On higher dimensional exact Courant algebroids

Camilo Rengifo Gutiérrez
Matemático, Ph.D.(c)
Code: 200322032

Universidad de los Andes
Facultad de Ciencias
Departamento de Matemáticas
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Advisor
Paul Bressler, Ph.D.
Profesor Asociado - Universidad de los Andes

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Abstract: In this thesis we establishes an equivalence of categories between the category of certain differential graded (dg-) Lie algebroids on the odd tangent bundle of a manifold and the category of (higher-dimensional) exact Courant algebroids on the manifold. In addition, it is shown that the above mentioned categories are equivalent to the category of dg-principal bundles with structure group the shifted additive group over the odd tangent bundle of the manifold.

Keywords: Lie algebroids. Courant algebroids. Differential graded Lie algebras. Derived bracket. Torsor of abelian groups. Sheaf of abelian groups. Sheaf of free generated modules. Cohomology of smooth manifolds. dg-manifolds

Título en español

Algebroides de Courant exactos de dimensión mayor

Resumen: En esta tesis se establece una equivalencia de categorías entre la categoría de cierta clases de algebroides diferenciales de Lie con graduación sobre el haz tangente impar a una variedad diferencial y la categría de algebroides de Courant exactos (de dimensión mayor) sobre la variedad diferencial. Adicionalmente, se demuestra que las categorías mencionadas anteriormente son equivalentes a la categoría de los haces principales diferenciales con graduación con grupo estructural el grupo aditivo corrido por un número natural sobre el haz tangente impar de la variedad diferencial.

Palabras clave: Algebroides de Lie. Algebroides de Courant. Álgebras de Lie diferenciales graduadas. Corchete derivado. Torsores de grupos abelianos. Gavilla de grupos abelianos. Gavilla de módulos finitamente generados. Cohomología de una variedad suave. Variedades diferenciales con graduación

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Aprobado

Jury
Henrique Bursztyn

Jury
Mikhail Malakhaltsev

Jury
Marco Zambon

Advisor
Paul Bressler

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To my family.

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Contents

Introduction

Courant algebroids were introduced in [15] in order to provide the analogue of the Drinfeld double for Lie bialgebras in the case of Lie bialgebroids. The first example of a Courant algebroid appeared in [8], where a bracket on the sections of the vector bundle $TX \oplus T^*X$ was defined in order to provide a geometric framework to study both Poisson structures and pre-symplectic structures on a manifold X, as Lagrangian isotropic vector sub-bundles with respect to the canonical pairing on $TX \oplus T^*X$.

Dmitry Roytenberg proposed an alternative point of view on the study of Courant algebroids in terms of super-manifolds, [19] taking as references several ideas found in [24], [25]. He proved that Courant algebroid structures on an Euclidean vector bundle on X are in one to one correspondence with degree two symplectic super-manifolds endowed with a cubic Hamiltonian which satisfies the Master Equation. One year earlier P. Severa and A. Weinstein proved that the Courant bracket is, in fact, a derived bracket [20]. Furthermore, they constructed a degree three cohomology class which characterized the Courant algebroid structure known as the *Severa class.*

More recently, in [23], using the derived algebra of symmetries of an $\mathbb{R}[2]$ dg-principal bundle over the odd tangent bundle, $T[1]X$, the author showed how to recover the Courant bracket as a derived bracket. In this thesis we provide a natural context for the study of Courant algebroids and their higher dimensional analogues in the framework of differential graded (dg-) geometry. The main result of this thesis is an explicit equivalence of categories which relates higher dimensional exact Courant algebroids on a smooth manifold with a special class of transitive Lie algebroids on the dg-manifold $T[1]X = (X, \Omega_X, d) = X^\sharp$, that we called $\mathcal{O}_{X^\sharp}[n]$-extensions of $\mathcal{T}_{X^\sharp}$.

Theorem .0.1. *Let $n \geq 1$. The category of $\mathcal{O}_{X^\sharp}[n]$-extension of $\mathcal{T}_{X^\sharp}$ is equivalent to the category of $(n-1)$-dimensional exact Courant algebroids over X.*

In other words, we related geometric data on X with geometric data on $X^\sharp$.

The thesis is organized in five chapters. In Chapter 1 we review the category of differential graded manifolds and prove that this category has finite products. In Chapter 2 we introduce the category of $\mathbb{R}[n]$ dg-principal bundles over $X^\sharp$. We show that this category is equivalent to the category of $\Omega_X^{n,cl}$-torsors which allows us to classify $\mathbb{R}[n]$ dg-principal bundles over $X^\sharp$. In Chapter 3 we construct the Atiyah algebroid of an $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$. The Atiyah algebroid as above provide examples of an $\mathcal{O}_{X^\sharp}[n]$-extension of $\mathcal{T}_{X^\sharp}$. In Chapter 4 we analyze the algebraic structure of an $\mathcal{O}_{X^\sharp}[n]$-extensions of $\mathcal{T}_{X^\sharp}$. In addition, we introduce a derived bracket and obtain a generalization of exact Courant algebroids. Finally, in Chapter 5 we introduce the category of higher dimensional Courant algebroids on X and prove that this category is equivalent to the category of $\mathcal{O}_{X^\sharp}[n]$-extensions of $\mathcal{T}_{X^\sharp}$. Moreover we give a cohomological classification of higher dimensional Courant algebroids.

CHAPTER 1

Preliminaries

Suppose that X is a smooth manifold. We denote by $\mathcal{C}_X^\infty$ the structure sheaf of smooth functions on X with values in $\mathbb{R}$. The sheaf of vector fields on X, i.e. sections of the vector bundle $TX \to X$ or derivations of the ring $\mathcal{C}_X^\infty$, will be denoted by $\mathcal{T}_X$. The sheaf of differential forms on X will be denote by Ω_X, i.e. k-differentials forms, Ω_X^k, are sections of the vector bundle $\bigwedge^k T^*X \to X$. Before we introduce the category of differential graded (dg-) manifolds we recall several facts about the category of graded vector spaces.

1.1 Graded vector spaces

The *local model* for a dg-manifold, as we will see, is given by the symmetric algebra of a graded vector space. In order to make things clear, we briefly recall several constructions that hold in the category of graded vector spaces. In addition, we recall the Koszul sign convention.

By definition a graded vector space is a real vector space of the form $W = \oplus_{i\in\mathbb{Z}} W^i$. For each $i \in \mathbb{Z}$, W^i is a real vector space whose elements are understood to have degree i. If $v \in W^i$, then $|v| = i$ will denote the degree of v. For any two graded vector spaces V and W, a *morphism*, is a linear map $f : V \to W$ such that $f(V^i) \subset W^i$ for each $i \in \mathbb{Z}$. The category of graded vector spaces is denoted by $Vect^{\mathbb{Z}}$. A graded vector space W is *concentrated in degree* k if the only non zero component is W^k. Let us describe additional structure in $Vect^{\mathbb{Z}}$.

For any integer n, there is the so called *shift functor* $[n] : Vect^{\mathbb{Z}} \to Vect^{\mathbb{Z}}$, whose effect on objects is given by $([n](V))^j =: V[n]^j = V^{j+n}$ for V a graded vector space. If $f : W \to V$ is a morphism in $Vect^{\mathbb{Z}}$, then $[n](f) := f[n] : W[n] \to V[n]$, i.e. $f[n]^j := f^{j+n}$.

A graded map of *degree* n between W and V, $g : W \to V$, is a family of $\mathbb{R}$-linear maps $g^i : W^i \to V^{i+n}$, i.e. a morphism $g : W \to V[n]$. The set of graded maps of degree n will be denoted by $\underline{\mathrm{Hom}}^n(W, V) := \mathrm{Hom}(W, V[n])$.

The dual space of W, denoted as $W^\vee$, is defined by the following formula:

$$(W^\vee)^n = \underline{\mathrm{Hom}}^n(W, \mathbb{R}) = \mathrm{Hom}(W, \mathbb{R}[n]) = \mathrm{Hom}(W^{-n}, \mathbb{R}).$$

It follows that degree n of the dual of W becomes $(W^\vee)^n = (W^{-n})^\vee$.

1.1.1 Tensor, Symmetric and Exterior algebras

The tensor product of W and V, $W \otimes V$, is the graded vector space whose degree n component is written as

$$(W \otimes V)^n = \bigoplus_{i+j=n} (W^i \otimes V^j).$$

Example 1.1.1. *For any $n \in \mathbb{Z}$, $V[n] = V \otimes \mathbb{R}[n]$.*

In a similar way, the n-fold tensor product of W with itself is the graded vector space

$$(W^{\otimes n})^k := \bigoplus_{i_1+...+i_n=k} W^{i_1} \otimes ... \otimes W^{i_n}.$$

The tensor algebra $T(W)$ of a graded vector space W is defined by

$$T(W) := \bigoplus_{n\geq 0} W^{\otimes n},\ W^{\otimes 0} := \mathbb{R},$$

the grading of $T(V)$ is given by

$$T(W)^n = \bigoplus_{k>0} \bigoplus_{i_1+...+i_k=n} (W^{i_1} \otimes ... \otimes W^{i_k}).$$

The tensor algebra $T(W)$ can be identified with a free associative algebra $\mathbb{R}(W)$ generated by W.

Remark 1.1.2. *For each $i \in \mathbb{Z}$ it follows that $(W^{\otimes n})^i \subset (T(W))^i$.*

From the tensor algebra on a graded vector space W, we can construct the symmetric algebra of W. Let us consider the two-sided ideal I_S of $T(W)$ generated by elements of the form

$$a \otimes b - (-1)^{|a||b|} b \otimes a,$$

for any homogeneous elements $a, b \in W$. The symmetric algebra of W, $S(W)$, is the quotient of the tensor algebra by I_S,

$$S(W) = T(W)/I_S.$$

The symmetric algebra of W, $S(W)$ is identified with a free associative algebra $\mathbb{R}\langle w\rangle$, with extra relations $w_i w_j = (-1)^{|w_i||w_j|} w_j w_i$, where $w_i, w_j \in W$. It follows that odd elements of $w_i \in W$ satisfy $w_i w_i = w_i^2 = 0$ in $\mathbb{R}(W)$. $S(W)$ is called the free commutative algebra generated by W.

In a similar manner, we can construct the exterior algebra out of the tensor algebra of W. Let us consider the two-sided ideal $I_\wedge$ of $T(W)$ generated by elements of the form,

$$a \otimes b + (-1)^{|a||b|} b \otimes a,$$

for any homogenous elements $a, b \in W$. Therefore the exterior algebra of W, $\bigwedge(W)$ is given by the quotient

$$\Lambda(W) = T(W)/I_{\wedge}.$$

Lemma 1.1.3. *[28] Suppose that V is a vector space concentrated in degree zero, then there is an isomorphism*

$$S(V[1]) = \bigoplus_i (\Lambda^i V)[-i].$$

□

Remark 1.1.4. *Suppose that W is a vector space concentrated in degree zero. If $dim(W) = 1$ then $S(W[n]) = T(W[n])$.*

As we mentioned above, in the graded setting there is a sign convention rule, *Koszul convention*, that is expressed by the formula $v \otimes w = (-1)^{|w||v|} w \otimes v$. In addition, when we multiply morphisms between graded vector spaces, we have to take into account signs. Namely if $f : W \to V$ and $g : W' \to V'$ are two graded maps, $f \otimes g : W \otimes W' \to V \otimes V'$ is given by $(f \otimes g)(w \otimes v) = (-1)^{|w||g|} f(w) \otimes g(v)$, where $|g|$ denotes the degree of g.

1.2 The category of dg-manifolds.

In this section, we define the category of dg-manifolds and prove that this category has finite products. In addition, we introduce several objects of differential calculus that extend to this category in a natural way.

Definition 1.2.1. *A graded manifold is a ringed space $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ where X is a smooth manifold and $\mathcal{O}_{\mathcal{X}}$ is a sheaf of graded algebras locally isomorphic to $\mathcal{C}^{\infty}_X \otimes S(W)$, for W a positively graded vector space.*

Definition 1.2.2. *A dg-manifold is a graded manifold* $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ *endowed with a degree one derivation* ∂ *of the algebra* $\mathcal{O}_{\mathcal{X}}$*, whose square is zero;* $\partial^2 = \partial \circ \partial = 0$.

Suppose that we are given two dg-manifolds $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}}, \partial)$ and $\mathcal{Y} = (Y, \mathcal{O}_{\mathcal{Y}}, \partial)$. A *morphism of dg-manifolds* $\mathcal{X} \to \mathcal{Y}$ is a pair of maps (f, f^*), where $f : X \to Y$ is a smooth map, and f^* is a map of sheaves of algebras $f^* : f^{-1}(\mathcal{O}_{\mathcal{Y}}) \to \mathcal{O}_{\mathcal{X}}$ which commutes with differentials; i.e. f^* is a map of differential graded algebras[1]. Given two dg-morphisms $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ the composition $g \circ f : \mathcal{X} \to \mathcal{Z}$ is given by the pair of maps $(g \circ f, (g \circ f)^*)$. At the level of spaces $g \circ f : X \to Z$. At the level of sheaves, from the map $g^* : g^{-1}\mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{Y}}$ we get a map of sheaves on X, by the pullback along f, namely $f^*(g^{-1}) : f^{-1} \circ g^{-1}\mathcal{O}_{\mathcal{Z}} \to f^{-1}\mathcal{O}_{\mathcal{Y}}$. Therefore our desired map

$$(g \circ f)^* : (f^{-1} \circ g^{-1})\mathcal{O}_{\mathcal{Z}} \to \mathcal{O}_{\mathcal{X}},$$

is obtained as the composition

$$f^* \circ f^*(g^{-1}) : f^{-1} \circ g^{-1}\mathcal{O}_{\mathcal{Z}} \to f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}.$$

For each $x \in X$, the compatibility of $(g \circ f)^*$ with the differentials is guaranteed since both morphisms f^* and g^* are maps of differential graded algebras:

$$\begin{array}{ccccc} \mathcal{O}_{\mathcal{Z},g(f(x))} & \xrightarrow{g^*} & \mathcal{O}_{\mathcal{Y},f(x)} & \xrightarrow{f^*} & \mathcal{O}_{\mathcal{X},x} \\ \downarrow \partial & & \partial \downarrow & & \downarrow \partial \\ \mathcal{O}_{\mathcal{Z},g(f(x))}[1] & \xrightarrow{g^*} & \mathcal{O}_{\mathcal{Y},f(x)}[1] & \xrightarrow{f^*} & \mathcal{O}_{\mathcal{X},x}[1]. \end{array}$$

The category of dg-manifolds will be denoted by *dgMnflds*.

1.2.1 Examples

Example 1.2.3. *Suppose that* $n > 0$*, and let* V *be a real vector space concentrated in degree zero. If we shift* V *by* n*,* $V[n]$*, it follows that* $\mathcal{O}_{V[n]} := S(V^\vee[-n])$*. In*

[1]Recall that $f^{-1}\mathcal{O}_{\mathcal{Y}}$ denotes the inverse image of $\mathcal{O}_{\mathcal{Y}}$ under f

particular, for $V = \mathbb{R}$*, we obtain that*

$$\mathcal{O}_{\mathbb{R}[n]} = S(\mathbb{R}[-n]) =: S[t],$$

where t *is the linear coordinate in* $\mathbb{R}^\vee$ *that generates* $S[t]$*. Therefore,* $\mathbb{R}[n] = (\{*\}, S[t], \partial = 0)$ *is a dg-manifold.*

Example 1.2.4. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. Let* $\mathfrak{g}[1]$ *the dg-manifold whose structure sheaf is equal to* $\mathcal{O}_{\mathfrak{g}[1]} = S(\mathfrak{g}^\vee[-1]) = \bigoplus_{i>0}(\bigwedge^i \mathfrak{g}^\vee)[-i]$*. The Chevalley-Eilenberg differential of* $\mathfrak{g}$*,* d_{ch}*, gives rise to the differential on* $\mathfrak{g}[1]$*.*

Example 1.2.5. *For any smooth manifold* X*, there is a canonical dg-manifold defined by* $T[1]X := (X, \Omega_X, d) =: X^\sharp$*, where* d *is the de Rham differential*[2]*.*

Example 1.2.6. *Any smooth manifold* X *gives rise to a dg-manifold concentrated in degree* 0 *with differential* $\partial = 0$*, namely* $(X, \mathcal{C}^\infty_X, 0)$*.*

Example 1.2.7. *Let* $\rho : E \to X$ *be a Lie algebroid on* X*. The dg-manifold* $E[1]$ *is defined by*

$$E[1] = (X, \mathcal{O}_{E[1]}, d_E),$$

where $\mathcal{O}_{E[1]} := \Omega_E = S(E^\vee[-1])$ *and* d_E *is the differential induced by the Lie algebroid structure on* $E \to X$*. Namely*

$$d_E(f)(e) = \rho(e)f,$$

$$d_E(\phi)(e_1, e_2) = \phi(e_1)e_2 - \phi(e_2)e_1 - \phi([e_1, e_2]),$$

for any $f \in \mathcal{C}^\infty_X$*,* $e_1, e_2 \in \Gamma(X, E)$*.*

Example 1.2.8. *Given a dg-manifold* $\mathcal{X} = (X, \mathcal{O}_\mathcal{X}, \partial)$*, by definition* $\mathcal{O}_\mathcal{X}$ *contains as the degree zero part the smooth functions on* X*,* $\mathcal{O}^0_\mathcal{X} = \mathcal{C}^\infty_X$*. Consider the morphism*

$$\mathcal{X} \to (X, \mathcal{C}^\infty_X, 0)$$

[2] We adopt different notation for $T[1]X$, for reasons which will become clear in Chapter 3.

which corresponds to $id_X : X \to X$ *and the map of algebras* $\mathcal{C}^\infty_X \hookrightarrow \mathcal{O}_\mathcal{X}$*. Notice that this map is not a dg-morphism. However, there is a dg-morphism in the opposite direction. Namely,* $(X, C^\infty_X, 0) \to \mathcal{X}$ *given by identity on* X id_X*, and the projection onto the degree zero part,* $\mathcal{O}_\mathcal{X} \twoheadrightarrow \mathcal{C}^\infty_X$*.*

Example 1.2.9. *There is a functor between the category of smooth manifolds and dgMnflds, denoted by* $T[1]$*, given by* $(X, \mathcal{C}^\infty_X) \mapsto (X, \Omega_X, d) = T[1]X = X^\sharp$*. The effect on morphisms is given by the pullback of differential forms. It is clear that* $T[1]$ *is a faithful functor.*

Example 1.2.10. *In the category of dg-manifolds there is a terminal object which is denoted by* $(\{*\}, \mathbb{R}, 0)$*. Notice that for any smooth manifold* X*,* $\mathbb{R} \subset \mathcal{C}^\infty_X$ *as the sheaf of locally constant functions on* X*. Therefore for any dg-manifold* $\mathcal{X} = (X, \mathcal{O}_\mathcal{X}, \partial)$ *there is the dg-morphism* $\mathcal{X} \to (\{*\}, \mathbb{R}, 0)$*.*

1.2.2 Products in *dgMnflds*

Let $\mathcal{X} = (X, \mathcal{O}_\mathcal{X}, \partial)$ and $\mathcal{Y} = (Y, \mathcal{O}_\mathcal{Y}, \partial)$ be two dg-manifolds. Recall that the box product between $\mathcal{O}_\mathcal{X}$ and $\mathcal{O}_\mathcal{Y}$ is given by $\mathcal{O}_\mathcal{X} \boxtimes \mathcal{O}_\mathcal{Y} := (pr_X^{-1}\mathcal{O}_\mathcal{X}) \otimes (pr_Y^{-1}\mathcal{O}_\mathcal{Y})$. We define the product, $\mathcal{X} \times \mathcal{Y}$, as $(X \times Y, \mathcal{O}_{\mathcal{X}\times\mathcal{Y}}, \partial)$, where

$$\mathcal{O}_{\mathcal{X}\times\mathcal{Y}} := \mathcal{C}^\infty_{X\times Y} \otimes_{(pr_X^{-1}\mathcal{C}^\infty_X \otimes pr_Y^{-1}\mathcal{C}^\infty_Y)} (\mathcal{O}_\mathcal{X} \boxtimes \mathcal{O}_\mathcal{Y}),$$

pr_X, pr_Y denote the canonical projections $Y \xleftarrow{pr_Y} X \times Y \xrightarrow{pr_X} X$. The differential on $\mathcal{X} \times \mathcal{Y}$ ∂ is such that the restriction to $\mathcal{O}_\mathcal{X} \boxtimes \mathcal{O}_\mathcal{Y}$ is equal to $\partial = \partial \otimes 1 + 1 \otimes \partial$. The canonical projections, $\mathcal{Y} \xleftarrow{pr_\mathcal{Y}} \mathcal{X} \times \mathcal{Y} \xrightarrow{pr_\mathcal{X}} \mathcal{X}$, are defined as follows; $pr_\mathcal{X}$ is prescribed by the projection onto X, $pr_X : X \times Y \to X$ and the map of differential graded algebras given by

$$pr_X^* : pr_X^{-1}\mathcal{O}_X \to \mathcal{C}^\infty_{X\times Y} \otimes_{pr_X^{-1}\mathcal{C}^\infty_X \otimes pr_Y^{-1}\mathcal{C}^\infty_Y} (\mathcal{O}_X \boxtimes \mathcal{O}_Y),$$

$g \mapsto \mathbf{1} \otimes (g \otimes 1)$. Similarly for the projection onto $\mathcal{Y}$, $pr_{\mathcal{Y}}$.

We point out two facts useful in the proof of the universal property for products.

Remark 1.2.11. *Recall that for any two smooth maps $f : Z \to X$ and $g : Z \to Y$, there is a unique map $f \times g : Z \to X \times Y$, such that $f = pr_X \circ (f \times g)$ and $g = pr_Y \circ (f \times g)$. The pullback along these three maps, $f^* : f^{-1}\mathcal{C}^\infty_X \to \mathcal{C}^\infty_Z$, $g^* : g^{-1}\mathcal{C}^\infty_Y \to \mathcal{C}^\infty_Z$ and $(f \times g)^* : (f \times g)^{-1}\mathcal{C}^\infty_{(X\times Y)} \to \mathcal{C}^\infty_Z$ are related by the formulas below,*

$$f^* = (f \times g)^* \circ pr_X^*, \;\; g^* = (f \times g)^* \circ pr_Y^*.$$

Remark 1.2.12. *Suppose that we are given two dg-morphisms, $f : \mathcal{Z} \to \mathcal{X}$ and $g : \mathcal{Z} \to \mathcal{Y}$. By multiplication there is a map of differential graded algebras,*

$$f^* \otimes g^* : f^{-1}(\mathcal{O}_\mathcal{X}) \otimes_{f^{-1}\mathcal{C}^\infty_X \otimes g^{-1}\mathcal{C}^\infty_Y} g^{-1}(\mathcal{O}_\mathcal{Y}) \to \mathcal{O}_\mathcal{Z} \otimes \mathcal{O}_\mathcal{Z} \to \mathcal{O}_\mathcal{Z},$$

where the last map is just the multiplication on the algebra $\mathcal{O}_\mathcal{Z}$.

Proposition 1.2.13. *In dgMnflds finite products exist.*

Proof: We show that the dg-manifold $(X \times Y, \mathcal{C}^\infty_{X\times Y} \otimes (\mathcal{O}_\mathcal{X} \boxtimes \mathcal{O}_\mathcal{Y}), \partial)$ with projections $pr_\mathcal{X}$, $pr_\mathcal{Y}$, satisfies the universal property for products. Suppose that $f : (Z, \mathcal{O}_\mathcal{Z}, \partial) \to (X, \mathcal{O}_\mathcal{X}, \partial)$ and $g : (Z, \mathcal{O}_\mathcal{Z}, \partial) \to (Y, \mathcal{O}_\mathcal{Y}, \partial)$ are two dg-morphisms. Then by Remark (1.2.12) there is a map $f^* \otimes g^* : f^{-1}\mathcal{O}_\mathcal{X} \otimes_{f^{-1}\mathcal{C}^\infty_X \otimes g^{-1}\mathcal{C}^\infty_Y} g^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{Z}$. Moreover, by Remark (1.2.11) there is a map $f \times g : Z \to X \times Y$ whose pullback is equal to $(f \times g)^* : (f \times g)^{-1}\mathcal{C}^\infty_{X\times Y} \to \mathcal{C}^\infty_Z$. We multiply both maps $f^* \otimes g^*$ and $(f \times g)^*$ to give rise to the following map,

$$(f \times g)^{-1}\mathcal{C}^\infty_{X\times Y} \otimes (f^{-1}\mathcal{O}_\mathcal{X} \otimes_{f^{-1}\mathcal{C}_X \otimes g^{-1}\mathcal{C}_Y} g^{-1}\mathcal{O}_\mathcal{Y}) \to \mathcal{O}_\mathcal{Z}. \qquad (1.2.1)$$

Using the compatibility between pullbacks and tensor products we obtain the identification below,

$$
\begin{aligned}
(f\times g)^{-1}\mathcal{C}^\infty_X \otimes (f^{-1}\mathcal{O}_\mathcal{X}\times g^{-1}\mathcal{O}_\mathcal{Y}) \cong & \\
&\cong (f\times g)^{-1}\mathcal{C}^\infty_{X\times Y}\otimes ((f\times g)^{-1}\circ pr_X^{-1}\mathcal{O}_\mathcal{X}\otimes (f\times g)^{-1}\circ pr_Y^{-1}\mathcal{O}_\mathcal{Y})\\
&\cong (f\times g)^{-1}\mathcal{C}^\infty_{X\times Y}\otimes ((f\times g)^{-1}(pr_X^{-1}\mathcal{O}_\mathcal{X}\otimes pr_Y^{-1}\mathcal{O}_\mathcal{Y}))\\
&\cong (f\times g)^{-1}(\mathcal{C}^\infty_{X\times Y}\otimes \mathcal{O}_\mathcal{X}\boxtimes \mathcal{O}_\mathcal{Y}).
\end{aligned}
$$

Then we rewrite (1.2.1) as follows:

$$(f\times g)^{-1}(\mathcal{C}^\infty_{X\times Y}\otimes \mathcal{O}_\mathcal{X}\boxtimes \mathcal{O}_\mathcal{Y})\to \mathcal{O}_\mathcal{Z}.$$

Therefore we obtain a dg-morphism $\mathcal{Z}\to\mathcal{X}\times\mathcal{Y}$. Conversely, given a dg-morphism

$$\phi:\mathcal{Z}\to\mathcal{X}\times\mathcal{Y}.$$

The map $\phi: Z\to X\times Y$ induces $\phi_X: Z\to X$ and $\phi_Y: Z\to Y$ by $\phi_X = pr_X\circ\phi$, $\phi_Y = pr_Y\circ\phi$. At the level of sheaves, ϕ gives rise to the map:

$$\phi^*: \phi^{-1}(\mathcal{C}^\infty_{X\times Y}\otimes_{f^{-1}\mathcal{C}^\infty_X\otimes g^{-1}\mathcal{C}^\infty_Y}\mathcal{O}_\mathcal{X}\boxtimes\mathcal{O}_\mathcal{Y})\to\mathcal{O}_\mathcal{Z}.$$

This map induces the map $\phi^{-1}(\mathcal{C}^\infty_{X\times Y})\otimes\phi^{-1}(\mathcal{O}_\mathcal{X}\boxtimes\mathcal{O}_\mathcal{Y})\to\mathcal{O}_\mathcal{Z}$. By restriction to identity elements in $\mathcal{C}^\infty_{X\times Y}$, $\mathcal{O}_\mathcal{X}$ and $\mathcal{O}_\mathcal{Y}$ respectively, we obtain the following two maps of differential graded algebras,

$$\phi_X^{-1}\mathcal{O}_\mathcal{X}\to 1\otimes\phi^{-1}(\mathcal{O}_\mathcal{X}\boxtimes 1)\to\mathcal{O}_\mathcal{Z},$$

$$f\mapsto 1\otimes f\otimes 1\mapsto \phi^*(1\otimes f\otimes 1).$$

$$\phi_Y^{-1}\mathcal{O}_\mathcal{Y}\to 1\otimes\phi^{-1}(1\boxtimes\mathcal{O}_\mathcal{Y})\to\mathcal{O}_\mathcal{Z},$$

$$g\mapsto 1\otimes 1\otimes g\mapsto \phi^*(1\otimes 1\otimes g).$$

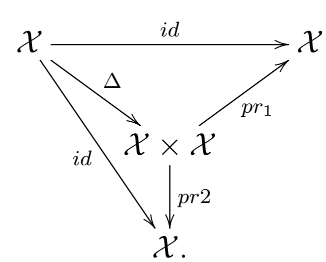
Therefore we obtain two morphisms from $(Z, \mathcal{O}_{\mathcal{Z}}, \partial)$ to $(X, \mathcal{O}_{\mathcal{X}}, \partial)$ and from $(Z, \mathcal{O}_{\mathcal{Z}}, \partial)$ to $(Y, \mathcal{O}_{\mathcal{Y}}, \partial)$ respectively. □

1.2.3 Calculus on *dgMnflds*

Suppose that $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}}, \partial)$ is a dg-manifold. We introduce several objects of differential calculus in the context of dg-manifolds. Namely, the sheaf of differentials 1-forms and the sheaf of derivations of the structure sheaf $\mathcal{O}_{\mathcal{X}}$.

1.2.3.1 Differentials

We will construct the sheaf of differential 1-forms over $\mathcal{X}$ will be denoted by $\Omega^1_{\mathcal{X}}$. Recall that locally $\mathcal{O}_{\mathcal{X}}$ is isomorphic to $\mathcal{C}^{\infty}_X \otimes S(W)$, where W is a positively graded vector space. As we are dealing with differential graded algebras over $\mathbb{R}$, whose degree zero component is equal to $\mathcal{C}^{\infty}_X$, we need to put together differential 1-forms for the commutative algebra $\mathcal{C}^{\infty}_X$ and differentials of the symmetric algebra $S(W)$. For our dg-manifold $\mathcal{X}$ consider the product $\mathcal{X} \times \mathcal{X}$. The identity map on $\mathcal{X}$ induces the *diagonal* map, $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by the formulas $pr_1 \circ \Delta = pr_2 \circ \Delta = id$,



Let I_Δ denote the kernel of $\Delta^{-1}\mathcal{O}_{\mathcal{X}\times\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$. It follows that I_Δ is a differential graded ideal, since $\Delta^{-1}\mathcal{O}_{\mathcal{X}\times\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is a homomorphism of differential graded algebras. Sections of I_Δ are functions on $\Delta^{-1}\mathcal{O}_{\mathcal{X}\times\mathcal{X}}$ which vanish along the diagonal. Let I^2_Δ denotes the ideal square, $I^2_\Delta \subset I_\Delta$. Notice that I^2_Δ is closed under ∂, since for any $a, b \in I_\Delta$, $\partial(a)$, $\partial(b) \in I_\Delta$ and by the Leibniz rule $\partial(ab) = a\partial(b) + (-1)^a \partial(a) b$ we conclude that $\partial(ab) \in I^2_\Delta$. We define the sheaf of differential 1-forms on $\mathcal{X}$, $\Omega^1_{\mathcal{X}}$

by,

$$\Omega^1_{\mathcal{X}} := (I_\Delta / I_\Delta^2).$$

The de Rham differential on $\mathcal{X}$, denoted by $d : \mathcal{O}_{\mathcal{X}} \to \Omega^1_{\mathcal{X}}$ is defined by

$$df = (f \otimes 1 - 1 \otimes f) + I_\Delta^2$$

for any $f \in \mathcal{O}_{\mathcal{X}}$. The calculation below,

$$\begin{aligned} (f \otimes 1)(dg) + (df)(1 \otimes g) &= (f \otimes 1)(g \otimes 1 - 1 \otimes g) + (f \otimes 1 - 1 \otimes f)(1 \otimes g) \\ &= fg \otimes 1 - f \otimes g + f \otimes g - 1 \otimes fg \\ &= d(fg). \end{aligned}$$

shows that, $d(fg) = (f \otimes 1)(dg) + (df)(1 \otimes g)$, i.e. d is a derivation. Moreover, d satisfy a universal property. Namely, any derivation of the algebra $\mathcal{O}_{\mathcal{X}}$ with values in a sheaf of differential $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{M}$, factors through d in a unique way,

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{M} \\ & {\scriptstyle d}\searrow & \uparrow {\scriptstyle \exists!} \\ & & \Omega^1_{\mathcal{X}}. \end{array}$$

Proposition 1.2.14. *The map* $\Psi : \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathcal{X}}}(\Omega^1_X, \mathcal{M}) \to \underline{Der}(\mathcal{O}_{\mathcal{X}}, \mathcal{M})$ *given by* $\phi \mapsto \phi \circ d$ *is an isomorphism.*

Proof: We provide an inverse of Ψ. Suppose that $D : \mathcal{O}_{\mathcal{X}} \to \mathcal{M}$ is a derivation. Let us consider the map $id \otimes D : I_\Delta \to \mathcal{M}$ given by $f \otimes g - g \otimes f \mapsto f \cdot D(g) - g \cdot D(f)$. It follows that D factors through I_Δ, since $D(1) = 0$,

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & \xrightarrow{\quad D \quad} & \mathcal{M} \\ & \searrow \qquad \nearrow {\scriptstyle id \otimes D} & \\ & I_\Delta. & \end{array}$$

By direct computation, (using the Leibniz rule), we obtain that $id \otimes D \mid_{I_\Delta^2} = 0$, hence $id \otimes D$ descends to the quotient $\Omega^1_{\mathcal{X}} \to \mathcal{M}$ this means that $id \otimes D \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\Omega^1_{\mathcal{X}}, \mathcal{M})$.

By definition $id \otimes D$ is $\mathcal{O}_{\mathcal{X}}$-linear. We show that the map $D \mapsto id \otimes D$ is the inverse of Ψ. Suppose that $\phi : \Omega^1_{\mathcal{X}} \to \mathcal{M}$ is $\mathcal{O}_{\mathcal{X}}$-linear map. The calculation

$$\begin{aligned} id \otimes (\phi \circ d)((f \otimes 1)dg) &= id \otimes (\phi \circ d)((f \otimes 1)(1 \otimes g - g \otimes 1 + I^2_{\Delta})) \\ &= id \otimes (\phi \circ d)(f \otimes g - fg \otimes 1 + I^2_{\Delta}) \\ &= f \cdot \phi(d(g)), \end{aligned}$$

shows that $id \otimes (\phi \circ d) = \phi$. Conversely, for any derivation $D : \mathcal{O}_{\mathcal{X}} \to \mathcal{M}$ we have

$$((id \otimes D) \circ d)(f) = (id \otimes D)(1 \otimes f - f \otimes 1 + I^2_{\Delta}) = D(f).$$

□

Suppose that $\phi : \mathcal{X} \to \mathcal{Y}$ is a morphism in *dgMnflds*.

Lemma 1.2.15. *ϕ induces a map on differential 1-forms, $\phi^* : \Omega^1_{\mathcal{Y}} \to \Omega^1_{\mathcal{X}}$*

Proof: Since ϕ^* is a map of differential graded algebras, ϕ maps both I_{Δ} and I^2_{Δ} in $\Delta^{-1}\mathcal{O}_{\mathcal{Y}}$ to the corresponding I_{Δ} and I^2_{Δ} in $\Delta^{-1}\mathcal{O}_{\mathcal{X}}$. □

1.2.3.2 The tangent sheaf

Suppose that $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}}, \partial)$ is a dg-manifold. Let us denote the sheaf of graded maps by $\bigoplus_{n\geq 0} \underline{\mathrm{Hom}}^n(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) := \oplus_{n\geq 0}\mathrm{Hom}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}[n])$. The given derivation ∂ of the algebra $\mathcal{O}_{\mathcal{X}}$, induces a degree one derivation on $\bigoplus_{n\geq 0} \underline{\mathrm{Hom}}^n(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}})$ by the formula,

$$\delta = [\partial, \ ].$$

The tangent sheaf of $\mathcal{X}$ is defined as the sheaf of derivations of the algebra $\mathcal{O}_X$,

$$\mathcal{T}_{\mathcal{X}} := \underline{\mathrm{Der}}(\mathcal{O}_{\mathcal{X}}) \subset \bigoplus_{n\geq 0} \underline{\mathrm{Hom}}^n(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}).$$

i.e. graded endomorphisms of $\mathcal{O}_{\mathcal{X}}$, D, which satisfy the Leibniz rule, $D(ab) := D(a)b + (-1)^{|a||D|}aD(b)$ for any (local) sections $a, b \in \mathcal{O}_{\mathcal{X}}$. Recall that $|a|$, $|D|$ denote the degree of a and D respectively. In this sense, $\partial \in \mathcal{T}^1_{\mathcal{X}}$.

1.2.3.3 The differential of a dg-morphism

Suppose that $\phi : \mathcal{X} \to \mathcal{Y}$ is a dg-morphism. There is an induced map $d\phi : \mathcal{T}_{\mathcal{X}} \to \phi^*\mathcal{T}_{\mathcal{Y}}$[3] defined as follows. Suppose that $\xi \in \mathcal{T}_{\mathcal{X}}$, $\omega \in \mathcal{O}_{\mathcal{Y}}$, the induced derivation along ϕ acts on $\phi^{-1}\mathcal{O}_{\mathcal{Y}}$ by the formula below,

$$(d\phi)(\xi)\omega := \xi(\phi^*\omega).$$

Lemma 1.2.16. *Suppose that $\phi : \mathcal{X} = (X, \mathcal{O}_{\mathcal{X}}, \partial) \to \mathcal{Y} = (Y, \mathcal{O}_{\mathcal{Y}}, \partial)$ is a dg-morphism. Then $d\phi : \mathcal{T}_{\mathcal{X}} \to \phi^*\mathcal{T}_{\mathcal{Y}}$ is a map of complexes, i.e.*

$$[\partial, \,] \circ d\phi = d\phi \circ [\partial, \,]$$

Proof: Pick $\xi \in \mathcal{T}_{\mathcal{X}}$ an $\omega \in \phi^{-1}\mathcal{O}_{\mathcal{Y}}$. By direct calculation the equality holds, namely:

$$[\partial, \,] \circ d(\phi)\xi(\omega) = [\partial, \,]\xi(\phi^*(\omega)) = [\partial, \xi](\phi^*\omega).$$

$$(d\phi \circ [\partial, \,]\xi)\omega = (d\phi)([\partial, \xi])\omega = [\partial, \xi]\phi^*\omega.$$

□

1.2.3.4 Duality pairing

We point out that $\mathcal{T}_{\mathcal{X}}$ and $\Omega^1_{\mathcal{X}}$ are dual to each other as sheaves of $\mathcal{O}_{\mathcal{X}}$-modules in the sense of the following lemma.

[3] As $\mathcal{T}_{\mathcal{Y}}$ is a $\mathcal{O}_{\mathcal{Y}}$-module $\phi^*\mathcal{T}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}} \otimes_{\phi^{-1}CC^\infty_Y} \phi^{-1}\mathcal{T}_{\mathcal{Y}}$

Lemma 1.2.17. *There is a canonical isomorphism between*

$$\mathcal{T}_{\mathcal{X}} \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega^1_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}).$$

Proof: This is a direct consequence of Proposition (1.2.14) applied to $\mathcal{M} = \mathcal{O}_{\mathcal{X}}$. The isomorphism gives rise to the canonical pairing $\langle\,,\,\rangle : \Omega^1_{\mathcal{X}} \to \mathcal{T}_{\mathcal{X}}$ defined by the formula $\alpha \otimes \xi \mapsto \alpha(\xi)$. □

CHAPTER 2

$\mathbb{R}[n]$ dg-principal bundles

In *dgMnflds* we will give the definition of a dg-principal bundle with a structure group $\mathbb{R}[n]$ and base dg-manifold $X^\sharp$. We will show an equivalence of categories that allows us to classify $\mathbb{R}[n]$ dg-principal bundle over the base $X^\sharp$.

2.1 Free actions

Given an arbitrary category $\mathcal{C}$, the definition of a group-object is stated as follows. Suppose that $\mathcal{C}$ is endowed with a terminal object 1 and finite products. A *group object* in $\mathcal{C}$ is an object G together with morphisms

$$m : G \times G \to G,\ e : 1 \to G,\ \iota : G \to G,$$

which satisfies the following formulas,

$$m \circ (m \times id) = m \circ (id \times m),\ m \circ (id \times e) = pr_1,\ m \circ (e \times id) = pr_2,$$

$$m \circ (id \times \iota) \circ diag = e_G,\ m \circ (\iota \times id) \circ diag = e_G$$

where $pr_1 : G \times 1 \to G$ and $pr_2 : 1 \times G \to G$ denotes the projection maps respectively, $diag : G \to G \times G$ denotes the diagonal map, i.e $id = pr_i \circ diag$ for $i = 1, 2$, and $e_G : G \xrightarrow{\exists !} 1 \xrightarrow{e} G$.

Recall that by definition $\mathcal{O}_{\mathbb{R}[n]}$ is equal to the symmetric algebra on the shifted vector space $\mathbb{R}^\vee[-n]$. Let $t \in \mathbb{R}^\vee$ denotes the linear coordinate on $\mathbb{R}$ of degree n, therefore t generates $S[t] = \mathcal{O}_{\mathbb{R}[n]}$; see Example (1.2.3). In this notation, we review the Hopf algebra structure on $S[t]$. In this case the tensor algebra and the symmetric algebra are equal, see Remark (1.1.4). The multiplication on $S[t]$ is denoted by ∇. The comultiplication is given by $\Delta(t) = 1 \otimes t + t \otimes 1$ and extended for all homogeneous elements as an algebra morphism. The unit map, η, is just the inclusion of $\mathbb{R}$ into $S[t]$, as long as the counit map, ϵ, is the projection of $S[t]$ to the field, $\epsilon(t) = 0$, $\epsilon(1) = 1$. The compatibility conditions are given by,

$$\Delta \circ \nabla = \nabla \otimes \nabla \circ (id \otimes \tau \otimes id) \circ \Delta \otimes \Delta, \ \tau(x \otimes y) = y \otimes x.$$

$$\epsilon \circ \nabla = \epsilon \otimes \epsilon, \quad \Delta \circ \eta = \eta \otimes \eta, \quad \epsilon \circ \eta = id_{\mathbb{C}}.$$

The antipode $s : S[t] \to S[t]$ is given by $s(t) = -t$ on the generator and for homogeneous elements extends as follows, $s(x_1 ... x_l) = (-1)^l x_l ... x_1$. Notice that the following equation holds

$$\nabla \circ (s \otimes id) \circ \Delta = \nabla \circ (id \otimes s) \circ \Delta = \eta \circ \epsilon.$$

Example 2.1.1. *The dg-manifold* $\mathbb{R}[n] = (\{*\}, S[t], 0)$ *is a group object in dg-Mnflds, i.e.* $(\mathbb{R}[n], m, \iota, e)$*, since the structure sheaf is a Hopf algebra. The maps* m, ι, e *are defined as follows,*

$$m : (\mathbb{R}[n], S[t], 0) \times (\mathbb{R}[n], S[t], 0) \to (\mathbb{R}[n], S[t], 0)$$

is the comultiplication $\Delta : S[t] \to S[t] \otimes S[t]$,

$$\iota : (\mathbb{R}[n], S[t], 0) \to (\mathbb{R}[n], S[t], 0)$$

is given by the antipode $S[t] \to S[t]$, *and*

$$e : (\{*\}, \mathbb{R}, 0) \to (\mathbb{R}[n], S[t], 0)$$

is given by the counit $\epsilon : S[t] \to \mathbb{R}$. *The following diagrams show that* $\mathbb{R}[n] = (\{*\}, S[t], 0)$ *satisfies the group object conditions.*

$$\begin{array}{ccc} S[t] & \xrightarrow{\Delta} & S[t] \otimes S[t] \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle id\otimes\Delta} \\ S[t] \otimes S[t] & \xrightarrow[\Delta\otimes id]{} & S[t] \otimes S[t] \otimes S[t] \end{array}$$

$$\begin{array}{ccc} S[t] \otimes \mathbb{R} & \xleftarrow{\otimes 1} & S[t] \\ & {\scriptstyle id\otimes\epsilon}\nwarrow & \downarrow{\scriptstyle\Delta} \\ & & S[t] \otimes S[t] \end{array}$$

$$\begin{array}{ccccc} S[t] & \xrightarrow{\epsilon} & \mathbb{R} & \xrightarrow{u} & S[t] \\ {\scriptstyle\Delta}\downarrow & & & & \downarrow{\scriptstyle\Delta} \\ S[t] \otimes S[t] & & \xrightarrow{id\otimes s} & & S[t] \otimes S[t]. \end{array}$$

Before giving the definition of a dg-principal bundle with fiber the group object $\mathbb{R}[n]$ and base $X^\sharp$, we will give the definition of a free action of $\mathbb{R}[n]$ on a dg-manifold $\mathcal{P} = (X, \mathcal{O}_{\mathcal{P}}, \partial)$.

Recall that without any grading, if G is a Lie group and X is a smooth manifold, an action of G on X is a group homomorphism $\alpha : G \to \text{Diff}(X)$, $g \mapsto \alpha_g$, such that $\alpha : G \times X \to X$ $\alpha(g, x) = \alpha_g x =: gx$ is smooth. Moreover, the action is called *free* if $gx = x$ for some $x \in X$ then $g = e$, (where e denotes the identity element of G). The freeness condition can be restate in terms of functions. Namely, an action $\alpha : G \times X \to X$ is called free if the coequalizer of the maps α^* and pr^*, where $pr : G \times X \to X,\ (g, x) \mapsto x$, is equal to (C^∞_X, id_X), since

$$Coeq(\alpha^*, pr^*) = (C^\infty_e \boxtimes C^\infty_X, i^*) \cong (C^\infty_X, id_X).$$

To simplify the notation, from now on, given a dg-morphism $f : \mathcal{X} \to \mathcal{Y}$, we denote the map between sheaves simply as $f^* : \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$. Suppose that $\mathcal{P} = (X, \mathcal{O}_\mathcal{P}, \partial)$ is a dg-manifold.

Definition 2.1.2. *A dg-action of* $\mathbb{R}[n]$ *over* $\mathcal{P}$ *is a dg-morphism* $\alpha : \mathcal{P} \times \mathbb{R}[n] \to \mathcal{P}$,

$$\alpha^*_t : \mathcal{O}_\mathcal{P} \to \mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]},$$

such that

$$(\alpha^*_t \otimes id) \circ \alpha^*_t = (id \otimes \Delta) \circ \alpha^*_t \tag{2.1.1}$$

$$id \otimes 1 = (id \otimes \epsilon) \circ \alpha^*_t. \tag{2.1.2}$$

The maps Δ *and* ϵ *denotes the comultiplication and the counit for the Hopf algebra* $S[t]$ *respectively.*[1]

For any $f \in \mathcal{O}_\mathcal{P}$, $\alpha^*_t(f)$ is a polynomial in t with coefficients in $\mathcal{O}_\mathcal{P}$,

$$\alpha^*_t(f) = f \otimes 1 + \alpha_1(f) \otimes t + \alpha_2(f) \otimes t^2 + ... \tag{2.1.3}$$

Furthermore, the first term in (2.1.3) is equal to the pullback by the projection $pr : \mathcal{P} \times \mathbb{R}[n] \to \mathcal{P}$ $pr^* : \mathcal{O}_\mathcal{P} \to \mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]}$, since by definition the equation $(id \otimes \epsilon)\alpha^*_t = id \otimes 1$ holds.

Let us consider the morphism in *dgMnflds* given by the projection map $pr : \mathcal{P} \times \mathbb{R}[n] \to \mathcal{P}$ and the map (in fact an inclusion) between sheaves

$$pr^* : \mathcal{O}_\mathcal{P} \to \mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]},$$

[1] In other words, a dg-action is a co-module structure on $\mathcal{O}_\mathcal{P}$ over the Hopf algebra $\mathcal{O}_{\mathbb{R}[n]}$ which is compatible with the differential.

defined by $pr^*(f) = f \otimes 1$. In addition, consider the morphism $i : \mathcal{P} \to \mathcal{P} \times \mathbb{R}[n]$, $p \mapsto (p, *)$, and the map of differential graded algebras $i^* : \mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_\mathcal{P}$ which corresponds to the projection onto the constant term.

Definition 2.1.3. *A dg-action α is called free if and only if the coequalizer of the maps α^*, pr^* is given by*

$$Coeq(\alpha^*, pr^*) = (\mathcal{O}_\mathcal{P}, i^*).$$

There is an equivalent way to state the freeness condition. For any element $f \in \mathcal{O}_\mathcal{P}$, the difference $\alpha^*(f) - pr^*(f)$ lands in the ideal generated by $1 \otimes t$, i.e. $(\alpha^* - pr^*)(f)$ does not have constant term. Let us denote $t \cdot (\mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ the ideal of generated by t inside $\mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]}$.

Lemma 2.1.4. *A dg-action α is free if and only if*

$$t \cdot (\mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]}) = \langle \alpha^*(f) - pr^*(f) \rangle_{f \in \mathcal{O}_\mathcal{P}}.$$

Proof: The coequalizer of α^* and pr^* is equal to

$$(\mathcal{O}_\mathcal{P} \otimes \mathcal{O}_{\mathbb{R}[n]}) / \langle \alpha^* - pr^*(f) \rangle_{f \in \mathcal{O}_\mathcal{P}} = \mathcal{O}_\mathcal{P}.$$

On the other hand, $t \cdot (\mathcal{O}_P \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ is precisely the kernel of the dg-morphism i^*:

$$\begin{array}{ccccc} \mathcal{O}_\mathcal{P} & \xrightarrow{\alpha^* - pr^*} & \mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]} & \xrightarrow{i^*} & \mathcal{O}_\mathcal{P} \\ & \searrow & \uparrow & & \\ & & t \cdot (\mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]}), & & \end{array}$$

It follows that

$$(\mathcal{O}_\mathcal{P} \otimes \mathcal{O}_{\mathbb{R}[n]}) / t \cdot (\mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]}) = \mathcal{O}_\mathcal{P}.$$

Therefore, we obtain the desired identification, $t \cdot (\mathcal{O}_\mathcal{P} \boxtimes \mathcal{O}_{\mathbb{R}[n]}) = \langle \alpha^*(f) - pr^*(f) \rangle_{f \in \mathcal{O}_\mathcal{P}}$.

□

We say some words about the properties of the dg-action that we already introduced. Recall that for any $f \in \mathcal{O}_{\mathcal{P}}$ the dg-action is written

$$\alpha_t^*(f) = f \otimes 1 + \alpha_1(f) \otimes t + \alpha_2(f) \otimes t^2 + ...$$

As t has degree $n \geq 1$, it follows that each α_i is a graded map from $\mathcal{O}_{\mathcal{P}}$ to itself which balance the total degree in each tensor product to zero. Namely,

$$\alpha_i : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}[-in].$$

Definition 2.1.5. *Suppose that α is a dg-action on a dg-manifold $\mathcal{P}$. The invariant element under the dg-action are equal to*

$$\mathcal{O}_{\mathcal{P}}^{\mathbb{R}[n]} = \{f \in \mathcal{O}_{\mathcal{P}} \mid \alpha^*(f) = f \otimes 1\}.$$

Notice that the invariant elements are precisely those elements whose pullback by the dg-action α is equal to the pullback by the projection $pr : \mathcal{P} \times \mathbb{R}[n] \to \mathcal{P}$.

Lemma 2.1.6. *For each $f \in \mathcal{O}_{\mathcal{P}}$ there is a compatibility condition between the graded maps α_j's given by the formula:*

$$\alpha_i \circ \alpha_j(f) = \binom{i+j}{j} \alpha_{i+j}(f).$$

Proof: The formula is a direct consequence of the equation $(\alpha_t^* \otimes id)\alpha_t^*(f) = (id \otimes \Delta)\alpha_t^*(f)$.

□

Proposition 2.1.7. *Properties of the free dg-action α_t^*.*

1. *An element $f \in \mathcal{O}_{\mathcal{P}}$ is invariant, $\alpha_t^*(f) = f \otimes 1$ if and only if $f \in ker(\alpha_1)$.*

2. *α_1 is a derivation of degree $-n$ of $\mathcal{O}_{\mathcal{P}}$.*

3. *There is an element $h \in \mathcal{O}_{\mathcal{P}}^n$ locally defined on P such that $\alpha_1(h) = 1$. Furthermore α_1 is linear with respect to $\mathcal{O}_{\mathcal{P}}^{\mathbb{R}[n]}$.*

4. The following formula holds,

$$\alpha_1(h^k) = \alpha_1(h^k) = kh^{k-1},$$

$$\alpha_1^k(h^k) = \alpha_1 \circ ... \circ \alpha_1(h^k) = k!$$

Proof: 1) From Lemma (2.1.6) it follows that $j\alpha_j(f) = \alpha_{j-1} \circ \alpha_1(f)$, for each j. Moreover $\alpha_j(f) = \frac{1}{j!}\alpha_1^j(f)$. Therefore, the co-action on $\mathcal{P}$ is rewritten as follows,

$$\alpha_t^*(f) = f \otimes 1 + \alpha_1(f) \otimes t + \frac{1}{2!}\alpha_1^2(f) \otimes t^2 + \frac{1}{3!}\alpha_1^3(f) \otimes t^3 + ... \tag{2.1.4}$$

2) As α^* is a multiplicative map, $\alpha_t^*(fg) = \alpha_t^*(f)\alpha_t^*(g)$, it follows that α_1 is a derivation.

3) Our chosen generator t of $\mathcal{O}_{\mathbb{R}[n]}$ gives rise to a generator of the ideal $t \cdot (\mathcal{O}_{\mathcal{P}} \otimes \mathcal{O}_{\mathbb{R}[n]})$, namely $1 \otimes t$. This element is written as a finite linear combination of the form

$$1 \otimes t = \sum_k (\alpha_t^* - pr^*(f_k)) = \sum_k \sum_n \alpha_n(f_k) \otimes t^n$$

$$= \sum_k (\alpha_1(f_k) \otimes t) = \alpha_1(\sum_k f_k) \otimes t.$$

Setting $h := \sum_k f_k$ we obtain an element $h \in \mathcal{O}_{\mathcal{P}}^n$ such that $\partial h = 0$ and $\alpha_1(h) = 1$.

4) As α_1 is a derivation on $\mathcal{O}_{\mathcal{P}}$, the equation follows by induction on k.

□

Since we required the compatibility between α_t^* and the differential on $\mathcal{P}$, it follows that $\alpha_i \partial(f) = \partial \alpha_i(f)$ for all i:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{P}} & \xrightarrow{\alpha_t^*} & \mathcal{O}_{\mathcal{P}} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial \otimes 1} \\ \mathcal{O}_{\mathcal{P}} & \xrightarrow[\alpha_t^*]{} & \mathcal{O}_{\mathcal{P}} \boxtimes \mathcal{O}_{\mathbb{R}[n]}. \end{array}$$

In particular, ∂ preserves the invariant elements since commutes with α_1.

2.1.1 Action on derivations

Suppose that $\mathbb{R}[n]$ acts on a dg-manifold $\mathcal{P}$. There is an induced action of $\mathbb{R}[n]$ on the sheaf of derivations $\underline{\mathrm{Der}}(\mathcal{O}_{\mathcal{P}})$. Recall that the action of $\mathbb{R}[n]$ on $\mathcal{P}$ is given by a map of differential graded algebras,

$$\alpha_t^* : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}} \boxtimes \mathcal{O}_{\mathbb{R}[n]}.$$

We extend the action α_t^* linearly on t to the map

$$\widetilde{\alpha_t^*} : \mathcal{O}_{\mathcal{P}} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}} \boxtimes \mathcal{O}_{\mathbb{R}[n]},$$

by the formula,

$$\widetilde{\alpha_t^*}(f \otimes 1 + \alpha_1(f) \otimes t + \frac{1}{2!}\alpha_1^2(f) \otimes t^2 + ...) = \alpha_t^*(f) \otimes 1 + \alpha_t^*(\alpha_1(f)) \otimes t + ...$$

Therefore the induced action of $\mathbb{R}[n]$ on $\underline{\mathrm{Der}}(\mathcal{O}_{\mathcal{P}})$ is given by

$$\widetilde{\alpha_t^*} D \widetilde{\alpha_{-t}^*} = D \otimes 1 + [\alpha_1, D] \otimes t + [\alpha_1, [\alpha_1, D]] \otimes t^2 + ... \qquad (2.1.5)$$

for any $D \in \underline{\mathrm{Der}}(\mathcal{O}_{\mathcal{P}})$. It follows that a derivation D is invariant with respect to the $\mathbb{R}[n]$ dg-action if and only if $[\alpha_1, D] = 0$

Remark 2.1.8. *By definition of the action on $\underline{\mathrm{Der}}(\mathcal{O}_{\mathcal{P}})$, it follows that the given derivation ∂ of $\mathcal{P}$ is invariant since $[\alpha_1, \partial] = 0$.*

2.2 Local triviality property

We apply the previous analysis to the following situation, namely the dg-morphism $\pi : \mathcal{P} \to X^\sharp$ with additional information. Suppose that there is a free dg-action on $\mathcal{P}$, $\alpha_t^* : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}} \boxtimes \mathcal{O}_{\mathbb{R}[n]}$. The trivial *bundle* is given by the product dg-manifold,

$$X^\sharp \times \mathbb{R}[n] = (X, \mathcal{O}_{X^\sharp \times \mathbb{R}[n]}, \partial) := (X, \mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]}, \partial \otimes 1).$$

The dg-action on the trivial bundle is given by the comultiplication of the Hopf algebra $\mathcal{O}_{\mathbb{R}[n]}$ on itself and $\pi : X^\sharp \times \mathbb{R}[n] \to X^\sharp$ is just the projection.

Definition 2.2.1. *The dg morphism $\pi : \mathcal{P} \to X^\sharp$ satisfies the local triviality property if and only if there is a local isomorphism of differential graded algebras between $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_P$.*

In order to verify whenever or not a dg-morphism $\pi : \mathcal{P} \to X^\sharp$ satisfies the local triviality property, we assume that the invariant elements under the dg-action α sits in a specific way inside $\mathcal{P}$. Namely

$$\mathcal{O}_{\mathcal{P}}^{\mathbb{R}[n]} \cong \mathcal{O}_{X^\sharp}. \tag{2.2.1}$$

Under the above assumption, (2.2.1), we will construct an isomorphism of differential graded algebras, $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_P$. From π we get the map, $\pi^* : \mathcal{O}_{X^\sharp} \to \mathcal{O}_{\mathcal{P}}$. Since we are given a free dg-action α on $\mathcal{P}$, there is a locally defined element on X, $h \in \mathcal{O}_{\mathcal{P}}^n$, such that $\partial h = 0$, $\alpha_1(h) = 1$. Let us define a map $\mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}}$ determined by the image of the generator t, as a map of differential graded algebras, i.e. $t \mapsto h$. By multiplication of both maps, we get our desired dg-morphism $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}} \otimes \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{\mathcal{P}}$.

Proposition 2.2.2. *The map $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}}$ is an isomorphism of differential graded algebras. Therefore π is locally trivial.*

Proof: Injectivity: Suppose that there is a polynomial in t with invariant coefficients $\mathcal{O}_{X^\sharp}$, $p(t) = \sum_i f_i \otimes t^i$ such that $p(h) = 0$ of minimal degree N. For the given $p(t) = f_0 + f_1 \otimes t + ... + f_N \otimes t^N$ it follows that $\alpha_1(p(h)) = p'(t)|_{t=h} = 0$, since $p(h) = 0$. Therefore,

$$p'(h) = f_1 + 2f_2 h + ... + N f_N h^{N-1} = \alpha_1(p(h)) = 0,$$

whose degree is less than N. This imply that $p(t) = 0$ by minimality of N. In other words, we show that h is algebraically independent of the invariant elements with respect to α.

Surjectivity: By degree reasons, for any $y \in \mathcal{O}_{\mathcal{P}}$ there is an $l > 0$, such that $\alpha_1^l(y) = 0$. We will show that every element $y \in \mathcal{O}_{\mathcal{P}}$ is written as a liner combination of the form:

$$y = \sum_{i=0}^{k} f_i \cdot h^i,$$

where $f_i \in \mathcal{O}_{X^\sharp}$. Suppose that for a given element $y \in \mathcal{O}_{\mathcal{P}}$ there is a minimal $k > 0$ for which $\alpha_1^k y \in \mathcal{O}_{X^\sharp}$. Minimality property of k means that $\alpha_1^{k-1}(y)$ is not an invariant element. We proceed by induction on k in order to show that y belongs to the image of $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}}$.

- If $k = 0$, then $\alpha_1(y) = 0$ so that y is an invariant element. It follows that y belongs to the image of the map $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}}$.

- If $k = 1$, then $\alpha_1(y)$ is an invariant element in $\mathcal{O}_{\mathcal{P}}$. Define $y_1 := y - \alpha_1(y) \cdot h$. As α_1 is a derivation, $\alpha_1(y_1) = \alpha_1(y) - \alpha_1(y) = 0$. Then y_1 is an invariant element and $y = y_1 + \alpha_1(y) \cdot h$ belongs to the image of the map $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}}$.

- Suppose that for any $j < k$ the statement holds. For the given $y \in \mathcal{O}_{\mathcal{P}}$, such that $\alpha_1^k(y)$ is an invariant element define y_1 as follows,

$$y_1 = y - \frac{1}{k!} \alpha_1^k(y) \cdot h^k.$$

  Notice that $\alpha_1^k(y_1) = 0$ since α_1 is a derivation. Therefore by induction on k $\alpha^{k-1}(y_1)$ is an invariant element. Hence, $y_1 = \sum_{i=0}^{k-1} f_i \cdot h^i$ belongs to the image of the map $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}}$. Therefore y also belongs to the image of the map $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{\mathcal{P}}$.

□

Keeping in mind the previous construction, we give the definition of an $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$. Also, we point out a nice property which holds for every $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$.

Definition 2.2.3. *Suppose that α is a free dg-action on $\mathcal{P}$. A dg-morphism $\pi : \mathcal{P} \to X^\sharp$ is called a $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$ if and only if the invariant elements under α are isomorphic to $\mathcal{O}_{X^\sharp}$.*

Proposition 2.2.4. *Every $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$ satisfied the local triviality property.*

Proof: It is a direct consequence of Proposition (2.2.2). □

Suppose that $\mathcal{P}$ and $\mathcal{P}'$ are two $\mathbb{R}[n]$ dg-principal bundles over $X^\sharp$. A morphism $\phi : \mathcal{P} \to \mathcal{P}'$ is a dg-morphism such that $\pi^* = \phi^* \circ \pi'^*$, and the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{P}} & \xrightarrow{\alpha_t^*} & \mathcal{O}_{\mathcal{P}} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \\ {\scriptstyle \phi^*}\uparrow & & \uparrow{\scriptstyle \phi^* \otimes id} \\ \mathcal{O}_{\mathcal{P}'} & \xrightarrow[\alpha_t'^*]{} & \mathcal{O}_{\mathcal{P}'} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \end{array}$$

Lemma 2.2.5. *Suppose that ϕ is a dg-$\mathbb{R}[n]$-bundle morphism, then ϕ is an isomorphism of differential graded algebras.*

Proof: As the dg-action on both $\mathcal{P}$ and $\mathcal{P}'$ is free, there are h and h' locally on X such that $\alpha_1(h) = \alpha_1'(h') = 1$. Therefore, $h' \mapsto h$ under ϕ. Recall that ϕ is identity on the invariant elements. □

The category of $\mathbb{R}[n]$ dg-principal bundles over $X^\sharp$ will be denoted by $\mathbb{R}[n]dgBn(X^\sharp)$. By Lemma (2.2.5) $\mathbb{R}[n]dgBn(X^\sharp)$ is a groupoid.

2.3 Torsors on X and dg-principal bundles on $X^\sharp$

In this section, we establish an equivalence of categories between the category of $\Omega_X^{n,cl}$-torsors on X and the category of $\mathbb{R}[n]dgBn(X^\sharp)$.

2.3.1 From $\mathbb{R}[n]dgBn(X^\sharp)$ to $\Omega_X^{n,cl}$-torsors

We will define the functor $\mathrm{Fr} : \{\mathbb{R}[n]dgBn(X^\sharp)\} \to \{\Omega_X^{n,cl} - \text{torsors on X}\}$.

Remark 2.3.1. *It is clear that for any given dg-manifold* $\mathcal{Y} = (Y, \mathcal{O}_{\mathcal{Y}}, \partial)$

$$\mathrm{Hom}(\mathcal{Y}, \mathbb{R}[n]) = Z^n(\mathcal{O}_{\mathcal{Y}}) = \{y \in \mathcal{O}^n_{\mathcal{Y}} | \partial y = 0\}.$$

In particular,

$$\mathrm{Hom}(X^\sharp, \mathbb{R}[n]) = \Omega^{n,cl}_X.$$

Lemma 2.3.2. *An element in* $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$ *gives rise to an element in* $\Omega^{n,cl}_X \cong \mathrm{Hom}(X^\sharp, \mathbb{R}[n])$.

Proof: Suppose that $\phi \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$. By definition, there is a map of differential graded algebras $\phi^* : \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]}$. Notice that ϕ^* is determined by the effect on generators. Namely, $\phi^*(\omega \otimes 1) = \omega \otimes 1$ and $\phi^*(1 \otimes t) = \phi_t \otimes 1 + 1 \otimes t$, where $\omega \in \mathcal{O}_{X^\sharp}$ and ϕ_t is a closed differential n-form. Let consider a section s of the canonical projection $pr_{X^\sharp} : X^\sharp \times \mathbb{R}[n] \to X^\sharp$, since $X^\sharp \times \mathbb{R}[n]$ is trivial. Recall that s as a map of differential graded algebras $s^* : \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp}$ is completely determined by $s^*(\omega \otimes 1) = \omega$ and $s^*(1 \otimes t) = 0$. The following composition

$$s^* \circ \phi^* : \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp},$$

gives rise to a map of differential graded algebras. Let us consider the projection map onto $\mathbb{R}[n]$ $pr : X^\sharp \times \mathbb{R}[n] \to \mathbb{R}[n]$, $pr^* : \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]}$, $t \mapsto 1 \otimes t$. Therefore we obtain a map

$$\bar{\phi} = s^* \circ \phi^* \circ pr^* : \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp},$$

which is compatible with differentials, i.e. $\bar{\phi}$ belongs to $\mathrm{Hom}(X^\sharp, \mathbb{R}[n]) \cong \Omega^{n,cl}_X$.
□

Lemma 2.3.3. *An element in* $\Omega^{n,cl}_X \cong \mathrm{Hom}(X^\sharp, \mathbb{R}[n])$ *gives rise to an element in* $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$

Proof: Suppose that $g \in \mathrm{Hom}(X^\sharp, \mathbb{R}[n])$, i.e. $g^* : \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp}$, then there is a map of $\mathbb{R}[n]$ dg-principal bundles $\phi : X^\sharp \times \mathbb{R}[n] \to X^\sharp \times \mathbb{R}[n]$ defined on generators by $\phi^*(\omega \otimes 1) = \omega \otimes 1$ and $\phi^*(1 \otimes t) = g(t) \otimes 1 + 1 \otimes t$. As a differential graded algebra morphism ϕ is extended by the formula,

$$\phi^*(\omega \otimes t) = \omega \wedge g(t) \otimes 1 + \omega \otimes t.$$

By direct computation it follows that ϕ^* fits in the commutative diagram below,

$$\begin{array}{ccc} \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} & \xrightarrow{id \otimes \Delta} & \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \otimes \mathcal{O}_{\mathbb{R}[n]} \\ \uparrow{\scriptstyle \phi^*} & & \uparrow{\scriptstyle \phi^* \otimes id} \\ \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} & \xrightarrow[id \otimes \Delta]{} & \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \otimes \mathcal{O}_{\mathbb{R}[n]}, \end{array}$$

i.e. $\phi \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$.

□

It is clear by the construction that both process are mutually inverses. Therefore, $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$ is isomorphic to $\Omega_X^{n,cl}$.

Lemma 2.3.4. *In $\mathbb{R}[n]dgBn(X^\sharp)$, $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n]) \cong \Omega_X^{n,cl}$ is a group homomorphism.*

Proof: For two given morphisms ψ, $\phi \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$, it follows that $\psi \circ \phi$ is determined by the addition of differential n-forms; namely:

$$\mathcal{O}_{\mathbb{R}[n]} \longrightarrow \mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \xrightarrow{\psi^*} \mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \xrightarrow{\phi^*} \mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]} \xrightarrow{s^*} \mathcal{O}_{X^\sharp}$$

$$t \mapsto 1 \otimes t \mapsto \psi(t) \otimes 1 + 1 \otimes t \mapsto \psi(t) \otimes 1 + \phi(t) \otimes 1 + 1 \otimes t \mapsto \psi(t) + \phi(t).$$

□

More generally, for any $\mathbb{R}[n]$ dg principal bundle $\pi : \mathcal{P} \to X^\sharp$, the set of sections of $\mathcal{P}$ is canonically isomorphic to $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$[2].

[2]In other words, local trivialization of $\mathcal{P}$ are in 1:1 correspondence with the sheaf of section on $\mathcal{P}$.

Lemma 2.3.5. *In* $\mathbb{R}[n]dgBn(X^\sharp)$

$$\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P}) \cong \mathrm{Hom}^\pi(X^\sharp, \mathcal{P}),$$

where $\mathrm{Hom}^\pi(X^\sharp, \mathcal{P})$ *denotes the sheaf of section on* $\mathcal{P}$.

Proof: Give an element $\phi \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$ let us consider a section s for the trivial object $X^\sharp \times \mathbb{R}[n]$ given by $s^*(\omega \otimes 1) = \omega \otimes 1$, $\omega \in \mathcal{O}_{X^\sharp}$ and $s^*(1 \otimes t) = 0$. Therefore there is a map of differential graded algebras, $s \circ \phi$,

$$\mathcal{O}_{\mathcal{P}} \xrightarrow{\phi^*} \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \xrightarrow{s^*} \mathcal{O}_{X^\sharp} ,$$

such that $s^* \circ \phi^* \circ \pi^* = id_{X^\sharp}$. Conversely, given a section $q \in \mathrm{Hom}^\pi(X^\sharp, \mathcal{P})$, let us define the map $\phi : X^\sharp \times \mathbb{R}[n] \to \mathcal{P}$ as follows,

$$\mathcal{O}_{\mathcal{P}} \xrightarrow{\alpha_t^*} \mathcal{O}_{\mathcal{P}} \otimes \mathcal{O}_{\mathbb{R}[n]} \xrightarrow{q^* \otimes id} \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} ,$$

where α_t^* is the co action on $\mathcal{P}$. By construction ϕ is compatible with differentials, therefore $\phi \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$ as desired.

□

For any pair of elements $\phi \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$ and $g \in \mathrm{Hom}(X^\sharp, \mathbb{R}[n])$ there is a map of differential graded algebras,

$$g^* \circ \phi^* : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]}.$$

In addition, for any pair of elements $\phi, \psi \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$, there is an automorphism of the trivial bundle given by $\phi^* \circ (\psi^*)^{-1} : \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]} \to \mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]}$. This map is given by an element $g \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$. Then $\phi^* \circ (\psi^*)^{-1} = g$, or $\phi^* = g \circ \psi^*$. Therefore $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$ is an $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n])$-torsor. By Lemma (2.3.4), we obtain that $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$ is an $\Omega_X^{n,cl}$-torsor.
For any $\pi : \mathcal{P} \to X^\sharp$ in $\mathbb{R}[n]dgBn(X^\sharp)$ we define

$$\mathrm{Fr}(\mathcal{P}) = \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P}).$$

2.3.2 From $\Omega_X^{n,cl}$-torsors to $\mathbb{R}[n]dgBn(X^\sharp)$

We will define the functor $B : \{\Omega_X^{n,cl} - \text{torsors on X}\} \to \{\mathbb{R}[n]dgBn(X^\sharp)\}$. Suppose that T is an $\Omega_X^{n,cl}$-torsor. Recall that the action of $\Omega_X^{n,cl}$ on $(\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ is given by algebra automorphism. On the product $T \times (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ the sheaf of groups $\Omega_X^{n,cl}$ acts diagonally. It means that for any $s \in T$ and $f \in (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$, (s, f) is identified with $(g.s, g.f)$ for any $g \in \Omega_X^{n,cl}$, where $g.s$ and $g.f$ denote the action of $\Omega_X^{n,cl}$ on T and $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]}$ respectively. Let us denote the set of equivalence classes by $\mathcal{O}_{\mathcal{P}_T} = T \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$. As the action of $\Omega_X^{n,cl}$ on $(\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ preserves the degree, $\mathcal{O}_{\mathcal{P}_T}$ has the induced grading from $(\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$. Notice that the differential on $(\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ is $\Omega_X^{n,cl}$ equivariant, then the differential on $(\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ descends to a differential on $\mathcal{O}_{\mathcal{P}_T}$. Therefore $\mathcal{P}_T = (X, T \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]}), \partial)$ is a dg-manifold.

Let us define the dg-morphism $\pi : \mathcal{P}_T \to X^\sharp$ as follows,

$$\pi^* : \mathcal{O}_{X^\sharp} \cong T \times_{\Omega_X^{n,cl}} \mathcal{O}_{X^\sharp} \to T \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]}).$$

Notice that the map $T \times \mathcal{O}_{X^\sharp} \to T \times (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$ is equivariant with respect to the $\Omega_X^{n,cl}$-action, therefore π^* is a well defined map. Moreover, on $\mathcal{O}_{\mathcal{P}_T}$ the $\mathbb{R}[n]$ free dg-action is induced from the dg-action on $\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]}$.

Lemma 2.3.6. *$\pi : \mathcal{P}_T \to X^\sharp$ is an $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$.*

Proof: Since T is locally trivial, it follows that $\pi : \mathcal{P}_T \to X^\sharp$ is locally trivial.
□

For any T $\Omega_X^{n,cl}$-torsor let us define $B(T) = \mathcal{P}_T$.

2.3.3 Cohomological classification of $\mathbb{R}[n]dgBn(X^\sharp)$

Proposition 2.3.7. *The assignment $Fr : \mathcal{P} \to \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P})$ establishes an equivalence of categories whose inverse is given by $B : T \to \mathcal{P}_T$.*

Proof: We check that both functors Fr and B are inverses to each other. Suppose that $\pi : \mathcal{P} \to X^\sharp$ is an object in $\mathbb{R}[n]dgBn(X^\sharp)$, we verify that $B \circ Fr \cong id$. By construction we have

$$B(Fr(\mathcal{P})) = (X, \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P}) \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]}), \partial).$$

Let us define the map

$$\Upsilon : \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P}) \times (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]}) \to \mathcal{O}_{\mathcal{P}}$$

$$(\phi, f) \mapsto (\phi^{-1})^*(f).$$

Notice that the action of $\Omega_X^{n,cl}$ on $\mathrm{Hom}(\mathcal{P}, X^\sharp \times \mathbb{R}[n])$ is given by $(g, \psi) \mapsto g.\psi = \psi^* \circ g^{-1}$ for any $\psi \in \mathrm{Hom}(\mathcal{P}, X^\sharp \times \mathbb{R}[n])$ and $g \in \mathrm{Hom}(X^\sharp \times \mathbb{R}[n], X^\sharp \times \mathbb{R}[n]) \cong \Omega_X^{n,cl}$. The following calculation

$$\begin{aligned} \Upsilon(g.\phi, f) &= ((g.\phi)^{-1})^*(f) = (\phi^{-1})^* \circ g^{-1}(f) \\ &= (\phi^{-1})^*(g^{-1}f) = \Upsilon(\phi, g^{-1}(f)) \end{aligned}$$

shows that Υ is well defined on the quotient $\mathrm{Hom}(X^\sharp \times \mathbb{R}[n], \mathcal{P}) \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \boxtimes \mathcal{O}_{\mathbb{R}[n]})$. By the definition of Υ it follows that $\Upsilon \circ \partial = \partial \circ \Upsilon$. In addition Υ commutes with π's, therefore Υ is a morphism in $\mathbb{R}[n]dgBn(X^\sharp)$, i.e. Υ is an isomorphism.
In order to verify that $Fr \circ B \cong id$ suppose that T is an $\Omega_X^{n,cl}$-torsor. Notice that the structure sheaf of the trivial bundle is isomorphic to $\Omega_X^{n,cl} \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \otimes \mathcal{O}_{\mathbb{R}[n]})$. By definition $T \cong \mathrm{Hom}_{\Omega_X^{n,cl}}(\Omega_X^{n,cl}, T)$ since the action on T is free and transitive. For any element in $\mathrm{Hom}_{\Omega_X^{n,cl}}(\Omega_X^{n,cl}, T)$ we obtain a morphism in $\mathbb{R}[n]dgBn$ that belongs to

$$\mathrm{Hom}(\Omega_X^{n,cl} \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \otimes \mathbb{R}[n]), T \times_{\Omega_X^{n,cl}} (\mathcal{O}_{X^\sharp} \otimes \mathbb{R}[n])).$$

Therefore we obtain an isomorphism $T \cong \mathrm{Fr} \circ B(T)$.

□

It is known that torsors under sheaves of groups on a space provided the universal language for classification purpose; see [6] [Chap. V Prop. 5.15], or [16] [Chap III

§4 Prop. 4.6]. Therefore, since $\Omega_X^{n,cl}$-torsors are in one to one correspondence with the group $H^1(X, \Omega_X^{n,cl}) \cong H_{dR}^{n+1}(X)$, it follows that isomorphism classes of objects in $\mathbb{R}[n]dgBn(X^\sharp)$ are in one to one correspondence with the group $H_{dR}^{n+1}(X)$, as it already known in [21] and [23].

CHAPTER 3

The Atiyah algebroid

Given a $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$, we will construct a graded object endowed with a structure of differential graded Lie algebra (dgla) that acts by derivations on $\mathcal{O}_{X^\sharp}$. This chapter follows several ideas found in [5].

3.1 The Atiyah algebra

We write explicitly all the ingredients that we need to construct the *Atiyah algebroid* associated to a given $\mathbb{R}[n]$ dg-principal bundle $\pi : \mathcal{P} \to X^\sharp$ in $\mathbb{R}[n]dgBn(X^\sharp)$. We proceed in three steps. First, we compute the Lie algebra, as invariant derivations, for our group object $\mathbb{R}[n]$. Secondly, we review the structure of the tangent sheaf $\mathcal{T}_{X^\sharp} = \underline{\mathrm{Der}}(\mathcal{O}_{X^\sharp})$ following [4]. To conclude, we produce the *Atiyah algebroid* for the given $\pi : \mathcal{P} \to X^\sharp$ as the derivations of $\mathcal{P}$ invariant with respect to the action of $\mathbb{R}[n]$, as in [5].

Lemma 3.1.1. *For the group object $\mathbb{R}[n] = (\{*\}, \mathcal{O}_{\mathbb{R}[n]}, 0)$, we have that the Lie algebra $Lie(\mathbb{R}[n]) = (Der(\mathcal{O}_{\mathbb{R}[n]}))^{\mathbb{R}[n]}$ is one dimensional real vector space.*

Proof: It is clear that $\mathbb{R}[n]$ acts on it self through the co-multiplication on $\mathcal{O}_{\mathbb{R}[n]}$. Namely

$$\alpha_t^*(t) := \Delta_t(t) = t \otimes 1 + 1 \otimes t,$$

$$\alpha_{-t}^*(t) := \Delta_t(t) = t \otimes 1 - 1 \otimes t.$$

In this case $\alpha_1 = \frac{\partial}{\partial_t}$. As the graded algebra $\mathcal{O}_{\mathbb{R}[n]}$ has just one generator, any derivation of $\mathcal{O}_{\mathbb{R}[n]}$ is written as follows,

$$D = g(t) \cdot \frac{\partial}{\partial_t},$$

for $g(t) \in \mathcal{O}_{\mathbb{R}[n]}$. Following Subsection (2.1.1), the induced co-action on the sheaf $\underline{\mathrm{Der}}(\mathcal{O}_{\mathbb{R}[n]})$ is given by the formula,

$$\widetilde{\Delta_t} D \widetilde{\Delta_{-t}} = D \otimes 1 + [\frac{\partial}{\partial_t}, D] \otimes t + ...$$

Therefore we obtain that the co-action is written as:

$$g(t)\frac{\partial}{\partial_t} \otimes 1 + [\frac{\partial}{\partial_t}, g(t)\frac{\partial}{\partial_t}] \otimes t + \frac{1}{2!}[\frac{\partial}{\partial_t}, [\frac{\partial}{\partial_t}, g(t)\frac{\partial}{\partial_t}]] \otimes t^2 + ...$$

$$g(t)\frac{\partial}{\partial_t} \otimes 1 + g'(t)\frac{\partial}{\partial_t} \otimes t + \frac{1}{2!}g''(t)\frac{\partial}{\partial_t} \otimes t^2 + ...$$

It follows that $D = g(t)\frac{\partial}{\partial_t}$ is an invariant derivation under the given co-action if and only of $g'(t) = 0$. Hence $g(t)$ is a constant function on $\mathbb{R}[n]$, i.e.

$$Lie(\mathbb{R}[n]) \cong \mathrm{sp}\langle \frac{\partial}{\partial_t} \rangle.$$

□

We briefly recall the description of the complex $\mathcal{T}_{X^\sharp}$ found in [4]. Consider the cone of the identity endomorphism of $\mathcal{T}_X$. It is a complex denote by $\widetilde{\mathcal{T}_X}$ concentrated in degree $i = -1, 0$, $\widetilde{\mathcal{T}_X}^i = \mathcal{T}_X$. This complex has a canonical structure of a dgla. In order to write down formulas for the dgla structure, we introduce an alternatively notation for $\widetilde{\mathcal{T}_X}$. Let be ϵ a formal variable of degree -1, ($\epsilon^2 = 0$). In this notation, elements in degree -1 part of $\widetilde{\mathcal{T}_X}$ are written as, $\xi\epsilon$ for $\xi \in \mathcal{T}_X$. We denote degree -1 part of $\widetilde{\mathcal{T}_X}$ by $\mathcal{T}_X \cdot \epsilon$. Similarly, elements in degree zero of $\widetilde{\mathcal{T}_X}$ reads simply as ξ, for any $\xi \in \mathcal{T}_X$. Then $\widetilde{\mathcal{T}_X}$ decomposes as follows:

$$\widetilde{\mathcal{T}_X}^0 \cong \mathcal{T}_X,\ \widetilde{\mathcal{T}_X}^{-1} \cong \mathcal{T}_X \cdot \epsilon.$$

There is a canonical action of $\widetilde{\mathcal{T}_X}$ on $\mathcal{O}_{X^\sharp}$ given by the assignment,

$$\xi \mapsto L_\xi,$$
$$\xi\epsilon \mapsto \iota_\xi,$$

for any vector field $\xi \in \mathcal{T}_X$. The differential on $\widetilde{\mathcal{T}_X}$ is just the identity map, namely $\xi\epsilon \mapsto \xi$. The Lie bracket on $\widetilde{\mathcal{T}_X}$ is given by the formulas below,

- degree -1: $[\xi\epsilon, \eta\epsilon] = 0$
- degree $-1, 0$: $[\xi\epsilon, \eta] = [\xi, \eta]\epsilon$
- degree zero: $[\xi, \eta]$,

for any ξ, η in $\mathcal{T}_X$. It follows directly that $\widetilde{\mathcal{T}_X}$ is a dgla. Since the action of $\widetilde{\mathcal{T}_X}$ on $\mathcal{O}_{X^\sharp}$ is given by Lie derivative and interior multiplication respectively, there is a map

$$\mathcal{T}_X \oplus_{\mathbb{R}} \mathcal{T}_X \cdot \epsilon \to \mathcal{T}_{X^\sharp},$$

which extends in a unique way to a map of $\mathcal{O}_{X^\sharp}$-modules, $\mathcal{O}_{X^\sharp} \otimes_{C^\infty_X} \widetilde{\mathcal{T}_X} \to \mathcal{T}_{X^\sharp}$. Let $\mathcal{T}_{X^\sharp/X}$ denotes the centralizer of $C^\infty_X \hookrightarrow \mathcal{O}_{X^\sharp}$, i.e. derivations of $\mathcal{O}_{X^\sharp}$ which are C^∞_X-linear. Since the action of $\mathcal{T}_X \cdot \epsilon$ on $\mathcal{O}_{X^\sharp}$ is C^∞_X-linear, it factors through $\mathcal{T}_{X^\sharp/X}$, i.e. there is a map,

$$\mathcal{O}_{X^\sharp} \otimes_{C^\infty_X} \mathcal{T}_X \cdot \epsilon \to \mathcal{T}_{X^\sharp/X}, \tag{3.1.1}$$

$$\beta \otimes \xi\epsilon \mapsto \beta \wedge \iota_\xi,$$

which is an isomorphism. Similarly, the action of degree zero part $\widetilde{\mathcal{T}_X}^0$ on $\mathcal{O}_{X^\sharp}$ is C^∞_X-linear modulo $\mathcal{T}_{X^\sharp/X}$ by the Cartan formulas: $L_{f\xi} = fL_\xi + df \wedge \iota_\xi$ and $L_\xi(f\omega) = fL_\xi\omega + \iota_\xi df \wedge \omega$. The map $\mathcal{T}_X \to \mathcal{T}_{X^\sharp}/\mathcal{T}_{X^\sharp/X}$, $\xi \mapsto L_\xi$, extends to an isomorphism

$$\mathcal{O}_{X^\sharp} \otimes_{C^\infty_X} \mathcal{T}_X \to \mathcal{T}_{X^\sharp}/\mathcal{T}_{X^\sharp/X}. \tag{3.1.2}$$

Therefore, we obtain the following short exact sequence of $\mathcal{O}_{X^\sharp}$-modules,

$$0 \to \mathcal{O}_{X^\sharp} \otimes_{C^\infty_X} \mathcal{T}_X \cdot \epsilon \to \mathcal{T}_{X^\sharp} \to \mathcal{O}_{X^\sharp} \otimes_{C^\infty_X} \mathcal{T}_X \to 0. \tag{3.1.3}$$

Since $\mathcal{T}^i_{X^\sharp} = 0$ for $i \leq -2$, its degree -1 part is isomorphic to $\mathcal{T}_X \cdot \epsilon$ and its degree zero part decomposes as, $\mathcal{T}_X \oplus (\Omega^1_X[-1] \otimes_{C^\infty_X} \mathcal{T}_X[\epsilon])$. In higher degrees we have that $\mathcal{T}_{X^\sharp}$ splits as an $\mathcal{O}_{X^\sharp}$-module: $\mathcal{T}^i_{X^\sharp} = (\Omega^i_X \otimes \mathcal{T}_X) \oplus (\Omega^{i+1}_X \otimes \mathcal{T}_X \cdot \epsilon)$.

Let $\pi : \mathcal{P} \to X^\sharp$ be a $\mathbb{R}[n]$ dg-principal bundle over $X^\sharp$. Following the construction of the Atiyah algebroid for principal bundles [5], we build up the *Atiyah algebroid* associated to $\pi : \mathcal{P} \to X^\sharp$. From the dg-morphism π, we obtain a map of complexes $d\pi : \mathcal{T}_\mathcal{P} \to \pi^*\mathcal{T}_{X^\sharp}$; see Lemma (1.2.16). We denote by $\mathcal{T}_{\mathcal{P}|X^\sharp}$ the kernel of $d\pi$. In other words, $\mathcal{T}_{\mathcal{P}|X^\sharp}$ is the centralizer of $\mathcal{O}_{X^\sharp}$ inside $\mathcal{O}_\mathcal{P}$. Therefore, there is a short exact sequence of $\mathcal{O}_\mathcal{P}$-modules,

$$0 \longrightarrow \mathcal{T}_{\mathcal{P}|X^\sharp} \longrightarrow \mathcal{T}_\mathcal{P} \xrightarrow{d\pi} \pi^*\mathcal{T}_{X^\sharp} \longrightarrow 0\ .$$

Passing to a short exact sequence of sheaves of modules over $X^\sharp$ we obtain,

$$0 \longrightarrow \pi_*\mathcal{T}_{\mathcal{P}|X^\sharp} \longrightarrow \pi_*\mathcal{T}_\mathcal{P} \xrightarrow{d\pi} \pi_*\pi^*\mathcal{T}_{X^\sharp} \longrightarrow 0\ .$$

By definition the Atiyah algebroid associated to $\pi : \mathcal{P} \to X^\sharp$, is the sheaf of derivations of $\pi_*\mathcal{T}_\mathcal{P}$ invariant under the induced action of $\mathbb{R}[n]$ as an $\mathcal{O}_{X^\sharp}$-module. Then we pass to the invariant elements and obtain the short exact sequence below

$$0 \longrightarrow (\pi_*\mathcal{T}_{\mathcal{P}|X^\sharp})^{\mathbb{R}[n]} \longrightarrow (\pi_*\mathcal{T}_\mathcal{P})^{\mathbb{R}[n]} \xrightarrow{d\pi} (\pi_*\pi^*\mathcal{T}_{X^\sharp})^{\mathbb{R}[n]} \longrightarrow 0. \tag{3.1.4}$$

Similarly as in [5], we will show that there are canonical identifications in both extremes of (3.1.4).

Lemma 3.1.2. *The map* $\upsilon : \mathcal{O}_{\mathcal{P}} \otimes Lie(\mathbb{R}[n]) \to \mathcal{T}_{\mathcal{P}|X^\sharp}$ $\omega \otimes \frac{\partial}{\partial_t} \mapsto \omega \cdot \alpha_1$ *is an isomorphism.*

Proof: Since the dg-action of $\mathbb{R}[n]$ on $\mathcal{P}$ is free, there is $h \in \mathcal{O}^n_{\mathcal{P}}$ locally defined on X such that $\alpha_1(h) = 1$. Then υ is injective. Since the rank$(\mathcal{T}_{\mathcal{P}|X^\sharp}) = 1$, υ is surjective. □

As a direct consequence, due to the abelian nature of $\mathbb{R}[n]$, we obtain that invariant elements of the kernel of $d\pi$ are identified with $\mathcal{O}_{X^\sharp}[n]$ since,

$$(\mathcal{O}_P)^{\mathbb{R}[n]} \otimes (Lie(\mathbb{R}[n]))^{\mathbb{R}[n]} \cong \mathcal{O}_{X^\sharp} \otimes \mathbb{R}[n] = \mathcal{O}_{X^\sharp}[n].$$

Lemma 3.1.3. *As invariant elements are identified with* $\mathcal{O}_{X^\sharp}$*, therefore*

$$(\pi_*\pi^*\mathcal{T}_{X^\sharp})^{\mathbb{R}[n]} = (\mathcal{O}_{\mathcal{P}})^{\mathbb{R}[n]} \otimes_{\mathcal{O}_{X^\sharp}} \mathcal{T}_{X^\sharp} \cong \mathcal{T}_{X^\sharp}.$$

□

After the above identifications, the Atiyah algebroid of $\pi : \mathcal{P} \to X^\sharp$ fits in the following exact sequence of $\mathcal{O}_{X^\sharp}$-modules,

$$0 \longrightarrow \mathcal{O}_{X^\sharp}[n] \xrightarrow{i} \mathcal{A}_{\mathcal{P}} \xrightarrow{\sigma} \mathcal{T}_{X^\sharp} \longrightarrow 0, \tag{3.1.5}$$

where $\mathcal{A}_{\mathcal{P}}$ stands for $(\pi_*\mathcal{T}_{\mathcal{P}})^{\mathbb{R}[n]}$.

Remark 3.1.4. *The homological vector field on* $\mathcal{P}$*,* ∂*, belongs to* $\mathcal{A}^1_{\mathcal{P}}$ *since we required* $[\alpha_1, \partial] = 0$.

Suppose that $a, b \in \mathcal{A}_{\mathcal{P}}$. The calculation below shows that $[a, b] \in \mathcal{A}_{\mathcal{P}}$, i.e. $\mathcal{A}_{\mathcal{P}}$ is closed under the bracket on $\pi_*\mathcal{T}_{\mathcal{P}}$.

$$\begin{aligned}[\alpha_1, [a, b]] &= [[\alpha_1, a], b] + [a, [\alpha_1, b]] \\ &= 0.\end{aligned}$$

Moreover $\mathcal{A}_{\mathcal{P}}$ is an $\mathcal{O}_{X^\sharp}$-submodule of $\pi_*\mathcal{T}_{\mathcal{P}}$. Since for any $\omega \in \mathcal{O}_{X^\sharp}$ and $a \in \mathcal{A}_{\mathcal{P}}$ it follows that,

$$\begin{aligned}[\alpha_1, \omega \cdot a] &= [\alpha_1, \omega] \cdot a + \omega \cdot [\alpha_1, a] \\ &= 0.\end{aligned}$$

Lemma 3.1.5. $(\mathcal{A}_{\mathcal{P}}, [\,,\,], \sigma)$ *is a sheaf of differential graded Lie algebras on* X

Proof: By construction it follows that $\mathcal{A}_{\mathcal{P}}$ is a differential graded Lie $\mathcal{O}_{X^\sharp}$-module. □

Example 3.1.6. *Let be* $\mathcal{P} = X^\sharp \times \mathbb{R}[n]$ *the trivial* $\mathbb{R}[n]$ *dg-principal bundle over* $X^\sharp$*. Then the Atiyah graded algebroid is the semi-direct product*

$$\mathcal{A}_{\mathcal{P}} = \mathcal{O}_{X^\sharp}[n] \ltimes \mathcal{T}_{X^\sharp}.$$

The Lie bracket is the $\mathbb{R}$*-linear extension of the bracket on* $\mathcal{T}_{X^\sharp}$ *to* $\mathcal{O}_{X^\sharp}[n] \otimes \mathcal{T}_{X^\sharp}$*, since* $\mathcal{T}_{X^\sharp}$ *acts by derivation on* $\mathcal{O}_{X^\sharp}$*. The anchor map is the projection onto* $\mathcal{T}_{X^\sharp}$*. The differential on* $\mathcal{P}$ *is given by* $d \otimes 1$*, therefore* $\delta = [d \otimes 1, \,]$ *on* $\mathcal{A}_{\mathcal{P}}$*.*

Example 3.1.7. *With the same notation as in Example* (3.1.6)*. A non trivial structure on* $\mathcal{A}_{\mathcal{P}}$ *is given by the differential* $d \otimes 1 + H \otimes \frac{\partial}{\partial_t}$*, where* $H \in \Omega_X^{n+1,cl} = \mathcal{O}^1_{X^\sharp}[n]$*.*

3.1.1 The algebraic structure of the Atiyah algebroid

The Atiyah algebroid $\mathcal{A}_{\mathcal{P}}$ acts by derivation on $\mathcal{O}_{X^\sharp}[n]$ through the map $\sigma : \mathcal{A}_{\mathcal{P}} \to \mathcal{T}_{X^\sharp}$. It is clear that the kernel of σ is equal to $\mathcal{O}_{X^\sharp}[n]$, therefore the Lie bracket $[\,,\,] : \mathcal{A}_{\mathcal{P}} \otimes \mathcal{A}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{P}}$ restricted to $\mathcal{O}_{X^\sharp}$ produced the following formulas

- $[a, i(\eta)] = i(\iota_{\sigma(a)}\eta)$, if $a \in \mathcal{A}^{-1}_{\mathcal{P}}$
- $[a, i(\eta)] = i(L_{\sigma(a)}\eta)$ if $a \in \mathcal{A}^0_{\mathcal{P}}$
- $\delta(i(\eta)) = [\partial, i(\eta)] = i(d\eta)$ d denotes the exterior differential

- For any $a, b \in \mathcal{A}_{\mathcal{P}}^{-1}$, $[a, b] \in \mathcal{A}_{\mathcal{P}}^{-2} \cong \Omega_X^{n-2}$

It will be useful to keep in mind the following diagram, from (3.1.5), that represents degree -2,-1 and 0 components of the Atiyah algebroid.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^n & \longrightarrow & \mathcal{A}_{\mathcal{P}}^0 & \xrightarrow{\sigma} & \mathcal{T}_{X^\sharp}^0 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle [d,]} & & \\ 0 & \longrightarrow & \Omega_X^{n-1} & \longrightarrow & \mathcal{A}_{\mathcal{P}}^{-1} & \xrightarrow{\sigma} & \mathcal{T}_{X^\sharp}^{-1} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle \delta} & & & & \\ 0 & \longrightarrow & \Omega_X^{n-2} & \xrightarrow{\cong} & \mathcal{A}_{\mathcal{P}}^{-2} & & & & \end{array} \tag{3.1.6}$$

As $\mathcal{A}_{\mathcal{P}}$ fits in a short exact sequence of $\mathcal{O}_{X^\sharp}$-modules we consider $\mathcal{O}_{X^\sharp}[n]$ as a subsheaf of $\mathcal{A}_{\mathcal{P}}$. For any $\eta \in \mathcal{O}_{X^\sharp}$, we will denoted $i(\eta)$ simply by η as a local section in $\mathcal{A}_{\mathcal{P}}$.

CHAPTER 4

$\mathcal{O}_{X^\sharp}[n]$-extensions of $\mathcal{T}_{X^\sharp}$

The Atiyah algebroid $\mathcal{A}$ that we constructed out of a given $\mathbb{R}[n]$ dg-principal bundle belongs to a special class of Lie algebroid over $X^\sharp$. In this chapter, we will analyze the algebraic structure of $\mathcal{A}$ in terms of generators and relations. In addition, we will show how to produce an exact Courant algebroid on X using the degree -1 part of $\mathcal{A}$.

Definition 4.0.8. *An $\mathcal{O}_{X^\sharp}[n]$-extension of $\mathcal{T}_{X^\sharp}$ $\mathcal{A}$ is a sheaf of $\mathcal{O}_{X^\sharp}$-modules over X endowed with the following structure*

- $(\mathcal{A}, [\ ,\ ], \delta)$ *is a differential graded Lie algebra*
- *An $\mathcal{O}_{X^\sharp}$-linear map of Lie algebras $\sigma : \mathcal{A} \to \mathcal{T}_{X^\sharp}$ called the anchor map.*
- *There is a central element of degree $-n$ given by $1 \mapsto c$, which gives rise to the short exact sequence of $\mathcal{O}_{X^\sharp}$-modules*

$$0 \longrightarrow \mathcal{O}_{X^\sharp}[n] \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} \mathcal{T}_{X^\sharp} \longrightarrow 0.$$

These data are required to satisfy

$$[a, fb] = f[a, b] + (\sigma(a)f)b.$$

For any $a, b \in \mathcal{A}$ and $f \in \mathcal{O}_{X^\sharp}$.

Suppose that $\mathcal{A}, \mathcal{A}'$ are two $\mathcal{O}_{X^\sharp}[n]$-extensions of $\mathcal{T}_{X^\sharp}$. A morphism is an $\mathcal{O}_{X^\sharp}$-linear map $\phi : \mathcal{A} \to \mathcal{A}'$, which preserved the brackets and it is compatible with the anchor maps. It follows that such a map should fit into the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X^\sharp}[n] & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{T}_{X^\sharp} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathcal{O}_{X^\sharp}[n] & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{T}_{X^\sharp} & \longrightarrow & 0. \end{array}$$

In other words ϕ is an isomorphism. The category of $\mathcal{O}_{X^\sharp[n]}$-extensions of $\mathcal{T}_{X^\sharp}$ will be denoted by $\mathcal{O}_{X^\sharp}[n]$-LA$(X^\sharp)$.

Remark 4.0.9. *Such a $\phi : \mathcal{A} \to \mathcal{A}'$ is in fact an isomorphism. Therefore $\mathcal{O}_{X^\sharp}[n]$-LA$(X^\sharp)$ is a groupoid.*

4.1 Generators for the extension

We will explain how the algebraic structure defined on $\mathcal{A}$ is completely determined by the degree -1 part, $\mathcal{A}^{-1}$. We will show that $\mathcal{A}^0$ is built up from $\mathcal{A}^{-1}$ using the differential and the $\mathcal{O}_{X^\sharp}$-module structure respectively.

4.1.1 Generators and relations

For the first set of generators, denoted by $d\mathcal{A}^{-1}$, consider the push-out along $d : \Omega_X^{n-1} \to \Omega_X^{n,cl}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^{n-1} & \longrightarrow & \mathcal{A}^{-1} & \xrightarrow{\sigma} & \mathcal{T}_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_X^{n,cl} & \longrightarrow & d\mathcal{A}^{-1} & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0. \end{array} \quad (4.1.1)$$

This means that local sections of $d\mathcal{A}^{-1}$ are written as ordered pairs (a,ω), where $a \in \mathcal{A}^{-1}$ and $\omega \in \Omega_X^{n,cl}$ modulo an equivalence relation. Two of those pairs are equivalent if their difference is given by differential $(n-1)$-form on X:

$$(a,\omega) \sim (a',\omega') \Leftrightarrow \exists \eta \in \Omega_X^{n-1} s.t.\ a - a' = -\eta,\ \omega - \omega' = d\eta.$$

In other words, an equivalence class is equal to $(a,\omega) = \{(a+\eta, \omega - d\eta) | \eta \in \Omega_X^{n-1}\}$, for $a \in \mathcal{A}^{-1}$ and $\omega \in \Omega_X^{n,cl}$. The minus sign in the above equivalence relation comes from the injective map,

$$\Upsilon : \Omega_X^{n-1} \to \mathcal{A}^{-1} \oplus \Omega_X^{n,cl},\ \eta \mapsto (-\eta, d\eta).$$

In this sense $d\mathcal{A}^{-1}$ is the cokernel of Υ,

$$d\mathcal{A}^{-1} = (\mathcal{A}^{-1} \oplus \Omega_X^{n,cl})/Im(\Omega_X^{n-1}) := (\mathcal{A}^{-1} \oplus \Omega_X^{n,cl})/\sim .$$

On $d\mathcal{A}^{-1}$ we define the following two maps. Since the kernel of σ is equal to Ω_X^{n-1} the anchor map $\bar{\sigma} : d\mathcal{A}^{-1} \to \mathcal{T}_X$, given by $(a,\omega) \mapsto \sigma(\delta(a))$ is well defined. The second map $\bar{\delta} : d\mathcal{A}^{-1} \to \mathcal{A}^0$ is given by $\bar{\delta}(a,\omega) = \delta a + \omega$. Notice that $\bar{\delta}$ is well define since $\bar{\delta}(a+\eta, \omega - d\eta) = \delta a + \omega = \bar{\delta}(a,\omega)$. By the definition $\bar{\delta}$ factors the differential δ on $\mathcal{A}$ through $d\mathcal{A}^{-1}$,

$$\mathcal{A}^{-1} \to d\mathcal{A}^{-1} \to \mathcal{A}^0.$$

The second set of generators, denoted $(\Omega_X^1 \wedge \mathcal{A}^{-1})$, is defined by the push out along the wedge product on Ω_X,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^n & \longrightarrow & \Omega_X^1 \wedge \mathcal{A}^{-1} & \longrightarrow & \Omega_X^1 \otimes \mathcal{T}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \wedge} & & \uparrow & & \uparrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \Omega_X^1 \otimes \Omega_X^{n-1} & \longrightarrow & \Omega_X^1 \otimes \mathcal{A}^{-1} & \longrightarrow & \Omega_X^1 \otimes \mathcal{T}_X & \longrightarrow & 0. \end{array} \tag{4.1.2}$$

Since the wedge product $\wedge$ is a surjective map, local sections of $\Omega_X^1 \wedge \mathcal{A}^{-1}$ are written as $\sum_l (\theta_l \otimes a_l)$ modulo an equivalence relation, for $\theta_l \in \Omega_X^1$ and $a_l \in \mathcal{A}^{-1}$. The equivalence relation is given by $\sum_l (\theta_l \otimes a_l) \sim \sum_m (\theta'_m \otimes a'_m) \Leftrightarrow \exists (\eta \otimes \gamma) \in Ker(\wedge)$

such that $\sum_m \theta'_m \otimes a'_m - \sum_l \theta_l \otimes a_l = \eta \otimes \gamma$. The $\mathcal{O}_{X^\sharp}$-module structure on $\mathcal{A}$ gives rise to the well defined map $\bar{\wedge} : \Omega^1_X \wedge \mathcal{A}^{-1} \to \mathcal{A}^0$, $\bar{\wedge}(\sum_l \theta_l \otimes a_l) = \sum_l \theta_l \cdot a_l = \bar{\wedge}(\sum_l \theta_l \otimes a_l - \eta \otimes \gamma)$, where $\eta \otimes \gamma \in Ker(\wedge)$.

4.1.2 The Lie algebra structure on generators

We construct a Lie algebra structure for both sets of generators $d\mathcal{A}^{-1}$ and $\Omega^1_X \wedge \mathcal{A}^{-1}$, respectively. Moreover we construct from $d\mathcal{A}^{-1}$ and $\Omega^1_X \wedge \mathcal{A}^{-1}$ the sheaf on X that would be isomorphic to $\mathcal{A}^0$.

On $\mathcal{A}^{-1} \oplus \Omega^{n,cl}_X$. For any pair of elements (a, ω) and (b, ζ), let

$$[(a, \omega), (b, \zeta)]_d := ([\delta a, b], L_{\delta(a)}\zeta - L_{\delta(b)}\omega)^1.$$

Consider the subsheaf $\{(\gamma, -d\gamma) | \gamma \in \Omega^{n-1}_X\}$ of $\mathcal{A}^{-1} \oplus \Omega^{n,cl}_X$.

Lemma 4.1.1. $\{(\gamma, -d\gamma) | \gamma \in \Omega^{n-1}_X\}$ *is a two sided ideal of the bracket on* $\mathcal{A}^{-1} \oplus \Omega^{n,cl}_X$.

Proof:

$$\begin{aligned}[(a, \omega), (\gamma, -d\gamma)]_d &= ([\delta(a), \gamma], L_{\delta(a)}(-d\gamma)) \\ &= (L_{\delta(a)}\gamma, -L_{\delta(a)}d\gamma) = (L_{\delta(a)}\gamma, -dL_{\delta(a)}\gamma)\end{aligned}$$

$$\begin{aligned}[(\gamma, -d\gamma), (a, \omega)]_d &= ([\delta(\gamma), a], -L_{\delta(a)}(-d\gamma) \\ &= (-[a, d\gamma], L_{\delta(a)}d\gamma) \\ &= (-\iota_a d\gamma, dL_{\delta(a)\gamma}) \\ &= (L_{\delta(a)}\gamma - d\iota_a\gamma, -dL_{\delta(a)}\gamma) = (\gamma_1, -d\gamma_1)\end{aligned}$$

[1]We abbreviate $\iota_{\sigma(a)} = \iota_a$

□

It follows that the bracket on $\mathcal{A}^{-1} \oplus \Omega_X^{n-1}$ induces a bracket in quotient $d\mathcal{A}^{-1}$ that will be denote by $[\ ,\ ]_d$ as well.

Lemma 4.1.2. *$d\mathcal{A}^{-1}$ is a sheaf of Lie algebras over X*

Proof: We verify that the bracket is skew-symmetric, the Jacobi identity is left to the reader.

$$\begin{aligned}
[(b,\zeta),(a,\omega)]_d &= ([\delta b, a], L_{\delta(b)}\omega - L_{\delta(a)}\zeta) \\
&= (\delta[a,b] + [b,\delta(a)], L_{\delta(b)}\omega - L_{\delta(a)}\zeta) \\
&= ([b,\delta(a)], -d^2[a,b] + L_{\delta(b)}\omega - L_{\delta(a)}\zeta) \\
&= (-[\delta(a),b], -(-L_{\delta(b)}\omega + L_{\delta(a)}\zeta)) \\
&= -[(a,\omega),(b,\zeta)]_d.
\end{aligned}$$

□

The bracket on $\Omega_X^1 \otimes \mathcal{A}^{-1}$ is defined using the Leibniz rule, since $\mathcal{A}^{-1}$ acts by derivations on $\mathcal{O}_{X^\sharp}$:

$$[\sum_i \alpha_i \otimes a_i, \sum_j \beta_j \otimes b_j]_\wedge = \sum_{i,j}(\iota_{a_i}\beta_j(\alpha_i \otimes b_j) - \iota_{b_j}\alpha_i(\beta_j \otimes a_i) + \beta_j \otimes \alpha_i \cdot [a_i, b_j])$$

Remark 4.1.3. *As the bracket $[\ ,\ ]_\wedge$ is $\mathbb{R}$-linear it is enough to consider expressions of the form $\alpha \otimes a$ instead than $\sum_i \alpha_i \otimes a_i$.*

Similarly to the construction of $d\mathcal{A}^{-1}$, the kernel of $\bar{\wedge}$ is a two sided ideal for the bracket $[\ ,\ ]$ on $\Omega_X^1 \otimes \mathcal{A}^{-1}$. Let us consider $\gamma_1, ..., \gamma_{n-1} \in \Omega_X^1$ such that $\beta \otimes \gamma_1 \wedge ... \wedge \gamma_{n-1} \in \operatorname{Ker}(\wedge)$,

$$\begin{aligned}
[\alpha \otimes a, \beta \otimes \gamma_1 \wedge ... \wedge \gamma_{n-1}]_\wedge &= (\iota_a\beta)\alpha \otimes \gamma_1 \wedge ... \wedge \gamma_{n-1} + \beta \otimes \alpha \wedge [a, \gamma_1 \wedge ... \wedge \gamma_{n-1}] \\
&= (\iota_a\beta)\alpha \otimes \gamma_1 \wedge ... \wedge \gamma_{n-1} + \beta \otimes \alpha \wedge \iota_a(\gamma_1 \wedge ... \wedge \gamma_{n-1})
\end{aligned}$$

$\bar{\wedge}((\iota_a\beta)\alpha\otimes\gamma_1\wedge...\wedge\gamma_{n-1}+\beta\otimes\alpha\wedge\iota_a(\gamma_1\wedge...\wedge\gamma_{n-1}))=0$ due to the fact that ι_a is a derivation on Ω_X and $\beta\otimes\gamma_1\wedge...\wedge\gamma_{n-1}\in Ker(\wedge)$. On the other hand, we obtain

$$\begin{aligned}[\beta\otimes\gamma_1\wedge...\wedge\gamma_{n-1},\alpha\otimes a]_\wedge &= -(\iota_a\beta)\alpha\otimes\gamma_1\wedge...\wedge\gamma_{n-1}+\alpha\otimes\beta\wedge[\gamma_1\wedge...\wedge\gamma_{n-1},a]\\ &= -(\iota_a\beta)\alpha\otimes\gamma_1\wedge...\wedge\gamma_{n-1}+\alpha\otimes\beta\wedge\iota_a(\gamma_1\wedge...\wedge\gamma_{n-1})\\ &= -\alpha\otimes(\iota_a\beta\wedge\gamma_1\wedge...\wedge\gamma_{n-1}-\beta\wedge\iota_a(\gamma_1\wedge...\wedge\gamma_{n-1}))\\ &= -\alpha\otimes\iota_a(\beta\wedge\gamma_1\wedge...\wedge\gamma_{n-1})=0\end{aligned}$$

It follows that the bracket on $\Omega^1_X\otimes\mathcal{A}^{-1}$ descends to the quotient and will be denoted by $[\ ,\ ]_\wedge$ as well.

Lemma 4.1.4. *$\Omega^1_X\wedge\mathcal{A}^{-1}$ is a sheaf of Lie algebras over X.*

Proof: On the quotient $\Omega^1_X\wedge\mathcal{A}^{-1}$, the bracket $[\ ,\ ]_\wedge$ is skew-symmetric and satisfies the Jacobi identity.

$$\begin{aligned}[\beta\otimes b,\alpha\otimes a]_\wedge &= \iota_b\alpha(\beta\otimes a)-\iota_a\beta(\alpha\otimes b)+\alpha\otimes\beta\cdot[b,a]\\ &= \iota_b\alpha(\beta\otimes a)-\iota_a\beta(\alpha\otimes b)+\alpha\otimes\beta\cdot[a,b],\ \text{and}\\ [\alpha\otimes a,\beta\otimes b]_\wedge &= \iota_a\beta(\alpha\otimes b)-\iota_b\alpha(\beta\otimes a)+\beta\otimes\alpha\cdot[a,b]\end{aligned}$$

If we add these two brackets we obtain

$$\begin{aligned}[\beta\otimes b,\alpha\otimes a]_\wedge+[\alpha\otimes a,\beta\otimes b]_\wedge &= \alpha\otimes\beta\cdot[a,b]+\beta\otimes\alpha\cdot[a,b]\\ &= 0\\ [\beta\otimes b,\alpha\otimes a]_\wedge &= -[\alpha\otimes a,\beta\otimes b]_\wedge\end{aligned}$$

In order to verify the Jacobi identity we compute separately the three terms involved in the identity.

$$[\alpha\otimes a,[\beta\otimes b,\gamma\otimes c]_\wedge]_\wedge=[[\alpha\otimes a,\beta\otimes b]_\wedge,\gamma\otimes c]_\wedge+[\beta\otimes b,[\alpha\otimes a,\gamma\otimes c]_\wedge]_\wedge$$

For the left-hand side of the Jacobi identity we obtain,

$$\begin{aligned}
[\alpha\otimes a,[\beta\otimes b,\gamma\otimes c]_\wedge]_\wedge &= [\alpha\otimes a,\iota_b\gamma(\beta\otimes c)-\iota_c\beta(\gamma\otimes b)+\gamma\otimes\beta\cdot[b,c]]_\wedge\\
&= \iota_b\gamma[\alpha\otimes a,\beta\otimes c]_\wedge-\iota_c\beta[\alpha\otimes a,\gamma\otimes b]_\wedge+[\alpha\otimes a,\gamma\otimes\beta\cdot[b,c]]_\wedge\\
&= \iota_b\gamma(\iota_a\beta(\alpha\otimes c)-\iota_c\alpha(\beta\otimes a)+\beta\otimes\alpha\cdot[a,c])\\
&\quad -\iota_c\beta(\iota_a\gamma(\alpha\otimes b)-\iota_b\alpha(\gamma\otimes a)+\gamma\otimes\alpha\cdot[a,b])\\
&\quad +\iota_a\gamma(\alpha\otimes\beta\cdot[b,c])+\iota_a\beta(\gamma\otimes\alpha\cdot[b,c])-\gamma\otimes\alpha\wedge\beta\cdot[a,[b,c]]
\end{aligned}$$

In a similar way, the right hand side in the Jacobi identity is given by,

$$\begin{aligned}
[[\alpha\otimes a,\beta\otimes b]_\wedge,\gamma\otimes c]_\wedge &= \iota_a\beta(\iota_b\gamma(\alpha\otimes c)-\iota_c\alpha(\gamma\otimes b)+\gamma\otimes\alpha\cdot[b,c])\\
&\quad -\iota_b\alpha(\iota_a\gamma(\beta\otimes c)-\iota_c\beta(\gamma\otimes a)+\gamma\otimes\beta\cdot[a,c])\\
&\quad -\iota_c\beta(\gamma\otimes\alpha\cdot[a,b])+\iota_c\alpha(\gamma\otimes\beta\cdot[a,b])-\gamma\otimes\beta\wedge\alpha\cdot[c,[a,b]]
\end{aligned}$$

$$\begin{aligned}
[\beta\otimes b,[\alpha\otimes a,\gamma\otimes c]_\wedge]_\wedge &= \iota_a\gamma(\iota_b\alpha(\beta\otimes c)-\iota_c\beta(\alpha\otimes b)+\alpha\otimes\beta\cdot[b,c])\\
&\quad -\iota_c\alpha(\iota_b\gamma(\beta\otimes a)-\iota_a\beta(\gamma\otimes b)+\gamma\otimes\beta\cdot[b,a])\\
&\quad +\iota_b\gamma(\beta\otimes\alpha\cdot[a,c])+\iota_b\alpha(\gamma\otimes\beta\cdot[a,c])-\gamma\otimes\beta\wedge\alpha\cdot[b,[a,c]]
\end{aligned}$$

For example, from the right-hand side of the Jacobi identity follows the left-hand side, since the bracket on $\mathcal{A}$ is a Lie a bracket

$$\begin{aligned}
-\gamma\otimes\alpha\wedge\beta\cdot([a,[b,c]]) &= -\gamma\otimes\beta\wedge\alpha\cdot(-[a,[b,c]])\\
&= -\gamma\otimes\beta\wedge\alpha\cdot(-([[a,b],c]-[b,[a,c]]))\\
&= -\gamma\otimes\beta\wedge\alpha\cdot(-[[a,b],c]+[b,[a,c]])\\
&= -\gamma\otimes\beta\wedge\alpha\cdot([c,[a,b]]+[b,[a,c]]).
\end{aligned}$$

□

So far, from the short exact sequence in degree -1 part of $\mathcal{A}$,

$$0\to\Omega_X^{n-1}\to\mathcal{A}^{-1}\to\mathcal{T}_X\to 0$$

we construct two short exact sequences of Lie algebras over X, namely:

$$0 \to \Omega_X^{n,cl} \to d\mathcal{A}^{-1} \to \mathcal{T}_X \to 0,$$

$$0 \to \Omega_X^n \to \Omega_X^1 \wedge \mathcal{A}^{-1} \to \Omega_X^1 \otimes \mathcal{T}_X \to 0.$$

If we add both sequences we obtain an extension whose middle term maps to $\mathcal{A}^0$ as follows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_X^{n,cl} \oplus \Omega_X^n & \longrightarrow & d\mathcal{A}^1 \oplus (\Omega_X^1 \wedge \mathcal{A}^{-1}) & \longrightarrow & \mathcal{T}_X \oplus (\Omega_X^1 \otimes \mathcal{T}_X) & \longrightarrow & 0 \\ & & & & \Big\downarrow \Phi & \overset{\sigma}{\nearrow} & & & \\ & & & & \mathcal{A}^0. & & & & \end{array} \tag{4.1.3}$$

The vertical map is given by $\Phi((a,\omega)\oplus(\theta\otimes b)) = \delta(a)+\omega+\theta\cdot b$. By construction Φ is a well defined map. The kernel of Φ is precisely the kernel of the addition between differential n-forms.

Lemma 4.1.5. *The map* Φ *in* (4.1.3) *is surjective.*

Proof: Since degree by degree the given $\mathcal{A}$ fits into a short exact sequence, any local section $a_0 \in \mathcal{A}^0$ we may write as $a_0 = x + \omega$, where $x \in \mathcal{T}_X \oplus \Omega_X^1 \otimes \mathcal{T}_X$ and $\omega \in \Omega_X^n$. By construction, there is an element $\bar{x} \in d\mathcal{A}^{-1} \oplus (\Omega^1 \wedge \mathcal{A}^{-1})$ such that $\bar{x} \mapsto x$ under the map $d\mathcal{A}^{-1} \oplus \Omega^1 \wedge \mathcal{A}^{-1} \to \mathcal{T}_X \oplus \Omega_X^1 \otimes \mathcal{T}_X$. It follows that, $\Phi(\bar{x} + (0,\omega)) = a_0$, viewing ω as an element in $d\mathcal{A}^{-1} \oplus \Omega^1 \wedge \mathcal{A}^{-1}$ as the pre-image of the addition map on differential n-forms composed with the inclusion $\Omega_X^{n,cl} \oplus \Omega_X^n \to d\mathcal{A}^{-1} \oplus \Omega^1 \wedge \mathcal{A}^{-1}$.
□

4.1.3 Additional structure on $d\mathcal{A}^{-1} \oplus (\Omega_X^1 \wedge \mathcal{A}^1)$

There is a natural way, (by linearity), to introduce a bracket on $d\mathcal{A}^{-1} \oplus (\Omega_X^1 \wedge \mathcal{A}^{-1})$ given by the formula:

$$[(a,\omega) + \theta \otimes b, (a',\omega') + \theta' \otimes b'] = [(a,\omega),(a',\omega')]_d + [\theta \otimes b, \theta' \otimes b']_\wedge + \tag{4.1.4}$$

$$+ [(a,\omega), \theta' \otimes b']_1 + [\theta \otimes b, (a',\omega')]_2 \tag{4.1.5}$$

where the last two terms of the bracket are defined using the Leibniz rule, namely:

$$[(a,\omega),\theta'\otimes b']_1 = -\theta'\otimes\iota'_b\omega + L_{\delta(a)}\theta'\otimes b' + \theta'\otimes[\delta(a),b'],$$

$$[\theta\otimes b,(a',\omega')]_2 = \theta\otimes\iota_b\omega' - L_{\delta(a')}\theta\otimes b - \theta\otimes[\delta(a'),b].$$

By definition of the bracket (4.1.4) on $d\mathcal{A}^1\oplus(\Omega^1_X\wedge\mathcal{A}^{-1})$ is skew-symmetric. On $d\mathcal{A}^1\oplus(\Omega^1_X\wedge\mathcal{A}^{-1})$ consider the subsheaf

$$K := \{(0,\omega_1\wedge\omega_{n-1}) + (-\omega_1)\otimes\omega_{n-1} | \omega_1\in\Omega^1_X, \omega_{n-1}\in\Omega^{n-1}_X\}.$$

The calculation below shows that K is an ideal for the bracket (4.1.4):

$$\begin{aligned}[(a,\omega)+\theta\otimes b,(0,\omega_1\wedge\omega_{n-1})+(-\omega_1)\otimes\omega_{n-1}] &= (0, L_{\delta(a)}(\omega_1\wedge\omega_{n-1}))\\ &- (L_{\delta(a)}\omega_1)\otimes\omega_{n-1} - \omega_1\otimes(L_{\delta(a)}\omega_{n-1}) + \theta\otimes\iota_b(\omega_1\wedge\omega_{n-1})\\ &- \iota_b(\omega_1)\theta\otimes\omega_{n-1} - \omega_1\otimes\theta\wedge\iota_b(\omega_{n-1})\\ &= (0, L_{\delta(a)}(\omega_1\wedge\omega_{n-1})) - (L_{\delta(a)}\omega_1)\otimes\omega_{n-1} - \omega_1\otimes(L_{\delta(a)}\omega_{n-1})\\ &\in K,\end{aligned}$$

Recall that $-\omega_1\otimes\theta\wedge\iota_b(\omega_{n-1}) = \theta\otimes\omega_1\wedge\iota_b(\omega_{n-1})$ in $\Omega^1_X\wedge\mathcal{A}^{-1}$. Therefore, the bracket descends to the quotient,

$$d\mathcal{A}^1\oplus(\Omega^1_X\wedge\mathcal{A}^{-1})/K,$$

and we will denoted by $[\,,\,]$ as well.

Remark 4.1.6. *By definition of the bracket* (4.1.4) *it follows that,* $[\theta\otimes b,(a,\omega)]_2 = -[(a,\omega),\theta\otimes b]_1$*, then we rewrite the bracket on* $d\mathcal{A}^{-1}\oplus(\Omega^1_X\wedge\mathcal{A}^{-1})/K$*,*

$$\begin{aligned}[(a,\omega)+\theta\otimes b,(a',\omega')+\theta'\otimes b'] &= [(a,\omega),(a',\omega')]_d + [\theta\otimes b,\theta'\otimes b']_\wedge +\\ &+ [(a,\omega),\theta'\otimes b']_1 - [(a',\omega'),\theta\otimes b]_1\end{aligned}$$

Lemma 4.1.7. $d\mathcal{A}^{-1}\oplus(\Omega^1_X\wedge\mathcal{A}^{-1})/K$ *is a sheaf of Lie algebras over* X*.*

Proof: The fist two terms of the bracket (4.1.4) satisfy the Jacobi identity as we shown previously. The remaining terms are computed separately.

$$\begin{aligned}[\theta_1 \otimes b_1, &[(a_2,\omega_2),\theta_3 \otimes b_3]_1]_\wedge = (\iota_{b_1} L_{\delta a_2}\theta_3)\theta_1 \otimes b_3 - (\iota_{b_3}\theta_1)L_{\delta a_2}\theta_3 \otimes b_1 \\ &+ L_{\delta a_2}\theta_3 \otimes \theta_1 \cdot [b_1,b_3] + (\iota_{b_1}\theta_3)\theta_1 \otimes [\delta a_2, b_3] - (\iota_{[\delta a_2,b_3]}\theta_1)\theta_3 \otimes b_1 \\ &+ \theta_3 \otimes \theta_1 \cdot [b_1,[\delta a_2,b_3]] - (\iota_{b_1}\theta_3)\theta_1 \otimes \iota_{b_3}\omega_2 - \theta_3 \otimes \theta_1 \cdot \iota_{b_1}\iota_{b_3}\omega_2\end{aligned}$$

$$\begin{aligned}[\theta_1 \otimes b_1, &[(a_3,\omega_3),\theta_2 \otimes b_2]_1]_\wedge = (\iota_{b_1} L_{\delta a_3}\theta_2)\theta_1 \otimes b_2 - (\iota_{b_2}\theta_1)L_{\delta a_3}\theta_2 \otimes b_1 \\ &+ L_{\delta a_3}\theta_2 \otimes \theta_1 \cdot [b_1,b_2] + (\iota_{b_1}\theta_2)\theta_1 \otimes [\delta a_3, b_2] - (\iota_{[\delta a_3,b_2]}\theta_1)\theta_2 \otimes b_1 \\ &+ \theta_2 \otimes \theta_1 \cdot [b_1,[\delta a_3,b_2]] - (\iota_{b_1}\theta_2)\theta_1 \otimes \iota_{b_2}\omega_3 - \theta_2 \otimes \theta_1 \cdot \iota_{b_1}\iota_{b_2}\omega_3\end{aligned}$$

$$\begin{aligned}[(a_1,\omega_1), &[\theta_2 \otimes b_2, \theta_3 \otimes b_3]_\wedge]_1 = L_{\delta a_1}(\iota_{b_2}\theta_3 \cdot \theta_2) \otimes b_3 + (\iota_{b_2}\theta_3)\theta_2 \otimes [\delta a_1, b_3] \\ &- (\iota_{b_2}\theta_3)\theta_2 \otimes \iota_{b_3}\omega_1 - L_{\delta a_1}(\iota_{b_3}\theta_2 \cdot \theta_3) \otimes b_2 - (\iota_{b_3}\theta_2)\theta_3 \otimes [\delta a_1, b_2] \\ &+ (\iota_{b_3}\theta_2)\theta_3 \otimes \iota_{b_2}\omega_1 + (L_{\delta a_1}\theta_3) \otimes \theta_2 \cdot [b_2,b_3] + \theta_3 \otimes (L_{\delta a_1}\theta_2) \cdot [b_2,b_3] \\ &+ \theta_3 \otimes \theta_2 \cdot [\delta a_1, [b_2,b_3]]\end{aligned}$$

$$\begin{aligned}[(a_1,\omega_1), [(a_2,\omega_2), \theta_3 \otimes b_3]_1]_1 &= L_{\delta a_1}(L_{\delta a_2}\theta_3) \otimes b_3 + L_{\delta a_2}\theta_3 \otimes [\delta a_1, b_3] \\ &- L_{\delta a_2}\theta_3 \otimes \iota_{b_3}\omega_1 + L_{\delta a_1}\theta_3 \otimes [\delta a_2, b_3] + \theta_3 \otimes [\delta a_1, [\delta a_2, b_3]] \\ &- \theta_3 \otimes \iota_{[\delta a_2,b_3]}\omega_1 - L_{\delta a_1}\theta_3 \otimes \iota_{b_3}\omega_2 - \theta_3 \otimes L_{\delta a_1}(\iota_{b_3}\omega_2)\end{aligned}$$

$$\begin{aligned}[(a_1,\omega_1), [(a_3,\omega_3), \theta_2 \otimes b_2]_1]_1 &= L_{\delta a_1}(L_{\delta a_3}\theta_2) \otimes b_2 + L_{\delta a_3}\theta_2 \otimes [\delta a_1, b_2] \\ &- L_{\delta a_3}\theta_2 \otimes \iota_{b_2}\omega_1 + L_{\delta a_1}\theta_2 \otimes [\delta a_3, b_2] + \theta_2 \otimes [\delta a_1, [\delta a_3, b_2]] \\ &- \theta_2 \otimes \iota_{[\delta a_3,b_2]}\omega_1 - L_{\delta a_1}\theta_2 \otimes \iota_{b_2}\omega_3 - \theta_2 \otimes L_{\delta a_1}(\iota_{b_2}\omega_3)\end{aligned}$$

$$\begin{aligned}[[(a_2,\omega_2),(a_3,\omega_3)]_d, \theta_1 \otimes b_1]_1 =& (L_{\delta[\delta a_2,a_3]}\theta_1) \otimes b_1 + \theta_1 \otimes [\delta[\delta a_2,a_3], b_1] \\ &- \theta_1 \otimes \iota_{b_1} L_{\delta a_2}\omega_3 + \theta_1 \otimes \iota_{b_1} L_{\delta a_3}\omega_2.\end{aligned}$$

For the right hand side of the Jacobi identity we need to compute two terms, namely

$$\begin{aligned}
&[[(a_1,\omega_1)+\theta_1\otimes b_1,(a_2,\omega_2)+\theta_2\otimes b_2],(a_3,\omega_3)+\theta_3\otimes b_3] = \\
&\qquad = [[(a_1,\omega_1),(a_2,\omega_2)]_d,(a_3,\omega_3)]_d + [[\theta_1\otimes b_1,\theta_2\otimes b_2]_\wedge,\theta_3\otimes b_3]_\wedge \\
&\qquad + [[(a_1,\omega_1),\theta_2\otimes b_2]_1,\theta_3\otimes b_3]_\wedge - [[(a_2,\omega_2),\theta_1\otimes b_1]_1,\theta_3\otimes b_3]_\wedge \\
&\qquad + [[(a_1,\omega_1),(a_2,\omega_2)]_d,\theta_3\otimes b_3]_1 - [(a_3,\omega_3),[\theta_1\otimes b_1,\theta_2\otimes b_2]_\wedge]_1 \\
&\qquad - [(a_3,\omega_3),[(a_1,\omega_1),\theta_2\otimes b_2]_1]_1 + [(a_3,\omega_3),[(a_2,\omega_2),\theta_1\otimes b_1]_1]_1
\end{aligned}$$

By direct computation we obtain the following expressions for each term in the above sum.

$$\begin{aligned}
[(a_1,\omega_1),\theta_2\otimes b_2]_1,\theta_3\otimes b_3]_\wedge &= (\iota_{b_2}\theta_3)L_{\delta a_1}\theta_2\otimes b_3 - \iota_{b_3}(L_{\delta a_1}\theta_2)\theta_3\otimes b_2 \\
&+ \theta_3\otimes(L_{\delta a_1}\theta_2)\cdot[b_2,b_3] + (\iota_{[\delta a_1,b_2]}\theta_3)\theta_2\otimes b_3 \\
&- (\iota_{b_3}\theta_2)\theta_3\otimes[\delta a_1,b_2] + \theta_3\otimes\theta_2\cdot[[\delta a_1,b_2],b_3] \\
&+ (\iota_{b_3}\theta_2)\theta_3\otimes\iota_{b_2}\omega_1 - \theta_3\otimes\theta_2\cdot\iota_{b_3}\iota_{b_2}\omega_1
\end{aligned}$$

$$\begin{aligned}
[(a_2,\omega_2),\theta_1\otimes b_1]_1,\theta_3\otimes b_3]_\wedge &= (\iota_{b_1}\theta_3)L_{\delta a_2}\theta_1\otimes b_3 - \iota_{b_3}(L_{\delta a_2}\theta_1)\theta_3\otimes b_1 \\
&+ \theta_3\otimes(L_{\delta a_2}\theta_1)\cdot[b_1,b_3] + (\iota_{[\delta a_2,b_1]}\theta_3)\theta_1\otimes b_3 \\
&- (\iota_{b_3}\theta_1)\theta_3\otimes[\delta a_2,b_1] + \theta_3\otimes\theta_1\cdot[[\delta a_2,b_1],b_3] \\
&+ (\iota_{b_3}\theta_1)\theta_3\otimes\iota_{b_1}\omega_2 - \theta_3\otimes\theta_1\cdot\iota_{b_3}\iota_{b_1}\omega_2
\end{aligned}$$

$$\begin{aligned}
[[(a_1,\omega_1),(a_2,\omega_2)]_d,\theta_3\otimes b_3]_1 &= L_{[\delta a_1,\delta a_2]}\theta_3\otimes b_3 + \theta_3\otimes[[\delta a_1,\delta a_2],b_3] \\
&- \theta_3\otimes\iota_{b_3}L_{\delta a_1}\omega_2 + \theta_3\otimes\iota_{b_3}L_{\delta a_2}\omega_1
\end{aligned}$$

$$\begin{aligned}
&[(a_3,\omega_3),[\theta_1\otimes b_1,\theta_2\otimes b_2]_\wedge]_1 = \\
&\quad = L_{\delta a_3}(\iota_{b_1}\theta_2\cdot\theta_1)\otimes b_2 + (\iota_{b_1}\theta_2)\theta_1\otimes[\delta a_3,b_2] - (\iota_{b_1}\theta_2)\theta_1\otimes\iota_{b_2}\omega_3\\
&\quad - L_{\delta a_3}(\iota_{b_2}\theta_1\cdot\theta_2)\otimes b_1 - (\iota_{b_2}\theta_1)\theta_2\otimes[\delta a_3,b_1] + (\iota_{b_2}\theta_1)\theta_2\otimes\iota_{b_1}\omega_3\\
&\quad + L_{\delta a_3}\theta_2\otimes\theta_1\cdot[b_1,b_2] + \theta_2\otimes(L_{\delta a_3}\theta_1)\cdot[b_1,b_2] + \theta_2\otimes\theta_1\cdot[\delta a_3,[b_1,b_2]]
\end{aligned}$$

$$\begin{aligned}
[(a_3,\omega_3),[(a_1,\omega_1),\theta_2\otimes b_2]_1]_1 &= L_{\delta a_3}(L_{\delta a_1}\theta_2)\otimes b_2 + L_{\delta a_1}\theta_2\otimes[\delta a_3,b_2]\\
&- (L_{\delta a_1}\theta_2)\otimes\iota_{b_2}\omega_3 + L_{\delta a_3}\theta_2\otimes[\delta a_1,b_2]\\
&+ \theta_2\otimes[\delta a_3,[\delta a_1,b_2]] - \theta_2\otimes\iota_{[\delta a_1,b_2]}\omega_3\\
&- L_{\delta a_3}\theta_2\otimes\iota_{b_2}\omega_1 - \theta_2\otimes L_{\delta a_3}\iota_{b_2}\omega_1
\end{aligned}$$

$$\begin{aligned}
[(a_3,\omega_3),[(a_2,\omega_2),\theta_1\otimes b_1]_1]_1 &= L_{\delta a_3}(L_{\delta a_2}\theta_1)\otimes b_1 + L_{\delta a_2}\theta_1\otimes[\delta a_3,b_1]\\
&- (L_{\delta a_2}\theta_1)\otimes\iota_{b_1}\omega_3 + L_{\delta a_3}\theta_1\otimes[\delta a_2,b_1]\\
&+ \theta_1\otimes[\delta a_3,[\delta a_2,b_1]] - \theta_1\otimes\iota_{[\delta a_2,b_1]}\omega_3\\
&- L_{\delta a_3}\theta_1\otimes\iota_{b_1}\omega_2 - \theta_1\otimes L_{\delta a_3}\iota_{b_1}\omega_2
\end{aligned}$$

For the other summand in the Jacobi identity, we obtain the following expressions,

$$\begin{aligned}
[\theta_2\otimes b_2,[(a_1,\omega_1),\theta_3\otimes b_3]_1]_\wedge &= (\iota_{b_2}L_{\delta a_1}\theta_3)\theta_2\otimes b_3 - (\iota_{b_3}\theta_2)L_{\delta a_1}\theta_3\otimes b_2\\
&+ L_{\delta a_1}\theta_3\otimes\theta_2\cdot[b_2,b_3] + (\iota_{b_2}\theta_3)\theta_2\otimes[\delta a_1,b_3] - (\iota_{[\delta a_1,b_3]}\theta_2)\theta_3\otimes b_2\\
&+ \theta_3\otimes\theta_2\cdot[b_2,[\delta a_1,b_3]] - (\iota_{b_2}\theta_3)\theta_2\otimes\iota_{b_3}\omega_1 - \theta_3\otimes\theta_2\cdot\iota_{b_2}\iota_{b_3}\omega_1
\end{aligned}$$

$$\begin{aligned}
[\theta_2\otimes b_2,[(a_3,\omega_3),\theta_1\otimes b_1]_1]_\wedge &= (\iota_{b_2}L_{\delta a_3}\theta_1)\theta_2\otimes b_1 - (\iota_{b_1}\theta_2)L_{\delta a_3}\theta_1\otimes b_2\\
&+ L_{\delta a_3}\theta_1\otimes\theta_2\cdot[b_2,b_1] + (\iota_{b_2}\theta_1)\theta_2\otimes[\delta a_3,b_1] - (\iota_{[\delta a_3,b_1]}\theta_2)\theta_1\otimes b_2\\
&+ \theta_1\otimes\theta_2\cdot[b_2,[\delta a_3,b_1]] - (\iota_{b_2}\theta_1)\theta_2\otimes\iota_{b_1}\omega_3 - \theta_1\otimes\theta_2\cdot\iota_{b_2}\iota_{b_1}\omega_3
\end{aligned}$$

$$\begin{aligned}[(a_2,\omega_2),&[\theta_1\otimes b_1,\theta_3\otimes b_3]_\wedge]_1 = L_{\delta a_2}(\iota_{b_1}\theta_3\cdot\theta_1)\otimes b_3 + (\iota_{b_1}\theta_3)\theta_1\otimes[\delta a_2,b_3]\\ &-(\iota_{b_1}\theta_3)\theta_1\otimes\iota_{b_3}\omega_2 - L_{\delta a_2}(\iota_{b_3}\theta_1\cdot\theta_3)\otimes b_1 - (\iota_{b_3}\theta_1)\theta_3\otimes[\delta a_2,b_1]\\ &+(\iota_{b_3}\theta_1)\theta_3\otimes\iota_{b_1}\omega_2 + (L_{\delta a_2}\theta_3)\otimes\theta_1\cdot[b_1,b_3] + \theta_3\otimes(L_{\delta a_2}\theta_1)\cdot[b_1,b_3]\\ &+\theta_3\otimes\theta_1\cdot[\delta a_2,[b_1,b_3]]\end{aligned}$$

$$\begin{aligned}[(a_2,\omega_2),[(a_1,\omega_1),\theta_3\otimes b_3]_1]_1 &= L_{\delta a_2}(L_{\delta a_1}\theta_3)\otimes b_3 + L_{\delta a_1}\theta_3\otimes[\delta a_2,b_3]\\ &- L_{\delta a_1}\theta_3\otimes\iota_{b_3}\omega_2 + L_{\delta a_2}\theta_3\otimes[\delta a_1,b_3] + \theta_3\otimes[\delta a_2,[\delta a_1,b_3]]\\ &- \theta_3\otimes\iota_{[\delta a_1,b_3]}\omega_2 - L_{\delta a_2}\theta_3\otimes\iota_{b_3}\omega_1 - \theta_3\otimes L_{\delta a_2}(\iota_{b_3}\omega_1)\end{aligned}$$

$$\begin{aligned}[(a_2,\omega_2),[(a_3,\omega_3),\theta_1\otimes b_1]_1]_1 &= L_{\delta a_2}(L_{\delta a_3}\theta_1)\otimes b_1 + L_{\delta a_3}\theta_1\otimes[\delta a_2,b_1]\\ &- L_{\delta a_3}\theta_1\otimes\iota_{b_1}\omega_2 + L_{\delta a_2}\theta_1\otimes[\delta a_3,b_1] + \theta_1\otimes[\delta a_2,[\delta a_3,b_1]]\\ &- \theta_1\otimes\iota_{[\delta a_3,b_1]}\omega_2 - L_{\delta a_2}\theta_1\otimes\iota_{b_1}\omega_3 - \theta_1\otimes L_{\delta a_2}(\iota_{b_1}\omega_3)\end{aligned}$$

$$\begin{aligned}[[(a_1,\omega_1),(a_3,\omega_3)]_d,\theta_2\otimes b_2]_1 =&(L_{\delta[\delta a_1,a_3]}\theta_2)\otimes b_2 + \theta_2\otimes[\delta[\delta a_1,a_3],b_2]\\ &- \theta_2\otimes\iota_{b_2}L_{\delta a_1}\omega_3 + \theta_2\otimes\iota_{b_2}L_{\delta a_3}\omega_1.\end{aligned}$$

Term by term, it is easy but long verify that the equality holds. For example, for the terms $\bullet\otimes b_3$ and $\bullet\otimes\theta_1\cdot[b_1,[\delta a_2,b_3]]$ respectively, it follows that,

$$\begin{aligned}(RHS) &= (\iota_{b_2}\theta_3)(L_{\delta a_1}\theta_2)\otimes b_3 + (L_{\delta a_1}\iota_{b_2}\theta_3)\theta_2\otimes b_3 - \iota_{b_2}(L_{\delta a_1}\theta_3)\theta_2\otimes b_3\\ &- (\iota_{b_1}\theta_3)(L_{\delta a_2}\theta_1)\otimes b_3 - (L_{\delta a_2}\iota_{b_1}\theta_3)\theta_1\otimes b_3 + \iota_{b_1}(L_{\delta a_2}\theta_3)\theta_1\otimes b_3\\ &+ L_{\delta a_1}(L_{\delta a_2}\theta_3)\otimes b_3 - L_{\delta a_2}(L_{\delta a_1}\theta_3)\otimes b_3 + \iota_{b_2}(L_{\delta a_1}\theta_3)\theta_2\otimes b_3\\ &+ L_{\delta a_2}(\iota_{b_1}\theta_3)\theta_1\otimes b_3 + (\iota_{b_1}\theta_3)(L_{\delta a_2}\theta_1)\otimes b_3 + L_{\delta a_2}(L_{\delta a_1}\theta_3)\otimes b_3\\ &= (\iota_{b_2}\theta_3)L_{\delta a_2}\theta_2\otimes b_3 + L_{\delta a_1}(\iota_{b_2}\theta_3)\theta_2\otimes b_3 + \iota_{b_1}(L_{\delta a_2}\theta_3)\theta_1\otimes b_3\\ &+ L_{\delta a_1}(L_{\delta a_2}\theta_3)\otimes b_3 = (LHS).\end{aligned}$$

$$\begin{aligned}(LHS) &= \theta_3 \otimes \theta_1 \cdot [b_1, [\delta a_2, b_3]] \\ &= -\theta_3 \otimes \theta_1 \cdot [[\delta a_2, b_1], b_3] + \theta_3 \otimes \theta_1 \cdot [\delta a_2, [b_1, b_3]] \\ &= (RHS)\end{aligned}$$

□

Lemma 4.1.8. *The map* Φ *in* (4.1.3) *is a morphism of Lie algebras.*

Proof: By direct computation we obtain that

$$\Phi([(a_1, \omega_1) + \theta_1 \otimes b_1, (a_2, \omega_2) + \theta_2 \otimes b_2]) = [\Phi((a_1, \omega_1) + \theta_1 \otimes b_1), \Phi((a_2, \omega_2) + \theta_2 \otimes b_2)].$$

□

Lemma 4.1.9. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{A}^{-1} & \longrightarrow & d\mathcal{A}^{-1} \oplus (\Omega^1_X \wedge \mathcal{A}^{-1})/K \\ & \searrow^{\delta} & \downarrow \Phi \\ & & \mathcal{A}^0. \end{array}$$

Proof: It follows by the construction of $d\mathcal{A}^{-1}$ and $\Omega^1 \wedge \mathcal{A}^{-1}$. Recall that any $a \in \mathcal{A}^{-1}$ goes to $(a, 0) \oplus 0$ in $d\mathcal{A}^{-1} \oplus (\Omega^1_X \wedge \mathcal{A}^{-1})/K$ □

Let us define a map $d\mathcal{A}^{-1} \oplus (\Omega^1_X \wedge \mathcal{A}^{-1})/K \to \mathcal{T}^0_{X^\sharp}$ by the formula $((a, \omega) + \theta \otimes b) \mapsto \sigma(\delta(a)) + \theta \cdot \sigma(b)$. Moreover, on $d\mathcal{A}^{-1} \oplus \Omega^1 \wedge \mathcal{A}^{-1}$ let us define a $\mathcal{C}^\infty_X$-module structure given by

$$f \cdot ((a, \omega) + (\theta \otimes b)) := (fa, f\omega) + f(\theta \otimes b) - df \otimes a.$$

Lemma 4.1.10. *The ideal* K *is a submodule.*

Proof:

$$\begin{aligned} f((a,\omega)-&(0,\omega_1\wedge\omega_{n-1})+\theta\otimes b+\omega_1\wedge\omega_{n-1}) = \\ &= f((a,\omega-\omega_1\wedge\omega_{n-1})+\theta\otimes b+\omega_1\wedge\omega_{n-1}) \\ &= (fa, f\omega-f\omega_1\wedge\omega_{n-1})+f\theta\otimes b+f\omega_1\wedge\omega_{n-1}-df\otimes a \\ &= (fa,f\omega)-(0,f\omega_1\wedge\omega_{n-1})+f\theta\otimes b+f\omega_1\wedge\omega_{n-1}-df\otimes a \\ &= (fa,f\omega)-f\omega_1\wedge\omega_{n-1}+f\theta\otimes b+f\omega_1\wedge\omega_{n-1}-df\otimes a \\ &= f((a,\omega)+\theta\otimes b) \end{aligned}$$

□

Therefore the quotient by K is endowed with a $\mathcal{C}^\infty_X$-module structure.

Lemma 4.1.11. *On $d\mathcal{A}^{-1}\oplus(\Omega^1_X\wedge\mathcal{A}^{-1})/K$ the Leinbiz rule holds, namely*

$$\begin{aligned} &[(a_1,\omega_1)+\theta_1\otimes b_1, f((a_2,\omega_2)+\theta_2\otimes b_2)] = \\ &\quad = f[(a_1,\omega_1)+\theta_1\otimes b_1,(a_2,\omega_2)+\theta_2\otimes b_2]+((a_1,\omega_1)+\theta_1\otimes b_1)f)((a_2,\omega_2)+\theta_2\otimes b_2) \end{aligned}$$

Proof: For the left hand side we obtain the following

$$\begin{aligned} (LHS) = {} & (L_{\delta a_1}f)a_2+f[\delta a_1,a_2],(L_{\delta a_1}f)\omega_2+f(L_{\delta a_1}\omega_2))-df\wedge\iota_{a_2}\omega_1-f(L_{\delta a_2}\omega_1) \\ & +f(\iota_{b_1}\theta_2)\theta_1\otimes b_2-f(\iota_{b_2}\theta_1)\theta_2\otimes b_1+f\theta_2\otimes\theta_1\cdot[b_1,b_2]-\iota_{b_1}(df)\theta_1\otimes a_2+(\iota_{a_2}\theta_1)df\otimes b_1 \\ & -df\otimes\theta_1[b_1,a_2]+(L_{\delta a_1}f)\theta_2\otimes b_2+f(L_{\delta a_1}\theta_2)\otimes b_2+f\theta_2\otimes[\delta a_1,b_2]-f\theta_2\otimes\iota_{b_2}\omega_1 \\ & -L_{\delta a_1}(df)\otimes a_2-df\otimes[\delta a_1,a_2]+df\otimes\iota_{a_2}\omega_1 \\ & -df(\iota_{a_2}\theta_1)\otimes b_1-fL_{\delta a_2}\theta_1\otimes b_1-\theta_1\otimes[df\cdot a_2+f\delta a_2,b_1]+f\theta_1\otimes\iota_{b_1}\omega_2 \end{aligned}$$

For the right hand side we have,

$$\begin{aligned} (RHS) = {} & ((L_{\delta a_1}f)a_2,(L_{\delta a_1}f)\omega_2)+(L_{\delta a_1}f)\theta_2\otimes b_2-L_{\delta a_1}(df)\otimes a_2 \\ & +(f[\delta a_1,a_2],fL_{\delta a_1}\omega_2-fL_{\delta a_2}\omega_1)+f(\iota_{b_1}\theta_2)\theta_1\otimes b_2-f(\iota_{b_2}\theta_1)\theta_2\otimes b_1 \\ & +f\theta_2\otimes\theta_1\cdot[b_1,b_2]+f(L_{\delta a_1}\theta_2)\otimes b_2+f\theta_2\otimes[\delta a_1,b_2]-f\theta_2\otimes\iota_{b_2}\omega_1-fL_{\delta a_2}\theta_1\otimes b_1 \\ & -f\theta_1\otimes[\delta a_2,b_1]+f\theta_1\otimes\iota_{b_1}\omega_2-df\otimes[\delta a_1,a_2] \end{aligned}$$

Term by term the (RHS) and (LHS) of the Leibniz rule cancels out and we obtain,

$$
\begin{aligned}
-f\theta_1 \otimes [\delta a_2, b_1] &= -df \otimes \iota_{a_2}\omega_1 - \iota_{b_1}(df)\theta_1 \otimes a_2 + (\iota_{a_2}\theta_1)df \otimes b_1 - df \otimes \theta_1 \cdot [b_1, a_2] \\
&\quad + df \otimes \iota_{a_2}\omega_1 - df(\iota_{a_2}\theta_1) \otimes b_1 - \theta_1 \otimes [df \cdot a_2, b_1] - \theta_1 \otimes [f\delta a_2, b_1] \\
&= -\iota_{b_1}(df)\theta_1 \otimes a_2 + (\iota_{a_2}\theta_1)df \otimes b_1 - df \otimes \theta_1 \cdot [b_1, a_2] - df(\iota_{a_2}\theta_1) \otimes b_1 \\
&\quad + \theta_1 \otimes [b_1, df] \cdot a_2 - \theta_1 \otimes df \cdot [b_1, a_2] - \theta_1 \otimes f[\delta a_2, b_1] \\
&= -df \otimes \theta_1 \cdot [b_1, a_2] - \theta_1 \otimes df \cdot [b_1, a_2] - \theta_1 \otimes f[\delta a_2, b_1] \\
&= -\theta_1 \otimes f[\delta a_2, b_1],
\end{aligned}
$$

The equality holds since $-df \otimes \theta_1 \cdot [b_1, a_2] - \theta_1 \otimes df \cdot [b_1, a_2]$ is zero in $\Omega^1_X \wedge \mathcal{A}^{-1}$. □

4.1.4 The isomorphism onto $\mathcal{A}^0$

Notice that $\mathcal{A}^0$ fits into the push-out diagram along the addition map between differential n-forms on X.

Proposition 4.1.12. *The diagram below is a co-cartesian square*

$$
\begin{array}{ccc}
\Omega^{n,cl}_X \oplus \Omega^n_X & \longrightarrow & d\mathcal{A}^{-1} \oplus (\Omega^1_X \wedge \mathcal{A}^{-1}) \\
{\scriptstyle +}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\Omega^n_X & \longrightarrow & \mathcal{A}^0
\end{array}
\tag{4.1.6}
$$

Proof: Notice that the ideal K is the image of the kernel of the addition map between differential n-forms on X under the injective map $\Omega^{n,cl}_X \oplus \Omega^n_X \to d\mathcal{A}^{-1} \oplus (\Omega^1_X \wedge \mathcal{A}^{-1})$. It means that the push-out along the addition map is precisely the quotient $d\mathcal{A}^{-1} \oplus (\Omega^1_X \wedge \mathcal{A}^{-1})/K$ □

In other words, the degree zero part of an $\mathcal{O}_{X^\sharp}$-extension of $\mathcal{T}_{X^\sharp}$ is completely determined by its -1 degree part.

4.1.5 What about $\mathcal{A}^k$?

The previous construction does not allows us to reproduce the brackets degree by degree, since $[\,,\,] : \mathcal{A}^i \otimes \mathcal{A}^{i+1} \to \mathcal{A}^{2i+1}$. However, as $\mathcal{A}$ fits in a short exact sequnce of $\mathcal{O}_{X^\sharp}$-modules

$$0 \to \mathcal{O}_{X^\sharp} \to \mathcal{A} \to \mathcal{T}_{X^\sharp},$$

and the algebraic structure of $\mathcal{T}_{X^\sharp}$ is completely determined by the degree -1 component as a differential graded $\mathcal{O}_{X^\sharp}$-module, we expect that algebraic structure on $\mathcal{A}$ is completely determine by $\mathcal{A}^{-1}$. We will prove in Proposition (5.1.5) that in fact

$$\mathcal{A}^k \cong \mathcal{O}^{i+1}_{X^\sharp} \cdot \mathcal{A}^{-1} \oplus \mathcal{O}^{i}_{X^\sharp} \cdot d\mathcal{A}^{-1}.$$

4.2 Derived bracket on $\mathcal{A}^{-1}$

Given an $\mathcal{O}_{X^\sharp}[n]$-extension of $\mathcal{T}_{X^\sharp}$ $\mathcal{A}$ we introduce the derived bracket on $\mathcal{A}^{-1}$ as follows,

$$\{a, b\} := [\delta a, b],$$

for any pair of (local) sections $a, b \in \mathcal{A}^{-1}$.

Proposition 4.2.1. *[12]. The derived bracket $\{\,,\,\}$ satisfies the following properties*

- $\{,\}$ *satisfies the Jacobi identity.*
- $\{\,,\,\}$ *satisfies the Leibniz rule with respect to the multiplication by $\mathcal{C}^\infty_X$.*

□

Remark 4.2.2. *[12] However the derived bracket $\{\,,\,\}$ is not skew-symmetric*

There are several geometric structures on X that we recover from $\mathcal{A}^{-1}$ for different values on n. For example for $n = 1$, we recover a special type of a Lie algebroid over X that was considered in [5]. Namely the Atiyah algebroid associated to a line bundle over X, $\mathcal{L} \to X$, has as kernel of the anchor map the structure sheaves of functions on X, $0 \to \mathcal{C}^\infty_X \to \mathcal{A}_\mathcal{L} \to \mathcal{T}_X \to 0$. Similarly, when $n = 2$ we obtain in terms of the derived bracket an *Exact Courant algebroid* structure on $\mathcal{A}^{-1}$ over X, see Section (4.3). It will be useful to write down the formulas that we obtain under the derived bracket on $\mathcal{A}^{-1}$. Let $\langle\, ,\, \rangle : \mathcal{A}^{-1} \times \mathcal{A}^{-1} \to \Omega^{n-2}_X$ denotes the Lie bracket $[\, ,\,]$.

Lemma 4.2.3. *The following formulas holds for $a, b, c \in \mathcal{A}^{-1}$, $f \in \mathcal{C}^\infty_X$, and $\eta \in \Omega^{n-1}_X \subset \mathcal{A}^{-1}$.*

$$\{a, b\} + \{b, a\} = \delta[a, b] := \delta\langle a, b\rangle \tag{4.2.1}$$

$$\langle a, fb\rangle = f\langle a, b\rangle \tag{4.2.2}$$

$$\langle\{a, b\}, c\rangle + \langle b, \{a, c\}\rangle = L_{\sigma(a)}\langle b, c\rangle \tag{4.2.3}$$

$$\{a, \eta\} = L_{\sigma(a)}\eta \tag{4.2.4}$$

$$\{\eta, a\} = -\iota_{\sigma(a)} d\eta \tag{4.2.5}$$

$$\{a, fb\} = f\{a, b\} + (L_{\sigma(a)} f)b \tag{4.2.6}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\} \tag{4.2.7}$$

Proof: Let $a, b \in \mathcal{A}^{-1}$, $\eta \in \Omega^{n-1}_X$ and $f \in \mathcal{C}^\infty_X$. The de Rham differential is denoted by d. Lets verify (4.2.1).

$$\begin{aligned}\{a, b\} + \{b, a\} &= [\delta a, b] + [\delta b, a] \\ &= [\delta a, b] - [a, \delta b] = \delta[a, b] \\ &= \delta\langle a, b\rangle.\end{aligned}$$

Equation (4.2.2) since

$$\begin{aligned}\langle a, fb\rangle &= [a, fb] = f[a, b] + (\iota_{\sigma(a)} f)b \\ &= f\langle a, b\rangle.\end{aligned}$$

To verify the compatibility between the pairing $\langle\ ,\ \rangle$ and derived bracket $\{\ ,\ \}$ (4.2.3), recall that, (by degree reasons), $[b,c] \in \mathcal{A}^{-2} \cong \Omega_X^{n-2}$.

$$\begin{aligned}\langle\{a,b\},c\rangle + \langle b,\{a,c\}\rangle &= \langle[\delta a,b],c\rangle + \langle b,[\delta a,c]\rangle \\ &= [[\delta a,b],c] + [b,[\delta a,c]] \\ &= [\delta a,[b,c]] \\ &= L_{\sigma(\delta a)}[b,c] \\ &= L_{\sigma(a)}\langle b,c\rangle.\end{aligned}$$

Equations (4.2.4) and (4.2.5) follows from

- $\{a,\eta\} = [\delta a,\eta] = L_{\sigma(\delta a)}\eta = L_{\sigma(a)}\eta.$
- $\{\eta,a\} = \delta\langle a,\eta\rangle - \{a,\eta\} = \delta[a,\eta] - L_{\sigma(\delta a)}\eta = d\iota_{\sigma(a)}\eta - L_{\sigma(\delta a)}\eta = -\iota_{\sigma(a)}d\eta.$

By definition of $\mathcal{A}$, the restriction to the first component of the bracket $[\ ,\ ]$ to Ω_X^{n-1} is given by interior product, i.e.

$$\mathcal{A}^{-1} \otimes \Omega^{n-1} \to \Omega_X^{n-2},$$

$$[a,\eta] = \iota_{\sigma(a)}\eta.$$

The Leibniz rule for derived bracket (4.2.6) follows from,

$$\begin{aligned}\{a,fb\} = [\delta,fb] &= f[\delta a,b] + (L_{\sigma(\delta a)}f)b \\ &= f\{a,b\} + (L_{\sigma(\delta a)}f)b.\end{aligned}$$

Recall that $\mathcal{A}^0$ acts on Ω_X^n by Lie derivative,

$$\mathcal{A}^0 \times \Omega_X^n \to \Omega_X^n,$$

$$[a,\eta] = L_{\sigma(a)}\eta.$$

The Jacobi identity for $\{\ ,\ \}$ (4.2.7) holds since $\delta \circ \delta = 0$. □

Lemma (4.2.3) motivates the definition of a *higher dimensional Courant algebroids over* X, that we will give in the next chapter.

Remark 4.2.4. *In terms of the derived bracket on $\mathcal{A}^{-1}$, it is clear that the Lie algebroid structure on $d\mathcal{A}^{-1} \oplus (\Omega^1_X \wedge \mathcal{A}^{-1})/K$ could be rewritten in terms of the derived bracket $\{\ ,\ \}$, the symmetric pairing $\langle\ ,\ \rangle$ and the anchor map $\sigma : \mathcal{A}^{-1} \to \mathcal{T}_X$.*

$$\begin{aligned}
[(a,\omega)+\theta\otimes b,(a',\omega')+\theta'\otimes b'] &= [(a,\omega),(a',\omega')]_d + [\theta\otimes b,\theta'\otimes b']_\wedge + \\
&+ [(a,\omega),\theta'\otimes b']_1 - [(a',\omega'),\theta\otimes b]_1 \\
&= (\{a,a'\},\{a,\omega'\}-\{a',\omega\}) + L_{\delta a},\theta\otimes b' + \theta'\otimes\{a,b'\} \\
&- \theta'\otimes\iota_{b'}\omega + (\iota_b\theta')\theta\otimes b' - (\iota_{b'}\theta)\theta'\otimes b + \theta'\otimes\theta\langle b,b'\rangle \\
&- L_{\delta a'}\theta\otimes b - \theta\otimes\{a',b\} + \theta\otimes\iota_b\omega' \\
&= (\{a,a'\},\{a,\omega'\}-\{a',\omega\}) + \{a,\theta'\}\otimes b' + \theta'\otimes\{a,b'\} \\
&- \theta'\otimes\iota_{b'}\omega + (\iota_b\theta')\theta\otimes b' - (\iota_{b'}\theta)\theta'\otimes b + \theta'\otimes\theta\langle b,b'\rangle \\
&- \{a',\theta\}\otimes b - \theta\otimes\{a',b\} + \theta\otimes\iota_b\omega'
\end{aligned}$$

Similarly, the bracket on $\mathcal{A}^0$ could be rewritten in terms of the derived bracket defined on $\mathcal{A}^{-1}$. In fact, for any given local sections $x, y \in \mathcal{A}^0$ there are suitable elements $(a,\omega)+\theta\otimes b$ and $(a',\omega')+\theta'\otimes b'$ such that $\Phi((a,\omega)+\theta\otimes b) = x$ and $\Phi((a',\omega')+\theta'\otimes b') = y$. It follows that

$$\begin{aligned}
[x,y] &= [\Phi((a,\omega)+\theta\otimes b),\Phi((a',\omega')+\theta'\otimes b')] \\
&= [\delta a+\omega+\theta\cdot b,\delta a'+\omega'+\theta'\cdot b'] \\
&= \delta\{a,a'\} + \{a,\omega'\} + \{a,\theta'\cdot\omega'\} - \{a',\omega\} \\
&- \{a',\theta\cdot b\} - \theta'\cdot\iota_{b'}\omega - (\iota_b\theta')\theta\cdot b' \\
&+ (\iota'_b\theta)\theta'\cdot b - \theta'\wedge\theta\wedge\langle b,b'\rangle
\end{aligned}$$

A similar computation shows that the Leibniz rule could be also written in terms of the derived bracket on $\mathcal{A}^{-1}$.

4.3 Example: the trivial case

4.3.1 For n=2

Fix $n = 2$. Suppose that $\mathcal{P} = X^\sharp \times \mathbb{R}[2]$ is the trivial $\mathbb{R}[2]$ dg-principal bundle. As we mention previously, the differential on $\mathcal{P}$ should be $\mathbb{R}[2]$-invariant. If we consider the differential $\partial = \partial \otimes 1$ we will recover the standard Courant-Dorfman bracket after passing through the derived bracket. Alternatively, if we consider a differential of the form $\partial = \partial \otimes 1 + \omega \otimes \partial_t$ where $\omega \in \Omega_X^{3,cl}$. since

$$[\partial, \partial] = 0 \Leftrightarrow \partial\omega = d\omega = 0,$$

we should recovered the twisted Courant-Dorfman bracket by the 3-form ω introducing the derived bracket. Recall that the structure on $\mathcal{A}_\mathcal{P}^{-1}$ is given by the following formulas

$$\text{pairing, } \langle\, , \rangle := [\, ,] : \mathcal{A}_\mathcal{P}^{-1} \times \mathcal{A}_\mathcal{P}^{-1} \to \mathcal{A}_\mathcal{P}^{-2}$$
$$\text{The Courant-Dorfman bracket, } \{\, ,\, \} : \mathcal{A}_\mathcal{P}^{-1} \times \mathcal{A}_\mathcal{P}^{-1} \to \mathcal{A}_\mathcal{P}^{-1}$$
$$\text{The anchor map, } \sigma : \mathcal{A}_\mathcal{P}^{-1} \to \mathcal{T}_{X^\sharp},\ \sigma(a) = \{a,\, \}.$$

In this case, local coordinates on $\mathcal{P}$ are given by local coordinates of the base $X^\sharp$ and a coordinate of the fiber $\mathbb{R}[2]$. Local coordinates for $X^\sharp$ are given by local coordinates of the manifold X, x_i, and odd coordinates ξ_i. The local coordinate on the fiber is denoted by t. Then, on $\mathcal{A}_\mathcal{P}$ we can choose local coordinates given by $(x_i, \xi_i, t, \partial_{x_i}, \partial_{\xi_i}, \partial_t)$ whose degrees are given by,

$$|x_i| = 0, |\xi_i| = 1, |t| = 2, |\partial_{x_i}| = 0, |\partial_{\xi_i}| = -1, |\partial_t| = -2$$

It follows that $\mathcal{A}_\mathcal{P}$ in degree 0 and -1 is given by

$$\mathcal{A}_\mathcal{P}^{-1} = \{f(x)\xi\partial_t + g(x)\partial_\xi\} \cong \Omega_X^1 \oplus \mathcal{T}_X,$$

$$\mathcal{A}_{\mathcal{P}}^{-2} = \{f(x)\partial_t\} \cong \mathcal{C}_X^\infty.$$

The expressions for the symmetric pairing are written as follows,

$$\langle f(x)\xi_i\partial_t, g(x)\xi_j\partial_t\rangle = 0 \tag{4.3.1}$$

$$\langle f(x)\xi_j\partial_t, g(x)\partial_{\xi_i}\rangle = f(x)g(x)\delta_{ij}\partial_t \tag{4.3.2}$$

$$\langle f(x)\partial_{\xi_i}, g(x)\partial_{\xi_j}\rangle = 0. \tag{4.3.3}$$

To define the bracket structure and the anchor map, we pick the simplest expression for the differential on $\mathcal{P}$, i.e. $\partial = \partial \otimes 1 = d \otimes 1$, where d denotes the de Rham differential on Ω_X. In local coordinates d is given by $d = \sum \xi_i\partial_{x_i}$. In this way we obtain expressions for the bracket and the anchor map,

$$\{a, b\} = [[\partial, a], b] \tag{4.3.4}$$

$$\sigma(a) = \{a, f\} = [[\partial, a], f]. \tag{4.3.5}$$

The anchor map σ are determined by the values on,

$$g(x)\partial_{\xi_k},\ g(x)\xi_k\partial_t,$$

since they generates $\mathcal{A}_{\mathcal{P}}^{-1}$. For any $f \in \mathcal{C}_X^\infty$ the anchor map is given by,

$$[[\xi_i\partial_{x_i}, g(x)\partial_{\xi_k}], f(x)] = g(x)\partial_{x_k}(f(x)),$$
$$[[\xi_i\partial x_i, g(x)\xi_j\partial_t], f(x)] = 0,$$

which means that $\{a,\ \}$ corresponds to the projection onto $\mathcal{T}_X$. Recall that we write elements in $\mathcal{A}_{\mathcal{P}}^{-1}$ as follows,

$$\alpha_1 \cdot \partial_t + \iota_{\xi_1} = f(x)\xi_m\partial_t + g(x)\partial_{\xi_j},$$

$$\alpha_2 \cdot \partial_t + \iota_{\xi_2} = h(x)\xi_k\partial_t + i(x)\partial_{\xi_l}.$$

By direct computation we obtain the expressions for the bracket $\{\, ,\, \}$,

$$
\begin{aligned}
&[[\xi_i\partial_{x_i}, f(x)\xi_m\partial_t + g(x)\partial_{\xi_j}], h(x)\xi_k\partial_t + i(x)\partial_l] \\
&\quad = [\xi_i\partial_{x_i}(f(x))\xi_m\partial_t + \xi_i\partial_{x_i}(g(x))\partial_{\xi_j} + g(x)\delta_{ij}\partial_{x_i}, h(x)\xi_k\partial_t + i(x)\partial_l] \\
&\quad = -\partial_{\xi_l}(\xi_i\xi_m)\partial_{x_i}(f(x))i(x)\partial_t + \xi_i\partial_{x_i}(g(x))h(x)\delta_{jk}\partial_t \\
&\quad - i(x)\delta_{il}\partial_{x_i}(g(x))\partial_{\xi_j} + g(x)\delta_{ij}\partial_{x_i}(h(x))\xi_k\partial_t + g(x)\delta_{ij}\partial_{x_i}(i(x))\partial_{\xi_l} \\
&\quad = (-\partial_{\xi_l}(\xi_i\xi_m)\partial_{x_i}(f(x))i(x) + \xi_i\partial_{x_i}(g(x))h(x)\delta_{jk} + g(x)\delta_{ij}\partial_{x_i}(h(x))\xi_k)\partial_t \\
&\quad + (-i(x)\delta_{il}\partial_{x_i}(g(x)) + g(x)\delta_{il}\partial_{x_i}(i(x)))\partial_{\xi_l} \\
&\quad = \iota_{[\xi_1,\xi_2]} + (L_{\xi_1}\alpha_2 - \iota_{\xi_2}d\alpha_1).
\end{aligned}
$$

4.3.2 For $n \geq 3$

For a $H \in \Omega_X^{n+1,cl} = \mathcal{O}^1_{X^\sharp}[n]$, the differential on $\mathcal{P} = X^\sharp \times \mathbb{R}[n]$ is given by $\partial = d\otimes 1 + H \otimes \partial_t$. Then, the twist H-bracket on sections of the vector bundle $TX \otimes \bigwedge^{n-1} T^*X$ is recovered from $\mathcal{A}^{-1}_{\mathcal{P}}$ through the derived bracket. In this case we write elements in $\mathcal{A}^{-1}_{\mathcal{P}}$ as, $\alpha_1\partial_t + \iota_{\xi_1}$, $\alpha_2\partial_t + \iota_{\xi_2}$. Then $\{\, ,\, \}$ is computed as follows,

$$
\begin{aligned}
&[[d\otimes 1 + H \otimes \partial_t, \alpha_1\partial_t + \iota_{\xi_1}], \alpha_2\partial_t + \iota_{\xi_2}] \\
&\qquad = [d\alpha_1\partial_t - \iota_{\xi_1}H\partial_t + L_{\xi_1}, \alpha_2\partial_t + \iota_{\xi_2}] \\
&\qquad = (L_{\xi_1}\alpha_2 - \iota_{\xi_2}d\alpha_1 + \iota_{\xi_2}\iota_{\xi_1}H)\partial_t + \iota_{[\xi_1,\xi_2]}.
\end{aligned}
$$

Remark 4.3.1. *Notice that our construction produce the brackets for the vector bundles $TX \oplus \bigwedge^{n-1} T^*X \to X$ as a derived bracket on degree -1 component of the Atiyah algebroid $\mathcal{A}$.*

CHAPTER 5

Higher dimensional Courant algebroids

We introduce the category of *higher dimensional Courant algebroids on* X motivated by Lemma (4.2.3). We will show that there is an equivalence of categories between the category of $\mathcal{O}_{X^\sharp}[n]$-LA$(X^\sharp)$ and the category of higher dimensional exact Courant algebroids on X.

Definition 5.0.2. *A k-dimensional Courant algebroid on* X *is a* $\mathcal{C}_X^\infty$*-module* $\mathcal{Q}$ *equipped with*

1. *a structure of a Leibniz* $\mathbb{R}$*-algebra*

$$\{\ ,\ \}\colon \mathcal{Q} \otimes_{\mathbb{R}} \mathcal{Q} \to \mathcal{Q};$$

2. *a* $\mathcal{C}_X^\infty$*-linear map of Leibniz algebras (the anchor map)*

$$\pi\colon \mathcal{Q} \to \mathcal{T}_X;$$

3. *a symmetric* $\mathcal{C}_X^\infty$*-bilinear pairing*

$$\mathcal{Q} \otimes_{\mathcal{C}_X^\infty} \mathcal{Q} \to \Omega_X^{k-1};$$

4. a $\mathcal{C}^\infty_X$-linear map

$$\pi^\dagger\colon \Omega^k_X \to \mathcal{Q}.$$

These data are required to satisfy

$$\begin{aligned} \pi\circ\pi^\dagger &= 0 && (5.0.1)\\ \{q_1, fq_2\} &= f\{q_1,q_2\} + \pi(q_1)(f)q_2 && (5.0.2)\\ \langle\{q,q_1\},q_2\rangle + \langle q_1,\{q,q_2\}\rangle &= L_{\pi(q)}\langle q_1,q_2\rangle && (5.0.3)\\ \{q,\pi^\dagger(\alpha)\} &= \pi^\dagger(L_{\pi(q)}(\alpha)) && (5.0.4)\\ \langle q,\pi^\dagger(\alpha)\rangle &= \iota_{\pi(q)}\alpha && (5.0.5)\\ \{q_1,q_2\}+\{q_2,q_1\} &= \pi^\dagger(d\langle q_1,q_2\rangle) && (5.0.6) \end{aligned}$$

for $f\in\mathcal{C}^\infty_X$ and $q, q_1, q_2 \in \mathcal{Q}$.

Notice that for $k = 1$ we recover the definition of Courant algebroid on X, [4]. Given two k-dimensional Courant algebroids on X, $\mathcal{Q}$, $\mathcal{Q}'$, a morphism is a map $\phi : \mathcal{Q} \to \mathcal{Q}'$ of C^∞_X-modules which preserves the brackets, the pairing and it is compatible with the anchor maps, $\pi'\circ\phi = \pi$. The category of k-dimensional Courant algebroids over X will be denoted by k-CA(X).

Definition 5.0.3. *A k-dimensional Courant algebroid $\mathcal{Q}$ is called exact if the sequence,*

$$0 \longrightarrow \Omega^k_X \xrightarrow{\pi^\dagger} \mathcal{Q} \xrightarrow{\pi} \mathcal{T}_X \longrightarrow 0.$$

is exact.

Notice that a morphism $\phi : \mathcal{Q} \to \mathcal{Q}'$ of k-dimensional exact Courant algebroids fits in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{n-1}_X & \xrightarrow{\pi^\dagger} & \mathcal{Q} & \xrightarrow{\pi} & \mathcal{T}_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \Omega^{n-1}_X & \xrightarrow{\pi^{\dagger\prime}} & \mathcal{Q}' & \xrightarrow{\pi'} & \mathcal{T}_X & \longrightarrow & 0, \end{array}$$

i.e. ϕ is an isomorphism. We denote by k-ECA(X) the category of k-dimensional exact Courant algebroids on X. Notice that k-ECA(X) is a groupoid.

Example 5.0.4. *Consider the sheaf of $\mathcal{C}^\infty_X$-modules $\mathcal{Q}_0 = \Omega^k_X \oplus \mathcal{T}_X$ whose local section are written as $(\omega \oplus \xi)$, where $\omega \in \Omega^k_X$ and $\xi \in \mathcal{T}_X$. The structure on $\mathcal{Q}_0$ is given by*

$$\pi(\omega \oplus \xi) = \xi,$$

$$\langle \omega_1 \oplus \xi_1, \omega_2 \oplus \xi_2 \rangle = \iota_{\xi_1}\omega_2 + \iota_{\xi_2}\omega_1$$

$$\{\omega_1 \oplus \xi_1, \omega_2 \oplus \xi_2\}_0 = L_{\xi_1}\omega_2 - \iota_{\xi_2}d\omega_1 \oplus [\xi_1, \xi_2]$$

$$\pi^\dagger(\omega) = \omega \oplus 0$$

Example 5.0.5. *Suppose that $H \in \Gamma(X; \Omega^{n+1,cl}_X)$. We denote by $\mathcal{Q}_H$ the H-twist of $\mathcal{Q}_0$. The bracket $\{\ ,\ \}_H$ on $\mathcal{Q}_H$ is given by the formula,*

$$\{\xi_1 + \omega_1, \xi_2 + \omega_2\}_H := \{\xi_1 + \omega_1, \xi_2 + \omega_2\}_0 + (0, \iota_{\xi_2}\iota_{\xi_1}H).$$

The remaining part of the structure on $\mathcal{Q}_H$ is the same as for $\mathcal{Q}_0$. H must be a closed form, since $\{\ ,\ \}_H$ has to satisfies the Jacobi identity.

These examples provided a geometric framework in [1], [2], [10], [27].

We will show that isomorphism classes of higher dimensional exact Courant algebroids are in one to one correspondence with the (n+1)-cohomology group of X. Therefore all higher dimensional exact Courant algebroids are of the form $\mathcal{Q}_H$ for some $H \in \Omega^{n+1}_X$.

5.1 The equivalence

In this section we will construct two functors in order to show an equivalence of categories between the category $(n-1)$-ECA(X) and the category $\mathcal{O}_{X^\sharp}[n]$-LA($X^\sharp$).

Namely we will construct two functor inverse to each other:

$$F : \mathcal{O}_{X^\sharp}[n]\text{-LA}(X^\sharp) \to (n-1)\text{-ECA}(X),$$

$$G : (n-1)\text{-ECA}(X) \to \mathcal{O}_{X^\sharp}[n]\text{-LA}(X^\sharp).$$

5.1.1 The functor F

Suppose that $0 \to \mathcal{O}_{X^\sharp}[n] \to \mathcal{A} \to \mathcal{T}_{X^\sharp} \to 0$ is an $\mathcal{O}_{X^\sharp}[n]$-extension of $\mathcal{T}_{X^\sharp}$. Let us define $F(\mathcal{A}) := (\mathcal{A}^{-1}, \{\ ,\ \}, \langle\ ,\ \rangle, \sigma)$ where $\{\ ,\ \}$ is the derived bracket, $\langle\ ,\rangle$ is the symmetric pairing, see Section (4.2). It follows by Lemma (4.2.3), that $(\mathcal{A}^{-1}, \{\ ,\ \}, \langle\ ,\ \rangle, \sigma)$ is an $(n-1)$-dimensional exact Courant algebroid on X.

5.1.2 The functor G

Suppose that $(\mathcal{Q}, \pi, \{\ ,\ \}, \langle\ ,\ \rangle)$ is an $(n-1)$-dimensional exact Courant algebroid on X. Let $d\mathcal{Q}$ be the sheaf defined by the pushout square,

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\ \delta\ } & d\mathcal{Q} \\ {\scriptstyle \pi^\dagger}\uparrow & & \uparrow \\ \Omega_X^{n-1} & \xrightarrow[d]{} & \Omega_X^{n,cl}. \end{array}$$

Let $\widetilde{\mathcal{Q}}$ denotes the complex $\widetilde{\mathcal{Q}^0} = d\mathcal{Q}$, $\widetilde{\mathcal{Q}^{-1}} = \mathcal{Q}$ and $\widetilde{\mathcal{Q}^i} = \Omega_X^{n+i}$ for $i \leq -2$. The differential $\delta : \widetilde{\mathcal{Q}} \to \widetilde{\mathcal{Q}}[1]$ is given by $\delta^k : \widetilde{\mathcal{Q}^k} \to \widetilde{\mathcal{Q}^{k+1}}$, where $\delta^k = d$ for $k \leq -3$, $\delta^{-2} = \pi^\dagger \circ d$ and $\delta^{-1} = \delta$,

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathcal{C}_X^\infty & \xrightarrow{d} & \Omega_X^1 & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega_X^{n-2} & \xrightarrow{\delta} & \mathcal{Q} & \xrightarrow{\delta} & d\mathcal{Q} \\ & & \uparrow{\scriptstyle \pi^\dagger = id} & & \uparrow{\scriptstyle \pi^\dagger = id} & & & & {\scriptstyle \pi^\dagger = id}\uparrow & \searrow^{d} & \uparrow{\scriptstyle \pi^\dagger} & & \uparrow \\ 0 & \longrightarrow & \mathcal{C}_X^\infty & \xrightarrow[d]{} & \Omega_X^1 & \xrightarrow[d]{} & \dots & \xrightarrow[d]{} & \Omega_X^{n-2} & \xrightarrow[d]{} & \Omega_X^{n-1} & \xrightarrow[d]{} & \Omega_X^{n,cl}, \end{array}$$

Define on $\widetilde{\mathcal{Q}}$ an operation $[\ ,\ ] : \widetilde{\mathcal{Q}^i} \times \widetilde{\mathcal{Q}^j} \to \widetilde{\mathcal{Q}^{i+j}}$ by the formulas,

- $[\ ,\ ] : \Omega_X^j \times \Omega_X^i \to \Omega_X^{i+j}$, is zero

- $[\,,\,] : \Omega^j_X \times \mathcal{Q} \to \Omega^{j-1}_X$, $[\gamma, q] = (-1)^{j+1}\iota_{\pi(q)}\gamma$
- $[\,,\,] : \mathcal{Q} \times \Omega^j_X \to \Omega^{j-1}_X$, $[q, \gamma] = \iota_{\pi(q)}\gamma$
- $[\,,\,] : d\mathcal{Q} \times \Omega^j_X \to \Omega^j_X$, $[\delta(q) + \omega, \gamma] = L_{\pi(q)}\gamma$
- $[\,,\,] : \Omega^j_X \times d\mathcal{Q} \to \Omega^j_X$, $[\gamma, \delta(q) + \omega] = -L_{\pi(q)}\gamma$
- $[\,,\,] : \mathcal{Q} \times \mathcal{Q} \to \Omega^{n-2}_X$, $[q_1, q_2] = \langle q_1, q_2\rangle$
- $[\,,\,] : \mathcal{Q} \times d\mathcal{Q} \to \mathcal{Q}$, $[q_1, \delta(q_2) + \omega] = -\{q_2, q_1\} + \iota_{\pi(q_1)}\omega_2$
- $[\,,\,] : d\mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$, $[\delta(q_2) + \omega_2, q_1] = \{q_2, q_1\} - \iota_{\pi(q_1)}\omega_2$
- $[\,,\,] : d\mathcal{Q} \times d\mathcal{Q} \to d\mathcal{Q}, [\delta(q_1) + \omega_1, \delta(q_2) + \omega_2] = \delta\{q_1, q_2\} + L_{\pi(q_1)}\omega_2 - L_{\pi(q_2)}\omega_1$.

It follows by direct computation that the operation $[\,,\,]$ define a structure of dgla on $\widetilde{\mathcal{Q}}$.

Let us define the map $\Phi : (\widetilde{\mathcal{Q}}, [\,,\,], \delta) \to (\widetilde{\mathcal{T}_X}, [\,,\,], id)$ given by $q \mapsto \pi(q)\epsilon$ and $\delta(q) + \omega \mapsto \pi(q)$ and zero in lower degrees. It follows that Φ is a surjective map of dgla's. By definition, the kernel of Φ is equal to $(\tau^{\leq n}\Omega_X)[n]$. Then we obtain a short exact sequence of dglas.

$$0 \to (\tau^{\leq n}\Omega_X)[n] \to \widetilde{\mathcal{Q}} \to \widetilde{\mathcal{T}_X} \to 0.$$

As we are looking for an $\mathcal{O}_{X^\sharp}[n]$-extension of $\mathcal{T}_{X^\sharp}$, we extend scalars multiplying by $\mathcal{O}_{X^\sharp}$,

$$0 \to \mathcal{O}_{X^\sharp} \otimes ((\tau^{\leq n}\Omega_X)[n] \to \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}} \to \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{T}_X} \to 0.$$

As $\widetilde{\mathcal{Q}}$ is a dgla that acts on $\mathcal{O}_{X^\sharp}$ by derivations, it follows that $\mathcal{O}_{X^\sharp} \otimes \tilde{\mathcal{Q}}$ is a dgla endowed with an $\mathcal{O}_{X^\sharp}$-module structure. Explicitly the structure on $\mathcal{O}_{X^\sharp} \otimes \tilde{\mathcal{Q}}$ is defined as follows. The differential of the complex $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$ is by definition $\partial \otimes 1 \pm 1 \otimes \delta$. The $\mathcal{O}_{X^\sharp}$-module structure on $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$ is given by the wedge product on $\mathcal{O}_{X^\sharp}$. The bracket on $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$ is defined as the extension of the Lie bracket on $\widetilde{\mathcal{Q}}$ by Leibniz rule. We write down the formulas for the extended bracket.

$$[\beta_1 \otimes q_1, \beta_2 \otimes q_2] = \beta_1 \wedge \iota_{\pi(q_1)}\beta_2 \otimes q_2 + (-1)^{(i-1)j}(\beta_2 \wedge \beta_1 \otimes \langle q_1, q_2\rangle + (-1)^i \beta_2 \wedge \iota_{\pi(q_2)}\beta_1 \otimes q_1),$$

for any $\beta_1 \in \Omega^i_X$, $\beta_2 \in \Omega^j_X$ and $q_1, q_2 \in \mathcal{Q}$.

$$\begin{aligned}[\beta_1 \otimes q_1, \beta_2 \otimes (\delta(q_2) + \omega_2)] &= \beta_1 \wedge \iota_{\pi(q_1)}\beta_2 \otimes (\delta(q_2) + \omega_2) \\ &+ (-1)^{(i-1)j+1}(\beta_2 \wedge \beta_1 \otimes (\{q_2, q_1\} - \iota_{\pi(q_1)}\omega_2) + \beta_2 \wedge L_{\pi(q_2)}\beta_1 \otimes q_1),\end{aligned}$$

for any $\beta_1 \in \Omega^i_X$, $\beta_2 \in \Omega^j_X$ $q_1 \in \mathcal{Q}$ and $\delta(q_2) + \omega_2 \in d\mathcal{Q}$.

$$\begin{aligned}[\beta_1 \otimes (\delta(q_1) + \omega_1), \beta_2 \otimes (\delta(q_2) + \omega_2)] &= \beta_1 \wedge L_{\pi(q_1)}\beta_2 \otimes (\delta(q_2) + \omega_2) \\ &+ (-1)^{ij}(\beta_2 \wedge \beta_1 \otimes (\delta\{q_1, q_2\} + L_{\pi(q_1)}\omega_2 - L_{\pi(q_2)}\omega_1) \\ &- \beta_2 \wedge L_{\pi(q_2)}\beta_1 \otimes (\delta(q_1) + \omega_1)),\end{aligned}$$

for any $\beta_1 \in \Omega^i_X$, $\beta_2 \in \Omega^j_X$ and $\delta(q_1) + \omega_1$, $\delta(q_2) + \omega_2 \in d\mathcal{Q}$.

Lemma 5.1.1. *The bracket on $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$ satisfies the Jacobi identity and the Leibniz rule.*

Proof: By direct calculation we obtain that the Jacobi identity and the Leibniz rule holds. The computation are practically the same as in Section (4.1.3) □

Let us denote by

$$\sigma : \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}} \to \mathcal{T}_{X^\sharp}$$

the composition $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}} \longrightarrow \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{T}_X} \longrightarrow \mathcal{T}_{X^\sharp}$ given by $\beta \otimes q \mapsto \beta \wedge \iota_{\pi(q)}\ \beta \otimes \delta(q) = \beta \wedge L_{\pi(q)}$. It follows that σ is $\mathcal{O}_{X^\sharp}$-linear since the $\mathcal{O}_{X^\sharp}$-module structure is the wedge product of forms. Moreover σ is a surjective map.

Lemma 5.1.2. *The map σ is a map of Lie algebras over $\mathbb{R}$.*

Proof: By construction $\sigma : \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}} \to \mathcal{T}_{X^\sharp}$ is compatible with brackets. □

Notice that by the definition of σ there are several expressions on $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$ that vanish. Namely,

$$1 \otimes \pi^\dagger(\omega)$$

$$\beta \wedge \omega \otimes \pi^\dagger(1) - \beta \otimes \pi^\dagger(\omega),$$

$$f\beta \otimes q - \beta \otimes fq,$$

$$\beta \otimes \delta(fq) - \beta \wedge df \otimes q - f\beta \otimes \delta(q),$$

$$\beta \otimes \pi^{\dagger}(\gamma_1 \wedge ... \wedge \gamma_{n-1}) - \beta \wedge \gamma_1 \wedge ... \wedge \gamma_k \otimes \pi^{\dagger}(\gamma_{k+1} \wedge ... \wedge \gamma_{n-1}),$$

for any $\omega \in \tau(\Omega_X^{\leq n}[n])$, $\beta \in \Omega_X$, $f \in \mathcal{C}_X^{\infty}$, $q \in \mathcal{Q}$ and $\gamma_l \in \Omega_X^1$, for $l = 1, ..., n-1$. Let K denote the subsheaf of vector subspaces generated by the above expressions.

Lemma 5.1.3. *K is an $\mathcal{O}_{X^\sharp}$-submodule and a differential graded ideal in $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$*

Proof: By direct calculation we obtain

$$\delta(K) \subset K, \ [K, \mathcal{O}_{X^\sharp} \otimes \tilde{\mathcal{Q}}] \subset K.$$

□

Notice that there is a map $\mathcal{O}_{X^\sharp}[n] \otimes 1 \to \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$ given by $\omega \otimes 1 \mapsto \omega \otimes \pi^{\dagger}(1)$. By the definition of K, the element $\omega \otimes \pi^{\dagger}(1)$ does not belong to K.

We define

$$\tau\mathcal{Q} := \mathcal{O}_{X^\sharp} \otimes \tilde{\mathcal{Q}}/K.$$

Lemma 5.1.4. *The structure on $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$ descends to $\tau\mathcal{Q}$.*

Proof: The result follows from Lemma (5.1.1), (5.1.2) and (5.1.3). The differential on $\tau\mathcal{Q}$ is induced from the differential $\mathcal{O}_{X^\sharp} \otimes \tilde{\mathcal{Q}}$ since K is a differential ideal.
□

We denote $(\tau\mathcal{Q}, [\ ,\ ], \sigma, \delta)$ the structure induced from $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}}$.

Proposition 5.1.5. *The sequence below,*

$$0 \longrightarrow \mathcal{O}_{X^\sharp}[n] \xrightarrow{i} \tau\mathcal{Q} \xrightarrow{\sigma} \mathcal{T}_{X^\sharp} \longrightarrow 0.$$

is exact.

Proof: By construction, $\sigma \circ i = 0$. Lets verify that the kernel of σ is equal to $(\mathcal{O}_{X^\sharp}[n], i)$. Notice that the image of $1 \otimes 1 \mapsto 1 \otimes \pi^\dagger 1$ is non zero. Moreover, if $\alpha \otimes \pi^\dagger(1) = 0$ for some $\alpha \in (\tau^{\leq n}\Omega_X)[n]$, it follows that $1 \otimes \pi^\dagger(\alpha) = 0$. Then $\alpha = 0$, since $\pi^\dagger$ is an injective map. Consider the submodule of $\tau\mathcal{Q}$ generated by elements of degree less than or equal to -1, denoted by $\mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1}$. Since $\sigma : \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}} \to \mathcal{T}_{X^\sharp}$ restricts to $\mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}^{\leq -1}} \to \mathcal{T}_{X^\sharp|X} = \mathcal{O}_{X^\sharp} \otimes \mathcal{T}_X[1]$, it follows that $\sigma : \mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1} \to \mathcal{T}_{X^\sharp|X}$. We claim that the kernel of $\sigma : \mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1} \to \mathcal{T}_{X^\sharp|X}$ is equal to $(\mathcal{O}_{X^\sharp}[n], i)$. Notice that any $\gamma \in \mathcal{O}_{X^\sharp}[n]$ is written as a linear combination $\sum_I f_I \cdot \gamma_{i_1} \wedge ... \wedge \gamma_{i_{|I|}}$, where I is a finite set. Hence $i(\gamma) = \sum_I f_I \cdot \gamma_{i_1} \wedge ... \wedge \gamma_{i_{|I|}} \otimes 1$ is identified with a linear combination of elements in $\mathcal{O}_{X^\sharp} \otimes \tau\mathcal{Q}^{\leq -1}$, namely, $\sum_I f_I \cdot \gamma_{i_1} \wedge ... \wedge \gamma_{i_{i_k}} \otimes \pi^\dagger(\gamma_{i_{k+1}} \wedge ... \wedge \gamma_{i_{|I|}})$ whose image is zero under σ. Therefore, the quotient of $\tau\mathcal{Q}$ by $\mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1}$ fits in the following diagram,

$$\begin{array}{ccccccccc}
 & & & & \tau\mathcal{Q}/(\mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1}) & \xrightarrow{\rho} & \mathcal{T}_{X^\sharp}/\mathcal{T}_{X^\sharp|X} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & \overset{i}{\nearrow} & \tau\mathcal{Q} & \xrightarrow{\sigma} & \mathcal{T}_{X^\sharp} & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_{X^\sharp}[n] & \longrightarrow & \mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1} & \xrightarrow{\sigma} & \mathcal{T}_{X^\sharp|X} & \longrightarrow & 0.
\end{array}$$

It is enough to show that the map ρ is an isomorphism. Notice that ρ is $\mathcal{C}^\infty_X$-linear since $\beta \otimes \delta(fq) - f\beta \otimes \delta(q) = \beta \wedge df \otimes q \in \mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1}$. In addition, for any $\omega \in \Omega^{n-1}_X$, $f \in \mathcal{C}^\infty_X$ and $\beta \in \mathcal{O}_{X^\sharp}$ the calculation,

$$\begin{aligned}
\beta \otimes \delta(f\pi^\dagger\omega) - f\beta \otimes \delta(\pi^\dagger\omega) &= \beta \otimes \delta(f\pi^\dagger\omega) - f\beta \otimes \pi^\dagger d\omega \\
&= \beta \otimes \delta(f\pi^\dagger\omega) - f\beta \wedge d(\omega) \otimes \pi^\dagger(1) \\
&= 0
\end{aligned}$$

shows that $\beta \otimes \delta(f\pi^\dagger\omega) \in \mathcal{O}_{X^\sharp} \cdot \tau\mathcal{Q}^{\leq -1}$. Therefore $ker(\rho) = 0$. The surjectivity of σ implies that ρ is surjective as well. $\square$

We define $G(\mathcal{Q}) = (\tau\mathcal{Q}, [\ ,\ ], \sigma)$.

5.1.3 The main result

Theorem 5.1.6. *The functors F and G are mutually inverse.*

Proof: First we check that $F \circ G = id$. Suppose that $(\mathcal{Q}, \{\ ,\ \}, \langle\ ,\ \rangle, \pi)$ is an $(n-1)$-dimensional exact Courant algebroid. It follows directly by construction that $F(G(\mathcal{Q})) = F(\tau\mathcal{Q}) = \mathcal{Q}$.

In order to verify $G \circ F \cong id$, suppose that

$$0 \longrightarrow \mathcal{O}_{X^\sharp}[n] \xrightarrow{i} \mathcal{A} \xrightarrow{\sigma} \mathcal{T}_{X^\sharp} \longrightarrow 0$$

is a Lie algebroid on $X^\sharp$. Let $\mathcal{Q} = F(\mathcal{A}) = (\mathcal{A}^{-1}, \{\ ,\ \}, \langle\ ,\ \rangle, \sigma)$ be an $(n-1)$-dimensional exact Courant algebroid, see Section (4.2). Consider the diagram below,

$$\begin{array}{ccc} \widetilde{\mathcal{Q}} & \longrightarrow & \mathcal{A} \\ \downarrow & \overset{\Phi}{\nearrow} & \uparrow \bar{\Phi} \\ \mathcal{O}_{X^\sharp} \otimes \widetilde{\mathcal{Q}} & \longrightarrow & \tau\mathcal{Q} \end{array}$$

By direct computation it follows that Φ is compatible with brackets, as in Proposition (4.1.12). We claim that $\bar{\Phi}$ is an isomorphism in $\mathcal{O}_{X^\sharp}[n]$-LA$(X^\sharp)$. This follows from Proposition (5.1.5) and the fact that the diagram below commutes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X^\sharp}[n] & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{T}_{X^\sharp} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow \Phi & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}_{X^\sharp}[n] & \longrightarrow & \tau(\mathcal{Q}) & \longrightarrow & \mathcal{T}_{X^\sharp} & \longrightarrow & 0. \end{array}$$

Therefore, $\bar{\Phi}$ is an isomorphism. □

Corollary 5.1.7. *For $n = 2$, the category of $ECA(X)$ is equivalent with the category of $\mathcal{O}_{X^\sharp}[2]$-$LA(X^\sharp)$.* □

5.2 Applications

In this section we study higher dimensional exact Courant algebroids introducing the notions of connection and the curvature. Let us consider an $(n-1)$-dimensional exact Courant algebroid $\mathcal{Q}$ on X.

Definition 5.2.1. *A connection ∇ is an isotropic splitting of the anchor map π, i.e. $\pi \circ \nabla = id$, and $\langle \nabla(\xi_1), \nabla(\xi_2) \rangle = 0$ for any $\xi_1, \xi_2 \in \mathcal{T}_X$.*

We denote the sheaf of locally defined connection by $C(\mathcal{Q})$. Notice that difference of two connections determines an element in $\mathrm{Hom}(\mathcal{T}_X, \Omega_X^{n-1}) \cong \Omega_X^{n-1} \otimes \Omega_X^1$. Recall that the automorphism of $\mathcal{Q}$ are written as,

$$q \mapsto q + \iota_{\pi(q)}\phi$$

where $\phi \in \Omega_X^{n-1} \otimes \Omega_X^1$. Such a ϕ has to preserve the given pairing $\langle \, , \, \rangle$. Then ϕ is totally skew-symmetric, i.e. $\phi \in \Omega_X^n$.

$$\begin{aligned} \langle a + \iota_a\phi, b + \iota_b\phi \rangle &= \langle a, b \rangle + \langle a, \iota_b\phi \rangle \\ &+ \langle \iota_a\phi, b \rangle + \langle \iota_a\phi, \iota_b\phi \rangle \\ &= \langle a, b \rangle. \end{aligned}$$

Lemma 5.2.2. *[4] Locally there exist connections on $\mathcal{Q}$, if $\mathcal{Q}$ is locally free $\mathcal{C}_X^\infty$-module on X*

Proof: Since $\mathcal{Q}$ is locally free as $\mathcal{C}_X^\infty$-module then there is a section $s : \mathcal{T_X} \to \mathcal{Q}$ of π, locally defined on X. Define $\phi : \mathcal{T}_X \to \Omega_X^1$ by the formula

$$\iota_\eta \phi(\xi) = -\frac{1}{2}\langle s(\xi), s(\eta) \rangle.$$

We claim that $\nabla = s + \phi$ is a connection on $\mathcal{Q}$. By definition of ∇ it follows that $\pi \circ \nabla = id$. The isotropic condition follows from the calculation below,

$$\begin{aligned}\langle (s+\phi)(\xi), (s+\phi)(\eta)\rangle &= \langle s(\xi)+\phi(\xi), s(\eta)+\phi(\eta)\rangle \\ &= \langle s(\xi), s(\eta)\rangle + \langle s(\xi), \phi(\eta)\rangle + \langle \phi(\xi), s(\eta)\rangle + \langle \phi(\xi), \phi(\eta)\rangle \\ &= \langle s(\xi), s(\eta)\rangle + \iota_\xi \phi(\eta) + \iota_\eta \phi(\xi) \\ &= \langle s(\xi), s(\eta)\rangle - \langle s(\xi), s(\eta)\rangle = 0.\end{aligned}$$

□

Lemma 5.2.3. *The sheaf $C(\mathcal{Q})$ is a Ω^n_X-torsor.*

Proof: For a given connection ∇ and $\phi \in \Omega^n_X$, it is clear that $\nabla + \phi$ is also a connection on $\mathcal{Q}$. By Lemma (5.2.2) $\mathcal{C}(\mathcal{Q})$ is non-empty. □

Definition 5.2.4. *The curvature of a connection ∇ is defined by the formula,*

$$c(\nabla)(\xi_1, \xi_2) = \{\nabla\xi_1, \nabla\xi_2\} - \nabla[\xi_1, \xi_2].$$

Since π is a map of Leibniz algebras, the curvature $c(\nabla)$ takes values in Ω^{n-1}_X. In addition, by the isotropic condition it follows that $c(\nabla)$ is skew-symmetric in the first two arguments, i.e. $c(\nabla) \in \Omega^{n-1}_X \otimes \Omega^2_X$. By a direct computation we show that the curvature is totally skew-symmetric, i.e. $c(\nabla) \in \Omega^{n+1}_X$.

$$\begin{aligned}\langle a + \iota_a \iota_{\xi_1} c(\nabla), \nabla\xi_2 + \iota_{\xi_2}\iota_{\xi_1} c(\nabla)\rangle &= \langle a, \nabla\xi_2\rangle + \langle a, \iota_{\xi_2}\iota_{\xi_1} c(\nabla)\rangle \\ &\quad + \langle \iota_a \iota_{\xi_1} c(\nabla), \nabla\xi_2\rangle \\ &= 0,\end{aligned}$$

therefore,

$$\iota_a \iota_{\xi_2} \iota_{\xi_1} c(\nabla) + \iota_{\xi_2} \iota_a \iota_{\xi_1} c(\nabla) = 0.$$

Lemma 5.2.5. *Suppose that ∇ is a connection on $\mathcal{Q}$ and $\phi \in \Omega^n_X$.*

1.

$$dc(\nabla) = 0.$$

2.

$$c(\nabla + \phi) = c(\nabla) + d\phi,$$

for any $a \in \mathcal{Q}$, $\xi_1, \xi_2 \in \mathcal{T}_X$.

Proof: 1) For a given ∇, we obtain an isomorphism $\mathcal{Q} \cong \Omega_X^{n-1} \oplus \mathcal{T}_X$ given by $q = \nabla(\xi) + \alpha$ for suitable $\xi \in \mathcal{T}_X$ and $\alpha \in \Omega_X^{n-1}$. For any pair of sections on $\mathcal{Q}$, $q = \nabla\xi + \alpha$, $q' = \nabla\eta + \beta$, the bracket and the pairing are rewritten as follows,

$$\begin{aligned}\langle q, q' \rangle &= \langle \nabla\xi + \alpha, \nabla\eta + \beta \rangle \\ &= \langle \nabla\xi, \beta \rangle + \langle \nabla\eta, \alpha \rangle \\ &= \iota_\xi \beta + \iota_\eta \alpha,\end{aligned}$$

$$\begin{aligned}\{q, q'\} &= \{\nabla\xi + \alpha, \nabla\eta + \beta\} \\ &= \{\nabla\xi, \nabla\eta\} + \{\nabla\xi, \beta\} + \{\alpha, \nabla\eta\} \\ &= \nabla[\xi, \eta] + \iota_\eta \iota_\xi c(\nabla) + L_\xi \beta - \iota_\eta d\alpha.\end{aligned}$$

The Jacobi identity for the bracket on $\mathcal{Q}$, implies that $dc(\nabla) = 0$, namely

$$\begin{aligned}&\{\nabla\xi_1 + \alpha_1, \{\nabla\xi_2 + \alpha_2, \nabla\xi_3 + \alpha_3\}\} \\ &\quad = \{\{\nabla\xi_1 + \alpha_1, \nabla\xi_2 + \alpha_2\}, \nabla\xi_3 + \alpha_3\} + \{\nabla\xi_2 + \alpha_2, \{\nabla\xi_1 + \alpha_1, \nabla\xi_3 + \alpha_3\}\}\end{aligned}$$

The right hand side of the Jacobi identity is given by,

$$\begin{aligned}\{\nabla\xi_1 + \alpha_1, \{\nabla\xi_2 + \alpha_2, \nabla\xi_3 + \alpha_3\}\} &= \nabla[\xi_1, [\xi_2, \xi_3]] \\ &+ \iota_{[\xi_2,\xi_3]} \iota_{\xi_1} c(\nabla) + L_{\xi_1} \iota_{\xi_3} \iota_{\xi_2} c(\nabla) + L_{\xi_1} L_{\xi_2} \alpha_3 \\ &- L_{\xi_1} \iota_{\xi_3} d\alpha_2 - \iota_{[\xi_2,\xi_3]} d\alpha_1\end{aligned}$$

The left hand side is given by the following formulas,

$$\begin{aligned}\{\{\nabla\xi_1+\alpha_1,\nabla\xi_2+\alpha_2\},\nabla\xi_3+\alpha_3\} &= \nabla[[\xi_1,\xi_2],\xi_3]\\ &+\iota_{\xi_3}\iota_{[\xi_1,\xi_2]}c(\nabla)+L_{[\xi_1,\xi_2]}\alpha_3-\iota_{\xi_3}d\iota_{\xi_2}\iota_{\xi_1}c(\nabla)\\ &-\iota_{\xi_3}dL_{\xi_1}\alpha_2+\iota_{\xi_3}d\iota_{\xi_2}d\alpha_1\end{aligned}$$

$$\begin{aligned}\{\nabla\xi_2+\alpha_2,\{\nabla\xi_1+\alpha_1,\nabla\xi_3+\alpha_3\}\} &= \nabla[\xi_2,[\xi_1,\xi_3]]\\ &+\iota_{[\xi_1,\xi_3]}\iota_{\xi_2}c(\nabla)+L_{\xi_2}\iota_{\xi_3}\iota_{\xi_1}c(\nabla)+L_{\xi_2}L_{\xi_1}\alpha_3\\ &-L_{\xi_2}\iota_{\xi_3}d\alpha_1-\iota_{[\xi_1,\xi_3]}d\alpha_2\end{aligned}$$

If we add these two terms and replace them in the Jacobi identity, it follows

$$0=\iota_{\xi_3}\iota_{\xi_2}\iota_{\xi_1}dc(\nabla).$$

2) By direct computation,

$$\begin{aligned}c(\nabla+\phi)(\xi_1,\xi_2) &= \{\nabla+\phi\xi_1,\nabla+\phi\xi_2\}-(\nabla+\phi)[\xi_1,\xi_2]\\ &= \{\nabla\xi_1+\iota_{\xi_1}\phi,\nabla\xi_2+\iota_{\xi_2}\phi\}-\nabla[\xi_1,\xi_2]-\iota_{[\xi_1,\xi_2]}\phi\\ &= c(\nabla)(\xi_1,\xi_2)+\{\nabla\xi_1,\iota_{\xi_2}\phi\}+\{\iota_{\xi_1}\phi,\nabla\xi_2\}-\iota_{[\xi_1,\xi_2]}\phi\\ &= c(\nabla)(\xi_1,\xi_2)+\{\nabla\xi_1,\iota_{\xi_2}\phi\}-\{\nabla\xi_2,\iota_{\xi_1}\phi\}+d\iota_{\xi_2}\iota_{\xi_1}\phi-\iota_{[\xi_1,\xi_2]}\phi\\ &= c(\nabla)(\xi_1,\xi_2)+L_{\xi_1}\iota_{\xi_2}\phi-L_{\xi_2}\iota_{\xi_1}\phi+d\iota_{\xi_2}\iota_{\xi_1}\phi-(L_{\xi_1}\iota_{\xi_2}-\iota_{\xi_2}L_{\xi_1})\phi\\ &= c(\nabla)(\xi_1,\xi_2)-L_{\xi_2}\iota_{\xi_1}\phi+\iota_{\xi_2}L_{\xi_1}\phi+d\iota_{\xi_2}\iota_{\xi_1}\phi\\ &= c(\nabla)(\xi_1,\xi_2)-\iota_{\xi_2}d\iota_{\xi_1}\phi+\iota_{\xi_2}L_{\xi_1}\phi\\ &= c(\nabla)(\xi_1,\xi_2)+\iota_{\xi_2}\iota_{\xi_1}d\phi\\ &= (c(\nabla)+d\phi)(\xi_1,\xi_2).\end{aligned}$$

□

Definition 5.2.6. *A connection ∇ is called flat if it is a morphism of Leibniz algebras, i.e. $c(\nabla)=0$.*

We denote the sheaf of sets of flat connections by $\mathcal{C}^\flat(\mathcal{Q})$. By definition $\mathcal{C}^\flat(\mathcal{Q})$ is a subsheaf of $\mathcal{C}(\mathcal{Q})$.

Lemma 5.2.7. *The sheaf of flat connections $\mathcal{C}^\flat(\mathcal{Q})$ is (locally) non empty.*

Proof: As we working in the $\mathcal{C}^\infty$ context, Poincaré lemma holds, then every closed form is locally exact. Therefore $\mathcal{C}^\flat(\mathcal{Q})$ is locally non empty. □

Corollary 5.2.8. *The action of Ω^n_X on $\mathcal{C}(\mathcal{Q})$ restricts to an action of $\Omega^{n,cl}_X$ on $\mathcal{C}^\flat(\mathcal{Q})$.*

Proof: By Lemma (5.2.5), if ∇ is a flat connection then $c(\nabla+\alpha) = c(\nabla)+d\alpha = 0$. Then, $d\alpha = 0$, i.e. α is a closed form. □

Example 5.2.9. *For the $(n-1)$-dimensional Courant algebroid $\mathcal{Q}_0$, see Example* (5.0.4)*, the inclusion $\mathcal{T}_X \to \mathcal{Q}_0$ is a flat connection. Since for any $\xi \in \mathcal{T}_X$, $\nabla_0(\xi) := 0 \oplus \xi$, it follows that $\{0\oplus\xi, 0\oplus\eta\} = [\xi,\eta]$.*

Remark 5.2.10. *Recall that a $(\Omega^n_X \to \Omega^{n+1,cl}_X)$-torsor is a pair (C,c) where C is a Ω^n_X-torsor and $c : C \to \Omega^{n+1,cl}_X$ which satisfies*

$$c(s+\alpha) = c(s) + d\alpha,$$

for any $\alpha \in \Omega^n_X$. A morphism of $(\Omega^n_X \to \Omega^{n+1,cl}_X)$-torsors (C_1,c_1), (C_2,c_2) is a map $\tau : C_1 \to C_2$ of Ω^n_X-torsors which commutes with c_i's, $c_1 = c_2 \circ \tau$.

Proposition 5.2.11. *The correspondence $\mathcal{Q} \to (\mathcal{C}(\mathcal{Q}), c)$, establishes an equivalence*

$$(n-1) - ECA(X) \to (\Omega^n_X \to \Omega^{n+1,cl}_X) - torsors.$$

Proof: Given $\mathcal{Q}$, $(\mathcal{C}(\mathcal{Q}), c)$ is an $(\Omega^n_X \to \Omega^{n+1,cl}_X)$-torsor. We construct a quasi-inverse for the functor $\mathcal{Q} \to \mathcal{C}(\mathcal{Q})$. Suppose that (C,c) is a $(\Omega^n_X \to \Omega^{n+1,cl}_X)$-torsor. Consider $\mathcal{Q}^c_0 := C \times_{\Omega^n_X} \mathcal{Q}_0$. Since the action of Ω^n_X on $\mathcal{Q}_0$, given by $\omega(q) = q + \iota_{\pi(q)}\omega$, preserves the symmetric pairing, then $\mathcal{Q}^c_0$ has the induced symmetric pairing. The Leibniz bracket is defined as follows,

$$\{(s_1,q_1),(s_2,q_2)\} := (s_1, \{q_1, q_2 + \iota_{\pi(q_2)}(s_1 - s_2)\}_0 + \iota_{\pi(q_2)}\iota_{\pi(q_1)}c(s_1)),$$

where $s_i \in C$, $q_i \in \mathcal{Q}_0$ and $s_1 - s_2 \in \Omega^n_X$ is the unique element such that $s_1 = s_2 + (s_1 - s_2)$. The bracket $\{\,,\,\}_0$ denotes the Courant bracket on $\mathcal{Q}_0$. For any $\omega \in \Omega^n_X$ we have

$$\{(s_1, q_1), (s_2, q_2 + \iota_{\pi(q_2)}\omega)\} = \{(s_1, q_1), (s_2 - \omega, q_2)\}^1$$

If we pick $\omega = -s_1 + s_2$ the bracket $\{\,,\,\}$ is written as,

$$\begin{aligned}\{(s_1, q_1), (s_2, q_2 + \iota_{\pi(q_2)}(-s_1 + s_2)\} &= \{(s_1, q_1), (s_1, q_2)\} \\ &= (s_1, \{q_1, q_2\}_0 + \iota_{\pi(q_2)}\iota_{\pi(q_1)}c(s_1)).\end{aligned}$$

Since $c(s)$ is skew-symmetric we have,

$$\{(s, q_1), (s, q_2)\} + \{(s, q_2), (s, q_1)\} = (s, d\langle q_1, q_2\rangle).$$

Since $c(s)$ is a closed form, the bracket $\{(s, q_1), (s, q_2)\}$ satisfies the Jacobi identity. The Leibniz rule follows by direct computation

$$\begin{aligned}\{(s, q_1), f(s, q_2)\} &= \{(s, q_1), (s, fq_2)\} \\ &= (s, \{q_1, fq_2\}_0 + f\iota_{\pi(q_2)}\iota_{\pi(q_1)}c(s)) \\ &= f\{(s, q_1), (s, q_2)\}_0 + (\pi(q_1)f)q_2.\end{aligned}$$

□

Lemma 5.2.12. *The assignment $(\mathcal{Q}, \nabla) \to c(\nabla)$ gives rise to an equivalence of categories,*

$$c : ECA\nabla(\mathcal{Q}) \to \Omega^{n+1,cl}_X,$$

where $\Omega^{n+1,cl}_X$ is viewed as a discrete category[2]

Proof: The quasi-inverse functor associates to $H \in \Omega^{n+1,cl}$, the H-twist $\mathcal{Q}_H$ structure on $\mathcal{Q}_0$. It follows that the curvature on $\mathcal{Q}_H$ is given by H. □

[1]since $\pi(q_2) = \pi(q_2 + \iota_{\pi(q_2)}\omega)$
[2]morphisms are just the identity maps.

Theorem 5.2.13. *The set of isomorphism classes of* $(n-1)$*-dimensional exact Courant algebroids is in one to one correspondence with* $H^{n+1}_{dR}(X)$*.*

Proof: This follows from Proposition (5.2.11), and the fact that the complex $\Omega^n_X \to \Omega^{n+1}_X$ is quasi-isomorphic to $\Omega^{n,cl}_X$. Since $(\Omega^n_X \to \Omega^{n+1}_X)$-torsors are classified by

$$H^1(X; \Omega^n_X \to \Omega^{n+1}_X) = H^1(X; \Omega^{n,cl}_X) \cong H^{n+1}_{dR}(X).$$

□

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