# *Vershik-Okounkov Approach to Representation Theory*

**Author:**
Verónica Calvo Cortés

**Advisor:**
Mauricio Velasco Gregory

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**Mathematician**

Universidad de los Andes

Facultad de Ciencias
Departamento de Matemáticas
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# Contents



## Agradecimientos

Gracias a Mauricio y a Carolina por el apoyo constante. A mi mamá, mi papá, mi abuela y mis hermanos les agradeceré siempre. A mis amigas gracias por la complicidad. Y a Sebas gracias por creer en mí incluso cuando no creía en mí misma.

# Introduction

The study of the representation theory of symmetric groups started over one hundred years ago with works of mathematicians such as Frobenius, Schur and Young. Today it is a well developed theory with connections to other areas of mathematics and physics. The classical approach to this theory is based in the appearance of a combinatorial object, Young tableaux, that allows us to build all irreducible representations of $S_n$. A detailed exposition can be found in [Sag01, Chapter 2]. In 1996 Russian mathematicians Anatoli Vershik and Andrei Okounkov proposed a different way to build the irreducible representations of symmetric groups with the goal of fixing some drawbacks of the conventional approach. Particularly, they wanted to amend that the presence of Young tableaux was only justified after developing most of the theory. The main advantages of this approach are that we obtain a spectral connection between Young tableaux and the irreducible representations of $S_n$, we provide a concise description of the irreducible representations of $S_n$ with a canonical basis and that these techniques have the potential to be naturally extended to other Coxeter groups.

This document is mainly based in the 2005 revision and translation of the original Russian paper [VO05]. With the goal of making this theory more accessible we present detailed proofs and concrete examples. We also propose an original algorithm based on the Vershik-Okounkov theory. We will first present some preliminary results regarding the Coxeter group structure of $S_n$ and $B_n$ (hyperoctahedral group) as well as some results from classical representation theory. In chapter 2, we will present the general technique to find the irreducible representations of a so called *multiplicity-free* chain of finite groups $G_1 \leq \ldots \leq G_n \leq \ldots$ . These chains will allow us to find orthogonal basis (GZ-basis) for the irreducible representations of each $G_n$ which behave well with respect to the restriction $\mathrm{Res}^{G_n}_{G_{n-1}}$. These bases are determined up to scalar multiplication by a maximal commutative subalgebra of $\mathbb{C}G_n$ which we will call the GZ-algebra. In the second part of the chapter we will give criteria for a chain to be multiplicy-free. These techniques based on centralizers and Gelfand pairs allow us to prove that the symmetric and hyperoctahedral groups are multiplicity-free chains.

In chapter 3 we focus on the case of symmetric groups. A particular set of generators of the GZ-algebra, the Young-Jucys-Murphy (YJM) elements, and the relations between them and the Coxeter generators of $S_n$ will be the key. Specifically, we will consider the spectrum of the GZ-vectors with respect to the YJM elements. This will be in bijection with standard Young tableaux through their content vectors. This bijection will allow us to obtain the classical branching theorem as usually stated in terms of Young tableaux. The construction of the irreducible representations is done in an inductive way through the degenerate Hecke algebra which allows us to understand how different vectors from the same GZ-basis relate. In the final section of the chapter we propose an explicit way to find a canonical GZ-basis, up to the choice of a particular vector, which gives us a very concise description of the irreducible representations of $S_n$. In our fourth and final chapter we present the original contribution of this work. We give an algorithm with its code in Julia that decomposes a representation of $S_n$ into irreducible representations and gives an explicit GZ-basis for each.

# Chapter 1: Preliminaries

## 1.1 Notation

- $\text{triv}_n$ is the trivial representation of $S_n$. That is, it is a one dimensional vector space $V$ where $\forall v \in V, \forall \sigma \in S_n$ we have $\sigma \cdot v = v$.

- $\text{sgn}_n$ is the sign representation of $S_n$. That is, it is a one dimensional vector space $V$ where $\forall v \in V, \forall \sigma \in S_n$ we have $\sigma \cdot v = \text{sgn}(\sigma) v$.

- $\text{span}\{\dots\}$ is the $\mathbb{C}$ vector space generated by what is inside the brackets.

- $\langle \dots \rangle$ is the $\mathbb{C}$ algebra generated by what is inside the brackets.

- For a finite group $G$ we denote $\widehat{G}$ as the set of all its irreducible representations (modulo isomorphism).

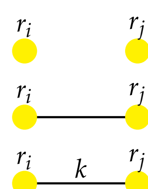
## 1.2 Coxeter Groups: $\mathbf{S_n}$ and $\mathbf{B_n}$

Throughout this document we will be working with two classical Coxeter groups: the symmetric group $S_n$ and the hyperoctahedral group $B_n$. In this section we will present some important results regarding these groups as in [Mus11, Sections 7.1 and 7.2] and [CSST10, Section 3.1.4]. We will start with a basic definition.

**Definition 1.1.** A *Coxeter group* is a group with presentation $\{r_1, \dots, r_n \mid (r_i r_j)^{m_{ij}} = id\}$ where $m_{ii} = 1$ and $m_{ji} = m_{ij} \geq 2$ for $i \neq j$. As a convention we denote $m_{ij} = \infty$ when we don't impose a relation of the type $(r_i r_j)^{m_{ij}} = id$. The set of generators $\{r_1, \dots, r_n\}$ are called *Coxeter generators*.

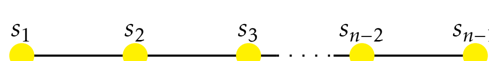
A nice way to understand Coxeter groups is by representing them as a weighted graph.The vertex set will be the generators $\{r_1, \dots, r_n\}$ and the edges correspond to the relations between them.

1. If $m_{ij} = 2$ there is not an edge between $r_i$ and $r_j$.
2. If $m_{ij} = 3$ there is an edge without weight between $r_i$ and $r_j$.
3. If $m_{ij} = k > 3$ there is an edge with weight $k$ between $r_i$ and $r_j$.




Our first example of a Coxeter group is the symmetric group $S_n$. We can prove this using the classical result which tell us that any permutation can be written as a product of transpositions.

**Proposition 1.2.** Let $s_i = (i, i+1) \in S_n$ for $i = 1, \dots, n-1$. Then $S_n$ is a Coxeter group with generators $\{s_i : i = 1, \dots, n-1\}$ and it corresponds to the following graph.




*Proof.* Note that $s_i s_i = id, (s_i s_{i-1})^3 = (i; i-1, i+1)^3 = id, (s_i s_{i+1})^3 = (i, i+1, i+2)^3 = id$ and $(s_i s_j)^2 = id$ for any other $j$. Thus, we have that the group generated by $\{s_i : i = 1, \dots n-1\}$ is a Coxeter subgroup of $S_n$. It remains it generates all $S_n$. Since every $\sigma \in S_n$ can be written as a product of transpositions it suffices to see that for any $(a, b) \in S_n$ we have that $(a, b)$ can be written as a product of $\{s_i : i = 1, \dots n-1\}$. Without loss of generality we assume that $a < b$. We will proceed by induction on $k = b - a$. For $k = 1$ we have that $(a, b) = (a, a+1) \in \langle s_i : i = 1, \dots n-1 \rangle$. Assume that the result holds for $k-1$. Then, $(a, a+k) = (a+k-1, a+k)(a, a+k-1)(a+k-1, a+k)$ can be written as a product of $\{s_i : i = 1, \dots n-1\}$ as desired. □

When we consider Coxeter groups we are distinguishing a particular set of generators. Thus, naturally for each element $g$ in the group we can also consider the minimal number of these generators needed to write $g$.

**Definition 1.3.** Let $G$ be a Coxeter group with generators $r_1, \dots, r_n$. We define the *Coxeter length (or reflexion length)* of an element $g \in G$ as the minimal number $k \in \mathbb{N}$ such that $g$ can be written as a product of $k$ Coxeter generators, i.e $g = r_{i_1} \cdots r_{i_k}$ for some (possibly repeated) indices $1 \leq i_1, \dots, i_k \leq n$. We denote the Coxeter length of $g$ as $l(g)$.

For our canonical example, the symmetric group, the reflection length can also be understood in a different way that later on will help us simplify its calculation.

**Definition 1.4.** Let $\sigma \in S_n$ be any permutation. We say that a pair $(i, j)$ with $i, j \in \{1, \dots, n\}$ and $i > j$ is an *inversion* for $\sigma$ if $\sigma(i) < \sigma(j)$. We denote the set of all inversions of $\sigma$ as $\mathcal{I}(\sigma)$.

**Proposition 1.5.** For any $\sigma \in S_n$ we have that $l(\sigma) = |\mathcal{I}(\sigma)|$.

*Proof.* First, we will prove a fundamental formula regarding the set of inversions. Let us consider $\tau \in S_n$ and $s_i$ with $i = 1, \dots, n-1$. We want to calculate $|\mathcal{I}(\tau s_i)|$ in relation to $|\mathcal{I}(\tau)|$. Clearly, if $j, k = 1, \dots, n$ with $j, k \neq i, i+1$ we have that $(j, k) \in \mathcal{I}(\tau s_i)$ if and only if $(j, k) \in \mathcal{I}(\tau)$. If $k \neq i, i+1$ we can also calculate $\tau s_i(i) = \tau(i+1), \tau s_i(i+1) = \tau(i)$ and $\tau s_i(k) = \tau(k)$. Thus,

$$(i, k) \in \mathcal{I}(\tau s_i) \Leftrightarrow (i+1, k) \in \mathcal{I}(\tau) \text{ and } (k, i) \in \mathcal{I}(\tau s_i) \Leftrightarrow (k, i+1) \in \mathcal{I}(\tau)$$
$$(i+1, k) \in \mathcal{I}(\tau s_i) \Leftrightarrow (i, k) \in \mathcal{I}(\tau) \text{ and } (k, i+1) \in \mathcal{I}(\tau s_i) \Leftrightarrow (k, i) \in \mathcal{I}(\tau).$$

This gives us a bijection between $\mathcal{I}(\tau) \setminus \{(i, i+1)\}$ and $\mathcal{I}(\tau s_i) \setminus \{(i, i+1)\}$. Finally we have that $(i, i+1) \in \mathcal{I}(\tau s_i)$ if and only if $(i, i+1) \notin \mathcal{I}(\tau)$. Hence, we obtain that

$$|\mathcal{I}(\tau s_i)| = \begin{cases} |\mathcal{I}(\tau)| - 1, & \text{if } (i, i+1) \in \mathcal{I}(\tau) \\ |\mathcal{I}(\tau)| + 1, & \text{if } (i, i+1) \notin \mathcal{I}(\tau). \end{cases} \tag{1.1}$$

We will now prove the proposition in two parts. First we will see by induction on $k = l(\sigma)$ that $|\mathcal{I}(\sigma)| \leq l(\sigma)$. For $k = 1$ we have that $\sigma = s_i$ for some $i = 1, \dots, n-1$. Then, $\mathcal{I}(\sigma) = \{(i, i+1)\}$ which has size 1. Now let us assume the result for $k-1$. Let $\sigma = s_{i_1} \cdots s_{i_k}$ be a minimal representation of $\sigma$. We will denote $\pi = s_{i_1} \cdots s_{i_{k-1}}$. We have that $l(\pi) = k-1$ because, otherwise, we could find a representation of $\sigma = \pi s_{i_k}$ with strictly less than $k$ Coxeter generators. Then, $|\mathcal{I}(\pi)| \leq k-1$ by induction hypothesis. By equation 1.1 we have that

$$|\mathcal{I}(\sigma)| = |\mathcal{I}(\pi s_{i_k})| \leq |\mathcal{I}(\pi)| + 1 \leq k - 1 + 1 = k = l(\sigma).$$

That is, the Coxeter length of $\sigma$ is at least its number of inversions. Now, let us see by induction on $n$ that we can always write $\sigma \in S_n$ as a product of $|\mathcal{I}(\sigma)|$ Coxeter generators. Denote $j = \sigma^{-1}(n)$ and consider $\sigma_n = \sigma s_j \cdots s_{n-1}$. Then, we have that $\sigma_n(n) = n$, i.e $\sigma_n \in S_{n-1}$, and $\sigma = \sigma_n s_{n-1} \cdots s_j$. Note that for any $k = 1, \dots, n-2$ we have that $\sigma_n s_{n-1} \cdots s_{k+1}(k) = \sigma_n(k) < n$ and $\sigma_n s_{n-1} \cdots s_{k+1}(k+1) = \sigma_n(n) = n$. That is, $(k, k+1)$ is a inversion of $\sigma_n s_{n-1} \cdots s_{k+1}$. Therefore we have that

$$|\mathcal{I}(\sigma)| = |\mathcal{I}(\sigma_n s_{n-1} \cdots s_j)| = |\mathcal{I}(\sigma_n s_{n-1} \cdots s_{j+1})| + 1 = \ldots = |\mathcal{I}(\sigma_n)| + (n-j).$$

By induction hypothesis we can write $\sigma_n$ as a product of $|\mathcal{I}(\sigma_n)|$ Coxeter generators which allows us to conclude the result. □

**Corollary 1.6.** Let $\pi \in S_{n-1} \subset S_n$ then for any $j = 1, \ldots, n-1$ we have that

$$l(\pi s_{n-1} s_{n-2} \cdots s_{j+1} s_j) = l(\pi) + (n-j).$$

*Proof.* It is analogous to the final part of the proof of proposition 1.5 just by changing $\sigma_n$ for $\pi$. □

A second example of a Coxeter group comes from a special subgroup of symmetric groups in an even number of elements. In order to distinguish this situation from the usual symmetric groups we will denote $S_{\overline{n}}$ as the group of permutations over $\{\pm 1, \ldots, \pm n\}$.

**Definition 1.7.** We say a permutation $\sigma \in S_{\overline{n}}$ is *signed* if for $i \in \{1, \ldots, n\}$, we have $\sigma(-i) = -\sigma(i)$. We denote $B_n$ as the set of all signed permutations.

**Proposition 1.8.** $B_n$ is a subgroup of $S_{\overline{n}}$ called the hyperotahedral group.

*Proof.* Let $i \in \{1, \ldots, n\}$. Clearly we have that $id(-i) = -i = -id(i)$ which means $id$ is a signed permutation. Let $\sigma, \tau \in S_{\overline{n}}$ be signed permutations. Denote $j := \sigma^{-1}(-i)$. Then we obtain

$$\begin{aligned}
&\sigma\tau(-i) = \sigma(-\tau(i)) = -\sigma\tau(i) \\
&\sigma^{-1}(-i) = j \Leftrightarrow -i = \sigma(j) = -\sigma(-j) \Leftrightarrow i = \sigma(-j) \Leftrightarrow \sigma^{-1}(i) = -j \Leftrightarrow -\sigma^{-1}(i) = \sigma^{-1}(-i).
\end{aligned}$$

□

**Definition 1.9.** We will give particular names to some special elements in $B_n$.

- For any $i, j \in \{\pm 1, \ldots, \pm n\}$ we will refer to $(i, j)(-i, -j)$ as a *positive reflection*.
- For $i = 1, \ldots, n-1$ we will refer to $r_i := (i, i+1)(-i, -i-1)$ as a *simple reflections*.
- For any $i \in \{1, \ldots, n\}$ we will call $(i, -i)$ a *negative reflection*.
- The non-redundant signed permutation $\sigma = (a_1, \ldots a_l)(-a_1, \ldots - a_l)$ will be called a *positive $l$-cycle*.
- The non-redundant signed permutation $\sigma = (a_1, \ldots a_l, -a_1, \ldots - a_l)$ will be called a *negative $l$-cycle*.

This vocabulary will be helpful in order to find analogous results to the decomposition of usual permutations in the case of $B_n$.

**Proposition 1.10.** Any signed permutation $\sigma \in B_n$ can be uniquely (up to changing the order of the factors) written as a product of disjoint positive and negative cycles.

*Proof.* Let $\sigma \in B_n$. Since $\sigma$ is also in $S_{\overline{n}}$ we can consider it is unique (up to changing the order of the factors) decomposition in disjoint cycles $\sigma = c_1 \cdots c_k$. Without loss of generality we can assume that there exists an $0 \leq r \leq k$ such that for every $j = 1, \ldots, r$ we have that $c_1, \ldots, c_r$ are all the non-negative cycles appearing in the cycle decomposition of $\sigma$. By convention, if $r = 0$ we have that all $c_i$ for $i = 1, \ldots, k$ are negative cycles. Let $j \in \{1, \ldots, r\}$. We will denote the cycle $c_j$ as $(a_1, \ldots, a_l)$. Assume by contradiction that for some $i = 2, \ldots, l$ we have that $a_i = -a_1$. Since $\sigma \in B_n$ we have that $a_{i+1} = \sigma(a_i) = \sigma(-a_1) = -\sigma(a_1) = -a_2$. We can apply the same argument again to get that $a_{i+2} = -a_3$. Continuing to do this to obtain that $a_{i+t} = -a_{t+1}$. This process stops for $t = i-1$ since we have $a_{2i-1} = -a_i = a_1$ which completes the cycle. What these equalities tell us is that

$c_j$ is $(a_1, a_2, a_3, \dots, a_{i-1}, -a_1, -a_2, -a_3, \dots, -a_{i-1})$. However, this means $c_j$ is a negative cycle which contradicts our choice of $j \leq r$. Thus, $-a_1 \notin \{a_1, \dots, a_l\}$. Moreover, note that we can reorder the cycle so that any $a_i$ for $i = 1, \dots, l$ is in the first position. We can conclude then that $\{-a_1, \dots, -a_l\} \cap \{a_1, \dots, a_l\} = \emptyset$. We also have that $\sigma(-a_i) = -\sigma(a_i) = -a_{i+1}$ for $i = 1, \dots, l-1$ and $\sigma(-a_l) = -a_1$ which implies $\overline{c_j} := (-a_1, \dots, -a_l)$ must appear in the cycle decomposition of $\sigma$. Thus, we have our desired result, i.e. $\sigma = c_1\overline{c_1} \cdots c_m\overline{c_m}\theta_1 \cdots \theta_s$ where $c_i\overline{c_i}$ are positive cycles and $\theta_i$ are negative cycles. □

**Lemma 1.11.** Every positive cycle is a product of positive reflections. And, every negative cycle is a product of one negative reflection and some positive reflections.

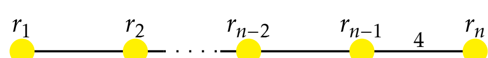
*Proof.* We will proceed by induction on the length of the cycles. For $l = 1$ it is clear that an negative $l$-cycle is actually the negative reflection $(a_1, -a_1)$. And, for $l = 2$ we have that a negative $l$-cycle is the positive reflection $(a_1, a_2)(-a_1, -a_2)$. Let us assume the results for $l-1$. Let $(a_1, \dots, a_l)(-a_1, \dots, -a_l)$ be a positive $l$-cycle and $(a_1, \dots, a_l, -a_1, \dots, -a_l)$ a negative $l$-cycle. Note that

$$(a_1, \dots, a_l)(-a_1, \dots, -a_l) = (a_1, a_2)(-a_1, -a_2)(a_2, \dots, a_l)(-a_2, \dots, -a_l)$$
$$(a_1, \dots, a_l, -a_1, \dots, -a_l) = (a_1, a_2)(-a_1, -a_2)(a_2, \dots, a_l, -a_2, \dots, -a_l).$$

Thus, we obtain the desired result. Note that what this gives us for negative $l$-cycles is that we can decompose it into a product of positive reflections until arriving to a final negative reflection. □

Similarly to what we showed on proposition 1.2 we can prove that the hyperoctahedral groups are also Coxeter groups.

**Proposition 1.12.** $B_n$ is a Coxeter group with generators $\{r_1, \dots, r_{n-1}, r_n\}$ where $r_i$ for $i = 1, \dots, n-1$ are the simple reflections and $r_n = (n, -n)$. Moreover, it corresponds to the following graph.




*Proof.* First, we will verify the relations between the generators. Clearly for every $i = 1, \dots, n$ we have $r_i^2 = id$. Simple calculations give us that $r_i r_{i-1} = (i, i-1, i+1)(-i, -(i-1), -(i+1))$ and $r_i r_{i+1} = (i, i+1, i+2)(-i, -(i+1), -(i+2))$ which both have order 3. We also have $r_{n-1} r_n = (n-1, n)(-(n-1), -n)(n, -n) = (n, -(n-1), -n, n-1)$ which has order 4. Finally, note that for $i \in \{1, \dots, n\}$ and $j \neq i, i-1, i+1$ we have that $r_i$ and $r_j$ are disjoint so $r_i r_j$ has order 2. Now, we will check that $\{r_1, \dots, r_n\}$ generates $B_n$. By lemma 1.11 and proposition 1.10, it suffices to check that all positive and negative reflections can be written as a product of $r_1, \dots, r_n$. Proving that the positive reflections are a product of $r_1, \dots, r_{n-1}$ is completely analogous to proposition 1.2 where we checked that $S_n$ was a Coxeter group. For the negative reflections let $j \in \{1, \dots, n-1\}$ then we have that $(j, -j) = (j, n)(-j, -n) r_n (-j, -n)(j, n) = (j, n)(-j, -n) r_n ((j, n)(-j, -n))^{-1}$. Since $(j, n)(-j, -n)$ is a positive reflection by what we argued before we have that it is a product of $r_1, \dots, r_{n-1}$. □

Now, since we are interested in the representation theory of $B_n$ we must figure out which are its conjugacy classes. And, as the reader may suspect the cycle decomposition from proposition 1.10 will be the key to determine them.

**Definition 1.13.** Let $\sigma \in B_n$ such that $\sigma = c_1\overline{c_1} \cdots c_m\overline{c_m}\theta_1 \cdots \theta_s$ where $c_i\overline{c_i}$ are positive cycles and $\theta_i$ are negative cycles as in proposition 1.10. For $i = 1, \dots, m$ and $j = 1, \dots, s$ let $\lambda_i$ be the length of the positive cycle $c_i\overline{c_i}$ and $\mu_j$ be the length of the negative cycle $\theta_j$. We call the *cycle type* of $\sigma$ the pair $(\lambda, \mu)$ where $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_s)$.

**Proposition 1.14.** Two elements in $B_n$ are conjugate if and only if they have the same cycle type.

*Proof.* Note that if $\sigma, \pi \in B_n$ such that $\sigma = c_1\overline{c_1}\cdots c_m\overline{c_m}\theta_1\cdots\theta_s$ then,

$$\pi\sigma\pi^{-1} = \pi c_1\overline{c_1}\pi^{-1}\cdots\pi c_m\overline{c_m}\pi^{-1}\pi\theta_1\pi^{-1}\cdots\pi\theta_s\pi^{-1}.$$

Thus, if suffices to check the statement for positive and negative cycles. Let $(a_1,\dots,a_l)(-a_1,\dots,-a_l)$ and $(a_1,\dots,a_l,-a_1,\dots,-a_l)$ be a positive and a negative $l$-cycle. Using a fundamental fact about symmetric groups together with the definition of $B_n$ we have that for any $\sigma, \pi \in B_n$ the conjugation $\pi\sigma\pi^{-1}$ corresponds to the permutation where $\{\pm 1,\dots,\pm n\}$ is changed for $\{\pm\pi(1),\dots,\pm\pi(n)\}$ in the cycle decomposition of $\sigma$. Then, for any $\pi \in B_n$ we have

$$\pi(a_1,\dots,a_l)(-a_1,\dots,-a_l)\pi^{-1} = (\pi(a_1),\dots,\pi(a_l))(-\pi(a_1),\dots,-\pi(a_l))$$
$$\pi(a_1,\dots,a_l,-a_1,\dots,-a_l)\pi^{-1} = (\pi(a_1),\dots,\pi(a_l),-\pi(a_1),\dots,-\pi(a_l))$$

which are also respectively a positive and a negative $l$-cycle. This means that conjugation does not change the length of positive and negative cycles. Now, let us check that two positive (or negative) $l$-cycles are conjugate. Let $(a_1,\dots,a_l)(-a_1,\dots,-a_l), (b_1,\dots,b_l)(-b_1,\dots,-b_l)$ and $(a_1,\dots,a_l,-a_1,\dots,-a_l)$, $(b_1,\dots,b_l;-b_1,\dots,-b_l)$ be respectively two positive and two negative $l$-cycles. Consider the permutation $\tau = (a_1,b_1)(-a_1,-b_1)\cdots(a_l,b_l)(-a_l,-b_l)$. Clearly this is a signed permutation and by our earlier remark regarding conjugation in $B_n$ we have that

$$\tau(a_1,\dots,a_l)(-a_1,\dots,-a_l)\tau^{-1} = (b_1,\dots,b_l)(-b_1,\dots,-b_l)$$
$$\tau(a_1,\dots,a_l,-a_1,\dots,-a_l)\tau^{-1} = (b_1,\dots,b_l,-b_1,\dots,-b_l).$$

□

Similarly to how we associate conjugacy classes with partitions of $n$ in the case of $S_n$ we can associate a multi-partition to the cycle type of signed permutations in $B_n$.

**Definition 1.15.** Consider $\lambda = (\lambda_1,\dots,\lambda_m)$ and $\mu = (\mu_1,\dots,\mu_s)$ such that $\lambda_1 \geq \ldots \geq \lambda_m \in \mathbb{N}^*$ and $\mu_1 \geq \ldots \geq \mu_s \in \mathbb{N}^*$. We say that the pair $(\lambda,\mu)$ is a *complementary partition* of a positive integer $n$ if $\lambda_1 + \ldots + \lambda_m + \mu_1 + \ldots + \mu_s = n$.

**Proposition 1.16.** The conjugacy classes of $B_n$, i.e. the cycle types $(\lambda,\mu)$, are in bijection with the complementary partitions of $n$.

*Proof.* Let us see first that every cycle type is a complementary partition of $n$. Note that a positive or negative $l$-cycle corresponds exactly to $2l$ elements from $\{\pm 1,\dots,\pm n\}$. Then, if $(\lambda = (\lambda_1,\dots,\lambda_m), \mu = (\mu_1,\dots,\mu_s))$ is a cycle type we get that $2\lambda_1 + \ldots + 2\lambda_m + 2\mu_1 + 2\mu_s = 2n$ and dividing by 2 we get our desired equality. Additionally, we can change the order of the factors in the cycle decomposition in such a way that the cycle lengths are increasing. Now, the same equality guarantees that a complementary partition $(\lambda,\mu)$ corresponds to the cycle decomposition $c_1\overline{c_1}\cdots c_m\overline{c_m}\theta_1\cdots\theta_s$ where $c_i\overline{c_i}$ is a positive $\lambda_i$-cycle and $\theta_j$ is a negative $\mu_j$-cycle. □

**Remark 1.17.** As a subgroup of $S_{\overline{n}}$ we have that the hyperoctahedral group $B_n$ has naturally two irreducible (since they are 1-dimensional) representations which are the trivial and sign representations. We will denote them as $\text{triv}_n^B$ and $\text{sgn}_n^B$.

**Example 1.18** (Character table of $B_2$)**.** First, using proposition 1.14 we have that the conjugacy classes of $B_2$ are

$C_1 = [id] = \{id\}$

$C_2 = [(1,-1)] = \{(1,-1),(2,-2)\}$

$C_3 = [(1,-1)(2,-2)] = \{(1,-1)(2,-2)\}$

$C_4 = [(1,2)(-1,-2)] = \{(1,2)(-1,-2),(1,-2)(-1,2)\}$

$C_5 = [(1,2,-1,-2)] = \{(1,2,-1,-2),(1,-2,-1,2)\}$.

By remark 1.17, we already know two irreducible representations of $B_2$ : $\text{triv}_2^B$ and $\text{sgn}_2^B$. To find the other three, let $\rho$ be the representation of $S_{\overline{4}}$ given by the natural action of permuting the coordinates in $\mathbb{C}^4 = \text{span}\{x_1,x_2,x_{-1},x_{-2}\}$. We are going to consider the restriction $\text{Res}_{B_2}^{S_4}\rho$. Since all permutations $\sigma \in B_2$ satisfy that $\sigma(-i) = -\sigma(i)$ for $i = 1,2$ we have that an invariant subspace of $\mathbb{C}^4$ with respect to the action of $B_2$ is $V = \text{span}\{x_1+x_{-1}, x_2+x_{-2}\}$. Clearly this is not an irreducible representation since it contains one copy of $\text{triv}_2^B : \text{span}\{x_1+x_{-1}+x_2+x_{-2}\}$. However, the complement of $\text{triv}_2^B$ in $V$, given by $\mu = \text{span}\{x_1+x_{-1}-x_2-x_{-2}\}$ is also a one-dimensional representation and therefore irreducible. Additionally, we also have that $\mu\otimes\text{sgn}_2^B$ is an irreducible representation of $B_2$. We can easily calculate the characters of these representations

$$\chi_{\text{Res}_{B_2}^{S_4}\rho} = (4,2,0,0,0), \quad \chi_V = (2,2,2,0,0), \quad ,\chi_\mu = (1,1,1,-1,-1) \quad \text{and } \chi_{\mu\otimes\text{sgn}_2^B} = (1,-1,1,-1,1)$$

where the $i$-coordinate correspond to the value of the character in the conjugacy class $C_i$. Then, we also have that the character of $V^\perp$ is $\chi_{V^\perp} = (2,0,-2,0,0)$. Calculating the inner product of $\chi_{V^\perp}$ with itself we obtain

$$\langle \chi_{V^\perp}, \chi_{V^\perp}\rangle = \frac{1}{|B_2|}\sum_{\sigma\in B_2}\chi_{V^\perp}^2(\sigma) = \frac{1}{8}(2^2+(-2)^2) = 1.$$

Thus, $W := V^\perp$ is an irreducible representation of $B_2$ and the character table of $B_2$ is the following.

Table 1.1: Irreducible characters of $B_2$

| $B_2$ | $[(id)]$ | $[(1,-1)]$ | $[(1,-1)(2,-2)]$ | $[(1,2)(-1,-2)]$ | $[(1,2,-1,-2)]$ |
|---|---|---|---|---|---|
| $\text{triv}_2^B$ | 1 | 1 | 1 | 1 | 1 |
| $\text{sgn}_2^B$ | 1 | −1 | 1 | 1 | −1 |
| $\mu$ | 1 | 1 | 1 | −1 | −1 |
| $\mu\otimes\text{sgn}_2^B$ | 1 | −1 | 1 | −1 | 1 |
| $W$ | 2 | 0 | −2 | 0 | 0 |

## 1.3 Classical Representation Theory

**Definition 1.19.** Let $G$ be a finite group and $\rho$ a representation. We say an inner product $\langle,\rangle$ in $V_\rho$ is *$G$-invariant* if for all $u,v\in V_\rho$ and $g\in G$ we have that $\langle\rho(g)(u),\rho(g)(v)\rangle = \langle u,v\rangle$.

**Proposition 1.20.** For any inner product $\langle,\rangle$ in $V_\rho$ we have that the inner product $\langle,\rangle_G$ given by $\langle u,v\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle\rho(g)(u),\rho(g)(v)\rangle$ is $G$-invariant.

*Proof.* it is clear that $\langle,\rangle_G$ is an inner product in $V_\rho$. Now let us check that it is $G$-invariant. Let $h\in G$. Then we have that

$$\langle\rho(h)(u),\rho(h)(v)\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle\rho(gh)(u),\rho(gh)(v)\rangle = \frac{1}{|G|}\sum_{g\in G}\langle\rho(g)(u),\rho(g)(v)\rangle = \langle u,v\rangle_G.$$

□

We recall that two fundamental tools to study the representations of a finite group $G$ are the group algebra $\mathbb{C}G = \text{span}\{e_g : g \in G\}$ with multiplication $e_h \cdot e_g = e_{gh}$. And the complex-valued function algebra $L(G) : \{f : G \to \mathbb{C}\}$ with convolution $f_1 * f_2(g) = \sum_{h \in G} f_1(gh) f_2(h^{-1})$ as multiplication.

**Proposition 1.21.** *$L(G)$ and $\mathbb{C}G$ are isomorphic as algebras.*

*Proof.* Recall that a basis for $L(G)$ is $\phi_g$ given by $\phi_g(h) = \delta_{g,h}$. Thus we can consider a linear map given by $\phi_g \mapsto e_g$. This is clearly an isomorphism between vector spaces. let us check that it is an algebra isomorphism. For any $g_1, g_2, h \in G$ we have that

$$\phi_{g_1} * \phi_{g_2}(h) = \sum_{k \in G} \phi_{g_1}(hk) \phi_{g_2}(k^{-1}) = \phi_{g_1}(hg_2^{-1}) = \begin{cases} 1, \text{ if } hg_2^{-1} = g_1 \\ 0, \text{ otherwise} \end{cases} = \phi_{g_1 g_2}(h).$$

Thus we have that $\phi_{g_1} * \phi_{g_2} \mapsto e_{g_1} e_{g_2}$ as desired. □

One of the most important results in classical representation theory is Schur's lemma which tells us that any representation morphism of finite groups between two complex irreducible representations is a multiple of the identity. Note that for a finite group $G$ and any $G$-representation $\rho$ the vector space $V_\rho$ has a natural $\mathbb{C}G$-submodule structure. Indeed, $\forall g \in G, \forall v \in V_\rho$ we define $e_g \cdot v := \rho(g)(v)$. This hints us that we can extend Schur's lemma to a wider context.

**Proposition 1.22** (Schur's Lemma for Simple Modules)**.** *Let $A$ be a $\mathbb{C}$-algebra and let $S$ be a simple and finite dimensional left $A$-module. If $\phi : S \to S$ is a non zero $A$-module homomorphism then $\phi = \lambda id_S$ for some $\lambda \in \mathbb{C}$.*

*Proof.* Let $\phi : S \to S$ be such a map. Recall that $\ker \phi$ and $\operatorname{im} \phi$ are submodules of $S$. Additionally, since $\phi$ is not the zero map we have that $\ker \phi \neq S$ and $\operatorname{im} \phi \neq \{0\}$. Then, by simplicity of $S$ (its only submodules are zero and itself) we obtain that $\ker \phi = \{0\}$ and $\operatorname{im} \phi = S$, i.e. $\phi$ is an isomorphism.

Furthermore, we can choose $\lambda \in \mathbb{C}$ an eigenvalue of $\phi$ (it exists since $S$ is a finite dimensional complex vector space). Clearly $\phi - \lambda id_S$ is a $\mathbb{C}$-linear map. Note that $\phi - \lambda id_S : S \to S$ satisfies $(\phi - \lambda id_S)(as) = \phi(as) - \lambda as = a\phi(s) - a(\lambda s) = a(\phi - \lambda id_S)(s)$. That is, it is an $A$-module morphism. Yet, by definition $\det(\phi - \lambda id_S) = 0$. Hence, it is not a $\mathbb{C}$-vector space (or $A$-module) isomorphism. Then, $\phi - \lambda id_S = 0$ which concludes the proposition. □

**Corollary 1.23.** *Let $A$ be a commutative algebra over $\mathbb{C}$. Then every non zero, simple and finite dimensional $A$-module is one dimensional.*

*Proof.* Let $S$ be a non zero, simple and finite dimensional $A$-module. For each $a \in A \setminus \{0\}$ define the $\mathbb{C}$-linear map $m_a : S \to S$ given by $m_a(s) = as$. Since $A$ is commutative we have that for every $b \in A : m_a(bs) = abs = bas = bm_a(s)$, i.e. it is an $A$-module morphism. Thus, by Schur's lemma for simple modules we have that $m_a = \lambda id_S$ for some $\lambda \in \mathbb{C}$. Now, let $v \in S \setminus \{0\}$. Note that for all $a \in A : m_a(v) = av \in \mathbb{C}v$. Thus, $\mathbb{C}v$ is a non zero submodule of $S$. Since $S$ is simple we obtain that $\mathbb{C}v = S$, i.e. $S$ is one dimensional. □

# Chapter 2: Inductive approach to Representations

In this chapter we will be working with inductive chains of finite groups, i.e.

$$\{1\} = G_1 \leq G_2 \leq \ldots \leq G_n \leq \ldots$$

in order to understand the representations of each $G_n$. In the first section we will introduce a technique to find an orthogonal basis for each $\rho \in \widehat{G_n}$ when the chain satisfies a certain condition. In the second section we will give some criteria to help determine when a particular chain satisfies that condition and we will use them to prove that we can use this technique for $S_n$ and $B_n$ . This approach was originally presented at the end of the twentieth century by Russian mathematicians Anatoly Vershik and Andrei Okounkov [VO05]. We will also be following [CSST10, Chapter 2] and [MM16, Sections 2.1, 2.2 and 2.3].

## 2.1 Gelfand-Tsetlin Algebra

Let us begin by explicitly defining what type of inductive group chains we want to work with.

**Definition 2.1.** Let $G$ be a finite group and $H \leq G$. We say $H$ is a *multiplicity-free subgroup* if for any irreducible representation of $G$, $\rho \in \widehat{G}$, we have that the restriction $\operatorname{Res}^G_H \rho$ is multiplicity-free, i.e. $\operatorname{Res}^G_H \rho = \sigma_1 \oplus \ldots \oplus \sigma_k$ where $\sigma_i$ are pairwise non isomorphic irreducible representations of $H$.

Following this definition, we can naturally define a multiplicity-free chain as one that satisfies that $G_n \leq G_{n+1}$ is a multiplicity-free subgroup for all $n \geq 1$. The next definition will motivate why we want to impose this condition.

**Definition 2.2.** Let us consider a multiplicity-free chain of finite groups. We define the *branching graph* of this chain as the directed graph with vertex set $\bigcup_{n\geq 1} \widehat{G_n}$, and edge set

$$\{(\rho, \sigma) : \rho \in \widehat{G}_{n+1},\ \sigma \in \widehat{G}_n,\ \sigma \text{ appears in the irreducible decomposition of } \operatorname{Res}^{G_{n+1}}_{G_n} \rho,\ n \geq 1\}.$$

We denote $\rho \to \sigma$ if $(\rho, \sigma)$ is an edge of the branching graph.

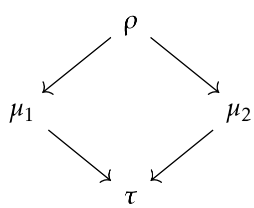
Observe that we can extend this definition to group chains that are not necessarily multiplicity-free by considering a multigraph (several edges between two vertex) or a weighted graph. However, we are interested in multiplicity-free chains, or simple branching chains, because they allow us to understand the irreducible decomposition of the restrictions canonically. Indeed, let us take $\rho$ an irreducible representation of $G_n$. Then,

$$\operatorname{Res}^{G_n}_{G_{n-1}} \rho = \bigoplus_{\rho \to \mu} \mu$$

where the $\mu \in \widehat{G}_{n-1}$ that appear in the irreducible decomposition are pairwise non-isomorphic. We can again restrict each $\mu$ to $G_{n-2}$ and obtain a multiplicity-free decomposition indexed by $\tau \in \widehat{G}_{n-2}$ such that $\mu \to \tau$. We have then the following irreducible decomposition

$$\operatorname{Res}^{G_n}_{G_{n-2}} \rho = \bigoplus_{\rho \to \mu \to \tau} \tau.$$

Here we cannot guarantee that the $\tau$ appearing in the decomposition are pairwise non-isomorphic. However if $\tau_1 \cong \tau_2$ appear in $\operatorname{Res}^{G_n}_{G_{n-2}} \rho$, since $G_{n-2} \leq G_{n-1}$ is a multiplicity-free subgroup, we can say that $\tau_1$ and $\tau_2$ appear, respectively, in $\mathrm{Res}^{G_{n-1}}_{G_{n-2}} \mu_1$ and $\mathrm{Res}^{G_{n-1}}_{G_{n-2}} \mu_2$ where $\rho \to \mu_1, \rho \to \mu_2$ and $\mu_1 \not\cong \mu_2$. The importance of this is that we can trace back each one of the $\tau$ in $\mathrm{Res}^{G_n}_{G_{n-2}} \rho$ to $\rho$ following a unique path $\rho \to \mu \to \tau$.




Continuing this process we obtain then a decomposition of $\rho$ in one-dimensional representations

$$\mathrm{Res}^{G_n}_{G_1} \rho = \bigoplus_{T \in \mathcal{T}(\rho)} \sigma_T, \quad V_\rho = \bigoplus_{T \in \mathcal{T}(\rho)} V_T.$$

Here $\mathcal{T}(\rho)$ is the set of all paths from $\rho$ to $\mathrm{triv}_1$ (the only representation of $G_1 = \{1\}$). We have indeed that each $\sigma_T$ corresponds to a unique path $T \in \mathcal{T}(\rho)$. The fact that the vector space $V_T$ corresponding to $\sigma_T$ is one dimensional allows us to consider a canonical basis.

**Definition 2.3.** Choosing a non-zero vector $v_T$ in each $V_T$ will give us a basis for $V_\rho$: $\{v_T : T \in \mathcal{T}(\rho)\}$ which we will call the *Gelfand-Tsetlin basis* (GZ-basis). And, since the spaces $V_T$ are one-dimensional each $v_T$ is unique up to scalar multiplication. We will call the vectors in the GZ-basis *Gelfand-Tsetlin vectors* (GZ-vectors).

**Remark 2.4.** We can chose the GZ-vectors in such a way that they are orthogonal with respect to a $G_n$-invariant inner product.

One of the most important examples of multiplicity-free group chains, as will be shown further ahead, are the symmetric groups. These can be considered as subgroups of each other in a natural way. We simply see $S_{n-1}$ as the subgroup of $S_n$ which consists of the permutations in $S_n$ that have $n$ as a fixed point. Another example we will explore in this work is the chain of hyperoctahedral groups $\{1\} = B_0 \leq B_1 \leq \ldots \leq B_n \leq \ldots$ where $B_{n-1}$ is the subgroup of $B_n$ that has $\pm n$ as fixed points.

**Example 2.5.** In this example we want to sketch the branching graph corresponding to the chain $S_1 \leq S_2 \leq S_3 \leq S_4$. $S_1$ has a unique irreducible representation $\mathrm{triv}_1$ and $S_2$ has two distinct irreducible representations $\mathrm{triv}_2$ and $\mathrm{sgn}_2$ which both trivially restrict to $\mathrm{triv}_1$. Also, from [FH91, Example 2.6 and Section 2.3] we have that the irreducible representations of $S_3$ and $S_4$ are given by the following character tables.

Table 2.1: Irreducible characters of $S_3$ and $S_4$

| $S_3$ | $[(id)]$ | $[(12)]$ | $[(123)]$ |
|---|---|---|---|
| $\mathrm{triv}_3$ | 1 | 1 | 1 |
| $\mathrm{sgn}_3$ | 1 | −1 | 1 |
| $W_3$ | 2 | 0 | −1 |

| $S_4$ | $[(id)]$ | $[(12)]$ | $[(123)]$ | $[(1234)]$ | $[(12)(34)]$ |
|---|---|---|---|---|---|
| $\mathrm{triv}_4$ | 1 | 1 | 1 | 1 | 1 |
| $\mathrm{sgn}_4$ | 1 | −1 | 1 | −1 | 1 |
| $W_4$ | 3 | 1 | 0 | −1 | −1 |
| $W_4 \otimes \mathrm{sgn}_4$ | 3 | −1 | 0 | 1 | −1 |
| $U$ | 2 | 0 | −1 | 0 | 2 |

It is clear that for the case of $S_3$ we have that $\mathrm{triv}_3$ restricts to $\mathrm{triv}_2$ in $S_2$ and similarly $\mathrm{sgn}_3$ restricts to $\mathrm{sgn}_2$. More generally, for $n \geq 3$, $\mathrm{triv}_n$ and $\mathrm{sgn}_n$ restrict to $\mathrm{triv}_{n-1}$ and $\mathrm{sgn}_{n-1}$ respectively.

Now, we want to understand the decomposition in irreducible representations of $\mathrm{Res}_{S_2}^{S_3} W_3$. For this let us recall that $W_3$ is the standard representation of $S_3$ that is obtained as the orthogonal complement of $(1,1,1)$ where $S_3$ acts naturally in $\mathbb{C}^3$ by permuting components. Thus, a possible basis for $W_3$ is $\{(1,1,-2),(1,-1,0)\}$. $S_2 \leq S_3$ acts trivially on $\mathrm{span}\{(1,1,-2)\}$ and acts as the sign representation on $\mathrm{span}\{(1,-1,0)\}$. Then, $\mathrm{Res}_{S_2}^{S_3} W_3 = \mathrm{triv}_2 \oplus \mathrm{sgn}_2$.

Similarly, $W_4$ is obtained as the orthogonal complement of $(1,1,1,1)$. And a possible basis for $W_4$ is $\{(1,1,1,-3),(1,-1,0,0),(1,0,-1,0)\}$. $S_3 \leq S_4$ acts trivially on $\mathrm{span}\{(1,1,1,-3)\}$. Now, to understand how $S_3$ acts on $W = \mathrm{span}\{(1,-1,0,0),(1,0,-1,0)\}$ we are going to use the characters tables 2.1. Note that

$$\rho_{W_4}\big|_{S_3}(23) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho_{W_4}\big|_{S_3}(123) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

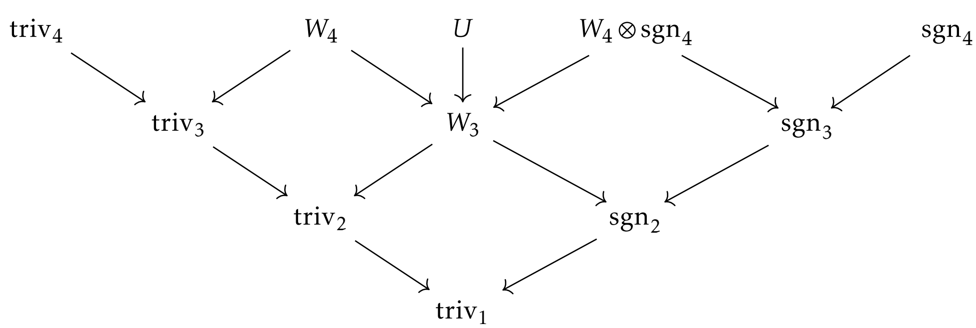
Then the characters of $W$ as a representation of $S_3$ are $(2,0,-1)$ which corresponds to the standard representation $W_3$. Thus, $\mathrm{Res}_{S_3}^{S_4} W_4 = \mathrm{triv}_3 \oplus W_3$.

Analogously, we have that $W_4 \otimes \mathrm{sgn}_4 = \mathrm{span}\{(1,1,1,-3)\otimes x, (1,-1,0,0)\otimes x, (1,0,-1,0)\otimes x\}$ where $\mathrm{span}\{x\}$ is endowed with the sign action of $S_4$. Clearly, $S_3 \leq S_4$ acts as $\mathrm{sgn}_3$ on $\mathrm{span}\{(1,1,1,-3)\otimes x\}$. And,

$$\rho_{W_4\otimes\mathrm{sgn}_4}\big|_{S_3}(23) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \rho_{W_4\otimes\mathrm{sgn}_4}\big|_{S_3}(123) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then the characters of $\{(1,-1,0,0)\otimes x, (1,0,-1,0)\otimes x\}$ as a representation of $S_3$ are $(2,0,-1)$. Thus, $\mathrm{Res}_{S_3}^{S_4} W_4 \otimes \mathrm{sgn}_4 = \mathrm{sgn}_3 \oplus W_3$.

Finally, for $\mathrm{Res}_{S_3}^{S_4} U$ it suffices to notice that the restriction of the characters in $S_4 : (2,0,-1,0,2)$ to $S_3$, which is $(2,0,-1)$, corresponds to an irreducible representation of $S_3 : W_3$. With this we obtain that the branching graph for the chain $S_1 \leq S_2 \leq S_3 \leq S_4$ is



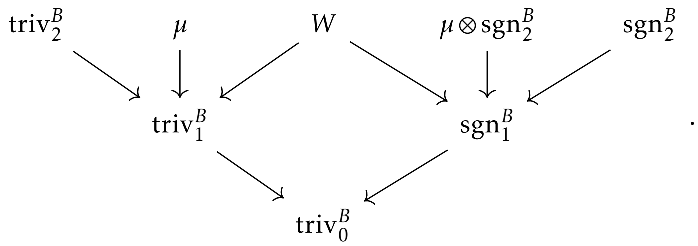

.

**Example 2.6.** Similarly, we would want to understand the branching graph corresponding to the hyperoctahedral groups $B_0 \leq B_1 \leq B_2$. $B_0 = S_1 = \{1\}$ and $B_1 = S_2$ have respectively one and two irreducible representations: $\mathrm{triv}_0^B, \mathrm{triv}_1^B$ and $\mathrm{sgn}_1^B$. We also have the character table of $B_2$ from example 1.18. Seeing $B_1 = \{id, (1,-1)\}$ we can easily restrict the irreducible characters of $B_2$ to $B_1$ and obtain that

$$\mathrm{Res}_{B_1}^{B_2} \mathrm{triv}_2^B = \mathrm{triv}_1^B \qquad \mathrm{Res}_{B_1}^{B_2} \mathrm{sgn}_2^B = \mathrm{sgn}_1^B$$

$$\mathrm{Res}_{B_1}^{B_2} \mu = \mathrm{triv}_1^B \qquad \mathrm{Res}_{B_1}^{B_2} \mu \otimes \mathrm{sgn}_2^B = \mathrm{sgn}_1^B \qquad \mathrm{Res}_{B_1}^{B_2} W = \mathrm{triv}_1^B \oplus \mathrm{sgn}_1^B.$$

And, thus we have that the part of the branching graph of $B_n$ with vertex set $\widehat{B_0} \cup \widehat{B_1} \cup \widehat{B_2}$ is




Now, the following algebra constructed out of these multiplicity-free chains will help us determine the GZ-vectors in terms of the action $\rho \in \widehat{G}_n$.

**Definition 2.7.** Recall that the center of an algebra is the set that commutes with all elements of the algebra. Denoting $Z(n)$ as the center of the group algebra $\mathbb{C}G_n$ we define $GZ(n)$ as the algebra generated by $Z(1), \dots, Z(n)$. This algebra is called the *Gelfand-Tsetlin Algebra* (GZ-Algebra) of the group chain $G_1 \leq \ldots \leq G_n$ .

This algebra can be explicitly described in terms of how it acts on the GZ-vectors.

**Proposition 2.8.** The GZ-algebra $GZ(n)$ is a maximal commutative subalgebra of the group algebra $\mathbb{C}G_n$. More specifically it is the algebra of operators that are diagonal with respect to the GZ-bases. That is,

$$GZ(n) = \left\{ f \in \mathbb{C}G_n : \forall \rho \in \widehat{G}_n, \forall T \in \mathcal{T}(\rho) \text{ we have that } f \cdot v_T \in \mathbb{C}v_T \right\}.$$

Here, the action $(f \cdot v)$ of the group algebra over the vector space of a representation of $G_n$ comes naturally from the following proposition.

**Lemma 2.9.** [FH91, Th. 3.29] For any group $G$ we have an algebra isomorphism $\mathbb{C}G \cong \bigoplus_{\rho \in \widehat{G}} \mathrm{End}(V_\rho)$ through the map

$$\varphi : \mathbb{C}G \to \bigoplus_{\rho \in \widehat{G}} \mathrm{End}(V_\rho), \quad e_g \mapsto \bigoplus_{\rho \in \widehat{G}} \rho(g) \tag{2.1}$$

where $\{e_g : g \in G\}$ is the basis of $\mathbb{C}G$ and $\alpha_g \in \mathbb{C}$.

*Proof.* Recall that

$$\dim \mathbb{C}G = \sum_{\rho \in \widehat{G}} (\dim V_\rho)^2 = \dim \bigoplus_{\rho \in \widehat{G}} \mathrm{End}(V_\rho).$$

Therefore it suffices to see that $\varphi$ is one-to-one. Let $\sum_{g \in G} \alpha_g e_g = f \in \ker \varphi$. Then, for any $\rho \in \widehat{G}$ we have that $\sum_{g \in G} \alpha_g \rho(g) = 0$. Since any representation of $G$ is isomorphic (as a representation) to a direct sum of irreducible representations this implies that $\sum_{g \in G} \alpha_g \mu(g) = 0$ where $\mu$ is any representation of $G$. Particularly, we can consider $\mu$ as the regular representation of $G$, i.e. $V_\mu = \mathbb{C}G$ and $G$ acts by left multiplication on the basis $\{e_g : g \in G\}$. However, note that

$$0 = \sum_{g \in G} \alpha_g \mu(g)(e_{id}) = \sum_{g \in G} \alpha_g e_g = f.$$

Thus, $\varphi$ is a vector space isomorphism.

Moreover, if we consider the natural multiplication in $\mathbb{C}G$ given by $e_g \cdot e_h = e_{gh}$ and the composition of endomorphisms in each component as a multiplication in $\bigoplus_{\rho \in \widehat{G}} \mathrm{End}(V_\rho)$ we have that $\varphi$ is an algebra isomorphism. Indeed, let $g, h \in G$, then

$$\varphi(e_g e_h) = \varphi(e_{gh}) = \bigoplus_{\rho\in\widehat{G}} \rho(gh) = \bigoplus_{\rho\in\widehat{G}} \rho(g)\rho(h) = \bigoplus_{\rho\in\widehat{G}} \rho(g) \bigoplus_{\rho\in\widehat{G}} \rho(h) = \varphi(e_g)\varphi(e_h).$$

□

This is better known as the semi-simple decomposition of the group algebra. Note that this is a different way of understanding the irreducible decomposition [FH91, Corollary 2.18] $\mathbb{C}G \cong \bigoplus_{\rho\in\widehat{G}} V_\rho^{\oplus \dim V_\rho}$. Fixing a basis $B$ of $V_\rho$ we obtain that $\mathrm{End}(V_\rho) \cong V_\rho^{\oplus \dim V_\rho}$ by mapping each $M \in \mathrm{End}(V_\rho)$ to $(M(v) : v \in B)$. In this way we obtain that in the semi-simple decomposition $\mathrm{End}(V_\rho)$ is precisely the isotypic component of $V_\rho$ in $\mathbb{C}G$.

Additionally, for each $\rho \in \widehat{G}$ we have that $\mathrm{End}(V_\rho)$ is an ideal of $\bigoplus_{\rho\in\widehat{G}} \mathrm{End}(V_\rho)$ generated (as an ideal) by a central idempotent $f_\rho$ which is the identity in $\mathrm{End}(V_\rho)$ and 0 in the other components. Lemma 2.9 allows us to understand this idempotent in terms of the group algebra.

**Lemma 2.10.** Let $G$ be a group. For every irreducible representation $\rho \in \widehat{G}$ we have that the central idempotent that generates $\rho$'s isotypic component in $\mathbb{C}G$ is

$$f_\rho = \frac{\dim V_\rho}{|G|} \sum_{g\in G} \overline{\chi_\rho(g)} e_g.$$

*Proof.* Let us fix $\rho \in \widehat{G}$. Since $\varphi$ in (2.1) is an algebra isomorphism it suffices to see that $\varphi(f)_\rho$ is the identity on $\mathrm{End}(V_\rho)$ and 0 in the other components of $\bigoplus_{\rho\in\widehat{G}} \mathrm{End}(V_\rho)$. Note that

$$\varphi(f)_\rho = \frac{\dim V_\rho}{|G|} \sum_{g\in G} \overline{\chi_\rho(g)} \bigoplus_{\mu\in\widehat{G}} \mu(g).$$

Let us see that $\frac{\dim V_\rho}{|G|} \sum_{g\in G} \overline{\chi_\rho(g)} \mu(g) \in \mathrm{End}(V_\mu)$ is a $G$-representation morphism. Let $h \in G$.

$$\begin{aligned}
\frac{\dim V_\rho}{|G|} \sum_{g\in G} \overline{\chi_\rho(g)} \mu(gh) &= \frac{\dim V_\rho}{|G|} \sum_{g'\in G} \overline{\chi_\rho(g'h^{-1})} \mu(g') && g' = gh \text{ as change of index} \\
&= \frac{\dim V_\rho}{|G|} \sum_{g'\in G} \overline{\chi_\rho(h^{-1}g')} \mu(g') && \text{characters are class functions} \\
&= \frac{\dim V_\rho}{|G|} \sum_{g\in G} \overline{\chi_\rho(g)} \mu(hg) && g = h^{-1}g' \text{ as change of index.}
\end{aligned}$$

Then, by Schur's lemma we have that $\frac{\dim V_\rho}{|G|} \sum_{g\in G} \overline{\chi_\rho(g)} \chi_\mu(g) = \lambda id_{V_\mu}$. Taking traces in both sides we obtain that $\frac{\dim V_\rho}{|G|} \sum_{g\in G} \overline{\chi_\rho(g)} \chi_\mu(g) = \dim V_\rho \lambda$. Using the character orthogonality formulas [FH91, Theorem 2.12] we get that $\lambda = \delta_{\rho,\mu}$ which allows us to conclude the lemma. □

*Proof of proposition 2.8.* Let $\mathcal{A}$ be the algebra of operators that are diagonal with respect to the GZ-basis. Note that if we take the matrix representation with respect to the GZ-basis of each component of $f$ we obtain that $\mathbb{C}G$ is isomorphic to a matrix algebra. Clearly, the image of $\mathcal{A}$ through this isomorphism is the subalgebra of diagonal matrices which is a maximal commutative subalgebra. Now, it remains to see that $\mathcal{A} \subset GZ(n)$. For this, we are going to build a basis for $\mathcal{A}$ that consists of elements in the GZ-algebra. Let $\rho \in \widehat{G}_n$ and $T = (\rho = \rho_n \to \rho_{n-1} \to \ldots \to \rho_1 = \mathrm{triv}_1) \in \mathcal{T}(\rho)$. For $i = 1, \ldots, n$ let $f_i$ be the central idempotent corresponding to the irreducible representation $\rho_i \in \widehat{G}_i$. We define $F_T = f_1 \cdot f_2 \cdots f_n$. Since each $f_i \in Z(i)$ then clearly $F_T \in GZ(n)$. By lemma 2.10 we particularly have that for $\mu \in \widehat{G}_n$, $(f_n)_\mu = 0$ if $\mu \neq \rho_n$. Thus, since $(F_T)_\mu$ is the composition of the endomorphisms $(f_i)_\mu$ we have that $(F_T)_\mu = 0$ if $\mu \neq \rho_n$. Finally, we want to understand what endomorphism is $(F_T)_{\rho_n}$. Let $\{v_S : S \in \mathcal{T}(\rho_n)\}$ be the GZ-basis for $\rho_n$. For a fixed $S = \rho_n \to \mu_{n-1} \to \ldots \to \mu_1$ we have by lemma 2.10 that for $i = 1, \ldots, n$

$$(f_i)_{\rho_n}(v_S) = \frac{\dim V_{\rho_i}}{|G_i|} \sum_{g \in G_i} \overline{\chi_{\rho_i}(g)} \rho_n(g)(v_S) = \frac{\dim V_{\rho_i}}{|G_i|} \sum_{g \in G_i} \overline{\chi_{\rho_i}(g)} \mathrm{Res}_{G_i}^{G_n} \rho_n(g)(v_S)$$
$$= \frac{\dim V_{\rho_i}}{|G_i|} \sum_{g \in G_i} \overline{\chi_{\rho_i}(g)} \mu_i(g)(v_S).$$

Similarly to the proof of lemma 2.10 we have that $\frac{\dim V_{\rho_i}}{|G_i|} \sum_{g \in G_i} \overline{\chi_{\rho_i}(g)} \mu_i(g) = \delta_{\rho_i, \mu_i} \mathrm{id}$. Hence, we obtain $(f_i)_{\rho_n}(v_S) = \delta_{\rho_i, \mu_i} v_S$. If $i$ is the first level in which $S$ and $T$ differ (i.e. such that $\rho_i \ncong \mu_i$) we have

$$(F_T)_{\rho_n}(v_s) = (f_1)_{\rho_n}(\ldots (f_n)_{\rho_n}(v_S)) = (f_1)_{\rho_n}(\ldots (f_i)_{\rho_n}(v_S)) = 0.$$

And if $T = S$ we will have $(F_T)_{\rho_n}(v_T) = v_T$. Finally, because an element $f \in \mathcal{A}$ is determined by its eigenvalues on each vector in the GZ-basis we have that $\{F_T : \rho \in \widehat{G}_n, T \in \mathcal{T}(\rho)\}$ is a basis for $\mathcal{A}$ which allows us to conclude the desired result. □

This proposition allows us to identify, up to scalar multiplication, the GZ-basis of an irreducible representation of $G_n$ from the eigenvalues and eigenvectors of the elements in the GZ-algebra.

**Corollary 2.11.** Let $\{v_T : T \in \mathcal{T}(\rho)\}$ be the GZ-basis for $\rho \in \widehat{S}_n$.

1. Let $v \in V_\rho$ for some $\rho \in \widehat{S}_n$. If for all $f \in GZ(n)$ we have that $f \cdot v \in \mathbb{C}v$ then $v = c v_T$ for some $c \in \mathbb{C}$ and $T \in \mathcal{T}(\rho)$.
2. If for all $f \in GZ(n)$ we have that $f \cdot v_T = f \cdot v_S$ then $T = S$.

*Proof.* Both statements follow from the basis we gave for $GZ(n)$ in the proof of proposition 2.8.

1. Let $\rho \in \widehat{G}_n$. Let us consider $v \in V_\rho$ such that for all $f \in GZ(n)$ we have $f_\rho(v) \in \mathbb{C}v$. In particular we have that $v$ is an eigenvector for the action of the basis of $GZ(n)$ defined in the proof of proposition 2.8. Since $v \in V_\rho$ we can write $v = \sum_{S \in \mathcal{T}(\rho)} a_S v_S$ where $a_S \in \mathbb{C}$. Then we obtain that for each $T \in \mathcal{T}(\rho)$, $(F_T)_\rho(v) = \sum\limits_{S \in \mathcal{T}(\rho)} a_S (F_T)_\rho(v_S) = a_T v_T$.

2. Let $v_T, v_S \in V_\rho$ be two GZ-vectors of $\rho \in \widehat{G}_n$ such that they have the same eigenvalues for all elements in the GZ-algebra. In particular, we have that $(F_T)_\rho(v_S) = v_S$ since the eigenvalue for $v_T$ under $F_T$ is 1. But, as shown in proposition 2.8 we also have that $(F_T)_\rho(v_S) = \delta_{T,S} v_S$. Thus, $v_T = v_S$.

□

**Example 2.12.** Proposition 2.8 allows us to explicitly calculate the GZ-algebra for $S_3$. Let us recall that $S_3$ has three irreducible representations $\mathrm{triv}_3, W_3$ and $\mathrm{sgn}_3$ with character table as in table 2.1. From example 2.5 we know that there are four possible paths for $S_3$,

$$T_1 = \mathrm{triv}_3 \to \mathrm{triv}_2 \to \mathrm{triv}_1 \qquad T_2^1 = W_3 \to \mathrm{triv}_2 \to \mathrm{triv}_1$$

$$T_2^2 = W_3 \to \mathrm{sgn}_2 \to \mathrm{triv}_1 \qquad\qquad T_3 = \mathrm{sgn}_3 \to \mathrm{sgn}_2 \to \mathrm{triv}_1 .$$

Additionally, following lemma 2.10 we have that the central idempotents corresponding to the irreducible representations of $S_1, S_2$ and $S_3$ are

$$f_{\mathrm{triv}_1} = e_{id} \qquad f_{\mathrm{triv}_2} = \frac{1}{2}(e_{id} + e_{(12)}) \qquad f_{\mathrm{triv}_3} = \frac{1}{6}\sum_{\sigma\in S_3} e_\sigma$$
$$f_{\mathrm{sgn}_2} = \frac{1}{2}(e_{id} - e_{(12)}) \qquad f_{\mathrm{sgn}_3} = \frac{1}{6}\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma) e_\sigma \qquad f_{W_3} = \frac{1}{6}(2e_{id} - e_{(123)} - e_{(132)}).$$

Then, we obtain that an explicit basis for the GZ-algebra of $S_3$, that is $GZ(3)$, which corresponds to the one used in the proof of proposition 2.8 is

$$F_{T_1} = f_{\mathrm{triv}_1} f_{\mathrm{triv}_2} f_{\mathrm{triv}_3} = e_{id}\left(\frac{1}{2}(e_{id} + e_{(12)})\right)\left(\frac{1}{6}\sum_{\sigma\in S_3} e_\sigma\right) = \frac{1}{12}\left(\sum_{\sigma\in S_3} e_\sigma + \sum_{\sigma\in S_3} e_{(12)\sigma}\right) = \frac{1}{6}\sum_{\sigma\in S_3} e_\sigma$$
$$F_{T_2^1} = f_{\mathrm{triv}_1} f_{\mathrm{triv}_2} f_{W_3} = e_{id}\left(\frac{1}{2}(e_{id} + e_{(12)})\right)\left(\frac{1}{6}(2e_{id} - e_{(123)} - e_{(132)})\right)$$
$$= \frac{1}{12}(2e_{id} - e_{(123)} - e_{(132)} + 2e_{(12)} - e_{(23)} - e_{(13)})$$
$$F_{T_2^2} = f_{\mathrm{triv}_1} f_{\mathrm{sgn}_2} f_{W_3} = e_{id}\left(\frac{1}{2}(e_{id} - e_{(12)})\right)\left(\frac{1}{6}(2e_{id} - e_{(123)} - e_{(132)})\right)$$
$$= \frac{1}{12}(2e_{id} - e_{(123)} - e_{(132)} - 2e_{(12)} + e_{(23)} + e_{(13)})$$
$$F_{T_3} = f_{\mathrm{triv}_1} f_{\mathrm{sgn}_2} f_{\mathrm{sgn}_3} = e_{id}\left(\frac{1}{2}(e_{id} - e_{(12)})\right)\left(\frac{1}{6}\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma) e_\sigma\right) = \frac{1}{12}\left(\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma) e_\sigma - \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma) e_{(12)\sigma}\right)$$
$$= \frac{1}{12}\left(\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma) e_\sigma + \sum_{\sigma\in S_3} \mathrm{sgn}((12)\sigma) e_{(12)\sigma}\right) = \frac{1}{6}\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma) e_\sigma$$

## 2.2 Criteria for multiplicity-free groups

As mentioned before, we are interested in group chains which are multiplicity-free. At this point, it is natural to ask what sort of criteria can help us to determine whether a particular chain has this property. In this section we will see that being multiplicity-free is determined by the commutativity of a particular centralizer. And, this commutativity can be checked via conjugacy conditions of the group. Before stating the main proposition we will give some basic definitions.

**Definition 2.13.** Let $M$ be an algebra and $N \leq M$ a subalgebra. We define the *centralizer* of $N$ as the subalgebra $Z(M,N) := \{m \in M : \forall n \in N,\ mn = nm\}$. That is, it consists of the elements of $M$ that commute (with respect to multiplication) with all the elements of $N$.

**Definition 2.14.** Let $G$ be a finite group and $H \leq G$. We say that $(G,H)$ is a *Gelfand pair* if the algebra of $H$ bi-invariant functions,

$$L(H/G\backslash H) = \{f : G \to \mathbb{C} \text{ such that } f(hgh') = f(g)\ \forall g \in G \text{ and } \forall h,h' \in H\},$$

is commutative under convolution.

Considering the subgroup $\tilde{H} = \{(h,h) : h \in H\} \leq G \times H$ we have the following result.

**Proposition 2.15.** [CSST10, Th. 2.1.20] Let $G$ be a finite group and $H \leq G$. The following statements are equivalent.

1. $H$ is a multiplicity-free subgroup of $G$.
2. $Z(\mathbb{C}G, \mathbb{C}H)$ is commutative.
3. $(G \times H, \tilde{H})$ is a Gelfand pair.

We will first prove that $1 \Leftrightarrow 2$. For this we need to understand the centralizer $Z(\mathbb{C}G, \mathbb{C}H) \subset \mathbb{C}G$ in terms of the isomorphism $\mathbb{C}G \cong \bigoplus_{\rho \in \widehat{G}} \operatorname{End}(V_\rho)$.

**Lemma 2.16.** Let $H \leq G$. Then, $\varphi(Z(\mathbb{C}G, \mathbb{C}H)) = \bigoplus_{\rho \in \widehat{G}} \operatorname{End}_H(V_\rho)$ where $\varphi$ is the algebra isomorphism in lemma 2.9.

*Proof.* " $\subseteq$": Let $f \in Z(\mathbb{C}G, \mathbb{C}H)$. In particular, $f$ commutes with the basis $\{e_h : h \in H\}$ from $\mathbb{C}H$. Then, for all $h \in H$ we have that $f e_h = e_h f$. That is, for each $\rho \in \widehat{G}$, $(f)_\rho \rho(h) = \rho(h)(f)_\rho$. Which by definition means that $(f)_\rho$ is an endomorphism of $V_\rho$ as an $H$-representation.

" $\supseteq$": Let $\psi \in \bigoplus_{\rho \in \widehat{G}} \operatorname{End}_H(V_\rho)$. In particular, $\psi$ is also an element of $\bigoplus_{\rho \in \widehat{G}} \operatorname{End}(V_\rho)$. Since $\varphi$ is an isomorphism there exists a unique $f \in \mathbb{C}G$ such that $\varphi(f) = \psi$. We want to see that $f \in Z(\mathbb{C}G, \mathbb{C}H)$. It suffices to see that $f$ commutes with $\mathbb{C}H$'s basis $\{e_h : h \in H\}$. Let $h \in H$. Since for each $\rho \in \widehat{G}$ we have that $\psi_\rho$ is an $H$-equivariant map, we obtain that $\varphi(f e_h) = \bigoplus_{\rho \in \widehat{G}} \psi_\rho \rho(h) = \bigoplus_{\rho \in \widehat{G}} \rho(h) \psi_\rho = \varphi(f e_h)$. □

*Proof of proposition 2.15. Part 1.* "1. $\Rightarrow$ 2.": For each $\rho \in \widehat{G}$ we can consider the decomposition of its $H$-restriction in irreducible $H$-representations $V_\rho \cong \bigoplus_{\substack{\mu \in \widehat{H} \\ \rho \to \mu}} V_\mu$. Thus, $\operatorname{End}_H(V_\rho) \cong \bigoplus_{\substack{\mu \in \widehat{H} \\ \rho \to \mu}} \operatorname{Hom}_H(V_\rho, V_\mu)$. Since $H \leq G$ is a multiplicity-free group, $\operatorname{Hom}_H(V_\rho, V_\mu) \cong \mathbb{C}$. In this way we obtain that $\varphi(Z(\mathbb{C}G, \mathbb{C}H))$ is isomorphic as an algebra to a direct sum of $\mathbb{C}$, which is commutative. Then $Z(\mathbb{C}G, \mathbb{C}H)$ is also commutative.

"2. $\Rightarrow$ 1.": Let $\rho \in \widehat{G}$ and $\mu \in \widehat{H}$. We will denote $k$ the multiplicity of $\mu$ in $\rho$, i.e. $V_\rho = V_\mu^{\oplus k} \oplus X$ where $X$ does not have any copies of $V_\mu$. Note that $\operatorname{Hom}_H(V_\mu, V_\rho)$ is a $Z(\mathbb{C}G, \mathbb{C}H)$-module with the action $f \cdot \psi := (f)_\rho \circ \psi$ where $f \in Z(\mathbb{C}G, \mathbb{C}H)$ and $\psi \in \operatorname{Hom}_H(V_\mu, V_\rho)$. We want to see that it is a simple $Z(\mathbb{C}G, \mathbb{C}H)$-module. For this, we are going to prove that none non-zero and proper $\mathbb{C}$-subspace of $\operatorname{Hom}_H(V_\mu, V_\rho)$ is invariant under the action of $Z(\mathbb{C}G, \mathbb{C}H)$. Let $\{0\} \neq M$ be a proper complex vector subspace of $\operatorname{Hom}_H(V_\mu, V_\rho)$. Fix $m \in M \setminus \{0\}$. Then, since $M$ is proper, we can choose a linear map $L \in \operatorname{End}_{\mathbb{C}}(\operatorname{Hom}_H(V_\mu, V_\rho))$ such that $L(m) \notin M$. Now, to conclude our desired result we previously need to understand the maps in $\operatorname{End}_{\mathbb{C}}(\operatorname{Hom}_H(V_\mu, V_\rho))$.

By Schur's lemma we have that $\dim \operatorname{Hom}_H(V_\mu, V_\rho) = k$. Moreover, we can consider a basis $\{\psi_1, \ldots, \psi_k\}$ for $\operatorname{Hom}_H(V_\mu, V_\rho)$ such that $\operatorname{im}(\psi_i) \cong V_\mu$ as $H$-representations. Thus, we can consider $H$-morphisms $\phi_i : V_\rho \cong \bigoplus_{j=1}^k \operatorname{im}(\psi_j) \oplus X \to V_\mu$ such that $\phi_i = \psi_i^{-1}$ on $\operatorname{im}(\psi_i)$ and equals zero in the other components of $V_\rho$. For each $F \in \operatorname{End}_{\mathbb{C}}(\operatorname{Hom}_H(V_\mu, V_\rho))$, we have that $\alpha_F := \sum_{i=1}^k F(\psi_i) \circ \phi_i \in \operatorname{End}_H(V_\rho)$ since it is the composition and sum of $H$-equivariant maps. Additionally, for $j = 1, \ldots, k$ we obtain that $\alpha_F \circ \psi_j = \sum_{i=1}^k F(\psi_i) \circ \phi_i \circ \psi_j = F(\psi_j)$. Finally, by lemma 2.16 we have that $\alpha_F = (f)_\rho$ for some $f \in Z(\mathbb{C}G, \mathbb{C}H)$. And thus, by linearity, we have that for any $\psi \in \operatorname{Hom}_H(V_\mu, V_\rho)$, $F(\psi) = f \cdot \psi$.

Going back to our main argument, we have that in particular there exists $l \in Z(\mathbb{C}G, \mathbb{C}H)$ such that $L(m) = (l)_\rho \circ m = l \cdot m$. Since $l \cdot m \notin M$ we obtain that $M$ is not a $Z(\mathbb{C}G, \mathbb{C}H)$-submodule of $\mathrm{Hom}_H(V_\mu, V_\rho)$. Then $\mathrm{Hom}_H(V_\mu, V_\rho)$ is simple as a $Z(\mathbb{C}G, \mathbb{C}H)$-module, and by corollary 1.23 it is one-dimensional which concludes the proposition. □

**Example 2.17.** With this first equivalence we can prove directly that the centralizer $Z(\mathbb{C}S_4, \mathbb{C}S_3)$ is commutative. This will allow us to conclude that for any $\rho \in \widehat{S_4}$ the $\mathrm{Res}^{S_4}_{S_3} \rho$ is multiplicity free, as previously seen in example 2.5. First, let us find a basis for $Z(\mathbb{C}S_4, \mathbb{C}S_3)$. Note that it suffices to find the $f \in \mathbb{C}S_4$ that commute with the basis $\{e_\sigma : \sigma \in S_3\}$. Writing $f = \sum_{\tau \in S_4} a_\tau e_\tau$ gives us that the following equations must be fulfilled for all $\sigma \in S_3$

$$e_\sigma f e_{\sigma^{-1}} = f \Leftrightarrow \sum_{\tau \in S_4} a_\tau e_{\sigma\tau\sigma^{-1}} = \sum_{\tau \in S_4} a_\tau e_\tau \Leftrightarrow \sum_{\tau \in S_4} a_{\sigma^{-1}\tau\sigma} e_\tau = \sum_{\tau \in S_4} a_\tau e_\tau.$$

Since the coefficients $(a_\tau : \tau \in S_4)$ corresponding to $f$ are unique we obtain that $Z(\mathbb{C}S_4, \mathbb{C}S_3)$ consists of elements $f = \sum_{\tau \in S_4} a_\tau e_\tau$ such that

$$\forall \sigma \in S_3, \forall \tau \in S_4 : a_{\sigma^{-1}\tau\sigma} = a_\tau.$$

Solving these equations reduces to finding the orbits of the action $S_3 \curvearrowright S_4$ by conjugation. With computational aid we obtain 7 orbits

$$\begin{aligned} &\mathcal{O}(id) = \{id\}, && \mathcal{O}((12)(34)) = \{(12)(34), (13)(24), (14)(23)\}, \quad \mathcal{O}(14) = \{(14), (24), (34)\}, \\ &\mathcal{O}(12) = \{(12), (13), (23)\}, && \mathcal{O}(124) = \{(124), (134), (142), (143), (234), (243)\}, \\ &\mathcal{O}(123) = \{(123), (132)\}, && \mathcal{O}(1234) = \{(1234), (1243), (1324), (1342), (1423), (1432)\}. \end{aligned}$$

Thus, a basis for $Z(\mathbb{C}S_4, \mathbb{C}S_3)$ is given by the vectors

$$\begin{aligned} v_1 &= e_{id} & v_5 &= e_{(12)(34)} + e_{(13)(24)} + e_{(14)(23)} \\ v_2 &= e_{(123)} + e_{(132)} & v_6 &= e_{(124)} + e_{(134)} + e_{(142)} + e_{(143)} + e_{(234)} + e_{(243)} \\ v_3 &= e_{(12)} + e_{(13)} + e_{(23)} & v_7 &= e_{(1234)} + e_{(1243)} + e_{(1324)} + e_{(1342)} + e_{(1423)} + e_{(1432)} \\ v_4 &= e_{(14)} + e_{(24)} + e_{(34)}. \end{aligned}$$

Then, by explicitly calculating the products of these vectors we can verify that $Z(\mathbb{C}S_4, \mathbb{C}S_3)$ is commutative.

Now, for the second part of the proof of proposition 2.15 we want to understand the commutativity of $Z(\mathbb{C}G, \mathbb{C}H)$ in terms of Gelfand pairs. This is interesting because it leads us to a well developed theory that can be extended to locally compact groups. The key is the following proposition.

**Proposition 2.18.** [CSST10, Lemma 2.1.1] The algebra of $\tilde{H}$ bi-invariant functions is isomorphic to the algebra of $H$-conjugacy invariant functions, $\mathcal{C}(G, H) := \{f : G \to \mathbb{C} \text{ such that } f(h^{-1}gh) = f(g), \forall g \in G, \text{ and } \forall h \in H\}$.

*Proof.* Consider the map $\Phi : L(\tilde{H}/(G \times H)\backslash\tilde{H}) \to \mathcal{C}(G, H)$ given by $\Phi(F)(g) = F(g, 1_G)$ where $1_G$ is the neutral element of $G$. For any $h \in H$ we have that $F(hgh^{-1}, 1_G) = F(hgh^{-1}, h1_Gh^{-1}) = F(g, 1_G)$ since $F$ is $\tilde{H}$ bi-invariant. Then, $\Phi$ is well defined and is clearly an algebra morphism. Letting $f := \Phi(F)$ note that for any $g \in G, h \in H$ by bi-invariance we have that

$$F(g, h) = F(gh^{-1}, 1_G) = f(gh^{-1}).$$

Which means that $f$ is uniquely determined by $F$. Also, for any $f \in \mathcal{C}(G,H)$ taking $F(g,h) = f(gh^{-1})$ then for any $h_1, h_2 \in H$ by $H$-conjugacy invariance we have that

$$F(h_1gh_2, h_1hh_2) = f(h_1gh_2(h_1hh_2)^{-1}) = f(h_1gh_2h_2^{-1}h^{-1}h_1^{-1}) = f(h_1gh^{-2}h_1) = f(gh^{-1}).$$

This guarantees that $\Phi$ is a bijection and thus an algebra isomorphism. □

The canonical isomorphism between $L(G)$ and $\mathbb{C}G$ (proposition 1.21) gives us an isomorphism between the function algebra $\mathcal{C}(G,H)$ and the centralizer $Z(\mathbb{C}G, \mathbb{C}H)$. And we obtain then a complete proof of proposition 2.15.

We would like to find some property of the group structure that help us verify one of the conditions in proposition 2.15. The following result comes from classical theory of Gelfand pairs and it hints us to such a simpler characterization.

**Proposition 2.19** (Gelfand's lemma)**.** Let $G$ be a finite group and $H \leq G$. If there exists an involutive automorphism $\theta$ of $G$ such that $\theta(g) \in Hg^{-1}H$ for all $g \in G$ then $(G,H)$ is a Gelfand pair.

To apply this in our context we are going to introduce a little of the basics of involutive algebras.

**Definition 2.20.** Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We say that an $F$-algebra is *involutive* if we can equip it with a conjugate anti-linear automorphism of order 2. Specifically, this means that there is a map $* : A \to A$ such that for all $x, y \in A$ and $\alpha \in F$,

$$(x+y)^* = x^* + y^*, \quad (xy) = y^*x^*, \quad (\alpha x)^* = \overline{\alpha}x^*, \quad \text{and} \quad (x^*)^* = x.$$

We call the element $x^*$ the *adjoint* of $x$. Additionally, we say that $x \in A$ is *normal* if it commutes with its adjoint, $xx^* = x^*x$ and that it is *self-adjoint* if $x = x^*$. Formally, involutive algebras are the pair $(A,*)$ and thus we usually refer to them as $*$-algebras over $\mathbb{F}$.

As we consider a $*$-algebra over $\mathbb{R}$ we can build a complex involutive algebra very naturally.

**Definition 2.21.** If $A$ is a $*$-algebra over $\mathbb{R}$ we define its *$*$-complexification* as the algebra $A \times A$ with naturally defined operations. Let us write elements $(x,y) \in A \times A$ as "complex elements" by denoting them $x + iy$. Then, the algebra operations are analogous to the operations in $\mathbb{C}$ and the map $*$ is given by $(x+iy)^* = x^* - iy^*$. Also, we say that $x$ in the $*$-complexification of $A$ is *real* if it has the form $x + i0$.

These definitions will allow us to give some conditions for determining the commutativity of an involutive algebra.

**Proposition 2.22.** [MM16, Th. 2.21] Let $A$ be a $*$-algebra over $\mathbb{C}$.

1. $x \in A$ is normal if and only if $x = y + iz$ where $y, z \in A$ are self-adjoint and commute.

2. $A$ is commutative if and only if it is normal, i.e. all elements in $A$ are normal.

3. Let $A$ be a $*$-complexification of a real involutive algebra. If every real element of $A$ is self-adjoint then $A$ is commutative.

*Proof.* 1. "$\Leftarrow$": Let $x = y + iz$ where $y, z$ are self-adjoint and commute. Then, $x^* = (y+iz)^* = y^* - iz^* = y - iz$. On one side, we have that $xx^* = (y+iz)(y-iz) = y^2 - iyz + izy + z^2 = y^2 + z^2$ and on the other hand, $x^*x = (y-iz)(y+iz) = y^2 + iyz - izy + z^2 = y^2 + z^2$. Thus, $x$ is normal.

" $\Rightarrow$": Let $x \in A$ be normal, i.e. $xx^* = x^*x$. Define $y, z \in A$ as $y = \frac{x+x^*}{2}$ and $z = \frac{i(x^*-x)}{2}$. Then, we have that

$$\begin{aligned}
y + iz &= \frac{x+x^*}{2} + i\frac{i(x^*-x)}{2} = \frac{x+x^*-x^*+x}{2} = x \\
y^* &= \left(\frac{x+x^*}{2}\right)^* = \frac{x^*+x}{2} = y \\
z^* &= \left(\frac{i(x^*-x)}{2}\right)^* = \frac{i(-x+x*)}{2} = z \\
yz &= \frac{x+x^*}{2}\frac{i(x^*-x)}{2} = \frac{i(xx^*-x^2+(x^*)^2-x^*x)}{4} = \frac{i((x^*)^2-x^2)}{4} \\
zy &= \frac{i(x^*-x)}{2}\frac{x+x^*}{2} = \frac{i(x^*x+(x^*)^2-x^2-xx^*)}{4} = \frac{i((x^*)^2-x^2)}{4}.
\end{aligned}$$

2. " $\Rightarrow$": If $A$ is commutative in particular every element commutes with its adjoint.
   " $\Leftarrow$": Let $A$ be normal. First, let us see that any two self-adjoint elements of $A$ commute. Let $a, b \in A$ be self-adjoint. By hypothesis we know that $a + ib \in A$ is normal. Then,

$$\begin{aligned}
&(a+ib)(a+ib)^* = (a+ib)^*(a+ib) \\
&\Rightarrow (a+ib)(a-ib) = (a-ib)(a+ib) \\
&\Rightarrow (a^2+b^2) + i(ba-ab) = (a^2+b^2) + i(ab-ba) \\
&\Rightarrow ba - ab = ab - ba \\
&\Rightarrow 2ba = 2ba \Rightarrow ba = ab.
\end{aligned}$$

   Let us consider $x_1, x_2 \in A$. Then, by 1. we have that for $i = 1, 2$ $x_i = y_i + iz_i$ where $y_i$ and $z_i$ are self-adjoint and commute. Note that by the previous argument we have that $y_1, y_2, z_1$ and $z_2$ all commute with each other. Thus,

$$\begin{aligned}
x_1x_2 &= (y_1+iz_1)(y_2+iz_2) = y_1y_2 - z_1z_2 + i(z_1y_2 + y_1z_2) \\
&= y_2y_1 - z_2z_1 + i(z_2y_1 + y_2z_1) = (y_2+iz_2)(y_1+iz_2) \\
&= x_2x_1.
\end{aligned}$$

3. Let $A$ be a $*$-complexification of a real involutive algebra where every real element is self-adjoint. By construction, every $x \in A$ is of the form $x = y + iz$ where $y$ and $z$ are real. Note that $yz \in A$ is also real. Then, $yz = (yz)^* = z^*y^* = zy$. By 1. we have that $x$ is normal. And, since $x \in A$ was arbitrary by 2. we have that $A$ is commutative.

□

This result interests us because group algebras are involutive algebras in a very natural way.

**Remark 2.23.** For any finite group $G$, the group algebra $\mathbb{F}G$ is involutive with the map

$$\left(\sum_{g\in G} a_g e_g\right)^* = \sum_{g\in G} \overline{a_g} e_{g^{-1}}.$$

Then, every subalgebra of $\mathbb{F}G$ is a $*$-algebra over $\mathbb{F}$. In particular, this will allow us to use proposition 2.22 to determine the commutativity of $Z(\mathbb{C}S_n, \mathbb{C}S_{n-1})$.

**Lemma 2.24.** Let $G$ be a group and $H \leq G$. $Z(\mathbb{C}G, \mathbb{C}H)$ is the $*$-complexification of $Z(\mathbb{R}G, \mathbb{R}H)$ where the map $*$ is the one defined in remark 2.23.

*Proof.* Each $f \in \mathbb{C}G$ can be written in a unique way as $f = \sum_{g\in G} \alpha_g e_g$ with $\alpha_g \in \mathbb{C}$. And, every $\alpha_g$ is uniquely written as $a_g + ib_g$ where $a_g, b_g \in \mathbb{R}$. Thus, we can decompose $f$ uniquely in its "real" and "imaginary" components: $f = a+ib$ with $a,b \in \mathbb{R}G$. Note that $f = a+ib \in Z(\mathbb{C}G, \mathbb{C}H)$ if and only if it commutes with the basis of $\mathbb{C}H$, $\{e_h : h \in H\}$. That is, for all $h \in H$, $e_h f = e_h a + ie_h b = ae_h + ibe_h = f e_h$. By uniqueness of the decomposition of the elements in $\mathbb{C}G$ we obtain that $f = a+ib \in Z(\mathbb{C}G, \mathbb{C}H)$ if and only if for all $h \in H$, $e_h a = ae_h$ and $e_h b = be_h$. Which is equivalent to, $a, b \in Z(\mathbb{R}G, \mathbb{R}H)$. Moreover, the involution operation of $Z(\mathbb{C}G, \mathbb{C}H)$ coincides with the complexified operation from $Z(\mathbb{R}G, \mathbb{R}H)$. Indeed, let $\alpha_g = a_g + ib_g \in \mathbb{C}$,

$$\left(\sum_{g\in G} \alpha_g e_g\right)^* = \left(\sum_{g\in G} (a_g + ib_g) e_g\right)^* = \sum_{g\in G} (a_g - ib_g) e_{g^{-1}} = \sum_{g\in G} \overline{\alpha_g} e_{g^{-1}}.$$

□

Inspired by Gelfand's lemma we can consider the special case when the involution $\theta$ is the identity (these are called symmetric Gelfand pairs). This gives us a way to address the question of being a multiplicity-free subgroup by checking directly the group structure as we wanted.

**Theorem 2.25.** Let $G$ be a finite group and $H \leq G$. If for any $g \in G$ there exists $h \in H$ such that $g^{-1} = hgh^{-1}$ then $H \leq G$ is a multiplicity-free subgroup.

*Proof.* By proposition 2.22 and lemma 2.24 it suffices to see that every real element of $Z(\mathbb{C}G, \mathbb{C}H)$ is self-adjoint. Let $f = \sum_{g\in G} a_g e_g \in Z(\mathbb{C}G, \mathbb{C}H)$ such that $a_g \in \mathbb{R}$. Then,

$$f^* = \left(\sum_{g\in G} a_g e_g\right)^* = \sum_{g\in G} a_g e_{g^{-1}}.$$

Let us fix a $l \in G$. We have that $l^{-1} = hlh^{-1}$ for some $h \in H$. Since $f$ is in the centralizer we have

$$e_h f = f e_h \Leftrightarrow f = e_{h^{-1}} f e_h \Leftrightarrow \sum_{g\in G} a_g e_{h^{-1}gh} = \sum_{g\in G} a_g e_g \Leftrightarrow \sum_{g\in G} a_{hgh^{-1}} e_g = \sum_{g\in G} a_g e_g \Leftrightarrow \forall g \in G : a_{hgh^{-1}} = a_g$$

and in particular for $l$ we have that $a_{l^{-1}} = a_l$. Since $l$ was an arbitrary element of $G$ we have that for all $g \in G : a_g = a_{g^{-1}}$. Thus,

$$f^* = \sum_{g\in G} a_g e_{g^{-1}} = \sum_{g\in G} a_{g^{-1}} e_{g^{-1}} = \sum_{g\in G} a_g e_g = f.$$

□

With this we can prove the result that motivates this whole chapter. That is, that the symmetric groups and hyperoctahedral groups are multiplicity-free chains.

**Lemma 2.26.** [VO05, Lemma 2.2] Let $\sigma \in S_n$. Then, there exists $\tau \in S_{n-1}$ such that $\sigma^{-1} = \tau\sigma\tau^{-1}$.

*Proof.* Let us recall a couple of basic facts about the symmetric group that will be used in this proof. First, every permutation in $S_n$ can be written as the product of disjoint cycles. Second, let $\sigma, \pi \in S_n$. Then, $\pi\sigma\pi^{-1}$ corresponds to the permutation obtained by changing $\{1, \ldots, n\}$ to

$\{\pi(1),\ldots,\pi(n)\}$ in a disjoint cycle notation of $\sigma$.

Let $\sigma \in S_n$. We are going to write this permutation in the following disjoint cycle notation

$$\sigma = c_1 c_2 \cdots c_k = (i_1,\cdots i_{m_1})(i_{m_1+1};\cdots i_{m_2})\cdots(i_{m_{k-1}+1};\cdots i_n).$$

Note that the inverse of an $l$-cycle $c = (a_1,\cdots a_{l-1},a_l)$ is the $l$-cycle $(a_{l-1},a_{l-2},\cdots a_1,a_l)$. Thus, we can obtain $c^{-1}$ conjugating $c$ by $\pi = (a_1,a_{l-1})(a_2,a_{l-2})\ldots(a_l)$. It is important to mention that we chose this particular way of writing the inverse cycle because $\pi$ has $a_l$ as a fixed point. Now, for each $c_i$ in the disjoint cycle notation of $\sigma$ we denote $\tau_i$ the permutation such that $c_i^{-1} = \tau_i c_i \tau_i^{-1}$ as constructed above. Because the cycles are disjoint the $\tau_i$ are also disjoint and in particular they commute, i.e. $\tau_i,\tau_i^{-1}$ and $c_i$ commute with all the $c_j,\tau_j$ and $\tau_j^{-1}$ such that $j \neq i$. Then, denoting $\tau = \tau_1\cdots\tau_k$ we get

$$\sigma^{-1} = c_1^{-1}\cdots c_k^{-1} = \tau_1 c_1 \tau_1^{-1}\cdots\tau_k c_k \tau_k^{-1} = \tau_1\cdots\tau_k c_1\cdots c_k \tau_k^{-1}\cdots\tau_1^{-1} = \tau\sigma\tau^{-1}.$$

Finally, without loss of generality, we can assume that $i_n = n$, it suffices to rearrange the cycles such that the cycle that contains $n$ is last and write that cycle with $n$ as its last element. Thus, by construction $n$ is a fixed point of $\tau$. That is, $\tau \in S_{n-1}$ as desired. □

**Lemma 2.27.** Let $\sigma \in B_n$. Then, there exists $\nu \in B_{n-1}$ such that $\sigma^{-1} = \nu\sigma\nu^{-1}$.

*Proof.* Note that the inverse of a positive $l$-cycle $c = (a_1,\ldots,a_l)(-a_1,\ldots,-a_l)$ is the positive $l$-cycle $c^{-1} = (a_{l-1};a_{l-2},\ldots,a_1,a_l)(-a_{l-1};-a_{l-2},\ldots,-a_1,-a_l)$. Then, by proposition 1.14 we have that $c^{-1} = \pi c \pi^{-1}$ where $\pi = (a_1,a_{l-1})(-a_1,-a_{l-1})(a_2,a_{l-2})(-a_2,-a_{l-2})\cdots(a_l)(-a_l)$ which has $\pm a_l$ as fixed points. For the case of negative cycles we will have to work a little more. Consider $\theta = (a_1,\ldots,a_l,-a_1,\ldots,-a_l)$. Its inverse is the negative $l$-cycle $\theta^{-1} = (a_l,a_{l-1},\ldots,a_1,-a_l,\ldots,-a_1)$. However, we are going to write it as $\theta^{-1} = (-a_{l-1};-a_{l-2},\ldots,-a_1,a_l,\ldots,a_1,-a_l)$. By considering these as elements of $S_{\overline{n}}$ we can consider the following permutation $\tau$ such that $\theta^{-1} = \tau\theta\tau^{-1}$.

$$\begin{array}{cccccccccc} a_1 & a_2 & \cdots & a_{l-1} & a_l & -a_1 & -a_2 & \cdots & -a_{l-1} & -a_l \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ -a_{l-1} & -a_{l-2} & \cdots & -a_1 & a_l & a_{l-1} & a_{l-2} & \cdots & a_1 & -a_l \end{array}$$

We can explicitly write it as $\tau = (a_1,-a_{l-1})(a_2,-a_{l-2})\cdots(-a_2,a_{l-2})(a_{l-1};-a_1)(a_l)(-a_l)$. Since the transpositions are disjoint we can rewrite it as $\tau = (a_1,-a_{l-1})(a_{l-1};-a_1)(a_2,-a_{l-2})(-a_2,a_{l-2})\cdots(a_l)(-a_l)$. And, since $(a_i;-a_j)(a_j;-a_i) \in B_n$ we obtain that $\tau \in B_n$ and has $\pm a_l$ as fixed points.

Going back to our statement let us write sigma in its cycle decomposition $\sigma = c_1\overline{c_1}\cdots c_m\overline{c_m}\theta_1\cdots\theta_s$. For each cycle $c_i$ or $\theta_j$ let us consider the signed permutations $\pi_i$ and $\tau_j$ as constructed above and let $\nu = \pi_1\cdots\pi_m\tau_1\cdots\tau_s$. As in the case of $S_n$ in lemma 2.26 we have that

$$\begin{aligned}\sigma^{-1} = (c_1\overline{c_1})^{-1}\cdots(c_m\overline{c_m})^{-1}\theta_1^{-1}\cdots\theta_s^{-1} &= \pi_1 c_1\overline{c_1}\pi_1^{-1}\cdots\pi_m c_m\overline{c_m}\pi_m^{-1}\tau_1\theta_1\tau_1^{-1}\cdots\tau_s\theta_s\tau_s^{-1} \\ &= \pi_1^{-1}\cdots\pi_m\tau_1\cdots\tau_s c_1\overline{c_1}\cdots c_m\overline{c_m}\theta_1\cdots\theta_s\tau_s^{-1}\cdots\tau_1^{-1}\pi_m^{-1}\cdots\pi_1^{-1} = \nu\sigma\nu^{-1}.\end{aligned}$$

We can also assume without loss of generality that $n$ is the "$a_l$" element of the cycle in which it appears (we just write it as the last element in the cycle). Then $\pm n$ are fixed point of the corresponding $\pi_i$ or $\tau_j$ which guarantees that $\nu \in B_{n-1}$. □

# Chapter 3: Vershik-Okounkov approach for $S_n$

In this chapter we will see the construction of the irreducible representations of the symmetric groups $S_n$ [VO05, CSST10, MM16]. We will first introduce a generating set of the GZ-algebra that was originally presented by Young but that can be exploited even further. Second, we will see the correspondence between this approach and the classical one. And, finally, we will prove some Young's formulas regarding irreducible representations of $S_n$ that in our context become fundamental.

## 3.1 YJM Elements

In this section we are going to introduce a very special generating set of the GZ-algebra for symmetric groups. This more detailed description will also allow us to understand better the relation between the centralizer $Z(\mathbb{C}S_n, \mathbb{C}S_{n-1})$ and the GZ-algebra.

**Definition 3.1.** Denote $X_i \in \mathbb{C}S_n$ the element $X_i = e_{(1,i)} + e_{(2,i)} + \ldots + e_{(i-1,i)}$ where $i = 1, \ldots, n$. By convention we take $X_1 = 0$. These will be called the *Young-Jucys-Murphy elements* (YJM elements) of the group algebra $\mathbb{C}S_n$.

In order to prove that these elements generate the GZ-algebra for $S_n$, i.e. $GZ(n)$, we will have to look at some classical results about $Z(n) := Z(S_n)$.

**Proposition 3.2.** [Lan02, Ch. 18 Prop. 4.1] For $n \in \mathbb{N}_{\geq 2}$ and $k = 1, \ldots, n$ denote $Y_k^n$ the sum of all $e_c \in \mathbb{C}S_n$ where $c \in S_n$ is a $k$-cycle. Then, $Z(n) = \langle Y_1^n, \ldots, Y_n^n \rangle$.

*Proof.* " $\supseteq$": It suffices to see that $Y_k^n \in Z(n)$. Let $\sigma \in S_n$ be any permutation. Then, for any $k$-cycle $c \in S_n$ there exists a unique $k$-cycle $c' \in S_n$ such that $\sigma c = c' \sigma$. Indeed if $c = (a_1, \cdots a_k)$ then $c' = (\sigma(a_1), \cdots \sigma(a_k))$. Thus, for all $\sigma \in S_n$, $e_\sigma Y_k^n = Y_k^n e_\sigma$.

" $\subseteq$": Let $f = \sum_{\sigma \in S_n} a_\sigma e_\sigma \in Z(n)$. Then for all $\tau \in S_n$ we have that

$$e_\tau f e_{\tau^{-1}} = f \quad \Leftrightarrow \quad \sum_{\sigma \in S_n} a_\sigma e_{\tau\sigma\tau^{-1}} = \sum_{\sigma \in S_n} a_\sigma e_\sigma \quad \Leftrightarrow \quad a_{\tau\sigma\tau^{-1}} = a_\sigma \quad \Rightarrow \quad f = \sum_{C \in \mathcal{C}(S_n)} a_C \sum_{\sigma \in C} e_\sigma$$

where $\mathcal{C}(S_n)$ is the set of conjugacy classes of $S_n$. This implies that $\{\sum_{\sigma \in C} e_\sigma : C \in \mathcal{C}(S_n)\}$ is a linear generating set of $Z(n)$. Now, recall that the conjugacy classes of $S_n$ are in bijection with the partitions of $n$: $P(n) = \{\lambda = (\lambda_1, \ldots, \lambda_m) : 1 \leq \lambda_1 \leq \ldots \leq \lambda_m \leq n \text{ and } \lambda_1 + \ldots + \lambda_m = n\}$. Specifically, a partition $\lambda$ corresponds with the type of decomposition in disjoint cycles of the permutations in a conjugacy class $C_\lambda$. That is, $\sigma \in C_\lambda$ if and only if the decomposition in disjoint cycles of $\sigma$ is of the form $c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_m}$ where each $c_{\lambda_i}$ is a $\lambda_i$-cycle. Then, to see that $Z(n) \subseteq \langle Y_1^n, \ldots, Y_n^n \rangle$ it suffices to check that for all $\lambda \in P(n)$, $P_\lambda := \sum_{\sigma \in C_\lambda} e_\sigma \in \langle Y_1^n, \ldots, Y_n^n \rangle$. In order to prove this we are going to do an induction on $k_\lambda$ the number of elements in $\{1, \ldots, n\}$ that all permutations in a conjugacy class $C_\lambda$ moves. Note that we can consider this number because for any $\lambda_i$-cycle either $\lambda_i = 1$ then $c_{\lambda_i} = id$ and does not move any elements or $\lambda_i > 1$ and we have that $c_{\lambda_i}$ moves exactly $\lambda_i$ elements. Thus, we obtain that if $\lambda_{i_1}, \ldots, \lambda_{i_s}$ are the $\lambda_j > 1$ in the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ then every $\sigma \in C_\lambda$ moves exactly $\lambda_{i_1} + \ldots + \lambda_{i_s}$ elements.
Proceeding with the induction, we have that for $k_\lambda = 0, 1$ the result is trivial because $P_\lambda = e_{id}$. Now assume that for all conjugacy classes $C_\mu$ that move $k_\mu < k_\lambda$ elements we have that $P_\mu \in \langle Y_1^n, \ldots, Y_n^n \rangle$. Let $C_\lambda$ be a conjugacy class that moves $k$ elements with $\lambda = (\lambda_1, \ldots, \lambda_m)$. Consider the product $Y_{\lambda_1}^n \cdots Y_{\lambda_m}^n$. For each summand $e_\rho$ in the product we have two cases. Either $\rho$ is a product of disjoint cycles or it is not. In the first case we have that $\rho \in C_\lambda$. And, in the second case we can

write $\rho = (a_1^1, \cdots a_{\lambda_1}^1) \cdots (a_1^m; \cdots a_{\lambda_m}^m)$ where for at least one pair $a_i^k, a_j^l$ with $i \neq j$ we have that $a_i^k = a_j^l$. Then, $\rho$ moves at most $k_\lambda - 1$ elements in $\{1, \dots, n\}$, i.e. $\rho \in C_\mu$ where $k_\mu < k_\lambda$. Furthermore, if $e_\rho$ is a summand of $Y_{\lambda_1}^n \cdots Y_{\lambda_m}^n$, i.e. $\rho = c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_m}$ with $c_{\lambda_i}$ a $\lambda_i$-cycle, then for all $\tau \in S_n$ we can write $\tau \rho \tau^{-1} = \tau c_{\lambda_1} \tau^{-1} \tau c_{\lambda_2} \tau^{-1} \cdots \tau c_{\lambda_m} \tau^{-1}$. Since for $i = 1, \dots, m$ we have that $\tau c_{\lambda_1} \tau^{-1}$ is also a $\lambda_i$-cycle we obtain that $e_{\tau\rho\tau^{-1}}$ also appears as a summand in the product $Y_{\lambda_1}^n \cdots Y_{\lambda_m}^n$. Thus,

$$Y_{\lambda_1}^n \cdots Y_{\lambda_m}^n = a_\lambda P_\lambda + \sum_{\substack{\mu \in P(n) \\ k_\mu < k_\lambda}} a_\mu P_\mu \quad \Rightarrow \quad a_\lambda P_\lambda = Y_{\lambda_1}^n \cdots Y_{\lambda_m}^n - \sum_{\substack{\mu \in P(n) \\ k_\mu < k_\lambda}} a_\mu P_\mu \in \langle Y_1^n, \dots, Y_n^n \rangle.$$

□

**Remark 3.3.** Recall the definition of the GZ-algebra in 2.7. For $i = 2, \dots, n$ we have that $X_i \in GZ(n)$ since another way to write the YJM elements is $X_i = Y_2^i - Y_2^{i-1}$. Which implies that $X_i$ is the difference between an element in $Z(i)$ and an element in $Z(i-1)$.

**Example 3.4.** For $S_3$ we have that the YJM elements are $X_1 = 0, X_2 = e_{(12)}$ and $X_3 = e_{(13)} + e_{(23)}$. Let us see that the subalgebra of $\mathbb{C}S_3$ generated by these elements is the GZ-algebra of $S_3$. Recall that a basis for this algebra was given in example 2.12. Note that

$$\begin{aligned}
X_2^2 + X_2 + X_3 + X_2X_3 &= e_{id} + e_{(12)} + e_{(13)} + e_{(23)} + e_{(123)} + e_{(132)} = \sum_{\sigma \in S_3} e_\sigma = 6F_{T_1} \\
2X_2^2 - X_2X_3 + 2X_2 - X_3 &= 2e_{id} - e_{(123)} - e_{(132)} + 2e_{(12)} - e_{(13)} - e_{(23)} = 12F_{T_2^1} \\
2X_2^2 - X_2X_3 - 2X_2 + X_3 &= 2e_{id} - e_{(123)} - e_{(132)} - 2e_{(12)} + e_{(13)} + e_{(23)} = 12F_{T_2^2} \\
X_2^2 - X_2 - X_3 + X_2X_3 &= e_{id} - e_{(12)} - e_{(13)} - e_{(23)} + e_{(123)} + e_{(132)} = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) e_\sigma = 6F_{T_3}.
\end{aligned}$$

Which allows us to conclude that $\langle X_1, X_2, X_3 \rangle = GZ(3)$.

**Lemma 3.5.** For any $n \in \mathbb{N}_{\geq 2}$ and $k \leq n$ we have that $Y_k^n \in \langle Z(n-1), X_n \rangle$ where $Y_k^n$ are defined as in proposition 3.2.

*Proof.* To simplify our notation denote $C_k^n$ the sum of all $e_c$ where $c \in S_n$ is a $k$-cycle containing $n$. Note that $Y_k^n = Y_k^{n-1} + C_k^n$. By proposition 3.2 we know that $Y_k^{n-1} \in Z(n-1)$. So, it suffices to see that $C_k^n \in \langle Z(n-1), X_n \rangle$. We will proceed by induction on $k$. For $k = 2$ we have $C_2^n = X_n \in \langle Z(n-1), X_n \rangle$. Now, let us assume that $C_m^n \in \langle Z(n-1), X_n \rangle$ for all $m < k$. Consider the product $X_n C_{k-1}^n$. Note that this product is the sum of all possible $e_{(j,n)c}$ where $j = 1, \dots, n-1$ and $c \in S_n$ is a $(k-1)$-cycle containing $n$. Without loss of generality, let us write $c = (i_1, \cdots i_{k-2}, n)$. To understand better the product $(j,n)c$ let us consider two cases.

1. If we have $j \neq i_s$ for all $s = 1, \dots, k-2$ then $(j,n)(i_1, \cdots i_{k-2}, n) = (i_1, \cdots i_{k-2}, j, n)$. Note that this could be any $k$-cycle containing $n$.

2. Else, we must have that $j = i_s$ for one specific $s \in \{1, \dots, k-2\}$. If $s = 1$ we have $(j,n)c = (i_1, n)(i_1, \cdots i_{k-2} n) = (i_1, \cdots i_{k-2})$ and if $s \geq 2$ we get

$$c = \quad n \longrightarrow i_1 \longrightarrow \dots \longrightarrow i_{s-1} \longrightarrow i_s = j \longrightarrow \dots \longrightarrow i_{k-2} \longrightarrow n$$

$$(j,n)c = \quad \underbrace{n \longrightarrow i_1 \longrightarrow \dots \longrightarrow i_{s-1}}_{i_{s-1} \to n} \qquad \underbrace{i_s = j \longrightarrow \dots \longrightarrow i_{k-2}}_{i_{k-2} \to j}.$$

That is, $(i_s, n)c = (i_s, \cdots i_{k-2})(i_1, \cdots i_{s-1}, n)$. Note that $(i_s, \cdots i_{k-2})$ can be any $(k-s-1)$-cycle that does not contain $n$, i.e. $(i_s, \cdots i_{k-2}) \in S_{n-1}$. And $(i_1, \cdots i_{s-1}, n)$ can be any $s$-cycle containing $n$.

Thus, we have that $X_n C^n_{k-1} = C^n_k + Y^{n-1}_{k-s-1} C^n_s$. Because $Y^{n-1}_{k-s-1} \in Z(n-1)$ and by the induction hypotheses ($C^n_{k-1}, C^n_s \in \langle Z(n-1), X_n \rangle$) we obtain that $C^n_k = X_n C^n_{k-1} - Y^{n-1}_{k-s-1} C^n_s \in \langle Z(n-1), X_n \rangle$. Which allows us to conclude the proposition. ☐

**Corollary 3.6.** Seeing $Z(n-1)$ as a subalgebra of $\mathbb{C}S_n$ we have that $Z(n) \subset \langle Z(n-1), X_n \rangle$.

*Proof.* By proposition 3.2 we have that $Z(n) = \langle Y^n_1, \ldots, Y^n_n \rangle$. By lemma 3.5 we have that each $Y^n_k \in \langle Z(n-1), X_n \rangle$. Thus, $Z(n) \subset \langle Z(n-1), X_n \rangle$. ☐

**Theorem 3.7.** For all $n \in \mathbb{N}_{\geq 2}$, $\langle X_1, \ldots, X_n \rangle = GZ(n)$. That is, the YJM elements generate (as an algebra) the GZ-algebra of $S_n$ defined in 2.7.

*Proof.* We will proceed by induction on $n$. For $n = 2$ we have that $\mathbb{C}S_2$ is commutative since $S_2$ is. Thus, $Z(2) = \mathbb{C}S_2$ and $GZ(2) = \mathbb{C}S_2 = \text{span}\{e_{id}, e_{(12)}\}$. And, $\langle X_1, X_2 \rangle = \text{span}\{X_2, X_2^2\} = GZ(2)$. Now, let us assume that $GZ(n-1) = \langle X_1, \ldots, X_{n-1} \rangle$. Clearly, $\langle GZ(n-1), X_n \rangle \subseteq GZ(n)$ since the YJM elements are contained in the GZ-algebra. To see the other inclusion observe that by definition $GZ(n) = \langle Z(1), \ldots, Z(n-1), Z(n) \rangle = \langle GZ(n-1), Z(n) \rangle$. And by corollary 3.6 we have $Z(n) \subset \langle Z(n-1), X_n \rangle \subset \langle GZ(n-1), X_n \rangle$. Then, $GZ(n) \subseteq \langle GZ(n-1), X_n \rangle$ which allows us to conclude the theorem. ☐

Using a similar argument to the one given in proposition 3.2 we obtain the following result that connects the YJM elements to the centralizer $Z(\mathbb{C}S_n, \mathbb{C}S_{n-1})$.

**Theorem 3.8.** For all $n \in \mathbb{N}_{\geq 2}$ we have that $Z(\mathbb{C}S_n, \mathbb{C}S_{n-1}) = \langle Z(n-1), X_n \rangle$.

*Proof.* "$\supseteq$": By definition $Z(n-1)$ commutes with $\mathbb{C}S_{n-1}$. Then, it suffices to see that $X_n$ commutes with $e_\sigma$ for all $\sigma \in S_{n-1}$. Note that for all $i = 1, \ldots, n-1$ we have that $(i, n)\sigma = \sigma(\sigma^{-1}(i), n)$. Thus,

$$X_n e_\sigma = \sum_{i=1}^{n-1} e_{(i,n)\sigma} = \sum_{i=1}^{n-1} e_{\sigma(\sigma^{-1}(i),n)} = \sum_{i=1}^{n-1} e_{\sigma(i,n)} = e_\sigma X_n.$$

"$\subseteq$": Note that a linear generating set of $Z(\mathbb{C}S_n, \mathbb{C}_{n-1})$ is $\{\sum_{\sigma \in C} e_\sigma$ : where $C$ is an orbit of the conjugation action $S_{n-1} \curvearrowright S_n\}$. We can split this into a generating set for $Z(n-1)$, corresponding to the orbits of elements in $S_{n-1} \subset S_n$, and another component corresponding to the orbits of elements in $S_n \setminus S_{n-1}$. It suffices to see that for all $\pi \in S_n \setminus S_{n-1}$ we have that $P_\pi := \sum_{\sigma \in \mathcal{O}(\pi)} e_\sigma \in \langle Z(n-1), X_n \rangle$. Writing $\pi$ as disjoint non trivial cycles $(a^1_1, \cdots, a^1_{l_1-1}, n)(a^2_1, \cdots, a^2_{l_2}) \cdots (a^k_1, \cdots, a^k_{l_k})$, we have that for $\tau \in S_{n-1}$ : $\tau\pi\tau^{-1} = (\tau(a^1_1), \cdots, \tau(a^1_{l_1-1}), n)(\tau(a^2_1), \cdots, \tau(a^2_{l_2})), \cdots, (\tau(a^k_1), \cdots, \tau(a^k_{l_k}))$. That is, if $n$ appears in a $l_1$-cycle in $\pi$ the orbit $\mathcal{O}(\pi)$ consists of permutations with the same cycle type as $\pi$ such that $n$ also appears in a $l_1$-cycle. Thus,

$$P_\pi = \sum_{\sigma \in \mathcal{O}(\pi)} e_\sigma = \sum_{\substack{a^j_i = 1 \\ \text{pairwise distinct}}}^{n-1} e_{(a^1_1, \cdots, a^1_{l_1-1}, n) \cdots (a^k_1, \cdots, a^k_{l_k})}.$$

In an analogous way to the proof of proposition 3.2 we can consider the product $C^n_{l_1} Y^{n-1}_{l_2} \cdots Y^{n-1}_{l_k}$ and obtain that $P_\pi \in \langle Z(n-1), C^n_1, \ldots, C^n_n \rangle$. Further, as we saw in the proof of lemma 3.5, we have that $C^n_k \in \langle Z(n-1), X_n \rangle$. Then, $P_\pi \in \langle Z(n-1), X_n \rangle$ which concludes the theorem. ☐

**Remark 3.9.** Note that by theorem 3.8 $X_n \in Z(\mathbb{C}S_n, \mathbb{C}S_{n-1})$ and in particular it commutes with every element in $Z(n-1) \subset \mathbb{C}S_{n-1}$. Which implies that $Z(\mathbb{C}S_n, \mathbb{C}S_{n-1}) = \langle Z(n-1), X_n \rangle$ is commutative. This gives us another proof of theorem 2.25 and therefore a new way to prove that the group chain of symmetric groups is multiplicity-free.

In honor of A. Young, who deeply studied the representations of symmetric groups, usually the GZ-basis of an irreducible representation $\rho \in \widehat{S_n}$ is called the Young basis and we will denote it by $B_\rho$. The GZ-vectors are also called Young vectors. In what follows, we will see how the YJM elements, specially their spectrum with respect to the Young bases, are key for understanding $\widehat{S_n}$.

**Definition 3.10.** Let $v \in B_\rho$ for some $\rho \in \widehat{S_n}$. We define the *weight* of $v$ as the vector $\alpha(v) = (a_1, \ldots, a_n) \in \mathbb{C}^n$ where $X_i \cdot v = a_i v$ for $i = 1, \ldots n$. We will also denote $\mathrm{Spec}(n)$ as the set of weights of the Young bases, i.e. $\mathrm{Spec}(n) = \{\alpha(v) : v \in B_\rho,\ \rho \in \widehat{S_n}\}$.

**Remark 3.11.** Corollary 2.11 gives us that the Young basis for any $\rho \in \widehat{S_n}$ is determined, up to scalar multiplication, by its eigenvalues corresponding to the YJM elements. Explicitly, if we know $\mathrm{Spec}(n)$ and the action of the YJM elements in an irreducible representation then, calculating the GZ-vectors reduces to solving a linear system. For each $(a_1, \ldots, a_n) \in \mathrm{Spec}(n)$ we find the solution to the equations $X_i \cdot v = a_i v$.

Note that by the construction of the Young basis (see definition 2.3) we have that $\mathrm{Spec}(n)$ is in bijection with the paths in the branching graph of $S_n$. Indeed, every path $T \in \mathcal{T}(\rho)$, $\rho \in \widehat{S_n}$ has its corresponding Young vector $v_T$ and we can define the map $T \mapsto \alpha(T) := \alpha(v_T)$. Conversely, every $\alpha \in \mathrm{Spec}(n)$ determines a unique Young vector which again corresponds to a path $T_\alpha$. This allows us to consider the following definition.

**Definition 3.12.** We define the following equivalence in relation in $\mathrm{Spec}(n)$. Let $\alpha, \beta \in \mathrm{Spec}(n)$. Then, $\alpha \sim \beta$ if and only if both paths $T_\alpha$ and $T_\beta$ start in the same irreducible representation $\rho_n \in \widehat{S_n}$. Or equivalently, $v_\alpha := v_{T_\alpha}$ and $v_{T_\beta} =: v_\beta$ belong to the same irreducible $S_n$-module: $V_{\rho_n}$.

**Example 3.13.** For the irreducible representations of $S_3$ we can find Young bases as follows. Since $\mathrm{triv}_3$ and $\mathrm{sgn}_3$ are one dimensional we can choose any non-zero vectors, let us say $v_1 \in \mathrm{triv}_3$ and $v_2 \in \mathrm{sgn}_3$. For $W_3 = \mathrm{span}(1,1,1)^\perp$ we can check that $\{v_3 = (1,1,-2), v_4 = (1,-1,0)\}$ is a Young basis. It might be useful for the reader to observe that the natural basis $\{(1,0,-1),(1,-1,0)\}$ is not a GZ-basis because its restriction to $S_2$ does not correspond to an irreducible decomposition. Then,

$$
\begin{aligned}
&X_1(v_1) = 0 && X_1(v_2) = 0 \\
&X_2(v_1) = \mathrm{triv}_3(12)(v_1) = v_1 && X_2(v_2) = \mathrm{sgn}_3(12)(v_2) = -v_2 \\
&X_3(v_1) = \mathrm{triv}_3(13)(v_1) + \mathrm{triv}_3(23)(v_1) = 2v_1 && X_3(v_2) = \mathrm{sgn}_3(13)(v_2) + \mathrm{sgn}_3(23)(v_2) = -2v_2
\end{aligned}
$$

$$
\begin{aligned}
&X_1(v_3) = 0 && X_1(v_4) = 0 \\
&X_2(v_3) = W_3(12)(v_3) = v_3 && X_2(v_4) = W_3(12)(v_4) = -v_4 \\
&X_3(v_3) = W_3(13)(v_3) + W_3(23)(v_3) && X_3(v_4) = W_3(13)(v_4) + W_3(23)(v_4) \\
&= (-2,1,1) + (1,-2,1) = -v_3 && = (0,-1,1) + (1,0,-1) = v_4.
\end{aligned}
$$

For $S_4$ with a little extra work we can also find Young bases. Similarly to the previous case we consider any non-zero vectors $v_1 \in \mathrm{triv}_4$ and $v_2 \in \mathrm{sgn}_4$. Also, a Young basis of $W_4 = \mathrm{span}(1,1,1,1)^\perp$ can be checked to be $\{v_3 = (1,1,1,-3), v_4 = (1,1,-2,0), v_5 = (1,-1,0,0)\}$. From this we also obtain a Young basis for $W_4 \otimes \mathrm{sgn}_4$ given by $\{v_6 = v_3 \otimes v_2, v_7 = v_4 \otimes v_2, v_8 = v_5 \otimes v_2\}$. Finally, for the 2-dimensional representation $U$ we use the following fact explained on pages 20-21 from [FH91].

$U$ as a vector space is simply $W_3$ and the action of $S_4$ on $U$ is given by the group isomorphism $S_3 \cong S_4/\{id, (12)(34), (13)(24), (14)(23)\}$. Then, a Young basis for $U$ is simply $\{v_9 = (1,1,-2), v_{10} = (1,-1,0)\}$. For computing the eigenvalues of these vectors with respect to $X_4 = e_{(14)} + e_{(24)} + e_{(34)}$ it is important to note that in $S_3$ as a quotient of $S_4$ we have that $\overline{(14)} = \overline{(23)}, \overline{(24)} = \overline{(13)}$ and $\overline{(34)} = \overline{(12)}$. Then, we obtain the following calculations

$X_4(v_1) = 3v_1$

$X_4(v_2) = -3v_2$

$X_1(v_3) = 0, \quad X_2(v_3) = v_3$
$X_3(v_3) = 2v_3$
$X_4(v_3) = (-3,1,1,1) + (1,-3,1,1) + (1,1,-3,1)$
$= (-1,-1,-1,3) = -v_4$

$X_1(v_4) = 0, \quad X_2(v_4) = v_4$
$X_3(v_4) = -v_4$
$X_4(v_4) = (0,1,-2,1) + (1,0,-2,1) + (1,1,0,-2)$
$= (2,2,-4,0) = 2v_4$

$X_1(v_5) = 0, \quad X_2(v_5) = -v_5$
$X_3(v_5) = (0,-1,1,0) + (1,0,-1,0) = v_5$
$X_4(v_5) = (0,-1,0,1) + (1,0,0,-1) + (1,-1,0,0)$
$= (2,-2,0,0) = 2v_5$

$X_1(v_6) = 0, \quad X_2(v_6) = -v_6$
$X_3(v_6) = -2v_6, \quad X_4(v_6) = v_6$

$X_1(v_7) = 0, \quad X_2(v_7) = -v_7$
$X_3(v_7) = v_7, \quad X_4(v_7) = -2v_7$

$X_1(v_8) = 0, \quad X_2(v_8) = v_8$
$X_3(v_8) = -v_8, \quad X_4(v_8) = -2v_8$

$X_4(v_9) = \overline{(14)}(v_9) + \overline{(24)}(v_9) + \overline{(34)}(v_9)$
$= (1,-2,1) + (-2,1,1) + (1,1,2) = 0$

$X_4(v_{10}) = \overline{(14)}(v_{10}) + \overline{(24)}(v_{10}) + \overline{(34)}(v_{10})$
$= (1,0,-1) + (0,-1,1) + (-1,1,0) = 0.$

Thus, we can represent Spec(3) and Spec(4) by the following tables. Additionally, by the definition of $\sim$ the eigenvalues' vectors corresponding to the same irreducible representation are in the same equivalence class.

Table 3.1: Spec(3) and Spec(4)

| **Irrep** | **Vector** | $\mathbf{X_1}$ | $\mathbf{X_2}$ | $\mathbf{X_3}$ |
|---|---|---|---|---|
| $\text{triv}_3$ | $v_1$ | 0 | 1 | 2 |
| $\text{sgn}_3$ | $v_2$ | 0 | −1 | −2 |
| $W_3$ | $v_3$ | 0 | 1 | −1 |
| | $v_4$ | 0 | −1 | 1 |

| **Irrep** | **Vector** | $\mathbf{X_1}$ | $\mathbf{X_2}$ | $\mathbf{X_3}$ | $\mathbf{X_4}$ |
|---|---|---|---|---|---|
| $\text{triv}_4$ | $v_1$ | 0 | 1 | 2 | 3 |
| $\text{sgn}_4$ | $v_2$ | 0 | −1 | −2 | −3 |
| $W_4$ | $v_3$ | 0 | 1 | 2 | −1 |
| | $v_4$ | 0 | 1 | −1 | 2 |
| | $v_5$ | 0 | −1 | 1 | 2 |
| $W_4 \otimes \text{sgn}_4$ | $v_6$ | 0 | −1 | −2 | 1 |
| | $v_7$ | 0 | −1 | 1 | −2 |
| | $v_8$ | 0 | 1 | −1 | −2 |
| $U$ | $v_9$ | 0 | 1 | −1 | 0 |
| | $v_{10}$ | 0 | −1 | 1 | 0 |

**Remark 3.14.** Note that if $\alpha = (a_1, \dots, a_n) \in \mathrm{Spec}(n)$ then $\alpha' = (a_1, \dots, a_{n-1}) \in \mathrm{Spec}(n-1)$. More specifically, if $T_\alpha = \rho_n \to \cdots \to \rho_1$ is the path corresponding to $\alpha$ then $\alpha'$ simply corresponds to the restriction of this path to $\widehat{S_{n-1}}$. That is, $T_{\alpha'} = \rho_{n-1} \to \cdots \to \rho_1$.

Clearly, the equivalence classes of $\sim$ are in one-to-one correspondence with the Young bases $B_\rho$ for $\rho \in \widehat{S_n}$. Thus, in particular we have that $|\mathrm{Spec}(n)/\sim| = |\widehat{S_n}|$. Additionally, we can also consider a "special" representative of each equivalence class.

**Definition 3.15.** For each $C_\rho$, an equivalence class of $\sim$, we define the *highest weight vector* as the vector $v \in B_\rho$ such that $\alpha(v) \in C_\rho$ is maximal with respect to the lexicographic order.

At first it may not seem obvious that such a vector even exists since we are considering complex representations. It will become clear in the following section when we give an explicit description of the highest weight vectors.

Another important tool to understand $\mathrm{Spec}(n)$ will be the Coxeter generators of $S_n$, $\{s_i = (i, i+1) : i = 1, \dots, n-1\}$ and their relation with the YJM elements and the Young bases. Henceforth we will also denote $s_i$ to refer to the element in $\mathbb{C}S_n$ corresponding to the transposition $(i, i+1)$.

**Lemma 3.16.** For $n \in \mathbb{N}^*$ we have the following relations between the YJM elements and the Coxeter generators.

- If $i, j = 1, \dots, n$ such that $j \neq i, i+1$ then $s_i X_j = X_j s_i$.
- For any $i = 1, \dots, n-1$ we have that $s_i X_i + e_{id} = X_{i+1} s_i$ and $s_i X_{i+1} - e_{id} = X_i s_i$.

*Proof.* For $j \neq i, i+1$ we can consider two cases: either $j < i$ or $j > i+1$. In the first case note that $(i, i+1)$ and $(k, j)$ are disjoint for $1 \leq k < j$. Then, $(i, i+1)(k, j) = (k, j)(i, i+1)$. In the second case we have that $s_i$ and $(k, j)$ are not disjoint only for $k = i, i+1$. We have that $(i, i+1)(i, j) = (i, j, i+1)$, $(i, i+1)(i+1, j) = (j, i, i+1)$, $(i, j)(i, i+1) = (j, i, i+1)$ and $(i+1, j)(i, i+1) = (i, j, i+1)$. Then, $s_i(e_{(i,j)} + e_{(i+1,j)}) = e_{(i,j,i+1)} + e_{(j,i,i+1)} = (e_{(i,j)} + e_{(i+1,j)})s_i$. Thus, $s_i X_j = X_j s_i$.

For the second part, note that

$$s_i X_i s_i + s_i = \sum_{j=1}^{i-1} e_{(i,i+1)(j,i)(i,i+1)} + e_{(i,i+1)} = \sum_{j=1}^{i-1} e_{(j,i+1)} + e_{(i,i+1)} = \sum_{j=1}^{i} e_{(j,i+1)} = X_{i+1}.$$

Then, multiplying by $s_i^{-1} = s_i$ on the right we obtain $s_i X_i + e_{id} = X_{i+1} s_i$. And, multiplying by $s_i$ on the left we obtain $s_i X_{i+1} - e_{id} = X_i s_i$. □

**Lemma 3.17.** For any path $T = \rho_n \to \dots \to \rho_1$ in the branching graph of $S_n$ and any $i = 1, \dots, n-1$ we have that $s_i(v_T)$ is a linear combination of $v_{T'}$ where $T' = \rho_n \to \dots \to \rho_i' \to \dots \to \rho_1$. That is, $s_i$ only moves the $i$-level of the path $T$. Or equivalently, $s_i(v_T) \in V_{\rho_j}$ for any $j \neq i$.

*Proof.* For $j \leq i-1$ because of lemma 3.16 we have that $X_j(s_i(v_T)) = s_i(X_j(v_T)) = a_j s_i(v_T)$. Thus, $v_T$ and $s_i(v_T)$ have the same eigenvalues with respect to $X_1, \dots, X_{i-1}$. Which implies that they correspond to the same path $\rho_{i-1} \to \dots \to \rho_1$. For $j \geq i+1$ we have that $s_i \in \mathbb{C}S_j$. Then, $s_i(v_T) \in \mathbb{C}S_j \cdot v_T$. Moreover, since $V_{\rho_j}$ is an irreducible representation of $S_j$ we have that $\mathbb{C}S_j \cdot v_T = V_{\rho_j}$. That is, $s_i(v_T) \in V_{\rho_j}$ which concludes the proof. □

The following result describes completely the action of the Coxeter generators of $S_n$ over the spectrum of the YJM elements $\mathrm{Spec}(n)$. As it will become clear ahead this proposition is the key of this theory since it will allow us to construct inductively the irreducible representations of $S_n$ using the Coxeter generators.

**Proposition 3.18.** Let $\alpha = (a_1, \dots, a_i, a_{i+1} \dots, a_n) \in \mathrm{Spec}(n)$. Then,

1. for all $i = 1, \dots, n$, $a_i \neq a_{i+1}$.
2. If $a_{i+1} = a_i \pm 1$, then $s_i(v_\alpha) = \pm v_\alpha$.
3. If $a_{i+1} \neq a_i \pm 1$ then $\alpha' := s_i \cdot \alpha = (a_1, \dots, a_{i+1}, a_i, \dots, a_n) \in \mathrm{Spec}(n)$, $\alpha' \sim \alpha$ and

   $$v_{\alpha'} = s_i(v_\alpha) - \frac{1}{a_{i+1} - a_i} v_\alpha.$$

   Additionally, $\mathrm{span}\{v_\alpha, v_{\alpha'}\}$ is invariant under the action of $X_i, X_{i+1}$ and $s_i$. These operators have as matrix representation (respectively) with respect to the basis $\{v_\alpha, v_{\alpha'}\}$

   $$\begin{bmatrix} a_i & 0 \\ 0 & a_{i+1} \end{bmatrix}, \quad \begin{bmatrix} a_{i+1} & 0 \\ 0 & a_i \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \frac{1}{a_{i+1}-a_i} & 1 - \frac{1}{(a_{i+1}-a_i)^2} \\ 1 & \frac{1}{a_i - a_{i+1}} \end{bmatrix}.$$

*Proof.* Note that by definition $X_i(v_\alpha) = a_i v_\alpha$ and $X_{i+1}(v_\alpha) = a_{i+1} v_\alpha$. By lemma 3.16 we also have that $X_i s_i(v_\alpha) = (s_i X_{i+1} - e_{id})(v_\alpha) = a_{i+1} s_i(v_\alpha) - v_\alpha$ and $X_{i+1} s_i(v_\alpha) = (s_i X_i + e_{id})(v_\alpha) = a_i s_i(v_\alpha) + v_\alpha$. Thus, the vector space $V = \mathrm{span}\{v_\alpha, s_i(v_\alpha)\}$ is invariant under the action of $X_i$ and $X_{i+1}$.

1. We are going to consider two cases. First, assume that $v_\alpha$ and $s_i(v_\alpha)$ are linearly dependent. That is, there exists $\lambda \in \mathbb{C}$ such that $s_i(v_\alpha) = \lambda v_\alpha$. Then, $v_\alpha = s_i^2(v_\alpha) = \lambda^2 v_\alpha$ implies that $\lambda = \pm 1$. Since $(s_i X_i s_i + s_i)(v_\alpha) = X_{i+1}(v_\alpha)$ we obtain that $a_i v_\alpha \pm v_\alpha = a_{i+1} v_\alpha$ and $a_{i+1} = a_i \pm 1$. In particular, we have that $a_i \neq a_{i+1}$. On the other hand, let us assume that $v_\alpha$ and $s_i(v_\alpha)$ are linearly independent. Then the matrix representation of $X_i, X_{i+1}$ and $s_i$, as operators on $V$, with respect to the basis $\{v_\alpha, s_i v_\alpha\}$ are respectively

   $$\begin{bmatrix} a_i & -1 \\ 0 & a_{i+1} \end{bmatrix}, \quad \begin{bmatrix} a_{i+1} & 1 \\ 0 & a_i \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Let $T_\alpha = \rho_n \to \dots \to \rho_1$. By lemma 3.17 we have that $s_i(v_\alpha) \in V_{\rho_{i+1}}$, i.e. $V$ is a vector subspace of $V_{\rho_{i+1}}$. Now, recall that $X_i, X_{i+1} \in GZ(i+1)$. In particular they are diagonalizable as operators of $V_{\rho_{i+1}}$ and since $V \leq V_{\rho_{i+1}}$ they are also diagonalizable as operators on $V$. Basic linear algebra allows us to see that matrices of the form $\begin{bmatrix} a & \pm 1 \\ 0 & b \end{bmatrix}$ with $a, b \in \mathbb{C}$ are diagonalizable if and only if $a \neq b$. Indeed, the eigenvalue(s) of such matrices are $a$ and $b$. If $a = b$ we have that $\dim\ker\left(\begin{bmatrix} a & \pm 1 \\ 0 & a \end{bmatrix} - aId_2\right) = 1$, which means $\begin{bmatrix} a & \pm 1 \\ 0 & a \end{bmatrix}$ is not diagonalizable. Thus, we must have that $a_i \neq a_{i+1}$.

2. Suppose that $a_{i+1} = a_i \pm 1$. By the previous part it suffices to see that $v_\alpha$ and $s_i(v_\alpha)$ are linearly dependent. Suppose by contradiction that they are not. We are going to consider now the algebra $H(2) := \langle X_i, X_{i+1}, s_i \rangle \leq \mathbb{C}S_n$. Since $V = \mathrm{span}\{v_\alpha, s_i(v_\alpha)\}$ is invariant under the action of its generators we have that $V$ has a structure of $H(2)$-module. Here the module operation naturally comes given by the actions of $X_i, X_{i+1}$ and $s_i$. Note that we can consider $\mathbb{C}S_n$ (and therefore any of its subalgebras) as a subalgebra of $M_{n! \times n!}(\mathbb{C})$ through the regular representation of $S_n$. Specifically, to each element $e_\sigma$ in the basis $B = \{e_\sigma : \sigma \in \sigma\}$ we associate the matrix representation (with respect to $B$) of the linear operator given by $e_\tau \mapsto e_{\sigma\tau}$ for all $\tau \in S_n$. We will denote such matrix as $P^\sigma$ and we will index its rows and columns by

elements of $S_n$. By definition, $P^\sigma$ is symmetric if and only if for all $\pi, \tau \in S_n$ we have that

$$P^\sigma_{\pi,\tau} = P^\sigma_{\tau,\pi} \Leftrightarrow \begin{cases} 1, \text{ if } \sigma\tau = \pi \\ 0, \text{ otherwise} \end{cases} = \begin{cases} 1, \text{ if } \sigma\pi = \tau \\ 0, \text{ otherwise} \end{cases} = \begin{cases} 1, \text{ if } \sigma^{-1}\tau = \pi \\ 0, \text{ otherwise} \end{cases} \Leftrightarrow \sigma = \sigma^{-1}.$$

Thus, in particular since $X_i, X_{i+1}$ and $s_i$ correspond to (integer) sums of transpositions we have that their corresponding matrices in $M_{n!\times n!}(\mathbb{C})$ are symmetric and real. In other words, they are self-adjoint.

**Claim.** Let $A$ be a self-adjoint subalgebra of a matrix algebra. Then, $A$ is semisimple.

*Proof.* We recall that an algebra $A$ is semisimple if and only if every $A$-module is completely reducible, i.e. it can be written as a direct sum of simple $A$-modules. Let $V$ be an $A$-module. We recall that since $A$ is a matrix algebra in particular we can endow it with an inner-product (for example Frobenius inner-product) such that the classical adjoint matrix (conjugate transpose) and the adjoint operator with respect to the inner product coincide. Let $W \subseteq V$ be an $A$-submodule. We can consider the vector space $W^\perp$ and we want to check that it is an $A$-submodule. Let $\alpha \in A$ and $w' \in W^\perp$. Then, for any $w \in W$ we have that $\langle \alpha w', w\rangle = \langle w', \alpha^* w\rangle$. Since $A$ is self-adjoint and $W$ is an $A$-submodule we have that $\alpha^* w \in W$. Then, $\langle \alpha w', w\rangle = 0$, i.e. $\alpha w' \in W^\perp$. Then $V = W \oplus W^\perp$ as $A$-submodules. If $W$ or $W^\perp$ are not simple we can continue to decompose them by taking orthogonal complements. □

By the previous claim we have that $H(2)$ is semisimple. Let us see what are the proper $H(2)$-submodules of $V$. Since we assumed that $V$ is 2-dimensional they must be 1-dimesional. Let $W = \text{span}\{bv_\alpha + cs_i(v_\alpha)\}$, for some $b, c \in \mathbb{C}$ not both zero, be an invariant (with respect to $X_i, X_{i+1}, s_i$) subspace of $V$. Then we must have $s_i(bv_\alpha + cs_i(v_\alpha)) = cv_\alpha + bs_i(v_\alpha) \in W$. That is, there exists $\varepsilon \in \mathbb{C}$ such that $c = \varepsilon b$ and $b = \varepsilon c$. This implies that $b = \pm c \neq 0$. Without loss of generality we will assume that $b = 1$. We will see the case for $a_{i+1} = a_i + 1$ since the case $a_{i+1} = a_i - 1$ is analogous. The other equations that $v_\alpha \pm s_i(v_\alpha)$ must satisfy so that $W$ is an $H(2)$-submodule are

$$\begin{aligned} X_i(v_\alpha \pm s_i(v_\alpha)) &= a_i v_\alpha \pm (a_{i+1}s_i(v_\alpha) - v_\alpha) &&= (a_i \mp 1)v_\alpha \pm (a_i + 1)s_i(v_\alpha) &&= C_i(v_\alpha \pm s_i(v_\alpha)) \\ X_{i+1}(v_\alpha \pm s_i(v_\alpha)) &= a_{i+1} v_\alpha \pm (a_i s_i(v_\alpha) + v_\alpha) &&= (a_i + 1 \pm 1)v_\alpha \pm a_i s_i(v_\alpha) &&= C_{i+1}(v_\alpha \pm s_i(v_\alpha)) \end{aligned}$$

for some $C_i, C_{i+1} \in \mathbb{C}$. Clearly, these are satisfied only for $v_\alpha - s_i(v_\alpha)$. Then, we would have that the complement of $W = \text{span}\{v_\alpha - s_i(v_\alpha)\}$ is not an $H(2)$-submodule of $V$ which contradicts the fact that $H(2)$ is semisimple.

3. Note that in this case we have that $v_\alpha$ and $s_i(v_\alpha)$ are linearly independent. Using again basic linear algebra we have that $(1,0)$ and $(\pm 1/(b-a), 1)$ are eigenvectors for $\begin{bmatrix} a & \pm 1 \\ 0 & b \end{bmatrix}$. Applying this to the matrix representations of $X_i$ and $X_{i+1}$ we have that $v = s_i(v_\alpha) - \frac{1}{a_{i+1}-a_i} v_\alpha$ is an eigenvector for $X_i$ and $X_{i+1}$ with respective eigenvalues $a_{i+1}$ and $a_i$. Additionally, because of the relations in lemma 3.16 we have that for $j \neq i, i+1$

$$X_j(v) = X_j\left(s_i(v_\alpha) - \frac{1}{a_{i+1} - a_i} v_\alpha\right) = a_j s_i(v_\alpha) - \frac{a_j}{a_{i+1} - a_i} v_\alpha = a_j v.$$

Then, $v$ is a GZ-vector such that $\alpha(v) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) = \alpha'$, i.e $\alpha' \in \text{Spec}(n)$ and $v = v_{\alpha'}$.

Since $v \in V \leq V_{\rho_{i+1}} \leq V_{\rho_n}$ we also have that $\alpha \sim \alpha'$. Finally, we easily calculate

$$s_i(v_\alpha) = \frac{1}{a_{i+1} - a_i} v_\alpha + v_{\alpha'} \quad \text{and}$$
$$s_i(v_{\alpha'}) = v_\alpha - \frac{1}{a_{i+1} - a_i} s_i(v_\alpha) = v_\alpha - \frac{1}{(a_{i+1} - a_i)^2} v_\alpha + \frac{1}{a_{i+1} - a_i} v_{\alpha'}$$

which gives us the desired matrix representation of $s_i$ with respect to the basis $\{v_\alpha, v_{\alpha'}\}$.

□

**Remark 3.19.** The algebra used in part 2 of proposition 3.18, $H(2) = \langle X_i, X_{i+1}, s_i \rangle$, is an example of degenerate Hecke algebras. For more details see [VO05, MM16, section 3]

**Remark 3.20.** Note that what we are doing in part 3 of proposition 3.18 can be understood as orthogonalizing the basis $\{v_\alpha, v_{\alpha'}\}$ with respect to a $S_n$-invariant inner-product that makes the GZ-vectors orthogonal.

**Corollary 3.21.** Let $\alpha = (a_1, \dots, a_n) \in \mathbb{C}^n$. If for some $i = 1, \dots, n-2$ we have that $a_i = a_{i+2} = a_{i+1} - 1$ then $\alpha \notin \text{Spec}(n)$. That is, a sequence of the form $(a, a+1, a)$ cannot be a part of a spectrum vector.

*Proof.* Consider such an $\alpha$ and assume that $\alpha \in \text{Spec}(n)$. Recall that by the Coxeter relations we have that $e_{id} = (s_i s_{i+1})^3 = s_i s_{i+1} s_i s_{i+1} s_i s_{i+1}$. Or equivalently, $s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$. However, note that by item 2. of proposition 3.18 we have that $s_i(v_\alpha) = v_\alpha$ and $s_{i+1}(v_\alpha) = -v_\alpha$. Then, $s_{i+1} s_i s_{i+1}(v_\alpha) = v_\alpha$ but $s_i s_{i+1} s_i(v_\alpha) = -v_\alpha$ which is a contradiction. □

**Definition 3.22.** We say that a Coxeter generator $s_i$ is an *admissible transposition* for $\alpha = (a_1, \dots, a_n) \in \text{Spec}(n)$ if $a_{i+1} \neq a_i \pm 1$.

The importance of this definition comes from proposition 3.18 since it guarantees that acting with admissible transpositions on spectrum vectors does not takes us out of $\text{Spec}(n)$.

## 3.2 Content vectors and Young tableaux

In this section we are going to prove the correspondence between the classical theory of Young tableaux [Sag01, Chapter 2] and the Vershik-Okounkov approach. For this we are going to associate standard Young tableaux with integer vectors that will end up being the spectrum of the YJM elements.

**Definition 3.23.** We call $(a_1, \dots, a_n) \in \mathbb{Z}^n$ a *content vector* if it satisfies the following conditions.

(i) $a_1 = 0$.

(ii) For all $i = 2, \dots, n$: $\{a_i - 1, a_i + 1\} \cap \{a_1, \dots, a_{i-1}\} \neq \emptyset$.

(iii) If $a = a_i = a_j$ for some $i < j$, then $\{a - 1, a + 1\} \subseteq \{a_{i+1}, \dots, a_{j-1}\}$.

We denote the set of all content vectors in $\mathbb{Z}^n$ as $\text{Cont}(n)$.

We are going to consider the natural action of $S_n$ over $\mathbb{Z}^n$ that permutes components to define an equivalence relation in $\text{Cont}(n)$. Notice that by condition (ii) we have that $a_2 = \pm 1$ then permuting $a_1$ and $a_2$ does not give us a content vector. In general, this $S_n$-action does not act invariantly over content vectors however we can characterize the permutations that do.

**Definition 3.24.** We define an equivalence relation on $\mathrm{Cont}(n)$ as follows. For $\alpha, \beta \in \mathrm{Cont}(n)$ we have that $\alpha \simeq \beta$ if and only if there exists $\sigma \in S_n$ such that $\sigma \cdot \alpha = \beta$.

**Proposition 3.25.** For any $\alpha = (a_1, \dots, a_n) \in \mathrm{Cont}(n)$, if $a_{i+1} \neq a_i \pm 1$ then $s_i \cdot \alpha \in \mathrm{Cont}(n)$.

*Proof.* Let $i$ be such that $a_{i+1} \neq a_i \pm 1$. Note that any $\alpha \in \mathrm{Cont}(n)$ satisfies that $a_2 = \pm 1 = a_1 \pm 1$, then $i \neq 1$. In particular this implies that $s_i \cdot \alpha = (a_1, \dots, a_{i+1}, a_i, \dots, a_n)$ satisfies condition (i) from the definition of content vectors. Clearly, conditions (ii) and (iii) are satisfied for any $j \neq i+1$ so it suffices to check them for $i+1$. Note that since $\alpha \in \mathrm{Cont}(n)$ we have that $\{a_{i+1} - 1, a_{i+1} + 1\} \cap \{a_1, \dots, a_i\} \neq \emptyset$. $a_{i+1} \pm 1 \neq a_i$ implies that $\{a_{i+1} - 1, a_{i+1} + 1\} \cap \{a_1, \dots, a_{i-1}\} \neq \emptyset$. Similarly, if for some $k < i+1$ we have that $a_k = a_{i+1}$ we have that $\{a_{i+1} - 1, a_{i+1} + 1\} \subseteq \{a_{k+1}, \dots, a_i\}$. And, since $a_{i+1} \pm 1 \neq a_i$ we have that $\{a_{i+1} - 1, a_{i+1} + 1\} \subseteq \{a_{k+1}, \dots, a_{i-1}\}$. □

Thus, similarly to how we defined admissible transpositions for $\mathrm{Spec}(n)$ we can also say that a Coxeter generator $s_i$ is *admissible* for $\alpha = (a_1, \dots, a_n) \in \mathrm{Cont}(n)$ if $a_{i+1} \neq a_i \pm 1$. And again, what motivates this definition is the fact that admissible permutations act invariantly over content vectors.

These vectors are our linking point to the usual way of understanding the representations of $S_n$ via standard Young tableaux. For this, to every standard Young tableau we will assign a particular vector. First, note that if we have a Ferrer's board of shape $\lambda = (\lambda_1, \dots, \lambda_k)$ each square can be associated to a coordinate $(i, j)$ where $i = 1, \dots, k$ and $j = 1, \dots, \lambda_1$. The following diagram indicating the coordinates of the Ferrer's board of shape $(3, 2, 1)$ is an example.

| | | |
|---|---|---|
| $(1,1)$ | $(1,2)$ | $(1,3)$ |
| $(2,1)$ | $(2,2)$ | |
| $(3,1)$ | | |

**Definition 3.26.** Denote $\mathrm{SYT}(n)$ the set of all standard Young tableaux for $n \in \mathbb{N}$. Let $T \in \mathrm{SYT}(n)$ be of shape $\lambda = (\lambda_1, \dots, \lambda_k)$. Consider the functions $i : \{1, \dots, n\} \to \{1, \dots, k\}$ and $j : \{1, \dots, n\} \to \{1, \dots, \lambda_1\}$ where $(i(m), j(m))$ are the coordinates of the box corresponding to the number $m \in \{1, \dots, n\}$. Then, we define the *content vector of* $T$ as

$$C(T) := (j(1) - i(1), \dots, j(n) - i(n)).$$

**Remark 3.27.** Note that the content of a tableau of shape $\lambda = (\lambda_1, \dots, \lambda_k)$ in $\mathrm{SYT}(n)$ consists of (possibly repeated) elements from $\{1-k, 2-k, \dots, 0, \dots, \lambda_1 - 1\}$. Particularly, we have that two different numbers $p < q \in \{1, \dots, n\}$ are such that $j(p) - i(p) = j(q) - i(q) \Leftrightarrow j(p) - j(q) = i(p) - i(q)$, if and only if they correspond to boxes in the same diagonal of the Ferrer's board. Then, we can associate each diagonal to a number $h$, which simply is $j - i$. Additionally, we have that the number of elements in $T \in \mathrm{SYT}(n)$ belonging to the diagonal $h$ is exactly how many times $h$ is repeated in the content vector $C(T)$.

**Example 3.28.** Let us calculate two examples of content vectors from standard Young tableaux.

$T =$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | | |
| 7 | 8 | | |

$S =$

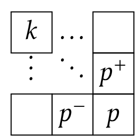
| | | |
|---|---|---|
| 1 | 4 | 6 |
| 2 | 5 | |
| 3 | | |

Then, $C(T) = (0,1,2,3,-1,0,-2,-1)$ and $C(S) = (0,-1,-2,1,0,2)$. Also, note that in $T$ the diagonals $0$ and $-1$ consist, respectively, of elements $1,6$ and $5,8$ while in $S$ they consist of $1,5$ and $2$.

As the name hints, we will see that the content vectors of standard Young tableaux are exactly the same as content vectors in $\mathrm{Cont}(n)$.

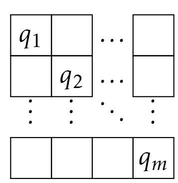
**Proposition 3.29.** The map $C : \mathrm{SYT}(n) \to \mathrm{Cont}(n)$, where $C(T)$ is the content vector of the standard Young tableau $T$, is a bijection. Moreover, for any $\alpha,\beta \in \mathrm{Cont}(n)$ with $\alpha = C(T)$ and $\beta = C(S)$ where $T,S \in \mathrm{SYT}(n)$ we have that $\alpha \simeq \beta$ if and only if $T$ and $S$ have the same shape.

*Proof.* First, let us check that the image of the function $C$ is indeed $\mathrm{Cont}(n)$. Let $T \in \mathrm{SYT}(n)$ and $C(T) = (a_1,\dots,a_n)$. Since $T$ is a standard tableau necessarily it contains $1$ in position $(1,1)$ then $a_1 = 0$. Take $k \in \{2,\dots,n\}$ in position $(i,j)$ in $T$. Since $k \neq 1$ we have that $i > 1$ or $j > 1$. In the first case we have that the $i-1$-th row of $T$ has at least $j$ columns. Then, we can consider the number $p$ which is in position $(i-1,j)$, i.e. $a_p = j-(i-1) = j-1+1 = a_k+1$. And, since $T$ is standard we must have that $p < k$. Thus, $a_k + 1 \in \{a_1,\dots,a_{k-1}\}$. Analogously, if $j > 1$ we can consider $p < k$ in position $(i,j-1)$ and obtain that $a_p = a_k - 1$. To verify the final condition let us consider $k < p \in \{1,\dots,n\}$ such that $a_k = a_p$. In particular this means that $k$ and $p$ are contained in the same diagonal of $T$ as in the following diagram.



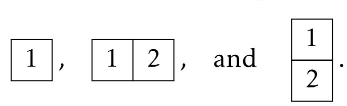

The important point is that $p$ cannot be in the up-most row or in the left-most column of $T$. Then, there exist $p^-,p^+$ as in the diagram, i.e. such that $a_{p^-} = j-1-i = a_p - 1$ and $a_{p^+} = j-(i-1) = a_p + 1$. And, since $T$ is standard we have that $k < p^-$, $p^+ < p$ therefore $C(T) \in \mathrm{Cont}(n)$.

Now, let us see that $C$ is an injective function. Let $T,S \in \mathrm{SYT}(n)$ such that $C(T) = C(S)$. This implies that $T$ and $S$ have not only diagonals of the same length (i.e. they have the same shape) but also exactly the same numbers in each diagonal. Take $h$ corresponding to any diagonal and let $q_1 < \dots < q_m \in \{1,\dots,n\}$ be the elements corresponding to that diagonal. Since $T$ and $S$ are standard we have that the diagonal must be contained in both tableaux as in the following diagram.



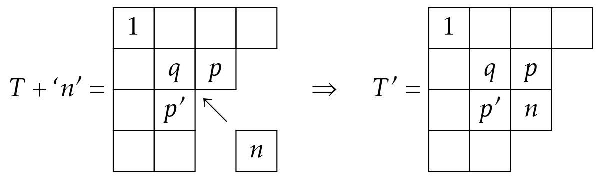

Then, we have that $T$ and $S$ have exactly the same diagonals. Since any number in a tableau is contained in one diagonal we obtain that $T = S$.

Finally, let us see that $C$ is surjective. We will proceed by induction on $n$. For $n = 1,2$ note that $\mathrm{Cont}(1) = \{(0)\}$ and $\mathrm{Cont}(2) = \{(0,1),(0,-1)\}$. Then we have that the corresponding standard tableaux that have those vectors as their content are respectively

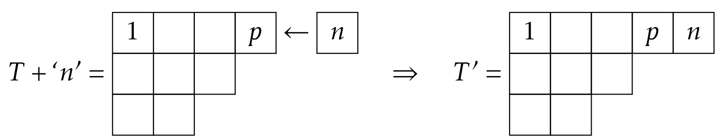



Assume that the result is true for $n-1$. Let $\alpha = (a_1,\dots,a_n) \in \mathrm{Cont}(n)$. By the definition of Cont it is clear that $\alpha' = (a_1,\dots,a_{n-1}) \in \mathrm{Cont}(n-1)$. Then there exists $T \in \mathrm{SYT}(n-1)$ such that $C(T) = \alpha'$. We want to add a box, which will contain $n$, to this $T$ to obtain $T' \in \mathrm{SYT}(n)$ that satisfies $C(T') = \alpha$.

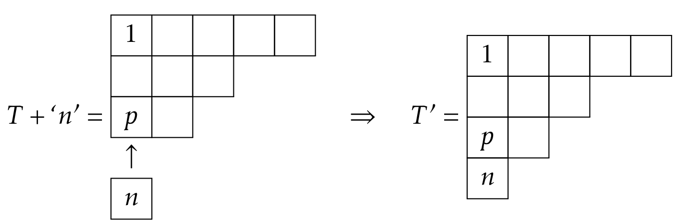
Let us consider two cases. First, assume that $a_n \in \{a_1, \dots, a_{n-1}\}$. Choose $q$ as the largest number such that $a_n = a_q$. This implies that $q$ is in the last box of its diagonal in $T$. The $T'$ we want to consider consists precisely of adding a box with $n$ to this diagonal. However, we must check that we do obtain a tableau. Note that by condition (iii.) of content vectors we have that $\{a_q - 1, a_q + 1\} \subseteq \{a_{q+1}, \dots, a_{n-1}\}$. Let $p, p' \in \{q+1, \dots, n-1\}$ such that $a_p = a_q - 1$ and $a_{p'} = a_q + 1$. Thus, we have that we can add the '$n$'-box to $T$ in the following way.



It is important to mention that $p$ and $p'$ cannot be in lower positions precisely because of the way we chose $q$ and that $T$ is standard. Then, $T' \in \mathrm{SYT}(n)$ and $C(T') = \alpha$ as desired. Second, let us assume that $a_n \notin \{a_1, \dots, a_{n-1}\}$. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ de the shape of $T'$. By condition (ii.) from the definition of Cont we have that $\{a_n - 1, a_n + 1\} \cap \{a_1, \dots, a_{n-1}\} \neq \emptyset$. Then, there exists $p \in \{1, \dots, n-1\}$ such that $a_p = a_n \pm 1$. Let us consider first the case when $a_p = a_n - 1$. Note that since $a_n$ is not a coordinate in $\alpha'$ we have that for all $1 \geq q \leq n-1$, $a_q \neq a_p + 1$. This implies that the box containing $p$ in $T'$ cannot have a box to its right or above it. Thus, $p$ is in position $(1, \lambda_1)$, i.e. it is contained in the top-right box of the Ferrer's board. This allows us to build $T'$ in the following way.



Analogously, if $a_p = a_n + 1$ we have that $p$ must be in position $(k, 1)$. And we obtain the following tableau.



We conclude then that $C : \mathrm{SYT}(n) \to \mathrm{Cont}(n)$ is bijective.

Additionally, to conclude the second part of the proposition it suffices to make the following remark. Observe that from how we defined the content of a standard tableau it is clear that the numbers that appear in their content vectors (including repetitions) are uniquely determined by the shape of the tableau. So, if we consider $\alpha, \beta \in \mathrm{Cont}(n)$ such that $\alpha = C(T), \beta = C(S)$ where $T, S \in \mathrm{SYT}(n)$ we have that $\alpha \simeq \beta$ if and only if there exists $\sigma \in S_n$ such that $\sigma \cdot \alpha = \beta$. This is equivalent to saying that $\alpha$ and $\beta$ contain the same numbers (including repetitions) in their coordinates. And, as we mentioned before, this happens if and only if $T$ and $S$ have the same shape. $\square$

**Remark 3.30.** The second part of proposition 3.29 tells us that each equivalence class in $\mathrm{Cont}(n)/\simeq$ corresponds with a shape $\lambda$, i.e. with a partition of $n$. Then, $|\mathrm{Cont}(n)/\simeq| = p(n)$ which is the number of partitions of $n$. Two classical results tell us that $p(n)$ is the number of conjugacy classes of $S_n$ and thus, it is also the number of irreducible representations of $S_n$.

Through the bijection from proposition 3.29 we can understand better the important role of admissible Coxeter generators for content vectors. First, we must ask how can we translate the condition "$a_{i+1} \neq a_i \pm 1$" to tableaux? For $T \in \mathrm{SYT}(n)$ with content $C(T) = (a_1, \ldots, a_n)$ we have that $a_{i+1} = a_i + 1$ or $a_{i+1} = a_i - 1$ when, respectively, the boxes

$$\begin{array}{|c|c|}\hline i & i+1 \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|}\hline i \\ \hline i+1 \\ \hline\end{array}$$

are contained in the tableau $T$. Thus, we say that $s_i$ is an *admissible transposition* for $T$ if $i$ and $i+1$ are on different rows and columns in $T$.

**Lemma 3.31.** Let $T, S \in \mathrm{SYT}(n)$ be tableaux of the same shape $\lambda = (\lambda_1, \ldots, \lambda_k)$. Then, $T$ can be obtained from $S$ by applying a sequence of admissible transpositions.

In order to prove this lemma we are going to consider a "special" standard Young tableau for each shape. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be any partition of $n$. Denote $T^\lambda$ as the following tableau

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & \cdots & \cdots & \lambda_1 \\ \hline \lambda_1+1 & \cdots & \cdots & \lambda_1+\lambda_2 & \multicolumn{3}{c}{} \\ \cline{1-4} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{\vdots} & \multicolumn{4}{c}{} \\ \cline{1-3} S_{k-1}+1 & \cdots & n & \multicolumn{4}{c}{} \\ \cline{1-3}\end{array}$$

where $S_{k-1} = \lambda_1 + \ldots + \lambda_{k-1}$. For any $T \in \mathrm{SYT}(\lambda)$ we will denote $\pi_T$ the unique permutation that transforms $T$ into $T^\lambda$

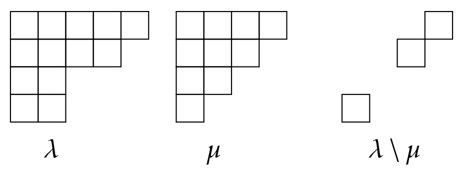
*Proof of lemma 3.31.* Since the inverse of a Coxeter generator $s_i$ is itself, and that if $s_i$ is admissible for $T$ then it is also admissible for $s_i \cdot T$, we have that if suffices to check that we can obtain $T^\lambda$ from any tableau of shape $\lambda$ by applying a sequence of admissible transpositions. We will prove this by induction on $n$. For $n = 1$ the result is trivial since $|\mathrm{SYT}(1)| = 1$. Assume the result for $n-1$. Let $T \in \mathrm{SYT}(n)$ be of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$. Consider first the case where the right-most box of the last row of $T$ (i.e. the box in position $(k, \lambda_k)$) contains $n$. By removing that box from $T$ we obtain $T' \in \mathrm{SYT}(n-1)$ of shape $\lambda' = (\lambda_1, \ldots, \lambda_k - 1)$. By induction hypothesis we have that there is a sequence of admissible transpositions that transform $T'$ into $T^{\lambda'}$. Since, by construction, $T^\lambda$ consists of adding a box containing $n$ to the right of the last row of $T^{\lambda'}$ we obtain that this same sequence of admissible transpositions also transforms $T$ into $T^\lambda$. Now, let us consider the case where $T$ has $j < n$ in the right-most box of its last row. Note that since $T$ is standard $j+1 \leq n$ cannot be in the same row or column as $j$. Thus, $s_j$ is admissible for $T$. If $j+1 < n$ we can apply the same argument to conclude that $s_{j+1}$ is admissible for $s_j \cdot T$. Continuing this process we obtain that $(s_{n-1} \cdots s_{j+1} s_j) \cdot T \in \mathrm{SYT}(n)$ contains $n$ in the right-most box of its last row. Therefore, we are again in the previous case which allows us to conclude the lemma. $\square$

Proposition 3.29 and lemma 3.31 give us another way to understand the equivalence relation $\simeq$.

**Corollary 3.32.** Let $\alpha, \beta \in \mathrm{Cont}(n)$. If $\alpha \simeq \beta$ then $\beta$ can be obtained from $\alpha$ via a sequence of admissible transpositions. That is, the $\sigma \in S_n$ such that $\sigma \cdot \alpha = \beta$ can be chosen as a product of admissible transpositions.

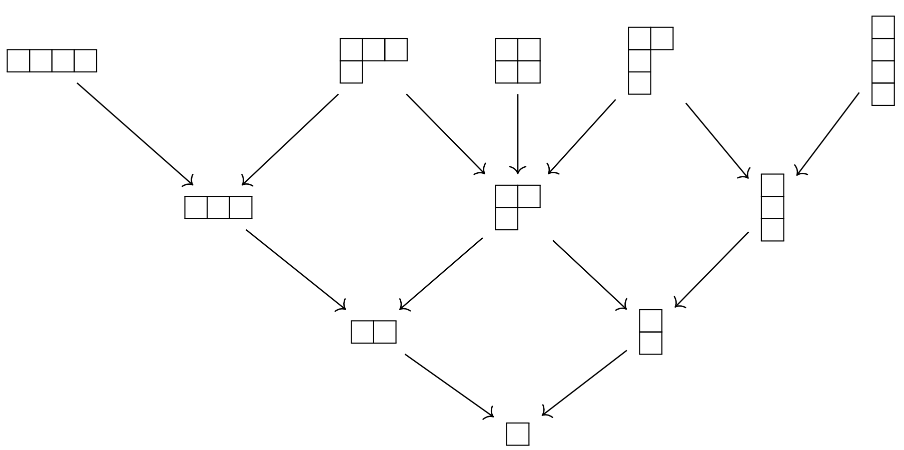
**Remark 3.33.** As explained in [CSST10, Theorem 3.1.5], [MM16, Proposition 1.84] and [VO05, Remark 5.6] the sequence of transpositions found on lemma 3.31 is minimal. That is, in the sense that the number of transpositions used corresponds exactly with the Coxeter length of the permutation which transforms a standard tableau $T$ into $T^\lambda$. This follows from corollary 1.6 since $l(\pi_{T'} s_{n-1} \cdots s_j) = l(\pi_{T'}) + (n-j)$. Furthermore, this minimal decomposition of $\pi_T$ can be explicitly written as $\sigma_k \dots \sigma_1$ for some $k \in \mathbb{N}$ where each $\sigma_i = s_{n-i} s_{n-i-1} \cdots s_{j_i}$. Here $j_i$ is the number that in $\sigma_{i-1} \cdots \sigma_1 T$ is in the box corresponding to $n-i$ in $T^\lambda$.

The final tool from Young tableaux that we will need relates all the possible partitions of $n$ for all $n \in \mathbb{N}^*$. Denote $\mathbb{Y} = \{\lambda : \lambda \text{ is a partition of } n \text{ for any } n \in \mathbb{N}^*\}$. Let us consider $\lambda, \mu \in \mathbb{Y}$ where $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of $n \in \mathbb{N}^*$ and $\mu = (\mu_1, \dots, \mu_l)$ is a partition of $m \in \mathbb{N}^*$. We say that $\mu \leq \lambda$ if $n \geq m, k \geq l$ and $\lambda_i \geq \mu_i$ for $i = 1, \dots k$. In terms of Ferrer's boards saying that $\mu \leq \lambda$ means that the frame of shape $\mu$ is contained in the frame of shape $\lambda$. Thus, we can also consider $\lambda \setminus \mu$ which corresponds to the boxes that remain when removing $\mu$ from $\lambda$. The following diagrams clarify these concepts.




**Definition 3.34.** We define the *Young graph* as the directed graph with vertex set $\mathbb{Y}$ and edge set $\{(\lambda, \mu) : \lambda, \mu \in \mathbb{Y},\ \mu \leq \lambda, \text{ and } \lambda \setminus \mu \text{ is a single box}\}$. Henceforth, we will denote $\mathbb{Y}$ as the Young graph.

**Example 3.35.** The partitions of $1, 2, 3$ and $4$ are $(1), (2), (1,1), (3), (2,1), (1,1,1), (4), (3,1), (2,2), (2,1,1)$ and $(1,1,1,1)$. An easy way to check if $\lambda = (\lambda_1, \dots, \lambda_k) \setminus \mu = (\mu_1, \dots, \mu_l)$ is a single box is that $\lambda_i - \mu_i = 0$ for $i = 1, \dots, k$ (here $\mu_j$ for $j > l$ is 0) except in one index $i_0$ where $\lambda_{i_0} - \mu_{i_0} = 1$. For example, note that for $\lambda = (2,1,1)$ and $\mu = (2,1)$ we have that $\lambda_i - \mu_i = 0$ for $i = 1, 2$ and $\lambda_3 - \mu_3 = 1$. Using this we can check that the bottom part of the $\mathbb{Y}$ is



Note that this graph is the same as the bottom part of the branching graph found on example 2.5. As the reader might suspect, this is not a coincidence and we will see ahead the bijection between these two graphs.

**Definition 3.36.** Fix $n \in \mathbb{N}^*$. Let $\lambda^{(i)}$ be partitions of $i$ for $i = 1, \dots, n$. We call $p = \lambda^{(n)} \to \dots \to \lambda^{(1)}$ a *n-path* in the Young graph if $\lambda^{(i)} \to \lambda^{(i-1)}$ is an edge in $\mathbb{Y}$ for $i = 2, \dots, n$. That is, if $\lambda^{(i)} \setminus \lambda^{(i-1)}$ is a single box. We will denote $\Pi_n(\mathbb{Y})$ all paths starting at a partition of $n$.

**Proposition 3.37.** Let us consider the map $\phi_n : \Pi_n(\mathbb{Y}) \to \mathrm{SYT}(n)$ where for each $n$-path $p = \lambda^{(n)} \to \ldots \to \lambda^{(1)}$ we define $\phi_n(p)$ as the tableau that contains the number $i$ in the box $\lambda^{(i)} \setminus \lambda^{(i-1)}$ for $i = 2, \ldots, n$ and the number 1 in the upper left-most box. Then, $\phi_n$ is a bijection.

**Example 3.38.** To clarify the definition of the map $\phi_n$ let us consider the 4-paths $p_1 = (2,2) \to (2,1) \to (1,1) \to (1)$ and $p_1 = (2,2) \to (2,1) \to (2) \to (1)$. Then we obtain that

$$\phi(p_1) = \quad \begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array} \quad \to \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$$\phi(p_2) = \quad \begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \to \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} \quad \to \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}.$$

*Proof of proposition 3.37.* We will prove this by induction on $n$. For $n = 1$ the result is trivial because the only possible path is $\lambda^{(1)} = (1)$ which clearly maps to the only standard tableau in $\mathrm{SYT}(1)$. Assume the result for $n-1$ and let us verify it for $n$. First, we will verify that $\phi_n$ is well defined. Let $p = \lambda^{(n)} \to \ldots \to \lambda^{(1)}$ be a $n$-path. Clearly, $p' = \lambda^{(n-1)} \to \ldots \to \lambda^{(1)}$ is an $(n-1)$-path. By induction hypothesis we particularly have that $\phi_{n-1}(p') \in \mathrm{SYT}(n-1)$. By definition, $\phi_n(p)$ is the tableau obtaining by adding the box $\lambda^{(n)} \setminus \lambda^{(n-1)}$ filled with $n$ to the standard tableau $\phi_{n-1}(p')$. Note that since $\lambda^{(n)}$ is a Ferrer's board the box $\lambda^{(n)} \setminus \lambda^{(n-1)}$ must be a "corner" box. That is, it must be the bottom box of its column and the right-most box of its row which guarantees that $\phi_n(p) \in \mathrm{SYT}(n)$. Second, we will check that $\phi_n$ is surjective. Similarly, for any $T \in \mathrm{SYT}(n)$ we must have that the box containing $n$ is a "corner" box. Then by removing that box from $T$ we obtain a tableau $T' \in \mathrm{SYT}(n-1)$. By the induction hypothesis we have that $T'$ has a unique pre-image through $\phi_{n-1}$ which we will denote as $p' = \lambda^{(n-1)} \to \ldots \to \lambda^{(1)} \in \Pi_{n-1}(\mathbb{Y})$. We can extend $p'$ to a $n$-path $p = \lambda^{(n)} \to \lambda^{(n-1)} \to \ldots \to \lambda^{(1)}$ where $\lambda^{(n)}$ is the shape of $T$. By construction, we have that $\phi_n(p) = T$. Finally, we will see that $\phi_n$ is injective by proving that this $p$ is unique. Let $\mu^n \to \ldots \to \mu^{(1)}$ be any pre-image of $T$ through $\phi_n$. Then, $\phi_{n-1}(\mu^{(n-1)} \to \ldots \to \mu^{(1)}) \in \mathrm{SYT}(n-1)$ is the tableau obtained by removing the box containing $n$ from $T$. That is, it must be the $T'$ from before. Then, $\mu^{(n-1)} \to \ldots \mu^{(1)} = p'$. Note that any pre-image of $T$ must be a path starting at $\lambda^{(n)}$ since the shape of $T$ is fixed which implies $\mu^n \to \ldots \to \mu^{(1)} = p$. Thus, we have obtained that $\phi_n$ is a bijection. $\square$

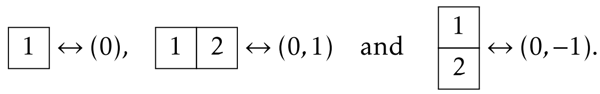
**Corollary 3.39.** There is a bijection between $\mathrm{Cont}(n)$ and $\Pi_n(\mathbb{Y})$. Moreover, if $\alpha, \beta \in \mathrm{Cont}(n)$ correspond to paths $p_\alpha = \lambda^{(n)} \to \ldots \to \lambda^{(1)}$ and $p_\beta = \mu^{(n)} \to \ldots \to \mu^{(1)}$ respectively then $\alpha \simeq \beta$ if and only if $\lambda^{(n)} = \mu^{(n)}$.

*Proof.* The explicit bijection comes from composing $\phi_n$ from proposition 3.37 with the content map $C$ from proposition 3.29. Explicitly, we obtain $C \circ \phi_n : \Pi_n(\mathbb{Y}) \to \mathrm{Cont}(n)$. Then, $\alpha = C(\phi_n(p_\alpha))$ and $\beta = C(\phi_n(p_\beta))$ correspond respectively with the tableaux $\phi_n(p_\alpha)$ and $\phi_n(p_\beta)$. From proposition 3.29 we can also conclude that $\alpha \simeq \beta$ if and only if $\phi_n(p_\alpha)$ and $\phi_n(p_\beta)$ have the same shape. Which is equivalent to saying that $\lambda^{(n)} = \mu^{(n)}$. $\square$

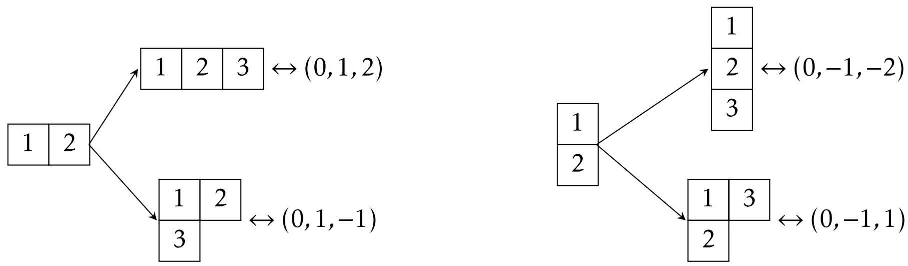
Above we have seen how content vectors codify completely standard Young tableaux. In the following part, we will introduce the fundamental results of the Vershik-Okounkov approach which relate the spectrum of the YJM elements with $\mathrm{Cont}(n)$. An important remark regarding content vectors is that if $(a_1, \ldots, a_n) \in \mathrm{Cont}(n)$ then $(a_1, \ldots, a_{n-1}) \in \mathrm{Cont}(n-1)$. This, together with remark 3.14, will allow us to cleverly deal with these sets inductively.

**Example 3.40.** In this example we will calculate $\mathrm{Cont}(n)$ for $n = 1, 2, 3, 4$. To make the argument easier to read we will be using the bijection $\mathrm{Cont}(n) \leftrightarrow \mathrm{SYT}(n)$. Recall that $\mathrm{Cont}(1)$ and $\mathrm{Cont}(2)$

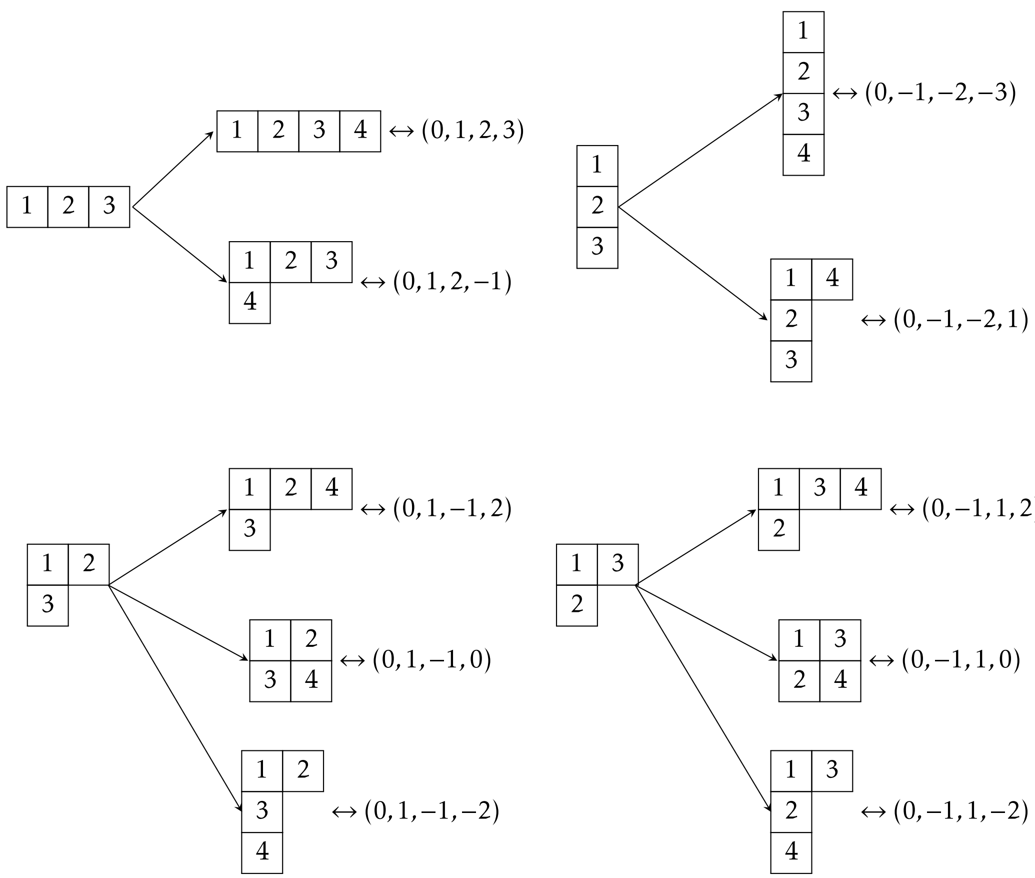
correspond respectively to




For each $T \in \mathrm{SYT}(2)$ we can add the box with 3 in two ways in order to obtain $\mathrm{Cont}(3)$.




We can do the same process for each $T \in \mathrm{SYT}(3)$ and add in all possible positions a box with 4. This will allow us to obtain $\mathrm{Cont}(4)$.




Let us recall from corollary 3.39 that the equivalence relation $\simeq$ corresponds in Young tableaux to having the same shape. Hence, we can classify the content vectors we just obtained for $n = 3, 4$ as in table 3.2. Comparing the results obtained with what we calculated in table 3.1 from example 3.13 we can conclude that $\mathrm{Spec}(3) = \mathrm{Cont}(3)$, $\mathrm{Spec}(4) = \mathrm{Cont}(4)$ and that in these two cases the equivalence relations $\sim$ and $\simeq$ coincide.

Table 3.2: Cont(3) and Cont(4)

| Partition | Content | Partition | Content |
|---|---|---|---|
| (3) | $(0,1,2)$ | (2,1) | $(0,1,-1)$ |
| (1,1,1) | $(0,-1,-2)$ | | $(0,-1,1)$ |

| Partition | Content | Partition | Content |
|---|---|---|---|
| (4) | $(0,1,2,3)$ | (2,2) | $(0,1,-1,0)$ |
| (1,1,1,1) | $(0,-1,-2,-3)$ | | $(0,-1,1,0)$ |
| (3,1) | $(0,1,2,-1)$ | (2,1,1) | $(0,-1,-2,1)$ |
| | $(0,1,-1,2)$ | | $(0,-1,1,-2)$ |
| | $(0,-1,1,2)$ | | $(0,1,-1,-2)$ |

**Proposition 3.41.** For any $n \in \mathbb{N}^*$, we have that $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$.

*Proof.* As we previously hinted, we are going to proceed by induction. The cases $n = 1,2,3,4$ are clear from the above example. Now let us assume that the result holds for $n-1$ and we will check that it still holds for $n$. Let $\alpha = (a_1,\dots,a_n) \in \mathrm{Spec}(n)$. By remark 3.14 we have that $(a_1,\dots,a_{n-1}) \in \mathrm{Spec}(n-1)$ and by induction hypothesis it also belongs to $\mathrm{Cont}(n-1)$. Thus, it remains to check that conditions (ii) and (iii) in definition 3.23 are satisfied for $a_n$. We will prove that both conditions are satisfied by contradiction. First, let us assume that $\{a_n-1, a_n+1\} \cap \{a_1,\dots,a_{n-1}\} = \emptyset$. In particular this implies that $a_n \neq a_{n-1} \pm 1$. That is, $s_{n-1}$ is admissible for $\alpha$. Then, $s_{n-1} \cdot \alpha = (a_1,\dots,a_{n-2},a_n,a_{n-1}) \in \mathrm{Spec}(n)$ and $(a_1,\dots,a_{n-2},a_n) \in \mathrm{Cont}(n-1)$. However, we have assumed that $\{a_n-1,a_n+1\} \cap \{a_1,\dots,a_{n-2}\} = \emptyset$ which leads to a contradiction. Second, let us assume that we have a $k < n$ such that $a = a_k = a_n$ but $\{a-1,a+1\} \not\subseteq \{a_{k+1},\dots,a_{n-1}\}$. Without loss of generality we can choose $k$ as large as possible, i.e. such that $a \notin \{a_{k+1},\dots,a_{n-1}\}$. We can also assume that $a_n - 1 \notin \{a_{k+1},\dots,a_{n-1}\}$. Note that $a+1$ can appear in $\{a_{k+1},\dots,a_{n-1}\}$ at most once. Indeed, since $(a_1,\dots,a_{n-1}) \in \mathrm{Cont}(n-1)$ if we had two indices $j,j'$, such that $k+1 \leq j < j' \leq n-1$ and $a_j = a_{j'} = a+1$ then by condition (iii) of definition 3.23 we should have that $\{a,a+2\} \subseteq \{a_{j+1},\dots,a_{j'-1}\} \subseteq \{a_{k+1},\dots,a_{n-1}\}$ which contradicts the way we chose $k$. Then we have that $\alpha$ has two possible forms, $(a_1,\dots,a,\dots,a)$ or $(a_1,\dots,a,\dots,a+1,\dots,a)$. In the first case, we have that $a \pm 1 \neq a_j$ for $j = k+1,\dots,n-1$ which in particular implies that $s_k$ is admissible for $\alpha$. Then, $s_k \cdot \alpha = (a_1,\dots,a_{k+1},a,\dots,a) \in \mathrm{Spec}(n)$. Again, we have that $a_{k+2} \neq a \pm 1$ thus $s_{k+1}$ is admissible for $s_k \cdot \alpha$. Continuing this process we obtain that $(s_{n-2}\cdots s_k)\cdot\alpha = (a_1,\dots,a,a) \in \mathrm{Spec}(n)$ which is a contradiction to 1. from proposition 3.18. In the second case we can do a similar procedure. Let $j$ be such that $a_j = a+1$. Then we have that $(s_{n-1}\cdots s_{j+1}s_{j-2}\cdots s_k)\alpha = (a_1,\dots,a,a+1,a,\dots,a_{n-1}) \in \mathrm{Spec}(n)$ which contradicts corollary 3.21. Thus, we have obtained by contradiction that $\alpha \in \mathrm{Cont}(n)$ which concludes the proof. □

**Lemma 3.42.** Let $\alpha \in \mathrm{Spec}(n)$ and $\beta \in \mathrm{Cont}(n)$ such that $\alpha \simeq \beta$. Then, $\beta \in \mathrm{Spec}(n)$ and $\alpha \sim \beta$.

*Proof.* By corollary 3.32 we have that $\beta$ can be obtained from $\alpha$ via a sequence of admissible transpositions. However, recall that the definition of admissibility was the same for $\mathrm{Spec}(n)$ and $\mathrm{Cont}(n)$. By 3. from proposition 3.18 we have that if $s_i$ is admissible for $\alpha$ then $s_i \cdot \alpha \in \mathrm{Spec}(n)$ and $s_i \cdot \alpha \sim \alpha$. Then, we have the desired result. □

Finally, we have all the ingredients to present the fundamental result from this work [CSST10, Theorem 3.3.7], [MM16, Theorem 5.1] and [VO05, Theorem 5.8].

**Theorem 3.43.** For all $n \in \mathbb{N}^*$, $\mathrm{Spec}(n) = \mathrm{Cont}(n)$ and the equivalence relations $\sim$ and $\simeq$ coincide. Furthermore, the branching graph from the inductive chain of symmetric groups is isomorphic to the Young graph $\mathbb{Y}$.

*Proof.* Let $C \in \mathrm{Cont}(n)/\simeq$ be any equivalence class. If there is an $\alpha \in \mathrm{Spec}(n)$ such that $\alpha \in C$ then, by lemma 3.42, we have that $[\alpha]_\simeq = C \subseteq [\alpha]_\sim \subset \mathrm{Spec}(n)$. So, a $\simeq$-equivalence class is either contained in $\mathrm{Spec}(n)$ or it does not intersect the spectrum set at all. By what we mentioned before ($[\alpha]_\simeq \subseteq [\alpha]_\sim$, i.e $\simeq$ is finer than $\sim$) we have that $|\mathrm{Spec}(n)/\sim| \leq |\mathrm{Spec}(n)/\simeq|$. And since $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$ we also have that $|\mathrm{Spec}(n)/\simeq| \leq |\mathrm{Cont}(n)/\simeq|$. But, by remark 3.30 it is clear that $|\mathrm{Cont}(n)/\simeq| = |\widehat{S_n}| = |\mathrm{Spec}(n)/\sim|$. By double inequality we obtain that $|\mathrm{Spec}(n)/\sim| = |\mathrm{Spec}(n)/\simeq| = |\mathrm{Cont}(n)/\simeq|$. The second equality implies that all $\simeq$-equivalence classes intersect $\mathrm{Spec}(n)$. Then, $\mathrm{Cont}(n) \subseteq \mathrm{Spec}(n)$. Together with proposition 3.41 we obtain the desired equality. Moreover, the first inequality tells us that there is the same amount of $\sim$-equivalence classes as $\simeq$-equivalence classes. Thus, since we already knew $\simeq$ was finer than $\sim$ we conclude that they coincide.

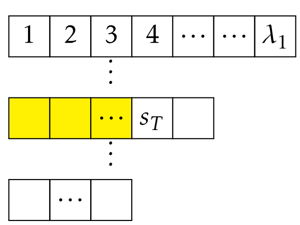
For the second part of the theorem, recall that the edges of the branching graph of symmetric groups are all $\bigcup_{n=1}^{\infty} \widehat{S_n}$. By the definition of $\mathrm{Spec}(n)$ and $\sim$ we can view each irreducible representation $\rho \in \widehat{S_n}$ as a equivalence class $C_\rho \in \mathrm{Spec}(n)/\sim$. Moreover, by what we just proved we can consider $C_\rho$ as a $\simeq$-equivalence class over $\mathrm{Cont}(n)$. Combining this with corollary 3.39 we obtain a bijection between irreducible representations of $S_n$ and Ferrer's boards of size $n$. That is, we have a bijective map between the vertices of the branching graph and the vertices of $\mathbb{Y}$. Note that by construction each element in $\mathrm{Spec}(n)$ corresponds to a unique path $T : \rho_n \to \ldots \to \rho_1$ in the branching graph. Thus, since the bijection in corollary 3.39 is actually a map between $\mathrm{Cont}(n) = \mathrm{Spec}(n)$ and paths in $\mathbb{Y}$ we obtain a graph isomorphism. $\square$

**Corollary 3.44.** Recall definition 3.15 of a highest weight vector. For any partition $\lambda$ of $n$ the highest weight vector of its corresponding equivalence class in $\mathrm{Spec}(n)$ is $C(T^\lambda)$.

*Proof.* Theorem 3.43 gives us that $\mathrm{Spec}(n) = \mathrm{Cont}(n) \subset \mathbb{Z}^n$. Then, it becomes clear that each equivalence class $C_\rho \in \mathrm{Spec}(n)/\sim$ with $\rho \in \widehat{S_n}$ has a highest weight vector. We have that

$$C(T^\lambda) = (0, 1, 2, \ldots, \lambda_1 - 1, -1, 0, 1, \ldots, \lambda_2, \ldots, 1-j, 2-j, \ldots, \lambda_j - j \ldots, 1-k, \ldots, \lambda_k - k).$$

For any tableau $T \in \mathrm{SYT}(\lambda)$ different from $T^\lambda$ set $s_T \in \{2, \ldots, n\}$ to be the smallest number such that $s_T$ is in a different box in $T$ and $T^\lambda$. Observe that $s_T$ must be in the first column of $T$. Indeed, if it wasn't the case since $T$ is standard we would have that the boxes left of $s_T$ in its same row (yellow boxes in the diagram below) should contain smaller numbers.




However, by how we chose $s_T$ all the smaller numbers are contained in the same box as they are in $T^\lambda$. This implies that $s_T$ is also in the same box both in $T$ and $T^\lambda$, which is a contradiction. Note also, that $s_T$ is clearly in a lower row in $T$ than $T^\lambda$ again because $T$ is standard. Thus, we obtain that the $s_T$-th coordinate of $C(T)$ is smaller than the $s_T$-th coordinate of $C(T^\lambda)$. And, since this is their first different coordinate we obtain the result. $\square$

What makes theorem 3.43 so important is that now we have a different proof of the classical results regarding irreducible representations of $S_n$ and Young tableaux, [Sag01, Theorem 2.2.4] and [Sag01, Theorem 2.5.2]. Indeed, each $\rho_n \in \widehat{S_n}$ corresponds to an equivalence class $C_{\rho_n} \in \mathrm{Spec}(n)/\sim$ (or $\mathrm{Cont}(n)/\simeq$) and, therefore, to the Ferrer's board of shape $\lambda$ (partition of $n$). Moreover, we also have that the Young basis of $\rho_n$ given by $\{v_\alpha : \alpha \in C_{\rho_n}\}$ can be parametrized by the standard tableaux of shape $\lambda$ since each $\alpha \in C_{\rho_n}$ corresponds to a unique $T \in \mathrm{SYT}(\lambda)$.

**Definition 3.45.** Let $\lambda$ be a partition of $n$ and let $C \in \mathrm{Spec}(n)/\sim$ be the equivalence class corresponding to $\lambda$. We will denote $S^\lambda$ to the irreducible representation of $S_n$ spanned by $\{v_\alpha : \alpha \in C\}$. These $S^\lambda$ will be called *Specht modules*.

Another fundamental result that we also obtain by using this approach are the branching formulas for the restricted and induced representations in terms of partitions or Ferrer's boards. As it is expected all of this information is codified into the branching (Young) graph.

**Corollary 3.46** (**Branching Rule**)**.** For all partitions $\lambda$ of $n$ and $\mu$ of $n-1$, we have that

$$\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\substack{\mu \text{ partition of } n-1 \\ \lambda \to \mu}} S^\mu, \quad \text{and} \quad \mathrm{Ind}^{S_n}_{S_{n-1}} S^\mu \bigoplus_{\substack{\lambda \text{ partition of } n \\ \lambda \to \mu}} S^\lambda.$$

*Proof.* The first formula is a direct consequence of the multiplicity freeness of the symmetric groups and the bijection between the Young graph and the branching graph. To see the second formula we will use Frobenius reciprocity. We have that $\langle \chi_{S^\mu}, \chi_{\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda} \rangle = \overline{\langle \chi_{S^\lambda}, \chi_{\mathrm{Ind}^{S_n}_{S_{n-1}} S^\mu} \rangle}$. Since $S^\lambda$ and $S^\mu$ are irreducible we have that $\langle \chi_{S^\lambda}, \chi_{\mathrm{Ind}^{S_n}_{S_{n-1}} S^\mu} \rangle$ and $\langle \chi_{S^\mu}, \chi_{\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda} \rangle$ are respectively the multiplicity of $S^\lambda$ in $\mathrm{Ind}^{S_n}_{S_{n-1}} S^\mu$ and of $S^\mu$ in $\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda$. So we have that

$$\langle \chi_{S^\lambda}, \chi_{\mathrm{Ind}^{S_n}_{S_{n-1}} S^\mu} \rangle = \langle \chi_{S^\mu}, \chi_{\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda} \rangle = \begin{cases} 1, & \text{if } \lambda \to \mu \\ 0, & \text{otherwise.} \end{cases}$$

□

## 3.3 Irreducible representations of $\mathbf{S_n}$

From the beginning we have argued that the GZ-vectors of a multiplicity-free chain could be chosen up to scalar multiplication. In this section we will give the choice of particular factors in the case of $S_n$ so that we obtain not only a concise description of the irreducible representations but also so that they are defined over $\mathbb{Q}$. This section contains the main results of this work.

**Proposition 3.47.** Let $\lambda$ be a partition of $n$. Choosing the highest weight vector $v_{T^\lambda}$ determines a particular way to chose the GZ basis that satisfies

$$\pi_T^{-1} v_{T^\lambda} = v_T + \sum_{\substack{S \in \mathrm{SYT}(\lambda) \\ l(\pi_S) < l(\pi_T)}} \gamma_S v_S$$

for any $T \in \mathrm{SYT}(\lambda)$ where $\pi_T$ is the unique permutation that transforms $T$ into $T^\lambda$ and each $\gamma_S \in \mathbb{Q}$.

*Proof.* We will proceed by induction on the Coxeter length of the permutation $\pi_T$. From lemma 3.31 and remark 3.33 recall that we can write $\pi_T$ in a "minimal" way as a product of admissible transpositions. For $l := l(\pi_T) = 1$ we have that $\pi_T = s_i$ where $s_i$ is admissible for $T$. Let $\alpha \in \mathrm{Spec}(n)$ be the content vector of $T^\lambda$. Then, by part 3 of proposition 3.18 we can choose $v_T$ as

$$v_T = v_{s_i \cdot \alpha} = s_i(v_{T^\lambda}) - \frac{1}{a_{i+1} - a_i} v_{T^\lambda}.$$

Since $\pi_{T^\lambda} = id$ has length 0 and that $\text{Spec}(n) = \text{Cont}(n) \subset \mathbb{Z}^n$ this choice of $v_T$ is as desired. Now let us assume the result for $l-1$ and let $\pi_T = s_{i_1} \cdots s_{i_l}$ be its "minimal" decomposition from lemma 3.31. Since these are admissible transpositions we have that $T' := s_{i_l} T \in \text{SYT}(\lambda)$ and $\pi_{T'} = s_{i_1} \cdots s_{i_{l-1}}$ (i.e. $l(\pi_{T'}) = l-1$). By induction hypothesis we can choose $v_{T'}$ such that

$$\pi_{T'}^{-1} v_{T^\lambda} = v_{T'} + \sum_{\substack{S \in \text{SYT}(\lambda) \\ l(\pi_S) < l(\pi_{T'})}} \gamma_S v_S.$$

We can apply again 3. from proposition 3.18 in order to chose $v_T$ as

$$v_{s_{i_l} T'} = s_{i_l}(v_{T'}) - \frac{1}{C(T')_{i_l+1} - C(T')_{i_l}} v_{T'}.$$

Denoting $\beta_{T'} = \frac{1}{C(T')_{i_l+1} - C(T')_{i_l}}$ we have the following equality

$$\pi_T^{-1} v_{T^\lambda} = s_j \pi_{T'}^{-1} v_{T^\lambda} = s_j(v_{T'}) + \sum_{\substack{S \in \text{SYT}(\lambda) \\ l(\pi_S) < l(\pi_{T'})}} \gamma_S s_j(v_S) = v_T + \beta_{T'} v_{T'} + \sum_{\substack{S \in \text{SYT}(\lambda) \\ l(\pi_S) < l(\pi_{T'})}} \gamma_S \left( v_{s_{i_l} S} + \beta_S v_S \right).$$

Note that we replaced $s_{i_l}(v_S)$ according to proposition 3.18 as well because by induction we are choosing the scalars in all the $v_S$ with $l(\pi_S) < l$ in that way. Finally, since $\pi_{s_{i_l} S}$ has Coxeter length at most $l(\pi_{T'}) = l-1$ and the $\beta_S \in \mathbb{Q}$ we have our desired result. □

Note that the "minimal" decomposition of $\pi_T$ we are using to choose the GZ-vectors $v_T$ could actually be any representation of length $l(\pi_T)$ and we would obtain the same result. However, what will interest us is that this choice does not really matter.

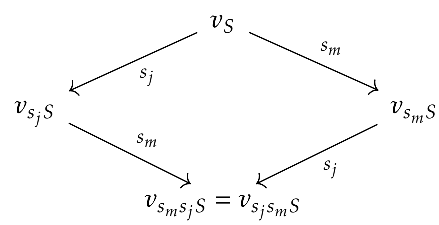
**Lemma 3.48.** Let $T \in \text{SYT}(\lambda)$ and $\pi_T = s_{i_1} \dots s_{i_l}$ be a decomposition of length $l = l(\pi_T)$. Then choosing the GZ-vector $v_T$ recursively using 3. from proposition 3.18 from $s_{j_1} \dots s_{j_l}$ or from it is "minimal" decomposition gives us exactly the same vector. In other words, the choice of the GZ-basis from proposition 3.47 is unique (up to the choice of $v_{T^\lambda}$).

*Proof.* We will proceed by induction on $l = l(\pi_T)$. For $l = 1$ there is a unique decomposition of length $l$. Now, let us assume the result for $l-1$. Let $\pi_T = s_{i_1} \dots s_{i_{l-1}} s_j$ be any decomposition of length $l = l(\pi_T)$. Denoting $T' = s_j T$ we have that $l(\pi_{T'}) = l-1$ and from induction hypothesis we have that the choice of $v_{T'}$ is unique. Thus, it suffices to check that choosing

$$v_T = s_j(v_{T'}) - \frac{1}{C(T')_{j+1} - C(T)_j} v_{T'}$$

is the same as choosing $v_T$ as in proposition 3.47. Recall from remark 3.33 that an explicit way of writing the "minimal" decomposition of $\pi_{T'}$ is $\sigma_k \dots \sigma_1$ for some $k \in \mathbb{N}$ where each $\sigma_i = s_{n-i} s_{n-i-1} \cdots s_{j_i}$. Since $\pi_T = \pi_{T'} s_j$ has length $l = l(\pi_{T'}) + 1$ we have that $j_1 \neq j$. Now, if $s_{j_1} = s_{j+1}$ then $\pi_{T'} = \sigma_k \dots \sigma_1 s_j$ is the "minimal" decomposition of $\pi_T$ and the result is trivial. If $s_{j_1} \neq s_j$ we will have to work a little more.

<u>Case 1:</u>
Let $s_m$ be any transposition that commutes with $s_j$. That is $m \neq j \pm 1$. Let $S \in \text{SYT}(n)$ such that $s_j$ and $s_m$ are both admissible for $S$. We want to verify that we are in the following situation.




Note that since $s_j$ and $s_m$ commute we have that $C(s_jS)_m = C(S)_m$, $C(s_jS)_{m+1} = C(S)_{m+1}$, $C(s_mS)_j = C(S)_j$ and $C(s_mS)_{j+1} = C(S)_{j+1}$. To simplify the calculations we will denote

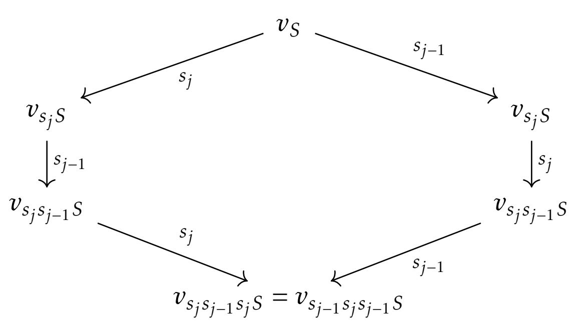
$$\beta_m = \frac{1}{C(S)_{m+1} - C(S)_m} = \frac{1}{C(s_jS)_{m+1} - C(s_jS)_m} \quad \text{and} \quad \beta_j = \frac{1}{C(S)_{j+1} - C(S)_j} = \frac{1}{C(s_mS)_{j+1} - C(s_mS)_j}.$$

Then we have that $v_{s_jS} = s_j(v_S) - \beta_j v_S$ and $v_{s_mS} = s_m(v_S) - \beta_m v_S$. Thus, we obtain that

$$\begin{aligned} v_{s_ms_jS} &= s_m(v_{s_jS}) - \frac{1}{C(s_jS)_{m+1} - C(s_jS)_m} v_{s_jS} \\ &= s_m(v_{s_jS}) - \beta_m v_{s_jS} \\ &= s_ms_j(v_S) - \beta_j s_m(v_S) - \beta_m s_j(v_S) + \beta_m\beta_j v_S \end{aligned}$$

$$\begin{aligned} v_{s_js_mS} &= s_j(v_{s_mS}) - \frac{1}{C(s_mS)_{j+1} - C(s_mS)_j} v_{s_mS} \\ &= s_j(v_{s_mS}) - \beta_j v_{s_mS} \\ &= s_js_m(v_S) - \beta_m s_j(v_S) - \beta_j s_m(v_S) + \beta_j\beta_m v_S. \end{aligned}$$

Case 2:
Note that $s_js_{j-1}s_j = s_{j-1}s_js_{j-1}$. Similarly to the previous case we want to check that we are in the following situation.




Let $(a_1, \ldots, a_n) = C(S)$. Then we have that

$$\begin{aligned} C(s_jS) &= (a_1, \ldots, a_{j-1}, a_{j+1}, a_j, \ldots, a_n) & C(s_{j-1}s_jS) &= (a_1, \ldots, a_{j+1}, a_{j-1}, a_j, \ldots, a_n) \\ C(s_{j-1}S) &= (a_1, \ldots, a_j, a_{j-1}, a_{j+1}, \ldots, a_n) & C(s_js_{j-1}S) &= (a_1, \ldots, a_j, a_{j+1}, a_{j-1}, \ldots, a_n). \end{aligned}$$

To simplify the following calculation we will denote $\alpha = \frac{1}{a_{j+1} - a_j}$, $\beta = \frac{1}{a_{j+1} - a_{j-1}}$ and $\gamma = \frac{1}{a_j - a_{j-1}}$. Thus, we have that

$$v_{s_{j-1}s_jS} = s_{j-1}(v_{s_jS}) - \frac{1}{C(s_jS)_j - C(s_jS)_{j-1}} v_{s_jS} = s_{j-1}s_j(v_S) - \alpha s_{j-1}(v_S) - \beta s_j(v_S) + \alpha\beta v_S$$

$$\begin{aligned} v_{s_js_{j-1}s_jS} &= s_j(v_{s_{j-1}s_jS}) - \frac{1}{C(s_{j-1}s_jS)_{j+1} - C(s_{j-1}s_jS)_j} v_{s_{j-1}s_jS} \\ &= s_js_{j-1}s_j(v_S) - \alpha s_js_{j-1}(v_S) - \gamma s_{j-1}s_j(v_S) + \beta(\alpha+\gamma)s_j(v_S) + \alpha\gamma s_{j-1}(v_S) - (\alpha\beta\gamma + \beta)v_S, \end{aligned}$$

$$v_{s_js_{j-1}S} = s_j(v_{s_{j-1}S}) - \frac{1}{C(s_{j-1}S)_{j+1} - C(s_{j-1}S)_j} v_{s_{j-1}S} = s_js_{j-1}(v_S) - \gamma s_j(v_S) - \beta s_{j-1}(v_S) + \beta\gamma v_S$$

$$\begin{aligned} v_{s_{j-1}s_js_{j-1}S} &= s_{j-1}(s_jv_{s_{j-1}S}) - \frac{1}{C(s_js_{j-1}S)_j - C(s_js_{j-1}S)_{j-1}} v_{s_js_{j-1}S} \\ &= s_{j-1}s_js_{j-1}(v_S) - \alpha s_js_{j-1}(v_S) - \gamma s_{j-1}s_j(v_S) + \alpha\gamma s_j(v_S) + \beta(\alpha+\gamma)s_{j-1}(v_S) - (\alpha\beta\gamma+\beta)v_S. \end{aligned}$$

Note that

$$\beta(\alpha+\gamma) = \beta\left(\frac{1}{a_{j+1}-a_j} + \frac{1}{a_j - a_{j-1}}\right) = \frac{1}{a_{j+1}-a_{j-1}} \frac{a_j - a_{j-1} + a_{j+1} - a_j}{(a_{j+1}-a_j)(a_j-a_{j-1})} = \frac{1}{(a_{j+1}-a_j)(a_j-a_{j-1})} = \alpha\gamma.$$

And thus, we have that $v_{s_js_{j-1}s_jS} = v_{s_{j-1}s_js_{j-1}S}$.

With both cases we just checked we have that we can accommodate $s_j$ in $\pi_{T'}$ without changing the GZ-vector we are choosing. Indeed, if $j_1 > j+1$ we have that all the transpositions in $s_{n-1}, \ldots, s_{j_1}$ commute with $s_j$. And, then it commutes with $\sigma_1$. If $j_1 < j$ then $\pi_{T'} = \sigma_k \cdots \sigma_2 s_{n-1} \cdots s_j s_{j-1} \cdots s_{j_1}$. Since $s_j$ commutes with $s_m$ for $m \neq j \pm 1$ we have that

$$\pi_{T'} = \sigma_k \cdots \sigma_2 s_{n-1} \cdots s_{j+1} s_j s_{j-1} s_j \cdots s_{j_1} = \sigma_k \cdots \sigma_2 s_{n-1} \cdots s_{j+1} s_{j-1} s_j s_{j-1} \cdots s_{j_1} = \sigma_k \cdots \sigma_2 s_{j-1} \sigma_1.$$

Let us write the "minimal" decomposition of $\pi_T$ as $\tau_p \ldots \tau_1$ for some $p \in \mathbb{N}$ where each $\tau_i = s_{n-i} \cdots s_{m_i}$. Note that if $j_1 \neq j+1$ then the element in the last box (i.e. the box of $n$ in $T^\lambda$) is the same in both $T$ and $T'$ and thus $\sigma_1 = \tau_1$. If $j_1 > j+1$ then $j$ is in the same box in $T$ and $\sigma_1 T$ and if $j_1 < j$ then the box of $j$ in $T$ will be filled with $j-1$ in $\sigma_1 T$. We can continue accommodating $s_j$ or $s_{j-1}$ in the same way with $\sigma_2$ and so on. Eventually we will end this process and obtain the "minimal" decomposition of $\pi_T$ which concludes the lemma. □

Proposition 3.47 is giving us a unique way of choosing all the GZ-vectors just by setting the highest weight vector $v_{T^\lambda}$. And as we will see ahead, this choice actually determines the action of $S_n$ on the Specth modules.

**Example 3.49.** Let us calculate what we would obtain as Young bases in the case of $S_3$ and $S_4$. Clearly, for the trivial and sign representations the choice of factors in the GZ-vectors is not very relevant because they are 1-dimensional. For the other irreducible representations we will choose $T^\lambda$ as the corresponding vector in example 3.13 and calculate the other vectors. For the regular representation of $S_3$, $S^{(2,1)}$ we chose $v_{T^{(2,1)}}$ as $v_3 := (1,1,-2)$. The only admissible transposition in this case is $s_2$ so let us find the factor for $v_{T'}$ with $T' = s_2 T^{(2,1)}$. According to proposition 3.47 we have

$$v_4 := v_{T'} = s_2(v_{T^{(2,1)}}) - \frac{1}{-1-1} v_{T^{(2,1)}} = (1,-2,1) + \frac{1}{2}(1,1,-2) = \frac{3}{2}(1,-1,0).$$

Now we can compute the action of the Coxeter generators of $S_3$, $s_1, s_2$ over this basis.

$$s_1 \cdot v_3 = (1,1,-2) \qquad s_2 \cdot v_3 = (1,-2,1) = \frac{-1}{2}(1,1,-2) + \frac{3}{2}(1,1,0)$$

$$s_1 \cdot v_4 = \frac{3}{2}(-1,1,0) \qquad s_2 \cdot v_4 = \frac{3}{2}(1,0,-1) = \frac{3}{4}(1,1,-2) + \frac{1}{2}\frac{3}{2}(1,-1,0)$$

We can do the same process for the irreducible representations of $S_4$. For $S^{(3,1)} = W_4$, consider $v_{T^{(3,1)}} = v_3 := (1,1,1,-3)$. We have only one admissible transposition for $T^{(3,1)}$, $s_3$. Let $T' = s_3 T^{(3,1)}$.

$$v_4 := v_{T'} = s_3(v_{T^{(3,1)}}) - \frac{1}{-1-2} v_{T^{(3,1)}} = (1,1,-3,1) + \frac{1}{3}(1,1,1,-3) = \frac{4}{3}(1,1,-2,0).$$

Note that $s_2$ is admissible for $T'$ so we can use again proposition 3.47 to find the final vector for our Young basis of $W_4$. Let $T'' = s_2 T' = s_2 s_3 T^{(3,1)}$. Then,

$$v_5 := v_{T''} = s_2(v_{T'}) - \frac{1}{-1-1} v_{T'} = \frac{4}{3}(1,-2,1,0) + \frac{1}{2}\frac{4}{3}(1,1,-2,0) = 2(1,-1,0,0).$$

Computing the action of the Coxeter generators $s_1, s_2$ and $s_3$ we obtain

$$\begin{aligned}
&s_1 \cdot v_3 = (1,1,1,-3) \quad s_1 \cdot v_4 = \frac{4}{3}(1,1,-2,0)\\
&s_1 \cdot v_5 = -2(1,-1,0,0) \quad s_2 \cdot v_3 = (1,1,1,-3)\\
&s_2 \cdot v_4 = \frac{4}{3}(1,-2,1,0) = \frac{-1}{2}\frac{4}{3}(1,1,-2,0) + 2(1,-1,0,0)\\
&s_2 \cdot v_5 = 2(1,0,-1,0) = \frac{3}{4}\frac{4}{3}(1,1,-2,0) + \frac{1}{2}2(1,-1,0,0)\\
&s_3 \cdot v_3 = (1,1,-3,1) = \frac{-1}{3}(1,1,1,-3) + \frac{4}{3}(1,1,-2,0)\\
&s_3 \cdot v_4 = \frac{4}{3}(1,1,0,-2) = \frac{8}{9}(1,1,1,-3) + \frac{1}{3}\frac{4}{3}(1,1,-2,0) \quad s_3 \cdot v_5 = 2(1,-1,0,0).
\end{aligned}$$

Now, for $S^{(2,1,1)} = W_4 \otimes \mathrm{sgn}_4$ we will chose $v_8 := (1,-1,0,0) \otimes v_2$ as $v_{T^{(2,1,1)}}$. The only admissible transposition for $T^{(2,1,1)}$ is $s_2$. Let $T' = s_2 T^{(2,1,1)}$ then,

$$v_7 := v_{T'} = s_2(v_{T^{(2,1,1)}}) - \frac{1}{-1-1} v_{T^{(2,1,1)}} = (-1,0,1,0) \otimes v_2 + \frac{1}{2}(1,-1,0,0) \otimes v_2 = \frac{-1}{2}(1,1,-2,0) \otimes v_2.$$

In this case, we have that $s_3$ is admissible for $T'$. Let $T'' = s_3 T' = s_3 s_2 T^{(2,2,1)}$ then,

$$v_6 := v_{T''} = s_3(v_{T'}) - \frac{1}{-2-1} v_{T'} = \frac{1}{2}(1,1,0,-2) \otimes v_2 - \frac{1}{3}\frac{1}{2}(1,1,-2,0) \otimes v_2 = \frac{1}{3}(1,1,1,-3) \otimes v_2.$$

Computing the action of the Coxeter generators $s_1, s_2$ and $s_3$ we obtain

$$\begin{aligned}
&s_1 \cdot v_6 = \frac{-1}{3}(1,1,1,-3) \otimes v_2 \quad s_1 \cdot v_7 = \frac{1}{2}(1,1,-2,0) \otimes v_2\\
&s_1 \cdot v_8 = (1,-1,0,0) \otimes v_2 \quad s_2 \cdot v_6 = \frac{-1}{3}(1,1,1,-3) \otimes v_2\\
&s_2 \cdot v_7 = \frac{1}{2}(1,-2,1,0) \otimes v_2 = \frac{1}{2}\frac{-1}{2}(1,1,-2,0) \otimes v_2 + \frac{3}{4}(1,-1,0,0) \otimes v_2\\
&s_2 \cdot v_8 = (-1,0,1,0) \otimes v_2 = \frac{-1}{2}(1,1,-2,0) \otimes v_2 + \frac{-1}{2}(1,-1,0,0) \otimes v_2\\
&s_3 \cdot v_6 = \frac{1}{3}(-1,-1,3,-1) \otimes v_2 = \frac{1}{3}\frac{1}{3}(1,1,1,-3) \otimes v_2 + \frac{8}{9}\frac{-1}{2}(1,1,-2,0) \otimes v_2\\
&s_3 \cdot v_7 = \frac{-1}{2}(1,1,0,-2) \otimes v_2 = \frac{1}{3}(1,1,1,-3) \otimes v_2 + \frac{1}{3}\frac{1}{2}(1,1,-2,0) \otimes v_2 \quad s_3 \cdot v_8 = -(1,-1,0,0) \otimes v_2
\end{aligned}$$

Finally, let us find the factors for $S^{(2,2)} = U$. Recall that as a vector space $U = W_3$ and the action of $S_4$ is given by the quotient $S_3 \cong S_4/\{id,(12)(34),(13)(24)\}$. Then, the Young basis is the same as for $W_3$: $v_9 := (1,1,-2)$ and $v_{10} := \frac{3}{2}(1,-1,0)$. We summarize the calculations made in table 3.3.

The results found on example 3.49 have a generalizing formula that gives us a concise description of the irreducible representations of $S_n$. This is the fundamental result of the Vershik-Okounkov approach to the representations of $S_n$ [VO05, Proposition 6.1] [CSST10, Theorem 3.4.2] and [MM16, Theorem 6.2]. What makes the following theorem so important is that the usual de-

Table 3.3: Irreducible representations of $S_3$ and $S_4$

| Irrep | Vector | $s_1$ | $s_2$ |
|---|---|---|---|
| $\mathrm{triv}_3$ | $v_1$ | $v_1$ | $v_1$ |
| $\mathrm{sgn}_3$ | $v_2$ | $-v_2$ | $-v_2$ |
| $W_3$ | $v_3 = (1,1,-2)$<br>$v_4 = \frac{3}{2}(1,-1,0)$ | $v_3$<br>$-v_4$ | $\frac{-1}{2}v_3 + \frac{3}{2}v_4$<br>$\frac{3}{4}v_3 + \frac{1}{2}v_4$ |

| Irrep | Vector | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|---|
| $\mathrm{triv}_4$ | $v_1$ | $v_1$ | $v_1$ | $v_1$ |
| $\mathrm{sgn}_4$ | $v_2$ | $-v_2$ | $-v_2$ | $-v_2$ |
| $W_4$ | $v_3 = (1,1,1,-3)$<br>$v_4 = \frac{4}{3}(1,1,-2,0)$<br>$v_5 = 2(1,-1,0,0)$ | $v_3$<br>$v_4$<br>$-v_5$ | $v_3$<br>$\frac{-1}{2}v_4 + v_5$<br>$\frac{3}{4}v_4 + \frac{1}{2}v_5$ | $\frac{-1}{3}v_3 + v_4$<br>$\frac{8}{9}v_3 + \frac{1}{3}v_4$<br>$v_5$ |
| $W_4 \otimes$ $\mathrm{sgn}_4$ | $v_6 = \frac{1}{3}(1,1,1,-3) \otimes v_2$<br>$v_7 = \frac{-1}{2}(1,1,-2,0) \otimes v_2$<br>$v_8 = (1,-1,0,0) \otimes v_2$ | $-v_6$<br>$-v_7$<br>$v_8$ | $-v_6$<br>$\frac{1}{2}v_7 + \frac{3}{4}v_8$<br>$v_7 + \frac{-1}{2}v_7$ | $\frac{1}{3}v_6 + \frac{8}{9}v_7$<br>$v_6 + \frac{-1}{3}v_7$<br>$-v_8$ |
| $U$ | $v_9 = (1,1,-2)$<br>$v_{10} = \frac{3}{2}(1,-1,0)$ | $v_9$<br>$-v_{10}$ | $\frac{-1}{2}v_9 + \frac{3}{2}v_{10}$<br>$\frac{3}{4}v_9 + \frac{1}{2}v_{10}$ | $v_9$<br>$-v_{10}$ |

scription of the representation theory of symmetric group is fairly more complicated.

**Theorem 3.50** (Young's seminormal form)**.** Let us consider that we chose our Young bases as in proposition 3.47. Let $\lambda$ be a partition of $n$ and $T \in \mathrm{SYT}(\lambda)$ such that $C(T) = (a_1, \dots, a_n)$. For any $j = 1, \dots, n-1$ we have that

(i) if $a_{j+1} = a_j \pm 1$ then $s_j v_T = \pm v_T$,

(ii) if $a_{j+1} \neq a_j \pm 1$ then for $T' = s_j T$ we have

$$s_j(v_T) = \begin{cases} \frac{1}{a_{j+1}-a_j} v_T + v_{T'}, & \text{if } l(\pi_{T'}) > l(\pi_T) \\ \frac{1}{a_{j+1}-a_j} v_T + \left(1 - \frac{1}{(a_{j+1}-a_j)^2}\right) v_{T'}, & \text{if } l(\pi_{T'}) < l(\pi_T). \end{cases}$$

*Proof.* Part (i) reduces to 2. from proposition 3.18. For part (ii) let $\pi_T = s_{i_1} \cdots s_{i_k}$ be the "minimal" decomposition of $\pi_T$. Note that $\pi_{T'} = \pi_T s_j$, then by equation (1.1) in the proof of proposition 1.5 we have that $l(\pi_{T'}) = l(\pi_T) \pm 1$. Let us consider first when $l(\pi_{T'}) = l(\pi_T) + 1 > l(\pi_T)$. Thus, $s_{i_1} \cdots s_{i_k} s_j$ is a decomposition of length $l(\pi_{T'})$. By lemma 3.48 we have that $v_{T'}$ as chosen in proposition 3.47 is equal to $s_j(v_T) - \frac{1}{a_{j+1}-a_j} v_T$ and thus the result follows directly. Similarly for the case $l(\pi_{T'}) = l(\pi_T) - 1 < l(\pi_T)$ we have that $v_T = v_{s_j T'}$. And using 3. from proposition 3.18 we obtain that

$$s_j(v_T) = \frac{1}{C(T')_j - C(T')_{j+1}} v_T + \left(1 - \frac{1}{(C(T')_j - C(T')_{j+1})^2}\right) v_{T'} = \frac{1}{a_{j+1} - a_j} v_T + \left(1 - \frac{1}{(a_{j+1} - a_j)^2}\right) v_{T'}.$$

□

Let $(V, \rho)$ be any representation of $S_n$ and $\lambda$ a partition of $n$. We will denote $S_0^\lambda$ as the one-dimensional space in the Specht module $S^\lambda$ that corresponds to the possible highest weight vectors and we will denote $V_0$ as the vector space that corresponds to the highest weight vectors from $\lambda$. That is, $V_0$ consists of the YJM elements' eigenvectors with eigenvalues precisely $C(T^\lambda)$.

**Corollary 3.51.** Let $\lambda$ be any partition of $n$ and $(V, \rho)$ a representation of $S_n$. Taking $k = \dim(\mathrm{Hom}_{S_n}(S^\lambda, V))$, we have a one-to-one correspondence.

$$\left\{\varphi : (S^\lambda)^k \to V, \text{ representation morphism}\right\} \quad \Leftrightarrow \quad \left\{f : (S_0^\lambda)^k \to V_0, \text{ linear map}\right\}.$$

*Proof.* It is clear that for every representation morphism $\varphi : (S^\lambda)^k \to V$ we get an induced linear map $(S_0^\lambda)^k \to V$ where each $v \in (S_0^\lambda)^k$ simply maps to $\varphi(v)$. Let $C(T^\lambda) = (a_1, \dots, a_n)$. By definition of $S_0^\lambda$ we know that $X_i(v) = a_i v$ and since $X_i$ is the sum of $\rho(\tau)$ for some $\tau \in S_n$ we have that $X_i$ and $\varphi$ commute. Thus, $X_i(\varphi(v)) = \varphi(X_i(v)) = a_i\varphi(v)$ which means that $\varphi(v)$ is an eigenvector for the YJM elements with eigenvalues $C(T^\lambda)$, i.e. $\varphi(v) \in V_0$.

For the other part of the correspondence we will start with a linear map $f : (S_0^\lambda)^k \to V_0$. By proposition 3.47 we have that fixing a non-zero vector in each component of $(S_0^\lambda)^k$ gives us a basis for $(S^\lambda)^k$. We can view this as $k$ copies of a GZ-basis of $S^\lambda$ and denote it as $\{v_T^i : T \in \mathrm{SYT}(\lambda),\ i = 1, \dots, k\}$. Let us define then a map $\varphi : (S^\lambda)^k \to V$ inductively on the Coxeter length of $\pi_T$. For $l(\pi_T) = 0$, i.e. $T = T^\lambda$, we define $\varphi(v^i_{T^\lambda}) = f(v^i_{T^\lambda})$. Now, let $\pi_T = s_{j_1} \cdots s_{j_l}$ be its "minimal" decomposition. Then $T' = s_{j_l} T$ has Coxeter length $l-1$. Assuming we have defined $\varphi(v^i_{T'})$ already we set

$$\varphi(v_T^i) = \rho(s_{j_l})(\varphi(v^i_{T'})) - \frac{1}{C(T')_{j_l+1} - C(T')_{j_l}}\varphi(v^i_{T'}).$$

Theorem 3.50 guarantees that the action of $\rho$ will be compatible with our choice of a GZ-basis. And, thus, $\varphi$ will be a representation morphism. □

Another Young's formula comes fixing an $S_n$-invariant inner-product over $S^\lambda$ and considering a orthonormal basis $\{w_T : T \in \mathrm{SYT}(\lambda)\}$ where $w_T = \frac{v_T}{\|v_T\|}$ for $v_T$ chosen as in proposition 3.47.

**Theorem 3.52** (Young's orthogonal form)**.** Denoting $C(T) = (a_1, \dots, a_n)$ and $r = a_{j+1} - a_j$ we have

$$s_j w_T = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{s_j T}.$$

*Proof.* We set again $T' = s_j T$. If $s_j$ is not admissible we have that $r = a_{j+1} - a_j = \pm 1$ and thus $s_j(w_T) = \pm w_T$ as desired. For $s_j$ admissible we will consider two cases. For $l(\pi_{T'}) > l(\pi_T)$ we have

$$\begin{aligned}
\|v_{T'}\|^2 &= \|s_j(v_T) - \frac{1}{r} v_T\|^2 \\
&= \|v_T\|^2 - \frac{1}{r}(\langle s_j(v_T), v_T\rangle + \langle v_T, s_j(v_T)\rangle) + \frac{1}{r^2}\|v_T\|^2 && \text{since } \langle,\rangle \text{ is } S_n\text{-invariant} \\
&= \|v_T\|^2 - \frac{1}{r}(\langle \frac{1}{r} v_T + v_{T'}, v_T\rangle + \langle v_T, \frac{1}{r} v_T + v_{T'}\rangle) + \frac{1}{r^2}\|v_T\|^2 \\
&= \|v_T\|^2 - \frac{2}{r^2}\|v_T\|^2 + \frac{1}{r^2}\|v_T\|^2 && \text{since } v_T \text{ and } v_{T'} \text{ are orthogonal} \\
&= \left(1 - \frac{1}{r^2}\right)\|v_T\|^2.
\end{aligned}$$

$$\Rightarrow \quad s_j(w_T) = \frac{1}{\|v_T\|} s_j(v_T) = \frac{1}{\|v_T\|}\left(\frac{1}{r} v_T + v_{T'}\right) = \frac{1}{r}\frac{v_T}{\|v_T\|} + \sqrt{1 - \frac{1}{r^2}}\frac{v_{T'}}{\|v_{T'}\|} = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{T'}.$$

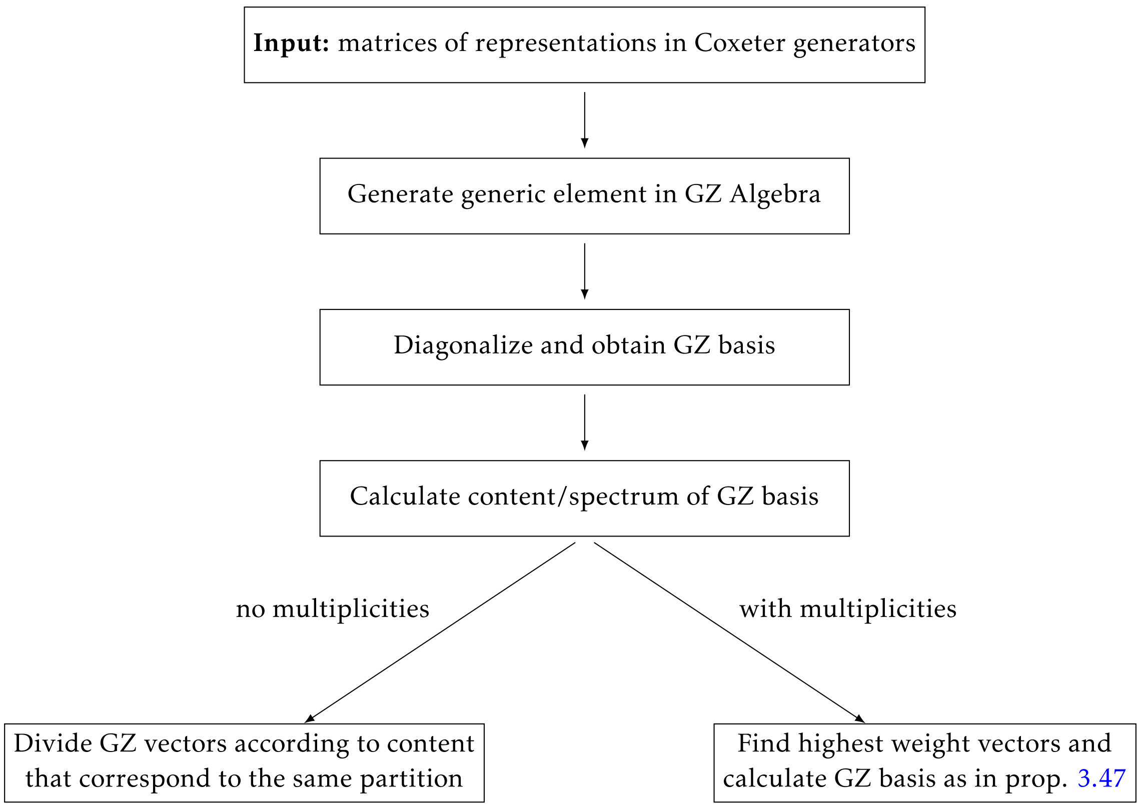
Similarly, for $l(\pi_{T'}) < l(\pi_T)$ we have that $\|v_T\|^2 = \left(1 - \frac{1}{r^2}\right)\|v_{T'}\|^2$. Thus, we have that

$$s_j(w_T) = \frac{1}{\|v_T\|} s_j(v_T) = \frac{1}{r}\frac{v_T}{\|v_T\|} + \left(1 - \frac{1}{r^2}\right)\frac{v_{T'}}{\sqrt{1 - \frac{1}{r^2}}\|v_{T'}\|} = \frac{1}{r} w_T + \sqrt{1 - \frac{1}{r^2}}\, w_{T'}.$$

□

# Chapter 4: Algorithm in Julia

In this chapter we will present an algorithm that decomposes representations of $S_n$ into its irreducible components giving also a GZ-basis for each (find the code in this GitHub repository). The goal of this part is to show the usefulness of thinking the representations of the symmetric groups (and eventually other Coxeter groups) in terms of the Vershik-Okounkov approach. The next diagram illustrates the general idea of the algorithm.




The first element of the program is to compute a generic element in the GZ-algebra. Here, we are actually working with the natural embedding of $GZ(n)$ into $\mathrm{End}_{\mathbb{C}}(W)$ where $(W,\rho)$ is the representation that was imputed into the algorithm. Explicitly we have the map

$$\psi : \sum_{\sigma\in S_n} a_\sigma e_\sigma \in GZ(n) \mapsto \sum_{\sigma\in S_n} a_\sigma \rho(\sigma) \in \mathrm{End}_{\mathbb{C}}(W).$$

We want to find a generic element in $\psi(GZ(n)) \subseteq \mathrm{End}_{\mathbb{C}}(W)$. Concretely we want to find $M \in \psi(GZ(n))$ whose eigenbasis is a GZ-basis for $W$.

**Theorem 4.1.** Let $M_1,\dots,M_m \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be simultaneously diagonalizable matrices. If the coefficients $(c_1,\dots,c_m)$ are "generic" enough (in the proof the exact condition will become clear) then a basis that diagonalizes $M = c_1M_1 + \ldots + c_mM_m$ also diagonalizes all $M_i$.

*Proof.* Let $Q$ be a matrix such that $Q^{-1}MQ = D$ a diagonal matrix with entries the eigenvalues of $M$. It is clear that if $C$ is the change of basis that simultaneously diagonalizes all $M_j$ then it also diagonalizes $M$. That is, $C^{-1}M_jC = D_j$ and $C^{-1}MC = D$ where again $D_j$ is a diagonal matrix with entries the eigenvalues of each $M_j$. Without loss of generality we will assume that the eigenvalues of all $D_j$ and $D$ are in non-decreasing order. Denoting $U = C^{-1}Q$ we have that

$$\begin{aligned} Q^{-1}MQ = Q^{-1}CDC^{-1}Q = U^{-1}DU = D \\ = \sum_{j=1}^{m} c_m Q^{-1} M_j Q = \sum_{j=1}^{m} c_m U^{-1} D_j U. \end{aligned}$$

The first line of equalities gives us that $DU = UD$. Hence, we obtain that

$$D_{jj}U_{ij} = \sum_{r=1}^{m} U_{ir}D_{rj} = (UD)_{ij} = (DU)_{ij} = \sum_{r=1}^{m} D_{ir}U_{rj} = D_{ii}U_{ij}.$$

If $D_{ii} \neq D_{jj}$ then $U_{ij} = 0$. Since we are assuming the diagonal of $D$ is in non-decreasing order the eigenvalues which are equal must adjacent. This implies that $U$ must be a block diagonal matrix where each block correspond to a basis change in each eigenspace of $M$. Now, we are going to impose a genericity condition on the coefficients $(c_1,\ldots,c_m)$. For $i = 1,\ldots,n$ and $j = 1,\ldots,m$ let $d_{ij} := (D_j)_{ii}$ be the eigenvalues of $M_j$. Note that the eigenvalues of $M$ are precisely $D_{ii} = c_1 d_{i1} + \ldots + c_m d_{im}$. If $D_{ii} = D_{kk}$ implies that $d_{ij} = d_{kj}$ for all $j$ we have that the block in $U$ that changes the basis of the eigenspace corresponding to $D_{ii} = D_{kk}$ in $M$ also leaves invariant the eigenspaces corresponding to $d_{ij} = d_{kj}$ in all $M_j$. Therefore, $Q^{-1}M_jQ = U^{-1}D_jU = D_j$ which is our desired result. □

**Remark 4.2.** For fixed matrices $M_1,\ldots,M_m$ with eigenvalues $d_{ij}$ as in theorem 4.1 we have that the genericity condition we imposed consists of choosing $(c_1,\ldots,c_m)$ outside a finite number of hyperplanes. Let $D_{ii} = D_{kk}$ and assume that there exists $j$ such that $d_{ij} \neq d_{kj}$. Then,

$$D_{ii} = D_{kk} \;\Leftrightarrow\; \sum_{p=1}^{m} c_p(d_{ip} - d_{kp}) = 0 \;\Leftrightarrow\; c_j(d_{kj} - d_{ij}) = \sum_{\substack{p=1 \\ p\neq j}}^{m} c_p(d_{ip} - d_{kp}) \;\Leftrightarrow\; c_j = \frac{\sum_{p\neq j} c_p(d_{ip} - d_{kp})}{d_{kj} - d_{ij}}.$$

This is a linear equation on the coefficients $(c_1,\ldots,c_m)$ that corresponds to a hyperplane $H(i,k,j)$ in $\mathbb{C}^m$. By taking into account all possible $i,k$ and $j$ we obtain a finite number of hyperplanes. And it suffices to take $(c_1,\ldots,c_m)$ outside of all of them to guarantee the result of theorem 4.1.

To guarantee that the element is generic we find a basis for $\psi(GZ(n))$, we pick random integers with the Random module of Julia and compute a random linear combination in $\psi(GZ(n))$. When we diagonalize this matrix we obtain a basis in which all $\psi(GZ(n))$ acts diagonally. By corollary 2.11 we have that this is precisely a GZ-basis of $W$. We propose two approaches to find a basis of $\psi(GZ(n))$.

**First approach:** In this strategy we will use that $\dim GZ(n)$ and $\dim W$ are bounds of the dimension of $\psi(GZ(n))$. The first bound follows directly from the fact that $\psi$ is an algebra morphism. The second bound comes from proposition 2.8. Indeed, note that $\psi(GZ(n))$ necessarily lives inside the maximal commutative subalgebra of $\mathrm{End}_{\mathbb{C}}(W)$ that corresponds to the linear operators that act diagonally on a GZ-basis of $W$ which has dimension $\dim W$. Thus, we start building our basis by taking the image of the YJM elements in $\psi(GZ(n))$. It may be necessary to eliminate some if they are not linearly independent. Then we start gradually adding the image of double products $X_iX_j$, triple products $X_iX_jX_k$ and so on. Each time we add an element we check first if it is linearly independent from the ones we already have in our list. We stop adding elements when we reach our bound $\min\{\dim GZ(n), \dim W\}$ or when we have added all possible linearly independent products of the YJM elements. The idea behind adding the products of the YJM elements is that that since they generate $GZ(n)$ as an algebra we have that every element in the GZ-algebra can be written as a linear combination of their products.

**Second approach:** For our second strategy we will also use the characterization of the GZ-algebra from proposition 2.8 but in a different way. Since $GZ(n)$ is a maximal commutative subalgebra and it is generated by the YJM elements we have that

$$\psi(GZ(n)) \subseteq C := \{B \in \mathrm{End}_{\mathbb{C}}(W) : B\psi(X_i) = \psi(X_i)B \quad \forall i = 1, \dots n\}.$$

Since $\dim \mathrm{End}_{\mathbb{C}}(W) = (\dim W)^2$ we have that the condition $B\psi(X_i) = \psi(X_i)B$ corresponds to $(\dim W)^2$ linear conditions that can be solved with basic linear algebra. Therefore, we can explicitly find a basis for $C$ and take a random linear combination of it.

Once we have our generic element in $\psi(GZ(n))$ we can diagonalize it and obtain its eigenbasis which will be a GZ-basis of $W$. For each eigenvector $v$ we can calculate the action of $X_i$ for $i = 1, \dots, n$ to obtain the weight of $v$, $\alpha(v) = (a_1, \dots, a_n)$ with the formula $a_i = \frac{\langle X_i \cdot v, X_i \cdot v \rangle}{\langle v, v \rangle}$. By theorem 3.43 we have that the spectrum of this GZ-basis corresponds to the content of the Young tableaux corresponding to the irreducible representations that appear in $W$. Moreover, we have that the equivalence relation $\alpha(v) \sim \alpha(w)$ ($v$ and $w$ come from the same irreducible representation) is the same as $\alpha(v) \simeq \alpha(w)$ (there exists a permutation $\sigma \in S_n$ such that $\sigma \cdot \alpha(v) = \alpha(w)$). This correspondence will allow us to decompose $W$ in its irreducible components.

**W is without multiplicities:** Using the equivalence relation $\simeq$ we check when two vectors have equivalent weights by comparing the set of their eigenvalues. Then, we divide the GZ-vectors in equivalence classes which correspond to basis for the irreducible representations of $S_n$ that appear in $W$. To determine the partition of $n$ that corresponds to each equivalence class we find its highest weight vector which is precisely the maximum with respect to the lexicographic order. Recall that if a $\simeq$ equivalence class corresponds to the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ then the highest weight vector is

$$C(T^\lambda) = (0, 1, 2, \dots, \lambda_1 - 1, -1, 0, 1, \dots, \lambda_2, \dots, 1 - j, 2 - j, \dots, \lambda_j - j \dots, 1 - k, \dots, \lambda_k - k).$$

Then, by checking the values of the entries in which the highest weight vector makes the jumps $\lambda_i - i, 1 - i$ we can find the corresponding partition.

**W may have multiplicities:** When the representation we want to decompose has multiplicities the previous approach does not work. Indeed, if $\rho \in \widehat{S_n}$ has two copies inside of $W$ when we find the GZ-basis we will have $2 \dim \rho$ vectors with equivalent weight vectors and there is not a direct way of checking which lie in which copy of $\rho$. The solution we propose to this inconvenience is to use the canonical GZ-basis that was determined in proposition 3.47. Similarly to the previous case we can still determine the highest weight vectors in each equivalence class and their corresponding partition $\lambda$. Additionally, using lemma 3.31 we can find for each weight vector $\alpha$ in the equivalence class of $\rho$ the minimal decomposition as a product of Coxeter generators of $\pi_\alpha$, the permutation that satisfies $\pi_\alpha \cdot \alpha = C(T^\lambda)$. With these ingredients we can inductively calculate the GZ-basis that corresponds to each highest weight vector of the copies $\rho$ in $W$ using the method in proposition 3.47.

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