}, \tag{75}$$

a bipartite MES of dimension $2\left(\begin{array}{c} n \\ k \end{array}\right)$. It can be proved that the expected entropy of entanglement of such states asymptotically ($n \rightarrow \infty$) is $nE - O(log_2 n)$, with E the entropy of entanglement of the original state.

The same problem was studied by Hayashi and Matsumoto [21] with a different focus. They used the Schur-Weyl duality in the Hilbert space of n-copies of an unknown bipartite state

$$\mathcal{H}_{AB}^{\otimes n} = \bigoplus_{\lambda \vdash n} V_{\lambda_A} \otimes V_{\lambda_B} \otimes [\lambda_A] \otimes [\lambda_B] \tag{76}$$

where V_{λ_i} are irreps of $GL(d)$ and $[\lambda_i]$ are the irreps of the symmetric group mentioned in Chapter 2. As the elements $|\psi\rangle_{AB}^{\otimes n}$ are invariant when the same permutation is applied to all parties, they belong to the invariant subspace of the symmetric group,

$$|\psi\rangle_{AB}^{\otimes n} \in \bigoplus_{\lambda \vdash n} V_{\lambda_A} \otimes V_{\lambda_B} \otimes ([\lambda_A] \otimes [\lambda_B])^{S_n}, \tag{77}$$

noting that $([\lambda_A] \otimes [\lambda_B])^{S_n}$ is the subspace to which bipartite Kronecker states belong, defining a subspace of MES that can be subtracted with local operations. The same idea can be extended to multipartite systems [2], in particular if the initial state is in the W SLOCC class, where the entanglement of the initial state is concentrated in the generalized Kronecker states.We want to better understand the structure of these states and to study how they appear in the entanglement concentration of general multipartite states.

4.5 Possible generalizations

The goal of this work is to develop framework for a formalism around Kronecker states, so we also will consider other spaces with similar properties to the invariant subspace of products of irreps of S_n, to which Kronecker states belong. For example, if we project the product space $[\alpha] \otimes [\beta] \otimes [\gamma]$ in the sign representation, the states $|\mathcal{S}\rangle$ belonging to this projection will have the next property

$$S_\alpha(\pi) \otimes S_\beta(\pi) \otimes S_\gamma(\pi) |\mathcal{S}\rangle = (-1)^{sgn(\pi)} |\mathcal{S}\rangle . \tag{78}$$

This means that these states are also ideal to study the case of superadditivity described in Section 3.1, due to the fact that the density matrix $|\mathcal{S}\rangle \langle\mathcal{S}|$ is invariant under the

action of the same elements presented in Equation 47. We want to find up to what point these subspaces are equivalent or different, studying the possibility of labelling them in a general subspace with general properties.
Besides the symmetric group, there are interesting invariant subspaces in other groups such as Lie groups, whose elements have similar properties to Kronecker states. It can be an interesting research topic to extend the knowledge of Kronecker states to other groups.

5 Goals

5.1 General Goal

The main purpose of this research is to start the construction of a formalism that establishes the properties and generalities of Kronecker states and their generalizations as discussed in Section 4.5, and their utility for quantum information processing.

5.2 Specific Goals

- Study the implications of the symmetry properties of Kronecker states when they are used as input states in quantum channels, in particular when the channels are composed of elements of the symmetric group.

- Develop a mechanism to recover secrets encoded with Kronecker states for multipartite systems identifying what advantages can be obtained using this mechanism.

- Analyze different quantum error correction codes, identifying the cases when Kronecker states appear as codewords and the possibility of making a generalization of such codes to higher dimensions using Kronecker states.

- Study how Kronecker states appear in the entanglement concentration method for general multipartite states and what conditions make possible the extraction of their entanglement.

- Summarize and relate the results obtained in each item in a way that allows one to understand in a general form Kronecker states.

6 Schedule

Trimester	1	2	3	4	5	6	7	8
Bibliographical revision	x	x	x		x	x		
Study of Kronecker states in Quantum Channels	x	x						
Study of Kronecker states in QSS		x	x	x				
Study of Kronecker states in QEC			x	x	x			
Study of Kronecker states in entanglement concentration					x	x	x	
Study of invariant states in other representations or groups						x	x	x
Analyze connections between the different cases in order to describe generally Kronecker states						x	x	x
Write and correct the final document							x	x

7 Ethical considerations

This project will be carried out with a theoretical approach. Some simulations will be developed to support theoretical conclusions. Simulations and results obtained from them will be published on a repository, which will be available for anyone who wants to reproduce the calculations.
This project has nothing to do with human or animal investigation. Due to the nature of the research and the absence of any possible conflict of interest, we consider that it does not have to be studied by the ethics committee.

References

[1] Mark Wilde. From classical to quantum shannon theory. 06 2011.

[2] Alonso Botero and José Mejía. Universal and distortion-free entanglement concentration of multiqubit quantum states in the $\mathcal{W}$ class. *Phys. Rev. A*, 98:032326, Sep 2018.

[3] J. I. de Vicente, C. Spee, and B. Kraus. Maximally entangled set of multipartite quantum states. *Phys. Rev. Lett.*, 111:110502, Sep 2013.

[4] K Audenaert. A digest on representation theory of the symmetric group. 01 2006.

[5] Matthias Christandl, Aram W. Harrow, and Graeme Mitchison. Nonzero kronecker coefficients and what they tell us about spectra. *Communications in Mathematical Physics*, 270:575–585, 03 2007.

[6] John Preskil. A course on quantum computation, 2004.

[7] Peter W. Shor. Capacities of quantum channels and how to find them. *Mathematical Programming*, 97(1):311–335, 2003.

[8] Peter W. Shor. The additivity conjecture in quantum information theory. *Current Developments in Mathematics*, 2005, 01 2007.

[9] Peter W. Shor. Equivalence of additivity questions in quantum information theory. *Communications in Mathematical Physics*, 246(3):453–472, Apr 2004.

[10] M. B. Hastings. Superadditivity of communication capacity using entangled inputs. *Nature Physics*, 5:255, 2009.

[11] Peter Cromwell, Elisabetta Beltrami, and Marta Rampichini. The mathematical tourist. *The Mathematical Intelligencer*, 20(1):53–62, Mar 1998.

[12] Karin Rietjens. An information theoretical approach to quantum secret sharing schemes. 2004.

[13] Amos Beimel. Secret-sharing schemes: A survey. In *Coding and Cryptology*, pages 11–46, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.

[14] M. Hillery, V. Buzek, and A. Berthiaume. Quantum secret sharing. *Physical Review A*, 59(3):1829–1834, 1999.

[15] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299:802, October 1982.

[16] K.P.T. Rietjens. An information theoretical approach to quantum secret sharing schemes. masther thesis, 2006.

[17] R. W. Hamming. Error detecting and error correcting codes. *The Bell System Technical Journal*, 29(2):147–160, April 1950.

[18] Simon J. Devitt, Kae Nemoto, and William J. Munro. Quantum Error Correction for Beginners. *Reports on Progress in Physics*, 76(7):076001, July 2013. arXiv: 0905.2794.

[19] Daniel Gottesman. Class of quantum error-correcting codes saturating the quantum hamming bound. *Phys. Rev. A*, 54:1862–1868, Sep 1996.

[20] Charles Bennett, Herbert Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations. *Physical Review A*, 53, 12 1995.

[21] Masahito Hayashi and Keiji Matsumoto. Universal distortion-free entanglement concentration. *Physical Review A*, 75, 10 2002.