Index theory and implementability of unitary representations of CAR algebras

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Dedicada a mi Mamá, mi Hermano y Minnie

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Contents

1 Introduction

The Algebra of Canonical Anticommutation Relations –*CAR* algebra– was introduced in 1928 by Jordan and Wigner in the quantization of electron field framework, and then it became more general studying fermionic systems [9]. For the study of these systems, an entire area of mathematics has been developed, namely C^*-algebras, which mixes tools from functional analysis with representation theory. Similarly, there exists another highly useful structure in the context of quantum field theory and quantum statistical mechanics, the one of Clifford algebra. This algebraic object was presented by William Kingdon Clifford, and was introduced by Dirac, Jordan and Wigner in these physical frameworks. Moreover, these algebras have had a strong impact in the development of areas such as spin geometry, non-commutative geometry and K-theory [21].

Alan Carey, Charles Hurst and Denis O'Brien, in [5] and [6], introduce the first connection between the study of the implementation of unitary symmetries in these classes of algebras with index theory, pointing out the topological properties of certain groups of automorphisms. The analysis of this topological description, together with the index theory adapted for these spaces, is the main object of study of this work. In particular, a description of the topological obstructions to the implementability of symmetries, from a point of view of K-theory, and its relationship with the theory of representations is discussed. This characterization is analyzed and described in the framework of topological insulators in the so-called *Bulk-Edge correspondence*, where these objects arise naturally in order to describe fermionic chains with periodic configuration and the so called *edge states*[22].

In chapter 2 we present the construction of the *CAR*-algebra and the definition of the Clifford algebra over a Hilbert space H. In addition, it is observed that any unitary bounded operator U on H induces an automorphism on the respective algebra, called the *Bogoliubov automorphism* associated to U. With these automorphisms, the problem of when they can be implemented in a fixed irreducible representation arises. The answer to this question turns out to be associated with topological properties of the group of unitary –complex case– and orthogonal –real case– operators on the Hilbert space. These topological properties are studied in detail in chapter 3, where it is observed that these topologies, which emerge inherently from the implementation problem, give rise to the use of a Fredholm index of type $\mathbb{Z}$ or $\mathbb{Z}_2$ according to the symmetry imposed, i.e. conditions that unitary operators must satisfy. We follow [5] and [6], supplying enough detail in the proofs there where we feel it is required.

Finally, in chapter 4, following [27] and [22], these tools are used to study a physical periodic model for a topological insulator, in particular the *Bulk-Edge correspondence*. In order to connect it with the theory developed, some generalities of K-theory for C^*-algebras are discussed, and the relation with Fredholm operators is addressed. The mentioned connection corresponds to study the *six term exact sequence of K groups* associated to the short exact sequence for the Toeplitz algebra. Thus, the Bulk-Edge correspondence can be understood from a topological invariant of the model. At the end of the chapter, two questions are formulated: Is the resulting invariant of type $\mathbb{Z}$ or type $\mathbb{Z}_2$?, and what is the operator associated with the invariant obtained according to the analysis carried out in chapter two and three?

2 CAR algebras, Clifford algebras and the Fock representation

In this chapter we shall present the construction of the algebra of *Canonical Anticommutation Relations*, called CAR-algebra, and the definition of the Clifford algebra associated to a separable Hilbert space. Additionally, we shall discuss some generalities about the irreducible representations of these algebras over the Hilbert space, starting with the called *Fock Representation*. Once the representation is constructed, we will introduce the *Bogoliubov automorphisms* over the algebra, which arise from unitary operators acting on the representation space. With these automorphisms, the problem of when they can be understood as automorphism of the Fock representation arises. We shall observe that the answer of this question is associated with the topological properties of the group of unitary –complex case– and orthogonal –real case– operators.
We will follow de la Harpe and Jones [9] (see §7 and §8) for the description of the CAR-algebra Then, we shall develop a similar characterization for the Clifford algebra, where we will follow Plymen and Robinson [21] (see §1 and §2).

2.1 CAR Algebra

2.1.1 Existence and Uniqueness

Before giving a formal definition for the CAR algebra, we will present a construction where this algebraic structure arises naturally. Let H_1, H_2 be complex Hilbert spaces, and let $H_1 \bigodot H_2$ denote the algebraic tensor product of these vector spaces. Let $\beta : H_1 \times H_2 \times H_1 \times H_2 \to \mathbb{C}$ be the multilinear form given by:

$$(\xi_1, \xi_2, \eta_1, \eta_2) \mapsto \langle \xi_1, \eta_1 \rangle_1 \cdot \langle \xi_2, \eta_2 \rangle_2,$$

where $\langle \cdot, \cdot \rangle_i$ denotes the inner product in H_i for $i = 1, 2$. We will use the convention: linearity in the second component of $\langle \cdot, \cdot \rangle_i$ and anti-linearity in the first component. By properties of the tensor product and using that this map is $\mathbb{C}$-bilinear in the two last factors and anti-bilinear in the two first factors, we can construct a sesquilinear form

$$\langle \cdot, \cdot \rangle : H_1 \bigodot H_2 \times H_1 \bigodot H_2 \to \mathbb{C}$$

given by

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle := \langle \xi_1, \eta_1 \rangle \cdot \langle \xi_2, \eta_2 \rangle.$$

This sesquilinear form defines an inner product on the tensor space $H_1 \bigodot H_2$. However, this inner product does not necessarily make the algebraic tensor product of Hilbert spaces a Hilbert space –in general, it is not complete. Thus, we must consider the completion of this space with respect to this scalar product, and we denote this new space by $H_1 \otimes H_2$, which will be now a Hilbert space with the scalar product given above. From this on, we refer to this space as the tensor Hilbert space. Given $n \in \mathbb{N}$ and n Hilbert spaces H_n, we can construct the tensor Hilbert space $H_1 \otimes H_2 \otimes \ldots \otimes H_n$ similarly.

Definition 2.1.1. *(The full Fock space of a Hilbert space) Let H be a Hilbert space. For each integer $n \geq 0$ set $H^{\otimes n} := H \otimes \ldots \otimes H$ (n copies), where $H^{\otimes 0} = \mathbb{C}$ and $H^{\otimes 1} = H$. The full Folk space associated to H is the Hilbert space direct sum*

$$EXP(H) := \bigoplus_{n \geq 0} H^{\otimes n}.$$

For each $\xi \in H$ define $Exp(\xi) := \bigoplus_{n \geq 0} \frac{1}{\sqrt{n!}} \xi^{\otimes n}$.

De la Harpe and Jones show that $\langle Exp(\xi), Exp(\eta) \rangle = e^{\langle \xi, \eta \rangle}$, for $\xi, \eta \in H$ (see [9] §7).

Let $\xi \in H$ and let $n \in \mathbb{N}$. Given the function $\eta_1 \otimes \ldots \otimes \eta_n \mapsto \xi \otimes \eta_1 \otimes \ldots \otimes \eta_n$ over irreducible tensors from $H^{\otimes n}$ to $H^{\otimes (n+1)}$, and extending by linearity, we get a bounded operator

$$l(\xi) : H^{\otimes n} \rightarrow H^{\otimes (n+1)}$$

of norm $||\xi||$ (see [9] §7). Since this norm is independent of the integer n, the direct sum of these over $n \geq 0$ is a bounded operator $l(\xi) : EXP(H) \rightarrow EXP(H)$ of norm $||\xi||$.
Furthermore, observe that

$$\langle \eta_0 \otimes \eta_1 \otimes \ldots \otimes \eta_n, \xi \otimes \eta_1' \otimes \ldots \otimes \eta_n' \rangle = \langle \langle \xi, \eta_0 \rangle \eta_1 \otimes \ldots \otimes \eta_n, \eta_1' \otimes \ldots \otimes \eta_n' \rangle;$$

thus the adjoint operator $l(\xi)^*$ is given by:

$$l(\xi)^*(\eta_0 \otimes \eta_1 \otimes \ldots \otimes \eta_n) = \langle \xi, \eta_0 \rangle \eta_1 \otimes \ldots \otimes \eta_n.$$

, we have the relation

$$l(\eta)^* l(\xi) = \langle \eta, \xi \rangle \mathbf{1}_{Exp(H)},$$

for all $\xi, \eta \in H$.
Now, let us consider, for $n \geq 1$, the unitary representation of the symmetric group $\mathcal{S}_n$ over $H^{\otimes n}$, $\sigma \mapsto u_\sigma$, defined by

$$u_\sigma(\eta_1 \otimes \ldots \otimes \eta_n) := \eta_{\sigma(1)} \otimes \ldots \otimes \eta_{\sigma(n)}.$$

We define the space $\bigwedge^n H$ as the subspace of $H^{\otimes n}$ of vectors on which $\mathcal{S}_n$ acts by signature, i.e, the vector subspace of $H^{\otimes n}$ given by:

$$\{\chi \in H^{\otimes n} \mid u_\sigma(\chi) = (-1)^\sigma \chi \ \forall \sigma \in \mathcal{S}_n\}.$$

Denoting by $P_n = \frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(-1)^\sigma u_\sigma$ the orthogonal projection of $H^{\otimes n}$ over $\bigwedge^n H$ (see [9] §7), we write

$$\xi_1 \wedge \ldots \wedge \xi_n = \sqrt{n!}P_n(\xi_1 \otimes \ldots \otimes \xi_n).$$

From this it is clear that

$$\xi_{\sigma(1)} \wedge \ldots \wedge \xi_{\sigma(n)} = (-1)^\sigma \xi_1 \wedge \ldots \wedge \xi_n.$$

Definition 2.1.2. *The Fock space of H is defined as the Hilbert space direct sum*

$$\mathcal{F}(H) := \bigoplus_{n\geq 0} \bigwedge^n H,$$

where $\bigwedge^0 H = \mathbb{C}$ by convention.

For the previous definition, we underline that the Hilbert space direct sum corresponds to the completion of the vector space given by the algebraic direct sum. Notice that this space inherits the inner product of $Exp(H)$. The following lemma will allow us to define the *creation* and *annihilation* operators (see [9] §7).

Lemma 2.1.3. *Let H_1, H_2 be two Hilbert spaces. Let $n \geq 1$ and let $L : H_1^{\otimes n} \to H_2$ be a bounded operator such that $L \circ u_\sigma = (-1)^\sigma L$ for all $\sigma \in \mathcal{S}_n$. Then $L(\xi_1 \wedge \ldots \wedge \xi_n) = \sqrt{n!}\; L(\xi_1 \otimes \ldots \otimes \xi_n)$.*

Let $\xi \in H$ and let $n \geq 0$. Define the linear operator $A_n(\xi) : H^{\otimes n} \to \bigwedge^{n+1} H$ by:

$$A_n(\xi)(\eta_1 \otimes \ldots \otimes \eta_n) = \frac{1}{\sqrt{n!}}\xi \wedge \eta_1 \wedge \ldots \wedge \eta_n = \sqrt{\frac{(n+1)!}{n!}}P_{n+1}(\xi \otimes \eta_1 \otimes \ldots \otimes \eta_n).$$

It is a bounded operator with norm at most $\sqrt{\frac{(n+1!)}{n!}}||\xi||$. According to previous lemma, since $A_n(\xi) \circ u_\sigma = (-1)^\sigma A_n(\xi)$, we have that

$$A_n(\xi)(\eta_1\wedge\ldots\wedge\eta_n) = \sqrt{n!}\; A_n(\xi)(\eta_1\otimes\ldots\otimes\eta_n) = \sqrt{(n+1)!}P_{n+1}(\xi\otimes\eta_1\otimes\ldots\otimes\eta_n) = \xi\wedge\eta_1\wedge\ldots\wedge\eta_n.$$

Let us denote by $a_n(\xi)$ the restriction of $A_n(\xi)$ to $\bigwedge^n H$, $a_n(\xi)(\eta_1 \wedge \ldots \wedge \eta_n) = \xi \wedge \eta_1 \wedge \ldots \wedge \eta_n$. Observe that

$$\begin{aligned} ||a_n(\xi)(\eta_1 \wedge \ldots \wedge \eta_n)|| &= ||A_n(\xi)(\sqrt{n!}P_n(\eta_1 \otimes \ldots \otimes \eta_n))|| \\ &\leq \sqrt{\frac{(n+1)!}{n!}}||\xi||\; ||\eta_1 \wedge \ldots \wedge \eta_n|| = \sqrt{n+1}||\xi||\; ||\eta_1 \wedge \ldots \wedge \eta_n||. \end{aligned}$$

Although this bound has a dependence on n, de la Harpe and Jones observe that it does not depend on it (see [9] §7). Moreover, they compute an explicit expression for the adjoint operator, $a_n(\xi)^*$, and some relations that these operators satisfy. We summarize these results in the following proposition.

Proposition 2.1.4. *Let $\xi, \eta \in H$. We have that the adjoint operator $a_n^*(\xi) : \bigwedge^{n+1} H \to \bigwedge^n H$ is given by:*

$$a_n(\xi)^*(\alpha_1 \wedge \ldots \wedge \alpha_{n+1}) = \sum_{j=1}^{n+1} (-1)^{j+1} \langle \xi, \alpha_j \rangle \alpha_1 \wedge \ldots \wedge \hat{\alpha_j} \wedge \ldots \wedge \alpha_{n+1},$$

where $\hat{\alpha_j}$ denotes absence of this term in the wedge product. Moreover, the following relations hold:

$$a_n(\xi)^* \circ a_n(\eta) + a_{n-1}(\eta) \circ a_{n-1}^*(\xi) = \langle \xi, \eta \rangle \mathbf{1}_{\bigwedge^n H},$$

$$a_{n+1}(\xi) \circ a_n(\eta) + a_{n+1}(\eta) \circ a_n(\xi) = 0.$$

As a consequence of this proposition, we have that for each $\xi \in H$, $\xi \neq 0$, and for all $n \geq 0$, the operator $\frac{1}{||\xi||^2} a_n(\xi)^* a_n(\xi)$ is a projection operator on $\bigwedge^n H$ and further $||a_n(\xi)|| = ||\xi||$ (see [9] §7). These results motivate the following definition:

Definition 2.1.5. *For each $\xi \in H$, the creation operator*

$$a(\xi) : \mathcal{F}(H) \to \mathcal{F}(H)$$

is defined as the direct sum of the operators $a_n(\xi)$ on the $\bigwedge^n H$'s subspaces. Its adjoint operator $a^*(\xi)$ is called annihilation operator.
The CAR algebra of H is defined as the C^*-algebra of operators on $\mathcal{F}(H)$ generated by the creation operators, and it is denoted $CAR(H)$.

De la Harpe and Jones show that the creation operator is bounded with norm equal to $||\xi||$ (see [9] §7). Thus, $a(\xi), a^*(\xi) \in \mathcal{B}(\mathcal{F}(H))$.

Proposition 2.1.6. *The CAR algebra $CAR(H)$ is a unital C^*-algebra and the map*

$$a : H \to CAR(H) \text{ given by } \xi \mapsto a(\xi)$$

is a linear isometry. Moreover, the creation and annihilation operators fulfill the relations:

$$a^*(\xi)a(\eta) + a(\eta)a^*(\xi) = \langle \xi, \eta \rangle \mathbf{1},$$

$$a(\xi)a(\eta) + a(\eta)a(\xi) = 0,$$

for all $\xi, \eta \in H$.

From a more general point of view, the CAR algebra can be defined as the C^*-algebra generated by the symbols $a(\xi)$, with $\xi \in H$, which satisfy the relations of the above proposition, and where

$$a : H \to CAR(H)$$

is a linear map. The construction before shows how to build in a canonical way the algebra of Canonical Anticommutation Relations, acting explicitly on the Fock space associated to the separable Hilbert space H. Actually, the existence and unicity of the CAR algebra can be ensured independently of this construction, for any separable Hilbert space H, as it is shown by de la Harpe and Jones (see [9] §7).

Theorem 2.1.7. *Let H be a complex Hilbert space. There exists a unital C^*-algebra, $CAR(H)$, and a linear map $a : H \to CAR(H)$ such that:*

i. as a C^*-algebra, $CAR(H)$ is generated by $a(H)$,

ii. we have the CAR relations:

$$a^*(\xi)a(\eta) + a(\eta)a^*(\xi) = \langle \xi, \eta \rangle \mathbf{1},$$

$$a(\xi)a(\eta) + a(\eta)a(\xi) = 0.$$

Moreover, the pair $(a, CAR(H))$ is unique in the following sense: For any pair $(a', CAR(H)')$ satisfying above properties, there exists an isomorphism $\phi : CAR(H) \to CAR(H)'$ such that $a' = \phi \circ a$.

Let us now consider a unitary operator u on H. Let $a_u : H \to CAR(H)$ be the linear map given by $\xi \mapsto a(u(\xi))$. Since u is unitary, it is straightforward to show that a_u fulfills the conditions of last theorem. Therefore, there exists an isomorphism

$$Bog(u) : CAR(H) \to CAR(H)$$

such that, for all $\xi \in H$, the following holds:

$$Bog(u)(a(\xi)) = a(u(\xi)).$$

The resulting map $Bog : \mathcal{U}(H) \to \mathrm{Aut}(CAR(H))$, where $\mathcal{U}(H)$ denotes the group of unitary bounded operators on H, can be understood as a representation of $\mathcal{U}(H)$ which is continuous if $\mathcal{U}(H)$ has the strong topology and $\mathrm{Aut}(CAR(H))$ has the pointwise convergence topology (see [9] §7).

Definition 2.1.8. *The automorphisms $Bog(u) : CAR(H) \to CAR(H)$ above are called Bogoliubov automorphisms.*

2.1.2 Representations

In this subsection we shall study some properties of the representations of the CAR algebra. In order to perform this analysis, we shall present first the main results related to the Gelfand-Naimark-Segal mechanism for the construction of irreducible representations. For this part we follow de la Harpe and Jones (see [9] §6) and Sunder (see [28] §3), where a detailed development of the theory can be found.

Let us start with some preliminary results. Let A be a C^*-algebra. A linear form $\phi : A \to \mathbb{C}$ is called positive if $\phi(a^*a) \geq 0$ for all $a \in A$. If A is unital these linear functionals are bounded and, moreover, they satisfy $||\phi|| = \phi(1)$ (see [28] §3) . Furthermore, any bounded operator $\phi : A \to \mathbb{C}$ with $||\phi|| = \phi(1)$ is positive (see [28] §3). The linear functionals with these properties are a fundamental component for the construction of representations over any C^*-algebra A, as we will see.

Definition 2.1.9. *A state ω on a C^*-algebra A is a linear form on A which is positive and of norm equal to 1. The state space S_A is the set of all states on A.*
A state ϕ is called pure if it has the following property: if ϕ_0, ϕ_1 are states on A and if $t \in (0,1)$ is a real number such that $\phi = (1-t)\phi_0 + t\phi_1$, then $\phi_0 = \phi_1$.

Example: Let H be a complex Hilbert space and let $A \subseteq \mathcal{B}(H)$ a C^*-subalgebra which contains the identity map $\mathbf{1}_H$. Let $\xi \in H$ be a vector of norm 1. Then the linear form $\omega_\xi : A \to \mathbb{C}$ given by $a \mapsto \langle \xi, a(\xi) \rangle$ is positive and $||\omega_\xi|| = 1$. We call ω_ξ a *vector state*.
A remarkable property for the states over a C^*-algebra A is that for any $a \in A$, $a \neq 0$, there exists one, say ϕ_a, which satisfies $\phi_a(a^*a) \geq 0$.

Definition 2.1.10. *Let A be a C^*-algebra. Two representations $\pi : A \to \mathcal{B}(H_\pi)$, $\rho : A \to \mathcal{B}(H_\rho)$ are called equivalent if there exists a surjective isometry $u : H_\pi \to H_\rho$ such that $\rho(a) = u\pi(a)u^*$ for all $a \in A$. Similarly, two states ϕ_1, ϕ_2 are equivalent if their GNS representations π_{ϕ_1} and π_{ϕ_2} are equivalent.*

Theorem 2.1.11. ***GNS Construction (see [28] §3).*** *Let A be a unital C^*-algebra and let $\phi : A \to \mathbb{C}$ be a state.*

i. Then there exist

a) a complex Hilbert space H_ϕ,

b) a representation $\pi_\phi : A \to \mathcal{B}(H_\phi)$,

c) a vector $\xi_\phi \in H_\phi$ of norm 1

such that $\phi(a) = \langle \xi_\phi, \pi_\phi(a)\xi_\phi \rangle$ for all $a \in A$, and such that ξ_ϕ is cyclic for π_ϕ, i.e. it satisfies that Closure $\pi_\phi(A)\xi_\phi = H_\phi$.

ii. The triple $(H_\phi, \pi_\phi, \xi_\phi)$ is unique modulo isomorphism in the following sense:
Let H be a Hilbert space, let $\pi : A \to \mathcal{B}(H)$ be a representation and let ξ be a unit vector such that $\phi(a) = \langle \xi, \pi(a)\xi \rangle$ for all $a \in A$ and such that ξ is cyclic for π. Then there exists an unitary isomorphism $u : H_\phi \to H$ such that $\pi(a) = u\pi_\phi u^*$ for all $a \in A$, and such that $u(\xi_\phi) = \xi$.

Two results of great importance follow from this theorem; they relate states with irreducible representations for a C^*-algebra A. Thus, in some sense, we can translate the problem of irreducible representations for a C^*-algebras into a language which involves states. The following two propositions show how it is possible to perform the mentioned translation. These are proved by de la Harpe and Jones [9] in §6.

Proposition 2.1.12. *Let A be an unital C^*-algebra, let ϕ be a state on A and let $\pi_\phi : A \to \mathcal{B}(H_\phi)$ be the representation obtained by the GNS construction. Then the representation π_ϕ is irreducible if and only if the state ϕ is pure.*

Proposition 2.1.13. *Let A be a C^*-algebra, let $\pi : A \to \mathcal{B}(H)$ a representation of A, let $\xi \in H$ be a unit vector and let $\phi : A \to \mathbb{C}$ the state given by $\phi(a) = \langle \xi, \pi(a)\xi \rangle$. Then the GNS construction associated to the state ϕ, π_ϕ, is a subrepresentation of π. In particular, if π is irreducible, then π and π_ϕ are unitarily equivalent.*

Example: The construction of the Hilbert space H_ϕ introduced above in the theorem 2.1.11 follows by identifying the closed left sided ideal $V_\phi = \{a \in A | \; \phi(a^*a) = 0\}$. With this subspace, the quotient A/V_ϕ becomes an inner product vector space whose inner product is given from ϕ as:

$$\langle a + V_\phi, b + V_\phi \rangle := \phi(a^*b).$$

The completion of A/V_ϕ with this inner product is what in this theorem is denoted as H_ϕ. Let us consider $A = M_n(\mathbb{C})$, and let $\phi : A \to \mathbb{C}$ be the state given by $\phi(a) = \text{Tr }(ap)$, where p is the projection

$$p = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

In this case, V_ϕ is given by the subspace of all matrices of the form:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix},$$

with $a_{ij} \in \mathbb{C}$. Since we are in a finite dimensional space, A/V_ϕ is complete with respect to the inner product defined as in theorem 2.1.11. Note that

$$\langle b + V_\phi, a + V_\phi \rangle = \sum_{k=1}^{n} (b_{k1})^* a_{k1}.$$

Let $\Psi : A \to \mathbb{C}^n$ be the linear map defined by $\Psi(a) = a(e_1)$, where e_1 denotes the first canonical basis vector of $\mathbb{C}^n$. This morphism is clearly surjective and its kernel is, precisely, V_ϕ. Thus, $A/V_\phi \cong \mathbb{C}^n$ isometrically. The representation π_ϕ is exactly the tautological representation of $M_n(\mathbb{C})$ on $\mathbb{C}^n$. This representation is known to be irreducible, therefore ϕ is pure.

These are the most technical results for this subsection. We will need them in order to understand the implementation problem of the Bogoliubov automorphisms –which will be introduced in the next subsection. We shall observe that this problem is solved by making use of the formalism of states associated to a C^* algebra A. Traditionally, these states on $CAR(H)$ are called, in the physics literature, as *quasi-free* states.

Given a state ω on the CAR algebra $CAR(H)$, we can associate a sesquilinear form on

H, $\langle\cdot,\cdot\rangle_\omega$, as follows: for $\xi,\eta\in H$, set $\langle\xi,\eta\rangle_\omega=\omega(a^*(\xi)a(\eta))$. It satisfies the following inequality:

$$|\langle\xi,\eta\rangle_\omega|\le||a^*(\xi)a(\eta)||\le||\xi||\,||\eta||,$$

where we have used that $||\omega||=1$ and the isometry property for a (see [9] §8). Hence, by the *Riesz representation theorem*, there exists a bounded linear operator $b\in\mathcal{B}(H)$ constructed as follows:
Fixing ξ, we have that there exists $\theta\in H$ such that the linear map $\langle\xi,\cdot\rangle_\omega$ is equal to $\langle\theta,\cdot\rangle$. Setting $\tilde{b}(\xi):=\theta$, we have then $\langle\xi,\eta\rangle_\omega=\langle\tilde{b}(\xi),\eta\rangle$. This operator $\tilde{b}$ is linear and bounded: by the Riesz theorem, $||\tilde{b}(\xi)||=||\langle\xi,\cdot\rangle||\le||\xi||$. Set $b:=\tilde{b}^*$; thus we obtain:

$$\langle\xi,\eta\rangle_\omega=\langle\xi,b(\eta)\rangle.$$

Furthermore, with these definitions, notice that $\langle\xi,b(\xi)\rangle=\omega(a^*(\xi)a(\xi))\ge0$ since ω is a state. In this order, the operator b is positive, and fulfills $0\le b\le\mathbf{1}$.

Proposition 2.1.14. *(see [9] §8) Let $b\in\mathcal{B}(H)$ a self-adjoint operator such that $0\le b\le\mathbf{1}$. Then there exists a unique state ϕ_b on $CAR(H)$ such that:*

$$\phi_b(a^*(\xi_m)a^*(\xi_{m-1})\cdots a^*(\xi_1)a(\eta_1)a(\eta_2)\cdots a(\eta_n))=\delta_{m,n}\det(\langle\xi_j,b(\eta_k)\rangle_{a\le j\le m,1\le k\le n}),$$

for all $\xi_1,\ldots,\xi_m,\eta_1,\ldots,\eta_n\in H$.

Definition 2.1.15. *The state ϕ_b on $CAR(H)$ is called the quasi-free state of covariance b.*

The proposition above is stronger than we need. In fact, we will be dealing with projections p on the Hilbert space H. In [9], de la Harpe and Jones prove the statement for a projection over a finite dimensional Hilbert space and, then, for the infinite dimensional case, they prove that it is possible to construct a dense tower of finite dimensional subspaces such that, in each level, there is the state ϕ_n.

Corollary 2.1.16. *(see [9] §8) Let $p\in\mathcal{B}(H)$ be a projection. Then the quasi-free state ϕ_p of covariance p is pure.*

In principle, the construction for the self-adjoint operator b from the state ω is not related with the state of covariance b, since in the first we only consider a two-point function. Nevertheless, using the CAR relations, the GNS construction for the state ω and the uniqueness expressed in the proposition 2.1.14, we can conclude the following assertions (see [9] §8):

Proposition 2.1.17. *A state ω on $CAR(H)$ with two-point function of the form $\omega(a^*(\xi)a(\eta))=\langle\xi,p(\eta)\rangle$, with p a projection on H, is exactly the quasi-free state ϕ_p of covariance p.*

We can then conclude that given a projection p on H, we can consider the pure state of covariance p, and thus obtain an irreducible representation for the algebra $CAR(H)$. Moreover, this state ϕ_p can be easily understood as a function acting on $H\otimes H$ as $(\xi,\eta)\mapsto(a^*(\xi),a(\eta))\mapsto\omega(a^*(\xi)a(\eta))$.

Although the GNS construction gives explicitly the representation space H_ϕ, we shall observe that this space can be constructed from the Fock space $\mathcal{F}(H)$. Given a complex Hilbert space K, let us denote by $\overline{K}$ the complex conjugate of K. This space $\overline{K}$ has the same elements and additive group structure than K, and its scalar product is given by: $\alpha \cdot x = \overline{\alpha} x$ for all $x \in K$ and $\alpha \in \mathbb{C}$. There exists an $\mathbb{R}$-linear bijection $K \to \overline{K}$, $x \mapsto \overline{x}$, such that $\overline{(zx)} = \overline{z}\ \overline{x}$ and $\langle \overline{x}, \overline{y} \rangle_{\overline{K}} = \langle \overline{y}, \overline{x} \rangle_K$ for all $z \in \mathbb{C}$ and $x, y \in K$. Moreover, there exists a canonical isomorphism $\mathcal{F}(\overline{K}) \cong \overline{\mathcal{F}(K)}$.

Let p be a projection on H. We define the Hilbert space

$$\Gamma_p(H) = \mathcal{F}(\overline{p^\perp(H)}) \bigotimes \mathcal{F}(p(H)),$$

where $p^\perp := \mathbf{1} - p$. Let $\Omega_p = 1 \otimes 1$ be the vector where 1 corresponds to the unit vector of the level zero for each factor.

Let D be the *parity operator* on $\mathcal{F}(\overline{p^\perp(H)})$ given by:

$$D(x) = \begin{cases} x & \text{if } x \in \bigoplus_{n\geq 0} \bigwedge^{2n}(\overline{p^\perp(H)}) \\ -x & \text{if } x \in \bigoplus_{n\geq 0} \bigwedge^{2n+1}(\overline{p^\perp(H)}) \end{cases}.$$

For $\zeta = \xi \oplus \eta \in (\mathbf{1} - p)(H) \bigoplus p(H)$, let us define $A_p(\zeta) \in \mathcal{B}(\Gamma_p(H))$ as follows:

$$A_p(\zeta) = A_p(\xi \oplus \eta) = a(\overline{\xi})^* \otimes \mathbf{1} + D \otimes a(\eta).$$

These operators fulfill the CAR relations:

$$A_p^*(\xi_1 \oplus \eta_1) A_p(\xi_2 \oplus \eta_2) + A_p(\xi_2 \oplus \eta_2) A_p^*(\xi_1 \oplus \eta_1) = \langle \xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2 \rangle \mathbf{1} \otimes \mathbf{1},$$
$$A_p(\xi_1 \oplus \eta_1) A_p(\xi_2 \oplus \eta_2) + A_p(\xi_2 \oplus \eta_2) A_p(\xi_1 \oplus \eta_1) = 0,$$

for all $\xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2 \in H$. In this order, the map $H \to \mathcal{B}(\Gamma_p(H))$ given by

$$\zeta = \xi \oplus \eta \mapsto A_p(\zeta),$$

extends to a representation

$$\pi_p : CAR(H) \to \mathcal{B}(\Gamma_p(H)).$$

The vector state $\omega_p \equiv \omega_{\Omega_p}$, defined by the representation π_p and by the unit cyclic vector Ω_p, is given by:

$$\omega_p(c(\zeta)) = \langle \Omega_p, \pi_p(c(\zeta)) \Omega_p \rangle_{\Gamma_p(H)},$$

for $c \in \{A, A^*\}$, and satisfies the following relation:

$$\begin{aligned} \omega_p(\pi_p(a(\xi_1 \oplus \eta_1))^* \pi_p(a(\xi_2 \oplus \eta_2))) &= \langle A_p(\xi_1 \oplus \eta_1) \Omega_p, A_p(\xi_2 \oplus \eta_2) \Omega_p \rangle_{\Gamma_p(H)} \\ &= \langle -1 \otimes \eta_1, -1 \otimes \eta_2 \rangle_{\Gamma_p(H)} = \langle \eta_1, \eta_2 \rangle = \langle \xi_1 \oplus \eta_1, p(\xi_2 \oplus \eta_2) \rangle, \end{aligned}$$

where $\xi_i \in (\mathbf{1} - p)(H)$ and $\eta_i \in p(H)$. Thus, according to proposition 2.1.17, ω_p is, precisely, the quasi-free state of covariance p. By uniqueness of the GNS construction stated in theorem 2.1.11, it follows that π_p and π_{ω_p} are unitarily equivalent, and they are irreducible. With these results, we have the following theorem (see [9] §8):

Theorem 2.1.18. *The action of the algebra $CAR(H)$ over the Fock space $\mathcal{F}(H)$ defines an irreducible representation.*

2.1.3 Equivalence and Implementation problem

In the previous subsection we have established a connection between projections on a Hilbert space H and irreducible representations of its CAR algebra $CAR(H)$. Now, we shall study the question of how to classify them modulo unitaryy equivalence. This question is studied in detail in [9], and we shall present the main results. Besides, this formalism will allow us to solve the problem of unitarily implementation of the Bogoliubov automorphisms in the CAR algebra $CAR(H)$. We shall find necessary conditions that unitary operators u on H must fulfill in order to have commutativity in the following diagram for a given projection p on H:

$$\begin{array}{ccc} \Gamma_p(H) & \xrightarrow{a(\xi)} & \Gamma_p(H) \\ \big\downarrow{\scriptstyle \Gamma_p(H)} & & \big\downarrow{\scriptstyle \Gamma_p(H)} \\ \Gamma_p(H) & \xrightarrow{a(u(\xi))} & \Gamma_p(H). \end{array}$$

We start with some technical results that will help answering the first question (see [9] §8).

Proposition 2.1.19. 1. *Let ϕ, ψ be two pure states on a C^*-algebra A such that $||\phi - \psi|| < 2$. Then ϕ and ψ are equivalent.*

2. *Let p, q be projections on H, and let ϕ_p, ϕ_q be the corresponding quasi-free states on $CAR(H)$. Then*
$$||\phi_p - \phi_q|| \leq 2||p - q||_2,$$
where $||\cdot||_2$ denotes the Hilbert-Schmidt norm.

We have then the first conclusion in order to answer the equivalence problem: given two projections p and q on H, if they differ by a Hilbert-Schmidt operator of norm less than 1, then their corresponding GNS representations are unitarily equivalent.
In order to solve the implementation problem of the Bogoliubov automorphisms, there are two technical lemmas (see [9] §6):

Lemma 2.1.20. *([9]) Let A be a unital C^*-algebra, α an automorphism of A, and ϕ a state on A. Let π_ϕ and $\pi_{\phi\alpha}$ be the GNS representations defined by the states ϕ and $\phi\alpha$ respectively. Then there exists a unitary operator $u : H_\phi \to H_{\phi\alpha}$ such that for all $a \in A$*

$$u\pi_\phi(\alpha(a)) = \pi_{\phi\alpha}(a)u.$$

Lemma 2.1.21. *Let A be a unital C^*-algebra, α an automorphism of A, and ϕ a state on A. Suppose that the representations π_ϕ and $\pi_{\phi\alpha}$ are equivalent. Then there exists a unitary operator v on H_ϕ such that for all $a \in A$*

$$v\pi_\phi(\alpha(a))v^* = \pi_\phi(a).$$

If $\tilde{v}$ is another unitary operator on H_ϕ such that $\tilde{v}\pi_\phi(\alpha(a))\tilde{v}^* = \pi_\phi(a)$, then there exists $z \in S^1 \subseteq \mathbb{C}$ such that $\tilde{v} = zv$.

These two lemmas are the core in the proof of the following theorem, which solves, in some sense, the implementation problem of the Bogoliubov automorphisms.

Theorem 2.1.22. *Let $p \in \mathcal{B}(H)$ be a projection and let $u \in \mathcal{B}(H)$ be a unitary operator such that the commutator $[p, u]$ is Hilbert-Schmidt. Then there exists a unitary operator $\Gamma_p(u)$ on $\Gamma_p(H)$ such that*

$$\Gamma_p(u)a(\xi)\Gamma_p(u)^* = a(u(\xi))$$

for all $\xi \in H$.

In this theorem, we have an abuse of notation which is common in the literature. To be precise with the previous subsection, we should write $\Gamma_p(u)A_p(\xi)\Gamma_p(u)^* = A_p(u(\xi))$.
This result motivates the following definition of the *restricted unitary group* associated to a projection p on the Hilbert space H, which will be used in the next chapter to characterize the problem of implementation of Bogoliubov automorphisms in terms of topological properties of the following subgroup.

Definition 2.1.23. *Let $p \in \mathcal{B}(H)$ be a projection. We define the restricted unitary group*

$$\mathcal{U}_{res}(H) = \{u \in \mathcal{B}(H) |\ u \text{ unitary and } ||\ [u, p]\ ||_2 < \infty\}.$$

2.2 Clifford Algebra

As we have done for the CAR algebra, we shall define the Clifford algebra associated to a Hilbert space and consider its automorphisms. We will see that, in this case, there are more structures to be considered. The main reference for this section is Robinson and Plymer [21], §1, §2 and §3.

2.2.1 Briefly generalities about Clifford Algebras

Definition 2.2.1. *Let E a real Hilbert space with inner product $(\cdot, \cdot)$. We will denote as Clifford map any real linear map $f : E \to B$ over an associative unital complex algebra B such that for all $v \in E,\ f(v)^2 = ||v||^2\mathbf{1}$.*
We define the complex Clifford algebra over E to be an associative unital complex algebra A together with a Clifford map $\phi : E \to A$ satisfying the universal mapping property: if $f : E \to B$ is a Clifford map, then there exists a unique homomorphism of algebras $F : A \to B$ such that $F \circ \phi = f$.

The problem of existence and uniqueness of this algebra is described by Husemoller in a highly detailed way (see [13] §12). The quotient of the tensor algebra $T(E^{\mathbb{C}})$ by the two sided ideal I generated by the subset $\{v \otimes v - (v, v)\mathbf{1} :\ \ v \in E \subseteq E^{\mathbb{C}}\}$ corresponds to the Clifford Algebra. From now on, we shall denote this algebra by $C(E)$. Note that the property of the Clifford map $\phi : E \to C(E)$ given by $\phi(v)^2 = ||v||^2\mathbf{1}$ implies that this map is one to one. Therefore, we identify E with its image in $C(E)$. Plymen and Robinson show that this complex Clifford algebra $C(E)$ is generated by its real subspace E satisfying the relations (see [21] §1):

$$xy + yx = 2(x, y)\mathbf{1}, \ \text{ for all } x, y \in E$$

which are called the *Clifford relations*.

Proposition 2.2.2. *(see [21] §1) Let $g : E \to E'$ be an isometric linear map. Then there exists a unique algebra homomorphism $\theta_g : C(E) \to C(E')$ such that $\theta_g \circ \phi = \phi' \circ g$, where ϕ and ϕ' denotes the corresponding Clifford maps.*

Let $O(E)$ be the group of orthogonal transformations on E. According to the previous proposition, for any $g \in O(E)$ there exists an algebra automorphism

$$\theta_g : C(E) \to C(E)$$

such that $\theta_g \circ \phi = \phi \circ g$. We shall refer to θ_g as the *Bogoliubov automorphism of $C(E)$ induced by g*. By uniqueness, it follows that given $g, h \in O(E)$, we have that $\theta_{gh} = \theta_g \circ \theta_h$. Thus, we can see this pairing as a group homomorphism $\theta : O(E) \to \mathrm{Aut}(C(E))$.

There exists an automorphism of special importance. Let γ denote the Bogoliubov automorphism induced by $-\mathbf{1}$. We have then $\gamma \circ \phi = -\phi$, so $\gamma(v) = -v$ for all $v \in E$. The automorphism γ is called *grading automorphism*. The subalgebra Ker $(\gamma - \mathbf{1}) \subseteq C(E)$ of fixed points by γ is called the even complex Clifford algebra $C^+(E)$; the subalgebra Ker $(\gamma + \mathbf{1}) \subseteq C(E)$, where γ acts as $-\mathbf{1}$, is called the odd complex Clifford algebra $C^-(E)$. We refer to elements of $C^+(E)$ as being even and to elements of $C^-(E)$ as being odd.

We shall observe that, in fact, $C(E)$ is an involutive algebra. In order to notice this, we will need two morphisms. Denote by $C(E)^0$ the algebra opposite to $C(E)$: $C(E)^0$ is $C(E)$ as a set, with the same linear structure, but it has reverse product $x \cdot y := yx$. The identity map $C(E) \to C(E)^0$ is an anti-isomorphism of algebras. The canonical inclusion $\tilde{\phi} = \mathbf{1} \circ \phi$ (seeing $\mathbf{1}$ as the anti-isomorphism mentioned before) is a Clifford map $E \to C(E)^0$. Using the universal property for $C(E)$ we get an algebra homomorphism $\alpha : C(E) \to C(E)^0$ which, restricted to E is the identity map. We can understand α as an anti-homomorphism from $C(E)$ to itself. Notice that $\alpha \circ \alpha : C(E) \to C(E)$ is an algebra homomorphism.

Let us denote by $\overline{C(E)}$ the algebra conjugate to $C(E)$: at the level of sets they are equal, with the same ring structure, but with conjugate scalar product. Thus, the identity map $C(E) \to \overline{C(E)}$ is an antilinear ring isomorphism. The canonical inclusion $E \to \overline{C(E)}$ is a Clifford map since E has a real vector space structure; therefore, by the universal property, there exists an algebra homomorphism $\kappa : C(E) \to \overline{C(E)}$ which, restricted to E is the identity map. Moreover, $\kappa^2 : C(E) \to C(E)$ is the identity map.

Notice that α and κ commute. Additionally, their product is the unique antilinear anti-automorphism of $C(E)$ which restricted to E is the identity map. Thus, $\alpha \circ \kappa = \kappa \circ \alpha$ is an involution (see [21] §1). We shall refer to it as the *main involution* of the Clifford algebra, and it will be denoted by a star *:

$$a^* = \alpha(\kappa(a)) = \alpha(\overline{a}) = \overline{\alpha(a)}.$$

Thus, $C(E)$ is an involutive algebra. Moreover, this algebra satisfies a further universal mapping property, but its statement requires the following definition.

Definition 2.2.3. *Let B be a unital involutive associative complex algebra; then the Clifford map $f : E \to B$ is called self-adjoint if $f(v)^* = f(v)$ for all $v \in E$.*

Proposition 2.2.4. *(see [21] §1) If $f : E \to B$ is a self-adjoint Clifford map, then the algebra homomorphism induced by f, $F : C(E) \to B$, is involution preserving.*

Notice that the Bogoliubov automorphism θ_g induced by $g \in O(E)$ is involution preserving: $\phi \circ g$ is self-adjoint since * acts like the identity over E.
There are two more ingredients which we will present in order to understand some other generalities of Clifford algebras. Let us consider the *left regular representation*

$$\lambda : C(E) \to \text{End }(C(E)),$$

given by $\lambda(\xi)(\eta) = \xi\eta$. Notice that $\lambda(\xi)$ leaves $C^+(E)$ and $C^-(E)$ invariant whenever $\xi \in C^+(E)$, and maps $C^+(E)$ to $C^-(E)$ - and vice-versa - whenever $\xi \in C^-(E)$.
If $\tau : C(E) \to \mathbb{C}$ is a linear functional on $C(E)$, we say that τ is *normalized* if $\tau(\mathbf{1}) = 1$, that it is *central* if $\tau(ab) = \tau(ba)$ for all $a, b \in C(E)$, and that it is *even* if $\tau = \tau \circ \gamma$, i.e. if it is invariant under the grading automorphism. In [21] §1, the authors show a detailed description of how this functional τ can be obtained, beginning from the finite dimensional case, and then generalizing for infinite dimensional Hilbert space E. They show the following proposition:

Proposition 2.2.5. *(see [21] §1) There exists a unique normalized even central linear functional τ on $C(E)$, which is constructed as follows. Given $a \in C(E)$, there exists a finite dimensional subspace $M \subseteq E$ such that $a \in M$. On M, consider the trace τ_M given by:*

$$\tau_M(b) = \frac{\text{Tr}(\lambda(\text{b}))}{\dim C(M)}.$$

Taking $\tau(a) := \tau_M(a)$, it fulfills the listed properties and does not depend on the subspace M. Moreover, τ satisfies

$$\tau(a^*) = \overline{\tau(a)},$$

and it is the unique normalized linear functional on $C(E)$ that is invariant under all Bogoliubov automorphisms induced by $g \in O(E)$.

As a consequence of this proposition we obtain an inner product on $C(E)$

$$\langle \cdot, \cdot \rangle : C(E) \times C(E) \to \mathbb{C}, \;\; \langle \xi, \eta \rangle := \tau(\eta^* \xi).$$

Now, we proceed to provide $C(E)$ with a structure of C^*-algebra. We shall construct a norm $|| \cdot ||$ that behaves well with respect to involution *. Let Rep E be the collection of all star-homomorphisms from $C(E)$ to the space $\mathcal{B}(H)$ of bounded linear operators for an arbitrary complex Hilbert space H. We define the function $|| \cdot ||_\infty : C(V) \to \mathbb{R}_{\geq 0}$ as follows:

$$||a||_\infty = \sup\{||\pi(a)|| \mid \pi \in \text{Rep } E\}.$$

First, let us observe that this supremum makes sense. Let $a \in C(E)$, and let $M \subseteq E$ a finite dimensional subspace such that $a \in M$. For a finite dimensional real Hilbert space M, the Clifford algebra $C(M)$ can be understood in terms of a finite set of generators: let $B = \{v_1, \ldots, v_m\}$ be an orthonormal basis for M; the Clifford algebra $C(M)$, seen as a vector space, is finite dimensional, and it has as a basis the set $\tilde{B} = \{v_S : \; S \subseteq [m]\}$, where $v_S := v_{j_1} v_{j_2} \cdots v_{j_k}$ with $S = \{j_1, \ldots, j_k\}$ and $[m] = \{1, \ldots, m\}$. In this order, $a = \sum_{S \subseteq [m]} \mu_S v_S$. And if $\pi \in$ Rep E is a nonzero star homomorphism from $C(V)$ to $\mathcal{B}(H)$ for some complex Hilbert space H,

$$\pi(v_S)\pi(v_S)^* = \pi(v_S v_S^*) = \pi(\mathbf{1}) = \mathbf{1},$$

and, since $\mathcal{B}(H)$ is a C^* algebra, $||\pi(v_S)\pi(v_S)^*|| = ||\pi(v_S)||^2 = 1$. Therefore,

$$||\pi(a)|| = \left|\left| \sum_{S \subseteq [m]} \mu_S \pi(v_S) \right|\right| \leq \sum_{S \subseteq [m]} |\mu_S| \; ||\pi(v_S)|| = \sum_{S \subseteq [m]} |\mu_S|.$$

Thus, this supremum is finite.
Another property that is not immediate from the definition of $|| \cdot ||_\infty$ is that for any nonzero $a \in C(E)$, $||\pi(a)||_\infty \neq 0$. To show this, we use the left regular representation. Considering the inner product $\langle \cdot, \cdot \rangle$ given by the trace functional τ, we can regard $C(E)$ as an inner product complex vector space, and thus define the complex Hilbert space $(H_\tau, \langle \cdot, \cdot \rangle)$ which results from the completion theorem for normed spaces. Furthermore, observe the following property:

$$\langle \lambda(a)\xi, \eta \rangle = \tau(\eta^*(a\xi)) = \tau((a^*\eta)^*\xi) = \langle \xi, \lambda(a^*)\eta \rangle.$$

The left regular representation can be extended by continuity to a morphism on H_τ for each $a \in C(E)$, i.e, we can extend $\lambda(a) : C(E) \to C(E)$ to an operator on H_τ, and consequently we have the representation $\lambda : C(E) \to \mathcal{B}(H_\tau)$. This representation is faithful: let $\Omega \in H_\tau$ be the unit in $C(E)$; then $\lambda(a)\Omega = a$. Therefore, $||a||_\infty \geq ||\lambda(a)|| > 0$.
In addition, this norm $|| \cdot ||_\infty$ satisfies the C^* property: $||a^*a||_\infty = ||a||_\infty^2$ since each norm over which the supremum is taken fulfills this property (see [21] §1).

Definition 2.2.6. *Considering the norm* $|| \cdot ||_\infty$*, we define* $C[E]$*, the* C^* *Clifford algebra associated to* E*, as the completion of* $C(E)$ *with this norm. With this construction, the canonical embedding* $\phi : E \to C[E]$ *is isometric.*

The next proposition gives a simplification of this C^*-algebra. It says that $||\cdot||_\infty$ is equivalent to any norm over which the supremum is taken.

Proposition 2.2.7. *(see [21] §1) If $\pi \in Rep\ E$ is nonzero, then $||\cdot||_\infty = ||\cdot||_\pi$, and the identity map on $C(E)$ extends continuously to a C^*-algebra isomorphism $C[V] \to C_\pi[E]$, where $C_\pi[E]$ denotes the completion of $C(E)$ in the norm $||a||_\pi := ||\pi(a)||$.*

Finally, we have the next proposition about the extension of Clifford maps to C^*-algebra homomorphisms (see [21] §1).

Proposition 2.2.8. *Let B be a unital C^* algebra and let $f : E \to B$ be a self-adjoint Clifford map. Then there exists a unique isometric C^* algebra homomorphism $F : C[V] \to B$ such that $F|_E = f$. In particular, each isometric linear map $g : E \to E'$ extends uniquely to an isometric C^* algebra map $\theta_g : C[E] \to C[E']$.*

2.2.2 Fock Representation

Let $(E, (\cdot,\cdot))$ be a real Hilbert space. By definition, a *unitary structure* is an orthogonal operator $J \in O(E)$ such that $J^2 = -\mathbf{1}$. Fixing a unitary structure J on E, we can turn E into a complex Hilbert space, denoted by E_J, by stipulating $iv := J(v)$ for all $v \in E$, and defining an inner product given by:

$$\langle u, v\rangle_J = (u, v) + i(u, J(v)).$$

When E is a separable infinite-dimensional real Hilbert, it admits a unitary structure J. It is known that, if $\dim_{\mathbb{R}} E = 2n < \infty$, then $\dim_{\mathbb{C}} E_J = n$. When E is infinite-dimensional, the dimension of E_J equals that of E.

Let us denote by $\mathbb{U}(E)$ the set of all unitary structures on E. The orthogonal group $O(E)$ acts transitively by conjugation on $\mathbb{U}(E)$. First, notice that for $g \in O(E)$ and $J \in \mathbb{U}(E)$, gJg^{-1} is orthogonal and $(gJg^{-1})^2 = -\mathbf{1}$. Second, if J and K are unitary structures on E, the complex spaces E_J and E_K have the same dimension, and thus they are isometrically isomorphic. Let $g : E_J \to E_K$ be the mentioned isomorphism; then for all $x \in E_J$, $g(J(x)) = g(ix) = ig(x) = Kg(x)$, i.e, $K = gJg^{-1}$. Moreover, g is orthogonal: $(g(x), g(y)) = \langle g(x), g(y)\rangle_K - i(g(x), Kg(y)) = \langle x, y\rangle_J - i(g(x), gJ(y)) = (x, y) + i(x, J(y)) - i(g(x), gJ(y))$, but $(\cdot,\cdot)$ is real, so $(g(x), g(y)) = (x, y)$.
Furthermore, the isotropy group for a fixed unitary structure $J \in \mathbb{U}(E)$, under this action, corresponds to the unitary group $\mathcal{U}(E, J)$ of the complex Hilbert space E_J.
Let us fix a unitary structure J. As we did in the construction of the CAR - algebra, we can consider the completion of the complex space $\bigoplus_{n\geq 0} \bigwedge^n(E_J)$. We denote this completion as $H_J(E)$. Additionally, we can consider the completion $\bigwedge^n[E_J]$ of the vector space $\bigwedge^n E_J$, and view $H_J(E)$ as the direct sum of these spaces.
Let $\zeta \in \bigwedge^2[E_J]$ and let

$$T_\zeta : E \times E \to \mathbb{C}$$

be the alternating bilinear map given by $(x, y) \mapsto \langle x \wedge y, \zeta \rangle_J$. Using the Cauchy Schwarz inequality, we have that $|T_\zeta(x, y)| \leq ||x \wedge y|| \, ||\zeta|| \leq ||x|| \, ||y|| \, ||\zeta||$. By the Riesz representation theorem, there exists a bounded real operator Z_ζ such that for all $x, y \in E$

$$\langle Z_\zeta x, y \rangle_J = \langle \zeta, x \wedge y \rangle_J.$$

Since T_ζ is alternating, we have that

$$\langle Z_\zeta \ x, y \rangle + \langle Z_\zeta \ y, x \rangle = \langle \zeta, x \wedge y \rangle + \langle \zeta, y \wedge x \rangle = 0.$$

This implies that $(Z_\zeta \ x, y) = -(Z_\zeta \ y, x)$ and $(Z_\zeta \ x, Jy) = -(Z_\zeta \ y, Jx)$. These relations show that $Z_\zeta^t = -Z_\zeta$ and $JZ_\zeta = -Z_\zeta J$. Moreover, Plymen and Robinson [21] §2 show that for any $\zeta \in \bigwedge^2[E_J]$ the operator Z_ζ is of Hilbert-Schmidt class and, moreover, that it satisfies that $||Z_\zeta||_2 = 2||\zeta||$.

Furthermore, it is possible to show that $\bigwedge^2[E_J]$ is isometrically isomorphic to the vector space of all real Hilbert-Schmidt operators Z on E that satisfying the condition:

$$\langle Z \ x, y \rangle + \langle Z \ y, x \rangle = 0.$$

This is known as the *antiskew property*. We denote this space as $\mathbb{S}(E_J)$.

These two properties are important in order to define the concept of *Gaussians*. Since these operators Z_ζ are of Hilbert-Schmidt class, they are compact. Thus, their spectrum consists of eigenvalues and $\{0\}$ as only possible limit point. The eigenspace decomposition and the J-antilinearity of Z_ζ in each finite dimensional λ-eigenspace ($\lambda \neq 0$), give rise to the following decomposition (see [21] §2):

Proposition 2.2.9. *For $\zeta \in \bigwedge^2[E_J]$ let $Z_\zeta \in \mathbb{S}(E_J)$. Then there exist a square summable positive sequence $(\lambda_j)_{j \geq 1}$ and a complete orthonormal basis $\{x_j, y_j\}_{j \geq 1}$ for $Ker Z_\zeta^\perp$ such that:*

$$\zeta = \sum_{n \geq 1} \lambda_j x_j \wedge y_j,$$

with $Z_\zeta \ x_j = \lambda_j \ x_j$ and $Z_\zeta \ y_j = -\lambda_j \ y_j$. Moreover,

$$||\zeta^n||^2 \leq n! ||\zeta||^{2n}, \ \textit{for } n > 0.$$

Definition 2.2.10. *For $\zeta \in \bigwedge^2[E_J]$, we define the Gaussian*

$$exp \ \zeta := \sum_{n \geq 0} \frac{1}{n!} \zeta^n \in H_J(E),$$

which satisfies the inequality

$$||exp \ \zeta||^2 \leq exp \ ||\zeta||^2.$$

Now, as we have done in the previous section, we can consider the CAR algebra over H_J. The construction of the creation and annihilation operators is equivalent. We have the following relation on how annihilation operators act over Gaussian elements (see [21] §2).

Proposition 2.2.11. *If $\zeta \in \bigwedge^2[E_J]$ and if $v \in E$, then*

$$a^*(v)\zeta = Z_\zeta(v),$$

and

$$a^*(v)(exp\ \zeta) = a(Z_\zeta\ v)(exp\ \zeta).$$

According to the construction of these operators, it can be shown that $a(J(v)) = Ja(v) = ia(v)$ and $a^*(J(v)) = -ia(v)$. Our next goal, once all these elements have been presented, is to construct an irreducible representation for the C^* Clifford algebra $C[E]$ determined by the specific choice of the unitary structure J on the real Hilbert space E. This representation will take place on the Fock space H_J. And, as we did for the CAR algebra, this will be defined in terms of the creation and annihilation operators, which will be denoted as a_J and a_J^* to underline the chosen unitary structure.
For $v \in E$ let us define a bounded linear operator $\pi_J(v)$ on H_J as $\pi_J(v) = a_J(v) + a_J^*(v)$. This operator is self-adjoint and also satisfies the relation:

$$\pi_J(v)^2 = a_J(v)a_J(v) + a_J^*(v)a_J(v) + a_J(v)a_J^*(v) + a_J^*(v)a_J^*(v) = ||v||^2\mathbf{1}.$$

Hence, this linear map from E to $\mathcal{B}(H_J)$ is a self-adjoint Clifford map. By proposition 2.2.8, we can extend π_J, in a unique way, to an isometric representation $\pi_J : C[E] \to \mathcal{B}(H_J)$ of the C^* Clifford algebra $C[E]$. It follows by a straight computation using the definition that:

$$a_J(v) = \frac{1}{2}\pi_J(v - iJv), \quad a_J^*(v) = \frac{1}{2}\pi_J(v + iJv) \text{ for all } v \in E.$$

These relations tell us that creation and annihilation operators lie in the image of this *Fock representation*. Therefore, the unit vector Ω_J is cyclic for π_J, in the sense that $\{\pi_J(a)\Omega_J|\ a \in C[E]\}$ is dense on H_J: the Hilbert space H_J was constructed as the completion of $\bigoplus_{n\geq 0}\bigwedge^n(E_J)$, and each decomposable vector here can be obtained from Ω_J by a successive application of creation operators. A consequence of this fact is the next theorem, which is the main result in this subsection.

Theorem 2.2.12. *(see [21] §2) The Fock representation π_J is irreducible.*

This theorem is proved by showing that the only operators which commute with the range of π_J are scalar multiples of the identity map. Indeed, the proof is straightforward: by density, it is enough to show that the only operators which commute with $\pi_J(v)$ for all $v \in E$ are the scalar multiples of the identity.
Let $\sigma_J : C[E] \to \mathbb{C}$ be the state associated to the representation π_J and the cyclic vector Ω_J:

$$\sigma_J(a) = \langle \pi_J(a)\Omega_J, \Omega_J\rangle_J.$$

Notice, in particular, that for all $u, v \in E$,

$$\sigma_J(uv) = \langle \pi_J(uv)\Omega_J, \Omega_J\rangle_J = \langle \pi_J(v)\Omega_J, \pi_J(u)\Omega_J\rangle_J = \langle v, u\rangle_J.$$

As a consequence of the GNS construction, we have the following proposition.

Proposition 2.2.13. *The state σ_J is pure. Moreover, σ_J is the unique state σ of $C[V]$ with the property that for all $v \in E$, $\sigma((v + iJv)(v - iJv)) = 0$.*

Consider a representation $\pi : C[E] \to \mathcal{B}(H)$. We refer to the nonzero vector $\Omega \in H$ as a *J-vacuum vector* for π if and only if satisfies the vacuum condition: for all $v \in E$, $\pi(v + iJv)\Omega = 0$. We have the next remarkable result, which tells us how to identify irreducible representations unitarily equivalent to π_J according to their J - vacuum vectors (see [21] §2).

Proposition 2.2.14. *Let $\pi : C[E] \to \mathcal{B}(H)$ be a representation with cyclic unit J-vacuum vector Ω. Then there exists a unique unitary isomorphism $U : H_J \to H$ such that $U(\Omega_J) = \Omega$ and, for $a \in C[E]$, $\pi(a) = U\pi_J(a)U^*$.*

This is, again, a consequence of the GNS construction. Given the cyclic unit J-vacuum vector Ω, and by uniqueness stated on proposition above for states σ on $C[E]$, the state σ_π associated with the representation π equals the state σ_J.

2.2.3 Orthogonal Restricted Group

In this subsection we shall study the equivalence and implementability problem of representations of the C^* Clifford algebra $C[V]$. Given two unitary structures J and K, the question arises when the Fock representations π_J and π_K are unitarily equivalent. In the same way, as we did for the CAR - algebra, we look for necessary and sufficient conditions on the orthogonal operator $g \in O(E)$ in order that the Bogoliubov automorphism θ_g can be implemented by an unitary operator U on H_J such that $\pi_J(\theta_g(a)) = U\pi_J(a)U^*$.
Let us start with some fundamental properties for orthogonal operators $g \in O(E)$. Given $g \in O(E)$, we can consider the decomposition $g = C_g + A_g$, where

$$C_g := \frac{1}{2}(g - JgJ), \quad A_g := \frac{1}{2}(g + JgJ).$$

These operators satisfy that $C_gJ = JC_g$ and $A_gJ = -JA_g$. Thus, we call C_g the J-linear part and A_g the J-antilinear part of g. Notice that $C_g \in \mathcal{B}(E_J)$. Moreover, for $g, h \in O(E)$, the following holds:

$$C_{gh} = C_gC_h + A_gA_h, \quad A_{gh} = C_gA_h + A_gC_h.$$

As a consequence of the second relation above, we have that $0 = C_gA_g^* + A_gC_g^*$. Additionally, we have an extra property associated to this decomposition for orthogonal operators: Let $g \in O(E)$ and $v \in E$. Then:

$$\langle C_gv, A_gv\rangle_J = 0, \quad ||C_gv||^2 + ||A_gv||^2 = ||v||^2.$$

Plymen and Robinson [21] construct another factorization for an orthogonal element $g \in O(E)$ given by:

$$g = u(\mathrm{abs}(C_g) + u^* A_g),$$

where $u \in \mathcal{U}(E, J)$ and $\mathrm{abs}(C_g)$ denotes de absolute value of the operator C_g. This is known as *multiplicative factorization.*

Let us recall some properties concerning to the absolute value of bounded operators as well as the so-called *polar decomposition.* The following definitions and propositions can be found in [28] §4 and [24] §6. Let H be a separable complex Hilbert space, and let $T \in \mathcal{B}(H)$ be a bounded operator on H. We say that T is positive if for all $x \in H$ it satisfies

$$\langle Tx, x \rangle \geq 0.$$

We denote the positiveness of an operator T as $T \geq 0$. For any $T \in \mathcal{B}(H)$, it can be shown that $T^*T \geq 0$. We have the following lemma, which is explained in detail by Simon and Reed [24] §6.

Lemma 2.2.15. *(The Square root lemma) Let $T \in \mathcal{B}(H)$ with $T \geq 0$. Then there exists a unique $A \in \mathcal{B}(H)$ with $A \geq 0$ and $A^2 = T$. Furthermore, A commutes with any bounded operator which commutes with T.*

Definition 2.2.16. *Let $T \in \mathcal{B}(H)$. Then the absolute value of T is defined as the square root of T^*T, and it is denoted as*

$$\mathrm{abs(T) := \sqrt{T^*T}.}$$

From [28] §4 we have the following characterization for *partial isommetries.*

Proposition 2.2.17. *Let $U \in \mathcal{B}(H)$. Then the following conditions are equivalent:*

i. $U = UU^*U$.

ii. $P = U^*U$ *is a projection.*

iii. $U|_{(\mathrm{KerU})^\perp}$ *is an isometry.*

An operator U which satisfies these equivalent properties is called a partial isometry.

We have now a technical proposition which will be useful in the next chapters. This can be found in [28] §4.

Proposition 2.2.18. *Let $T \in \mathcal{B}(H)$. Then*

$$\ker T = \ker(T^*T) = \ker \sqrt{T^*T} = (\mathrm{ran}\, T^*)^\perp.$$

In particular, we have that $(\ker T)^\perp = \overline{\mathrm{ran}(T^*)}$.

Combining these two last propositions, observe the following. Let $U \in \mathcal{B}(H)$ be a partial isometry, and set $\mathcal{M} = (\ker U)^{\perp}$ and $\mathcal{N} = \operatorname{ran} U$ –which is closed since U is a partial isometry. Notice that U is zero on $\mathcal{M}^{\perp}$, and it maps isometrically $\mathcal{M}$ to $\mathcal{N}$. We will refer to $\mathcal{M}$ as the *initial space*, and to $\mathcal{N}$ as the *final space*, of the partial isometry U. It can be derived from these propositions that U^* is also a partial isometry, and that $\mathcal{N}$ is the initial space and $\mathcal{M}$ the final space of U^*. We will now state the main result concerning to this brief exposition of partial isometries and the square root lemma. The proof of the next theorem can be found in [28] §4 and [24] §6.

Theorem 2.2.19. *(Polar decomposition) (a) Any operator $T \in \mathcal{B}(H)$ admits a decomposition $T = UA$ such that*

i. *$U \in \mathcal{B}(H)$ is a partial isometry;*

ii. *$A \in \mathcal{B}(H)$ is a positive operator;*

iii. $\ker T = \ker A = \ker U$.

(b) If $T = VB$ is another decomposition of T as a product of a partial isometry V and a positive operator B such that $\ker V = \ker B$, then $U = V$ and $B = A = abs(T)$.
This unique decomposition is called the polar decomposition of T.

Although we have presented the problem of equivalence and implementability as two different problems, we shall observe that there is a relation between them. We have observed that the orthogonal group $O(E)$ acts transitively by conjugation on $\mathbb{U}(E)$: given two unitary structures J and K, there exists $g \in O(E)$ such that $K = gJg^{-1}$ which, in turn, it is also an isometric isomorphism from E_J to E_K. Extending this linear map to a bounded linear map $\bigwedge_g : H_J \to H_K$, which in each level acts on decomposable vectors as follows:

$$\bigwedge\nolimits_g(v_1 \wedge \ldots \wedge v_n) = g(v_1) \wedge \ldots \wedge g(v_n).$$

We have the following computations for this isomorphism $\bigwedge_g$: let $v \in E$ and $\zeta \in H_J$, then

$$\bigwedge\nolimits_g \circ a_J(v)(\zeta) = \bigwedge\nolimits_g(v \wedge \zeta) = g(v) \wedge \bigwedge\nolimits_g(\zeta) = a_K(g(v)) \circ \bigwedge\nolimits_g(\zeta).$$

Thus, for any $v \in E$, we have that $a_K(g(v)) = \bigwedge_g \circ a_J(v) \circ \bigwedge_g^*$. Taking adjoint, we have a similar relation for annihilation operators: $a_K^*(g(v)) = \bigwedge_g \circ a_J^*(v) \circ \bigwedge_g^*$. Adding these two equations, we set the next result:

$$\pi_K(g(v)) = \bigwedge\nolimits_g \pi_J(v) \bigwedge\nolimits_g^*.$$

Now, in virtue of proposition 2.2.8 we have the next statement.

Proposition 2.2.20. *Let $K = gJg^{-1}$ as before. If $a \in C[E]$, then $\pi_K(\theta_g a) = \bigwedge_g \circ \pi_J(a) \circ \bigwedge_g^*$. The following diagram is commutative:*

$$\begin{array}{ccc} H_J & \xrightarrow{\bigwedge_g} & H_K \\ \downarrow{\scriptstyle \pi_J(a)} & & \downarrow{\scriptstyle \pi_K(\theta_g a)} \\ H_J & \xrightarrow{\bigwedge_g} & H_K \end{array} .$$

In terms of the Fock states, given $a \in C[E]$, we get that:

$$\sigma_K(\theta_g a) = \langle \pi_K(\theta_g a)\Omega_K, \Omega_K\rangle_K = \langle \bigwedge_g \circ \pi_J(a) \circ \bigwedge_g^*(\Omega_K), \Omega_K\rangle_K = \langle \pi_J(a)\Omega_J, \Omega_J\rangle_J = \sigma_J(a).$$

We observe then that $\sigma_K \circ \theta_g = \sigma_J$, so we conclude that $O(g)$ acts transitively on the Fock states by Bogoliubov automorphisms.
We can now relate both problems. Let $U : H_J \to H_J$ be a unitary automorphism that implements the Bogoliubov automorphism θ_g in the Fock representation π_J; thus, for any $a \in C[E]$, we have that $\pi_J(\theta_g(a)) = U\pi_J(a)U^*$. Let us define $T = \bigwedge_g \circ U^*$. In this order, we have obtained a unitary isomorphism from H_J to H_K such that for all $a \in C[E]$:

$$T\pi_J(a)T^* = \bigwedge_g U^*\pi_J(a)U\bigwedge_g^* = \bigwedge_g \pi_J(\theta_g^{-1}a)\bigwedge_g^* = \pi_K(a).$$

In the other direction, let $T : H_J \to H_K$ be a unitary isomorphism intertwining the Fock representations π_J and π_K in the following sense: given $a \in C[E]$, $\pi_K(a) = T\pi_J(a)T^*$. Let us define $Y = T^* \circ \bigwedge_g$. Thus, we get a unitary operator on H_J such that for any $a \in C[E]$ the following holds:

$$U\pi_J(a)U^* = T^* \circ \bigwedge_g \circ \pi_J(a) \circ \bigwedge_g^* \circ T = T^* \circ \pi_K(\theta_g a) \circ T = \pi_J(\theta_g a).$$

We summarize these results in the next proposition.

Proposition 2.2.21. *Let $K = gJg^{-1}$ with $J \in \mathbb{U}(E)$ a unitary structure and $g \in O(E)$ be an orthogonal operator. The equation $T \circ U = \bigwedge_g$ sets up a bijective correspondence between unitary isomorphisms $T : H_J \to H_K$ intertwining the Fock representations π_J and π_K, and unitary operators $U \in Aut\ H_J$ implementing θ_g on π_J.*

As a consequence of the above proposition, each unitary operator $g \in \mathcal{U}(E, J)$ is canonically implemented in the Fock representation π_J by the unitary operator $\bigwedge_g$ on H_J. This answers partially the implementation problem. We shall observe that this question is linked to a topological property of the orthogonal operators.
Furthermore, the authors in [21] §2 show that, in the case when E is an infinite-dimensional Hilbert space, taking $J \in \mathbb{U}(E)$, the Fock representations π_J and π_{-J} are inequivalent. This property tell us that there are at least two *components* in the space of unitary structures which generates inequivalent Fock representations.

The next theorem, proved in detail in [21] §3, answers the implementability problem of the Bogoliubov automorphisms in the Fock representation pi_J.

Theorem 2.2.22. *Let $g \in O(E)$. The Bogoliubov automorphism θ_g on $C[E]$ is unitarily implemented in the Fock representation if and only if the J-antilinear part A_g is Hilbert Schmidt.*

This solution for the implementation problem allows us to introduce the *restricted orthogonal group* $\mathcal{O}_{\mathrm{res}}(E)$, associated to J, as the space of all orthogonal operators $g \in O(E)$ such that the commutator $[g, J]$ is Hilbert Schmidt:

$$\mathcal{O}_{\mathrm{res}}(E) := \{g \in O(E) | \ || \ [g, J] \ ||_2 < \infty\}.$$

Theorem 2.2.22 tells us that $\mathcal{O}_{\mathrm{res}}(E)$ contains the orthogonal operators on E for which their associated Bogoliubov automorphisms on $C[E]$ are implemented in the Fock representation π_J.
As we have seen in previous theorems, solving the implementation problem gives a solution for the equivalence problem. Thus, we have the following theorem (see [21] §3).

Theorem 2.2.23. *Let J and K be unitary structures on E. The Fock representations π_J and π_K of the C^* Clifford algebra $C[E]$ are unitarily equivalent if and only if the difference $K - J$ is of Hilbert-Schmidt class.*

The constructions developed throughout this chapter have shown how these restricted groups arise naturally, both for the CAR algebra and for the Clifford algebra, in order to answer the problem of the implementability of the Bogoliubov automorphisms on the irreducible representations induced by a projection p —in the case of the CAR algebra, or by a complex structure J —in the case of the Clifford algebra. In the next chapter we will provide these restricted groups with some topologies where (commutators with p or J, respectively) being Hilbert-Schmidt highlights this property. Furthermore, we will see that the connected components of these groups can be labeled by the index of an appropriately constructed Fredholm operator.

3 Automorphisms of the CAR and Clifford algebras and the Fredholm Index

In the previous chapter we have obtained some important results about the implementability and equivalence problems of unitary representations for the CAR algebra associated to a complex Hilbert space H and the C^* Clifford algebra associated to a real Hilbert space E. With these properties studied previously, it arises a natural question about the structures and topologies of these subgroups of orthogonal (real case) and unitary (complex case) operators, respectively, which can be implemented in the corresponding representation associated to a projection p –complex case– or a complex unitary structure J –real case. Although the equivalence problem could be understood as a distinct problem, we have observed a bijection between these two characteristic problems.
In this chapter we shall follow [6] and [5], where some natural topologies are given to the restricted unitary and orthogonal groups, looking for conditions on the operators in order to define unitarily implemented automorphisms of representations of the C^*-algebras defined in the chapter before. The topology of such groups will be studied using index theoretical techniques, labeling connected components with Fredholm-like indexes. We will present, in a sufficiently detailed way, the main results exposed by Carey, Hurst and O'Brian in the cited articles, developing those points that are not immediate in the published proofs.

3.1 The Fredholm Index Theory and the CAR Algebra

3.1.1 Group structure and topology

Let H be a complex Hilbert space and let $CAR(H)$ be the C^*-algebra associated to H, generated by the set:

$$\{a^*(v), a(v) | v \in H\},$$

whose generators satisfy the canonical anticommutation relations given by:

$$\begin{aligned} a(u)a(v) + a(v)a(u) &= 0 \\ a(u)a^*(v) + a^*(v)a(u) &= \langle u, v \rangle \mathbf{1}. \end{aligned}$$

Let P_+ be a projection on H, and P_- its orthogonal complement, i.e. $P_+ + P_- = \mathbf{1}$. We shall need some definitions which generalize the special case of Hilbert-Schmidt operators which arose naturally in the previous chapter. The first notion concerns symmetric norms and symmetrically normed ideal (see [12] §2 and §3). We consider the following definitions and properties related with these objects.

Definition 3.1.1. *Let R be a ring. A subset I is called an algebraic two sided ideal of the ring R if it has the following properties:*

i. *For any elements $a, b \in I$, $a + b \in I$;*

ii. *For any $a \in I$ and any $r \in R$ we have $a \cdot r, r \cdot a \in I$;*

iii. $I \neq 0$ *and* $I \neq R$.

Definition 3.1.2. *A functional $|\cdot|_{\mathcal{Q}}$ defined over a two sided ideal $\mathcal{Q}$ of the ring $\mathcal{B}(H)$ is called a* symmetric norm *if it satisfies the following properties:*

i. $0 \leq |X|_{\mathcal{Q}} < \infty$ *with equality if and only if* $X = 0$.

ii. $|\lambda X|_{\mathcal{Q}} = |\lambda|\ |X|_{\mathcal{Q}}$ *with* $\lambda \in \mathbb{C}$.

iii. *For all $X, Y \in \mathcal{Q}$, $|X + Y|_{\mathcal{Q}} \leq |X|_{\mathcal{Q}} + |Y|_{\mathcal{Q}}$.*

iv. *For all $X \in \mathcal{Q}$, $A, B \in \mathcal{B}(H)$, $|AXB|_{\mathcal{Q}} \leq ||A||\ |X|_{\mathcal{Q}}\ ||B||$.*

v. *For any one dimensional operator $X \in \mathcal{Q}$, $|X|_{\mathcal{Q}} = s_1(X)$, where $s_1(X)$ refers to the first singular value of X.*

In this order, according to these definitions, we call a pair $(\mathcal{Q}, |\cdot|_{\mathcal{Q}})$ as *symmetrically normed ideal*, where $\mathcal{Q}$ is a two sided ideal of $\mathcal{B}(H)$ and $|\cdot|_{\mathcal{Q}}$ is a symmetric norm which makes $\mathcal{Q}$ a Banach space. From now, we will denote as $\mathcal{Q}$ for the pair $(\mathcal{Q}, |\cdot|_{\mathcal{Q}})$. Gohberg and Krein [12] prove some properties concerning these spaces. There are two main results which will be useful for the rest of the chapter.
Let $\mathcal{Q}$ be a symmetrically normed ideal. Then, for any $X \in \mathcal{Q}$, $X^* \in \mathcal{Q}$ and

$$|X^*|_{\mathcal{Q}} = |X|_{\mathcal{Q}} = |(X^*X)^{1/2}|_{\mathcal{Q}} = |(XX^*)^{1/2}|_{\mathcal{Q}}.$$

Moreover, any two sided ideal $\mathcal{Q}$ of $\mathcal{B}(H)$ contains the ideal of finite rank operator $C_0(H)$ and it is contained in the ideal of compact operators $C_\infty(H)$ (see [20] §7).
Additionally, observe that for any rank one operator X, we have that $|X|_{\mathcal{Q}} = ||X||$. According to (v) in the above definition, the symmetric norm of X corresponds to the first singular value of X, but these values are defined, for compact operators, as the absolute value of the eigenvalues of its absolute value, abs$X$ sorted in descending order. Since abs$X$ is positive and compact, its first singular value is precisely $||X||$.
Notice that the class of Hilbert-Schmidt operators, with the Hilbert-Schmidt norm, is a special case of a symmetrically normed ideal.

In the previous chapter we have seen that given an orthogonal projection P_+ on the complex Hilbert space H, we can construct a state ω on $CAR(H)$ defined by

$$\omega(a^*(u_1)\cdots a^*(u_m)a(v_1)\cdots a(v_n)) = \delta_{m,n}\text{det}\langle u_i, P_+(v_j)\rangle.$$

According to the previous chapter, since ω is precisely the state of covariance P_+, the GNS construction generates an irreducible representation of $CAR(H)$, and it will be denoted as π.
Given a unitary operator U on H, we construct the Bogoliubov automorphism on the CAR algebra via generators

$$a(v) \mapsto a(Uv), \;\; a(v)^* \mapsto a^*(Uv).$$

Let us consider the subset of unitary operators U determined by the conditions:

$$P_+UP_- \in \mathcal{Q}, \;\; P_-UP_+ \in \mathcal{Q},$$

where $\mathcal{Q}$ denotes a symmetrically normed ideal in $\mathcal{B}(H)$.

Proposition 3.1.3. *The Conditions above are equivalent to the requirement that* $[P_+, U] = [U, P_-] \in \mathcal{Q}$.

Proof. Observe that $[P_+, U] = P_+U(P_+ + P_-) - (P_+ + P_-)UP_+ = P_+UP_- - P_-UP_+$. Thus, we have obtained one direction. Now, if $[P_+, U] \in \mathcal{Q}$, since $\mathcal{Q}$ is an ideal, we have that $P_-[P_+, U], [P_+, U]P_- \in \mathcal{Q}$. Therefore, computing each bracket, we obtain that P_-UP_+ and P_+UP_- lie in $\mathcal{Q}$. □

As we have observed in the previous chapter, the above family of unitary operators, in the case that $\mathcal{Q}$ corresponds to the Hilbert-Schmidt class operators, contains the family of unitary automorphisms on H which can be unitarily implemented in the representation π by a unitary operator on the representation space $\Gamma(U)$ which satisfies $\Gamma(U)\pi(a(v))\Gamma(U)^* = \pi(a(U(v)))$ for each $v \in H$. This is what we mean when we refer to generalize to any symmetric ideal $\mathcal{Q}$.
With all these tools and motivation in mind, our objective will be now to provide a topology to this group that highlights the property that the mentioned commutator belongs to the symmetric normed ideal. Once we construct this topology, we shall observe, using some results concerning to Fredholm operators, that the index Fredholm of P_+UP_- labels the path connected component which contains U.

Definition 3.1.4 (Restricted unitary group)**.** *Let*

$$\mathcal{U}_{res} := \{U \in \mathcal{B}(H) | U \text{ unitary}, \;\; P_+UP_- \in \mathcal{Q}, \;\; P_-UP_+ \in \mathcal{Q}\}.$$

Let $\rho : \mathcal{U}_{res} \times \mathcal{U}_{res} \to \mathbb{R}_{\geq 0}$ *be the function defined by*

$$\rho(U, V) = ||U - V|| + \frac{1}{2}|P_+(U - V)P_-|_{\mathcal{Q}} + \frac{1}{2}|P_-(U - V)P_+|_{\mathcal{Q}}.$$

The pair $(\mathcal{U}_{res}, \rho)$ *is called the restricted unitary group of* H.

Remark. Notice that $(\mathcal{U}_{\text{res}}, \rho)$ is a topological group, with the topology inherited from ρ.

i. **Group structure:** To see that $\mathcal{U}_{\text{res}}$ has structure of group, let us observe that $\mathcal{U}_{\text{res}}$ is a subgroup of the group of unitary operators on H. Let $U, V \in \mathcal{U}_{\text{res}}$. Notice that:

$$P_\pm UV^* P_\mp = P_\pm U(P_\mp + P_\pm)V^* P_\mp = P_\pm UP_\mp(V^* P_\mp) + (P_\pm U)P_\pm V^* P_\mp.$$

Thus, by definition of $\mathcal{Q}$, since it is an ideal and $|\cdot|_{\mathcal{Q}}$ respects adjoint it follows that $UV^* \in \mathcal{U}_{\text{res}}$.

ii. **Continuity of group operations:** To check continuity of multiplication $m : \mathcal{U}_{\text{res}} \times \mathcal{U}_{\text{res}} \to \mathcal{U}_{\text{res}}$, with $m(U, V) = UV$, we can provide a metric to $\mathcal{U}_{\text{res}} \times \mathcal{U}_{\text{res}}$, called d, by setting

$$d((U_1, V_1), (U_2, V_2)) = \rho(U_1, U_2) + \rho(V_1, V_2).$$

It is clear that this metric generates the product topology on $\mathcal{U}_{\text{res}} \times \mathcal{U}_{\text{res}}$.
Thus, to prove continuity will be enough to consider sequences $(U_n)_{n\geq 1} \xrightarrow{\rho} U$, $(V_n)_{n\geq 1} \xrightarrow{\rho} V$, and observe that $(U_n V_n)_{n\geq 1}$ converges in ρ to UV. It will suffice to show that $|P_+(U_n V_n - UV)P_-|_{\mathcal{Q}} \to 0$. According to the definition of the metric ρ and since U_n converges to U and V_n converges to V, we have that $|P_+(U - U_n)P_-|_{\mathcal{Q}} \to 0$, $|P_+(V - V_n)P_-|_{\mathcal{Q}} \to 0$, $||U_n - U|| \to 0$ and $||V_n - V|| \to 0$. Let $\epsilon > 0$. We have:

$$\begin{aligned}
&|P_+UVP_- - P_+U_nV_nP_-|_{\mathcal{Q}} \\
&= |P_+UP_-VP_- + P_+UP_+VP_- - P_+U_nV_nP_- - P_+U_nP_-VP_- + P_+U_nP_-VP_-|_{\mathcal{Q}} \\
&\leq |(P_+(U - U_n)P_-)VP_-|_{\mathcal{Q}} + |P_+UP_+VP_- - P_+U_nV_nP_- + P_+U_nP_-VP_-|_{\mathcal{Q}},
\end{aligned}$$

but the first term goes to zero as n goes to ∞, so from certain N_1, we have that for all $n \geq N_1$, the expression above satisfies:

$$\begin{aligned}
&< \frac{\epsilon}{4} + |P_+UP_+VP_- - P_+U_nV_nP_- + P_+U_nP_-VP_- - P_+UP_+V_nP_- + P_+UP_+V_nP_-|_{\mathcal{Q}} \\
&\leq \frac{\epsilon}{4} + |P_+U(P_+(V - V_n)P_-)|_{\mathcal{Q}} + |P_+UP_+V_nP_- - P_+U_nV_nP_- + P_+U_nP_-VP_-|_{\mathcal{Q}},
\end{aligned}$$

and, as before, noting again that the first term tend to zero, there exists $N_2 \geq N_1$ such that for all $n \geq N_2$:

$$\begin{aligned}
&< \frac{\epsilon}{2} + |P_+UP_+V_nP_- - P_+U_nP_+V_nP_- - P_+U_nP_-V_nP_- + P_+U_nP_-VP_-|_{\mathcal{Q}} \\
&\leq \frac{\epsilon}{2} + |P_+(U - U_n)P_+V_nP_-|_{\mathcal{Q}} + |P_+U_nP_-(V - V_n)P_-|_{\mathcal{Q}}.
\end{aligned}$$

Since $P_+V_nP_- \xrightarrow{|\cdot|_{\mathcal{Q}}} P_+VP_-$ (same for U_n), this sequence is $|\cdot|_{\mathcal{Q}}$ - bounded. Thus, using the properties of the symmetric norm, $|P_+(U - U_n)P_+V_nP_-|_{\mathcal{Q}} \leq ||U - U_n||\ |P_+V_nP_-|_{\mathcal{Q}} \leq ||U - U_n||M \to 0$. Then we can find $N_3 \geq N_2$ such that for all $n \geq N_3$ the expression is less than ϵ.

In view of the fact that multiplication is continuous in operator norm $||\cdot||$, the computation above then shows that the sequence $(U_nV_n)_{n\geq 1}$ converges to UV in the metric ρ. Therefore, continuity of multiplication map has been proved.
For the inverse map, which in this case is simply taking adjoints, we can use that $|\cdot^*|_{\mathcal{Q}} = |\cdot|_{\mathcal{Q}}$.

Definition 3.1.5. *Let $i\mathcal{A}_{res}$ denote the subset of self-adjoint operators given by*

$$\{iA | A \text{ bounded, self-adjoint on } H, P_+AP_- \in \mathcal{Q}\}.$$

Let $|||\cdot||| : \mathcal{A}_{res} \to \mathbb{R}_{\geq 0}$ be a norm defined on $i\mathcal{A}_{res}$ given by:

$$|||A||| = ||A|| + |P_+AP_-|_{\mathcal{Q}}.$$

We call the pair $(i\mathcal{A}_{res}, |||\cdot|||)$ the Lie algebra of the restricted group $\mathcal{U}_{res}$.

Given these topological structures, we shall show some remarkable properties.

Let $(U_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{U}_{\text{res}}$ in the ρ-topology. By definition of this metric, we have that $(U_n)_{n\geq 1}$ is also a Cauchy sequence in $\mathcal{B}(H)$ with respect to the operator-norm topology. Since this last space is a Banach space, the sequence converges to some U, which is a unitary operator since taking adjoint is continuous as a map $\mathcal{B}(H) \to \mathcal{B}(H)$. Thus, $(P_\pm U_n P_\mp)_{n\geq 1}$ converges in $\mathcal{B}(H)$ to $P_\pm U P_\mp$. Besides, we know that $(P_\pm U_n P_\mp)_{n\geq 1}$ is a Cauchy sequence in the $|\cdot|_{\mathcal{Q}}$-norm, therefore, by completeness, this sequence converges.

There is a relevant property associated to symmetrically normed ideals (see [12] §2 and §3, and [20] §7): for all $X \in \mathcal{Q}$, $||X|| \leq |X|_{\mathcal{Q}}$. Consequently, being V the limit of $P_+U_nP_- \xrightarrow{|\cdot|_{\mathcal{Q}}} V$, we have that $||V - P_+U_nP_-|| \leq |V - P_+U_nP_-|_{\mathcal{Q}} \xrightarrow{n\to\infty} 0$. Then, by uniqueness of the limit in $||\cdot||$, we must have that $V = P_+UP_-$, and again by completeness property of symmetrically normed ideals, $V \in \mathcal{Q}$. Similarly, we observe that $P_-UP_+ \in \mathcal{Q}$. As a result, $U_n \xrightarrow{\rho} U$:

$$\rho(U_n, U) = ||U - U_n|| + \frac{1}{2}|P_+U_nP_- - P_+UP_-|_{\mathcal{Q}} + \frac{1}{2}|P_-U_nP_+ - P_-UP_+|_{\mathcal{Q}} \xrightarrow{n\to\infty} 0.$$

Since $P_\pm UP_\mp \in \mathcal{Q}$, the sequence $(U_n)_{n\geq 1}$ converges to U in $\mathcal{U}_{\text{res}}$. Thus, we have observed that the restricted group $\mathcal{U}_{\text{res}}$, equipped with the ρ-topology, is a complete metric space. In a similar way it is possible to show that $\mathcal{A}_{\text{res}}$ is a complete metric space with respect to $|||\cdot|||$.

Now, let us define a product on $i\mathcal{A}_{\text{res}}$ as follows:

$$A \circ B := i[A, B],$$

where $[\cdot,\cdot]$ is the standard commutator. Notice that this product is well defined: for $A, B \in \mathcal{A}_{\text{res}}$, since they are self-adjoint, $[A, B]^* = BA - AB = -[A, B]$. Thus, $(A \circ B)^* = (i[A, B])^* =$

$i[A, B] = A \circ B$.
Now, we observe the following: let $A, B \in \mathcal{A}_{\text{res}}$, then:

$$\begin{aligned} |||AB||| &= ||AB|| + |P_+ABP_-|_{\mathcal{Q}} = ||AB|| + |P_+A(P_+ + P_-)BP_-|_{\mathcal{Q}} \\ &\leq ||A||\ ||B|| + |P_+AP_-BP_-|_{\mathcal{Q}} + |P_+AP_+BP_-|_{\mathcal{Q}} \\ &\leq ||A||\ ||B|| + ||B||\ |P_+AP_-|_{\mathcal{Q}} + ||A||\ |P_+BP_-|_{\mathcal{Q}} \\ &\leq (||A|| + |P_+AP_-|_{\mathcal{Q}})\,(||B|| + |P_+BP_-|_{\mathcal{Q}}) \\ &= |||A|||\ |||B|||. \end{aligned}$$

Thus, since $P_+AP_-, P_+BP_- \in \mathcal{Q}$, we get that $|||\ i[A, B]\ ||| \leq 2|||A|||\ |||B||| < \infty$. Therefore, $P_+(i[A, B])P_- \in \mathcal{Q}$, and so the product is well defined. Moreover, since the definition of $A \circ B$ involves the usual bracket for operators, it satisfies the Lie bracket properties.

Proposition 3.1.6. *For any $U_0 \in \mathcal{U}_{res}$ and any $A \in \mathcal{A}_{res}$, the curve $U_s = U_0 e^{isA}$, with $0 \leq s \leq 1$, lies in $\mathcal{U}_{res}$. Moreover, the curve U_s in continuous in the $\rho-$topology.*

Proof. To see that U_s lies in $\mathcal{U}_{\text{res}}$, we will use power series. By induction, suppose that $P_+A^nP_- \in \mathcal{Q}$ (base case is straightforward by definition). Then we have:

$$P_+A^{n+1}P_- = P_+A(P_+ + P_-)A^nP_- = P_+AP_+A^nP_- + P_+AP_-A^nP_-.$$

By assumptions, the right hand side lies in $\mathcal{Q}$ (it is an ideal). Therefore, for all $n \geq 0$, $P_+A^nP_- \in \mathcal{Q}$. Now, we observe that:

$$|P_+A^2P_-|_{\mathcal{Q}} \leq |P_+AP_+AP_-|_{\mathcal{Q}} + |P_+AP_-AP_-|_{\mathcal{Q}} \leq 2||A||\ \ |P_+AP_-|_{\mathcal{Q}}\,,$$

and using induction in n, we obtain:

$$|P_+A^nP_-|_{\mathcal{Q}} \leq n||A||^{n-1}\ \ |P_+AP_-|_{\mathcal{Q}}\,.$$

Let us define the sequence $(B_j)_{j\geq 1}$ as follows:

$$B_j := \sum_{n=0}^{j} \frac{(is)^n}{n!} P_+A^nP_-.$$

Using the results above, we observe that $\{B_j\}_{j\geq 1}$ is a Cauchy sequence:

$$|B_j - B_k|_{\mathcal{Q}} = \left| \sum_{n=j+1}^{k} \frac{(is)^n}{n!} P_+A^nP_- \right|_{\mathcal{Q}} \leq \sum_{n=j+1}^{k} \frac{|s|^n}{n!} ||A||^{n-1}\ \ |P_+AP_-|_{\mathcal{Q}} \xrightarrow{j,k\to\infty} 0$$

since the real convergence of exponential map. This limit is $P_+e^{isA}P_-$: to see this we use a similar argument given above for the uniqueness of the limit when we showed that $\mathcal{U}_{\text{res}}$ is complete. Moreover, since $\mathcal{Q}$ is complete, this limit lies in $\mathcal{Q}$.
Therefore, $U_s \in \mathcal{U}_{\text{res}}$ for all $0 \leq s \leq 1$:

i. The unitarity of U_s is trivial.

ii. $P_+U_sP_- \in \mathcal{Q}$

iii. $P_-U_sP_+ \in \mathcal{Q}$: To observe this, we use the self-adjointness condition on A: Since $P_+AP_- \in \mathcal{Q}$, its adjoint also belongs to $\mathcal{Q}$, i.e, $|P_-AP_+|_{\mathcal{Q}} \in \mathcal{Q}$. The argument follows analogously, using the sequence $(\tilde{B}_j)_{j\geq 1}$ given by:

$$\tilde{B}_j := \sum_{n=0}^{j} \frac{(is)^n}{n!} P_-A^nP_+.$$

Thus, we see that $e^{isA} \in \mathcal{U}_{\text{res}}$ (furthemore, it is true for all $s \in \mathbb{R}$).
Now, for the continuity statement, first we observe the following:

$$\left|P_+e^{isA}P_-\right|_{\mathcal{Q}} \leq \sum_{n=1}^{\infty} \frac{|s|^n}{n!} n||A||^{n-1} \ |P_+AP_-|_{\mathcal{Q}} = |s|e^{s||A||}\,|P_+AP_-|_{\mathcal{Q}}\,. \tag{3-1}$$

The equation (3-1) will be helpful for the following computation:

$$\begin{aligned} |P_+U_{s+t}P_- - P_+U_tP_-|_{\mathcal{Q}} &= |P_+U_t(P_+ + P_-)U_t{}^*U_{s+t}P_- - P_+U_tP_-|_{\mathcal{Q}} \\ &= |P_+U_tP_+U_t{}^*U_{s+t}P_- + P_+U_tP_-U_t{}^*U_{s+t}P_- - P_+U_tP_-|_{\mathcal{Q}}\,, \end{aligned}$$

and using the self-adjointness of A and that $U_t{}^*U_{s+t} = U_s$, we get:

$$\begin{aligned} &= \left|P_+U_tP_+e^{isA}P_- + P_+U_tP_-(e^{isA} - \mathbf{1})P_-\right|_{\mathcal{Q}} \\ &\leq \left|P_+e^{isA}P_-\right|_{\mathcal{Q}} + |P_+U_tP_-|_{\mathcal{Q}} \ ||e^{isA} - \mathbf{1}||. \end{aligned}$$

Using again relation (3-1), for $s \to 0$ we have that the first right hand term tends to 0. Moreover, with $s \to 0$, $||e^{isA} - \mathbf{1}||$ tends to 0 because

$$||e^{isA} - \mathbf{1}|| \leq \sum_{n\geq 1} \frac{|s|^n}{n!} ||A||^n = e^{|s| \ ||A||} - 1.$$

Hence, $P_+U_sP_-$ is continuous in $(\mathcal{Q}, |\cdot|_{\mathcal{Q}})$. Similarly, notice that $P_-U_sP_+$ is continuous in $(\mathcal{Q}, |\cdot|_{\mathcal{Q}})$. Finally, we see that $||U_{s+t} - U_t|| \leq ||e^{i(s+t)A} - e^{itA}|| \xrightarrow{s\to 0} 0$. Therefore, the curve U_t is continuous in the ρ-topology. □

Now, let us derive some identities for $U \in \mathcal{U}_{\text{res}}$ which will be useful to observe several algebraic properties for $P_\pm UP_\pm$. Since U is unitary, we have:

$$\begin{aligned} (A) \qquad & (P_\pm UP_\pm)(P_\pm U^*P_\pm) + (P_\pm UP_\mp)(P_\mp U^*P_\pm) \\ &= P_\pm UP_\pm U^*P_\pm + P_\pm UP_\mp U^*P_\pm \\ &= P_\pm U(P_\pm + P_\mp)U^*P_\pm = P_\pm. \end{aligned}$$

$$
\begin{aligned}
(B) \qquad & (P_\pm U^* P_\pm)(P_\pm U P_\pm) + (P_\pm U^* P_\mp)(P_\mp U P_\pm) \\
& = P_\pm U^* P_\pm U P_\pm + P_\pm U^* P_\mp U P_\pm \\
& = P_\pm U^* (P_\pm + P_\mp) U P_\pm = P_\pm .
\end{aligned}
$$

Furthermore, from these, we obtain:

$$
\begin{aligned}
(I) \qquad & (P_\pm U P_\pm)(P_\pm U^* P_\pm) = P_\pm - (P_\pm U P_\mp)(P_\mp U^* P_\pm), \\
(II) \qquad & (P_\pm U^* P_\pm)(P_\pm U P_\pm) = P_\pm - (P_\pm U^* P_\mp)(P_\mp U P_\pm).
\end{aligned}
$$

Thus, since $P_\pm U P_\mp \in \mathcal{Q}$, the second terms of the right hand side of (I) and (II) belong to $\mathcal{Q}$, and hence they are compact operators. When we restrict to $H_\pm := P_\pm(H)$, we have that they are inverse of each other modulus compact operator, and by Atkinson's theorem (see [26] §7), they are Fredholm operators (on $H_\pm$).

Lemma 3.1.7. *Let $U \in \mathcal{U}_{res}$ and denote $H_\pm := P_\pm(H)$. Then $P_\pm U P_\pm$ is a Fredholm operator on $H_\pm$. Moreover, dim Ker $P_\pm U P_\pm =$ dim Ker $P_\mp U^* P_\mp$.*

Proof. We already have shown the first part of the lemma. Now, for the second statement, we will show that $U^*\text{Ker } (P_\mp U^* P_\mp) = \text{Ker } (P_\pm U P_\pm)$. To do that, observe the following computation:

$$
\begin{aligned}
& (P_\pm U P_\pm)(P_\pm U^* P_\mp) + (P_\pm U P_\mp)(P_\mp U^* P_\mp) \\
& = P_\pm U P_\pm U^* P_\mp + P_\pm U P_\mp U^* P_\mp \\
& = P_\pm U (P_\pm + P_\mp) U^* P_\mp = P_\pm P_\mp = 0.
\end{aligned} \tag{3-2}
$$

For $f \in H_\mp \cap \text{Ker } (P_\mp U^* P_\mp)$, we have $0 = P_\mp U^* P_\mp(f) = P_\mp U^*(f)$, so $U^*(f) \in \text{Ker } P_\mp = H_\pm$. Notice that we want the part of Ker $(P_\mp U^* P_\mp)$ which lies in $H_\mp$; if we take the entire subspace, we must consider $H_\pm$, but we are analyzing these operators $P_\mp U^* P_\mp$ in each component $H_\mp$.

Using relation (3-2), we obtain the following result:

$$0 = (P_\pm U P_\pm)(P_\pm U^* P_\mp)(f) + (P_\pm U P_\mp)(P_\mp U^* P_\mp)(f),$$

and since $f \in H_\mp$, we conclude that $P_\mp(f) = f$. Thus, $0 = P_\pm U P_\pm(U^*(f)) + P_\pm U P_\mp(U^*(f))$, where the last term vanishes.

Consequently, seeing these operators as operators on $H_\pm$ respectively, we have shown that:

$$U^*\text{Ker } (P_\mp U^* P_\mp) \subseteq \text{Ker } (P_\pm U P_\pm).$$

Again, we must be careful to understand Ker $(P_\mp U^* P_\mp)$ as a subspace of $H_\mp$ and not as subspace of the whole space H.

To show the other inclusion, we consider a similar relation given in (3-2) above. It is given by:

$$(P_\mp U^* P_\mp)(P_\mp U P_\pm) + (P_\mp U^* P_\pm)(P_\pm U P_\pm) = 0. \tag{3-3}$$

Take $g \in H_{\pm} \cap \text{Ker } (P_{\pm}UP_{\pm})$. Hence, we have $0 = P_{\pm}U(g)$, so $U(g) \in \text{Ker } P_{\pm} = H_{\mp}$. Using (3-3), we obtain $0 = (P_{\mp}U^*P_{\mp})U(g)$. Consequently, seeing Ker $(P_{\pm}UP_{\pm})$ as a subspace of $H_{\pm}$, we observe that $U\text{Ker } (P_{\pm}UP_{\pm}) \subseteq \text{Ker } (P_{\mp}U^*P_{\mp})$.
Thus, we have obtained the equality $\text{Ker } (P_{\pm}UP_{\pm}) = U^*(\text{Ker } (P_{\mp}U^*P_{\mp}))$. Actually, what we have shown is:

$$U^*(H_{\mp} \cap \text{Ker } (P_{\mp}U^*P_{\mp})) \subseteq H_{\pm} \cap \text{Ker } (P_{\pm}UP_{\pm}) \text{ and}$$
$$U(H_{\pm} \cap \text{Ker } (P_{\pm}UP_{\pm})) \subseteq H_{\mp} \cap \text{Ker } (P_{\mp}U^*P_{\mp}),$$

and thus the equality $H_{\pm} \cap \text{Ker } (P_{\pm}UP_{\pm}) = U^*(H_{\mp} \cap \text{Ker })(P_{\mp}U^*P_{\mp}))$ holds.
With this last observation, we understand Ker $(P_{\pm}UP_{\pm})$ as subspace of $H_{\pm}$ and Ker $(P_{\mp}U^*P_{\mp})$ as subspace of $H_{\mp}$.
Now, since $P_{\pm}UP_{\pm}$ is a Fredholm operator on $H_{\pm}$, dim (Ker $(P_{\pm}UP_{\pm})$) is finite. Therefore, dim Ker $(P_{\mp}U^*P_{\mp}) = \text{dim } U^*(\text{Ker } (P_{\mp}U^*P_{\mp}))$ is finite. □

This result motivates a new definition which will relate the notion of a Fredholm index with the topological group $\mathcal{U}_{\text{res}}$.

Definition 3.1.8. *For $U \in \mathcal{U}_{res}$, define*

$$n_{\pm}(U) = dim\ Ker\ (P_{\pm}UP_{\pm}),$$

and

$$i(U) = n_{+}(U) - n_{-}(U).$$

Remark: From [26] §7, we have that for any Fredholm operator X on a Hilbert space H, its Kernel and Range are closed subspaces of H. According to this and the relations found in Lemma 3.1.7, we can rewrite the number $i(U)$ as follows:

$$\begin{aligned} i(U) = n_{+}(U) - n_{-}(U) = n_{+}(U) - n_{+}(U^*) &= \text{dim Ker } P_{+}UP_{+} - \text{dim Ker } (P_{+}U^*P_{+}) \\ &= \text{dim Ker } P_{+}UP_{+} - \text{dim Coker } (P_{+}UP_{+}). \end{aligned}$$

Therefore, we see that $i(U)$ corresponds to the Fredlholm index of the operator $P_{+}UP_{+} : H_{+} \to H_{+}$. In fact, we can consider i as a homomorphism $i : \mathcal{U}_{\text{res}} \to \mathbb{Z}$. To show this, first notice the following. Let $U_0, U_1 \in \mathcal{U}_{\text{res}}$, then:

$$\begin{aligned} P_{+}(U_0U_1)P_{+} = P_{+}U_0(P_{+} + P_{-})U_1P_{+} &= P_{+}U_0P_{+}U_1P_{+} + P_{+}U_0P_{-}U_1P_{+} \\ &= (P_{+}U_0P_{+})(P_{+}U_1P_{+}) + (P_{+}U_0P_{-})U_1P_{+}. \end{aligned}$$

Now, observe that the last term is a compact operator because $P_{+}U_0P_{-} \in \mathcal{Q}$. Then, since $i(U)$ is the index Fredholm of $P_{+}UP_{+}$, $i_F(P_{+}UP_{+})$, we have that:

$$\begin{aligned} i(U_0U_1) &= i_{\text{F}}(P_{+}U_0U_1P_{+}) = i_{\text{F}}((P_{+}U_0P_{+})(P_{+}U_1P_{+}) + C) \\ &= i_{\text{F}}((P_{+}U_0P_{+})) + i_{\text{F}}((P_{+}U_1P_{+})) = i(U_0) + i(U_1), \end{aligned}$$

where we have used the invariance under addition of compact operators and the fact that the Fredholm index is a homomorphism (see [26] §7).

Theorem 3.1.9. *If U_0, U_1 are points of $\mathcal{U}_{res}$ with $||U_0 - U_1|| < 2$, then there exists an operator $A \in \mathcal{A}_{res}$ such that $U_s = U_0 e^{isA}$, $0 \leq s \leq 1$, is a path in $\mathcal{U}_{res}$ connecting the two elements U_0, U_1. The operator will be uniquely determined by the extra condition that its spectrum should be contained in $(-\pi, \pi)$. The curve is continuous in the ρ-topology.*

Proof. In order to prove this theorem, we will use some results concerning to the continuous functional calculus and the measurable calculus for a C^*-algebra. In [9] §4 and [28] §3 there is a highly detailed description of these results.
Let $V = U_0^* U_1$. Since this operator is unitary, by the measurable functional calculus, we can find a self-adjoint operator A uniquely determined by its spectrum, which is contained in $[-\pi, \pi]$, and satisfies $V = e^{iA}$. We see that, in fact, its spectrum must be contained in $(-\pi, \pi)$. First, observe the following reltions:

$$||V - \mathbf{1}|| \leq ||U_0^*(U_1 - U_0)|| \leq ||U_1 - U_0|| < 2,$$

and

$$||V - \mathbf{1}|| \geq \underbrace{r(V - \mathbf{1})}_{\text{spectral radious}} = \sup\{|\lambda| \mid \lambda \in \sigma(V - \mathbf{1})\}.$$

Thus, using the continuous functional calculus with the function $e^{i\lambda}$ defined on $\Sigma = \sigma(A)$, if $\sigma(A)$ contains either π or $-\pi$, we would have that:

$$\sup\{|\lambda| \mid \lambda \in \sigma_{C(\Sigma)}(e^{ix} - 1)\} = \sup_{\lambda \in \Sigma} |e^{i\lambda} - 1| = 2,$$

where $\sigma_{C(\Sigma)}(f)$ denotes the spectrum of an element $f \in C(\Sigma)$.
For this reason, we must have $\sigma(A) \subseteq (-\pi, \pi)$. Now, since $\sigma(A)$ is a closed compact subset in $\mathbb{R}$, we can bound $\sigma(A)$ with a closed interval $[m_-, m_+]$ such that $-\pi < m_- \leq m_+ < \pi$.
Our goal will be to construct A as a trigonometric series from V using Fourier analysis. To do that, consider a function f with the following properties:

i. f must be an odd and periodic function with period of 2π.

ii. $f \in C^\infty(\mathbb{R})$, where all its derivatives must vanish at $\pm\pi$.

iii. For $m_- \leq x \leq m_+$, $f(x) = x$.

For instance, the function f might have the graph as:

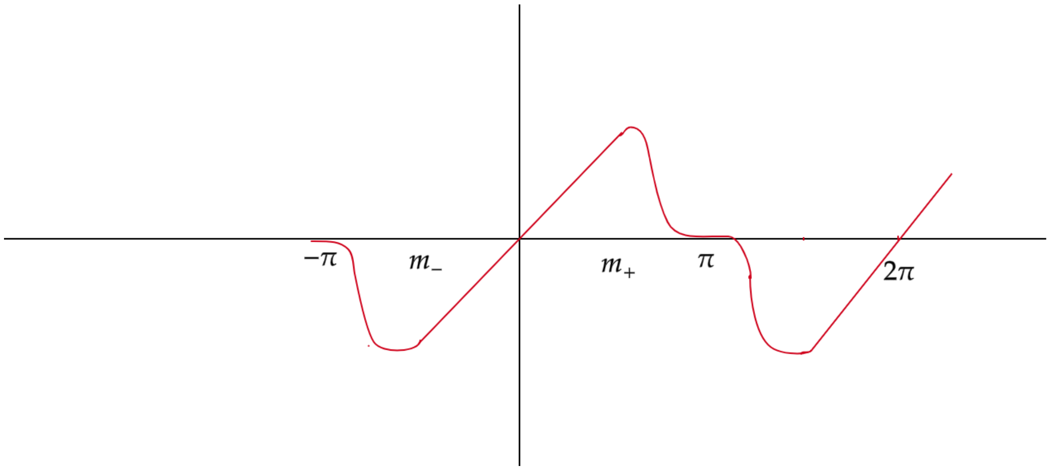


Figure 3-1: Graph of a function f which satisfies conditions 1, 2 and 3.

Since this function is well behaved, its Fourier series converges uniformly in $[-\pi,\pi]$, and it is given by:

$$f(x)=\sum_{n=0}^{\infty}b_n\sin nx,\quad b_n=\frac{1}{\pi}\sin nx f(x)\ dx,$$

(cos-terms vanish because odd condition of the function). Due to the C^∞ condition on f and the vanishing of all derivatives in $\pm\pi$, we have that $|n^k b_n|\xrightarrow{n\to\infty}0$ for all $k\geq 0$: for k even,

$$0=|f^{(k)}(\pm\pi)|=|\sum_{n\geq 1}n^k b_n(-1)^n|,$$

and by convergence of the series, $|n^k b_n|\xrightarrow{n\to\infty}0$; now, if k is odd, $k=2l+1$, we have that $2l\leq k\leq 2(l+1)$, so $|n^k b_n|\leq|n^{2(l+1)}b_n|\xrightarrow{n\to\infty}0$.
Since $\mathcal{Q}$ is an ideal, and using the relation:

$$P_\pm V^{n+1}P_\mp=P_\pm V(P_++P_-)V^nP_\mp=P_\pm VP_-V^nP_\mp+P_\pm VP_+V^nP_\mp,$$

we see via induction that for all $n\geq 0$, $P_\pm V^nP_\mp\in\mathcal{Q}$ and, moreover:

$$|P_\pm V^nP_\mp|_{\mathcal{Q}}\leq n|P_\pm VP_\mp|_{\mathcal{Q}}.$$

Now, by the continuous functional calculus applied to the self-adjoint operator A, and considering the function $\sin nx=\frac{1}{2i}\left(e^{inx}-e^{-inx}\right)$, we see that $\sin(nA)=\frac{1}{2i}(V^n-V^{*n})$. Therefore, we obtain the following relation:

$$|P_+\sin nAP_-|_{\mathcal{Q}}\leq\frac{1}{2}\left(|P_+V^nP_-|_{\mathcal{Q}}+|P_+V^{*n}P_-|_{\mathcal{Q}}\right)=\frac{1}{2}\left(|P_+V^nP_-|_{\mathcal{Q}}+|P_-V^nP_+|_{\mathcal{Q}}\right)$$
$$\leq\frac{n}{2}\left(|P_+VP_-|_{\mathcal{Q}}+|P_-VP_+|_{\mathcal{Q}}\right)=na,$$

where $a = \frac{1}{2}\left(|P_+VP_-|_{\mathcal{Q}} + |P_-VP_+|_{\mathcal{Q}}\right)$.
With these results, we can analyze the sequence $(T_j)_{j\geq 1}$ defined as:

$$T_j := P_+\left(\sum_{n=1}^{j} b_n \sin nA\right) P_-.$$

Observe that this sequence is a Cauchy sequence in $\mathcal{Q}$:

$$\begin{aligned}|T_j - T_k|_{\mathcal{Q}} &= \left|\sum_{j+1}^{k} b_n P_+ \sin nAP_-\right|_{\mathcal{Q}}, \quad (\text{with } k > j)\\ &\leq a \sum_{n=j+1}^{k} n|b_n| \leq a \sum_{n=j+1}^{\infty} n|b_n| \xrightarrow{j\to\infty} 0,\end{aligned}$$

where the last limit follows since $|n^k b_n| \xrightarrow{n\to\infty} 0$ for all $k \geq 1$ (there must exist an integer $N > 0$ such that for all $m \geq N$ $|b_m| \leq 1/m^3$, otherwise the sequence $n^4|b_n|$ would not converge, contradicting our observation above).
Therefore, by completeness of $\mathcal{Q}$, sequence $(T_j)_{j\geq 1}$ converges in $\mathcal{Q}$. Thus,

$$P_+f(A)P_- = P_+\sum_{n=1}^{\infty} b_n \sin nA\ P_-$$

lies in $\mathcal{Q}$ (using one more time the continuous functional calculus for A).
Applying now the measurable functional calculus, there exists a unique spectral measure $\mathcal{B}_\Sigma \ni E \to P(E)$, with $\Sigma = \sigma(A)$ and $\mathcal{B}_\Sigma$ the Borel sigma algebra on Σ, such that $A = \int_\Sigma \lambda\, dE(\lambda)$. By property 3 of the constructed function f, we have that $f(\lambda) = \lambda$ on Σ, so $f(A) = \int_\Sigma f(\lambda)\, dE(\lambda) = \int_\Sigma \lambda\, dE(\lambda) = A$. Consequently, $P_+AP_- \in \mathcal{Q}$ and, thus, $A \in \mathcal{A}_{\text{res}}$.
Consider the path $U_s = U_0 e^{isA}$ with $0 \leq s \leq 1$. According to lemma 3.1.6, this path is continuous in the ρ-topology on $\mathcal{U}_{\text{res}}$, and it connects U_0 with U_1. □

This theorem has multiple important corollaries which will help with the understanding of the path components of this topological group.

Corollary 3.1.10. *Suppose that $U_s = e^{isA}$ is a uniformly continuous one parameter group. Then $U_s \in \mathcal{U}_{res}$ for all s if and only if $A \in \mathcal{A}_{res}$.*

Proof. Lemma 3.1.6 covers the case when $A \in \mathcal{A}_{\text{res}}$. Now, suppose that $U_s = e^{isA}$ is a uniformly continuous one parameter group. Since $U_s \in \mathcal{U}_{\text{res}}$, the operator sA must be self-adjoint. According to the continuous functional calculus, there exists an isometric C^* isomorphism $C(\Sigma) \to C^*(\{\mathbf{1}, A\})$, with $\Sigma = \sigma(A)$. Take the function $x \mapsto e^{ix} - 1$, which has norm $= \sup_{\lambda\in\Sigma} |e^{i\lambda} - 1|$. Choosing a small value of s we can attain this norm < 2: by the spectral mapping theorem (see [28] §3), $\sigma(sA) = s\sigma(A)$, so if $\sigma(A) \subseteq [a, b]$ then for $s > 0$, $s\sigma(A) \subseteq [sa, sb]$, reducing the interval. Therefore, with an appropriate value of s, we see $||e^{isA} - \mathbf{1}|| < 2$. In this order, by theorem 3.1.9, there exists a unique self-adjoint operator $\tilde{A} \in \mathcal{A}_{\text{res}}$ such that the path $V_t = e^{it\tilde{A}}$, with $0 \leq t \leq 1$, connects $\mathbf{1}$ with e^{isA}. Therefore, $e^{isA} = e^{i\tilde{A}}$; thus $sA = \tilde{A} + 2\pi k\mathbf{1}$, and then, clearly, $A \in \mathcal{A}_{\text{res}}$. □

Corollary 3.1.11. *The connected component of $\mathcal{U}_{res}$ containing U is the set of all finite products*

$$Ue^{iA_1}e^{iA_2}\cdots e^{iA_n},$$

with $A_j \in \mathcal{A}_{res}$.

Proof. Suppose that V is connected to U by a continuous path in the ρ-topology. This path is the image of a compact set under a continuous map, so it is compact. Therefore, we can cover it by a finite number of balls of ratio 2. Since the ρ-norm involves the operator norm $||\cdot||$ in its definition, it follows that elements in these balls satisfy $||\cdot|| < 2$.

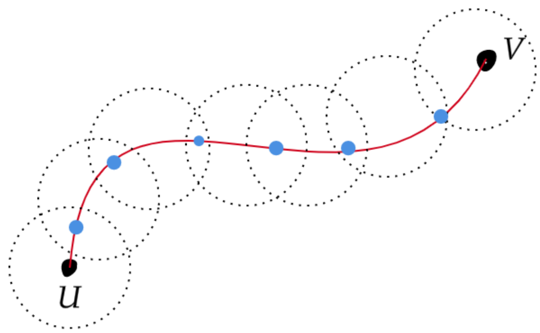


Figure 3-2: Given U, V connected through a curve, by compactness we can cover the path by a finite number of ρ-balls of ratio 2. Blue dots denote points in the intersection of these balls.

Hence, we have a collection of points $U = U_0, U_1, \ldots, U_n = V$ on the curve such that $||U_k - U_{k+1}|| < 2$. By theorem 3.1.9, there exists operators $A_{k+1} \in \mathcal{A}_{\text{res}}$ such that $U_{k+1} = U_k e^{iA_{k+1}}$, $k = 0, 1, \ldots, n-1$. Thus, $V = Ue^{iA_1}\cdots e^{iA_n}$.
Conversely, suppose that $V = Ue^{iA_1}\cdots e^{iA_n}$, with $A_i \in \mathcal{A}_{\text{res}}$. Define the path

$$U_s := U_k e^{i(s-k)A_{k+1}}, \text{ with } k \leq s \leq k+1,$$

for $k = 0, \ldots, n-1$.By lemma 3.1.6 these are continuous paths on $\mathcal{U}_{\text{res}}$ which connect $U_0, U_1, \ldots, U_n$. Then, this path connects U with V:
Take $k = 0$ and consider the continuous path $U_s = U_0 e^{isA_1}$ with $0 \leq s \leq 1$ which connects U_0 with $U_0 e^{iA_1}$. Now, take other path $U_0 e^{iA_1} e^{i(s-1)A_2}$ with $1 \leq s \leq 2$, which connects $U_0 e^{iA_1}$ with $U_0 e^{iA_1} e^{iA_2}$, and repeat the process $n-1$ times. □

The proof of the corollary above shows a more general characteristic of the topological group $\mathcal{U}_{\text{res}}$. Although we have been developed some properties for the topological space $(\mathcal{U}_{\text{res}}, \rho)$, we can show that $\mathcal{U}_{\text{res}}$ endowed with the subspace topology $||\cdot||$ has the same path components of $(\mathcal{U}_{\text{res}}, \rho)$.

Proposition 3.1.12. *Two points in $\mathcal{U}_{res}$ can be connected by a continuous path in the $||\cdot||$-topology if and only if they can be connected by a continuous path in the ρ-topology.*

Proof. One direction is trivial: If $U, V \in \mathcal{U}_{\text{res}}$ can be connected by a continuous path γ in the ρ-topology, this path is also continuous in the $||\cdot||$-topology since, by definition of the norm ρ, $||A - B|| \leq \rho(A, B)$.
Conversely, if $U, V \in \mathcal{U}_{\text{res}}$ can be connected by a continuous path γ in the $||\cdot||$-topology, we can construct a new path using a similar argument given in the proof of the previous corollary. By continuity and compactness, we can find n points $U = U_0, U_1, \ldots, U_n = V$ on the curve such that $||U_k - U_{k+1}|| \leq 2$. Thus, we can construct a continuous path in the ρ-topology which connects U and V. □

Continuing with the corollaries from theorem 3.1.9, we have the following:

Corollary 3.1.13. *If U and V are in different connected components of $\mathcal{U}_{res}$, then $||U - V|| = 2$.*

Proof. Since U and V are unitary operators, $||U - V|| \leq 2$. We have already proved the contrapositive statement: if $||U - V|| < 2$, then U and V are in the same path component. □

Why is this property necessary? Let us consider the following example where we will obtain a contradiction.

Example: Let $H^2(\mathbb{T}, \mathbb{C})$ be the Hardy space of the Hilbert space $L^2(\mathbb{T}, \mathbb{C})$, where $\mathbb{T}$ denotes the unitary circle, and it is identified with the interval $[-\pi, \pi]$. Let $\{e^{-inx}\}_{n\in\mathbb{Z}}$ a basis for the Hilbert space. Hence, we expand each element f by its Fourier series:

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f_n e^{-inx}, \quad f_n = \int_{-\pi}^{\pi} f(x) e^{inx}\, dx.$$

Let P_+ be the orthogonal projection over $H^2(\mathbb{T}, \mathbb{C})$ given by:

$$(P_\pm f)_n = \Theta(\pm(n + 1/2)) f_n, \text{ where } \Theta(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

Let V denote the operation of multiply by a function v with Fourier coefficients $\{v_n\}_{n\in\mathbb{Z}}$. Computing V over each basis element:

$$V(e^{-inx}) = v e^{-inx} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} v_k e^{-i(n+k)x},$$

it is straightforward how we can represent, on the chosen basis, the operator V with the matrix:

$$V_{mn} = \frac{1}{2\pi} v_{m-n}.$$

Using a similar argument, it follows that:

$$P_\pm V P_\mp (e^{-inx}) = \frac{1}{2\pi} \Theta(\mp(n + 1/2)) \sum_{k\in\mathbb{Z}} v_k \Theta(\pm(k + n + 1/2)) e^{-i(k+n)x}.$$

Thus, we have:

$$(P_\pm V P_\mp)_{mn} = \frac{1}{2\pi}\Theta(\mp(n+1/2))v_{m-n}\Theta(\pm(m+1/2)).$$

Now, considering the symmetric normed ideal of Hilbert-Schmidt operators, and its symmetric norm, we see that:

$$\begin{aligned}|P_\pm V P_\mp|_2^2 &= \text{Tr }((P_\pm V P_\mp)^*(P_\pm V P_\mp)) = \sum_{n\in\mathbb{Z}}\langle P_\pm V P_\mp(e^{-inx}), P_\pm V P_\mp(e^{-inx})\rangle \\ &= \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}}\Theta(\mp(n+1/2))^2\sum_{k,l\in\mathbb{Z}} v_k\bar{v}_l\Theta(\pm(k+n+1/2))\Theta(\pm(l+n+1/2))\langle e^{-i(k+n)x}, e^{-i(l+n)x}\rangle \\ &= \frac{1}{4\pi^2}\sum_{n,k\in\mathbb{Z}}\Theta(\mp(n+1/2))\Theta(\pm(k+n+1/2))|v_k|^2.\end{aligned}$$

Thus, opening the sum and expanding, we obtain:

$$|P_\pm V P_\mp|_2^2 = \frac{1}{4\pi^2}\sum_{k\geq 1} k|v_{\pm k}|^2.$$

In particular, consider the operator U_s of multiplication by the function e^{-isx}. When $s=1$, we have the bilateral shift, with matrix coefficients given by:

$$(U_1)_{mn} = \frac{1}{2\pi}\delta_{1,m-n}.$$

Observe that U_1 is a unitary operator and, moreover, it satisfies the relation:

$$|P_\pm U_1 P_\mp|_2^2 = \frac{1}{4\pi^2} < \infty.$$

Denote by X the operator of multiplication by x. This is a self-adjoint operator and satisfies $\sigma(X) = [-\pi,\pi]$. Therefore, applying the functional calculus over X, we have that

$$||U_1 - \mathbf{1}|| = ||e^{iX} - \mathbf{1}|| = \sup_{\lambda\in\sigma(X)} |e^{i\lambda} - 1| = 2.$$

Therefore, we can not apply 3.1.9. The function x can be expanded as $x = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{in} e^{-inx}$ in the interval $-\pi < x < \pi$. Using this and the relations above, we observe that:

$$|P_+ X P_-|_2^2 = \frac{1}{4\pi^2}\sum_{j=1}^{\infty} j|x_j|^2 = \frac{1}{4\pi^2}\sum_{j\geq 1}\frac{1}{j} = \infty.$$

Now, consider the unitary operator $U_{1/2} = e^{-ix/2}$. Its Fourier series is given by:

$$e^{-ix/2} = \sum_{n\in\mathbb{Z}} \frac{\sin\pi(n-1/2)}{\pi(n-1/2)} e^{-inx}.$$

Thus, applying the formulas above, we have that:

$$|P_+ U_{1/2} P_-|_2^2 = \frac{1}{4\pi^2}\sum_{k\geq 1} k\frac{1}{\pi(k-1/2)^2} = \infty.$$

Hence, the curve e^{isX} fails to be contained in the restricted group $\mathcal{U}_{\text{res}}$.

3.1.2 $\mathbb{Z}$ Index

Throughout the previous subsection we have developed some topological tools and properties associated with the unitary restricted group $\mathcal{U}_{\text{res}}$ for a given symmetrically normed ideal $\mathcal{Q}$ and a projection P_+. Our next objective will be to understand how the Fredholm index of P_+UP_+ classifies the path component of our unitary restricted group $\mathcal{U}_{\text{res}}$.
To start with, consider the decomposition $H = H_+ \bigoplus H_-$. Given $U \in \mathcal{U}_{\text{res}}$, using this decomposition, we write U as:

$$U = \begin{pmatrix} P_+UP_+ & P_+UP_- \\ P_-UP_+ & P_-UP_- \end{pmatrix}.$$

Using the polar decomposition for the operator $P_\pm UP_\mp$ we have

$$P_\pm UP_\mp = I_\pm \text{abs } (P_\pm UP_\mp),$$

with $I_\pm$ a partial isometry with the property ker $(I_\pm) =$ ker $(P_\pm UP_\mp)$.
By proposition 2.2.17, we have that $I_\pm|_{(\text{Ker } I_\pm)^\perp} : (\text{ker } I_\pm)^\perp \to \text{ran } I_\pm$ is an isometry; therefore, there exist operators $K_\pm$ such that $I_\pm^* I_\pm = \mathbf{1} - K_\pm$, where $K_\pm$ are the orthogonal projections over ker $I_\pm$. Similarly, we can find orthogonal projections $C_\pm$ over (ran $I_\pm)^\perp$ such that $I_\pm I_\pm^* = \mathbf{1} - C_\pm$.
Now, according to lemma 3.1.7, dim ker $P_\pm UP_\pm < \infty$, so dim ker $P_\pm UP_\pm =$ dim ker $I_\pm =$ dim ran $K_\pm < \infty$.
Remark. Let $A = J$ abs(A) be the polar decomposition of A, with ker $A =$ ker $J =$ ker abs(A). Since J^*J is the projection operator on (ker $J)^\perp =$ (ker (abs$(A)))^\perp = \overline{\text{ran abs}(A)}$, we have that J^*J abs$(A) =$ abs(A). Additionally, abs$(A)J^*J = (J^*J \text{ abs}(A))^* =$ abs(A). Let $S = J$ abs$(A)J^*$, and notice that this operator is positive:

$$\langle J \text{ abs}(A)J^*(x), x\rangle = \langle \text{abs}(A)J^*(x), J^*(x)\rangle = \langle (\text{abs}(A))^{1/2}J^*(x), (\text{abs}(A))^{1/2}J^*(x)\rangle \ \geq 0.$$

Moreover, observe that the following identity holds:

$$(\text{abs}(A^*))^2 = AA^* = J\text{abs}(A) \text{ abs}(A)J^* = S^2,$$

and since both of them are positive, by the square root lemma 2.2.15, abs$(A^*) = S$. Therefore, we obtain that:

$$A^* = \text{abs}(A)J^* = J^*J \text{ abs}(A)J^* = J^*S = J^* \text{ abs}(A^*).$$

From the first equality, we see that ker $J^* \subseteq$ ker A^*. Now, if $x \in$ ker A^*, we have that $J^*(x) \in$ ker abs $(A) =$ ker $A =$ ker J, i.e, $JJ^*(x) = 0$, and therefore

$$0 = \langle JJ^*(x), x\rangle = \langle J^*(x), J^*(x)\rangle \Leftrightarrow J^*(x) = 0 \Leftrightarrow x \in \text{ker } J^*.$$

Thus, ker $J^* =$ ker A^*.
Using the remark above and lemma 3.1.7 we get the following series of implications: since

$$\text{dim ker } P_\pm UP_\pm = \text{dim ker } P_\mp U^* P_\mp,$$

and considering the polar decomposition $P_\mp U P_\mp = I_\mp \text{ abs}(P_\mp U P_\mp)$, we have that ker $I_\mp^* =$ ker $P_\mp U^* P_\mp$, and thus dim ker $P_\mp U^* P_\mp =$ dim ker $I_\mp^* = (\text{dim ran} I_\mp)^\perp =$ dim ran $C_\mp$. As a result, we have the identity

$$\text{dim ran } K_\pm = \text{dim ran } C_\mp.$$

Since these spaces have equal dimensions, we can construct unitary operators

$$S_\pm : \text{ran } K_\mp \to \text{ran } C_\pm.$$

To do that, just consider an orthonormal basis for each subspace, $\{\phi_i\}_{i=1}^n$ for ran $K_\mp$ and $\{\psi_j\}_{j=1}^n$ for ran $C_\pm$, and define $S_\pm(\phi_i) = \psi_i$.

Notice that $S_+ \circ K_- : H_- \xrightarrow{K_-} \text{ran } K_- \xrightarrow{S_+} \text{ran } C_+$, and $C_+ \circ S_+ : \text{ran } K_- \xrightarrow{S_+} \text{ran } C_+ \xrightarrow{C_+} \text{ran } C_+$, where the last map acts as the identity map. We can be more general with the definition of $S_\pm$: Since ran $C_\pm$ and ran $K_\pm$ are closed subspaces, we can define $S_\pm : H_\mp \to H_\pm$ in such a way that for $x \in (\text{ran } K_\mp)^\perp$, $S_\pm(x) = 0$.
In this order, for $x \in (\text{ran } K_-)^\perp$ we have that $S_+ \circ K_-(x) = 0$ and $C_+ \circ S_+(x) = 0$, and if $x \in \text{ran } K_-$ then $S_+ \circ K_-(x) = S_+(x)$. On the other hand, $C_+ \circ S_+(x) = S_+(x)$ because $S_+(x) \in \text{ran } C_+$. Similarly, we notice that $S_- \circ K_+ = C_- \circ S_-$.
With these definitions, we construct a unitary operator $\overline{U}$ on $H = H_+ \bigoplus H_-$ as follows:

$$\overline{U} = \begin{pmatrix} I_+ & S_+K_- \\ S_-K_+ & I_- \end{pmatrix}$$

First, note that $\overline{U}$ is, in effect, a bounded operator on H: by the polar decomposition, we can interpret $I_\pm$ as operators $H_\pm \to H_\pm$, and by construction of $S_\pm$, we know that $S_\pm K_\mp : H_\mp \to H_\pm$. In the same order, since all these operators are bounded, we can conclude that $\overline{U}$ is a bounded operator on H. Now, notice that indeed $\overline{U}$ is unitary:

$$\overline{U}\,\overline{U}^* = \begin{pmatrix} I_+ & S_+K_- \\ S_-K_+ & I_- \end{pmatrix} \begin{pmatrix} I_+^* & K_+S_-^* \\ K_-S_+^* & I_-^* \end{pmatrix} = \begin{pmatrix} I_+I_+^* + S_+K_-S_+^* & I_+K_+S_-^* + S_+K_-I_-^* \\ S_-K_+I_+^* + I_-K_-S_+^* & S_-K_+S_-^* + I_-I_-^* \end{pmatrix},$$

and knowing that $S_\pm$ are local isometries by definition, i.e, $S_\pm^* \circ S_\pm|_{\text{ran } K_\mp} = \mathbf{1}$ and $S_\pm \circ S_\pm^*|_{\text{ran } C_\pm} = \mathbf{1}$, each matrix element above simplifies as follows:

- $S_+K_-S_+^* = C_+S_+S_+^*$ according to the relations established previously, and since ker $S_+^* = (\text{ran } S_+)^\perp = (\text{ran } C_+)^\perp$ we obtain $C_+S_+S_+^* = C_+$: for $z = x + y$ with $x \in \text{Ran } C_+$ and $y \in \text{Ran}^\perp C_+$, it follows that

  $$C_+S_+S_+^*(z) = C_+S_+S_+^*(x) = C_+(x) = C_+(x+y) = C_+(z).$$

  Therefore, the first element is $\mathbf{1}_{H_+}$.

- The other diagonal entry is similar.

- $I_+K_+S_-^* = 0$ since K_+ is a projection over ker I_+, and $S_+K_-I_-^* = S_+(I_-K_-)^* = 0$ by the same reason. The remaining entry is analyzed identically, and we obtain that it is 0.

Therefore, we have seen that $\overline{U}\,\overline{U}^* = \mathbf{1}$. Using the same arguments, the relation $\overline{U}^*\,\overline{U} = \mathbf{1}$ also holds.

In addition to the properties shown above for $\overline{U}$, we also have that $\overline{U} \in \mathcal{U}_{\text{res}}$: To see this, we shall show that $P_\pm \overline{U} P_\mp$ has finite rank. Writing $P_\pm$ in matrix form, we have:

$$P_+\overline{U}P_- = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_+ & S_+K_- \\ S_-K_+ & I_- \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & S_+K_- \\ 0 & 0 \end{pmatrix},$$

and since rank $S_+K_- =$ rank $C_+ < \infty$ we obtain the desired result. For $P_-\overline{U}P_+$ is analogous. In order to associate properties of the Fredholm index to the topological group $\mathcal{U}_{\text{res}}$, we shall need a main result from the Fredholm index theory, whicn can be found in [26] §7:

Proposition 3.1.14. *Let H be a separable Hilbert space, and denote by Fred (H) the space of Fredholm operators on H. Then Fred (H) is an open subset of $\mathcal{B}(H)$ with the operator norm topology. Moreover, the index function*

$$\textit{Index: Fred } (H) \to \mathbb{Z}$$

is constant on the path connected components of Fred (H), and two Fredholm operators belong to the same path component of Fred (H) if and only if they have the same index. If dim $H = \infty$, the index map is surjective.

Additionally, we shall analyze several intermediate steps:

i. Any $U \in \mathcal{U}_{\text{res}}$ with $n_\pm(U) = 0$ is connected to the identity map $\mathbf{1}$.

ii. For all $U \in \mathcal{U}_{\text{res}}$, $n_\pm(U) = n_\pm(\overline{U})$ and $n_\pm(\overline{U}^* U) = 0$.

iii. Any $\overline{U} \in \mathcal{U}_{\text{res}}$ with $i(\overline{U}) = 0$ can be connected to the identity $\mathbf{1}$.

Results (i) and (ii) tell us that U is connected to $\overline{U}$: since $n_\pm(\overline{U}^* U) = 0$, by (i) there exists a ρ-continuous path $\phi : [0,1] \to \mathcal{U}_{\text{res}}$ such that $\phi(0) = \mathbf{1}$ and $\phi(1) = \overline{U}^* U$, and thus $\overline{U}\phi(t)$ is a continuous curve connecting $\overline{U}$ with U. Consequently, if $i(U) = 0$, from (ii) we have that $i(\overline{U}) = 0$ and then, by (iii), there exists a ρ-continuous path $\gamma : [0,1] \to \mathcal{U}_{\text{res}}$ such that $\gamma(0) = \overline{U}$ and $\gamma(1) = \mathbf{1}$, so U is connected to $\mathbf{1}$. Now, if $U_0, U_1 \in \mathcal{U}_{\text{res}}$ satisfy $i(U_0) = i(U_1)$, we have that $0 = i(U_0) - i(U_1)$, and hence $0 = i(U_0^*) + i(U_1) = i(U_0^* U_1)$. Then $\overline{U_0^* U_1}$ is connected to $\mathbf{1}$; therefore, $U_0^* U_1$ is connected to $\mathbf{1}$ and thus U_0 is in the same path connected component as U_1. On the other hand, if U_0 is connected to U_1 in the ρ-topology, by proposition 3.1.12 we can find a continuous path in the $||\cdot||$-topology which connects U_0 with U_1, and multiplying by $P_\pm$ at both left and right sides, the resulting curve connects $P_\pm U_0 P_\pm$ with $P_\pm U_1 P_\pm$. As

a result, applying proposition 3.1.14, these operators have the same Fredholm index; thus $i(U_0) = i(U_1)$.
Once we demonstrate (i), (ii) and (iii), we shall have proved the following theorem:

Theorem 3.1.15. *(i) Let $U_0, U_1 \in \mathcal{U}_{res}$. Then U_0 is connected with U_1 in the ρ-topology if and only if $i(U_0) = i(U_1)$.*

(ii) The path connected components of $\mathcal{U}_{res}$ are the sets:

$$\mathcal{U}_k = \{U|\ U \in \mathcal{U}_{res},\ i(U) = k\},\ \ k \in \mathbb{Z}.$$

Now, in order to prove these assertions, we shall need a more general result (see [10]):

Proposition 3.1.16. *Let P_+ be an orthogonal projection over a Hilbert space H, and denote $H_\pm = ran\ P_\pm$. Let A be a bounded operator on H and consider the operator $P_+AP_+ : H+ \to H_+$. If P_+AP_+ is invertible (as an operator on H_+), then $\rho_P(A) := \sup\{|\langle Ap, q\rangle| \mid p \in H_+,\ q \in H_-,\ ||Ap|| = 1 = ||q||\} < 1$.*

Remark. The notation used in the proposition is suggested by Devinatz and Shinbrot in their paper *General Weiner-Hopf operators* where they studied necessary and sufficient conditions for invertibility for general Toeplitz operators denoted by $T_P(A)$.

Proof. First of all, we observe, using Cauchy-Schwarz inequality, that $\rho_P(A) \leq 1$ since $|\langle Ap, q\rangle| \leq ||Ap||\ ||q|| = 1$.
Now, suppose that $\rho_P(A) = 1$. Then, for any $\epsilon > 0$ there exists $p \in H_+$, $q \in H_-$ with $||Ap|| = ||q|| = 1$ such that $1 - \epsilon < |\langle Ap, q\rangle| \leq 1$.
Let us study some results from a functional analysis course. Given a normed space X, and denoting as X^* its dual space (bounded linear operators from X to the field $\mathbb{K}$), for any $l \in X^*$ we have the norm $||l|| = \sup\{|l(y)|\ y \in X, ||y|| = 1\}$. Now, if we take $l : H \to \mathbb{C}$ defined as $l(y) = \langle P_-A(p), y\rangle$, by the Riesz Representation theorem, we have that $||l|| = ||P_-A(p)||$, so

$$||P_-A(p)||^2 = \left(\sup\{|\langle P_-A(p), y\rangle|\ y \in H, ||y|| = 1\}\right)^2.$$

Moreover, observe that $\langle P_-A(p), y\rangle = \langle A(p), P_-(y)\rangle$, and we can take the supremum over the elements with $||P_-(y)|| = 1$ (any y can be decomposed as the sum $x_+ + x_-$ with $x_\pm \in H_\pm$, and thus $P_-(y) = P_-(x_-) = x_-$, with $||x_-|| \leq 1$, but by properties of operator norm, taking the supremum over elements with $||\cdot|| \leq 1$ is the same as taking supremum over elements with $||\cdot|| = 1$). Therefore, $||P_-A(p)||^2 = (\sup\{|\langle A(p), q'\rangle|\ q' \in H_-, ||q'|| = 1\})^2$.
With this assertion, we have that $||P_-A(p)||^2 \geq |\langle Ap, q\rangle^2 > (1-\epsilon)^2$. Now, observe that $1 = ||Ap||^2 = ||P_+Ap||^2 + ||P_-Ap||^2 > ||P_+Ap||^2 + (1-\epsilon)^2$. Therefore, $||P_+Ap||^2 < 2\epsilon - \epsilon^2 \leq 2\epsilon$.
Secondly, since by hypothesis the operator P_+AP_+ is invertible, it is bounded below, i.e, there exists $0 < m < \infty$ such that for all $x \in H_+$, $||P_+AP_+(x)|| \geq m||x||$. Thus, we have the inequalities:

$$m^2||p||^2 \leq ||P_+AP_+(p)||^2 = ||P_+A(p)||^2 < 2\epsilon.$$

In addition, we have trivially that $1 = ||Ap|| \leq ||A||\,||p|| \Rightarrow ||p|| \geq 1/||A||$.
With these observations we can get a contradiction: Taking $\epsilon < m^2/(2||A||^2)$, we have $||p||^2 < 1/||A||^2$ and $||p||^2 \geq 1/||A||^2$.
Consequently, $\rho_P(A) < 1$. □

We shall start now by showing the three main statements above.

Lemma 3.1.17. *Let $U \in \mathcal{U}_{res}$ with $n_\pm(U) = 0$. Then U can be connected to the identity map.*

Proof. If $n_\pm(U) = 0$, we have that dim ker $P_\pm U P_\pm = 0 =$ dim ker $P_\mp U^* P_\mp$ since $n_\pm(U) = n_\mp(U^*)$. Therefore, the orthogonal projections $K_\pm$ and $C_\pm$ are equal to 0. In this order, $\overline{U} = \begin{pmatrix} I_+ & 0 \\ 0 & I_- \end{pmatrix}$, and $I_\pm$ are unitary operators. Let us consider the following operators D and X:

$$D = \begin{pmatrix} \text{abs }(P_+UP_+) & 0 \\ 0 & \text{abs }(P_-UP_-) \end{pmatrix}, \quad X = \begin{pmatrix} 0 & I_+^* P_+UP_- \\ I_-^* P_-UP_+ & 0 \end{pmatrix}.$$

Now, observe that:

$$\overline{U}^* U = \begin{pmatrix} I_+ & 0 \\ 0 & I_- \end{pmatrix} \begin{pmatrix} P_+UP_+ & P_+UP_- \\ P_-UP_+ & P_-UP_- \end{pmatrix} = \begin{pmatrix} \text{abs }(P_+UP_+) & I_+^* P_+UP_- \\ I_-^* P_-UP_+ & \text{abs }(P_-UP_-) \end{pmatrix} = D + X.$$

In the above equality we have used the relations given by the polar decomposition and the partial isometries. Furthermore, we compute $D^2 + X^*X$:

$$\begin{aligned} D^2 + X^*X &= \begin{pmatrix} \text{abs }(P_+UP_+)^2 & 0 \\ 0 & \text{abs }(P_-UP_-)^2 \end{pmatrix} + \begin{pmatrix} 0 & P_+U^*P_-I_- \\ P_-U^*P_+I_+ & \end{pmatrix} \begin{pmatrix} 0 & I_+^* P_+UP_- \\ I_-^* P_-UP_+ & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_+U^*P_+UP_+ & 0 \\ 0 & P_-U^*P_-UP_- \end{pmatrix} + \begin{pmatrix} P_+U^*P_-UP_+ & 0 \\ 0 & P_-U^*P_+UP_- \end{pmatrix} \\ &= \begin{pmatrix} P_+U^*(P_+ + P_-)UP_+ & 0 \\ 0 & P_-U^*(P_- + P_+)UP_- \end{pmatrix} = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} = \mathbf{1}. \end{aligned}$$

Summarizing, we have obtained the identities $\overline{U}^* U = D + X$ and $D^2 + X^*X = \mathbf{1}$. Now, given that dim ker $(P_\pm U^* P_\pm) = 0$ and ker $(P_\pm U^* P_\pm) = (\text{ran }(P_\pm U P_\pm))^\perp$, we have that ran $(P_\pm U P_\pm) = H_\pm$, but we already know that $P_\pm U P_\pm$ is a Fredholm operator, so its range is closed. Therefore, since its kernel is trivial and its range is all $H_\pm$ respectively, we have that $P_\pm U P_\pm$ is invertible on $H_\pm$ (with bounded inverse on $H_\pm$).
According to the previous proposition, we have that $\rho_{P_\pm}(U) < 1$. Now, notice that $\langle U(p), q \rangle$, with $p \in H_+$ and $q \in H_-$ is equal to $\langle UP_+(p), P_-(q) \rangle$, and therefore $\rho_{P_\pm}(U) = \sup \{|\langle U(p), q\rangle| : p \in H_\pm, q \in H_\mp, ||U(p)|| = 1 = ||q||\} = \sup \{|\langle P_\mp U P_\pm(p), q\rangle| : ||p|| = 1 = ||q||\} = ||P_\mp U P_\pm|| < 1$.
In the computation at the beginning of the proof, we have showed that:

$$X^*X = \begin{pmatrix} P_+U^*P_-UP_+ & 0 \\ 0 & P_-U^*P_+UP_- \end{pmatrix}.$$

Thus, given $z \in H$ decomposed as $y_+ + y_-$ with $y_\pm \in H_\pm$, we have that $X^*X(z) = P_+U^*P_-UP_+(y_+) + P_-U^*P_+UP_-(y_-)$. In this way,

$$\begin{aligned}||X^*X(z)|| &\leq ||P_+U^*P_-UP_+(y_+)|| + ||P_-U^*P_+UP_-(y_-)|| \leq ||P_-UP_+||\ ||y_+|| + ||P_+UP_-||\ ||y_-|| \\ &\leq \max\{||P_-UP_+||, ||P_+UP_-||\}\ (||y_+|| + ||y_-||) \\ &= \max\{||P_-UP_+||, ||P_+UP_-||\}\ ||z|| < ||z||.\end{aligned}$$

Consequently, $||X||^2 = ||X^*X|| \leq \max\{||P_-UP_+||, ||P_+UP_-||\} < 1$, and so $||X|| < 1$. Moreover, we notice that $1 - X^*X$ is positive:

$$\langle(\mathbf{1} - X^*X)z, z\rangle = ||z||^2 - ||X(z)||^2 \geq 0.$$

Using the relation $D^2 = \mathbf{1} - X^*X$, the positiveness of D –which is clear by construction, and the uniqueness of the square root operator, we have that $D = (\mathbf{1} - X^*X)^{1/2}$. Since X^*X is positive and its norm is less than 1, we see that $\sigma(X^*X) \subseteq [0, 1)$ (here we use the spectral radius formula for normal operators, see [28] §3).
Let $A = \mathbf{1} - X^*X$; by the spectral mapping theorem, we have that $\sigma(A) \subseteq (0, 1]$. Consider the continuous function $f : \Sigma := \sigma(A) \to \mathbb{C}$ given by $f(x) = x^{1/2}$. Now, making use of the continuous functional calculus, we can conclude that $\sigma((\mathbf{1} - X^*X)^{1/2}) = \sigma(A^{1/2}) = \sigma_{C(\Sigma)}(f) = f(\Sigma) \subseteq (0, 1]$. Using, one more time, the spectral mapping theorem applied over $A^{1/2}$, we see that $\sigma(A^{1/2} - \mathbf{1}) \subseteq (-1, 0]$, and since this operator is self-adjoint, we have that $-||A^{1/2} - \mathbf{1}||$ lies in $\sigma(A^{1/2} - \mathbf{1})$. Therefore, $||A^{1/2} - \mathbf{1}|| < 1$, i.e. $||(\mathbf{1} - X^*X)^{1/2} - \mathbf{1}|| < 1$.
Hence, we have obtained the following inequalities:

$$||\overline{U}^*U - \mathbf{1}|| = ||D + X - \mathbf{1}|| \leq ||D - \mathbf{1}|| + ||X|| < 2.$$

Now, according to theorem 3.1.9, there exists a ρ-continuous path $\gamma : [0, 1] \to \mathcal{U}_{\text{res}}$ with $\gamma(0) = \mathbf{1}$ and $\gamma(1) = \overline{U}^*U$ and, thus, we can construct a ρ-continuous path connecting $\overline{U}$ with U.
It remains a last step in the proof of this lemma. Let us consider a unitary operator V on H_+, and define a new operator W on H given by $W = \begin{pmatrix} V & 0 \\ 0 & I_- \end{pmatrix}$. Observe that this operator is unitary because I_- is unitary on H_-, and, moreover, lies in $\mathcal{U}_{\text{res}}$:

$$P_+WP_- = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I_- \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} = 0, \quad P_-WP_+ = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & I_- \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

We have a main result concerning unitary operators on a complex Hilbert space H, which states that this group is path connected in the $||\cdot||-$ topology. In fact, this result is an easy consequence of the measurable functional calculus (see [28] §4).
Using these last two assertions, we are ready to construct a ρ-continuous path which connects $\overline{U}$ with $\mathbf{1}$. Since $I_\pm$ are unitary operators on $H_\pm$, we can connect both of them with $\mathbf{1}_{H_\pm}$,

respectively. Therefore, there exists a $||\cdot||-$ continuous path contained in $\mathcal{U}_{\text{res}}$ which connects $\overline{U}$ with $\mathbf{1}$ joining:

$$\overline{U} = \begin{pmatrix} I_+ & 0 \\ 0 & I_- \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{1}_{H_+} & 0 \\ 0 & I_- \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{1}_{H_+} & 0 \\ 0 & \mathbf{1}_{H_-} \end{pmatrix} = \mathbf{1}.$$

Finally, by proposition 3.1.12, we see that U is connected with the identity map. □

Lemma 3.1.18. *For any $U \in \mathcal{U}_{res}$, $n_\pm(\overline{U}) = n_\pm(U)$ and $n_\pm(\overline{U}^* U) = 0$. Thus, U and $\overline{U}$ are in the same path connected component of $\mathcal{U}_{res}$.*

Proof. Considering the following computation:

$$P_+\overline{U}P_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_+ & S_+K_- \\ S_-K_+ & I_- \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_+ & 0 \\ 0 & 0 \end{pmatrix},$$

we observe that dim ker (P_+UP_+) $=$ dim ker I_+ $=$ dim ker $(P_+\overline{U}P_+)$. Analogously, it can be shown that $P_-\overline{U}P_- = \begin{pmatrix} 0 & 0 \\ 0 & I_- \end{pmatrix}$ and, thus, we conclude that dim ker (P_-UP_-) $=$ dim ker I_- $=$ dim ker $(P_-\overline{U}P_-)$. Therefore, $n_\pm(\overline{U}) = n_\pm(U)$.
For the second part of the proof, let us take the operator $V = \overline{U}^* U$. By the first part, we notice that $i(V) = i(U) - i(\overline{U}) = 0$. Consequently, since by definition $i(V) = n_+(V) - n_-(V)$, we obtain that $n_+(V) = n_-(V)$. Therefore, it suffices to show that $n_+(V) = 0$ since, by lemma 3.1.17, if it holds then V can be connected with the identity map.

Computing the entry P_+VP_+ for the matrix form of V, we obtain $P_+VP_+ = I_+^* P_+UP_+ + K_+S_-^* P_-UP_+$. Using the polar decomposition $P_+UP_+ = I_+\text{abs}(P_+UP_+)$ and the relation $I_+^* I_+ = \mathbf{1} - K_+$, we simplify the expression for P_+VP_+ and obtain $(\mathbf{1} - K_+)\ \text{abs}(P_+UP_+) + K_+S_-^* P_-UP_+ = \text{abs}(P_+UP_+) + K_+S_-^* P_-UP_+$. Notice that $K_+\ \text{abs}(P_+UP_+) = 0$ since K_+ is the projection over ker $I_+ =$ ker (P_+UP_+), and that the following holds:
If $\text{abs}(P_+UP_+)(x) \neq 0$, we have that $0 \neq (\ \text{abs}(P_+UP_+))^2(x) = (P_+U^*P_+)(P_+UP_+)(x)$ because for any bounded operator T, ker $(T) =$ ker (T^*T), and thus $\text{abs}(P_+UP_+)(x)$ does not belong to ker $\text{abs}(P_+UP_+) =$ ker (I_+). Therefore, applying the projection K_+, this expression vanishes.

Let $f \in H_+$ such that $P_+VP_+(f) = 0$. Given that K_+ is a projection, we must have that $(\mathbf{1} - K_+)\text{abs}\ (P_+UP_+)(f) = 0$ and $K_+S_-^* P_-UP_+(f) = 0$ (ranges of K_+ and $\mathbf{1} - K_+$ are mutually orthogonal spaces). Therefore, $\text{abs}(P_+UP_+)(f) = 0$ and then $f \in$ ker (P_+UP_+). Since $f \in H_+$ and U is unitary, we have that $U(f) \in H_-$, and thus $P_-UP_+(f) = U(f)$. In this order, we obtain the following expressions:

$$0 = K_+S_-^* P_-UP_+(f) = K_+S_-^* U(f) = S_-^* C_-U(f),$$

where in the last equality we conjugated the relation $S_-K_+ = C_-S_-$.

In addition to the properties of the polar decomposition we have used until now, we refer to another one related with the behavior of the ranges: Let $A = U\text{abs}(A)$ be the polar decomposition for a bounded operator A on a Hilbert space H; then ran $(U) = \overline{\text{ran } A}$ (see [24] §6).
With these results and the relation showed in the proof of lemma 3.1.7,

$$\ker (P_\pm U P_\pm) = U^*(\ker (P_\mp U^* P_\mp)),$$

since $f \in \ker (P_+ U P_+)$ we observe that $U(f) \in \ker (P_- U^* P_-) = (\text{Ran } (P_- U P_-))^\perp = (\text{Ran } I_-)^\perp = \ker I_-^* = \text{Ran } C_-$. Therefore, $0 = S_-^* C_- U(f) = S_-^* U(f)$.
Observe now that, if we restrict S_- to a map Ran $K_+ \to$ Ran C_- –it is unitary on these restrictions, the map S_-^* has kernel equal to 0. Thus, since $U(f) \in \text{ran } C_-$ and $S_-^* U(f) = 0$, we conclude that $U(f) = 0 \Rightarrow f = 0$. Consequently, $0 = \dim \text{Ker } (P_+ V P_+) = n_+(V)$. □

Lemma 3.1.19. *If $i(U) = 0$ then U can be connected to the identity.*

Proof. If $i(U) = 0$ then $n_+(U) = n_-(U)$ and, thus, by the previous lemma, $n_+(\overline{U}) = n_-(\overline{U})$. By definition of $\overline{U}$, we have that $P_+ \overline{U} P_+ = I_+$; therefore,

$$n_+(\overline{U}) = \dim \ker I_+ = \dim \text{ran } K_+ = \dim \text{ran } C_-.$$

In a similar way, we see that

$$n_-(\overline{U}) = \dim \ker I_- = \dim \text{ran } K_- = \dim \text{ran } C_+.$$

Thus, we obtain the equivalences $\dim \text{ran } K_\pm = \dim \text{ran } C_\pm$.
Both spaces, ran $K_\pm$ and ran $C_\pm$, are contained in $H_\pm$ respectively. By equality of dimensions, we can then construct a unitary operator $T_\pm : H_\pm \to H_\pm$ such that $T_\pm K_\pm = C_\pm T_\pm$: Since these ranges are closed finite dimensional subspaces, we can consider orthonormal basis $B_\pm^{(1)}, B_\pm^{(2)}$ for $H_\pm$ such that the first n elements –being $n = \dim \text{ran } C_\pm = \dim \text{ran } K_\pm < \infty$– form an orthonormal basis for ran $K_\pm$ and ran $C_\pm$ respectively. For instance, let us consider

$$B_\pm^{(1)} = \{\varphi_{\pm 1}^{(1)}, \varphi_{\pm 2}^{(1)}, \ldots, \varphi_{\pm n}^{(1)}, \varphi_{\pm n+1}^{(1)}, \ldots\},\ B_\pm^{(2)} = \{\psi_{\pm 1}^{(2)}, \psi_{\pm 2}^{(2)}, \ldots, \psi_{\pm n}^{(2)}, \psi_{\pm n+1}^{(2)}, \ldots\},$$

with $\{\varphi_{\pm i}^{(1)}\}_{i=1}^n$, an orthonormal basis for ran $K_\pm$ and $\{\psi_{\pm i}^{(2)}\}_{i=1}^n$ an orthonormal basis for ran $C_\pm$. We define $T_\pm(\varphi_{\pm i}^{(1)}) = \psi_{\pm i}^{(2)}$. Now, notice that with this definition we get the relation $T_\pm K_\pm = C_\pm T_\pm$. Given $x_\pm \in H_\pm$ we can decompose it in a unique way as a sum $x_\pm = y_\pm + z_\pm$ where $y_\pm \in \text{ran } K_\pm$ and $z_\pm \in (\text{ran } K_\pm)^\perp$; therefore, $T_\pm K_\pm(x_\pm) = T_\pm(y_\pm)$ and, on the other hand, $C_\pm T_\pm(x_\pm) = C_\pm T_\pm(y_\pm) + C_\pm T_\pm(z_\pm)$, but the last term is 0 because $T_\pm(z_\pm) \in (\text{ran } C_\pm)^\perp$, so $= C_\pm T_\pm(y_\pm) = T_\pm(y_\pm)$.
Define $T := \begin{pmatrix} T_+ & 0 \\ 0 & T_- \end{pmatrix}$ on H. It is clear that T is unitary. Moreover, we have that $T \in \mathcal{U}_{\text{res}}$:

$$P_+ T P_- = 0 = P_- T P_+.$$

Consider now the product $T^*\overline{U}$:

$$T^*\overline{U} = \begin{pmatrix} T_+^* & 0 \\ 0 & T_-^* \end{pmatrix} \begin{pmatrix} I_+ & S_+K_- \\ S_-K_+ & I_- \end{pmatrix} = \begin{pmatrix} T_+^*I_+ & T_+^*S_+K_- \\ T_-^*S_-K_+ & T_-^*I_- \end{pmatrix}.$$

Now, observe the following properties: Given $x \in H_+$, we can decompose it as $x = y + z$ with $y \in \ker I_+$ and $z \in (\ker I_+)^\perp$, so $T_+^*I_+(x) = T_+^*I_+(z)$ and, in this way, applying K_+, we obtain

$$K_+T_+^*I_+(z) = (T_+K_+)^*I_+(z) = (C_+T_+)^*I_+(z) = T_+^*C_+I_+(z) = 0,$$

since $\mathbf{1} - C_+$ is an orthogonal projection over ran I_+, i.e. C_+ is an orthogonal projection over (ran $I_+)^\perp$. Therefore, $T_+^*I_+ = (\mathbf{1} - K_+)T_+^*I_+(\mathbf{1} - K_+)$.
Analogously, we obtain a similar relation:

$$T_-^*I_- = (\mathbf{1} - K_-)T_-^*I_-(\mathbf{1} - K_-).$$

Moreover, we notice that:

$$K_-T_-^*S_-K_+ = (T_-K_-)^*S_-K_+ = (C_-T_-)^*S_-K_+ = T_-^*C_-S_-K_+ = T_-^*S_-K_+,$$

and

$$K_+T_+^*S_+K_- = (T_+K_+)^*S_+K_- = (C_+T_+)^*S_+K_- = T_+^*C_+S_+K_- = T_+^*S_+K_-.$$

Consequently, we have attained an expression for $T^*\overline{U}$:

$$T^*\overline{U} = \begin{pmatrix} (\mathbf{1} - K_+)T_+^*I_+(\mathbf{1} - K_+) & K_+T_+^*S_+K_- \\ K_-T_-^*S_-K_+ & (\mathbf{1} - K_-)T_-^*I_-(\mathbf{1} - K_-) \end{pmatrix}.$$

Thus, we have obtained a significant property of this operator: It leaves invariant the subspaces $(\mathbf{1} - K_+)H_+$, $K_+(H_+) \bigoplus K_-(H_-)$ and $(\mathbf{1} - K_-)H_-$. Moreover, it is unitary on each of these subspaces:

- Let $x \in (\mathbf{1} - K_+)(H_+)$. Then we have:

  $$\begin{aligned}(\mathbf{1} - K_+)I_+^*T_+(\mathbf{1} - K_+)T_+^*I_+(\mathbf{1} - K_+)(x) &= (\mathbf{1} - K_+)I_+^*T_+T_+^*I_+(x) \\ &= (\mathbf{1} - K_+)I_+^*I_+(x) = (\mathbf{1} - K_+)^2(x) = x.\end{aligned}$$

  On the other hand, we have:

  $$(\mathbf{1} - K_+)T_+^*I_+(\mathbf{1} - K_+)I_+^*T_+(\mathbf{1} - K_+)(x) = T_+^*I_+I_+^*T_+(x),$$

  but $x \in (\mathbf{1} - K_+)(H_+) = (\text{ran } K_+)^\perp$, which implies that $T_+(x) \in (\text{ran } (C_+))^\perp$, and using the relation $I_+I_+^* = \mathbf{1} - C_+$, we obtain $I_+I_+^*T_+(x) = T_+(x)$. Thus, $T_+^*I_+I_+^*T_+(x) = x$.

- In the same way as we have done for the entry $(1, 1)$, we see that the entry $(2, 2)$ is unitary in the respective subspace.

- Let $(x,y) \in K_+(H_+) \bigoplus K_-(H_-)$. Computing $(T^*\overline{U})^*$, we obtain:

$$\begin{aligned}(T^*\overline{U})^*(T^*\overline{U})(x \oplus y) &= (T^*\overline{U})^* \begin{pmatrix} K_+T_+^*S_+K_-(y) \\ K_-T_-^*S_-K_+(x) \end{pmatrix} \\ &= K_+S_-^*C_-T_-T_-^*S_-K_+(x) \oplus K_-S_+^*C_+T_+T_+^*S_+K_-(y) \\ &= K_+S_-^*S_-K_+(x) \oplus K_-S_+^*S_+K_-(y),\end{aligned}$$

  but S_- is unitary on Ran K_+ and S_+ is unitary on Ran K_-, so the last expression is equal to $x \oplus y$.
  Applying these observations, we see that $(T^*\overline{U})(T^*\overline{U})^*$ is the identity when it is restricted to $K_+(H_+) \bigoplus K_-(H_-)$.

Since $K_+(H_+) \bigoplus K_-(H_-)$ is finite dimensional, its unitary group of operator is path connected, and thus we can continuously deform $T^*\overline{U}$ to the identity map on $K_+(H_+) \bigoplus K_-(H_-)$: Observe that, for any bounded operator $W = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the entries P_+WP_- and P_-WP_+ correspond to B and C respectively; therefore, in our case, since for $T^*\overline{U}$ these entries are finite rank operators, this continuous deformation, restricted to $K_+(H_+) \bigoplus K_-(H_-)$, will be contained in $\mathcal{U}_{\text{res}}$ and will leave invariant $T^*\overline{U}$ on $(\mathbf{1} - K_\pm)(H_\pm)$. In principle, the path obtained is continuous in the $||\cdot||$-topology, but again, by proposition 3.1.12, we can construct a continuous path in the ρ-topology which performs this deformation. Thus, $T^*\overline{U}$ is connected to:

$$V = \begin{pmatrix} (\mathbf{1} - K_+)T_+^*I_+(\mathbf{1} - K_+) + K_+ & 0 \\ 0 & (\mathbf{1} - K_-)T_-^*I_-(\mathbf{1} - K_-) + K_- \end{pmatrix}.$$

Now, observe that $n_\pm(V) = 0$: Entry $(1,1)$ above (resp. $(2,2)$) has trivial kernel since $(\mathbf{1} - K_+)T_+^*I_+(\mathbf{1} - K_+)$ has inverse on $(\mathbf{1} - K_+)(H_+)$ and, over K_+, we have the identity map. Therefore, by lemma 3.1.17, V is connected with the identity.
Finally, we note that T can be connected with the identity. Since $T_\pm$ are unitary operators on $H_\pm$, by the measurable functional calculus, we can find self-adjoint operators $A_\pm$ on $H_\pm$, respectively, such that $T_\pm = e^{iA_\pm}$. Then, considering the curve

$$s \mapsto T_s = \begin{pmatrix} e^{isA_+} & 0 \\ 0 & e^{isA_-} \end{pmatrix},$$

which is continuous in the $|| \cdot ||$-topology and is contained in $\mathcal{U}_{\text{res}}$ $(P_\pm T_s P_\mp = 0)$, we have connected T with the identity map $\mathbf{1}$. One more time, by proposition 3.1.12, we can construct a ρ-continuous path connecting T with $\mathbf{1}$. Therefore, $\overline{U} = T(T^*\overline{U})$ is connected with $\mathbf{1}$. Thus, by lemma 3.1.18, U is connected with the identity map. □

These three lemmas 3.1.17, 3.1.18 and 3.1.19 are necessary in the proof of theorem 3.1.15 as we have seen. Nevertheless, we do not have a complete characterization of an index theory for $\mathcal{U}_{\text{res}}$ in the sense that its path components are not completely classified. To do that, we present a last result where each path component is labeled with an integer number.

Theorem 3.1.20. *Let $\hat{i} : \mathcal{U}_{res}/Ker\ i \to \mathbb{Z}$ be the induced homomorphism by $i : \mathcal{U}_{res} \to \mathbb{Z}$. Then $\hat{i}$ is an isomorphism.*

Proof. It is clear that $\hat{i}$ is a one-to-one homomorphism by the First Isomorphism Theorem. Moreover, as we have seen, ker $i = \mathcal{U}_0$, the path connected component of the identity $\mathbf{1}$. To prove the isomorphism, we shall show that it is surjective.
Let $\{e_j : j \in \mathbb{Z}\}$ be an adapted orthonormal basis for H in the sense that $\{e_j : j \geq 0\}$ and $\{e_j : j < 0\}$ are orthonormal basis for H_+ and H_- respectively. Consider the right unilateral shift on H, $B : H \to H$ given by $B(e_j) = e_{j+1}$. It is clear that B is unitary and, moreover, it satisfies dim ker $(P_+BP_+) = 0$ (right shift has no problem on H_+), and dim ker $(P_-BP_-) = 1$ since $B(e_{-1}) = e_0$.
Notice that P_+BP_- has finite rank: for $j \geq 0$ then $P_+BP_-(e_j) = 0$, and for $j < 0$ $P_+BP_-(e_j) = P_+B(e_j)$, being $\neq 0$ if and only if $j = -1$. In the same way, we observe that P_-BP_+ is 0. Therefore, $P_\pm BP_\mp \in \mathcal{Q}$, and thus $B \in \mathcal{U}_{\text{res}}$.
As we have noticed at the beginning, $n_+(B) = 0$ and $n_-(B) = 1$, so $i(B) = -1$. Now, for any $k \geq 0$, $i(B^k) = ki(B) = -k$, and $i(B^*k) = ki(B^*) = k$. Thus, $\hat{i}$ is surjective. □

We finish this section with the above theorem, where we now have an understanding of the group $\mathcal{U}_{\text{res}}$. In Chapter 3 we shall examine an example where this value can be calculated making use of characteristic classes of the Grothendieck group −*K-Theory*.

3.2 An Index Theory for Clifford Algebra

In this section, we shall present a similar analysis to the one we did with the CAR algebra for the Clifford algebra, following the description given by Carey and O'Brien in [5]. Nevertheless, in this description we shall consider a real Hilbert space $(E, (\cdot,\cdot))$ together with a complex structure J. All of the definitions and results presented before for a symmetrically normed ideal also apply for the real case: according to Gohberg and Krein (see [12] §1 and §2), the symmetric norm of the symmetric ideal $\mathcal{Q}$ depends exclusively of the singular values of the bounded operator $S \in \mathcal{Q}$, which are defined as the absolute values of the eigenvalues of abs(S). Again, notice that this definition makes sense because S is compact.
In the previous chapter we have noticed that when $\mathcal{Q} = \mathcal{Q}_2$, the space of Hilbert-Schmidt operators, given $g \in \mathcal{O}(E)$, an orthogonal operator, the Bogoliubov automorphism θ_g on the Clifford algebra $C[E]$ can be unitarily implemented in the Fock representation if and only if the $J-$ antilinear part $A_g = \frac{1}{2}(g + JgJ)$ is a Hilbert-Schmidt operator. In this order, we will define a new topological group such that this property is considered, but for a more general symmetric normed ideal $\mathcal{Q}$.

Definition 3.2.1. *(Orthogonal restricted group) Let $\mathcal{O}_{res}$ denote the subset of orthogonal operators O on E such that $O + JOJ \in \mathcal{Q}$, i.e*

$$\mathcal{O}_{res} := \{O \in \mathcal{O}(E)|\ O + JOJ \in \mathcal{Q}\}.$$

Let $\sigma : \mathcal{O}_{res} \times \mathcal{O}_{res} \to \mathbb{R}_{\geq 0}$ be the norm function defined by

$$\sigma(g,h) = ||(g-h) - J(g-h)J|| + |(g-h) + J(g-h)J|_{\mathcal{Q}}.$$

The pair $(\mathcal{O}_{res}, \sigma)$ is called the orthogonal restricted group of E.

Remark. Using the notation of the section 2.2.3, the function σ can be written as

$$\sigma(g,h) = ||C_g - C_h|| + |A_g - A_h|_{\mathcal{Q}}.$$

Remark. Notice that $(\mathcal{O}_{\text{res}}, \sigma)$ is a topological group, with the topology induced by σ.

i. **Group structure:** Given $g, h \in \mathcal{O}_{\text{res}}$, let us consider the relations:

$$A_{gh} = C_g A_h + A_g C_h, \quad C_{gh} = C_g C_h + A_g A_h.$$

With the first identity, since $\mathcal{Q}$ is an ideal, we see then that $\mathcal{O}_{\text{res}}$ is closed under product operation. Now, since $A_{g^{-1}} = (A_g)^*$, $\mathcal{O}_{\text{res}}$ is closed under taking inverse.

ii. **Continuity of group operations:** To check continuity of the multiplication map $m : \mathcal{O}_{\text{res}} \times \mathcal{O}_{\text{res}} \to \mathcal{O}_{\text{res}}$, we can provide, as we did for $\mathcal{U}_{\text{res}}$, a metric to $\mathcal{O}_{\text{res}} \times \mathcal{O}_{\text{res}}$, called d, given by

$$d((g_1,h_1),(g_2,h_2)) = \sigma(g_1,g_2) + \sigma(h_1,h_2).$$

This metric induces the product topology on $\mathcal{O}_{\text{res}} \times \mathcal{O}_{\text{res}}$.
With this in mind, to prove continuity will be sufficient to consider sequences $(g_n)_{n\geq 1} \xrightarrow{\sigma} g$, $(h_n)_{n\geq 1} \xrightarrow{\sigma} h$, and show that $(g_n h_n)$ converges, in σ, to gh. Let us observe that $||C_{g_n h_n} - C_{gh}|| \xrightarrow{n\to\infty} 0$ and $|A_{g_n h_n} - A_{gh}|_{\mathcal{Q}} \xrightarrow{n\to\infty} 0$. The first convergence is clear since the operations involved in C_g are continuous with respect to the $||\cdot||$-topology. For the second one, we must be more careful. Let $\epsilon > 0$. Observe that the following computation holds:

$$\begin{aligned}|A_{gh} - A_{g_n h_n}|_{\mathcal{Q}} &= |A_g C_h + C_g A_h - A_{g_n} C_{h_n} - C_{g_n} A_{h_n}|_{\mathcal{Q}} \\ &= |A_g C_h - A_{g_n} C_h + A_{g_n} C_h + C_g A_h - A_{g_n} C_{h_n} - C_{g_n} A_{h_n}|_{\mathcal{Q}} \\ &\leq |A_g C_h - A_{g_n} C_h|_{\mathcal{Q}} + |A_{g_n} C_h + C_g A_h - A_{g_n} C_{h_n} - C_{g_n} A_{h_n}|_{\mathcal{Q}},\end{aligned}$$

and notice that the first term in the last inequality tends to zero since $A_{g_n} \xrightarrow{|\cdot|_{\mathcal{Q}}} A_g$. Thus, there exists N_1 such that for all $n \geq N_1$ the whole expression above satisfies

$$\begin{aligned}&< \frac{\epsilon}{4} + |A_{g_n} C_h + C_g A_h - A_{g_n} C_{h_n} - C_{g_n} A_{h_n} + C_{g_n} A_h - C_{g_n} A_h|_{\mathcal{Q}} \\ &\leq \frac{\epsilon}{4} + |A_{g_n} C_h + C_g A_h - A_{g_n} C_{h_n} - C_{g_n} A_h|_{\mathcal{Q}} + |C_{g_n} A_{h_n} - C_{g_n} A_h|_{\mathcal{Q}}.\end{aligned}$$

Again, observe that the third term in the last inequality goes to zero with $n \to \infty$ since $||C_{g_n}|| \leq 1$ for all n. Hence, there exists $N_2 \geq N_1$ such that for all $n \geq N_2$ we obtain the inequality:

$$\begin{aligned}&< \frac{\epsilon}{2} + |A_{g_n} C_h + C_g A_h - A_{g_n} C_{h_n} - C_{g_n} A_h|_{\mathcal{Q}} \\ &\leq \frac{\epsilon}{2} + |A_{g_n}(C_h - C_{h_n})|_{\mathcal{Q}} + |(C_g - C_{g_n}) A_h|_{\mathcal{Q}}.\end{aligned}$$

Now, $|A_{g_n}(C_h - C_{h_n})|_{\mathcal{Q}} \leq ||C_h - C_{h_n}||\,|A_{g_n}|_{\mathcal{Q}}$, but since $A_{g_n} \xrightarrow{|\cdot|_{\mathcal{Q}}} A_g$ (because $g_n \xrightarrow{\sigma} g$), we obtain that there exists $0 < M < \infty$ such that $|A_{g_n}|_{\mathcal{Q}} \leq M$ for all n, and therefore $A_{g_n}C_h - A_{g_n}C_{h_n} \xrightarrow{|\cdot|_{\mathcal{Q}}} 0$. Similarly, $C_g A_h - C_{g_n} A_h \xrightarrow{|\cdot|_{\mathcal{Q}}} 0$. Thus, $A_{g_n h_n} \xrightarrow{|\cdot|_{\mathcal{Q}}} A_{gh}$.

For the inverse map, $\mu : \mathcal{O}_{\text{res}} \to \mathcal{O}_{\text{res}}$, due to the fact that $\mu(g) = g^t$, if $(g_n)_{n\geq 1} \xrightarrow{\sigma} g$ and considering that $||A^t|| = ||A||$ and same for $|\cdot|_{\mathcal{Q}}$, we obtain easily that $(g_n^t) \xrightarrow{\sigma} g^t$. Therefore, μ is continuous.

Similarly, as we did for $\mathcal{U}_{\text{res}}$, we can observe that $\mathcal{O}_{\text{res}}$ equipped with the $\sigma-$topology is a complete metric space. Let $(g_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{O}_{\text{res}}$. According to the definition of the metric σ, we notice that $(C_{g_n})_{n\geq 1}$ is a Cauchy sequence with respect to the $||\cdot||$-metric and $(A_{g_n})_{n\geq 1}$ is also a Cauchy sequence with respect to the $|\cdot|_{\mathcal{Q}}$-metric. Since $(\mathcal{B}(E), ||\cdot||)$ is a complete metric space, there exists an element A such that $A = \lim_{n\to\infty} C_{g_n}$, and by a similar reasoning, using completeness of the symmetric normed ideal $\mathcal{Q}$, there exists an element B such that $B = \lim_{n\to\infty} A_{g_n}$. Although, in principle, the last convergence happens in the symmetric ideal, notice that $||\cdot|| \leq |\cdot|_{\mathcal{Q}}$ implies that B is the limit of convergence in $||\cdot||$ as well. Let $g = A + B$; by continuity on $||\cdot||$, A commutes with J and B anticommutes with J. Therefore, by uniqueness of this decomposition (see [21] §3), we have:

$$A = \frac{1}{2}(g - JgJ), \quad B = \frac{1}{2}(g + JgJ).$$

Moreover, in the metric $||\cdot||$, $g = \lim_{n\to\infty} g_n$; hence, g is an orthogonal operator, and by construction, $g \in \mathcal{O}_{\text{res}}$.

At this point, we have introduced two distinct groups with no related structures, $\mathcal{U}_{\text{res}}$ and $\mathcal{O}_{\text{res}}$. These two groups arise in the attempt to answer the problem of implementability and equivalence of unitary automorphisms and unitary representations, respectively, for two different algebras. However, we shall observe that, requesting a little more structure, there exists a relation between these two restricted groups.

Let $H := E \bigoplus E$ be the real Hilbert space obtained from the direct sum of E with itself, and with inner product given by:

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right)_H = (x_1, x_2)_E + (y_1, y_2)_E.$$

Let us consider the complex structure on H given by

$$\tilde{J} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}.$$

With this structure, we can equip H with a new inner product $\langle\langle \cdot, \cdot \rangle\rangle := (\cdot, \cdot)_H + i(\cdot, \tilde{J}\cdot)_H$, and then view $(H_{\tilde{J}}, \langle\langle \cdot, \cdot \rangle\rangle)$ as a complex Hilbert space, with multiplication by i given by:

$$i \begin{pmatrix} x \\ y \end{pmatrix} := \tilde{J} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let P_+, P_- be the orthogonal projection operators on the first and the second factor of H, respectively, and let us consider the operator $\Gamma := \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$. With these definitions, Γ fulfills the following properties:

i. $\langle\langle \Gamma(\vec{u}), \Gamma(\vec{v})\rangle\rangle = \langle\langle \vec{v}, \vec{u}\rangle\rangle$. This is a direct computation: let $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and observe that:

$$\begin{aligned}
\langle\langle \Gamma(\vec{u}), \Gamma(\vec{v})\rangle\rangle &= (\Gamma(\vec{u}), \Gamma(\vec{v}))_H + i(\Gamma(\vec{u}), \tilde{J}\Gamma(\vec{v}))_H \\
&= (x_1, y_1) + (x_2, y_2) + i\left(\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)_H \\
&= (x_1, y_1) + (x_2, y_2) + i(x_2, Jy_2) - i(x_1, Jy_1) \\
&= (x_1, y_1) + (x_2, y_2) - i(Jx_2, y_2) + i(Jx_1, y_1) \\
&= (y_1, x_1) + (y_2, x_2) + i(y_2, -Jx_2) + i(y_1, Jx_1) \\
&= (y_1, x_1) + (y_2, x_2) + i(\vec{v}, \tilde{J}\vec{u})_H = \langle\langle \vec{v}, \vec{u}\rangle\rangle.
\end{aligned}$$

ii. $\Gamma^2 = \mathbf{1}$.

iii. $\Gamma i = -i\Gamma$, i.e, $\Gamma\tilde{J} = -\tilde{J}\Gamma$. This property follows from the next computation:

$$\Gamma\tilde{J} + \tilde{J}\Gamma = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix} + \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} = 0.$$

iv. $\Gamma P_+ = P_-\Gamma$. This expression results from a direct calculation considering the matrix form for P_+ and P_-:

$$P_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Now, observe that, over the real Hilbert space H, the Banach space of bounded linear operators $H \to H$ is given by $\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A, B, C, D \in \mathcal{B}(E)\right\}$. We can consider a norm over this space defined as

$$\left\|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right\| := \max\{||A||, ||B||, ||C||, ||D||\}.$$

From [27] §1, using the language of C^*-algebras, we see that this norm makes $\mathcal{B}(H)$ a Banach space. Analogously, we can extend the definition of the symmetric normed ideal $\mathcal{Q}$ to the space $\mathcal{B}(H)$ applying the symmetric norm $|\cdot|_{\mathcal{Q}}$ over each matrix entry. Let us denote this new symmetric normed ideal as $\tilde{\mathcal{Q}}$. With this consideration, $\tilde{\mathcal{Q}}$ fulfills the properties of a symmetric normed ideal.

We shall need a one last observation in order to understand the relation between these

two restricted groups. In order that a bounded operator A on H can be understood as an operator on $H_{\tilde{J}}$, we require that $[A, \tilde{J}] = 0$. Hence, seeing $\mathcal{B}(H_{\tilde{J}})$ as a subspace of $\mathcal{B}(H)$, the construction of $\tilde{\mathcal{Q}}$ still works.
Consider the following groups:

$$\mathcal{U}_{\text{res}} = \{U | U \text{ unitary on } H_{\tilde{J}}, P_{\pm} U P_{\mp} \in \tilde{\mathcal{Q}}\},\ \mathcal{U}_{\Gamma} = \{U | U \in \mathcal{U}_{\text{res}},\ \Gamma U = U \Gamma\}.$$

Observe that the condition $P_{\pm} U P_{\mp} \in \tilde{\mathcal{Q}}$ is equivalent to $P_{\pm} U P_{\mp} \in \mathcal{Q}$. Moreover, notice that U_{Γ} is a subgroup of $\mathcal{U}_{\text{res}}$.
With all these definitions, we are ready to prove the next proposition.

Proposition 3.2.2. *Let $\mu : \mathcal{O}_{res} \to \mathcal{U}_{\Gamma}$ be the group homomorphism given by:*

$$\mu(g) = \frac{1}{2} \begin{pmatrix} g - JgJ & g + JgJ \\ g + JgJ & g - JgJ \end{pmatrix} = \begin{pmatrix} C_g & A_g \\ A_g & C_g \end{pmatrix}.$$

Then μ is a group isomorphism in the category of topological groups.

Proof. First, notice that μ really is a group homomorphism, and that $\mu(g) \in \mathcal{U}_{\Gamma}$ for all $g \in \mathcal{O}_{\text{res}}$. Let $g, h \in \mathcal{O}_{\text{res}}$; then:

i.

$$\mu(g)\mu(h) = \begin{pmatrix} C_g & A_g \\ A_g & C_g \end{pmatrix} \begin{pmatrix} C_h & A_h \\ A_h & C_h \end{pmatrix} = \begin{pmatrix} CgC_h + A_g A_h & C_g A_h + A_g C_h \\ C_g A_h + A_g C_h & CgC_h + A_g A_h \end{pmatrix} = \begin{pmatrix} C_{gh} & A_{gh} \\ A_{gh} & C_{gh} \end{pmatrix} = \mu(gh).$$

ii.

$$\mu(g)\tilde{J} = \begin{pmatrix} C_g & A_g \\ A_g & C_g \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} = \begin{pmatrix} C_g J & -A_g J \\ A_g J & -C_g J \end{pmatrix}$$
$$= \begin{pmatrix} JC_g & JA_g \\ -JA_g & -JC_g \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \begin{pmatrix} C_g & A_g \\ A_g & C_g \end{pmatrix} = \tilde{J}\mu(g).$$

From these two computations, it also follows that $\mu(g)$ is unitary on $H_{\tilde{J}}$.

iii. For all $g \in \mathcal{O}_{\text{res}}$, according to the matrix form, $P_{\pm}\mu(g)P_{\mp}$ corresponds to A_g; therefore, $P_{\pm}\mu(g)P_{\mp} \in \mathcal{Q}$.

iv.

$$\mu(g)\Gamma = \begin{pmatrix} C_g & A_g \\ A_g & C_g \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} C_g & A_g \\ A_g & C_g \end{pmatrix} = \Gamma\mu(g).$$

To check the continuity of the morphism μ, we shall need the metric ρ given in the previous section for $\mathcal{U}_{\text{res}}$. For $g, h \in \mathcal{O}_{\text{res}}$, we have:

$$\rho(\mu(g), \mu(h)) = \left\| \begin{pmatrix} C_g - C_h & A_g - A_h \\ A_g - A_h & C_g - C_h \end{pmatrix} \right\| + |A_g - A_h|_{\mathcal{Q}}.$$

Given $\epsilon > 0$, let us consider the $\sigma-$ball centered on g with radius $\epsilon/2$. Thus, $||C_g - C_h|| < \epsilon/2$ and $|A_g - A_h|_{\mathcal{Q}} < \epsilon/2$. Therefore, $\rho(\mu(g), \mu(h)) < \epsilon$ whenever $\sigma(g,h) < \epsilon/2$.
To prove that μ is an homeomorphism, let us consider the map $T : \mathcal{U}_\Gamma \to \mathcal{O}_{\text{res}}$ given by $T(U) = A + B$, where $U = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$. Again, we have to see that this map is a well defined group homomorphism. Given $U \in \mathcal{U}_\Gamma$, we can express U in matrix form as follows:

$$U = \begin{pmatrix} P_+UP_+ & P_+UP_- \\ P_-UP_+ & P_-UP_- \end{pmatrix},$$

and using $\Gamma U = U\Gamma$, we get that $P_-UP_+ = P_+UP_-$ and $P_-UP_- = P_+UP_+$, i.e. U has the form written above. Since U is unitary, we obtain the relations:

$$AA^* + BB^* = \mathbf{1},\ AB^* + BA^* = 0,\ A^*A + B^*B = \mathbf{1}\ A^*B + B^*A = 0.$$

Moreover, due to complexification of H with respect to the complex structure $\tilde{J}$, we have that $U\tilde{J} = \tilde{J}U$. Using the matrix form, we get these other relations:

$$AJ = JA,\ \ BJ = -JB.$$

Let us observe that $A+B \in \mathcal{O}_{\text{res}}$. Using the expressions above, and considering $A, B \in \mathcal{B}(E)$ we see that:

$$(A+B)(A+B)^* = AA^* + AB^* + BA^* + BB^* = \mathbf{1},$$

and, analogously, $(A+B)^*(A+B) = \mathbf{1}$. In addition, we need to show that $(A+B) + J(A+B)J \in \mathcal{Q}$, but this also follows from above expressions:

$$(A+B) + J(A+B)J = A + BJAJ + JBJ = A + B + AJ^2 - BJ^2 = A + B - A + B = 2B,$$

and by definition of U, $B \in \mathcal{Q}$. Thus, T is well defined and, by similar arguments, we can show that it is a continuous group homomorphism. Finally, note that μ and T are inverses of each other:

$$T \circ \mu(g) = \frac{1}{2}(g - JgJ + g + JgJ) = g,$$

and

$$\mu \circ T(U) = \mu(A+B) = \frac{1}{2}\begin{pmatrix} A + B - J(A+B)J & A + B + J(A+B)J \\ A + B + J(A+B)J & +B - J(A+B)J \end{pmatrix},$$

but since $A+B-JAJ-JBJ = 2A$ and $A+B+J(A+B)J = 2B$, we obtain $\mu \circ T(U) = U$. □

Our goal in this section 3.2 is to construct an *index theory* for the restricted group $\mathcal{O}_{\text{res}}$. To do that, we shall need a continuous group homomorphism $i_{\mathcal{O}_{\text{res}}} : \mathcal{O}_{\text{res}} \to \mathbb{Z}/n\mathbb{Z}$, for some $n \geq 0$, with the property that two elements $g, h \in \mathcal{O}_{\text{res}}$ can be connected by a continuous path $s \mapsto g_s$, $0 \leq s \leq 1$ if and only if $i_{\mathcal{O}_{\text{res}}}(g) = i_{\mathcal{O}_{\text{res}}}(h)$. A map with these properties brings with it the following immediate conclusions:

- Path connected components of $\mathcal{O}_{\text{res}}$ are the subsets where $i_{\mathcal{O}_{\text{res}}}$ is constant.

- The path connected component which contains the identity element is $\mathcal{O}_{\text{res}0} := \ker\, i_{\mathcal{O}_{\text{res}}}$.
- The canonical map $\mathcal{O}_{\text{res}}/\mathcal{O}_{\text{res}0} \to i_{\mathcal{O}_{\text{res}}}(\mathcal{O}_{\text{res}})$ is a group isomorphism.

These results have been shown in the previous section for the unitary restricted group $\mathcal{U}_{\text{res}}$. Although these results will also be apply for $\mathcal{O}_{\text{res}}$, the path intended for the construction will be different. Here, we shall understand some topological properties of this group acting over a homogeneous space.

3.2.1 Homotopy type of $\mathcal{O}_{\text{res}}$

Let χ be the set of complex structures on E which differ from J by an element on $\mathcal{Q}$, i.e the set of complex structures K such that $|K-J|_{\mathcal{Q}} < \infty$. Note that this set, in the case where $\mathcal{Q} = \mathcal{Q}_2$, corresponds to the complex structures which are unitarily equivalent to the representation π_J. We give χ the topology that results from the metric $|J_1 - J_2|_{\mathcal{Q}}$.
For all $J_1 \in \chi$ and $g \in \mathcal{O}_{\text{res}}$, notice that $gJ_1g^* - J_1 = (gJ_1 - J_1g)g^* = -J_1(g + J_1gJ_1)g^*$. Now, since $J_1 \in \chi$, it satisfies $J_1 = J + M$ where $M \in \mathcal{Q}$. In this way, we have:

$$g + J_1gJ_1 = g + (J+M)g(J+M) = g + JgJ + JgM + MgJ + MgM,$$

and thus $g + J_1gJ_1 \in \mathcal{Q}$ because $\mathcal{Q}$ in an ideal. Therefore, fixing $g \in \mathcal{O}_{\text{res}}$, we have a well defined map from χ to itself given by $J_1 \mapsto gJ_1g^*$: $gJ_1g^* - J = gJ_1g^* - J_1 + J_1 - J$, and both terms belong to $\mathcal{Q}$.
With these observations, we define a group action $\alpha : \mathcal{O}_{\text{res}} \times \chi \to \chi$ given by conjugation, as above.

Proposition 3.2.3. *The group action $\alpha : \mathcal{O}_{res} \times \chi \to \chi$ is transitive and continuous.*

Proof. From [21] §3, we know that given two unitary structures J, K, their complexified spaces E_J and E_K have the same dimensions (in the finite case they will have the same dimension $\dim E/2$, and in the infinite case, both of them will be infinite dimensional separable Hilbert spaces). Thus, there exists an isometric isomorphism $g : E_J \to E_K$, and according to the complex linearity, this map satisfies $gJ = Kg$, i.e, $K = gJg^*$. Thus, given $J_2 \in \chi$, we can find an orthogonal operator g on E such that $gJ_2g^* = J_1$. Now, notice that $g + J_1gJ_1 = g + gJ_2J_1 = g(J_2 - J_1)J_1 \in \mathcal{Q}$. Therefore, we get that $g \in \mathcal{O}_{\text{res}}$ because

$$2A_g = g + JgJ = g + (J_1+M)g(J_1+M) = (g + J_1gJ_1) + J_1gM + MgJ_1 + MgM \in \mathcal{Q},$$

where $M \in \mathcal{Q}$.
Now, to check continuity, let us define a function $d : \mathcal{O}_{\text{res}} \times \chi \to \mathbb{R}_{\geq 0}$ given by

$$d((g_1, J_1), (g_2, J_2)) := \sigma(g_1, g_2) + |J_1 - J_2|_{\mathcal{Q}}\,.$$

It is not hard to see that d is a metric for $\mathcal{O}_{\text{res}} \times \chi$ and that it induces the product topology on this product space. Let $(g_n, \tilde{J}_n)_{n\geq 1}$ be a sequence in $\mathcal{O} \times \chi_{\text{res}}$ which converges to (g, J_1), and

let us observe that $(g_n\tilde{J}_n g_n)_{n\geq 1}$ converges to gJ_1g in χ, i.e observe that $\left|gJ_1g^* - g_n\tilde{J}_n g_n^*\right|_{\mathcal{Q}} \to 0$.
Since $||h|| \leq ||C_h|| + ||A_h||$ for all $h \in \mathcal{B}(E)$ and $g_n \xrightarrow{\sigma} g$, we get that $g_n \xrightarrow{||\cdot||} g$. In addition, we have that $[g - g_n, J] \xrightarrow{|\cdot|_{\mathcal{Q}}} 0$: In order to see this assertion, it will be enough to observe that $g \in \mathcal{O}_{\text{res}}$ if and only if $[g, J] \in \mathcal{Q}$, but this is a direct computation

$$g \in \mathcal{O}_{\text{res}} \Leftrightarrow g + JgJ \in \mathcal{Q} \Leftrightarrow -gJJ + JgJ \in \mathcal{Q} \Leftrightarrow [J, g]J \in \mathcal{Q} \Leftrightarrow [g, J] \in \mathcal{Q}.$$

Thus, the commutator $[g - g_n, J]$ tends to zero in the operator norm and it also fulfills $|[g - g_n, J]|_{\mathcal{Q}} \xrightarrow{|\cdot|_{\mathcal{Q}}} 0$. Additionally, observe that, by transitivity of the action, we can find $\tilde{g}$ such that $J_1 = \tilde{g}J\tilde{g}^*$; therefore, since $\mathcal{O}_{\text{res}}$ is a topological group, we can take, without loss of generality, J instead of J_1 (we would have a sequence $(g_n\tilde{g})_{n\geq 1}$ which converges to $g\tilde{g}$). We have the following computation:

$$\begin{aligned}\left|gJg^* - g_n\tilde{J}_n g_n^*\right|_{\mathcal{Q}} &= \left|gJg^* - g_n\tilde{J}_n g_n^* + g_nJg_n^* - g_nJg_n^*\right|_{\mathcal{Q}} \\ &\leq |gJg^* - g_nJg_n^*|_{\mathcal{Q}} + \left|g_nJg_n^* - g_n\tilde{J}_n g_n^*\right|_{\mathcal{Q}}.\end{aligned}$$

Notice that the second term in the above inequality tends to 0 when $n \to \infty$ since $\left|\tilde{J}_n - J\right|_{\mathcal{Q}} \xrightarrow{|\cdot|_{\mathcal{Q}}} 0$ and $||g_n|| = 1$ for all n. For $|gJg^* - g_nJg_n^*|_{\mathcal{Q}}$ we have:

$$\begin{aligned}&|gJg^* - g_nJg_n^*|_{\mathcal{Q}} \\ &= |gJg^* - g_nJg_n^* - Jgg^* + Jgg^* - g_nJg^* + g_nJg^* + Jg_ng^* - Jg_ng^*|_{\mathcal{Q}} \\ &= |(gJ - Jg - g_nJ + Jg_n)g^*|_{\mathcal{Q}} + |J - g_nJg_n^* + g_nJg^* - Jg_ng^*|_{\mathcal{Q}},\end{aligned}$$

and observe that the first term in this inequality corresponds to the Lie Bracket, so it tends to zero as we have shown previously. Thus, given $\epsilon > 0$, there exists N_1 such that for all $n \geq N_1$ we get:

$$\begin{aligned}|gJg^* - g_nJg_n^*|_{\mathcal{Q}} &< \frac{\epsilon}{4} + |J - g_nJg_n^* + g_nJg^* - Jg_ng^*|_{\mathcal{Q}} \\ &= \frac{\epsilon}{4} + |g_n(Jg^* - Jg_n^*) + J - Jg_ng^*|_{\mathcal{Q}} \\ &= \frac{\epsilon}{4} + |g_n(Jg^* - Jg_n^* - g^*J + g_n^*J) + J - Jg_ng^* + g_ng^*J - J|_{\mathcal{Q}} \\ &\leq \frac{\epsilon}{4} + |g_n(Jg^* - Jg_n^* - g^*J + g_n^*J)|_{\mathcal{Q}} + |Jg_ng^* - g_ng^*J|_{\mathcal{Q}},\end{aligned}$$

which, again, by convergence of the Lie Bracket, there exists $N_2 \geq N_1$ such that for all

$n \geq N_2$, the above expression fulfills the inequality

$$\begin{aligned}
&< \frac{\epsilon}{2} + ||[g_n g^*, J]||_{\mathcal{Q}} \\
&= \frac{\epsilon}{2} + ||[g_n g^*, J] - [\mathbf{1}, J]||_{\mathcal{Q}} \\
&= \frac{\epsilon}{2} + ||[g_n g^* - \mathbf{1}, J]||_{\mathcal{Q}} \\
&= \frac{\epsilon}{2} + ||[(g_n - g)g^*, J]||_{\mathcal{Q}} \\
&\leq \frac{\epsilon}{2} + ||[g_n - g, J]g^*|_{\mathcal{Q}} + |(g_n - g)[g, J]||_{\mathcal{Q}} \\
&\leq \frac{\epsilon}{2} + ||[g_n - g, J]||_{\mathcal{Q}} + ||g_n - g||\,||[g, J]||_{\mathcal{Q}} \to 0,
\end{aligned}$$

where this last convergence is satisfied thanks to the properties previously mentioned. □

Recall that given an element $K \in \chi$, its isotropy group is defined as

$$\mathcal{O}_K := \{g \in \mathcal{O}_{\text{res}}|\ gKg^* = K\}.$$

In the case $K = J$, $\mathcal{O}_J$ is exactly the group of unitary operators on E_J. We denote this group as $\mathcal{U}(E, J)$. We shall observe that this space is contractible, but first we will need a crucial result proved by Kuiper [15].

Theorem 3.2.4. *Let H a separable infinite dimensional Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Then the group of bounded invertible operators $GL(H)$ is contractible in the norm topology.*

From this main result, contractibility of $\mathcal{U}(H)$ is an straight corollary.

Corollary 3.2.5. *With the same hypothesis as in the previous theorem, the subgroup of unitary operators $\mathcal{U}(H)$ is contractible in the norm topology.*

Proof. To prove this statement, it will be enough to see that $U(H)$ is a deformation retract of $\text{GL}(H)$. Consider the continuous map $F : \text{GL}(H) \times [0, 1] \to \text{GL}(H)$ given by

$$F(P, t) \equiv f_t(P) := P((1 - t)\mathbf{1} + t \text{ abs } (P)^{-1}).$$

This map is well defined: if $P \in \text{GL}(H)$, then $P^*P \in \text{GL}(H)$ and, consequently, $0 \notin \sigma(P^*P)$; thus, according to the continuous functional calculus and the positivity of P^*P, using the square root function $g(t) = \sqrt{t}$, we observe that $0 \notin \sigma(\text{abs}(P))$, i.e abs$(P)$ is a positive invertible operator.
Now, notice that if A is a positive invertible operator, then $tA + (1 - t)\mathbf{1}$, for $t \in [0, 1]$, is also invertible: for $t = 0$ and $t = 1$, statement is trivial; for $t \in (0, 1)$, we have that

$$tA + (1 - t)\mathbf{1} = A - (1 - 1/t)\mathbf{1};$$

but $(1-1/t)<0$, and since A is positive, $(1-1/t)\notin\sigma(A)$. Thus, $f_t(P)\in \mathrm{GL}(H)$.
Let us observe that F is precisely the retract we are looking for. For $t=1$ and $P\in\mathrm{GL}(H)$, we have that $F(P,1)=P(\operatorname{abs}(P))^{-1}$ is an unitary operator:

$$P(\operatorname{abs}(P))^{-1}(P(\operatorname{abs}(P))^{-1})^*=P(\operatorname{abs}(P))^{-1}(\operatorname{abs}(P))^{-1}P^*=P(P^*P)^{-1}P^*=\mathbf{1},$$

and, in a similar way, we get that

$$(P(\operatorname{abs}(P))^{-1})^*(P(\operatorname{abs}(P))^{-1})=\mathbf{1}.$$

For $U\in\mathcal{U}(H)$, we obtain $F(U,1)=U$ because $\operatorname{abs}(U)=\mathbf{1}$. □

Remark. In his article, Kupier shows this result proving first the theorem for $\mathrm{GL}(H)$, which requires more machinery.

Now, let us observe that there exists a bijection $\chi\cong\mathcal{O}_{\text{res}}/\mathcal{U}(E,J)$. To do this, consider the morphism $\phi:\mathcal{O}_{\text{res}}\to\chi$ given by $g\mapsto gJg^*$. By transitivity of the action, this morphism is surjective. Now, if $g_1Jg_1^*=g_2Jg_2^*$, then $g_2^*g_1J=Jg_2^*g_1$, i.e $g_2^{-1}g_1\in\mathcal{U}(E,J)$. Therefore, taking the quotient map, $\tilde{\phi}:\mathcal{O}_{\text{res}}/\mathcal{U}(E,J)\to\chi$ is an isomorphism. The well definition follows from the following remark: if the classes $[g]$ and $[h]$ are equal in this quotient, we will have that $h^{-1}g=U\in\mathcal{U}(E,J)$, and therefore, if $J_1=gJg^*$ and $J_2=hJh^*$, we get that $J_1=hh^*gJg^*hh^*=hUJU^*h^*=hJh^*=J_2$.
Furthermore, let us observe that this isomorphism in the category of sets is, in reality, an isomorphism in the category of topological spaces. That is, we shall observe that χ is the homogeneous space of the transitive group action of $\mathcal{O}_{\text{res}}$ with the isotropy group over $J\in\chi$. To do that, we will require some definitions and results concerning group actions for Banach manifolds.

Definition 3.2.6. *A Banach manifold M is a topological space with the property: for every point $p\in M$ there exists an open neighborhood U over p such that it is homeomorphic to an open set in a Banach space.*

With this definition, we see that $\mathcal{O}_{\text{res}}$ and χ are Banach manifolds.

Definition 3.2.7. *Let $\mathcal{G}$ a topological group. If $\mathcal{G}$ is a separable and metrizable topological space, we call $\mathcal{G}$ a Polish group.*

Proposition 3.2.8. *Every $g\in\mathcal{O}_{res}$ can be written as $g=U(\mathbf{1}+X)$, with $X\in\mathcal{Q}$ and $U\in\mathcal{U}(E,J)$.*

To prove the previous proposition, we shall need some tools associated with *singular values.* We have followed Gohberg and Krein [12] §2 and §3, and Pandey-Satish [20] §7 in the understanding of some main results associated to this topic. From [24] §6, we know that given a separable Hilbert space and a positive compact operator $A:H\to H$, its eigenvalues are greater than 0 and, additionally, they have finite multiplicity if they are different from 0. Moreover, its spectrum consists of the eigenvalues and 0 as the unique accumulation point. We also know that for a compact operator T, its square root $\text{abs}(T)$ is compact and positive. We can organize the eigenvalues of $\text{abs}(T)$ as a decreasing sequence $\lambda_j(\text{abs }(T))$. The singular values of T are then defined as $S_j(T) = \lambda_j(\text{abs}(T))$. Although we have not mentioned it before, the $|\cdot|_{\mathcal{Q}}$ is constructed using these singular values (see [12] §3). We can extend the definition of singular values for non-compact operators, but to do this let us consider first the following definition.

Definition 3.2.9. *Let A be a bounded operator. A point $\lambda \in \sigma(A)$ is called a condensed spectrum point if it is an accumulation point in $\sigma(A)$ or if it is an eigenvalue of H with infinite multiplicity.*
If, in addition, A is a positive bounded operator, denote by μ the supremum of $\sigma(A)$. If μ belongs to the condensed spectrum, define $\lambda_j(A) = \mu$ for $j = 1, 2, \ldots$. If μ does not belong to the condensed spectrum, then it must be an eigenvalue of finite multiplicity; in this case, define $\lambda_j(A) = \mu$ for $j = 1, 2, \ldots, p$, where p is the multiplicity of μ. For the other singular values $\lambda_j(A)$, define $\lambda_{p+j}(A) := \lambda_j(A_1)$, where $A_1 = A - \mu P$, being P the orthogonal projection over the eigenspace associated to μ. Then, the singular values are defined as $S_n(T) = \lambda_n(abs(T))$.

Remark. This construction of the singular values for a general bounded operator coincides with the case when the operator is positive and compact. Moreover, the way as we define each λ_j coincides with the standard proof of the spectral theorem for positive compact operators: once an eigenvalue is found, we remove its eigenspace and repeat inductively to find a new eigenvalue over the resulting subspace (see [24] §6). From [12] §2, we have the following lemma.

Lemma 3.2.10. *From the definition of the values $\lambda_j(A)$ for a bounded operator, it follows that if $0 \leq A \leq B$, with A, B bounded operators, then $\lambda_j(A) \leq \lambda_j(B)$. Thus, for positive bounded operators with $0 \leq A \leq B$, $S_j(A) \leq S_j(B)$.*
Now, consider $A \in \mathcal{Q}$. Then every operator B for which $S_j(B) \leq cS_j(A)$ for all j, with c a positive constant, belongs to $\mathcal{Q}$ and, moreover, $|B|_{\mathcal{Q}} \leq c\,|A|_{\mathcal{Q}}$.

With this lemma we are ready to prove proposition 3.2.8.

Proof. (Proposition 3.2.8) According to Plymen [21] §3, every orthogonal operator g on E can be written as $g = U(\text{abs}(C_g) + U^* A_g)$, where $U \in \mathcal{U}(E, J)$. We also have the relation $A_g^* A_g + C_g^* C_g = \mathbf{1}$, so $\mathbf{1} - A_g^* A_g = C_g^* C_g \geq 0$. Therefore, due to the uniqueness of the square root, we have that $\text{abs}(C_g) = (\mathbf{1} - A_g^* A_g)^{1/2}$. Moreover, we can understand this operator as an element in $\mathcal{B}(E_J)$, i.e as an operator on the complexified Hilbert space E_J, because $A_g^* A_g$ is $J-$linear
We also know, according to the decomposition $g = A_g + C_g$, that $||A_g|| \leq 1$, and thus $||A_g^* A_g|| = \mu \leq 1$. By the continuous functional calculus, and since $\sigma(A_g^* A_g) \subseteq [0, \mu]$, we

have that $||(\mathbf{1} - A_g^* A_g)^{1/2} - \mathbf{1}|| \leq ||A_g^* A_g|| \leq ||A_g||$: to see this, take the continuous functions $f(x) = (1-x)^{1/2} - 1$ and $g(x) = x$, which are elements of $C(\sigma(A_g^* A_g))$ and satisfy $|f| \leq |g|$, then by the isometric isomorphism which arises from this theorem, we obtain the mentioned inequality.
Observe that for any $\mu \in [0,1]$ the inequality $1 - (1-\mu)^{1/2} \leq \mu$ holds. Now, by compactness and positiveness of $A_g^* A_g$, its spectrum is composed by eigenvalues and 0 as an accumulation point (as long as it is not an eigenvalue). Moreover, note that the singular values of $A_g^* A_g$ are, exactly, its eigenvalues: For a positive operator A, given an eigenvalue λ, and an eigenvector x associated to λ, the following holds: $A^2 x = \lambda^2 x \equiv A^* A = \lambda^2 x$, and using the square root and uniqueness, $\text{abs}(A)x = \lambda x$.

Furthermore, notice that $\mathbf{1} - (\mathbf{1} - A_g^* A_g)^{1/2}$ is positive. Then, again by the continuous functional calculus, its spectrum corresponds to $h(\mu) := 1 - (1-\mu)^2$, with $\mu \in \sigma(A_g^* A_g)$. Since this function is continuous, 0 is also an accumulation point of the spectrum of this operator.
As we have mentioned before, a main result of the spectral theorem for compact bounded operators is that its eigenvalues $\lambda \neq 0$ are isolated points in the spectrum. Therefore, by continuity of h, for $\mu \in \sigma(A_g^* A_g) \setminus \{0\}$, we get that $h(\mu)$ is isolated too.

There is a relevant result about the spectrum of normal operators over a complex Hilbert space that we will need: *Every isolated point in the spectrum of a bounded normal operator T is an eigenvalue.* This result is a consequence of the Spectral Theorem (see [28] §4). From this theorem, given a normal operator T, and denoting by $P : \mathcal{B}_\Sigma \to \mathcal{P}(E_J)$ its spectral measure (with $\Sigma := \sigma(T)$, $\mathcal{B}_\Sigma$ the Borel algebra on Σ and $\mathcal{P}(E_J)$ the set of orthogonal projections over E_J), we have that a point λ belongs to Σ if and only if $P(U) \neq 0$ for every open neighborhood U over λ. If λ is an isolated point of $\sigma(T)$, then $U = \{\lambda\}$ is an open neighborhood over λ, so $P(U) \neq 0$. Now, considering the continuous function $f = \mathbf{1}_U$ ($f(t) = 1$ if $t = \lambda$ and 0 otherwise), we get that $f(T) = P(U)$, and since $tf(t) = \lambda f(t)$ by definition of f, we have that $TP = \lambda P$.
Thus, we have obtained the eigenvalues of $\mathbf{1} - (\mathbf{1} - A_g^* A_g)^{1/2}$. Furthermore, if μ is an eigenvalue of $A_g^* A_g$ and x is an eigenvector associated to μ, note that $((\mathbf{1} - A_g^* A_g)^{1/2})(x) = f(\mu)x = (1 - (1-\mu)^{1/2})x$. Furthermore, dimensions of the eigenspaces of $f(\mu)$ and μ are equal for the operators $f(A_g^* A_g)$ and $A_g^* A_g$ respectively. To see this statement, first observe that, by compactness and self-adjointness of the operator $A_g^* A_g$, we can construct an orthonormal basis for E_J of eigenvectors. Now, supposing that $\tilde{x}$ is an eigenvector associated to $f(\mu)$ but not to μ, we get a contradiction: by Parseval's identity $\tilde{x} = \sum_{n \geq 1} \langle x_n, \tilde{x} \rangle x_n$, but each factor $\langle x_n, \tilde{x} \rangle = 0$ since $\tilde{x}$ is orthogonal to all eigenvectors associated to eigenvalues $\lambda \neq \mu$ ($f(\lambda) \neq f(\mu)$), and by construction, they are zero for each x_n associated to the eigenspace of μ. Therefore, they have the same multiplicity.

We have already all the ingredients to conclude the proof. Since the eingenspaces of $f(\mu)$

and μ have the same dimension, $A_g^*A_g$ and $f(A_g^*A_g)$ are positive operators and $f(\mu) \leq \mu$ for $\mu \in [0,1]$, we have that $S_j(f(A_g^*A_g)) \leq S_j(A_g^*A_g)$; therefore $\left|f(A_g^*A_g)\right|_{\mathcal{Q}} \leq \left|A_g^*A_g\right|_{\mathcal{Q}} < \infty$, i.e, $(\mathbf{1} - A_g^*A_g)^{1/2} - \mathbf{1} \in \mathcal{Q}$. □

With this characterization of the restricted group $\mathcal{O}_{\text{res}}$, which will be very useful in the understanding of its homotopy type, we shall observe now how the group action can restrict to a subgroup which its topological properties have been studied by de la Harpe [8].
According to the last proposition, we can see the restricted orthogonal group as $\mathcal{O}_{\text{res}} = \mathcal{U}(E,J)\cdot\mathcal{O}_{\mathcal{Q}}$, where $\mathcal{O}_{\mathcal{Q}}$ is the set of orthogonal operators that can be written as $\mathbf{1}+X$ with $X \in \mathcal{Q}$. Although proposition 3.2.8 only tells us that $\mathcal{O}_{\text{res}} \subseteq \mathcal{U}(E,J)\cdot\mathcal{O}_{\mathcal{Q}}$, we can easily see the other inclusion: let $U \in \mathcal{U}(E,J)$ and $X \in \mathcal{Q}$ such that $\mathbf{1}+X$ is an orthogonal operator, then

$$U(\mathbf{1}+X) + JU(\mathbf{1}+X)J = U + UX + JUJ + JXJ = UX + JXJ$$

since $[U,J] = 0$, and thus it belongs to $\mathcal{Q}$. Moreover, observe that $\mathcal{O}_{\mathcal{Q}}$ has the structure of a topological group, where its topology arises from the condition of belonging to $\mathcal{Q}$.
Additionally, $\mathcal{O}_{\mathcal{Q}}$ acts transitively on χ. Given $J_1 \in \chi$, we have seen that there exists $g \in \mathcal{O}_{\text{res}}$ such that $gJg^* = J_1$, and according to our characterization, $g = U(\mathbf{1}+X)$; therefore

$$\begin{aligned} J_1 &= (U+UX)J(U^* + X^*U^*) = UJU^* + UJX^*U^* + UXJU^* + UXJX^*U^* \\ &= (\mathbf{1}+UXU^*)J + J(UX^*U^*) + UXU^*JUX^*U^* \\ &= (\mathbf{1}+UXU^*)J + (\mathbf{1}+UXU^*)JUX^*U^* \\ &= (\mathbf{1}+UXU^*)J(\mathbf{1}+UXU^*)^*. \end{aligned}$$

It is clear that $UXU^* \in \mathcal{Q}$, and, moreover, that $\mathbf{1}+UXU^*$ is an orthogonal operator since it is equal to $U(\mathbf{1}+X)U^*$.
According to Carey [4] and Espinoza and Bernardo [11], the restricted orthogonal group is a Polish group. In these two references, they show this property using the strong operator topology. The hardest part is to show that this space is separable (observe that we already have completeness), and these articles prove that. Although, in principle, it does not apply for our purposes, observe that the strong operator topology is weaker than the operator norm topology. Hence, if we have separability in the topology with more open sets (weaker) then we have this property in the set with less open sets. In his article, Carey uses a similar topology for the restricted orthogonal group, but he changes the condition $||C_g - C_h||$ by strong convergence. Now, using the proposition above, he asserts that the morphism $\mathcal{U}(E,J)\times\mathcal{O}_{\mathcal{Q}} \to \mathcal{O}_{\text{res}}$ is surjective and continuous when $\mathcal{U}(E,J)$ is equipped with the strong operator topology. Thus, $\mathcal{O}_{\text{res}}$ is the product of two separable spaces. Notice that this argument applies for $\mathcal{O}_{\mathcal{Q}}$ when it is equipped with the resulting topology from the condition of belonging to $\mathcal{Q}$, since $||\cdot|| \leq |\cdot|_{\mathcal{Q}}$, and applying the same argument, we obtain the mentioned conclusion.
Now, we have a main result about Polish group and homogeneous spaces. The following theorem is due to Ramsey [23].

Theorem 3.2.11. *Let $\mathcal{G}$ be a Polish group acting on the space X, let R be the equivalence relation induced by $\mathcal{G}$ on X, let Y be the orbit space $X/\mathcal{G}$ and let π be the quotient map from X to Y. Then the following are equivalent:*

i. For each $x \in X$ the map of $\mathcal{G}/\mathcal{G}_x$ (the quotient by the stabilizer of x) to the orbit of x, $[x]$, is an homeomorphism.

ii. Y is a T_0 space.

Thus, since our action is transitive and there is only one orbit, we can conclude that $\chi \cong \mathcal{O}_{\text{res}}/\mathcal{U}(E, J)$.

Remark. We do not have to take the full version of the above theorem. Ramsey gives four distinct statements which are equivalent. Moreover, he proves a more general result in the context of grupoids.

In a similar way, it is possible to show that the group $\mathcal{O}_\mathcal{Q}$ acts continuously over χ. Defining $\mathcal{U}_\mathcal{Q}$ as $\mathcal{U}(E, J) \cap \mathcal{O}_\mathcal{Q}$, we can conclude, by analogous arguments, that $\chi \cong \mathcal{O}_\mathcal{Q}/\mathcal{U}_\mathcal{Q}$ in the category of topological spaces. These groups have been studied by Pierre de la Harpe [8], where the autohor shows some topological and homotopycal properties of them. We shall summarize some of the main results studied by de la Harpe.
Let V be an infinite dimensional Banach space over $\mathbb{X}$ $(= \mathbb{R}, \mathbb{C}, \mathbb{Q})$. Let $\mathcal{P}(V)$ be a subspace of $\mathcal{B}(V)$ equipped with a norm (not necessarily complete with respect to this norm) with the following properties:

i. If $A \in \mathcal{P}(V)$, then $\mathbf{1} + A$ is a Fredholm operator on V.

ii. If $A \in \mathcal{P}(V)$, then $A + C_o(V) \subseteq \mathcal{P}(V)$, where $C_o(V)$ denotes the subspace of finite rank operators.

iii. The multiplication map $C_o(V) \times \mathcal{P}(V) \to \mathcal{P}(V)$ is continuous, where $C_o(V)$ is equipped with the operator norm.

Let $\mathrm{GL}(V, \mathcal{P})$ be the subset of $\mathrm{GL}(V)$ which consists of those invertible operators on V that can be written as $\mathbf{1} + X$ with $X \in \mathcal{P}(E)$. Endow $\mathrm{GL}(V, \mathcal{P})$ with the topology inherited from the norm given on $\mathcal{P}(V)$. We have the following theorem:

Theorem 3.2.12. *Let V and $GL(V, \mathcal{P})$ as above. Then $GL(V, \mathcal{P})$ is homotopically equivalent to the stable general linear group $GL(\infty, \mathbb{X}) := \varinjlim GL(n, \mathbb{X})$.*

This theorem is proved by de la Harpe [8] §2. Notice that $\mathcal{Q}$ satisfies each of the properties mentioned above for $\mathcal{P}(E)$ (the first one follows from Atkinson's theorem because A is compact, so $\mathbf{1} + A$ is invertible modulo compact operators). De la Harpe generalizes this theorem to the case where we are dealing with a symmetric normed ideal $\mathcal{Q}$ and orthogonal and unitary operators. We have then the equivalence homotopies:

$$\mathcal{O}_{\mathcal{Q}} \simeq \mathcal{O}(\infty), \quad \mathcal{U}_{\mathcal{Q}} \simeq \mathcal{U}(\infty).$$

Let $B = (e_n)_{n\geq 1}$ be an orthonormal basis for E_J. Then $B \cup J(B)$ is an orthonormal basis for the real Hilbert space E. For each $n \in \mathbb{N}$, let us consider the complex Hilbert subspace $E_{J,n}$ spanned by the first n elements of the base B. Notice that $\{e_1, \ldots, e_n, Je_1 \ldots, Je_n\}$ is then a basis for the real Hilbert subspace E_n. We can consider the groups of orthonormal and unitary operators $O(n)$ and $U(n/2)$ acting over E_n and $E_{J,n}$ respectively. Observe that $U(n/2) \subseteq O(n)$ because $E_{J,n}$ is the complexification of the subspace E_n. The direct limit of these two structures are, precisely, $\mathcal{O}(\infty)$ and $\mathcal{U}(\infty)$.
Milnor, in his book *Morse Theory* [18], presents a proof from a geometric point of view of the Bott's periodicity applied, exactly, to the groups $\mathcal{O}(\infty)$ and $\mathcal{U}(\infty)$. To be precise, he shows the following isomorphisms for the homotopy groups π_i:

$$\pi_i(\mathcal{U}(\infty)) = \begin{cases} 0 & \text{if } i \text{ even} \\ \mathbb{Z} & \text{if } i \text{ odd} \end{cases},$$

for the complex case, and for the real case, we have:

i modulo 8	$\pi_i(\mathcal{O}(\infty))$
0	$\mathbb{Z}_2$
1	$\mathbb{Z}_2$
2	0
3	$\mathbb{Z}$
4	0
5	0
6	0
7	$\mathbb{Z}$

Now let us consider the following diagram:

$$\begin{array}{ccccccc}
\mathcal{U}(n/2) & \hookrightarrow & \mathcal{U}((n+1)/2) & \dashrightarrow & \mathcal{U}(\infty) & \xrightarrow{\alpha} & \mathcal{U}_{\mathcal{Q}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}(n) & \hookrightarrow & \mathcal{O}(n+1) & \dashrightarrow & \mathcal{O}(\infty) & \xrightarrow{\beta} & \mathcal{O}_{\mathcal{Q}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}(n)/\mathcal{U}(n/2) & \longrightarrow & \mathcal{O}(n+1)/\mathcal{U}((n+1)/2) & \dashrightarrow & \mathcal{O}(\infty)/\mathcal{U}(\infty) & \xrightarrow{\gamma} & \chi
\end{array}.$$

where α and β correspond to the equivalence homotopies between these two groups. Vertical lines corresponds to fibrations, and therefore each column gives rise to a large exact sequence in the homotopy groups π_i. Thus, we have the following commutative diagram:

$$\begin{array}{ccccccccc} \pi_i(\mathcal{U}(\infty)) & \longrightarrow & \pi_i(\mathcal{O}(\infty)) & \longrightarrow & \pi_i(\mathcal{O}(\infty)/\mathcal{U}(\infty)) & \longrightarrow & \pi_{i-1}(\mathcal{U}(\infty)) & \longrightarrow & \pi_{i-1}(\mathcal{O}(\infty)) \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \beta_i} & & \vdots{\scriptstyle \gamma_i} & & \downarrow{\scriptstyle \alpha_{i-1}} & & \downarrow{\scriptstyle \beta_{i-1}} \\ \pi_i(\mathcal{U}_\mathcal{Q}) & \longrightarrow & \pi_i(\mathcal{O}_\mathcal{Q}) & \longrightarrow & \pi_i(\chi) & \longrightarrow & \pi_{i-1}(\mathcal{U}_\mathcal{Q}) & \longrightarrow & \pi_{i-1}(\mathcal{O}_\mathcal{Q}) \end{array}.$$

Since α and β are homotopy equivalences, α_i and β_i are group isomorphisms for each i. Thus, applying the *V Lemma*, we obtain that γ_i is a group isomorphism for each i. In this order, we observe that γ is a weak homotopy equivalence between $\mathcal{O}(\infty)/\mathcal{U}(\infty)$ and χ. De la Harpe presents a more elaborated argument to show that γ actually is a homotopy equivalence (see [8] §2).

Note that by exactness of the rows and commutativity of the diagram for the homotopy groups, $\pi_0(\chi)$ is isomorphic to $\mathbb{Z}_2$. Now, using the homeomorphism $\chi \cong \mathcal{O}_{\text{res}}/\mathcal{U}(E,J)$, we can use the fibration $\mathcal{U}(E,J) \hookrightarrow \mathcal{O}_{\text{res}} \twoheadrightarrow \chi$, and then by the long exact sequence on the homotopy groups, we see that $\pi_0(\mathcal{O}_{\text{res}}) \cong \mathbb{Z}_2$ because $\mathcal{U}(E,J)$ is contractible, so all its homotopy groups are zero. Consequently, we have proved the following theorem:

Theorem 3.2.13. *The group $\mathcal{O}_{res}$ has two connected components.*

3.2.2 Index map

In this subsection we shall construct an index map for the group $\mathcal{O}_{\text{res}}$. This map will correspond to $i_{\mathcal{O}_{\text{res}}} : \mathcal{O}_{\text{res}} \to \mathbb{Z}_2$ given by $i_{\mathcal{O}_{\text{res}}}(g) = \dim_{\mathbb{C}}\text{Ker }(g - JgJ) \bmod 2$. We know, [21] §3, that given $g \in \mathcal{O}_{\text{res}}$, C_g is a Fredholm operator with index zero, so the map $i_{\mathcal{O}_{\text{res}}}$ makes sense.

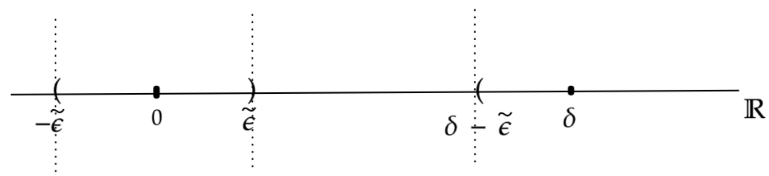
By definition of $A_g := \frac{1}{2}(g + JgJ)$, we observe that $||A_g|| \leq 1$. Notice the following: if $1 \notin \sigma(A_g^*A_g)$, then $A_g^*A_g - \mathbf{1}$ is invertible, i.e $C_g^*C_g$ is invertible, so $0 \notin \sigma(C_g^*C_g)$; if $1 \in \sigma(A_g^*A_g)$, then by compactness of $A_g^*A_g$, 1 is an eigenvalue, and according to the continuous functional calculus, using the function $f(x) = 1 - x$, 0 turns out to be an eigenvalue of $C_g^*C_g$, which is an isolated point because 1 is isolated on $\sigma(A_g^*A_g)$ and f preserves this property. In both cases, we can find $\delta > 0$ such that $(0,\delta] \cap \sigma(C_g^*C_g) = \emptyset$.

Let us consider the *inclusion map* $(\mathcal{O}_{\text{res}}, \sigma) \hookrightarrow \mathcal{B}(E)$ given by $g \mapsto C_g$. This map is continuous: let $\epsilon > 0$, and let $B_{||\cdot||}(C_g, \epsilon)$ be the ball in the operator norm centered on C_g with radius ϵ. Notice that $i^{-1}(B_{||\cdot||}(C_g,\epsilon)) = \{h \in \mathcal{O}_{\text{res}} |\ ||C_g - C_h|| < \epsilon\}$. Thus, considering the ball $B_\sigma(g,\epsilon) \subseteq \mathcal{O}_{\text{res}}$, by definition of the metric σ, we obtain that $B_\sigma(g,\epsilon) \subseteq i^{-1}(B_{||\cdot||}(C_g,\epsilon))$, i.e i is continuous.

We can then consider the continuous map $\mathcal{O}_{\text{res}} \overset{i}{\hookrightarrow} \mathcal{B}(E) \to \mathcal{B}(E)$ given by $g \mapsto C_g \mapsto C_g^*C_g$. Let $\epsilon > 0$ and consider the open neighborhood $\mathcal{N}_1$ over $g \in \mathcal{O}_{\text{res}}$ as the inverse image of the open ball $B_{||\cdot||}(C_g^*C_g, \epsilon)$ of this map. We shall need the following result proved by Murphy in his article [19]:

Theorem 3.2.14. *Let A be a Banach algebra, let $x \in A$ and let $U \in \mathbb{C}$ be an open set. If $\sigma(x) \subset U$, then there exists $\delta > 0$ such that $||y - x|| < \delta$ implies that $\sigma(y) \subset U$.*

We should underline that this theorem applies for complex Banach algebras. Therefore, in order to use this result, we must take the Banach algebra $\mathcal{B}(E_J)$. However, it will not be an issue since $C_h^*C_h$ and $A_h^*A_h$ commute with J, so they are elements of $\mathcal{B}(E_J)$. Moreover, in our case, since we are studying positive operators, their spectrum is contained in $[0, \infty)$.
Let $\tilde{\epsilon} > 0$ such that $\delta - \tilde{\epsilon} > \tilde{\epsilon}$. Let $U \in \mathbb{C}$ an open set such that it contains the open set $(-\tilde{\epsilon}, \tilde{\epsilon}) \cup (\delta - \tilde{\epsilon}, \infty)$ in $\mathbb{R}$.



Observe that $\sigma(C_g^*C_g) \subset U$. Thanks to the above theorem, there exists ϵ' such that if $||A - C_g^*C_g|| < \epsilon'$ then $\sigma(A) \subseteq U$. Therefore, for any $h \in \mathcal{O}_{\text{res}}$ such that $||C_h^*C_h - C_g^*C_g|| < \epsilon'$, we get that $\sigma(C_h^*C_h) \subseteq U$, but since this operator is positive, we have that it is contained in $(-\tilde{\epsilon}, \tilde{\epsilon}) \cup (\delta - \tilde{\epsilon}, \infty)$. In particular, $C_h^*C_h$ has no spectrum in $(\tilde{\epsilon}, \delta - \tilde{\epsilon})$ (the last affirmation follows from the fact that the spectrum is a closed set, so it contains neither $\tilde{\epsilon}$ nor $\delta - \tilde{\epsilon}$). By continuity of our map above we can find an open set $\mathcal{N} \subset \mathcal{O}_{\text{res}}$ such that for any $h \in \mathcal{N}$ the following two conditions hold:

i. $C_h^*C_h$ has no spectrum in $(\tilde{\epsilon}, \delta - \tilde{\epsilon})$.

ii. $||C_h^*C_h - C_g^*C_g|| < \epsilon'$.

Let us suppose that for any $\tilde{\epsilon}$, the found ϵ' is less than $\tilde{\epsilon}$ –if ϵ' is greater than $\bar{\epsilon}$, we take $\bar{\epsilon}$ and the following proof works for this case. Let Q_h be the spectral projection of $C_h^*C_h$, with $h \in \mathcal{O}_{\text{res}}$, over $[0, \tilde{\epsilon}]$. Let K be the orthogonal projection of E over ker $(C_g^*C_g) =$ ker C_g. We shall observe that K defines an isomorphism of vector spaces between $Q_h(E)$ and ker C_g. Let $v \in Q_h(E) \cap K(E)^\perp$ with $||v|| = 1$, and let us compute

$$\langle v, (C_g^*C_g - C_h^*C_h)v\rangle = \langle v, C_g^*C_g v\rangle - \langle v, C_h^*C_h v\rangle.$$

Since $v \in K(E)^\perp$, we have that $v \notin$ Ker C_g, therefore $\langle v, C_g^*C_g v\rangle > 0$. Moreover, according to [21] §3, when A_g is compact (as in our case), C_g restricted as a homomorphism C_g : Ker ${C_g}^\perp \to$ Ker ${C_g^*}^\perp$ is, in fact, an isomorphism. In this order, C_g^* restricted to Ker ${C_g^*}^\perp$ is also an isomorphism, and thus $C_g^*C_g$ restricted to ker ${C_g}^\perp$ is an automorphism. We will now consider a result from [17] which is presented as an exercise:

Lemma 3.2.15. *Let H be a complex Hilbert space and let $T \in \mathcal{B}(H)$ a self-adjoint operator. Let $M := \sup_{||x||=1} |\langle Tx, x\rangle|$ and $m := \inf_{||x||=1} |\langle Tx, x\rangle|$. Then $\sigma(T) \subseteq [m, M]$. Moreover, $m, M \in \sigma(T)$.*

Thus, since $C_g^*C_g$ restricted to ker ${C_g}^\perp$ is invertible, we have that $\langle v, C_g^*C_g v\rangle \geq \delta$: since $\sigma(C_g^*C_g)$ must be contained between its infimum and supremum, it is not possible to have

that $\langle v, C_g^*C_g\rangle < \delta$ because this would imply that the infimum is less than δ, so there would exist an element between 0 and δ in the spectrum.
Now, $Q_h(E)$ is the image of the projection over the subspace of vectors with spectrum in $[0, \tilde{\epsilon}]$. In particular, if $v \in Q_h(E)$, and since the spectrum here corresponds to eigenvalues, we have that $C_h^*C_h(v) = \alpha v$ with $0 \leq \alpha \leq \tilde{\epsilon}$, so $\langle v, C_h^*C_h v\rangle \leq \tilde{\epsilon}$. Therefore, $\langle v, (C_g^*C_g - C_h^*C_h)v\rangle \geq \delta - \tilde{\epsilon} > \tilde{\epsilon} > \epsilon'$.
With this computation, we have showed that K is one to one on $Q_h(E)$: if $v \in Q_h(E) \cap (\mathbf{1} - K)(E)$ then we get a contradiction on the second property for $h \in \mathcal{N}$. If $v \in Q_h(E)$ is such that $K(v) = 0$, then $v \in Q_h(E) \cap (\mathbf{1} - K)(E)$, but then we conclude that $v = 0$.
Suppose now that K, viewed as a map from $Q_h(E)$ to ker C_g, is not surjective. Thus, there exists $v \in \ker C_g \cap (Q_h(E))^\perp$ with $||v|| = 1$ ($K : E \to \ker C_g$ is surjective, so there must exist such an element in $(Q(E))^\perp$). In this case,

$$\langle v, (C_h^*C_h - C_g^*C_g)v\rangle = \langle v, C_h^*C_h(v)\rangle.$$

By properties of the spectral measure (see [28] §4), if v does not belong to $Q_h(E) = P([0, \tilde{\epsilon}])(E)$, then $v \in P([0, \tilde{\epsilon}]^c)(E)$, where P denotes the spectral projection $P : \mathcal{B}_{\sigma(C_h^*C_h)} \to \mathcal{P}(E)$, and $[0, \tilde{\epsilon}] \cup [0, \tilde{\epsilon}]^c = \sigma(C_h^*C_h)$ –as subsets of $\sigma(C_h^*C_h)$. Therefore, $\langle v, C_h^*C_h v\rangle > \tilde{\epsilon} > \epsilon'$, and again we obtain a contradiction with the properties that $h \in \mathcal{N}$ must satisfy.
We conclude then that K defines an isomorphism between $Q_h(E)$ and ker C_g.

Considering the identities

$$\mathbf{1} = C_g^*C_g + A_g^*A_g = C_gC_g^* + A_gA_g^*, \quad 0 = C_g^*A_g + A_g^*C_g = C_gA_g^* + A_gC_g^*,$$

notice that $A_g^*C_g$ is skew adjoint, i.e $(A_g^*C_g)^* = -A_g^*C_g$. Moreover, note that the operator $A_h^*C_h$ over Ker $C_h{}^\perp \cap Q_h(E)$ has trivial kernel and is J-antilinear:
Let $v \in (\ker C_h^*C_h)^\perp \cap Q_h(E)$; thus v is an eigenvector with $C_h^*C_h(v) = \alpha v$ where $0 < \alpha \leq \tilde{\epsilon}$. Suppose that $A_h^*C_h(v) = 0$, then $C_h(v) \in \ker A_h^*$. Applying $A_hA_h^* + C_hC_h^*$ over $C_h(v)$, we get $C_hC_h^*C_h(v) = C_h(v)$, but $C_h^*C_h(v) = \alpha v$ where $0 < \alpha \leq \tilde{\epsilon} \leq 1$. Therefore $C_h(v) = 0$, but $v \in (\ker C_h)^\perp$, so $v = 0$.
Considering $A_h^*C_h$ restricted to the subspace $(\ker C_h)^\perp \cap Q_h(E)$, and applying the polar decomposition, we obtain:

$$A_h^*C_h = J' \operatorname{abs}(A_h^*C_h),$$

where $\ker J' = 0$. Additionally, according to the skew adjointness property, we get that $A_h^*C_h = -\text{abs } (A_h^*C_h)J'^*$. We observed previously how it is the polar decomposition for the adjoint operator, so we get that

$$A_h^*C_h = -J'^*\text{abs } (-(A_h^*C_h)).$$

Now, observe that $\operatorname{abs}(A_h^*C_h) = (-(A_h^*C_h)^2)^{1/2} = \operatorname{abs}((A_h^*C_h)^*) = \operatorname{abs}(-(A_h^*C_h))$. Hence, we have observed that J'^* commutes with $\operatorname{abs}(A_h^*C_h)$, and therefore it commutes with $-(A_h^*C_h)^2$.

Since $\mathrm{abs}(A_h^*C_h)$ is positive, J' also commutes with this.

Now, according to the properties of the polar decomposition, ker $\mathrm{abs}(A_h^*C_g)$ is trivial, therefore this map is one to one; thus we can find a left inverse function, and then, by the relation

$$J'\mathrm{abs}(A_h^*C_h) = -\mathrm{abs}(A_h^*C_h)J'^*,$$

we conclude that $J'^* = -J'$. Reasoning in a similar way and observing that

$$(A_h^*C_h)^2 = J'\mathrm{abs}\ (A_h^*C_h)J'\mathrm{abs}(A_h^*C_h) = J'^2(-(A_h^*C_h)^2),$$

we conclude that $J'^2 = -\mathbf{1}$.
Finally, we analyze the following relation:

$$J(A_h^*C_h) = JJ'\mathrm{abs}(A_h^*C_h) = -(A_h^*C_h)J,$$

where the last equality follows from the $J-$antilinear property. Hence, $JJ'\mathrm{abs}(A_h^*C_h) = -J'\mathrm{abs}(A_h^*C_h)J$. Since J commutes with $\mathrm{abs}(A_h^*C_h)^2$, it commutes with $\mathrm{abs}(A_h^*C_h)$. This last statement is a consequence of the continuous functional calculus, where we use the density of polynomial functions in the set of continuous functions on $\sigma(\mathrm{abs}(A_h^*C_h))$. In this order, we conclude that $JJ' = -J'J$.

We have obtained two operators, J and J', which satisfy the Clifford relations. We can view them as generators of a Clifford algebra represented on $(\ker C_H)^\perp \cap Q_h(E)$. Notice that this subspace if finite dimensional: we have shown that $Q_h(E) \cong \ker\, C_g$, and this one is finite dimensional since C_g is a Fredholm operator. The algebra generated by these operators corresponds, modulo isomorphism, to $\mathbb{H}$ (see [13] §12). Thus, as a real subspace, the dimension of $(\ker\, C_h)^\perp \cap Q_h(E)$ is a multiple of 4, and as complex subspace it is even dimensional.

Clearly $Q_h(E) \supseteq \mathrm{Ker}\ C_h$. If this is an equality, then both dimensions are equal. Otherwise, observe that $\mathrm{Ker}\ C_g \cong Q_h(E) = (\mathrm{Ker}\ C_h) \bigoplus ((\mathrm{Ker}\ C_h)^\perp \cap Q_h(E))$, therefore we have the equality $\dim_{\mathbb{C}} \mathrm{Ker}\ C_g = \dim_{\mathbb{C}} \mathrm{Ker}\ C_h \bmod 2$.
This construction tell us that the map $i_{\mathcal{O}_{\mathrm{res}}}$ is locally constant. Therefore, we have proved the following theorem:

Theorem 3.2.16. *The map $i_{\mathcal{O}_{res}} : \mathcal{O}_{res} \to \mathbb{Z}_2$ is an index map.*

Remark We must emphasize that we have not shown yet that the above map is really a group homomorphism. We will see in the case $\mathcal{Q} = \mathcal{Q}_2$, i.e when we are dealing with the symmetric ideal of Hilbert-Schmidt operators. Although it is a less general statement than the one proposed in the above theorem, it corresponds precisely to our case of interest, as we have seen in chapter 2. For this part, we shall follow [21] §3.
Let $\pi_J : C[E] \to \mathcal{H}_J$ be the irreducible representation of the Clifford algebra associated to

the complex structure J. Let $g \in \mathcal{O}_{\text{res}}$ and let U be the unitary operator which implements the Bogoliubov automorphism θ_g over $C[V]$, i.e U is the unitary operator which makes the diagram commute:

$$\begin{array}{ccc} \mathcal{H}_J & \xrightarrow{U} & \mathcal{H}_J \\ \Big\downarrow{\scriptstyle \pi_J(v)} & & \Big\downarrow{\scriptstyle \pi_J(gv)} \\ \mathcal{H}_J & \xrightarrow{U} & \mathcal{H}_J \end{array} .$$

Recall, from chapter 2, the definition of the grading operator Γ_J which implements the grading automorphism γ on $C[V]$: $\Gamma_J|_{\bigwedge^n(E_J)} = (-\mathbf{1})^n$ for $n \geq 1$. Thus, considering the subspaces:

$$\mathcal{H}_J^+ := \text{Closure} \bigoplus_{m \geq 0} \bigwedge^{2m}(E_J), \quad \mathcal{H}_J^- := \text{Closure} \bigoplus_{m \geq 0} \bigwedge^{2m+1}(E_J),$$

we get $\Gamma|_{\mathcal{H}_J^\pm} = \pm\mathbf{1}$, and thus $\mathcal{H}_J^\pm = (\mathbf{1} \pm \Gamma_J)\mathcal{H}_J$. Notice that Γ_J is unitary and self-adjoint. Let $v \in E$ and $n \in \mathbb{N}$. We have observed that the "creation operator" $c(v)$ maps $\bigwedge^n(E_J)$ to $\bigwedge^{n+1}(E_J)$, and the "annihilation operator" $a(v)$ maps $\bigwedge^n(E_J)$ to $\bigwedge^{n-1}(E_J)$. Therefore, $\pi_J(v) = c(v) + a(v)$ maps $\mathcal{H}_J^\pm$ to $\mathcal{H}_J^\mp$. Moreover, $\pi_J(v)$ anti commutes with Γ_J. Hence, the following holds: $\Gamma_J \pi_J(v) \Gamma_J = -\pi_J(v)$. More generally, the map Γ_J implements the grading automorphism $\gamma = \theta_{-\mathbf{1}}$ over $C[E]$ on the Fock space. Therefore, given $v \in E$, we get:

$$\Gamma_J U \Gamma_J \pi_J(v) \Gamma_J U^* \Gamma_J = -\Gamma_J U \pi_J(v) U^* \Gamma_J = -\Gamma_J \pi_J(gv) \Gamma_J = \pi_J(gv).$$

Thus, $\Gamma_J U \Gamma_J$ also implements θ_g. Since π_J is irreducible, by Schur's lemma we get that $\Gamma_J U \Gamma_J = \mu U$ where $\mu \in \mathbb{S}^1$. But $\Gamma_J^2 = \mathbf{1}$, and therefore $U = \mu \Gamma_J U \Gamma_J$, so we conclude that $\mu \in \{-1, 1\}$.
Hence, if $U \in \text{Aut}\, \mathcal{H}_J$ implements θ_g on π_J, then U satisfies $\Gamma_J U \Gamma_J = U$ or $\Gamma_J U \Gamma_J = -U$. In the first case we say that U is even, and if the second case holds we say that U is odd. Notice that U is even if and only if it maps $\mathcal{H}_J^\pm$ to $\mathcal{H}_J^\pm$, i.e. if and only if it commutes with Γ_J; and U is odd if and only if it maps $\mathcal{H}_J^\pm$ to $\mathcal{H}_J^\mp$, i.e. if and only if it anti commutes with Γ_J. We shall present two propositions from [21], which are the main results in order to show the group homomorphism structure:

Proposition 3.2.17. *Let $g \in \mathcal{O}_{res}$, and let $u \in \mathcal{U}(E, J)$ such that C_{gu^*} is self adjoint. If U is an unitary operator on $\mathcal{H}_J$ implementing θ_g in π_J, then*

$$U(\Omega_J) = \zeta \wedge exp(\eta),$$

for some ζ in the top power of $u(ker\ C_g)$ and some $\eta \in \bigwedge^2[u(ker\ C_g)^\perp]$.

This proposition tell us that $U(\Omega_J)$ belongs to $\mathcal{H}_J^+$ or $\mathcal{H}_J^-$ depending if the complex dimension of $\ker C_g$ is even or odd ($\exp(\eta)$ belongs to the even Fock subspace). Thus, the parity of the unitary operator U implementing θ_g in π_J is equal to the parity of the complex dimension of $\ker C_g$.

Proposition 3.2.18. *Let $g \in \mathcal{O}_{res}$, and let U be an unitary operator on the Fock space $\mathcal{H}_J$ implementing the Bogoliubov automorphism θ_g of $C[E]$. Then $\Gamma_J U \Gamma_J = \epsilon_g U$, where ϵ_g is ± 1 according to the complex dimension of ker C_g. Moreover, $\epsilon : \mathcal{O}_{res} \to \mathbb{Z}_2$ is a group homomorphism.*

4 $K-$Theory for C^*-algebras and the Bulk-Edge correspondence

In this chapter we shall study an application of the theory developed previously. We will consider a physical periodic model for a topological insulator and analyze the *bulk boundary correspondence*, where we will require some generalities from K-theory for C^*-algebras. We will obtain a topological invariant which, in turn, could be understood as the Fredholm Index for some linear operator. The main references for this chapter are [27] and [22] §1. After discussing the example and the connection between the so called Bulk-Edge correspondence and Fredholm index from K-theory, two natural questions arise concerning the results obtained in previous chapter: Is it an invariant type $\mathbb{Z}$ or type $\mathbb{Z}_2$?, and how can we construct the operator associated with the invariant obtained according to the development presented in chapter two and three?

4.1 Elements of K-Theory for C^*-algebras

Before giving some definitions and results concerning $K-$theory, let us introduce some preliminary aspects regarding C^*-algebras.
Let A be a C^*-algebra. Let us associate a unique unital C^*-algebra to A, say $\hat{A}$, that contains A as an ideal and with the property that $\hat{A}/A$ is isomorphic to $\mathbb{C}$. Let $\pi : \hat{A} \to \mathbb{C}$ be the quotient map, and let $\lambda : \mathbb{C} \to \hat{A}$ be defined as $\lambda(\alpha) = \alpha \mathbf{1}_{\hat{A}}$. Then

$$0 \longrightarrow A \xrightarrow{i} \hat{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0 \ .$$

The algebra $\hat{A}$ is called *the unitization of A*.
Adjoining a unit is a well behaved operation at the level of $*$-homomorphisms. Let A, B be C^*-algebras, and $\phi : A \to B$ be a $*-$homomorphism. Then there exists a unique $\hat{\phi} : \hat{A} \to \hat{B}$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & \hat{A} & \xrightarrow{\pi} & \mathbb{C} & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow \hat{\phi} & & \downarrow \mathbf{1} & & \\ 0 & \longrightarrow & B & \xrightarrow{i} & \hat{B} & \xrightarrow{\pi} & \mathbb{C} & \longrightarrow & 0 \end{array} .$$

This map $\hat{\phi}$ is given by $\hat{\phi}(a + \mathbf{1}_{\hat{A}}) := \phi(a) + \mathbf{1}_{\hat{B}}$. Notice that $\hat{\phi}$ is unit preserving.
The unitization $\hat{A}$ when A is a unital C^*-algebra can be understood as follows. Let $\mathbf{1}_A$ and

$\mathbf{1}_{\hat{A}}$ be the units of A and $\hat{A}$ respectively. Then, the element $f := \mathbf{1}_A - \mathbf{1}_{\hat{A}}$ is a projection in $\hat{A}$ and

$$\hat{A} = \{a + \alpha f \mid a \in A,\ \alpha \in \mathbb{C}\}.$$

Moreover, the $*$-homomorphism $A \bigoplus \mathbb{C} \to \hat{A}$ given by $(a, \alpha) \mapsto a + \alpha f$ is, in fact, an isomorphism (see [27] §1).

4.1.1 K_0 Group

In this subsection, following [27] §3 and §4, we shall define the K_0 group for any C^*-algebra A. This group is defined in terms of the projections in the algebra. We will first describe the construction of this group for unital C^*-algebras, and then, using the unitization, we will present this construction for any C^*-algebra.
Let A be a C^*-algebra. Let $\mathcal{P}_n(A)$ be the projections of the C^*-algebra $M_n(\mathbb{C}) \otimes A \cong M_n(A)$. Let

$$\mathcal{P}_\infty(A) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(A),$$

where the sets $\mathcal{P}_n(A)$ are viewed as being pairwise disjoint. Define the equivalence relation $\sim_0$ on $\mathcal{P}_\infty(A)$ as follows: Let $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$ for some $m, n \in \mathbb{N}$; then $p \sim_0 q$ if there exists an element $v \in M_{m,n}(A)$ –the set of $m \times n$ matrices whose entries are elements in A– with $p = v^*v$ and $q = vv^*$, viewing the adjoint v^* as an element of $M_{n,m}(A)$ obtained by transposing the matrix and taking adjoints on each entry.
Define a binary operation $\oplus$ on $\mathcal{P}_\infty(A)$ by setting:

$$p \oplus q := \operatorname{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathcal{P}_{n+m}(A),$$

for $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$. This operation $\oplus$ is well behaved with respect to the equivalence relation $\sim_0$ in the following sense:

i. $p \sim_0 p \oplus 0_n$ for all $p \in \mathcal{P}_\infty(A)$ and any $n \in \mathbb{N}$.

ii. If $p \sim_0 p'$ and $q \sim_0 q'$ then $p \oplus q \sim_0 p' \oplus q'$ for all $p, q, p', q' \in \mathcal{P}_\infty(A)$.

iii. $p \oplus q \sim_0 q \oplus p$ for all $p, q \in \mathcal{P}_\infty(A)$.

iv. $(p \oplus q) \oplus r \sim_0 p \oplus (q \oplus r)$ for all $p, q, r \in \mathcal{P}_\infty(A)$.

v. If p, q are projections in $\mathcal{P}_n(A)$ such that $pq = 0$, then $p + q$ is a projection in $\mathcal{P}_n(A)$ and $p + q \sim_0 p \oplus q$.

Now, set

$$\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0,$$

and denote by $[p]_0$ the equivalence class of p in $\mathcal{D}(A)$. Define addition on $\mathcal{D}(A)$ by:

$$[p]_0 + [q]_0 = [p \oplus q]_0, \quad p, q \in \mathcal{P}_\infty(A).$$

With this operation, $(\mathcal{D}(A), +)$ is an abelian semi-group.

Definition 4.1.1. ***The*** K_0 ***Group for unital*** C^****-algebras:*** *Let A be a unital C^*-algebra, and let $(\mathcal{D}(A), +)$ be the abelian semi-group of the construction above. Define the K_0 group to be the Grothendieck group of $\mathcal{D}(A)$.*

Although this definition also makes sense for non unital C^*-algebras, we will define the K_0 group for non unital C^*-algebras in a different way. This is because this definition, for non unital C^*-algebras, fails to be half exact (see [27] §3).
There is a more standard description for this group K_0 for unital C^*-algebras. We call this the *standard picture of* K_0, and it is given by:

$$K_0(A) = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_\infty(A)\} = \{[p]_0 - [q]_0 \mid p, q \in \mathcal{P}_n(A),\ n \in \mathbb{N}\}.$$

Furthermore, we have these properties (see [27] §3):

- $[p \oplus q]_0 = [p]_0 + [q]_0$.
- $[0]_0 = 0$.
- If $p, q \in \mathcal{P}_n(A)$ for some n, and if there exists a continuous path $\gamma : [0, 1] \to \mathcal{P}_n(A)$ connecting these elements, then $[p]_0 = [q]_0$.
- If p, q are mutually orthogonal projections in $\mathcal{P}_n(A)$, then $[p + q]_0 = [p]_0 + [q]_0$.
- Let us define a new equivalence relation $\sim_s$ on $\mathcal{P}_\infty(A)$, called *stable relation*, as follows: if $p, q \in \mathcal{P}_\infty(A)$ then $p \sim_s q$ if and only if there exists a projection $r \in \mathcal{P}_\infty(A)$ such that $p \oplus r \sim_0 q \oplus r$. In this way, since A is unital by hypothesis, then $p \sim_s q$ if and only if $p \oplus \mathbf{1}_n \sim_0 1 \oplus \mathbf{1}_n$ for some $n \in \mathbb{N}$. Then $p \sim_s q$ if and only if $[p]_0 = [q]_0$.

Moreover, we have a sort of universal property for K_0.

Proposition 4.1.2. *Given a unital C^*-algebra A, an abelian group G and $\mu : \mathcal{P}_\infty(A) \to G$ such that (i) μ is additive, (ii) $\mu(0_A) = 0$, and (iii) $\mu(p) = \mu(q)$ for $p, q \in \mathcal{P}_n(A)$ which are connected by a continuous path, then there exists a unique group homomorphism $\alpha : K_0(A) \to G$ which makes the diagram*

$$\begin{array}{ccc} \mathcal{P}_\infty(A) & \xrightarrow{\mu} & G \\ \downarrow & \nearrow_{\alpha} & \\ K_0(A) & & \end{array},$$

commutative. Furthermore, K_0 is a functor from the category of unital C^*-algebras to the category of abelian groups.

We shall see later that, in fact, K_0 is a functor from the category of C^*-algebras to the category of Abelian groups. But first, let us state some properties for the unital case, which can be found in [27] §3.

Proposition 4.1.3. *For every unital C^*-algebra A, the split sequence:*

$$0 \longrightarrow A \xrightarrow{\ i\ } \hat{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0 \ ,$$

induces a split exact sequence in the K_0 groups:

$$0 \longrightarrow K_0(A) \xrightarrow{K_0(i)} K_0(\hat{A}) \underset{K_0(\lambda)}{\overset{K_O(\pi)}{\rightleftarrows}} K_0(\mathbb{C}) \longrightarrow 0 \ .$$

Remark. Husemoller [13] presents the construction of the K_0 ring as the Grothendieck extension for the semi ring $\mathrm{Vect}(X)$ when X is a compact Hausdorff space. In these spaces, the set $C(X)$ of continuous functions from X to $\mathbb{C}$ is a unital C^*-algebra with the supremum norm $||\cdot||_\infty$. Rørdam, Larsen and Laustsen [27] show that these two constructions, as abelian groups, are isomorphic: $K_0(X) \cong K_0(C(X))$.

We now extend this construction of the functor K_0 to arbitrary C^*-algebras, unital or not. Let A be a non-unital C^*-algebra, and consider the exact split sequence:

$$0 \longrightarrow A \xrightarrow{\ i\ } \hat{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0 \ .$$

Define $K_0(A)$ as the kernel of the group homomorphism $K_0(\pi) : K_0(\hat{A}) \to K_0(\mathbb{C})$, which makes sense according to the construction for unital algebras. Thus, $K_0(\hat{A})$ is an abelian group. Now, given $p \in \mathcal{P}_\infty(A)$, and considering the equivalence class $[p]_0$ in $K_0(\hat{A})$, note that $K_0(\pi)([p]_0) = [\pi(p)]_0 = [\pi \circ i(p)]_0 = 0$ since the sequence is exact. Hence, we have a morphism $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$.
For each C^*-algebra A, unital or not, we can consider the short exact sequence:

$$0 \longrightarrow K_0(A) \longrightarrow K_0(\hat{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0 \ ,$$

where the map $K_0(A) \to K_0(\hat{A})$ is $K_0(i)$ when A is unital, and it is the inclusion map of the subgroup $K_0(A)$ when A is non unital. By proposition 4.1.3, this sequence is exact in the unital case; however, this assertion fails when we consider non unital C^*-algebras. In the unital case $K_0(A)$ is then isomorphic to its image in $K_0(\hat{A})$ under $K_0(i)$, which maps $[p]_0$ in $K_0(A)$ to $[p]_0$ in $K_0(\hat{A})$. And by exactness, this image is equal to the kernel of $K_0(\pi)$. Therefore, the definition $K_0(A) = \ker\,(K_0(\pi))$ agrees with the previous construction for unital C^*-algebras.
This more general definition of K_0 also satisfies functorial properties, and thus we can understand K_0 as a functor from the category of C^*-algebras to the category of abelian groups. The exactness for this functor is summarized in the following proposition (see [27] §3)

Proposition 4.1.4. *Let* $0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$ *be an exact sequence of* C^**-algebras. This sequence induces an exact sequence of abelian groups:*

$$K_0(I) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B) \ .$$

If, in addition, the sequence splits, i.e if there exists a $*$*-homomorphism* $\lambda : B \to A$ *such that* $\lambda \circ \psi = \mathbf{1}_B$*, then there is an induced split exact sequence of abelian groups:*

$$0 \longrightarrow K_0(I) \xrightarrow{K_0(\phi)} K_0(A) \underset{\overset{\curvearrowleft}{K_0(\lambda)}}{\xrightarrow{K_O(\psi)}} K_0(B) \longrightarrow 0 \ .$$

Again, as we have done for the unital case, we can consider a more descriptive point of view, in terms of computation, for the K_0 group. We will call this description the *standard picture of the group* K_0. Considering the split exact sequence given by the unitization of the algebra A, let us define the *scalar mapping* $s = \lambda \circ \pi : \hat{A} \to \hat{A}$, $s(a + \alpha\mathbf{1}) = \alpha\mathbf{1}$, for all $a \in A$ and $\alpha \in \mathbb{C}$. Extending this morphism to a morphism $M_n(\hat{A}) \to M_n(\hat{A})$, acting on each matrix entry, we call an element $x \in M_n(\hat{A})$ a *scalar element* if $x = s(x)$. The mentioned standard picture of K_0 is then given by:

$$K_0(A) = \{[p]_0 - [s(p)]_0 \mid p \in \mathcal{P}_\infty(\hat{A})\}.$$

Furthermore, we have the next properties:

- Given $p, 1 \in \mathcal{P}_\infty(\hat{A})$, the following are equivalent:
  i. $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,
  ii. there exists $k, l \in \mathbb{N}$ such that $p \oplus \mathbf{1}_k \sim_0 q \oplus \mathbf{1}_l$ in $\mathcal{P}_\infty(\hat{A})$,
  iii. there exists scalar projections r_1 and r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$.
- If $p \in \mathcal{P}_\infty(\hat{A})$ satisfies $[p]_0 - [s(p)]_0 = 0$, then there exists a natural number m with $p \oplus \mathbf{1}_m \sim_0 s(p) \oplus \mathbf{1}_m$.
- Given a $*$-homomorphism $\phi : A \to B$ between C^* algebras, then

$$K_0(\phi)([p]_0 - [s(p)]_0) = [\hat{\phi}(p)]_0 - [s(\hat{\phi}(p))]_0.$$

Examples: (a) The group $K_0(M_n(\mathbb{C}))$ is isomorphic to $\mathbb{Z}$ for each positive integer n. The isomorphism is induced by the classical trace $\mathrm{Tr} : P_k(M_n(\mathbb{C})) \to \mathbb{Z}$. Thus, $K_0(\mathbb{C}) \cong \mathbb{Z}$.

(b) From (a) and the split exactness of K_0, we see for any C^* algebra A that

$$K_0(\hat{A}) \cong K_0(A) \oplus \mathbb{Z}.$$

(c) For any separable complex Hilbert H space with $\dim_{\mathbb{C}} H = \infty$, $K_0(\mathcal{B}(H)) = 0$.

(d) Let $\mathcal{K}$ denote the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space. Let A be a C^*-algebra, and let us construct a new C^*-algebra $\mathcal{K}A$, called the *stabilization* of A (see [27] §5 and §6). Consider the sequence of C^* algebras

$$A \xrightarrow{\varphi_1} M_2(A) \xrightarrow{\varphi_2} M_3(A) \xrightarrow{\varphi_3} \cdots ,$$

where $\varphi_n : M_n(A) \to M_{n+1}(A)$ is given by:

$$\varphi_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $(\mathcal{K}A, \{\kappa_n\})$ be the C^*-algebra resulting from the inductive limit of this sequence, with $\kappa_n : M_n(A) \to \mathcal{K}(A)$. The functor K_0 satisfies a property of stabilization which is given by the following statement:
Let $\kappa = \kappa_1 : A \to \mathcal{K}A$ be the canonical inclusion of the C^* algebra A into its stabilization $\mathcal{K}A$. Then $K_0(\kappa) : K_0(A) \to K_0(\mathcal{K}A)$ is an isomorphism.
It is possible to show that $\mathcal{K}\mathbb{C} \cong \mathcal{K}$. Thus, $K_0(K)$, the K_0 group of the C^* of compact operators, is isomorphic to $\mathbb{Z}$.

4.1.2 K_1 Group

Now, we shall define the K_1 group for any C^*-algebra A. This group is constructed in terms of the unitary elements in the unitization of the algebra. For this, we will follow [27] §8.
Let A be a unital C^*-algebra, and let $\mathcal{U}(A)$ denote its group of unitary elements. Set $\mathcal{U}_n(A) = \mathcal{U}(M_n(A))$ and $\mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$. Define a binary operation $\bigoplus$ on $\mathcal{U}(A)$ by:

$$u \oplus v = \text{diag } (u, v) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_{n+m}(A), \ \text{ with } u \in \mathcal{U}_n(A), v \in \mathcal{U}_m(A).$$

In addition, define a relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows: for $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, we say that $u \sim_1 u$ if and only if there exists a natural number $k \geq \max\{m, n\}$ such that $u \oplus \mathbf{1}_{k-n}$ and $v \oplus \mathbf{1}_{k-m}$ are connected by a continuous path in $\mathcal{U}_k(A)$ –here we use the convention $u \oplus \mathbf{1}_0 = u$. The binary operation $\oplus$ and the equivalence relation $\sim_1$ are compatible in the following sense:

i. $u \sim_1 u \oplus \mathbf{1}_n$ for all $u \in \mathcal{U}_\infty(A)$ and $n \in \mathbb{N}$.

ii. $u \oplus v \sim_1 v \oplus u$ for all $u, v \in \mathcal{U}_\infty(A)$.

iii. If $u, v, u', v' \in \mathcal{U}_\infty(A)$ with $u \sim_1 u'$ and $v \sim_1 v'$, then $u \oplus v \sim_1 u' \oplus v'$.

iv. If $u, v \in \mathcal{U}_n(A)$ for some n, then $uv \sim_1 vu \sim_1 u \oplus v$.

v. $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ for all $u, v, w \in \mathcal{U}_\infty(A)$.

Definition 4.1.5. ***The*** K_1***-Group for*** C^****-algebras*** *Let* A *be a* C^**-algebra. Define*

$$K_1(A) := \mathcal{U}_\infty(\hat{A})/\sim_1 .$$

Let $[u]_1 \in K_1(A)$ *denote the equivalence class containing* $u \in \mathcal{U}_\infty(\hat{A})$*. Define the binary operation* $+ : K_1(A) \times K_1(A) \to K_1(A)$ *as* $[u]_1 + [v]_1 = [u \oplus v]_1$*. By the properties above, this operation is well defined, commutative, associative and has a zero element* $[1]_1$*. Moreover*

$$0 = [1]_1 = [\mathbf{1}_k]_n = [uu^*]_1 = [u]_1 + [u^*]_1,$$

where $u \in \mathcal{U}_k(\hat{A})$*. Thus,* $(K_1(A), +)$ *is an abelian group.*

Again, as we have seen with the K_0 group, given a C^*-algebra A, the $K_1(A)$ groups satisfies a universal property.

Proposition 4.1.6. *Let* G *be an abelian group, and let* $\nu : \mathcal{U}_\infty(\hat{A}) \to G$ *be a map such that (i)*$\nu(u \oplus v) = \nu(u) + \nu(v)$*, (ii)* $\nu(\mathbf{1}) = 0$ *and (iii)* $\nu(u) = \nu(v)$ *if* $u, v \in \mathcal{U}_n(\hat{A})$ *for some* n *and there exists a continuous curve* γ *connecting them. Then there exists a unique group homomorphism* $\alpha : K_1(A) \to G$ *which makes the diagram commutative:*

$$\begin{array}{ccc} \mathcal{U}_n(\hat{A}) & \xrightarrow{\nu} & G \\ \downarrow & \nearrow_{\alpha} & \\ K_1(A) & & \end{array} .$$

Consider the case when A is unital. Define $\mu : \mathcal{U}_\infty(\hat{A}) \to \mathcal{U}_\infty(A)$ as follows. Since A is unital, $\hat{A} \cong A \bigoplus f\mathbb{C}$, where $f = \mathbf{1}_A - \mathbf{1}_{\hat{A}}$, thus $\mu(a + \alpha f) := a$. This morphism is a unital $*$-homomorphism, and we can extend it to a unital $*$-homomorphism $M_n(\hat{A}) \to M_n(A)$, and then obtain μ as a map between the mentioned spaces. We have the following proposition (see [27] §8).

Proposition 4.1.7. *Let* A *be a unital* C^**-algebra. Then there exists an isomorphism* $\rho : K_1(A) \to \mathcal{U}_\infty(A)/\sim_1$ *such that the diagram commutes.*

$$\begin{array}{ccc} \mathcal{U}_\infty(\hat{A}) & \xrightarrow{\mu} & \mathcal{U}_\infty(A) \\ \downarrow{\scriptstyle [\cdot]_1} & & \downarrow \\ K_1(A) & \xrightarrow{\rho} & \mathcal{U}_\infty(A)/\sim_1 \end{array}$$

Thus, when A is unital, we identify $K_1(A)$ with $\mathcal{U}_\infty(A)/\sim_1$. From this proposition, it is also immediate that $K_1(\hat{A}) \cong K_1(A)$ for any C^*-algebra A.

As we expect in analogy to the functor K_0, K_1 has functorial properties, being a functor from the category of C^*-algebras to the category of abelian groups (see [27] §8).

Proposition 4.1.8. *Let* $0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$ *be a short exact sequence of* C^**-algebras. Then the sequence*

$$K_1(I) \xrightarrow{K_1(\phi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B)$$

is exact. If, in addition, the sequence splits, i.e if there exists a $*$*-homomorphism* $\lambda : B \to A$ *such that* $\lambda \circ \psi = \mathbf{1}_B$*, then there is an induced split exact sequence of abelian groups:*

$$0 \longrightarrow K_1(I) \xrightarrow{K_1(\phi)} K_1(A) \underset{K_1(\lambda)}{\overset{K_1(\psi)}{\rightleftarrows}} K_1(B) \longrightarrow 0\ .$$

Examples: (a) For any Hilbert space H, $K_1(\mathcal{B}(H)) = 0$. Let H be a finite dimensional Hilbert space. We have that the unitary group of $M_k(M_n(\mathbb{C})) = M_{kn}(\mathbb{C})$ is path connected for any n and k. Thus, $\mathcal{U}_\infty(M_n(\mathbb{C}))/\sim_1$ is the trivial group for all $n \in \mathbb{N}$. For the infinite dimensional case, it is possible to show, using the measurable functional calculus, that $u \sim_1 \mathbf{1}_n$ for every unitary $u \in \mathbb{U}_n(\mathcal{B}(H))$. With this, the assertion follows.

(b) The functor K_1 satisfies a *stability property.* Let A be a C^*-algebra and $n \in \mathbb{N}$. Define the map $\lambda_{n,A} : A \to M_n(A)$ as:

$$\lambda_{n,A}(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\kappa : A \to \mathcal{K}A$ the inclusion map of A into its stabilization $\mathcal{K}A$. Then the group homomorphisms $K_1(\lambda_{n,A}) : K_1(A) \to K_1(M_n(A))$ and $K_1(\kappa) : K_1(A) \to K_1(\mathcal{K}A)$ are isomorphisms. Therefore, being $\mathcal{K}$ the C^*-algebra of all compact operators, we conclude that $K_1(\mathcal{K}) = 0$.

4.1.3 Index Map and Bott Periodicity

In this subsection we introduce the *index map* associated to a short exact sequence

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

of C^*-algebras. The index map is a group homomorphism $\delta_1 : K_1(B) \to K_0(I)$ that gives rise to an exact sequence:

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(\phi)} & K_1(A) & \xrightarrow{K_1(\psi)} & K_1(B) \\ & & & & \downarrow{\scriptstyle \delta_1} \\ K_0(B) & \xleftarrow[K_0(\psi)]{} & K_0(A) & \xleftarrow[K_0(\phi)]{} & K_0(I) \end{array}.$$

The construction and definition of the index map is based on the following lemma (see [27] §9).

Lemma 4.1.9. *Let* $0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$ *be a short exact sequence of* C^*-*algebras, and let* $u \in \mathcal{U}_n(\hat{B})$ *be given. Then:*

i. There exists a unitary $v \in \mathcal{U}_{2n}(\hat{A})$ *and a projection* $p \in \mathcal{P}_{2n}(\hat{I})$ *such that:*

$$\hat{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \hat{\phi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad s(p) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}.$$

ii. If v *and* p *are as above, and if* $w \in \mathcal{U}_{2n}(\hat{A})$ *and* $q \in \mathcal{P}_{2n}(\hat{I})$ *satisfy:*

$$\hat{\psi}(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \hat{\phi}(q) = w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^*,$$

then $s(q) = diag\,(\mathbf{1}_n, 0_n)$ *and* $p \sim_u q$ *in* $\mathcal{P}_{2n}(\hat{I})$, *where* $\sim_u$ *denotes the equivalence relation on* $\mathcal{P}_{2n}(\hat{I})$ *given by:* $p \sim_u q$ *if and only if there exists a unitary element* $u \in \mathcal{U}_{2n}(\hat{I})$ *with* $q = upu^*$.

Additionally, define $\nu : \mathcal{U}_\infty(\hat{B}) \to K_0(I)$ *by* $\nu([u]_1) = [p]_0 - [s(p)]_0$ *where* $p \in \mathcal{P}_{2n}(\hat{I})$ *as above. This map* ν *satisfies the following properties:*

i. $\nu(u_1 \oplus u_2) = \nu(u_1) + \nu(u_2)$ *for all* $u_1, u_2 \in \mathcal{U}_\infty(\hat{B})$,

ii. $\nu(\mathbf{1}) = 0$,

iii. if u_1, u_2 *belong to* $\mathcal{U}_n(\hat{B})$ *and they are connected by a continuous path on* $\mathcal{U}_n(\hat{B})$, *then* $\nu(u_1) = \nu(u_2)$,

iv. $\nu(\hat{\psi}(u)) = 0$ *for every* $u \in \mathcal{U}_\infty(\hat{A})$,

v. $K_0(\phi)(\nu(u)) = 0$ *for all* $u \in \mathcal{U}_\infty(\hat{B})$.

Definition 4.1.10. *Suppose we are given the short exact sequence of above. Let* $\nu : \mathcal{U}_\infty(\hat{B}) \to K_0(I)$ *be the map from the previous lemma. By the universal property of* K_1, *there exists a unique group homomorphism* $\delta_1 : K_1(B) \to K_0(I)$ *such that* $\delta_1([u]_1) = [p]_0 - [s(p)]_0$, *where* $p \in \mathcal{P}_{2n}(\hat{I})$ *as above for a given* $u \in \mathcal{U}_n(\hat{B})$. *The map* δ_1 *is called the index map associated to the short exact sequence of* C^*-*algebras* $0 \to I \xrightarrow{\phi} A \xrightarrow{\psi} B \to 0$. *This map satisfies (i)* $\delta_1 \circ K_1(\psi) = 0$ *and (ii)* $K_0(\phi) \circ \delta_1 = 0$.

Although the previous definition gives rise an explicit form of how the index map is applied, there is an alternative way which makes use of partial isometries. Given a C^*-algebra A, we say that an element $v \in A$ is a partial isometry if v^*v is a projection. If this occurs, it is possible to show that vv^* is also a projection. The next construction is based on this elements (see [27] §2 and §9).

Let $0 \to I \xrightarrow{\phi} A \xrightarrow{\psi} B \to 0$ be a short exact sequence of C^*-algebras and let $m \leq n$ natural numbers. Given a unitary element $u \in \mathcal{U}_n(\hat{B})$, there exists a partial isometry $v \in M_m(\hat{A})$ such that

$$\hat{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & 0_{m-n} \end{pmatrix}.$$

Then it is possible to obtain projections $p, q \in \mathcal{P}_m(\hat{I})$ with $\mathbf{1}_m - v^*v = \hat{\phi}(p)$ and $\mathbf{1}_m - vv^* = \hat{\phi}(q)$, and the index map will be given by:

$$\delta_1([u]_1) = [p]_0 - [q]_0.$$

When A is unital, this morphism has an even easier description. In this case, since ψ is surjective, B is unital too, and ψ is unit preserving. Define the $*$ homomorphism $\bar{\phi} : \hat{I} \to A$ by $\bar{\phi}(x + \alpha \mathbf{1}_{\hat{I}}) = \phi(x) + \alpha \mathbf{1}_A$, and let u be a unitary element in B. If v is a unitary element in $M_{2n}(A)$ and p is a projection in $M_{2n}(\hat{I})$ such that

$$\bar{\phi}(p) = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad \psi(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix},$$

then $\delta_1([u]_1) = [p]_0 - [s(p)]_0$.

Proposition 4.1.11. *(see [27] §9) Consider a short exact sequence of C^*-algebras as above. We have the following properties:*

i. The kernel of the index map $\delta_1 : K_1(B) \to K_0(I)$ is contained in the image of the map $K_1(\psi) : K_1(A) \to K_1(B)$.

ii. The kernel of the map $K_0(\phi) : K_0(I) \to K_0(A)$ is contained in the image of the index map $\delta_1 : K_1(B) \to K_0(I)$.

Thus, we have that the following sequence of K-groups is exact:

$$\begin{array}{ccccc} K_1(I) & \xrightarrow{K_1(\phi)} & K_1(A) & \xrightarrow{K_1(\psi)} & K_1(B) \\ & & & & \downarrow \delta_1 \\ K_0(B) & \xleftarrow[K_0(\psi)]{} & K_0(A) & \xleftarrow[K_0(\phi)]{} & K_0(I) \end{array}.$$

Example: There is an analogy between the index map and the Fredholm index for Fredholm operators. Let H be an infinite dimensional separable Hilbert space; let $\mathcal{K}$ denote the C^*-algebra of compact operators on H, and let $\mathcal{Q}(H) := \mathcal{B}(H)/\mathcal{K}$ be the so called Calkin algebra of H. Let us denote by π the quotient map $\mathcal{B}(H) \to \mathcal{Q}(H)$. Consider the short exact sequence:

$$0 \longrightarrow \mathcal{K} \xhookrightarrow{i} \mathcal{B}(H) \xrightarrow{\pi} \mathcal{Q}(H) \longrightarrow 0 .$$

We have noticed that $K_0(\mathcal{K}) \cong \mathbb{Z}$. This isomorphism is given by the map $K_0(\mathrm{Tr}) : K_0(\mathcal{K}) \to \mathbb{Z}$ with $K_0(\mathrm{Tr})([E]_0) = \mathrm{Tr}\ (E) = \dim\ (E(H))$ –it is well defined since compact projections

must have finite dimensional range by Atkinson's theorem.
The aforementioned relation between these two *indexes* is given by the equation:

$$\text{Index }(T) = (K_0(\text{Tr}) \circ \delta_1)([\pi(T)]_1),$$

where T is a Fredholm operator. The Atkinson's theorem allows us to see $\pi(T)$ as a unitary element in the Calkin algebra. Now, we can consider the six term exact sequence:

$$\begin{array}{ccccc} K_1(\mathcal{K}) & \xrightarrow{K_1(i)} & K_1(\mathcal{B}(H)) & \xrightarrow{K_1(\pi)} & K_1(\mathcal{Q}(H)) \\ & & & & \big\downarrow \delta_1 \\ K_0(\mathcal{Q}(H)) & \xleftarrow[K_0(\pi)]{} & K_0(\mathcal{B}(H)) & \xleftarrow[K_0(i)]{} & K_0(\mathcal{K}) \end{array},$$

and since $K_i(\mathcal{B}(H)) = 0$ in the infinite dimensional case, δ_1 is, in fact, an isomorphism. Thus, $K_1(\mathcal{Q}(H)) \cong \mathbb{Z}$.

We will now introduce two new constructions over C^*-algebras, the *cone* and the *suspension*. Let A be a C^* algebra. The cone CA and the suspension SA of A are defined as follows:

$$CA = \{f \in C([0,1], A) : \ f(0) = 0\}, \quad SA = \{f \in C([0,1], A) : \ f(0) = 0, f(1) = 0\}.$$

We can consider the short exact sequence

$$0 \longrightarrow SA \xrightarrow{\ i\ } CA \xrightarrow{\ \pi\ } A \longrightarrow 0\ ,$$

where i is the inclusion map and $\pi(f) = f(1)$ - notice that π is surjective since A is path connected.
In the category of C^*-algebras, we have the notion of homotopy equivalence. Given two C^*-algebras A, B and two $*$-homomorphisms $\phi, \psi : A \to B$, we say that the maps are homotopic, $\phi \sim_h \psi$, if there exists a path of $*$-homomorphisms $\phi_t : A \to B$, $t \in [0,1]$, such that $t \mapsto \phi_t(a)$ is a continuous path from $[0,1]$ to B for each $a \in A$, with $\phi_0 = \phi$ and $\phi_1 = \psi$. The path $t \mapsto \phi_t$ is point-wise continuous. We say that A and B are *homotopy equivalent* if there exists morphisms $\phi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \phi \sim_h \mathbf{1}_A$ and $\phi \circ \psi \sim_h \mathbf{1}_B$. The functors K_0 and K_1 are homotopy invariant in the sense that given $\phi, \psi : A \to B$ with $\phi \sim_h \psi$, then $K_i(\phi) = K_i(\psi) : K_i(A) \to K_i(B)$ (see [27] §4 and §8).
Now, note that CA is homotopy equivalent to the algebra $\{0\}$. Consider $\phi_t : CA \to CA$ with $\phi_t(f)(s) = f(st)$ for all $f \in CA$ and $s, t \in [0,1]$. Fixing $f \in CA$, the map $t \mapsto \phi_t(f)$ is continuous, $\phi_0 = 0$ and $\phi_1 = \mathbf{1}_{CA}$. Therefore, $K_i(C(A)) = 0$ for $i = 0, 1$.
We can understand S as a functor from C^*-algebras to C^*-algebras, associating the C^*-algebra SA to each A, and for $\phi : A \to B$, a $*$-homomorphism, the map $S\phi : SA \to SB$ given by $(S\phi(f))(t) = \phi(f(t))$. This functor is exact and, moreover, given any C^*-algebra A, S satisfies the following property:

$$K_1(A) \cong K_0(SA).$$

This isomorphism is constructed as follows. Let $u \in \mathcal{U}_n(\hat{A})$ with $s(u) = \mathbf{1}_n$ and let $v \in C([0,1], \mathcal{U}_n(\hat{A}))$ such that $v(0) = \mathbf{1}_{2n}$, $v(1) = \text{diag}\,(u, u^*)$, and $s(v(t)) = \mathbf{1}_{2n}$ for all $t \in [0,1]$ (it is possible to get u and v which fulfill these properties). Set

$$p = v \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} v^*,$$

which is an element in $\mathcal{P}_{2n}(\widehat{SA})$ with $s(p) = \text{diag}\,(\mathbf{1}_n, 0_n)$. Then, let $\theta_A : K_1(A) \to K_0(SA)$ given by $\theta_A([u]_1) = [p]_0 - [s(p)]_0$.
Thus, thanks to this property, we can define, inductively, the higher K functors K_n for $n \geq 2$ as $K_n = K_{n-1} \circ S$. This functor is a half exact functor from the category of C^*-algebras to the category of abelian groups (see [27] §10). With these functors, we can construct a long exact sequence of K-groups from a short exact sequence $0 \to I \xrightarrow{\phi} A \xrightarrow{\psi} B \to 0$ of C^*-algebras. First, let us observe the higher index maps arising from this short exact sequence. By exactness of S, the sequence

$$0 \to S^n I \xrightarrow{S^n\phi} S^n A \xrightarrow{S^n\psi} S^n A \to 0$$

is exact, and by the previous statement, we have the isomorphism

$$\theta_{S^{n-1}I} : K_n(I) = K_1(S^{n-1}I) \to K_0(S^n I).$$

Denoting by $\bar{\delta}$ the index map associated with this short exact sequence, we note that there exists only one group homomorphism δ_{n+1} which makes the next diagram commutative:

$$\begin{array}{ccc} K_{n+1}(B) & \xrightarrow{\delta_{n+1}} & K_n(I) \\ \Big\| & & \Big\downarrow \theta_{S^{n-1}I} \\ K_1(S^n B) & \xrightarrow{\bar{\delta}} & K_0(S^n I) \end{array} .$$

Proposition 4.1.12. *Every short exact sequence of C^*-algebras*

$$0 \longrightarrow I \xrightarrow{\phi} A \xrightarrow{\psi} B \longrightarrow 0$$

induces a long exact sequence of K-groups:

$$\cdots \xrightarrow{K_{n+1}(\psi)} K_{n+1}(B) \xrightarrow{\delta_{n+1}} K_n(I) \xrightarrow{K_n(\phi)} K_n(A) \xrightarrow{K_n(\psi)} K_n(B) \xrightarrow{\delta_n} K_{n-1}(I) \xrightarrow{K_{n-1}(\phi)} \cdots$$

$$\cdots \xrightarrow{\delta_1} K_0(I) \xrightarrow{K_0(\phi)} K_0(A) \xrightarrow{K_0(\psi)} K_0(B),$$

where δ_1 is the index map and δ_n, for $n \geq 2$, its higher analogues.

We will now state one of the main results associated with K-theory, which, in essence, tell us that we should only worry about the K_0 and K_1 groups. This theorem was proved by Rauol Bott in his paper [2]. A more detailed proof can be found in [27] §11.

Theorem 4.1.13. ***Bott periodicity*** *There exists an isomorphism* $\beta_A : K_0(A) \to K_1(SA)$ *for every* C^**-algebra* A*. This isomorphism is called the Bott map.*

Corollary 4.1.14. *For every* C^**-algebra* A *and every integer* $n \geq 0$*,*

$$K_{n+2}(A) \cong K_n(A).$$

Examples: (a) Note that the suspension SA for any C^*-algebra is isomorphic to $C_0(\mathbb{R}, A)$, since $\mathbb{R}$ is homeomorphic to the open interval $(0,1)$ and $SA = C_0((0,1), A)$ –using the one point compactification, $SA = C(\mathbb{S}^1, A)$. In addition, since $C_0(X, C_0(Y))$ is isomorphic to $C_0(X \times Y)$ for any locally compact Hausdorff spaces X and Y, note that $K_n(\mathbb{C}) \cong K_0(C_0(\mathbb{R}^n))$ and $K_1(\mathbb{C}) \cong K_1(C_0(\mathbb{R}^n))$. Thus, according to the previous corollary, we conclude that:

$$K_0(C_0(\mathbb{R}^n)) \cong \begin{cases} K_0(\mathbb{C}) \cong \mathbb{Z} & \text{if } n \text{ even} \\ K_1(\mathbb{C}) = \{0\} & \text{if } n \text{ odd} \end{cases}, \quad K_1(C_0(\mathbb{R}^n)) \cong \begin{cases} 0 & \text{if } n \text{ even} \\ \mathbb{Z} & \text{if } n \text{ odd} \end{cases}.$$

(b) Let $\mathbb{S}^n$ denote the n-sphere. The one point compactification of $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n$, and it is possible to show that the unitization $\widehat{C_0(\mathbb{R}^n)}$ is isomorphic to $C(\mathbb{S}^n)$. Thus, using (a) and previous results for the K_0 functor, we have that:

$$K_0(C(\mathbb{S}^n)) \cong K_0(C_0(\mathbb{R}^n)) \bigoplus \mathbb{Z} \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \text{ even} \\ \mathbb{Z} & \text{if } n \text{ odd} \end{cases},$$

$$K_1(C(\mathbb{S}^n)) \cong K_1(C_0(\mathbb{R}^n)) \cong \begin{cases} 0 & \text{if } n \text{ even} \\ \mathbb{Z} & \text{if } n \text{ odd} \end{cases}.$$

(c) **The Toeplitz algebra:** Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for a separable infinite dimensional Hilbert space H. Let $S \in \mathcal{B}(H)$ denote the unilateral shift operator, $Se_n = e_{n+1}$. This operator is an isometry, $S^*S = \mathbf{1}_H$, where $S^*e_n = e_{n-1}$ with $e_0 = 0$. Define the Toeplitz algebra $\mathcal{T}$ as the sub-C^*-algebra generated by S, i.e $\mathcal{T} := C^*(S)$.
For $i, j \in \mathbb{N}$, let E_{ij} be the bounded rank one operator on H given by $E_{ij}(v) = \langle v, e_j \rangle e_i$, and let $F_n := \sum_{n=1}^{n} E_{ii}$. Thus, F_n is the projection onto the subspace H_n spanned by $\{e_1 \ldots, e_n\}$. Observe that $\mathcal{B}(H_n) = \text{span}\,\{E_{ij} \mid 1 \leq i, j \leq n\} = F_n\mathcal{B}(H)F_n$. Furthermore, the following hold:

$$F_1 = E_{11} = \mathbf{1} - SS^*, \;\; E_{ij} = S^{i-1}F_1(S^*)^{j-1}, \text{ for } i, j \in \mathbb{N}.$$

Thus, each E_{ij} belongs to $\mathcal{T}$. On the other hand, the C^*-algebra of compact operators $\mathcal{K}$ on H is the closure of the ideal of finite rank operators, i.e it is the closure of the union $\bigcup_{n=1}^{\infty} F_n\mathcal{B}(H)F_n$ (see [26] §7); therefore, $\mathcal{K}$ is contained in $\mathcal{T}$ and, moreover, it is an ideal in $\mathcal{T}$. Consider the quotient mapping $\pi : \mathcal{K} \to \mathcal{Q}(H)$; since $F_1 = \mathbf{1} - SS^*$, and F_1 is compact, we have that $\pi(S)$ is, in fact, a unitary element in $\mathcal{Q}(H)$. Moreover, by the Atkinson's theorem, S is a Fredholm operator because $\pi(S)$ is invertible in the Calkin algebra. By definition, ker $S = \{0\}$ and ker $S^* =$ Span $\{e_1\}$, so its Fredholm index is equal to -1, and by the relation between the index map on the K-groups and the Fredholm index studied previously, we must have that $[\pi(S)]_1$ is a non-zero element in $K_1(\mathcal{Q}(H))$. Thus, $\pi(S)$ can not be connected to the unit in $\mathcal{Q}(H)$. When this happens for a unitary element u in a unital C^*-algebra, its spectrum is equal to $\mathbb{S}^1$ (see [27] §2). In conclusion, $\sigma_{\mathcal{Q}(H)}(\pi(S)) = \mathbb{S}^1$. Therefore, by the continuous functional calculus, $C^*(\pi(S)) \cong C(\mathbb{S}^1)$.

Note also that $\mathcal{T}/\mathcal{K} = C^*(\pi(S))$. Thus, we can consider the short exact sequence:

$$0 \longrightarrow \mathcal{K} \xrightarrow{\ i\ } \mathcal{T} \xrightarrow{\ \psi\ } C^*(\pi(S)) \longrightarrow 0,$$

where ψ is the composition of π with the isomorphism from $C^*(\pi(S))$ to $C(\mathbb{S}^1)$, which maps $\pi(S)$ to the identity function in $C(\mathbb{S}^1)$. Now, notice that $[F_1]_0$ generates the group $K_0(\mathcal{K})$ because $K_0(\mathrm{Tr})([F_1]_0) = 1$. Likewise, according to the definition of the index map, and the fact that one of the morphisms in the short exact sequence is the inclusion map, we see that:

$$\delta_1([\psi(S)]_1) = [\mathbf{1} - S^*S]_0 - [\mathbf{1} - SS^*]_0 = -[F_1]_0.$$

We know the K-groups for $\mathcal{K}$ and $C^*(\pi(S))$. In order to obtain the K-groups for the Toeplitz algebra, we will need an additional map which will allow us to understand, in a better way, the six term exact sequence which involves the index map. We shall come back to this example in the next subsection.

4.1.4 The Exponential Map

Consider the short exact sequence of C^*-algebras

$$0 \to I \overset{\phi}{\to} A \overset{\psi}{\to} B \to 0,$$

and define the morphism $\delta_0 : K_0(B) \to K_1(I)$ as the composition of the maps

$$K_0(B) \xrightarrow{\ \beta_B\ } K_2(B) \xrightarrow{\ \delta_2\ } K_1(I),$$

where β_B denotes the Bott map and δ_2 the respective higher index map.
If $\bar{\delta_1}$ denotes the index map associated with the short exact sequence

$$0 \to SI \xrightarrow{S(\phi)} SA \xrightarrow{S(\psi)} SB \to 0,$$

then δ_0 is defined as the unique group homomorphism which makes the following diagram commutative:

$$\begin{array}{ccc} K_0(B) & \xrightarrow{\ \delta_0\ } & K_1(I) \\ \downarrow{\scriptstyle \beta_B} & & \downarrow{\scriptstyle \theta_I} \\ K_1(SB) & \xrightarrow{\ \bar{\delta_1}\ } & K_0(SI) \end{array} .$$

This homomorphism δ_0 is called exponential map. With this map, we can obtain a six term exact sequence between K_0 and K_1 groups for a short exact sequence of C^*-algebras (see [27] §12).

Proposition 4.1.15. *For every short exact sequence of C^*-algebras as above, the associated six term sequence:*

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0(\phi)} & K_0(A) & \xrightarrow{K_0(\psi)} & K_0(B) \\ {\scriptstyle\delta_1}\big\uparrow & & & & \big\downarrow{\scriptstyle\delta_0} \\ K_1(B) & \xleftarrow[K_1(\psi)]{} & K_1(A) & \xleftarrow[K_1(\phi)]{} & K_1(I) \end{array}$$

is exact.

Example: Let us return to the last example of the previous subsection, the Toeplitz algebra. With this exact sequence, we can compute the K-groups of $\mathcal{T}$. This sequence is given by:

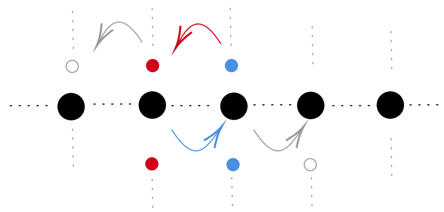
$$\begin{array}{ccccc} K_0(\mathcal{K}) & \xrightarrow{K_0(i)} & K_0(\mathcal{T}) & \xrightarrow{K_0(\psi)} & K_0(C^*(\pi(S))) \\ {\scriptstyle\delta_1}\big\uparrow & & & & \big\downarrow{\scriptstyle\delta_0} \\ K_1(C^*(\pi(S))) & \xleftarrow[K_1(\psi)]{} & K_1(\mathcal{T}) & \xleftarrow[K_1(i)]{} & K_1(\mathcal{K}) \end{array} .$$

Since $K_1(\mathcal{K}) = 0$, the exponential map is the zero map, and also $K_1(i)$. Thus, $K_0(\psi)$ is surjective and $K_1(\psi)$ is injective. We have observed that $[F_1]_0$ generates $K_0(\mathcal{K})$, so in order to compute $K_0(i)$, it is enough to calculate $K_0(i)([F_1]_0)$. In $\mathcal{T}$, we have that $\mathbf{1}-SS^* = F_1$ and $S^*S = \mathbf{1}$. Thus, according to the equivalence relation $\sim_0$ used in the construction of the K_0 group, we have that $[\mathbf{1}-F_1]_0 = [\mathbf{1}]_0$, and $[i(F_1)]_0 = [\mathbf{1}]_0-[\mathbf{1}-F_1]_0 = 0$. Hence, $K_0(i) = 0$, and, by exactness, this implies that $K_0(\psi)$ is injective; therefore, $K_0(\mathcal{T}) \cong K_0(C^*(\pi(S))) \cong \mathbb{Z}$. In addition, since $K_0(i) = 0$, the index map δ_1 is surjective, and since it is a map from $\mathbb{Z}$ to $\mathbb{Z}$, it must be an isomorphism, in particular injective. We conclude then that $K_1(\psi)$ is the zero map, so $K_1(\mathcal{T}) = 0$.

4.2 Bulk-Edge Correspondence

In this section we will introduce a physical example which will be the main application of the theory developed in the previous chapters. This example corresponds to the Su-Schriefer-Heeger model –SSH model– of a conducting polymer with a non-trivial topology where it will arise a winding number associated with a physical invariant in the model. We will be following Prodan and Schulz-Baldes [22] §1 throughout this section. For the study of these invariants, they use an algebraic approach passing through the K-theory formalism in order to connect what they called topological invariants with Chern characteristic classes. We will see that, although the analysis carried out by these authors in this model uses different tools than those we have developed in the previous chapters, there are common ingredients that allow us to conjecture the relevance of our analysis to describe this type of physical systems.

For the physical description of this fermionic model, consider an infinite chain labeled by an integer. On each "box", there will be N different non-interacting species of fermions, and each specie can be of two different *colors*: blue (B) or red (R). Moreover, all species have the same mass m, and there will be an additional potential interacting between colors. Additionally, there will be interactions between neighbors. A blue particle will be annihilated when it interacts with the right neighbor –if it moves to the right, it is annihilated– and it changes its color to red if it interacts with the left position. In a similar way, a red particle is annihilated when it interacts with the left neighbor, and it changes its color with right interactions.



We can then model this physical system using, as model space, the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z})$; and the physical dynamics of this chain can be described by the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes \mathbf{1}_N \otimes S + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes \mathbf{1}_N \otimes S^* + m\sigma_2 \otimes \mathbf{1}_N \otimes \mathbf{1},$$

where $\mathbf{1}_N$ and $\mathbf{1}$ are the identity operators on $\mathbb{C}^N$ and $\ell^2(\mathbb{Z})$ respectively, σ_i correspond to the Pauli matrices, and S the right shift on $\ell^2(\mathbb{Z})$. Using the definition of the Pauli matrices, it can be shown that:

$$\frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which, in fact, describe the respective annihilation and color changes.

Before discussing the development presented in [22] §1, let us observe a brief description of the discrete Fourier transform, which will help us to understand the description of this physical model from a geometric point of view and, in addition, it will be useful in the diagonalization problem. The discrete Fourier transform

$$\mathcal{F} : L^2(\mathbb{S}^1) \to \ell^2(\mathbb{Z})$$

is defined as

$$g \mapsto (g_n)_{n\in\mathbb{Z}}, \text{ with } g_n := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} g(x)e^{-inx}dx.$$

This transformation is, in fact, a linear isometric isomorphism between these two Hilbert spaces, with inverse given, in an analogous way, by:

$$(\mathcal{F}^{-1}((\phi)_{n\in\mathbb{Z}}))(k) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \phi_n e^{-ink}.$$

Let $g \in L^2(\mathbb{S}^1)$ with coefficients g_n such that $g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} g_n e^{-inx}$, and let us consider the linear operator $\mathcal{F}^{-1}S\mathcal{F}$. Applying this operator on g, we obtain that:

$$(\mathcal{F}^{-1}S\mathcal{F}(g))(k) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} g_{n-1}e^{-ink}.$$

Thus, omitting the normalization factors, we observe that $(\mathcal{F}^{-1}S\mathcal{F}(g))_n$ is, precisely, the $n-1$ component of g, i.e $= g_{n-1}$. Analogously, $\mathcal{F}^{-1}S^*\mathcal{F}$ gives the opposite shift, i.e g_n is mapped to g_{n+1}. In this order, applying $\mathcal{F}^{-1}$ and $\mathcal{F}$ to the hamiltonian $\mathcal{H}$ which describes the dynamics of the particles in the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z})$, we obtain a new operator $\int_{\mathbb{S}^1}^{\oplus} \tilde{\mathcal{H}}_k \, dk$, where $\tilde{\mathcal{H}}_k$ is given by:

$$\tilde{\mathcal{H}}_k = \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes \mathbf{1}_N e^{-ik} + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes \mathbf{1}_N e^{ik} + m\sigma_2 \otimes \mathbf{1}_N,$$

understanding its action on the k-th component of $g \in L^2(\mathbb{S}^1)$, as follows:

$$g_k \mapsto \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes \mathbf{1}_N g_{k-1} + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes \mathbf{1}_N g_{k+1} + m\sigma_2 \otimes \mathbf{1}_N g_k.$$

With this description, we can write each $\tilde{\mathcal{H}}_k$ as:

$$\mathcal{H}_k = \begin{pmatrix} 0 & e^{-ik} - im \\ e^{ik} + im & 0 \end{pmatrix} \otimes \mathbf{1}_N.$$

Smooth geometric description of the infinite chain. With these transformations and relations, we can model a geometrical description of the system in terms of smooth bundles over the circle. Using the Fourier transformation, we have obtained the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z}) & \xrightarrow{\mathcal{H}} & \mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z}) \\ \Big\downarrow {\scriptstyle \mathbf{1}\otimes\mathbf{1}\otimes\mathcal{F}^{-1}} & & {\scriptstyle \mathbf{1}\otimes\mathbf{1}\otimes\mathcal{F}} \Big\uparrow \\ \mathbb{C}^2 \otimes \mathbb{C}^N \otimes L^2(\mathbb{S}^1) & \xrightarrow{\tilde{\mathcal{H}}} & \mathbb{C}^2 \otimes \mathbb{C}^N \otimes L^2(\mathbb{S}^1). \end{array}$$

Notice that, according to the matrix description for $\mathcal{H}_k$, we can consider the trivial vector bundle:

$$\begin{array}{c} \mathbb{S}^1 \times (\mathbb{C}^2 \otimes \mathbb{C}^N) \\ \Big\downarrow {\scriptstyle \pi} \\ \mathbb{S}^1 \end{array}$$

and thus, $\mathcal{H}$ can be understood as a vector bundle morphism –to be more precise, as a morphism on the total space, where in each $k \in \mathbb{S}^1$ the linear operator corresponds to $\tilde{\mathcal{H}}_k$. To see this assertion, notice the following:

Let $\psi \in \Gamma(E)$, with $E = \mathbb{S}^1 \times (\mathbb{C}^2 \otimes \mathbb{C}^N)$. ψ can be viewed as an element of $L^2(\mathbb{S}^1, \mathbb{C}^2 \otimes \mathbb{C}^N)$ and, thus, $\tilde{\mathcal{H}}$ can be understood as a map from $\Gamma(E)$ to itself, i.e $\tilde{\mathcal{H}} : \Gamma(E) \to \Gamma(E)$. Given $\psi \in \Gamma(E)$ and $k \in \mathbb{S}^1$ (with $\mathbb{S}^1 = [0, 2\pi]/\sim$), we have

$$\tilde{\mathcal{H}}(\psi)(k) = \tilde{\mathcal{H}}_k \psi(k).$$

From now on, we will write only $\mathcal{H}$ to refer to the Hamiltonian.

According to the matrix form for $\mathcal{H}_k$, we can find the respective point spectrum diagonalizing the matrix. It is a straightforward computation to see that these eigenvalues, for each $k \in \mathbb{S}^1$, are given by:

$$E_{\pm}(k) = \pm\sqrt{m^2 + 1 + 2m \sin k},$$

which are N-fold degenerate. Note that there exists a gap around zero which corresponds to $\Delta = [-E_g, E_g]$ where $E_g = |\ |m| - 1\ |$, and is open as long as $m \notin \{-1, 1\}$.

Furthermore, we can extract important information from the algebraic form of the Hamiltonian $\mathcal{H}$. Let us consider the next commutative diagram:

$$\begin{array}{ccc} \mathbb{S}^1 \times (\mathbb{C}^2 \otimes \mathbb{C}^N) & \xrightarrow{\ \tau\ } & \mathbb{S}^1 \times (\mathbb{C}^2 \otimes \mathbb{C}^N) \\ \big\downarrow \pi & & \big\downarrow \pi \\ \mathbb{S}^1 & \xrightarrow{\ \mathbf{1}\ } & \mathbb{S}^1 \end{array} ,$$

where the morphism τ is given by:

$$\tau(k, v \otimes m) = (k, \sigma_3(v) \otimes m),$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that $\tau^* = \tau$ and $\tau^2 = \mathbf{1}$ and, moreover, this morphism satisfies that $\tau^* \mathcal{H} \tau = -\mathcal{H}$. This operator is called, in the physical literature, as *chiral symmetry operator* [22].

Now, in order to highlight this chiral symmetry, let us construct the spectral projection

$$P_F := \chi(\mathcal{H} \leq \mu = 0),$$

where the subscript F and the value μ refer to the *Fermi projection* and *Fermi level*, respectively, and this Fermi level is established, in this models of topological insulators, at 0. Notice that this Fermi projection satisfies, for each $k \in \mathbb{S}^1$, that $\tau_k^*\ (P_F)_k\ \tau_k = \mathbf{1} - (P_F)_k$; in other words, as a bundle morphism, $\tau^* P_F \tau = \mathbf{1} - P_F$.
Set

$$Q := \mathbf{1} - 2P_F,$$

which corresponds to $sign(\mathcal{H})$. Notice that this operator satisfies

$$\tau^* Q \tau = -Q \text{ and } Q^2 = \mathbf{1}.$$

Furthermore, from the definition of Q and its invertibility property, it can be shown that $\sigma_Q = \{-1, 1\}$. Combining these ingredients, we can conclude the existence of a morphism of sections

$$Q : \Gamma(\mathbb{S}^1 \times (\mathbb{C}^2 \otimes \mathbb{C}^N)) \to \Gamma(\mathbb{S}^1 \times (\mathbb{C}^2 \otimes \mathbb{C}^N)),$$

such that

$$Q(k) = \begin{pmatrix} 0 & U_F^*(k) \\ U_F(k) & 0 \end{pmatrix},$$

where the $U_F(k)$ is a unitary operator on $\mathbb{C}^N$ and it is given by:

$$U_F(k) = \frac{e^{ik} + im}{|e^{ik} + im|}\mathbf{1}_N.$$

The morphism $U_F : \Gamma(\mathbb{S}^1 \times \mathbb{C}^N) \to \Gamma(\mathbb{S}^1 \times \mathbb{C}^N)$ is well defined as long as $m \notin \{-1, 1\}$. In the physics literature, this unitary operator is known as the *Fermi unitary operator*.

Remark. In the previous definition for the morphism Q, we have made an abuse of notation. Actually, this map acts on sections $\psi : \mathbb{S}^1 \to \mathbb{S}^1 \times (\mathbb{C}^2 \otimes \mathbb{C}^N)$ where for any $k \in \mathbb{S}^1$, we have that $(Q(\psi))(k) \in \mathbb{C}^2 \otimes \mathbb{C}^N$. Hence, we can understand this morphism as a morphism where, on each fiber, acts as a linear operator. In a similar way, we can think U_F like Q. In this sense, this morphism, viewed from $\mathbb{S}^1$ to $M_N(\mathbb{C})$, is smooth.

Definition 4.2.1. *Let $\tilde{U} : \mathbb{S}^1 \to M_N(\mathbb{C})$ be a smooth map such that for all $k \in \mathbb{S}^1$, $U(k)$ is a unitary element. Let U be the unitary operator given by:*

$$U = \begin{pmatrix} 0 & \tilde{U}^* \\ \tilde{U} & 0 \end{pmatrix}.$$

The winding number associated to the unitary operator

$$Ch_1(U) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} dk \ \ Tr\ (U_k^* \partial_k U_k),$$

is called the first odd Chern number.

Computing this winding number for the operator U associated to the Fermi unitary operator, it follows that (see [22] §1):

$$\mathrm{Ch}_1(U_F) = \begin{cases} -N & \text{if } m \in (-1, 1) \\ 0 & \text{if } m \notin [-1, 1]. \end{cases}$$

This integer $\mathrm{Ch}_1(U_F)$ is called *the bulk invariant* associated to the ground state of the Hamiltonian $\mathcal{H}$. According to [22], the existence of the Fermi unitary operator –from which this invariant comes– is related to the existence of the gap in the spectrum of the Hamiltonian $\mathcal{H}$. Moreover, the authors state that U_F can be constructed entirely from the ground state of the system. In this sense, we find the first relation with the results presented in the previous chapters. Once a ground state is established, we can construct a projection operator over the subspace generated by the eigenvectors with positive energy –positive eigenvalues. Thus, we can consider the associated Fock representation and then apply the formalism carried out in [5] and [6] to this projection. With these elements, it arises naturally the question: *Can the bulk invariant be obtained as a Fredholm index for some unitary operator (in the sense of the previous chapter)?*

An algebraic model for the infinite chain with boundary. Now, following Prodan and Schulz-Baldes, we will modify this model introducing a boundary condition on the Hamiltonian $\mathcal{H}$. This corresponds geometrically to a restriction over a non-compact subspace of the circle which can be modeled by the constraint half-Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{N})$, restricting from $\mathbb{Z}$ to $\mathbb{N}$ with the boundary condition:

$$\hat{\mathcal{H}} = \frac{1}{2}(\sigma_1 + i\sigma_2) \otimes \mathbf{1}_N \otimes \hat{S} + \frac{1}{2}(\sigma_1 - i\sigma_2) \otimes \mathbf{1}_N \otimes \hat{S}^* + m\sigma_2 \otimes \mathbf{1}_N \otimes \mathbf{1}.$$

Here, as we expected, $\hat{S}$ denotes the unilateral right shift on $\ell^2(\mathbb{N})$ with the property that $\hat{S}^*\hat{S} = \mathbf{1}$. Notice that we have no longer a smooth fibration over the circle, but instead we can consider the morphism τ as a linear operator between Hilbert spaces:

$$\tau : \mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z}) \to \mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z}).$$

Thus, restricting to $\ell^2(\mathbb{N})$, we have the operator $\hat{\tau} = \sigma_3 \otimes \mathbf{1}_N \otimes \mathbf{1}$ called *the half-space chirality operator* which also satisfies

$$\hat{\tau}\hat{\mathcal{H}}\hat{\tau} = -\hat{\mathcal{H}},$$

and we can associate to it a projection which will allow us to implement the theory described in previous chapter. Indeed, this chirality condition implies that $\sigma(\hat{\mathcal{H}}) = -\sigma(\hat{\mathcal{H}})$ and, furthermore, according to [22], the direct sum $\hat{\mathcal{H}} \bigoplus \hat{\mathcal{H}}$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z})$ can be understood as a finite dimensional perturbation of $\mathcal{H}$ and, hence, the essential spectra will coincide $\sigma_{Ess}(\mathcal{H}) = \sigma_{Ess}(\hat{\mathcal{H}})$. However, this finite perturbation can generate additional point spectrum corresponding to those states which are annihilated in the boundary. These are called *boundary states*.

Let $\psi \in \mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{N})$ be a normalized boundary state with energy ε, i.e $\hat{\mathcal{H}}\psi = \varepsilon\psi$. Note that

$$\hat{\mathcal{H}}\hat{\tau}\psi = -\varepsilon\hat{\tau}\psi.$$

Hence, the subspace $\mathcal{E}$ spanned by all eigenvectors with eigenvalues in $[-\delta, \delta] \subseteq \Delta$ is invariant under the action of $\hat{\tau}$. Therefore, we can diagonalize $\hat{\tau}$ on $\mathcal{E}$ as $\mathcal{E} = \mathcal{E}_+ \bigoplus \mathcal{E}_-$ such that $\hat{\tau}$ is ± 1 on $\mathcal{E}_\pm$. Denote by $\tilde{P}(\delta)$ the spectral projection $\chi([-\delta, \delta])$ for the self-adjoint operator $\hat{\mathcal{H}}$. This operator splits into an orthogonal sum $\tilde{P}_+(\delta) \oplus \tilde{P}_-(\delta)$. Additionally, note that this operator satisfies that

$$\hat{\tau}\tilde{P}(\delta) = \tilde{P}_+(\delta) - \tilde{P}_-(\delta),$$

where $\mathcal{E}_\pm = \text{Im } \tilde{P}_\pm(\delta)$. The difference between the dimensions of the subspaces $\mathcal{E}_\pm$ is called *the boundary invariant of the system.* This is computed as:

$$\text{Tr}(\hat{\tau}\tilde{P}(\delta)) = N_+ - N_-,$$

with $N_\pm = \dim \mathcal{E}_\pm$. This invariant is independent of the choice of δ whenever $\delta \leq E_g$ [22]. Therefore, its value is entirely determined by by the spectral subspace of the zero eigenvalue, which is called *the space of zero modes.* The following theorem gives a relation between the previous physical model where there was no restriction, and this model where a restriction, understood as a border, has been placed (see [22] §1).

Theorem 4.2.2. *Consider the Hamiltonian $\mathcal{H}$ on $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \ell^2(\mathbb{Z})$ given at the beginning of the section, and let $\hat{\mathcal{H}}$ denote its half-space restriction. If U_F is the Fermi unitary operator and $Ch_1(U_F)$ denotes its winding number, then:*

$$Ch_1(U_F) = -Tr(\hat{\tau}\tilde{P}(\delta)),$$

whenever $[-\delta, \delta] \subseteq \Delta$.

This relation is known as the *Bulk-Edge correspondence.* Prodan and Schulz-Baldes present a proof of this theorem making use of some results of K-theory for C^*-algebras. We will end this section with a sketch of the proof using the tools of K-theory for C^*-algebras presented in the previous section.

To start with, we will observe that the first model, which has been presented from a geometrical point of view, can be translate into a more algebraic language. The main ingredient for this point of view in the unilateral right shift S involved in the Hamiltonian $\mathcal{H}$. The C^*-algebra generated by this operator, $C^*(S)$, is isomorphic to the C^*-algebra of the continuous complex valued functions on $\mathbb{S}^1$, $C(\mathbb{S}^1)$. To see this, we can consider the natural representation of $C(\mathbb{S}^1)$ over the Hilbert space $L^2(\mathbb{S}^1)$:

$$\pi : C(\mathbb{S}^1) \to \mathcal{B}(L^2(\mathbb{S}^1)),$$

given by multiplication, i.e

$$\pi(f)(g)(z) := f(z)g(z).$$

As we have observed above, this Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{Z})$, where the isomorphism corresponds to the discrete Fourier transform. Hence, the multiplication operator by z –the generator of $C(\mathbb{S}^1)$ as a C^*-algebra– is mapped to the operator S. Thus, we construct the respective isomorphism between C^*-algebras sending generators to generators (see [1]).
Recall that the Toeplitz algebra $\mathcal{T}$ is the smallest C^*-algebra which contains the operator $\hat{S}$ –note that we have changed the notation used in previous section for convenience. Now, we shall observe a different construction following [26] §7. For this, we will consider the

Toeplitz operators on $L^2(\mathbb{S}^1)$. Let $H^2(\mathbb{S}^1)$ be the Hardy space of $L^2(\mathbb{S}^1)$, and denote by $P : L^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$ the orthogonal projection onto this subspace. Given $g \in C(\mathbb{S}^1)$, consider the operator $\pi(g) \in \mathcal{B}(L^2(\mathbb{S}^1))$, and construct the operator $T_g := P\pi(g)$. This operator is called the Toeplitz operator with symbol g. Note that $\hat{S} = T_z$, where the right side operator can be understood as the generator of all Toeplitz operators. In [26] §7 there is a detailed description of these operators and, in addition, it is shown, using this description, that the finite rank operators are contained in the C^*-algebra generated by these Toeplitz operators: $C^*(T_f : \ f \in C(\mathbb{S}^1))$.

We shall now consider the example of the Toeplitz algebra developed in the previous section. Using the identifications and the changes of notation, we obtain the short exact sequence of C^*-algebras:

$$0 \longrightarrow \mathcal{K} \xrightarrow{\ i\ } C^*(\hat{S}) \xrightarrow{\ \psi\ } C^*(S) \longrightarrow 0.$$

According to the example, and using these identifications, we have the six term exact sequence for the Toeplitz algebra, and the index homomorphism

$$\delta_1 : K_1(C^*(\pi(S))) \to K_0(\mathcal{K})$$

which is, in fact, an isomorphism. Since $C^*(S)$ is generated as C^*-algebra by S, $K_1(C^*(S))$ is generated by $[S]_1$.

According to the short exact sequence of the Toeplitz algebra, and the second description of the index map, we can describe δ_1 explicitly in this case. By construction, given $u \in M_n(C^*(S))$ unitary, we can construct a lifting $w \in M_{2n}(\mathcal{T})$ such that

$$\psi(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

Using the description for δ_1 for unital algebras, we can obtain a projection $\tilde{p} \in M_{2n}(\hat{\mathcal{K}})$ such that

$$\bar{i}(\tilde{p}) = w \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix} w^*.$$

Moreover, this projection $\tilde{p}$ is such that its scalar part, $s(\tilde{p})$, is equal to $\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{pmatrix}$. Thus, $\delta_1([u]_1) = [w \text{ diag } (\mathbf{1}_n, 0)\ w^*]_0 - [\text{diag } (\mathbf{1}_n, 0)]_0$. With these results, we can compute $\delta_1([S^n]_1)$. A unitary lift, for $n \geq 0$, is

$$\begin{pmatrix} \hat{S}^n & F_n \\ 0 & (\hat{S}^*)^n \end{pmatrix},$$

where F_n have been defined in the previous section and they satisfy $\hat{S}^n(\hat{S}^*)^n = \mathbf{1} - F_n$ and $F_n\hat{S}^n = 0$. Thus, $\delta_1([\hat{S}^n]) = -[F_n]_0 = -n[F_1]_0$.
We will present a technical statement in order to conclude the main result of this section (see [22] §1).

Proposition 4.2.3. *Let $U_F \in M_N(C(\mathbb{S}^1))$ be the Fermi unitary operator. Then for $0 < \delta < E_g$,*

$$\delta_1([U_F]_1) = [\tilde{P}_+(\delta)]_0 - [\tilde{P}_-(\delta)]_0.$$

Now, let us consider the pairings:

$$\mathrm{Ch}_0 : K_0(\mathcal{K}) \to \mathbb{Z}, \quad \mathrm{Ch}_0([\tilde{P}]_0 - [\tilde{P}']_0) = \mathrm{Tr}\ (\tilde{P}) - \mathrm{Tr}\ (\tilde{P}'),$$

$$\mathrm{Ch}_1 : K_1(C(\mathbb{S}^1)) \to \mathbb{Z}, \quad \mathrm{Ch}_1([u]_1) = \frac{i}{2\pi}\int_{\mathbb{S}^1} dk\ \mathrm{Tr}\ (u(k)^*\partial_k u(k)).$$

These pairings are fundamental results which associate the K-theory with cohomology theory. However, we will not go deeper into this connection. Notice that, in the definition of Ch_1, we require that $k \mapsto u(k)$ be differentiable. Although we can not guarantee this directly, we use the fact that any continuous path $k \mapsto u(k)$ can be approximated by a differentiable one. Since $C^\infty(\mathbb{S}^1)$ is dense in $C(\mathbb{S}^1)$, for each representation class in K_1, we can consider a smooth representative [22]. In this way, we obtain the following theorem which shows the Bulk-Edge correspondence in terms of characteristic classes of the K groups. Theorem 4.2.2 follows as a corollary of this.

Theorem 4.2.4. *([22]) The maps Ch_0 and Ch_1 are well defined group homomorphisms and*

$$Ch_1([u]_1) = -Ch_0(\delta_1([u]_1)).$$

Remark. The previous result shows us how the Bulk-Edge can be understood using tools from K-theory. In this case, the winding number associated with the unitary Fermi operator is determined by the dimension of the boundary eigenfunctions. An important fact is that, according to previous results on K-theory, this invariant can be understood as the Fredholm index of some operator. This leads us to the question of how this operator can be obtained, which can be partially answered once the type of index, $\mathbb{Z}$ or $\mathbb{Z}_2$, is identified.

In [29] the authors consider an example which could guide us to give an answer to this question. If we consider a complex Hilbert space $\mathcal{H}$, it is possible to consider its realification where $Q := i$ is now a complex structure, and additionally consider an additional complex structure J. According to the results studied in chapter three, an index of type $\mathbb{Z}_2$ arises with the structure J and the respective orthogonal operators, and the structure Q, which can be understood as a charge generator, will allow us, with a careful analysis, to understand the appearance of a index type $\mathbb{Z}$. What leads us to think about the relationship of this example presented in this book with the development carried out by Prodan and Schulz-Baldes, is the way of how the index is calculated, where according to a decomposition into orthogonal subspaces given by the structure J, say E_+ and E_-, the index of S_{--}, where S is unitary with respect to the structure Q, is obtained using a trace form –very similar to the winding number. In [7] the authors consider an interpretation of $\mathbb{Z}_2$ valued invariants for this type of physical systems using the notion of spectral flow for Skew-adjoint Fredholm operators. This could be another promising direction of research in order to understand the relevance of topology in the description of infinite chains of this type.

4.3 Physical example for the $\mathbb{Z}_2$ index

In this section we shall describe a physical example where the $\mathbb{Z}_2$ arises labeled the connected components of the orthogonal group $O(E)$, where E is a finite dimensional real Hilbert space. Additionally, we will describe the connection of this index with the existence of *edge states*. Kitaev [14] performs a description of a chain consisting of L sites, where each site can be either empty or occupied by an electron, and he gives a mathematical description in terms of the Majorana fermions. In order to perform this characterization, we will follow [3].

4.3.1 Majorana Fermions

Let us consider an even finite dimensional Hilbert space E with the standard inner product denoted by $g : E \times E \to \mathbb{R}$. Let J be a complex structure on E, and denote by E_J the complexification of E with this structure. In order to simplify the computation, considering the standard basis for $E \cong \mathbb{R}^{2n}$, let us take J given by:

$$Je_k = e_{k+n}, \;\; Je_{k+n} = -e_k \text{ for } k \in [n].$$

Thus, the complexified vector space E_J corresponds to $\text{span}_{\mathbb{C}}\{e_1, \dots e_n\}$. Hence, the CAR algebra associated to E_J is generated by the elements:

$$a_i \equiv a(e_i), \;\; a_i^* \equiv a^*(e_i), \;\; \text{for } i \in [n],$$

which satisfy the CAR relations. With these elements, we can construct the Fock representation for the Clifford algebra $C[E]$, $\pi_J : C[E] \to \mathcal{B}(\bigwedge E_J)$, which recall, from chapter 1, it is given by:

$$\pi_J(v) := a_J(v) + ia_J^*(v),$$

where $v \in E$. From this, notice that:

$$a_J(iv) = ia_J(v), \;\; a_J^*(iv) = ia_J^*(v), \;\; a_J(Jv) = ia_J(v), \;\; a_J^*(Jv) = -ia_J^*(v).$$

Notice that for any operator $T \in \mathcal{B}(\bigwedge E_J)$ we have the decomposition:

$$T = \frac{T + T^*}{2} + i\frac{T - T^*}{2i},$$

and, moreover, $\frac{T+T^*}{2}$ and $\frac{T-T^*}{2i}$ are self-adjoint operators. In this order, for the creation and annihilation operators we get the decomposition:

$$a_j = \gamma_j^A + i\gamma_j^B, \quad a_j^* = \gamma_j^A - i\gamma_j^B.$$

These operators $\gamma_j^{(A)}, \gamma_j^{(B)}$ are known as the *Majorana fermions*.

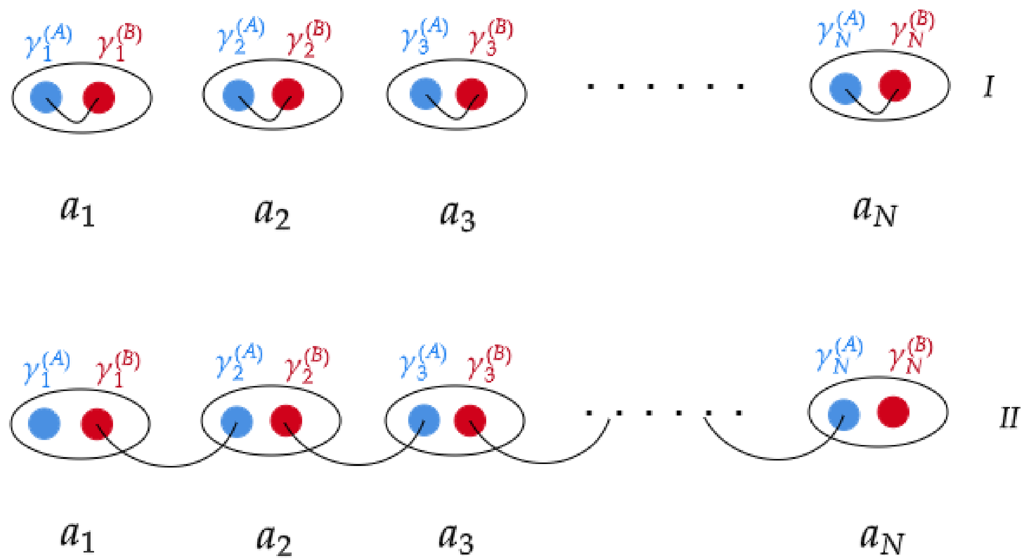


Figure 4-1: Majorana Fermions. Figure (I) corresponds to an extreme case where there is not *intracell interaction* in the fermionic chain. Figure (II) illustrates a second extreme case, where there is not *intercell interaction*. In this case, we have an illustration of the so called *edge states*.

Figure $4-1$ illustrates a fermionic chain subject to an interaction described by the operators a_j and a_j^*. According to the decomposition presented, this interaction can be written in terms of the Majorana fermions. Furthermore, the edge states can be understood as the zero level energy of this interaction once it is written in terms of the Majorana fermions.

Let us observe how these edge states are related with the $\mathbb{Z}_2$ index. First of all, recall from chapter 3 that given an orthogonal operator $g \in O(E)$, we can consider the Bologiubov automorphism on the Clifford algebra associated to g, θ_g, and find an unitary operator $U \in \mathcal{B}(\bigwedge E_J)$ such that for all $v \in E$ the following diagram commutes:

$$\begin{array}{ccc} \bigwedge E_J & \xrightarrow{\pi_J(v)} & \bigwedge E_J \\ \big\downarrow U & & \big\downarrow U \\ \bigwedge E_J & \xrightarrow{\pi_J(\theta_g(v))} & \bigwedge E_J. \end{array}$$

We can view this as a new representation of the Clifford algebra, $\tilde{\pi}_J : C[E] \to \mathcal{B}(\bigwedge E_J)$ given by:

$$\tilde{\pi}_J(v) := c(v) + c^*(v),$$

where the elements $c(v), c^*(v)$ fulfill the CAR relations. Using the commutativity of the diagram, it follows that:

$$c(v) + c^*(v) = U\pi_J(v)U^* = Ua_J(v)U^* + Ua_J^*(v)U^*$$

From [21] §3, we have the following proposition:

Proposition 4.3.1. *If $g \in O(E)$ and if $v \in E$ then*

$$\frac{1}{2}\pi_J \circ \theta_g(v + iJv) = a_J(A_g(v)) + a_J^*(C_g(v)).$$

It is not difficult to observe that the right hand side corresponds to

$$\frac{1}{2}U\pi_J(v + iJv)U^*.$$

In this order, we get that:

$$c(v) = Ua_J(v)U^*, \quad c^*(v) = Ua_J^*(v)U^*.$$

Moreover, from these expressions and the above proposition, we also get that:

$$c(v) = a_J(A_g(v)) + a_J^*(C_g(v)).$$

4.3.2 Two site chain

In this subsection let us study a Hamiltonian for a two site fermionic chain and the diagonalization problem, as we have done for the SSH model. Again, we will observe that the orthogonal operator arising from the diagonalization will give information about the existence of edge states.

Let $(\mathbb{R}^4, g)$ be our real Hilbert space with g the standar inner product. Let J be the complex structure given in the canonical basis by:

$$J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

According to the form of the structure J, the complexified Hilbert space E_J corresponds to $\text{span}_{\mathbb{C}}\{e_1, e_2\}$. On $\bigwedge E_J$ let us consider the Hamiltonian given in terms of the creation and annihilation operators by:

$$\mathcal{H} = t(a_1 a_2^* + a_2 a_1^*) + \Delta(a_1 a_2 - a_1^* a_2^*) - 2\mu(a_1 a_1^* + a_2 a_2^*),$$

where $t, \Delta, \mu \in \mathbb{R} \setminus \{0\}$. This Hamiltonian can be written as a quadratic form as:

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix} a & a^* \end{pmatrix} \Lambda \begin{pmatrix} a^* \\ a \end{pmatrix} + c,$$

where c is a constant and Λ is a block matrix

$$\Lambda = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix},$$

where

$$A = \begin{pmatrix} -2\mu & t \\ t & -2\mu \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}.$$

Notice that $A^T = A$ and $B^T = -B$. Our objective is to find an orthogonal transformation $H \in O(4)$ such that the implemented Bogoliubov automorphism θ_H allows us to write $\mathcal{H}$ in diagonal form as:

$$\mathcal{H} = \sum_k \lambda_k c_k c_k^*,$$

where the elements c_k, c_k^* fulfill the CAR relations.

In order to solve the problem, let us consider the following ansatz for the operators c_k, c_k^*:

$$c_k = \sum_{i=1}^{2} g_{ki} a_i^* + h_{ki} a_i, \;\; c_k = \sum_{i=1}^{2} g_{ki} a_i + h_{ki} a_i^*.$$

Using the fact that these operators must satisfy the CAR relations, we obtain the relations for the matrices g and h:

$$gg^T + hh^T = \mathbf{1}_2, \;\; gh^T + hg^T = 0.$$

In addition, computing the commutator $[c_k, \mathcal{H}]$ using the above expressions for $\mathcal{H}$, we obtain the equations (see [25]):

$$\lambda_k \; g_{ki} = \sum_{j=1}^{2} \left(g_{kj} A_{ji} - B_{ji} h_{kj} \right), \quad \lambda_k \; h_{ki} = \sum_{j=1}^{2} \left(g_{kj} B_{ji} - A_{ij} h_{kj} \right).$$

Let us consider the matrices $\Phi := g + h$ and $\Psi = g - h$. For any $k \in \{1, 2\}$, let us consider the vectors $\Phi_k, \; \Psi_k$ as the kth row of the respective matrices. In this order, the equations above can be written as

$$(A - B)\Psi_k = \lambda_k \Phi_k, \quad (A + B)\Phi_k = \lambda_k \Psi_k.$$

These two equations take the following form in matrix notation:

$$\begin{pmatrix} 0 & A - B \\ A + B & 0 \end{pmatrix} \begin{pmatrix} \Phi_k \\ \Psi_k \end{pmatrix} = \lambda_k \begin{pmatrix} \Phi_k \\ \Psi_k \end{pmatrix}.$$

Notice the following properties of this system:

i. It is diagonalizable since the matrix is symmetric.

ii. If α is an eigenvalue of the system with eigenvector $\begin{pmatrix} \Phi_k \\ \Psi_k \end{pmatrix}$, then $-\alpha$ is also an eigenvalue with eigenvector $\begin{pmatrix} \Phi_k \\ -\Psi_k \end{pmatrix}$. To show this statement, consider the block matrix $\mathcal{P} = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}$, and notice that

$$\mathcal{P} \begin{pmatrix} 0 & A - B \\ A + B & 0 \end{pmatrix} \mathcal{P} = - \begin{pmatrix} 0 & A - B \\ A + B & 0 \end{pmatrix}.$$

A straightforward computation of these eigenvalues and eigenvectors yields to the following values:

$$
\begin{aligned}
\lambda_1^{(1)} &= -(\alpha+t), \quad & v_1^T &= \begin{pmatrix} 2\mu, & -(\alpha+\Delta), & \alpha+\Delta, & -2\mu \end{pmatrix} \\
\lambda_1^{(2)} &= \alpha+t, & v_2^T &= \begin{pmatrix} 2\mu, & -(\alpha+\Delta), & -(\alpha+\Delta), & 2\mu \end{pmatrix} \\
\lambda_2^{(1)} &= -(\alpha-t), & v_3^T &= \begin{pmatrix} 2\mu, & \alpha-\Delta, & \alpha-\Delta, & 2\mu \end{pmatrix} \\
\lambda_2^{(2)} &= \alpha-t, & v_4^T &= \begin{pmatrix} 2\mu, & \alpha-\Delta, & -(\alpha-\Delta), & -2\mu \end{pmatrix},
\end{aligned}
$$

where $\alpha := \sqrt{\Delta^2+4\mu^2}$. Hence, notice the presence of a minus sign. Let $\beta_\pm := \sqrt{2\alpha(\alpha\pm\Delta)}$, $\theta_\pm := \operatorname{sgn}(\alpha\pm t)$. With these eigenvectors, and considering the respective minus signs, we construct the matrices:

$$
\Phi = \begin{pmatrix} 2\mu/\beta_+ & -\beta_+/2\alpha \\ 2u/\beta_- & \beta_-/2\alpha \end{pmatrix}, \quad \Psi = \begin{pmatrix} -\theta_+\beta_+/2\alpha & \theta_+ 2\mu/\beta_+ \\ -\theta_-\beta_-/2\alpha & -\theta_- 2\mu/\beta_- \end{pmatrix}.
$$

With these matrices, it follows that the orthogonal transformation $H \in O(4)$ which performs the diagonalization via the Bogoliubov automorphism is given by:

$$
H = \begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix}.
$$

According to the index formula studied in the previous chapter for these complex structures, we get that:

$$
i_{\mathcal{O}_{\text{res}}}(H) = \dim_{\mathbb{C}} \ker\,(H-JHJ)(\operatorname{mod} 2) = \dim_{\mathbb{C}} \ker \begin{pmatrix} \Phi+\Psi & 0 \\ 0 & \Phi+\Psi \end{pmatrix} \operatorname{mod} 2.
$$

Thus, the index of H depends of the determinant of $\Phi+\Psi$. If this is different from zero, then the dimension of the kernel, modulo 2, is zero. If this determinant is equal to zero, then the dimension of the kernel, as a complex subspace, is equal to one (since the third and fourth column are complex multiples of the first and second column). Notice that multiply by J does not affect the complex dimension. Therefore, performing this operation, we obtain that:

$$
i_{\mathcal{O}_{\text{res}}} = \dim_{\mathbb{C}} \ker \begin{pmatrix} 0 & -\Phi-\Psi \\ \Phi+\Psi & 0 \end{pmatrix}.
$$

Again, it only depends of the determinant on one entry. If we multiply one entry by $\Psi^T = \Psi^{-1}$, we reduce the problem and compute the determinant of $\mathbf{1}_2+\Psi^T\Phi$. We get the expression:

$$
\theta_+\theta_- + 1 - \frac{2\mu}{\alpha}(\theta_+ + \theta_-).
$$

Setting $\Delta = 1$, we can consider the regions in the plane $\mu - t$ where the index is 0 or 1.

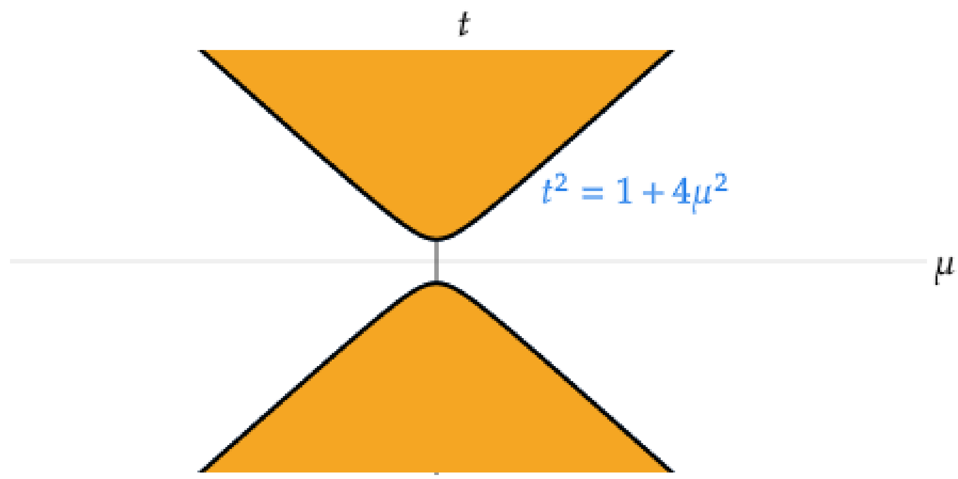


Figure 4-2: The shaded region corresponds to the values of t and μ such that $i_{\mathcal{O}_{\text{res}}} = 1$. The other region is for $i_{\mathcal{O}_{\text{res}}} = 0$. The limit curve is given by $t^2 = \alpha^2 = 1 + 4\mu^2$.

It is also possible to observe that this index labels precisely the path connected components of the group $O(4)$, where the regions of the Figure 4-2 correspond exactly to those where H belongs (or not) to $SO(4)$. The shaded region labels the path component of the identity, and the other labels the other path connected component.

Remark. This index is related with the presence of edge states as in the previous example in section 4.2 (see [3]). These edge states can be characterized in terms of the Majorana fermions. However, there is not a precise proof that this Bulk edge correspondence is always presented. In this models, known as the *N-site Kitaev chain*, it is conjectured that this correspondence exists. The authors in [3] compute these regions for larger N and show that this index, in fact, labels those regions where the edge states appear. The authors in [7] have studied this characterization using methods from spectral flow for Fredholm operators. This is a future work in order to stablish some connection with the $\mathbb{Z}_2$ index and real K-theory, as we have done for the SSH model.

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