MCKAY CORRESPONDENCE IN QUASITORIC ORBIFOLDS

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McKay correspondence in quasitoric orbifolds

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Abstract

We prove that every positively omnioriented quasitoric orbifold admits a torus-invariant almost complex structure. We construct blowdown maps of quasitoric orbifolds and study their properties. We define the Chen-Ruan cohomology of any omnioriented quasitoric orbifold. We define the notion of quasi-SL quasitoric orbifold. We show that the Betti numbers of Chen-Ruan cohomology of a quasi-SL quasitoric orbifold are preserved under crepant blowdown.

Contents

1. Introduction

McKay correspondence has been studied widely for complex algebraic SL (also called Gorenstein) orbifolds [24]. A cohomological version of this correspondence says that the Hodge numbers (and Betti numbers) of Chen-Ruan cohomology [8] are preserved under crepant resolution. This was proved in [18] and [25] following fundamental work of [4] and [12]. Chen-Ruan cohomology is defined originally for any almost complex orbifold and it makes sense to ask if such a correspondence holds for Betti numbers when the orbifold has almost complex structure only.

In this article we define Chen-Ruan cohomology for any omnioriented quasitoric orbifold which is not necessarily almost complex. We find a corresponding generalization of the SL condition which we call quasi-SL. We construct blowdown maps between omnioriented quasitoric orbifolds and define crepant blowdown in this context. We find a correspondence for Betti numbers of Chen-Ruan cohomology of omnioriented quasitoric quasi-SL orbifolds under crepant blowdowns. This generalizes the known correspondence for algebraic toric orbifolds.

A quasitoric orbifold(in [10] [23]) is a $2n$-dimensional differentiable orbifold equipped with a smooth action of the n-dimensional compact torus such that the orbit space is diffeomorphic as manifold with corners to an n-dimensional simple polytope. (An n- dimensional polytope is called simple if every vertex is the intersection of exactly n codimension one faces.) The preimage of every codimension one face is a torus invariant $(2n-2)$-dimensional quasitoric orbifold which is stabilized by a circle subgroup of the form $\{(e^{2\pi a_1 t}, \dots e^{2\pi a_n t}) : t \in \mathbb{R}\}$. The vector $(a_1, \dots a_n)$ is a primitive integral vector called the characteristic vector associated with this codimension one face. In general a codimension k face is the intersection of k codimension one faces and its characteristic set consists of the characteristic vectors of these codimension one faces. The characteristic set of every face is linearly independent over $\mathbb{R}$.

An omniorientation is a choice of orientation for the quasitoric orbifold as well as for each invariant suborbifold of codimension two. When these orientations are compatible, the quasitoric orbifold is called positively omnioriented, see section 4.9 for details. We prove the existence of invariant almost complex structure on positively omnioriented quasitoric orbifolds (Theorem 6.2) by adapting the technique of Kustarev [15] for quasitoric manifolds.

Chen-Ruan cohomology was originally defined for almost complex orbifolds in [8]. There the almost complex structure on normal bundle of singular strata is used to determine the grading of the cohomology. In quasitoric case, an omniorientation together with the torus action determine a complex structure on the normal bundle of every invariant suborbifold. Moreover the singular locus is a subset of the union of invariant suborbifolds. Thus we can define Chen-Ruan cohomology groups for any omnioriented quasitoric orbifold, see section 9. We also define a ring structure for this cohomology in section 10 following the approach of [7]. The Chen-Ruan cohomology of the same quasitoric orbifold is in general different for different omniorientations. For a positively omnioriented quasitoric orbifold with the almost complex structure of Theorem 6.2, our definition of Chen-Ruan cohomology ring agrees with that of [8].

We construct equivariant blowdown maps of omnioriented quasitoric orbifolds, see section 7. These maps are continuous and they are diffeomorphism of orbifolds away from the exceptional set. They are not morphisms of orbifolds (see [1] for definition). In some cases they are analytic near the exceptional set, see Lemma 7.4. (In these cases they are pseudoholomorphic in a natural sense, see Definition 7.2.) For these we can compute the pull-back of the canonical sheaf and test if the blowdown is crepant in the sense of complex geometry: The pull back of the canonical sheaf of the blowdown is the canonical sheaf of the blowup. However the combinatorial condition this corresponds to, makes sense in general and may be applied to an arbitrary blowdown. We work with this generalized notion of crepant blowdown, see section 8.

We prove the conservation of Betti numbers of Chen-Ruan cohomology under crepant blowdowns (see section 12). By McKay correspondence in quasitoric orbifolds, we mean this correspondence. We give an example in section 13. This example is particularly interesting as it corresponds to the weighted projective space $\mathbb{P}(1,1,3,3,3)$ which is not a Gorenstein or SL orbifold. Hence McKay correspondence as studied in complex algebraic geometry does not apply to it. However under suitable choice of omniorientation it is quasi-SL and McKay correspondence holds. Note that the blowup is not a toric variety.

We briefly summarize our main contributions:

(1) Invariant almost complex structure in quasitoric orbifolds (see section 6.2)
(2) Construction of blowdown in the quasitoric category and its basic properties (see section 7)
(3) Definition of Chen-Ruan cohomology ring for omnioriented quasitoric orbifolds (see sections 9 and 10)
(4) Mackay Correspondence for Betti numbers of quasi-SL quasitoric orbifolds under crepant blowdown (see section 12)

Parts of this thesis have appeared in the articles [13] and [14] authored jointly with my advisor M. Poddar.

2. **Orbifolds**

We give a brief introduction to orbifolds following the treatment in [1]

Definition 2.1. *Let X be a topological space, and fix n*≥ 0.

(1) *An n-dimensional orbifold chart on X is given by a connected open subset* $\tilde{U} \subset R^n$, *a finite group G of smooth diffeomorphisims of* $\tilde{U}$, *and a map* $\phi : \tilde{U} \to X$ *so that* ϕ *is G-invariant and induces a homeomorphism of* $\tilde{U}/G$ *onto an open subset* $U \subset X$
(2) *An embedding* $\lambda(\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ *between two such charts is a smooth embedding* $\lambda : \tilde{U} \to \tilde{V}$ *with* $\psi\lambda = \phi$
(3) *An orbifold atlas on X is a family of such charts* $\mathcal{U} = (\tilde{U}, G, \phi)$, *which cover X and satisfy the compatibility condition: given any two charts* $(\tilde{U}, G, \phi)$ *for* $U = \phi(\tilde{U}) \subset X$ *and* $(\tilde{V}, H, \psi)$ *for* $V = \psi(\tilde{V}) \subset X$ *and a point* $x \in U \cap V$ *there exists an open neighborhood* $W \subset U \cap V$ *of x and a chart* $(\tilde{W}, K, \mu)$ *for W such that there are embeddings* $\lambda_1 : (\tilde{W}, K, \mu) \to (\tilde{U}, G, \phi)$ *and* $\lambda_2 : (\tilde{W}, K, \mu) \to (\tilde{V}, H, \phi)$. *Every embedding of charts induces injective homomorphisms* $\lambda : K \to G$.
(4) *An atlas* $\mathcal{U}$ *is said to refine another atlas* Υ *if for every chart in* $\mathcal{U}$ *there exists an embedding into some chart of* Υ. *Two orbifold atlases are said to be equivalent if they have a common refinement.*

Definition 2.2. *An effective orbifold X of dimension n is a paracompact Hausdorff spaceX equipped with an equivalence class* $[\mathcal{U}]$ *of n-dimensional orbifold atlases.*

Throughout this thesis we will always assume that our orbifolds are effective

(1) We are assuming that for each chart $(\tilde{U}, G, \phi)$ the group G is acting smoothly and effectively on $\tilde{U}$. In particular G will act freely on a dense open subset of $\tilde{U}$.
(2) If the finite group actions on all the charts are free, then X is locally Euclidean, hence a manifold. Points in an orbifold which are not locally Euclidean form the singular locus

Definition 2.3. *Let* $\mathbf{X} = (X, \mathcal{U})$ *and* $\mathbf{Y} = (Y, \Upsilon)$ *be orbifolds. A map* $f : X \to Y$ *is said to be smooth if for any point* $x \in X$ *there are charts* $(\tilde{U}, G, \phi)$ *around x and* $(\tilde{V}, G, \psi)$, *around* $f(x)$ *with the property that f maps* $U = \phi(\tilde{U})$ *into* $V = \psi(\tilde{U})$*and can be lifted to a smooth map* $\tilde{f} : \tilde{U} \to \tilde{V}$ *satisfying* $\psi\tilde{f} = f\phi$

Definition 2.4. *The isotropy subgroup of a point* $\tilde{x}$ *in the chart* $(\tilde{U}, G, \phi)$ *is* $G_{\tilde{x}} = \{g \in G \mid g\tilde{x} = \tilde{x}\}$.*If* $\tilde{y}$ *is a point in the G-orbit of* $\tilde{x}$, *then* $G_{\tilde{y}}$ *is conjugate to* $G_{\tilde{x}}$ *in G. If G is Abelian* ,$G_{\tilde{x}} = G_{\tilde{y}}$ *and we denote* $G_{\tilde{x}}$ *by* G_x *where* $x = \phi(\tilde{x})$. G_x *is called the isotropy group of x.*

2.1. **Tangent Bundle of an Orbifold.**

We describe a tangent bundle for an orbifold $\mathbf{X} = (X, \mathcal{U})$. Given a chart $(\tilde{U}, G, \phi)$ we consider a tangent bundle $\mathcal{T}\tilde{U}$. Since G acts smoothly on $\tilde{U}$, hence it acts smoothly on $\mathcal{T}\tilde{U}$. Indeed if $(\tilde{u}, v)$ a typical element of $\mathcal{T}\tilde{U}$ then $g(\tilde{u}, v) = (g\tilde{u}, dg_{\tilde{u}}(v))$. Moreover the projection map $\mathcal{T}\tilde{U} \to \tilde{U}$

is equivariant from which we get a projection $p : \mathcal{T}\tilde{U}/G \to U$ by using the map ϕ. . If $x = \phi(\tilde{x})$

$$p^{-1}(x) = \{G(\tilde{x}, v) \subset \mathcal{T}\tilde{U}/G\} \tag{2.1}$$

It can be easily seen this fiber is homeomorphic to $\mathcal{T}_{\tilde{x}}\tilde{U}/G_{\tilde{x}}$ where as $G_{\tilde{x}}$ is the isotropy subgroup of the G action at $\tilde{x}$. This means we have constructed locally a bundle like object where the fiber is not necessarily a vector space, but rather a quotient of the form $\mathbb{R}^n/G_0$ where $G_0 \subset Gl_n(R)$. To construct the tangent bundle on an orbifold $\tilde{X} = (X, \mathcal{U})$,we simply need to glue together the bundles defined over the charts . Our resulting space will be an orbifold, with an atlas $\mathcal{TU}$ comprising local charts $(\mathcal{T}\tilde{U}, G, \phi)$ over $\mathcal{T}U = \mathcal{T}\tilde{U}/G$ for each $(\tilde{U}, G, \phi) \in \mathcal{U}$: We observe that the gluing maps $\lambda_{12} = \lambda_2{\lambda_1}^{-1}$ are smooth, so we can use the transition functions $d\lambda_{12} : \mathcal{T}\lambda_1(\tilde{W}) \to \mathcal{T}\lambda_2(\tilde{W})$ to glue $\mathcal{T}\tilde{U}/G \to U$ to $\mathcal{T}\tilde{U}/H \to H$. In other words, we define the space $\mathcal{T}X$ as an identification space $\bigsqcup_{\tilde{U}\in v}(\mathcal{T}\tilde{U}/G)/\sim)$ where we give it the minimal topology that will make the natural maps $\mathcal{T}\tilde{U}/G \to \mathcal{T}X$ homeomorphisms onto open subsets of $\mathcal{T}X$. We summarize this in the next remark.

Remark 2.1. *The tangent bundle of an n-dimensional orbifold* $\mathbf{X}$ *denoted by* $\mathcal{T}\mathbf{X} = (\mathcal{T}X, \mathcal{TU})$*has the structure of a 2n-dimensional orbifold. Moreover, the natural projectionp* $: \mathcal{T}X \to X$ *defines a smooth map of orbifolds, with fibers* $p^{-1}(x) = \mathcal{T}_{\tilde{x}}\tilde{U}/G_x$

2.2. **Example: Teardrop orbifold.** A teardrop orbifold is a topological sphere with an orbifold singularity at north pole. Consider the standard embedding of the sphere in $\mathbb{R}^3$ with$(0, 0, 1)$ as the north pole and $(0, 0, -1)$as the south pole. We give two orbifold charts.Let $U = \{(x, y, z) \in S^2 \mid z < \frac{1}{3}\}$ and $V = \{(x, y, z) \in S^2 \mid z > \frac{-1}{3}\}$ are two open sets around the south pole and north pole respectively. We define a chart $(\tilde{U}, G, \phi)$ where $\tilde{U} = U$, G is the trivial group and ϕ identity map. Around north pole we take the open set V and define an orbifold chart $(\tilde{V}, H, \psi)$ where $\tilde{V} = V$, $H = \mathbb{Z}_3$ and ψ is a triple cover branched at the north pole. Now if $z \in U \cap V$ we take neighborhood so small that the maps ϕ and ψ cover these neighborhood evenly. Let us call this neighborhood W_z and with the chart (W_z, I, id) and the maps λ_1 and λ_2 are mere inclusions.Thus we have an orbifold structure which is different from the manifold structure of the sphere and we call it a teardrop orbifold. From the above description it is fairly easy to construct its tangent bundle.

3. McKay correspondence in complex geometry

Suppose Y is an n-dimensional complex algebraic or analytic variety with SL orbifold singularities, i.e. every singularity is isomorphic to the quotient $\mathbb{C}^n/G$ where G is a finite subgroup of $SL(n,\mathbb{C})$. Let K_Y denote the canonical sheaf of Y. The sections of this sheaf are locally generated by G-invariant $(n,0)$ forms. As $G \subset SL(n,\mathbb{C})$, K_Y is in fact a vector bundle. A (partial) resolution of singularities $\rho : \widehat{Y} \to Y$ is called crepant if $\rho^*(K_Y) = K_{\widehat{Y}}$.

The story of McKay correspondence begins with the work [19], where McKay considers smooth crepant resolutions of $Y = \mathbb{C}^2/G$. The exceptional divisor E is the preimage of the singular point 0, i.e. $E = \rho^{-1}\{0\}$. McKay shows that the topology of E (and therefore of $\widehat{Y}$) is determined by the representation theory of G. In particular, the nontrivial conjugacy classes of G correspond to irreducibe components of E or equivalently to generators of $H_2(E) \cong H^2(\widehat{Y})$.

For $Y = \mathbb{C}^n/G$ where $n > 2$, it is hard in general to establish a natural correspondence between conjugacy classes and generators of $H^*(\widehat{Y})$. However the rank of $H^{2k}(\widehat{Y})$ equals the number of congugacy classes of G having age or weight $\iota(g) = k$. This was proved for $n = 3$ in [16]; and later for general n independently in [4] and [12] using sophisticated motivic integration techniques. Here $\iota(g) = \sum_{j=1}^{n} \alpha_j$ where $e^{2\pi i \alpha_j}$ denote the eigen values of g and $0 \le \alpha_j < 1$.

For a general complex SL orbifold Y, there may be many nontrivial local groups G corresponding to different orbifold charts. In this case, to state the correspondence we need a correct generalization of the notion of conjugacy classes. This is provided by the Chen-Ruan cohomology [8]. (The stringy Hodge numbers of Batyrev [4] work for even more general singularities, but they agree with Hodge numbers of Chen-Ruan cohomology in the orbifold case.) Then, McKay correspondence translates to the statement that the Hodge numbers (and Betti numbers) of Y and $\widehat{Y}$ are equal. This is proved in [18] and [25] for any complete complex algebraic SL orbifold Y. Note that this correspondence is valid even when $\widehat{Y}$ is a partial crepant resolution of Y, that is, $\widehat{Y}$ need not be smooth.

There are other forms of McKay correspondence, for instance at the level of derived categories. We refer to the article [24] of Reid for details. The version discussed above and later in this thesis may be referred to as a cohomological version.

4. **Quasitoric orbifolds**

In this section we review the combinatorial construction of quasitoric orbifolds. We also construct an explicit orbifold atlas for them and list a few important properties. The notations established here will be important for the rest of the article.

4.1. **Construction.** Fix a copy N of $\mathbb{Z}^n$ and let $T_N := (N \otimes_{\mathbb{Z}} \mathbb{R})/N \cong \mathbb{R}^n/N$ be the corresponding n-dimensional torus. A primitive vector in N, modulo sign, corresponds to a circle subgroup of T_N. More generally, suppose M is a submodule of N of rank m. Then

$$T_M := (M \otimes_{\mathbb{Z}} \mathbb{R})/M \tag{4.1}$$

is a torus of dimension m. Moreover there is a natural homomorphism of Lie groups $\xi_M : T_M \to T_N$ induced by the inclusion $M \hookrightarrow N$.

Definition 4.1. *Define T(M) to be the image of T_M under ξ_M. If M is generated by a vector $\lambda \in N$, denote T_M and $T(M)$ by T_λ and $T(\lambda)$ respectively.*

Usually a polytope is defined to be the convex hull of a finite set of points in $\mathbb{R}^n$. To keep our notation manageable, we will take a more liberal interpretation of the term polytope.

Definition 4.2. *A polytope P will denote a subset of $\mathbb{R}^n$ which is diffeomorphic, as manifold with corners, to the convex hull Q of a finite number of points in $\mathbb{R}^n$. Faces of P are the images of the faces of Q under the diffeomorphism.*

Let P be a simple polytope in $\mathbb{R}^n$, i.e. every vertex of P is the intersection of exactly n codimension one faces (facets). Consequently every k-dimensional face F of P is the intersection of a unique collection of $n-k$ facets. Let $\mathcal{F} := \{F_1, \ldots, F_m\}$ be the set of facets of P.

Definition 4.3. *A function $\Lambda : \mathcal{F} \to N$ is called a characteristic function for P if $\Lambda(F_{i_1}), \ldots, \Lambda(F_{i_k})$ are linearly independent whenever $F_{i_1}, \ldots, F_{i_k}$ intersect at a face in P. We write λ_i for $\Lambda(F_i)$ and call it a characteristic vector.*

Remark 4.1. *In this article we assume that all characteristic vectors are primitive. Corresponding quasitoric orbifolds have been termed primitive quasitoric orbifold in [23]. They are characterized by the codimension of singular locus being greater than or equal to four.*

Definition 4.4. *For any face F of P, let $\mathcal{I}(F) = \{i | F \subset F_i\}$. Let Λ be a characteristic function for P. The set $\lambda_F := \{\lambda_i : i \in \mathcal{I}(F)\}$ is called the characteristic set of F. Let $N(F)$ be the submodule of N generated by λ_F. Note that $\mathcal{I}(P)$ is empty and $N(P) = \{0\}$.*

For any point $p \in P$, denote by $F(p)$ the face of P whose relative interior contains p. Define an equivalence relation $\sim$ on the space $P \times T_N$ by

$$(p, t) \sim (q, s) \text{ if and only if } p = q \text{ and } s^{-1}t \in T(N(F(p))) \tag{4.2}$$

Then the quotient space $X := P \times T_N / \sim$ can be given the structure of a $2n$-dimensional quasitoric orbifold. Moreover it can be seen $2n$-dimensional primitive quasitoric orbifold defined in previous section can be obtained in this way. We refer to the pair (P, Λ) as a model for the quasitoric orbifold. The space X inherits an action of T_N with orbit space P from the natural action on $P \times T_N$. Let $\pi : X \to P$ be the associated quotient map.

The space X is a manifold if the characteristic vectors $\lambda_{i_1}, \ldots, \lambda_{i_k}$ generate a unimodular subspace of N whenever the facets $F_{i_1}, \ldots, F_{i_k}$ intersect. The points $\pi^{-1}(v) \in X$, where v is any vertex of P, are fixed by the action of T_N. For simplicity we will denote the point $\pi^{-1}(v)$ by v when there is no confusion.

4.2. **Orbifold charts.** Consider open neighborhoods $U_v \subset P$ of the vertices v such that U_v is the complement in P of all edges that do not contain v. Let

$$X_v := \pi^{-1}(U_v) = U_v \times T_N / \sim \tag{4.3}$$

For a face F of P containing v there is a natural inclusion of $N(F)$ in $N(v)$. It induces an injective homomorphism $T_{N(F)} \to T_{N(v)}$ since a basis of $N(F)$ extends to a basis of $N(v)$. We will regard $T_{N(F)}$ as a subgroup of $T_{N(v)}$ without confusion. Define an equivalence relation $\sim_v$ on $U_v \times T_{N(v)}$ by $(p, t) \sim_v (q, s)$ if $p = q$ and $s^{-1}t \in T_{N(F)}$ where F is the face whose relative interior contains p. Then the space

$$\widetilde{X}_v := U_v \times T_{N(v)} / \sim_v \tag{4.4}$$

is θ-equivariantly diffeomorphic to an open set in $\mathbb{C}^n$, where $\theta : T_{N(v)} \to U(1)^n$ is an isomorphism, see [10]. This means that there exists a diffeomorphism $f : \widetilde{X}_v \to B \subset \mathbb{C}^n$ such that $f(t \cdot x) = \theta(t) \cdot f(x)$ for all $x \in \widetilde{X}_v$. This will be evident from the subsequent discussion.

The map $\xi_{N(v)} : T_{N(v)} \to T_N$ induces a map $\xi_v : \widetilde{X}_v \to X_v$ defined by $\xi_v([(p,t)]^{\sim_v}) = [(p, \xi_{N(v)}(t))]^{\sim}$ on equivalence classes. The kernel of $\xi_{N(v)}$, $G_v = N/N(v)$, is a finite subgroup of $T_{N(v)}$ and therefore has a natural smooth, free action on $T_{N(v)}$ induced by the group operation. This induces smooth action of G_v on $\widetilde{X}_v$. This action is not free in general. Since $T_N \cong T_{N(v)}/G_v$, X_v is homeomorphic to the quotient space $\widetilde{X}_v/G_v$. An orbifold chart (or uniformizing system) on X_v is given by $(\widetilde{X}_v, G_v, \xi_v)$.

Let $(p_1, \ldots, p_n)$ denote the standard coordinates on $\mathbb{R}^n \supset P$. Let $q_1, \ldots, q_n$ be the coordinates on $N \otimes \mathbb{R}$ with respect to the standard basis of N. They correspond to standard angular coordinates on T_N. Let $\{u_1, \ldots, u_n\}$ be the standard basis of N. Suppose the characteristic vectors u_i are assigned to the facets $p_i = 0$ of the cone $\mathbb{R}^n_{\geq}$. In this case there is a homeomorphism $\phi : (\mathbb{R}^n_{\geq} \times T_N / \sim) \to \mathbb{R}^{2n}$ given by

$$x_i = \sqrt{p_i}\cos(2\pi q_i), \quad y_i = \sqrt{p_i}\sin(2\pi q_i) \;\text{ where } i = 1 \ldots n. \tag{4.5}$$

Remark 4.2. *The square root over p_i is necessary to ensure that the orbit map $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n_{\geq}$ is smooth.*

We define a homeomorphism $\phi(v) : \widetilde{X}_v \to \mathbb{R}^{2n}$ as follows. Assume without loss of generality that $F_1, \ldots, F_n$ are the facets of U_v. Let the equation of F_i be $p_i(v) = 0$.

Assume that $p_i(v) > 0$ in the interior of U_v for every i. Let $\Lambda_{(v)}$ be the corresponding matrix of characteristic vectors

$$\Lambda_{(v)} = [\lambda_1 \dots \lambda_n]. \tag{4.6}$$

If $\mathbf{q}(v) = (q_1(v), \dots, q_n(v))^t$ are angular coordinates of an element of T_N with respect to the basis $\{\lambda_1, \dots, \lambda_n\}$ of $N \otimes \mathbb{R}$, then the standard coordinates $\mathbf{q} = (q_1, \dots, q_n)^t$ may be expressed as

$$\mathbf{q} = \Lambda_{(v)}\mathbf{q}(v). \tag{4.7}$$

Then define the homeomorphism $\phi(v) : \widetilde{X}_v \to \mathbb{R}^{2n}$ by
(4.8)
$$x_i = x_i(v) := \sqrt{p_i(v)}\cos(2\pi q_i(v)), \quad y_i = y_i(v) := \sqrt{p_i(v)}\sin(2\pi q_i(v)) \quad \text{for } i = 1, \dots, n$$

We write

$$z_i = x_i + \sqrt{-1}y_i, \quad \text{and} \quad z_i(v) = x_i(v) + \sqrt{-1}y_i(v) \tag{4.9}$$

Now consider the action of $G_v = N/N(v)$ on $\widetilde{X}_v$. An element g of G_v is represented by a vector $\sum_{i=1}^n a_i\lambda_i$ in N where each $a_i \in \mathbb{Q}$. The action of g transforms the coordinates $q_i(v)$ to $q_i(v) + a_i$. Therefore

$$g \cdot (z_1, \dots, z_n) = (e^{2\pi\sqrt{-1}a_1} z_1, \dots, e^{2\pi\sqrt{-1}a_n} z_n). \tag{4.10}$$

We may identify G_v with the cokernel of the linear map $\Lambda_{(v)} : N \to N$. Then standard arguments using the Smith normal form of the matrix $\Lambda_{(v)}$ imply that

$$o(G_v) = |\det \Lambda_{(v)}|. \tag{4.11}$$

4.3. **Compatibility of charts.** We show the compatibility of the charts $(\widetilde{X}_v, G_v, \xi_v)$. Let v_1 and v_2 be two vertices so that the minimal face S of P containing both has dimension $s \geq 1$. Then $X_{v_1} \cap X_{v_2}$ is nonempty. Assume facets $(F_1, \dots, F_s, F_{s+1}, \dots, F_n)$ meet at vertex v_1 and facets $(F_{n+1}, \dots, F_{n+s}, F_{s+1}, \dots, F_n)$ meet at v_2. We take

$$\begin{aligned} \Lambda_{(v_1)} &= [\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_n] \text{ and} \\ \Lambda_{(v_2)} &= [\lambda_{n+1}, \dots, \lambda_{n+s}, \lambda_{s+1}, \dots, \lambda_n]. \end{aligned} \tag{4.12}$$

Then

$$\mathbf{q}(v_2) = \Lambda_{(v_2)}^{-1}\Lambda_{(v_1)}\mathbf{q}(v_1) \tag{4.13}$$

Suppose

$$\lambda_k = \sum_{j=s+1}^{n+s} c_{j,k}\lambda_j, \; 1 \leq k \leq s. \tag{4.14}$$

Then by (4.13),

$$\begin{aligned} q_j(v_2) &= \textstyle\sum_{k=1}^s c_{n+j,k}\, q_k(v_1) & &\text{if } 1 \leq j \leq s \\ q_j(v_2) &= \textstyle\sum_{k=1}^s c_{j,k}\, q_k(v_1) + q_j(v_1) & &\text{if } s+1 \leq j \leq n. \end{aligned} \tag{4.15}$$

Let the facets F_j, $j=1,\ldots,n+s$, be defined by $\widehat{p}_j=0$ such that $\widehat{p}_j>0$ in the interior of the polytope P. Then the coordinates (4.8) on $\widetilde{X}_{v_2}$ and $\widetilde{X}_{v_1}$ are related as follows.

$$(4.16)\qquad \begin{array}{ll} z_j(v_2)=\prod_{k=1}^{s} z_k(v_1)^{c_{n+j,k}}\sqrt{\widehat{p}_{n+j}\prod_{k=1}^{s}\widehat{p}_k^{\;-c_{n+j,k}}} & \text{if } 1\le j\le s\\ z_j(v_2)=z_j(v_1)\prod_{k=1}^{s} z_k(v_1)^{c_{j,k}}\sqrt{\prod_{k=1}^{s}\widehat{p}_k^{\;-c_{j,k}}} & \text{if } s+1\le j\le n. \end{array}$$

Take any point $x\in X_{v_1}\bigcap X_{v_2}$. Let $\widetilde{x}$ be a preimage of x with respect to ξ_{v_1}. Suppose $\pi(x)$ belongs to the relative interior of the face $F\subset S$. Suppose F is the intersection of facets $F_{i_1},\ldots,F_{i_t}$ where $s+1\le i_1<\ldots<i_t\le n$. Then the coordinate $z_j(v_1)(\tilde{x})$ is zero if and only if $j\in\mathcal{I}(F)=\{i_1,\ldots,i_t\}$. Consider the isotropy subgroup G_x of $\tilde{x}$ in G_{v_1}. It consists of all elements that do not affect the nonzero coordinates of $\tilde{x}$,

$$(4.17)\qquad G_x=\{g\in G_{v_1}\,:\,g\cdot z_j(v_1)=z_j(v_1)\text{ if } j\notin\mathcal{I}(F)\}$$

It is clear that G_x is independent of the choice of $\tilde{x}$ and

$$(4.18)\qquad G_x=\{[\eta]\in N/N(v_1)\,:\,\eta=\sum_{j\in\mathcal{I}(F)}a_j\lambda_j\}.$$

Note that $j\in\mathcal{I}(F)$ if and only if $\lambda_j\in N(F)$. It follows from the linear independence of $\lambda_1,\ldots,\lambda_n$ that

$$(4.19)\qquad G_x\cong G_F:=((N(F)\otimes_{\mathbb{Z}}\mathbb{Q})\cap N)/N(F).$$

Note that G_P is the trivial group.

Choose a small ball $B(\widetilde{x},r)$ around $\widetilde{x}$ such that $(g\cdot B(\widetilde{x},r))\bigcap B(\widetilde{x},r)$ is empty for all $g\in G_{v_1}-G_x$. Then $B(\widetilde{x},r)$ is stable under the action of G_x and $(B(\widetilde{x},r),G_x,\xi_{v_1})$ is an orbifold chart around x induced by $(\widetilde{X}_{v_1},G_{v_1},\xi_{v_1})$. We show that for sufficiently small value of r, this chart embeds into $(\widetilde{X}_{v_2},G_{v_2},\xi_{v_2})$ as well.

Note that the rational numbers $c_{j,k}$ in (4.14) are integer multiples of $\frac{1}{\Delta}$ where $\Delta=\det(\Lambda_{(v_2)})$. Choose a branch of $z_k(v_1)^{\frac{1}{\Delta}}$ for each $1\le k\le s$, so that the branch cut does not intersect $B(\widetilde{x},r)$. Assume r to be small enough so that the functions $z_k(v_1)^{c_{j,k}}$ are one-to-one on $B(\widetilde{x},r)$ for each $s+1\le j\le n+s$ and $1\le k\le s$. Then equation (4.16) defines a smooth embedding ψ of $B(\widetilde{x},r)$ into $\widetilde{X}_{v_2}$. Note that $\widehat{p}_k$, $1\le k\le s$, and $\widehat{p}_{n+j}$, $1\le j\le s$ are smooth non-vanishing functions on $\xi_{v_1}^{-1}(X_{v_1}\bigcap X_{v_2})$. Let $i_{v_2}:G_x\to G_{v_2}$ be the natural inclusion obtained using equation (4.19). Then $(\psi,i_{v_2}):(B(\widetilde{x},r),G_x,\xi_{v_1})\to(\widetilde{X}_{v_2},G_{v_2},\xi_{v_2})$ is an embedding of orbifold charts.

4.4. **Independence of shape of polytope.**

Lemma 4.3. *Suppose* $\mathbf{X}$ *and* $\mathbf{Y}$ *are quasitoric orbifolds whose orbit spaces* P *and* Q *are diffeomorphic and the characteristic vector of any edge of* P *matches with the characteristic vector of the corresponding edge of* Q*. Then* $\mathbf{X}$ *and* $\mathbf{Y}$ *are equivariantly diffeomorphic.*

Proof. Pick any vertex v of P. For simplicity we will write p_i for $p_i(v)$, and q_i for $q_i(v)$. Suppose the diffeomorphism $f : P_1 \to P_2$ is given near v by $f(p_1, p_2 \ldots, p_n) = (f_1, f_2 \ldots, f_n)$. It induces a map of local charts $\widetilde{X}_v \to \widetilde{Y}_{f(v)}$ by
(4.20)

$$(\sqrt{p_i}\cos(2\pi q_i), \sqrt{p_i}\sin(2\pi q_i)) \mapsto (\sqrt{f_i}\cos(2\pi q_i), \sqrt{f_i}\sin(2\pi q_i)) \quad \text{for } i = 1, \ldots, n.$$

This is a smooth map if the functions $\sqrt{f_i/p_i}$ are smooth functions of $p_1, \ldots, p_n$. Without loss of generality let us consider the case of $\sqrt{f_1/p_1}$. We may write

$$f_1(p_1, p_2 \ldots p_n) = f_1(0, p_2 \ldots p_n) + p_1 \frac{\partial f_1}{\partial p_1}(0, p_2 \ldots p_n) + p_1^2 g(p_1, p_2 \ldots p_n) \tag{4.21}$$

where g is smooth, see section 8.14 of [11]. Note that $f_1(0, p_2 \ldots p_n) = 0$ as f maps the edge $p_1 = 0$ to the edge $f_1 = 0$. Then it follows from equation (4.21) that f_1/p_1 is smooth. We have

$$\frac{f_1}{p_1} = \frac{\partial f_1}{\partial p_1}(0, p_2, \ldots p_n) + p_1 g(p_1, p_2, \ldots p_n) \tag{4.22}$$

Note that $\frac{f_1}{p_1}$ is nonvanishing away from $p_1 = 0$. Moreover we have

$$\frac{f_1}{p_1} = \frac{\partial f_1}{\partial p_1}(0, p_2 \ldots p_n) \text{ when } p_1 = 0. \tag{4.23}$$

Since $f_1(0, p_2 \ldots p_n)$ is identically zero, $\frac{\partial f_1}{\partial p_j}(0, p_2, \ldots, p_n) = 0$ for each $2 \leq j \leq n$. As the Jacobian of f is nonsingular we must have

$$\frac{\partial f_1}{\partial p_1}(0, p_2, \ldots, p_n) \neq 0 \tag{4.24}$$

Thus $\frac{f_1}{p_1}$ is nonvanishing even when $p_1 = 0$. Consequently $\sqrt{f_1/p_1}$ is smooth. Therefore the map (4.20) is smooth and induces an isomorphism of orbifold charts. □

4.5. **Torus action.** An action of a group H on an orbifold $\mathbf{Y}$ is an action of H on the underlying space Y with some extra conditions. In particular for every sufficiently small H-stable neighborhood U in Y with uniformizing system (W, G, π), the action should lift to an action of H on W that commutes with the action of G.

4.6. **Metric.** By a torus invariant metric on $\mathbf{X}$ we will mean a metric on $\mathbf{X}$ which is $T_{N(F)}$-invariant in some uniformizing neighborhood of x for any point $x \in \pi^{-1}(F^\circ)$.

Any cover of X by T_N-stable open sets induces an open cover of P. Choose a smooth partition of unity on the polytope P subordinate to this induced cover. Composing with the projection map $\pi : X \to P$ we obtain a partition of unity on X subordinate to the given cover, which is T_N-invariant. Such a partition of unity is smooth as the map π is smooth, being locally given by maps $p_j = x_j^2 + y_j^2$. For instance, choose a $T_{N(v)}$-invariant metric on each $\widetilde{X}_v$. Then using a partition of unity as above we can define an invariant metric on $\mathbf{X}$.This is a valid orbifold metric because the the uniformizing elements are subset of $T_{N(v)}$.

4.7. **Invariant suborbifolds.** The T_N-invariant subset $X(F) = \pi^{-1}(F)$, where F is a face of P, has a natural structure of a quasitoric orbifold [23]. This structure is obtained by taking F as the polytope for $\mathbf{X}(F)$ and projecting the characteristic vectors to $N/N^*(F)$ where $N^*(F) = (N(F) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap N$. With this structure $\mathbf{X}(F)$ is a suborbifold of $\mathbf{X}$. It is called a characteristic suborbifold if F is a facet. Suppose λ is the characteristic vector attached to the facet F. Then $\pi^{-1}(F)$ is fixed by the circle subgroup $T(\lambda)$ of T_N. We denote the relative interior of a face F by F° and the corresponding invariant space $\pi^{-1}(F^\circ)$ by $X(F^\circ)$. Note that $v^\circ = v$ if v is a vertex.

4.8. **Orientation.** Note that for any vertex v, $dp_i(v) \wedge dq_i(v) = dx_i(v) \wedge dy_i(v)$. Therefore $\omega(v) := dp_1(v) \wedge \ldots \wedge dp_n(v) \wedge dq_1(v) \wedge \ldots \wedge dq_n(v)$ equals $dx_1(v) \wedge \ldots \wedge dx_n(v) \wedge dy_1(v) \wedge \ldots \wedge dy_n(v)$. The standard coordinates $(p_1, \ldots, p_n)$ are related to $(p_1(v), \ldots, p_n(v)$ by a diffeomorphism. The same holds for $(q_1, \ldots, q_n)$ and $(q_1(v), \ldots, q_n(v))$. Therefore $\omega := dp_1 \wedge \ldots \wedge dp_n \wedge dq_1 \wedge \ldots \wedge dq_n$ is a nonzero multiple of each $\omega(v)$. The action of G_v on $\widetilde{X}_v$, see equation (4.10), preserves $\omega(v)$ for each vertex v as $dx_i(v) \wedge dy_i(v) = \frac{\sqrt{-1}}{2} dz_i(v) \wedge d\bar{z}_i(v)$. The action of G_v affects only the angular coordinates. Since $dq_1 \wedge \ldots \wedge dq_n = \det(\Lambda_{(v)}) dq_1(v) \wedge \ldots \wedge dq_n(v)$ and the right hand side is G_v-invariant, we conclude that ω is G_v-invariant. Therefore ω defines a nonvanishing $2n$-form on $\mathbf{X}$. Consequently a choice of orientations for $P \subset \mathbb{R}^n$ and T_N induces an orientation for $\mathbf{X}$.

4.9. **Omniorientation.** An omniorientation is a choice of orientation for the orbifold as well as an orientation for each characteristic suborbifold. At any vertex v, the G_v-representation $\mathcal{T}_0 \widetilde{X}_v$ splits into the direct sum of n G_v-representations corresponding to the normal spaces of $z_i(v) = 0$. Thus we have a decomposition of the orbifold tangent space $\mathcal{T}_v \mathbf{X}$ as a direct sum of the normal spaces of the characteristic suborbifolds that meet at v. Given an omniorientation, we say that the sign of a vertex v is positive if the orientations of $\mathcal{T}_v(\mathbf{X})$ determined by the orientation of $\mathbf{X}$ and orientations of characteristic suborbifolds coincide. Otherwise we say that sign of v is negative. An omniorientation is then said to be positive if each vertex has positive sign.

It is easy to verify that reversing the sign of any number of characteristic vectors does not affect the topology or differentiable structure of the quasitoric orbifold. There is a circle action of T_{λ_i} on the normal bundle of $\mathbf{X}(F_i)$ producing a complex structure and orientation on it. This action and orientation varies with the sign of λ_i. Therefore, given an orientation on $\mathbf{X}$, omniorientations correspond bijectively to choices of signs for the characteristic vectors. We will assume the standard orientations on P and T^n so that omniorientations will be solely determined by signs of characteristic vectors. Also under this choice the sign of v equals the sign of $\det(\Lambda_{(v)})$.

5. **Invariant almost complex structure on quasitoric manifolds**

Kustarev [15] showed that the obstruction to existence of a torus invariant almost complex structure on a quasitoric manifold, which is furthermore orthogonal with respect to a torus invariant metric, reduces to the obstruction to its existence on a section of the orbit map.

Definition 5.1. *An almost complex structure in a manifold is a continuous assignment of linear operators J in the fibers of the tangent bundle such that its restriction to each fiber satisfies $J^2 = -I$. The tangent bundle along with this operator becomes a complex vector bundle.*

Definition 5.2. *An almost complex structure J is torus invariant if $dgJdg^{-1} = J$ for every $g \in T_N$ and dg is the differential of g.This condition makes J commuting with the isotropy subgroups.*

Let X be a positively omnioriented (for definition see the previous section) quasitoric manifold. We give a sketch of the proof of Kustarev which shows existence of an invariant orthogonal almost complex structure on X. Denote the set of all i-dimensional faces of P by $sk_i(P)$. We refer to $\pi^{-1}(sk_i(P))$ as the i-th skeleton of X where$\pi : X \to P$ is the orbit map. We fix an embedding $\iota : P \to X$ that satisfies

$$\begin{aligned} &\pi \circ \iota = id \quad \text{and} \\ &\iota|_{int(F)} \text{ is smooth for any face } F \subset P. \end{aligned} \tag{5.1}$$

A choice of ι is given by the composition $P \xrightarrow{i} P \times T_N \xrightarrow{j} X$ where i is the inclusion given by $i(p_1, \ldots, p_n) = (p_1, \ldots, p_n, 1, \ldots, 1)$ and j is the quotient map that defines X.

Kustarev [15] defined a orthogonal J on the restriction of the tangent bundle of a quasitoric manifold X on the section and then transferred the almost complex structure to whole of X by the T_N action. Let V be a real oriented Euclidean vector space of even dimension. Denote by $J(V)$ the space of all almost complex structures of V preserving orientation and metric. Let F be a face of the polytope. In our case $V = \mathcal{T}(\pi^{-1}(F))_{\iota(x)}$ for a given $x \in F$. Choosing a basis in V identifies $J(V)$ with $SO(2i)/U(i)$ where $i = dim(V)$. In particular $J(V)$ is simply connected. We denote by J_F the bundle with fiber$J(\mathcal{T}(\pi^{-1}(F))_y)$ over $\iota(F)$ where $y \in \iota(F)$. Clearly J_F is trivial.

5.1. **Zero and one skeletons.**

Consider any vertex v of P. It corresponds to a unique T_N-fixed point in X which we denote by v as well. The tangent space $\mathcal{T}_v X$ decomposes into the direct sum of normal spaces at v to the characteristic submanifolds corresponding to the facets incident at v. Each of these normal spaces has an action of the isotropy circle whose direction is specified by the sign of the characteristic vector of the facet. This circle action endows the normal space with a rotation by $\pi/2$, in other words a complex structure. Therefore the omniorientation induces a canonical complex structure J_v on $\mathcal{T}_v X$. We may assume, without loss of generality, that this complex structure is orthogonal with respect to the invariant metric μ on X.

Let E be an edge of P. Suppose E has vertices u and v. Suppose $\{\lambda_1, \ldots, \lambda_{n-1}\}$ is the characteristic set of E. The characteristic manifold corresponding to E is a sphere which we denote by S^2. Let ν denote the normal bundle to S^2. Then $\nu = \oplus_{i=1}^{n-1} \nu_i$ where ν_i is the restriction of the normal bundle to the characteristic submanifold corresponding to the i-th facet incident on E. Note that ν_i supports nontrivial action of the circle $T(\lambda_i)$.

Take any $x \in S^2$. Observe, after Lemma 3.1 of [15], that ν_x and $\mathcal{T}_x S^2$ are orthogonal with respect to the metric. This may be verified as follows. Take any nonzero vectors $\eta_1 \in \nu_x$ and $\eta_2 \in \mathcal{T}_x S^2$. Let θ be an element of the subgroup $\prod T(\lambda_i)$ that acts on ν_x as multiplication by -1. Since $\prod T(\lambda_i)$ acts trivially on $\mathcal{T}_x S^2$, we have $\mu(\eta_1, \eta_2) = \mu(\theta \cdot \eta_1, \eta_2) = \mu(-\eta_1, \eta_2) = 0$. Thus we may split the construction of an orthogonal J into constructions of orthogonal almost complex structures on ν and $\mathcal{T}S^2$.

Similarly, it follows from Lemma 3.1 of [15] that the splitting $\nu = \oplus \nu_i$ is orthogonal. Define the restriction of J to ν_i as rotation by the angle $\frac{\pi}{2}$ with respect to the metric μ as specified by the orientation of ν_i obtained from the $T(\lambda_i)$ action. Then $J|_\nu$ is torus invariant as μ is preserved by torus action.

Recall that the space of all orthogonal complex structures on the oriented vector space $\mathbb{R}^2$ is parametrized by $SO(2)/U(1)$ which is a single point. We orient $\mathcal{T}S^2$ according to the given omniorientation. Consider the path $\iota(E) \in S^2$ given by the embedding ι in (5.1). Since this path is contractible, the restriction of $\mathcal{T}S^2$ on it is trivial. Thus there is a canonical choice of an orthogonal almost complex structure on $\mathcal{T}S^2|_{\iota(E)}$. We want this structure to agree with the complex structures $J_u|_{\mathcal{T}S^2}$ and $J_v|_{\mathcal{T}S^2}$ at the vertices, chosen earlier. This is possible since the omniorientation is positive. Then we use the torus action to define $J|_{\mathcal{T}S^2}$ on each point x in S^2. Find y in $\iota(E)$ and $\theta \in T_N/(\prod T(\lambda_i)$ such that $x = \theta \cdot y$. Then define $J(x) = d\theta \circ J(y) \circ d\theta^{-1}$. This completes the construction of a torus invariant orthogonal J on the 1-skeleton of X.

5.2. **Higher skeletons.** For brevity, from now on we refer to an invariant almost complex structure as structure.

Use induction. Suppose the structure is defined on skeleton sk_{i-1}(P) . Try to extend it to sk_i(P). Note the bundle $\mathcal{T}(\pi^{-1}(F))|_{\iota(F)}$ is trivial(where F is a i dimensional face). Let us fix a point $x \in \iota(F)$ and a trivialization of the bundle $\mathcal{T}(\pi^{-1}(F))|_{\iota(F)}$. Then the bundle J_F is also trivialized. Since J is already defined over $\mathcal{T}(X)|_{\iota(\delta F)}$and $\mathcal{T}(\pi^{-1}(F))|_{\iota(\delta F)} \subset \mathcal{T}(X)|_{\iota(\delta F)}$ is J invariant sub-bundle, we obtain a continuous map $f : \delta F \to J(\mathcal{T}(\pi^{-1}(F))_{\iota(x)}$. Denote by C_F the homotopy class of the spheroid f in $\pi_{i-i}(J(\mathcal{T}(\pi^{-1}(F))_{\iota(x)})$. In [15] this class has been shown to be well defined. The following lemma (Lemma 3.4 [15]) summarizes the above statements.

Lemma 5.1. *Let J be a structure on $i(sk_{i-i}(P))$ respecting the omniorientation. Then J may be extended over $\mathcal{T}((\pi^{-1}(F))_{i(sk_{i-i}(P))\cup F}$ if and only if $C_F = 0$ in the group $\pi_{i-i}(SO(2i)/U(i))$*

Remark 5.2. *The above lemma ensures structure on $\mathcal{T}((\pi^{-1}(F))_{\iota(F)}$. For defining on $\mathcal{T}(X)_{\iota(F)}$ define J on the normal bundle by $\pi/2$ rotation along the characteristic directions*

Every polytope P has a canonical cellular decomposition, with faces playing the role of cells. Kustarev defines cellular cochain $\sigma^i_J \in C^i(P, \pi_{i-i}(SO(2i)/U(i)))$ by the rule $\sigma^i_J(F) = C_F$. By definition σ^i_J is zero if and only if J may be extended from $\iota(sk_{i-1}(P))$ to $sk_i(P)$. The cochain σ^i_J is proved to be well defined and in lemma 3.8 in [15] it is proved to be a cocycle. The following lemma 3.10 in [15] helps us to extend the structure to $i(sk_i(P))$.

Lemma 5.3. *Suppose that J is a structure on $sk_{i-1}(P)$ that respects the omniorientation and σ^i_J is a coboundary. Then one can change J on $\iota(sk_{i-1}(P)) - \iota(sk_{i-2}(P))$ and obtain a new structure $J^{'}$ on $\iota(sk_{i-1}(P))$ such that $\sigma^i_{J'} = 0$*

We extend the structure on $\mathcal{T}(X)|\iota(P)$ in the following manner. We extend to the image of one skeleton as done in the previous sub section due to positivity of the omniorientation. Then we proceed by induction. Let the structure be defined in the image of the $i-1$ skeleton. By lemma 5.1 the obstruction to extension to image of skeleton i is the cocycle σ^i_J. Since the polytope is contractible it is a coboundary. And since it is a coboundary by lemma 5.3 there exists a structure $J^{'}$ on the image of $i-1$ skeleton that can be extended to the image of i skeleton.

Remark 5.4. *Since $SO(2i)/U(i)$ is simply connected for all i there are no obstructions for defining on 2-skeleton as a result the structure on 1-skeleton defined above remains intact.*

Remark 5.5. *A close inspection of lemma5.3 (lemma 3.10 [15]) shows the structure does not change in the normal direction of the face. That is it remains $\pi/2$ rotation along the characteristic directions.*

6. **Invariant almost complex structure in quasitoric orbifolds**

Definition 6.1. *Let $\mathcal{T}\mathbf{X}$ be the tangent bundle of the orbifold $\mathbf{X} = (X,\mathcal{U})$. An almost complex structure J is a base preserving continuous map from $\mathcal{T}\mathbf{X} \to \mathcal{T}\mathbf{X}$ satisfying the following:*

1. *It lifts to a fiberwise linear map $J : \mathcal{T}\tilde{U} \to \mathcal{T}\tilde{U}$ for every chart $(\tilde{U}, G, \phi) \in \mathcal{U}$*
2. *$J^2 \mid (fiber) = -I$*
3. *For every $g \in G$ where $(\tilde{U}, G, \phi) \in \mathcal{U}$, $dgJdg^{-1} = J$*
4. *For any gluing map λ_{12} of orbifold charts, $d\lambda_{12}Jd\lambda_{12}{}^{-1} = J$*

Definition 6.2. *We say that an almost complex structure on a quasitoric orbifold $\mathbf{X}$ is torus invariant if it is $T_{N(F)}$-invariant in some uniformizing neighborhood of each point $x \in X(F^\circ)$.*

Remark 6.1. *This definition gives an orbifold almost structure since the elements g of the uniformizing groups are elements of $T_{N(F)}$.*

Theorem 6.2. *Let $\mathbf{X}$ be a positively omnioriented quasitoric orbifold and μ an invariant metric on it. Then there exists an orthogonal invariant almost complex structure on $\mathbf{X}$ that respects the omniorientation.*

Proof. Consider the subset $R_v \subset \widetilde{X}_v$ consisting of points whose coordinates (4.9) are real and nonnegative,

$$R_v = \{x \in \widetilde{X}_v : z_j(v)(x) \in \mathbb{R}_{\geq} \, \forall 1 \leq j \leq n\} \tag{6.1}$$

In other words,

$$R_v = \{x \in \widetilde{X}_v : z_j(v)(x) = \sqrt{p_j(v)(x)},\ j = 1, \ldots, n\} \tag{6.2}$$

We glue the spaces R_v according to the transition maps (4.16), choosing the branch cuts uniformly as $-\pi < q_k(v) < \pi$. We obtain a manifold with boundary R.

Let x be any point in R_{v_1} such that $\xi_{v_1}(x) \in X_{v_1} \cap X_{v_2}$. Then the transition maps (4.16), with above choice of cuts, define a local diffeomorphism ϕ_{12} from a neighborhood of x in $\widetilde{X}_{v_1}$ to a neighborhood of the image of x in $\widetilde{X}_{v_2}$.

Let $\mathcal{E}_v$ denote the restriction of $\mathcal{T}\widetilde{X}_v$ to R_v. The last paragraph shows that these bundles glue to form a smooth rank $2n$ real vector bundle $\mathcal{E}$ on R. The metric μ on $\mathcal{T}\mathbf{X}$ induces a metric on the bundle $\mathcal{E}$.

The restriction of the quotient map $\xi_v|_{R_v} : R_v \to X_v$ is a homeomorphism onto its image. As a result the space R is homeomorphic to the subspace $\iota(P)$ of X used by Kustarev [15]. The map $\iota : P \to X$ is a homeomorphism given by the composition $P \xrightarrow{i} P \times T_N \xrightarrow{j} X$ where i is the inclusion given by $i(p_1, \ldots, p_n) = (p_1, \ldots, p_n, 1, \ldots, 1)$ and j is the quotient map that defines X. For any face F of P we denote its image in R under the composition of above homeomorphisms as $R(F)$. The restriction of this homeomorphism to the relative interior of F is smooth, and we denote the image by $R(F^\circ)$.

Let $\widetilde{X}_v(F)$ be the preimage of $X(F)$ in $\tilde{X}_v$. If F is the intersection of facets $F_{i_1}, \ldots, F_{i_t}$, then $\widetilde{X}_v(F)$ is the submanifold of $\widetilde{X}_v$ defined by the equations $z_{i_j}(v) = 0$,

$1 \leq j \leq t$. Then arguments similar to the case of $\mathcal{E}$ show that the restrictions $\mathcal{T}\widetilde{X}_v(F)|_{R_v \cap R(F)}$ glue together to produce a subbundle $\mathcal{E}_F$ of $\mathcal{E}|_{R(F)}$.

It is easy to check from (4.16) that

$$\frac{\partial}{\partial z_{i_j}(v_1)} = \frac{\partial}{\partial z_{i_j}(v_2)} \tag{6.3}$$

at any point in $R_{v_1} \cap R_{v_2} \cap R(F)$. Therefore we obtain a subbundle $\mathcal{N}_F$ of $\mathcal{E}|_{R(F)}$ corresponding to the normal bundles of $\widetilde{X}_{F,v}$ in $\widetilde{X}_v$. The bundle $\mathcal{N}_F$ obviously splits into the direct sum of the rank 2 bundles $\mathcal{N}_{F_k}$ where $k \in \mathcal{I}(F) := \{i_1, \ldots, i_t\}$.

Recall the torus $T_{N(F)}$ corresponding to the face F of P from equation (4.1) and definition 4.1. For any vertex v of F, the module $N(F)$ is a direct summand of the module $N(v)$. Consequently, $T_{N(F)}$ injects into $T_{N(v)}$. Suppose x is a point in $R(F^\circ)$. Then $T_{N(F)}$ is the stabilizer of any preimage of x in $\widetilde{X}_v$.

$T_{N(F)}$ is the product of the circles T_{λ_k}, $k \in \mathcal{I}(F)$. The circle T_{λ_k} acts nontrivially on $\mathcal{N}_{F_k}$ and induces an almost complex structure on it corresponding to rotation by $\frac{\pi}{2}$. Note that this structure depends on the sign of λ_k or, in other words, the specific omniorientation. Thus the $T_{N(F)}$ action induces an almost complex structure on $\mathcal{N}_F$.

Using the method of Kustarev [15] it is possible to construct an orthogonal almost complex structure J on $\mathcal{E}$ that satisfies the following condition: $(\star)$ For any face F of P of dimension less than n, the restriction of J to $\mathcal{N}_F|_{R(F^\circ)}$ agrees with the complex structure induced by the $T_{N(F)}$ action and the omniorientation. For future use, we give a brief outline of the proof of existence of such a structure.

An orthogonal almost complex structure on $\mathcal{E}$ may be regarded as a map $J : R \to SO(2n)/U(n)$. We proceed by induction. Let $sk_i(R)$ denote the union of all i-dimensional faces of R. For $i = 0$, existence of J is trivial. Extension to $sk_1(R)$ is possible due to positivity of omniorientation. For $i \geq 2$, suppose J is a structure on $sk_{i-1}(R)$ satisfying the condition $(\star)$. Then J may be regarded as a map from $sk_{i-1}(R)$ to $SO(2i-2)/U(i-1)$ as it is fixed in the normal directions by the torus action. Construct a cellular cochain $\sigma_J^i \in C^i(R, \pi_{i-1}(SO(2i)/U(i))$ by defining the value of σ_J^i on an i-dimensional face of R to be the homotopy class of the value of J on the boundary of the face, composed with a canonical isomorphism between $\pi_{i-1}(SO(2i-2)/U(i-1))$ and $\pi_{i-1}(SO(2i)/U(i))$. J extends to $sk_i(R)$ if and only if $\sigma_J^i = 0$. Following [15], one proves that σ_J^i is a cocycle. Therefore, by contractibility of R it is a coboundary. Suppose $\sigma_J^i = \delta\beta$, where $\beta \in C^{i-1}(R, \pi_{i-1}(SO(2i)/U(i))$. Note that $\delta\beta(Q) = \pm \sum_{G \subset \partial Q} \beta(G)$. For each $H \in sk_{i-1}(R)$, one perturbs J in the interior of H by a factor of $-\beta(H)$. This makes $\sigma_J^i = 0$. (Note that if $\beta(H) = 0$, no change is required for face H. This will be used crucially in Lemma 6.3.)

By $(\star)$ the structure J on $\mathcal{E}_v$ is invariant under the action of isotropy groups. We can therefore use the action of $T_{N(v)}$ to produce an invariant almost complex structure on $\mathcal{T}\widetilde{X}_v$ as follows,

$$J(t \cdot x) = dt \circ J(x) \circ dt^{-1} \; \forall x \in R_v, \text{ and } \forall t \in T_{N(v)} \tag{6.4}$$

The local group G_v of orbifold chart $(\widetilde{X}_v, G_v, \xi_v)$ is a subgroup of $T_{N(v)}$. Thus J is G_v-invariant on $\widetilde{X}_v$.

The compatibility of J across charts may be verified as follows. Take any point $x \in X_{v_1} \cap X_{v_2}$. Let $\widetilde{x} \in \widetilde{X}_{v_1}$ be a preimage of x under ξ_{v_1}. Suppose $\widetilde{x} = t_1 \cdot x_0$ where $x_0 \in R$ and $t_1 \in T_{N(v_1)}$. Choose an embedding $\widetilde{\phi}_{12}$ of a small G_x-stable neighborhood of $\widetilde{x}$ into $\widetilde{X}_{v_2}$ as outlined in section 4.3. Suppose $\widetilde{\phi}_{12}(\widetilde{x}) = t_2 \cdot x_0$ where $t_2 \in T_{N(v_2)}$. Then

$$\widetilde{\phi}_{12} = t_2 \circ \phi_{12} \circ {t_1}^{-1} \tag{6.5}$$

where ϕ_{12} is the local diffeomorphism that describes the gluing of R_{v_1} and R_{v_2} (see page 19).

By construction of J on $\mathcal{E}$, J commutes with $d\phi_{12}|_R$. J commutes with dt_i and dt_i^{-1} by its construction on $\widetilde{X}_{v_i}$. Therefore J commutes with $d\widetilde{\phi}_{12}$, as desired. □

The following result gives us better control on the invariant almost complex structures in the manifold as well as orbifold cases.

Theorem 6.3. *Suppose an orthogonal invariant almost complex structure is given on a characteristic suborbifold* $\mathbf{X}(F)$*. Then it can be extended to* $\mathbf{X}$*.*

Proof. We follow the notation of the previous theorem. J has been already specified on $\mathbf{X}(F)$ where $\dim(F) = n-1$. This determines J on the subbundle $\mathcal{E}_F$ of $\mathcal{E}$ over $R(F)$. We use the torus action and omniorientation to extend J to $\mathcal{E}|_{R(F)}$.

We construct an extension of J to R skeleton-wise. Extension up to $sk_1(R) \cup F$ is achieved using positivity of omniorientation. For extension to higher skeletons we need to use obstruction theory. We need to take care so that J is preserved on sub-faces of F. We use induction. Suppose J has been extended to $sk_{d-1}(R) \cup F$, where $d < n$. (We will deal with the $d = n$ case separately.)

Let $\sigma^d \in C^d(R, \pi_{d-1}(SO(2d)/U(d)))$ be the obstruction cocycle. Let $i : R(F) \hookrightarrow R$ be inclusion map. Restriction to F produces a cochain

$$i^*(\sigma^d) \in C^d(R(F), \pi_{d-1}(SO(2d)/U(d))).$$

Then $i^*(\sigma^d) = 0$ since we know that J extends to $R(F)$. Since $\sigma^d = \delta\beta$, $i^*(\beta)$ is a cocycle. As $R(F)$ is contractible $i^*(\beta)$ is a coboundary. Let $i^*(\beta) = \delta\beta_1$ where $\beta_1 \in C^{d-2}(R(F))$. Define a chain $\beta_2 \in C^{d-2}(R)$ such that

$$\beta_2(H) = \begin{cases} \beta_1(H) & \text{for any } (d-2) \text{ face } H \subset R(F) \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

Then define $\beta_3 = \beta - \delta(\beta_2)$. This new cochain has the property that $\delta(\beta_3) = \sigma^d$ and its action on $(d-1)$-dimensional faces of $R(F)$ is zero. So we can now extend the structure to $sk_d \cup R(F)$ without affecting the sub-faces of $R(F)$.

By induction, we may assume that J has been extended to $sk_{n-1}(R) \cup R(F)$. Let $\sigma^n \in C^n(R, \pi_{n-1}(SO(2n)/U(n))$ be the corresponding obstruction cochain for extension to sk_n. Since R is contractible we have $\sigma^n = \delta\beta$. We modify β as follows.

Suppose K is a facet adjacent to F. Define $\beta' \in C^{n-1}$ as follows.

$$\beta'(H) = \begin{cases} 0 & \text{if } H = R(F) \\ \beta(R(F)) + \beta(R(K)) & \text{if } H = R(K) \\ \beta(H) & \text{otherwise} \end{cases} \tag{6.7}$$

Then $\delta\beta' = \delta\beta = \sigma^n$ and $\beta'(R(F)) = 0$. So we may extend J to R without changing it on $R(F)$. □

Corollary 6.4. *Suppose an orthogonal invariant almost complex structure is given on a suborbifold* $\mathbf{X}(F)$ *where* F *is any face of* P*. Then it can be extended to* $\mathbf{X}$.

Proof. Consider a nested sequence of faces $F = H_0 \subset H_1 \ldots \subset H_k = P$ where $\dim(H_i) = \dim(F) + i$. Extend the structure inductively from $\mathbf{X}(H_i)$ to $\mathbf{X}(H_{i+1})$ using Theorem 6.3. □

7. Blowdowns

Topologically the blowup will correspond to replacing an invariant suborbifold by the projectivization of its normal bundle. Combinatorially we replace a face by a facet with a new characteristic vector. Suppose F is a face of P. We choose a hyperplane $H = \{\widehat{p}_0 = 0\}$ such that $\widehat{p}_0$ is negative on F and $\widehat{P} := \{\widehat{p}_0 > 0\} \cap P$ is a simple polytope having one more facet than P. Suppose $F_1, \ldots, F_m$ are the facets of P. Denote the facets $F_i \cap \widehat{P}$ by F_i without confusion. Denote the extra facet $H \cap P$ by F_0.

Without loss of generality let $F = \bigcap_{j=1}^{k} F_j$. Suppose there exists a primitive vector $\lambda_0 \in N$ such that

$$\lambda_0 = \sum_{j=1}^{k} b_j \lambda_j,\ b_j > 0 \,\forall\, j. \tag{7.1}$$

Then the assignment $F_0 \mapsto \lambda_0$ extends the characteristic function of P to a characteristic function $\widehat{\Lambda}$ on $\widehat{P}$. Denote the omnioriented quasitoric orbifold derived from the model $(\widehat{P}, \widehat{\Lambda})$ by $\mathbf{Y}$.

Consider a small open neighborhood $U := \{x \in P : \widehat{p}_0(x) < \epsilon\}$ of the face F, where $0 < \epsilon < 1$. Denote $U \cap \widehat{P}$ by $\widehat{U}$. By lemma 4.3 we may assume that

$$f : U = F \times [0,1)^k \tag{7.2}$$

We also assume without loss of generality that the defining function $\widehat{p}_j$ of the facet F_j equals the j-th coordinate p_j of $\mathbb{R}^n$ on U, for each $1 \le j \le k$.

Choose small positive numbers $\epsilon_1 < \epsilon_2 < \epsilon$ and a smooth non-decreasing function $\delta : [0, \infty) \to \mathbb{R}$ such that

$$\delta(t) = \begin{cases} t & \text{if } t < \epsilon_1 \\ 1 & \text{if } t > \epsilon_2 \end{cases} \tag{7.3}$$

Then define $\tau : \widehat{P} \to P$ to be the map given by

$$\tau(p_1, \ldots, p_k, p_{k+1}, \ldots, p_n) = (\delta(\widehat{p}_0)^{b_1} p_1, \ldots, \delta(\widehat{p}_0)^{b_k} p_k, p_{k+1}, \ldots, p_n). \tag{7.4}$$

The blow down map $\rho : (\widehat{P} \times T_N / \sim) \to (P \times T_N / \sim)$ is defined by

$$\rho(\mathbf{p}, \mathbf{q}) = (\tau(\mathbf{p}), \mathbf{q}). \tag{7.5}$$

Since $\delta = 1$ if $\widehat{p}_0 > \epsilon_2$, ρ is a diffeomorphism of orbifolds away from a tubular neighborhood of $X(F)$. We study the map ρ near $X(F)$.

Let $w = \bigcap_{j=1}^{n} F_j$ be a vertex of F. Suppose v be a vertex of F_0 such that $\tau(v) = w$. Then the edge joining v and w is the intersection of $n-1$ facets common to both which must include $F_{k+1}, \ldots, F_n$. Therefore there are k choices for v, namely $v_i = \bigcap_{0 \le j \ne i \le n} F_j$ with $1 \le i \le k$.

Let $\widehat{p}_j = 0$ be the defining equation of the facet F_j for $k+1 \le j \le n$. Order the facets at w as $F_1, \ldots, F_n$, and those at v_i as $F_1, \ldots, F_{i-1}, F_0, F_{i+1}, \ldots, F_n$. Let $z_j(w)$ and $z_j(v_i)$ be the coordinates on $\widetilde{X}_w$ and $\widetilde{Y}_{v_i}$ defined according to (4.8) and (4.9).

Then by using a process similar to the one used for (4.16), we obtain the following description of ρ near Y_{v_i},

$$\begin{array}{lll} & z_i(w)\circ\rho = z_i(v_i)^{b_i}\sqrt{p_i\delta(\widehat{p}_0)^{b_i}(\widehat{p}_0)^{-b_i}} & \\ (7.6) & z_j(w)\circ\rho = z_i(v_i)^{b_j}z_j(v_i)\sqrt{\delta(\widehat{p}_0)^{b_j}(\widehat{p}_0)^{-b_j}} & \text{if } 1\le j\ne i\le k \\ & z_j(w)\circ\rho = z_j(v_i) & \text{if } k+1\le j\le n \end{array}$$

We define a new coordinate system on $\widetilde{Y}_{v_i}$, for each $1 \le i \le k$, as follows.

$$\begin{array}{lll} & z'_i(v_i) = z_i(v_i)(\sqrt{p_i})^{1/b_i}\sqrt{\delta(\widehat{p}_0)(\widehat{p}_0)^{-1}} & \\ (7.7) & z'_j(v_i) = z_j(v_i)(\sqrt{p_i})^{-b_j/b_i} & \text{if } 1\le j\ne i\le k \\ & z'_j(v_i) = z_j(v_i) & \text{if } k+1\le j\le n \end{array}$$

This is a valid change of coordinates as p_i is positive on $\widetilde{Y}_{v_i}$ and $\delta(\widehat{p}_0)(\widehat{p}_0)^{-1}$ is identically one near $\widehat{p}_0 = 0$.

In these new coordinates, ρ can be expressed as

$$\begin{array}{lll} & z_i(w)\circ\rho = z'_i(v_i)^{b_i} & \\ (7.8) & z_j(w)\circ\rho = z'_i(v_i)^{b_j}z'_j(v_i) & \text{if } 1\le j\ne i\le k \\ & z_j(w)\circ\rho = z'_j(v_i) & \text{if } k+1\le j\le n \end{array}$$

Lemma 7.1. *The restriction $\rho : \mathbf{Y} - \mathbf{Y}(F_0) \to \mathbf{X} - \mathbf{X}(F)$ is a diffeomorphism of orbifolds.*

Proof. This is obvious outside $\pi^{-1}(U)$. On $\pi^{-1}(U) - X(F)$, by formula (7.8), ρ is locally equivalent to a blowup in complex geometry. Therefore ρ is an analytic isomorphism on $\pi^{-1}(U) - X(F)$. However since our quasitoric orbifolds are primitive, there is no complex reflection in our orbifold groups. Hence using the results of [21], analytic isomorphism yields diffeomorphism of orbifolds. □

Lemma 7.2. *If $\mathbf{X}$ is positively omnioriented, then so is a blowup $\mathbf{Y}$.*

Proof. Note that the matrix $\Lambda_{(v_i)}$ is obtained by replacing the i-th column of $\Lambda_{(w)}$, namely λ_i, by $\lambda_0 = \sum_{j=1}^k b_j\lambda_j$. Therefore $\det\Lambda_{(v_i)} = b_i \det\Lambda_{(w)}$. □

Definition 7.1. *A blowdown ρ is said to be a resolution if for any vertex w of the exceptional face F and any vertex $v_i \in \rho^{-1}(F)$ we have $o(G_{v_i}) < o(G_w)$.*

Lemma 7.3. *A blowdown ρ is a resolution if $b_i < 1$ for each i.*

Proof. The lemma holds since by (4.11) we have $o(G_{v_i}) = |\det\Lambda_{(v_i)}| = b_i|\det\Lambda_{(w)}| = b_i o(G_w)$. □

7.1. Pseudoholomorphic blowdowns.

Lemma 7.4. *Let $\rho : Y \to X$ be a blowdown along a subset $X(F)$. Suppose there exist holomorphic coordinate systems $z_1^*(w), \ldots, z_n^*(w)$ on the uniformizing chart $\widetilde{X}_w$ for every vertex w of F, which produce an analytic structure on a neighborhood $\pi^{-1}(U)$ of $X(F)$. Assume further that this analytic structure extends to an almost complex structure on $\mathbf{X}$. Then the blowup induces an almost complex structure on $\mathbf{Y}$ which* *is analytic near the exceptional set $Y(F_0)$. Moreover, with respect to these structures ρ is analytic near $Y(F_0)$ and an almost complex diffeomorphism of orbifolds away from $Y(F_0)$.*

Proof. Note that for two vertices w_1, w_2 of F, the coordinates must be related as

$$z_j^*(w_2) = \prod_{i=1}^{n} z_i^*(w_1)^{d_{ij}} \tag{7.9}$$

where the d_{ij}s are rational numbers determined from the matrix $\Lambda_{(w_2)}^{-1}\Lambda_{(w_1)}$, see (4.13) and (4.16).

Also the coordinates $z_j^*(w)$ have to relate to the coordinates defined in (4.8) and (4.9) as follows,

$$z_j^*(w) = z_j(w)f_j, \; 1 \leq j \leq n \tag{7.10}$$

where each f_j is smooth and non-vanishing on $\widetilde{X}_w$. For each $v_i \in \rho^{-1}(w)$ we define coordinates in its neighborhood, by modifying the coordinates of (7.7) as follows,

$$\begin{array}{ll} z_i^*(v_i) = z_i'(v_i)(f_i \circ \tau)^{1/b_i} & \\ z_j^*(v_i) = z_j'(v_i)(f_j \circ \tau)(f_i \circ \tau)^{-b_j/b_i} & \text{if } 1 \leq j \neq i \leq k \\ z_j^*(v_i) = z_j'(v_i) & \text{if } k+1 \leq j \leq n \end{array} \tag{7.11}$$

In these coordinates ρ takes the following form near v_i,

$$\begin{array}{ll} z_i^*(w) \circ \rho = z_i^*(v_i)^{b_i} & \\ z_j^*(w) \circ \rho = z_i^*(v_i)^{b_j} z_j^*(v_i) & \text{if } 1 \leq j \neq i \leq k \\ z_j^*(w) \circ \rho = z_j^*(v_i) & \text{if } k+1 \leq j \leq n \end{array} \tag{7.12}$$

We define an almost complex structure $\widehat{J}$ on $\mathbf{Y}$ by defining the coordinates $z_j^*(v_i)$ to be holomorphic near $Y(F)$ and by $\widehat{J} = d\rho^{-1} \circ J \circ d\rho$ away from it. This is consistent as ρ is a diffeomorphism of orbifolds on the complement of Y_F.

By (7.9) and (7.12), for any two vertices u_1 and u_2 of F_0, we have

$$z_j^*(u_2) = \prod_{i=1}^{n} z_i^*(u_1)^{e_{ij}} \tag{7.13}$$

for some rational numbers e_{ij}. But these numbers are determined by the matrix $\Lambda_{(u_2)}^{-1}\Lambda_{(u_1)}$. It is then obvious from the arguments about compatibility of charts in section 4.2 that the patching of the charts Y_{u_1} and Y_{u_2} is holomorphic. □

Examples of blowdowns that satisfy the hypothesis of lemma 7.4 include blowdowns of four dimensional positively omnioriented quasitoric orbifolds constructed in [13] and toric blow-ups of simplicial toric varieties.

Definition 7.2. *A function f on X is said to be smooth if $f \circ \xi$ is smooth for every uniformizing system $(\widetilde{U}, G, \xi)$. A complex valued smooth function f on an almost complex orbifold $(\mathbf{X}, J)$ is said to be J-holomorphic if the differential $d(f \circ \xi)$ commutes with J for every chart $(\widetilde{U}, G, \xi)$. We denote the sheaf of J-holomorphic functions on $\mathbf{X}$ by $\Omega^0_{J,X}$. A continuous map $\rho : Y \to X$ between almost complex* *orbifolds* $(\mathbf{Y}, J_2)$ *and* $(\mathbf{X}, J_1)$ *is said to be pseudo-holomorphic if* $f \circ \rho \in \Omega^0_{J_2,Y}(\rho^{-1}(U))$ *for every* $f \in \Omega^0_{J_1,X}(U)$ *for any open set* $U \subset X$*; that is,* ρ *pulls back pseudo-holomorphic functions to pseudo-holomorphic functions.*

Lemma 7.5. *Blowdowns that satisfy the hypothesis of lemma 7.4 are pseudoholomorphic.*

Proof. Suppose $\rho : Y \to X$ is such a blowdown. Since ρ is an almost complex diffeomorphism of orbifolds away from the exceptional set $Y(F_0)$, it suffices to check the statement near $Y(F_0)$. Pick any vertex w of F. Define $W = X_w \cap \pi^{-1}(U)$. For any vertex $v_i \in \rho^{-1}(w)$, let $V_i = Y_{v_i} \cap \rho^{-1}(\pi^{-1}(U))$. We will denote the characteristic vectors at v_i by $\widehat{\lambda}_j$, $j = 1, \dots, n$. Note that

$$\widehat{\lambda}_j = \begin{cases} \lambda_j & \text{if } j \neq i \\ \lambda_0 & \text{if } j = i. \end{cases} \tag{7.14}$$

The ring $\Omega^0_{J_1,X}(W)$ is the G_w-invariant subring of convergent power series in variables $z^*_j(w)$. It is generated by monomials of the form

$$f = \prod_{j=1}^{n} z^*_j(w)^{d_j} \tag{7.15}$$

where the d_js are integers such that $\sum a_j d_j$ is an integer whenever the vector $\sum a_j \lambda_j \in N$. This last condition follows from invariance under action of the element $g \in G_w$ corresponding to $\sum a_j \lambda_j$.

Using (7.12) and $\lambda_0 = \sum_{j=1}^n b_j \lambda_j$ with $b_j = 0$ for $j \geq k+1$, we get

$$f \circ \rho = z^*_i(v_i)^{\sum b_j d_j} \prod_{j \neq i} z^*_j(v_j)^{d_j}. \tag{7.16}$$

Take any element h in G_{v_i}. Suppose h is represented by $\sum c_j \widehat{\lambda}_j \in N$. The action of h on $f \circ \rho$ is multiplication by $e^{2\pi\sqrt{-1}\alpha}$, where

$$\alpha = c_i \sum_j b_j d_j + \sum_{j \neq i} c_j d_j = c_i b_i d_i + \sum_{j \neq i} (c_j + c_i b_j) d_j \tag{7.17}$$

Note that $\eta := c_i b_i \lambda_i + \sum_{j \neq i} (c_j + c_i b_j) \lambda_j = c_i \sum_j b_j \lambda_j + \sum_{j \neq i} c_j \lambda_j = \sum c_j \widehat{\lambda}_j$. Hence this is an element of N.

Suppose f is a generator of $\Omega^0_{J_1,X}(W)$ as in (7.15). Consider the action of the element of G_w corresponding to η on f. It is multiplication by $e^{2\pi\sqrt{-1}\alpha}$. Since f is G_w-invariant, α is an integer. Hence $f \circ \rho$ is G_{v_i} invariant. The ring $\Omega^0_{J_1,Y}(V_i)$ is the G_{v_i}-invariant subring of convergent power series in variables $z^*_j(v_i)$. Therefore $f \circ \rho \in \Omega^0_{J_1,Y}(V_i)$. □

The proof of the following corollary of lemma 7.4 is straightforward.

Corollary 7.6. *Consider a sequence of blowups* $\rho_i : Y_i \to Y_{i-1}$ *where* $1 \leq i \leq r$ *and* ρ_1 *satisfies the hypothesis of lemma 7.4. Assume that the locus of the* i*-th blowup is contained in the exceptional set of the* $(i-1)$*-st blowup for every* i*. Then we* *can inductively choose almost complex structures so that each blowdown map in the sequence is pseudoholomorphic.*

Proposition 7.7. *Suppose* $\mathbf{X}$ *is a* 4*-dimensional positively omnioriented primitive quasitoric orbifold. Then* $\mathbf{X}$ *admits an invariant almost complex structure that is analytic in the neighborhood of every vertex (i.e.* T_N*-fixed point).*

Proof. We also fix a torus invariant metric μ on $\mathbf{X}$ as follows. Choose an open cover of P such that each vertex of P has a neighborhood which is contained in exactly one open set of the cover. This induces a cover of X. On each open set W_v of this cover, corresponding to the vertex v, choose the standard metric with respect to the coordinates in (4.8). On the remaining open sets, choose any T_N-invariant metric. Then use a T_N-invariant partition of unity, subordinate to the cover, to glue these metrics and obtain μ.

Choose small orbifold charts $(\widetilde{X}'_v, G_v, \xi_v)$ around each vertex v where $X'_v \subset W_v$. Choose coordinates $x_i(v)$, $y_i(v)\, i = 1, 2$ on $\widetilde{X}'_v$ according to (4.8). Declare $z_i(v) = x_i(v) + \sqrt{-1}y_i(v)$. Choose the standard complex structure J_v on $\widetilde{X}'_v$ with respect to these coordinates, i.e. $z_1(v), z_2(v)$ are holomorphic coordinates under J_v. Since J_v commutes with action of $T_{N(v)}$ and G_v is a subgroup of $T_{N(v)}$, we may regard J_v as a torus invariant complex structure on a neighborhood of v in the orbifold $\mathbf{X}$. Note that these local complex structures are orthogonal with respect to μ near the vertices.

Recall the embedding $\iota : P \to X$ and the corresponding decomposition of X into skeletons. The construction of almost complex structure on the first skeleton proceeds exactly as described in section 5.1. Details may be found in [13].

Finally, choose a simple loop γ in P that goes along the edges for the most part but avoids the vertices. By the previous step of our construction, J is given on $\iota(\gamma)$. Let D be the disk in P bounded by γ. The set $X_0 := \pi^{-1}(D) \subset X$ is a smooth manifold with boundary. The restriction of $\mathcal{T}X_0$ to $\iota(D)$ is a trivial vector bundle. Fix a trivialization. Recall that the space of all orthogonal complex structures on oriented vector space $\mathbb{R}^4$, up to isomorphism, is homeomorphic to $SO(4)/U(2)$. This is a simply connected space. Thus J may be extended from $\iota(\gamma)$ to $\iota(D)$. Then we produce a T_N-invariant orthogonal J on $\mathcal{T}X_0$ by using the T_N action. This completes the proof. ∎

Theorem 7.8. *There exists a pseudoholomorphic resolution of singularity for any primitive positively omnioriented four dimensional quasitoric orbifold.*

Proof. For any primitive positively omnioriented four dimensional quasitoric orbifold, Proposition 7.7 produces an almost complex structure that satisfies the hypothesis of lemma 7.4 for every vertex. The singularities are all cyclic and concentrated at the vertices. We can resolve them by applying a sequence of blow-ups as in corollary 7.6. ∎

8. Crepant blowdowns

Definition 8.1. *A blowdown is called crepant if* $\sum b_j = 1$.

This has the following geometric interpretation.

Definition 8.2. *Given an almost complex* $2n$*-dimensional orbifold* $(\mathbf{X}, J)$*, we define the canonical sheaf* K_X *to be the sheaf of continuous* $(n,0)$*-forms on* X*; that is, for any orbifold chart* $(\widetilde{U}, G, \xi)$ *over an open set* $U \subset X$*,* $K_X(U) = \Gamma(\wedge^n \mathcal{T}^{1,0}(\widetilde{U})^*)^G$ *where* Γ *is the functor that takes continuous sections.*

An almost complex orbifold is called Gorenstein or SL orbifold if the linearization of every local group element g belongs to $SL(n,\mathbb{C})$. For an SL-orbifold $\mathbf{X}$, the canonical sheaf is a complex line bundle over X.

Lemma 8.1. *Suppose* $\rho : Y \to X$ *is a pseudoholomorphic blowdown of* SL *quasitoric orbifolds along a face* F *satisfying the hypothesis of lemma 7.4. Then* ρ *is crepant if and only if* $\rho^* K_X = K_Y$.

Proof. We consider the canonical sheaf K_X as a sheaf of modules over the sheaf of continuous functions $\mathcal{C}^0_X$. Since ρ is an almost complex diffeomorphism away from the exceptional set it suffices to check the equality of the $\rho^* K_Y$ and K_X on the neighborhood $\rho^{-1}(\pi^{-1}(U)) \subset Y$ of the exceptional set. Choose any vertex w of F. On $X_w \cap \pi^{-1}(U)$, the sheaf K_X is generated over the sheaf $\mathcal{C}^0_X$ by the form $dz_1^*(w) \wedge \ldots \wedge dz_n^*(w)$, see (7.10). Let v_i be any preimage of w under ρ. Similarly on $Y_{v_i} \cap \rho^{-1}(\pi^{-1}(U))$, K_Y is generated over the sheaf $\mathcal{C}^0_Y$ by the form $dz_1^*(v_i) \wedge \ldots \wedge dz_n^*(v_i)$.

Using (7.12) we have

$$
\begin{aligned}
&\rho^* dz_i^*(w) = b_i z_i^*(v_i)^{b_i - 1} dz_i^*(v_i) && \\
&\rho^* dz_j^*(w) = z_i^*(v_i)^{b_j} dz_j^*(v_i) + b_j z_i^*(v_i)^{b_j - 1} z_j^*(v_i) dz_i^*(v_i) && \text{if } 1 \le j \neq i \le k \\
&\rho^* dz_j^*(w) = dz_j^*(v_i) && \text{if } k+1 \le j \le n
\end{aligned}
\tag{8.1}
$$

Therefore we have

$$
\rho^*(dz_1^*(w) \wedge \ldots \wedge dz_n^*(w)) = b_i z_i^*(v_i)^{b_1 + \ldots + b_k - 1} dz_1^*(v_i) \wedge \ldots \wedge dz_n^*(v_i) \tag{8.2}
$$

The lemma follows. ∎

9. **Chen-Ruan Cohomology**

The Chen-Ruan cohomology group is built out of the ordinary cohomology of certain copies of singular strata of an orbifold called twisted sectors. The twisted sectors of orbifold toric varieties was computed in [22]. The determination of such sectors for quasitoric orbifolds is similar in essence. Another important feature of Chen-Ruan cohomology is the grading which is rational in general. In our case the grading will depend on the omniorientation.

Let $\mathbf{X}$ be an omnioriented quasitoric orbifold. Consider any element g of the group G_F (4.19). Then g may be represented by a vector $\sum_{j \in \mathcal{I}(F)} a_j \lambda_j$. We may restrict a_j to $[0, 1) \cap \mathbb{Q}$. Then the above representation is unique. Then define the degree shifting number or age of g to be

$$\iota(g) = \sum a_j. \tag{9.1}$$

For faces F and H of P we write $F \leq H$ if F is a sub-face of H, and $F < H$ if it is a proper sub-face. If $F \leq H$ we have a natural inclusion of G_H into G_F induced by the inclusion of $N(H)$ into $N(F)$. Therefore we may regard G_H as a subgroup of G_F. Define the set

$$G_F^{\circ} = G_F - \bigcup_{F<H} G_H \tag{9.2}$$

Note that $G_F^{\circ} = \{\sum_{j \in \mathcal{I}(F)} a_j \lambda_j | 0 < a_j < 1\} \cap N$, and $G_P^{\circ} = G_P = \{0\}$.

Definition 9.1. *We define the Chen-Ruan orbifold cohomology of an omnioriented quasitoric orbifold* $\mathbf{X}$ *to be*

$$H^*_{CR}(\mathbf{X}, \mathbb{R}) = \bigoplus_{F \leq P} \bigoplus_{g \in G_F^{\circ}} H^{*-2\iota(g)}(X(F), \mathbb{R}).$$

Here H^* refers to singular cohomology or equivalently to de Rham cohomology of invariant forms when $X(F)$ is considered as the orbifold $\mathbf{X}(F)$. The pairs $(X(F), g)$ where $F < P$ and $g \in G_F^{\circ}$ are called twisted sectors of $\mathbf{X}$. The pair $(X(P), 1)$, i.e. the underlying space X, is called the untwisted sector. We denote the Betti number rank$(H^d_{CR}(\mathbf{X}))$ by h^d_{CR}.

9.1. **Poincaré duality.** Poincaré duality is established in a similar fashion as for compact almost complex orbifolds. We need to distinguish the copies of $X(F)$ corresponding to different twisted sectors. Therefore for $g \in G_F^{\circ}$, we define the space

$$S(F, g) = \{(x, g) : x \in X(F)\}. \tag{9.3}$$

Of course $S(F, g)$ is homeomorphic to $X(F)$. It is denoted by $\mathbf{S}(F, g)$ when endowed with an orbifold structure which is the structure of $\mathbf{X}(F)$ with an additional trivial action of G_F at each point. With this structure, it is a suborbifold of $\mathbf{X}$ in a natural way. The untwisted sector is denoted by $S(P, 1)$. In this notation the Chen-Ruan groups may be written as

$$H^*_{CR}(\mathbf{X}, \mathbb{R}) = \bigoplus_{F \leq P} \bigoplus_{g \in G_F^{\circ}} H^{*-2\iota(g)}(S(F, g), \mathbb{R}) \tag{9.4}$$

Lemma 9.1. *Suppose $g \in G_F^\circ$. Then $2\iota(g) + 2\iota(g^{-1}) = 2n - \dim(X(F))$.*

Proof. When $F = P$, $G_P^\circ = \{0\}$ and the result is obvious. Suppose $F = \bigcap_{i=1}^k F_i$. Then $g = \sum_{i=1}^k a_i\lambda_i$ where each $0 < a_i < 1$. Then g^{-1} is represented by the vector $\sum_{i=1}^k -a_i\lambda_i$ in N modulo $N(F)$. Therefore g^{-1} may be identified with the vector $\sum_{i=1}^k (1-a_i)\lambda_i$. Note that $0 < 1 - a_i < 1$ for each i. Therefore the age of g^{-1}, $\iota(g^{-1}) = \sum_{i=1}^k (1-a_i)$. Hence $2\iota(g) + 2\iota(g^{-1}) = 2\sum_{i=1}^k a_i + 2\sum_{i=1}^k (1-a_i) = 2k = 2n - \dim(X(F))$. □

For any compact orientable orbifold, there exists a notion of orbifold integration $\int^{orb}$ for invariant top dimensional forms which gives Poincaré duality for the de Rham cohomology of the orbifold, see [8]. For a chart $\mathbf{U} = (\widetilde{U}, G, \xi)$ orbifold integration for an invariant form ω on $\widetilde{U}$ is defined by

$$\int_{\mathbf{U}}^{orb} \omega = \frac{1}{o(G)} \int_{\widetilde{U}} \omega. \tag{9.5}$$

Let $I : \mathbf{S}(F, g) \to \mathbf{S}(F, g^{-1})$ be the diffeomorphism of orbifolds defined by $I(x, g) = (x, g^{-1})$. We define a bilinear pairing

$$\langle , \rangle_{(F,g)}^{orb} : H^{d-2\iota(g)}(S(F,g)) \times H^{2n-d-2\iota(g^{-1})}(S(F,g^{-1})) \tag{9.6}$$

for every $0 \le d \le 2n$ by

$$\langle \alpha, \beta \rangle_{(F,g)}^{orb} = \int_{\mathbf{S}(F,g)}^{orb} \alpha \wedge I^*(\beta). \tag{9.7}$$

This pairing is nondegenerate because of lemma 9.1. By taking a direct sum of the pairing (9.6) over all pairs of sectors $((F, g), (F, g^{-1}))$ for $F \le P$, we get a nonsingular pairing for each $0 \le d \le 2n$

$$\langle , \rangle^{orb} : H_{CR}^d(\mathbf{X}) \times H_{CR}^{2n-d}(\mathbf{X}). \tag{9.8}$$

10. Ring structure of Chen-Ruan cohomology

We will follow [7] and define the structure of an associative ring on Chen-Ruan cohomology of an omnioriented quasitoric orbifold.

The normal bundle of a characteristic suborbifold has an almost complex structure determined by the omniorientation. More generally suppose $F = \bigcap_{i=1}^{k} F_i$ is an arbitrary face of P. The normal bundle of the suborbifold $\mathbf{S}(F, g)$, see section 9.1, decomposes into the direct sum of complex orbifold line bundles L_i which are restrictions of the normal bundles corresponding to facets F_i that contain F. Each of these line bundles L_i have a Thom form θ_i. (Note that the Thom forms of $\mathbf{X}(F)$ and $\mathbf{S}(F, g)$ in $\mathbf{X}$ may differ at most by a constant factor.) For any $g = \sum_{0 \leq i \leq k} a_i \lambda_i \in Box_F^{\circ}$ define the formal form (twist factor)

$$t(g) = \prod_{1 \leq i \leq k} \theta_i^{a_i}. \tag{10.1}$$

The order of the θ_is in the above product is not important. The degree of $t(g)$ is defined to be $2\iota(g)$. For any invariant form ω on $\mathbf{S}(F, g)$ define a corresponding twisted form $\omega t(g)$. Define the degree of $\omega t(g)$ to be the sum of the degrees of ω and $t(g)$. Define

$$\Omega_{CR}^{p}(F, g) = \{\omega t(g) \mid \omega \in \Omega^{*}(\mathbf{S}(F, g)), \deg(\omega t(g)) = p\}. \tag{10.2}$$

Define the de Rham complex of twisted forms by

$$\Omega^{p}{}_{CR} = \bigoplus_{F \leq P, g \in Box_F^{\circ}} \Omega_{CR}^{p}(F, g) \tag{10.3}$$

with differential

$$d(\sum \omega_i t(g_i)) = \sum d(\omega_i) t(g_i). \tag{10.4}$$

It is easy to see that the cohomology of this complex coincides with the Chen-Ruan cohomology defined in section 9.

Now we define a product $\star : \Omega_{CR}^{p_1}(K_1, g_1) \times \Omega_{CR}^{p_2}(K_2, g_2) \to \Omega_{CR}^{p_1+p_2}(K, g_1 g_2)$ of twisted forms as follows,

$$\omega_1 t(g_1) \star \omega_2 t(g_2) = i_1^{*}\omega_1 \wedge i_2^{*}\omega_2 \wedge \Theta(g_1, g_2) t(g_1 g_2). \tag{10.5}$$

Here K is the unique face such that $(K_1 \cap K_2) \leq K$ and $g_1 g_2 \in G_K^{\circ}$. The map i_j is the inclusion of $\mathbf{X}(K_1 \cap K_2)$ in $\mathbf{X}(K_j)$. The form $\Theta(g_1, g_2)$ is obtained as follows.

Consider the product $t(g_1)t(g_2)$. We can think of the g_js as elements of Box_v where v is a vertex of $K_1 \cap K_2$. Write $g_j = \sum_{i=1}^{n} a_{ij}\lambda_i$. Write the twist factor $t(g_j)$ as $\prod_{1 \leq i \leq n} \theta_i^{a_{ij}}$. A term in the product $t(g_1)t(g_2)$ looks $\theta_i^{a_{i1}+a_{i2}}$. We may ignore the i's for which both a_{i1} and a_{i_2} are zero. Then there can be three cases:

1. $a_{i1} + a_{i2} < 1$. Then $\theta_i^{a_{i1}+a_{i2}}$ contributes to $t(g_1 g_2)$.
2. $a_{i1} + a_{i2} > 1$. Then fractional part $\theta_i^{a_{i1}+a_{i2}-1}$ contributes to $t(g_1 g_2)$ and the integral part is the Thom form θ_i which contributes as an invariant 2-form to $\Theta(g_1, g_2)$.

(3) $a_{i1} + a_{i2} = 1$. When this happens $g_1 g_2 \in Box^\circ_K$ where $(K_1 \cap K_2) < K$ and θ_i contributes to $\Theta(g_1, g_2)$.

If case (3) does not occur for any i, then $K = K_1 \cap K_2$ and $i_1^*\omega_1 \wedge i_2^*\omega_2 \wedge \Theta(g_1, g_2)$ restricts to $\mathbf{S}(K, g_1 g_2)$ without problem. If case (3) occurs for some i's then the product of the restrictions of corresponding θ_is to $\mathbf{X}(K)$ is, up to a constant factor, the Thom form of the normal bundle of $\mathbf{X}(K_1 \cap K_2)$ in $\mathbf{X}(K)$. The wedge of this Thom form with $i_1^*\omega_1 \wedge i_2^*\omega_2$ and the restriction of the contributions from case (2) to $\mathbf{X}(K)$ defines a form on $\mathbf{X}(K)$. Thus the star product is well-defined.

We extend the star product to a product on $\Omega^*{}_{CR}$ by bi-linearity. The differential acts on the star product as follows,

(10.6)

$$d(\omega_1 t(g_1) \star \omega_2 t(g_2)) = d(\omega_1 t(g_1)) \star \omega_2 t(g_2) + (-1)^{\deg(\omega_1)+\deg(\omega_2)} \omega_1 t(g_1) \star d(\omega_2 t(g_2)).$$

Hence the star product induces a product on the Chen-Ruan cohomology.

Observe that the form $i_1^*\omega_1 \wedge i_2^*\omega_2 \wedge \Theta(g_1, g_2)$ is supported in a small neighborhood of $X(K_1 \cap K_2)$. Therefore the star product of three forms $\omega_i t(g_i) \in \Omega^{p_i}_{CR}(K_i, g_i)$, $1 \le i \le 3$, is nonzero only if $K_1 \cap K_2 \cap K_3$ is nonempty. Now it is fairly straightforward to check that the star product is associative.

11. Correspondence for Euler characteristic

First we introduce some notation. Consider a codimension k face $F = F_1 \cap \ldots \cap F_k$ of P where $k \geq 1$. Define a k-dimensional cone C_F in $N \otimes \mathbb{R}$ as follows,

$$C_F = \{\sum_{j=1}^{k} a_j \lambda_j : a_j \geq 0\} \tag{11.1}$$

The group G_F can be identified with the subset Box_F of C_F, where

$$Box_F := \{\sum_{j=1}^{k} a_j \lambda_j : 0 \leq a_j < 1\} \cap N. \tag{11.2}$$

Consequently the set G_F° is identified with the subset

$$Box_F^{\circ} := \{\sum_{j=1}^{k} a_j \lambda_j : 0 < a_j < 1\} \cap N \tag{11.3}$$

of the interior of C_F. We define $Box_P = Box_P^{\circ} = \{0\}$.

Suppose $v = F_1 \cap \ldots \cap F_n$ is a vertex of P. Then $Box_v = \bigsqcup_{v \leq F} Box_F^{\circ}$. This implies

$$G_v = \bigsqcup_{v \leq F} G_F^{\circ} \tag{11.4}$$

11.1. **Euler characteristic.** An almost complex orbifold is SL if the linearization of each g is in $SL(n, \mathbb{C})$. This is equivalent to $\iota(g)$ being integral for every twisted sector. Therefore, to suit our purposes, we make the following definition.

Definition 11.1. *A quasitoric orbifold is said to be quasi-SL if the age of every twisted sector is an integer.*

Lemma 11.1. *Suppose* $\mathbf{X}$ *is a quasi-SL quasitoric orbifold. Then the Chen-Ruan Euler characteristic of* $\mathbf{X}$ *is given by*

$$\chi_{CR}(\mathbf{X}) = \sum_{v} o(G_v)$$

where v varies over all vertices of P.

Proof. Note that each $X(F)$ is a quasitoric orbifold. So its cohomology is concentrated in even degrees, see [23]. Since $\mathbf{X}$ is quasi-SL, the shifts $2\iota(g)$ in grading are also even integers. Therefore the Euler characteristic of Chen-Ruan cohomology is given by

$$\chi_{CR}(\mathbf{X}) = \sum_{F \leq P} \chi(X(F)) \cdot o(G_F^{\circ}). \tag{11.5}$$

Each $X(F)$ admits a decomposition into locally closed subsets as follows

$$X(F) = \bigsqcup_{H \leq F} X(H^{\circ}) \tag{11.6}$$

where H° is the relative interior of H and $X(H^\circ) = \pi^{-1}(H^\circ)$. We have

$$\chi(X(F)) = \sum_{H \le F} \chi(X(H^\circ)) \tag{11.7}$$

However $X(H^\circ)$ is homeomorphic to the product of H° with $(S^1)^{\dim(H)}$. Therefore $\chi(X(H^\circ)) = 0$ unless H is a vertex. Hence

$$\chi(X(F)) = \text{number of vertices of } F. \tag{11.8}$$

Using (11.4), (11.5) and (11.8), we have the desired formula. □

Lemma 11.2. *The crepant blowup of a quasi-SL quasitoric orbifold is quasi-SL.*

Proof. Suppose the blowup is along a face $F = F_1 \cap \ldots \cap F_k$. The new sectors that appear correspond to G_H° where $H < F_0$. Take any vertex v in H. Suppose v projects to the vertex w of F under the blowdown. Without loss of generality assume $w = \bigcap_{j=1}^n F_j$. Then $v = \bigcap_{0 \le j \ne i \le n} F_j$ for some $1 \le i \le k$. Without loss of generality assume $i = 1$. Since $v \le H$, $\mathcal{I}(H) \subset \{0, 2, \ldots, n\}$. Therefore any $g \in G_H^\circ$ may be represented by an element $\eta = c_0\lambda_0 + \sum_{j=2}^n c_j\lambda_j$ of N where each $c_j \in [0,1) \cap \mathbb{Q}$. We need to show that the age of g, namely $c_0 + \sum_{j=2}^n c_j$, is an integer.

But using $\lambda_0 = \sum_{j=1}^k b_j\lambda_j$ we get that $\eta \in C_w$. In fact

$$\eta = c_0 b_1 \lambda_1 + \sum_{j=2}^{k} (c_0 b_j + c_j)\lambda_j + \sum_{j=k+1}^{n} c_j \lambda_j \tag{11.9}$$

We may write $\eta = \sum_{j=1}^n (m_j + a_j)\lambda_j$ where each m_j is an integer and each $a_j \in [0,1) \cap \mathbb{Q}$. Then $\sum_{j=1}^n a_j\lambda_j$ corresponds to an element of G_w. Since $\mathbf{X}$ is quasi-SL, $\sum_{j=1}^n a_j$ must be an integer. Therefore $\sum_{j=1}^n (m_j + a_j)$ is an integer. Hence $c_0 b_1 + \sum_{j=2}^k (c_0 b_j + c_j) + \sum_{j=k+1}^n c_j$ is an integer. Using $\sum_{j=1}^k b_j = 1$, this yields that $c_0 + \sum_{j=2}^n c_j$ is an integer. □

Theorem 11.3. *The Euler characteristic of Chen-Ruan cohomology is preserved under a crepant blowup of a quasi-SL quasitoric orbifold.*

Proof. Let $\rho : Y \to X$ be a crepant blowdown along a face $F = \bigcap_{j=1}^k F_j$ of P. Let w be any vertex of P and let $v_1, \ldots, v_k$ be the vertices of $\widehat{P}$ such that $\rho(v_i) = w$. Suppose $w = \bigcap_{1 \le j \le n} F_j$. Then $v_i = F_0 \cap \bigcap_{1 \le j \ne i \le n} F_j$.

The contribution of w to $\chi_{CR}(\mathbf{X})$ is $o(G_w) = |\det \Lambda_{(w)}|$, see (4.11). The contribution of each v_i to $\chi_{CR}(\mathbf{Y})$ is $o(G_{v_i}) = |\det \Lambda_{(v_i)}| = b_i |\det \Lambda_{(w)}| = b_i o(G_w)$. As the blowdown is crepant, we have $o(G_w) = \sum_{i=1}^k o(G_{v_i})$. The theorem follows. □

11.2. **Betti numbers.** Some partial results on Betti numbers are not hard to prove. We describe them below.

Theorem 11.4. *Suppose $\rho : Y \to X$ is a crepant blowdown of quasi-SL quasitoric orbifolds of dimension ≤ 6. Then the Betti numbers of Chen-Ruan cohomology of $\mathbf{X}$ and $\mathbf{Y}$ are equal.*

Proof. Assume that $\dim(\mathbf{X}) = 6$. Note that there are no facet sectors as every characteristic vector is primitive. Therefore the twisted sectors correspond to either vertices or edges. The age of a vertex sector is either 1 or 2 and such a sector contributes a generator to H^2_{CR} or H^4_{CR} respectively. An edge sector always has age 1. Since such a sector is a sphere it contributes a generator to H^2_{CR} as well as H^4_{CR}. There is only one generator in H^0_{CR} and H^6_{CR} coming from the untwisted sector. Therefore h^0_{CR} and h^6_{CR} are unchanged under blowup. If h^2_{CR} changes under blowup then by Poincaré duality, h^4_{CR} must change by the same amount. That would contradict the conservation of Euler characteristic. Therefore all Betti numbers are unchanged.

The proof for dimension four is similar. □

Lemma 11.5. *Suppose $\rho : Y \to X$ is a crepant blowdown of quasi-SL quasitoric orbifolds of dimension ≥ 8. Then $h^2_{CR}(\mathbf{Y}) \geq h^2_{CR}(\mathbf{X})$.*

Proof. The sectors that contribute to h^2_{CR} are the untwisted sector and twisted sectors of age one. Each age one sector contributes one to h^2_{CR}. The untwisted sector contributes h^2. It is proved in [23] that $h^2 = m - n$ where m is the number of facets and n is the dimension of the polytope.

Suppose the blowup is along a face F. The twisted sectors that may get affected by the blowup are the ones that intersect $X(F)$. These must be of the form (S, g) where g belongs to $\bigcup_w G_w$ where w varies over vertices of F. Consider any such w. Suppose $\lambda_1, \ldots, \lambda_n$ are the corresponding characteristic vectors. Note that the age one sectors of X coming from G_w belong to the set

$$A_w = \{\sum_{j=1}^{n} a_j\lambda_j : \sum_{j=1}^{n} a_j = 1\} \tag{11.10}$$

Since $\lambda_1, \ldots, \lambda_n$ are linearly independent, there exists a unique vector v such that the dot product $\langle \lambda_i, v\rangle = 1$ for each i. Hence A_w is a hyperplane given by

$$A_w = \{x \in N \otimes \mathbb{R} : \langle x, v\rangle = 1\}. \tag{11.11}$$

Note that since the blowup is crepant, $\lambda_0 \in A_w \cap C_F \cap N$. The sector corresponding to λ_0 is lost under the blowup. However the loss in h^2_{CR} because of it is compensated by the contribution from the untwisted sector on account of the new facet F_0.

Consider any other age one sector g of $\mathbf{X}$ in G_w. C_w is partitioned into n sub-cones by the introduction of λ_0. Accordingly g may be represented by $\sum_{0\leq j\neq i\leq n} c_j\lambda_j$ with each $c_j \geq 0$, for some $1 \leq i \leq n$. This means that g becomes a sector of Y coming from G_{v_i} where $v_i = \bigcap_{0\leq j\neq i\leq n} F_j$. Now $g \in A_w$ as it is an age one sector of $\mathbf{X}$. Also each $\lambda_j \in A_w$. Therefore by (11.11), $\sum_{0\leq j\neq i\leq n} c_j = 1$. This implies that each $0 \leq c_j < 1$ and age of g as a sector of $\mathbf{Y}$ is one as well. The lemma follows. □

12. Correspondence of Betti numbers

12.1. **Singularity and lattice polyhedron.** Following the discussions in sections 9 and 11, a singularity of a face F is defined by a cone C_F formed by positive linear combinations of vectors in its characteristic set $\lambda_1, \ldots, \lambda_d$ where d is the codimension of the face in the polytope. The elements of the local group G_F are of the form $g = diag(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_d})$, where $\sum_{i=1}^{d} \alpha_i \lambda_i \in N$, and $0 \leq \alpha_i < 1$. Recall that the age

$$\iota(g) = \alpha_1 + \ldots + \alpha_d \tag{12.1}$$

is integral in quasi-SL case by definition 11.1.

The singularity along the interior of F is of the form $\mathbb{C}^d/G_F$. These singularities are same as Gorenstein toric quotient singularities in complex algebraic geometry. Now let N_v be the lattice formed by $\{\lambda_1, \ldots, \lambda_n\}$, the characteristic vectors at a vertex v contained in the face F. Let m_v be the element in the dual lattice of N_v such that its evaluation on each λ_i is one. Now from Lemma 9.2 of [9] we know that the cone C_v contains an integral basis, say $e_1, \ldots, e_n$. Suppose $e_i = \sum a_{ij}\lambda_j$. By (11.2) e_i corresponds to an element of G_v, and since the singularity is qausi-SL, $\sum a_{ij}$ is integral. Hence m_v evaluated on each e_j is integral. So m_v an element of the dual of the integral lattice N.

Consider the $(n-1)$-dimensional lattice polyhedron Δ_v defined as $\{x \in C_v \mid \langle$ x $, m_v\rangle = 1\}$. Note that $\Delta_v = \{\sum_{i=1}^{n} a_i \lambda_i \mid a_i \geq 0, \ \sum a_i = 1\}$. For any face F containing v we define $\Delta_F = \Delta_v \cap C_F$. If $\{\lambda_i, \ldots, \lambda_d\}$ denote the chracteristic set of F, then $\Delta_F = \{\sum_{i=1}^{d} a_i\lambda_i \mid a_i \geq 0, \ \sum a_i = 1\}$. Hence Δ_F is independent of the choice of v.

Remark 12.1. *An element $g \in G$ of an SL orbifold singularity can be diagonalized to the form g= $diag(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_d})$, where $0 \leq \alpha_i < 1$ and $\iota(g) = \alpha_1 + \ldots + \alpha_d$ is integral.*

We make some definitions following [5].

Definition 12.1. *Let G be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\psi_i(G)$ the number of the conjugacy classes of G having $\iota(g) = i$. Define*

$$W(G; uv) = \psi_0(G) + \psi_1(G)uv + \ldots + \psi_{d-1}(G)(uv)^{d-1} \tag{12.2}$$

Definition 12.2. *We define height(g) = rank(g-I)*

Definition 12.3. *Let G be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\tilde{\psi}_i(G)$ the number of the conjugacy classes of G having the height $= d$ and $\iota(g) = i$.*

$$\widetilde{W}(G; uv) = \tilde{\psi}_0(G) + \tilde{\psi}_1(G)uv + \ldots + \tilde{\psi}_{d-1}(G)(uv)^{d-1} \tag{12.3}$$

Definition 12.4. *For a lattice polyhedron Δ_F defining a SL singularity $\mathbb{C}^d/G_F$, we define the following:*

$$W(\Delta_F; uv) = W(G_F; uv) \tag{12.4}$$

$$\psi_i(\Delta_F) = \psi_i(G_F) \tag{12.5}$$

$$\widetilde{W}(\Delta_F; uv) = \widetilde{W}(G_F; uv) \tag{12.6}$$

$$\tilde{\psi}_i(\Delta_F) = \tilde{\psi}_i(G_F) \tag{12.7}$$

Definition 12.5. *A finite collection $\tau = \{\theta\}$ of simplices with vertices in $\Delta_F \cap N$ is called a triangulation of Δ_F if the following properties are satisfied.*

(1) *If θ' is a face of $\theta \in \tau$ then $\theta' \in \tau$*
(2) *The intersection of any two simplices θ', $\theta'' \in \tau$ is either empty, or a common face of both of them;*
(3) $\Delta_F = \cup_{\theta\in\tau}\theta$

12.2. **Blowdown and triangulation of polyhedron.** A crepant blowup gives rise to triangulation of the polyhedrons corresponding to some of the faces. Suppose we blow up about a face F. Then it is clear that new characteristic vector is an integral vector lying in the interior of the polyhedron Δ_F. Note that Δ_F is a simplex. The crepant blow up induces a barycentric subdivision of Δ_F with the new characteristic vector as barycenter. We denote this triangulation of Δ_F by τ_F. For the faces F' contained in F, $\Delta_{F'}$ is triangulated as follows. Let $K_{F'} = \lambda_{F'} - \lambda_F$ be difference of two characteristic sets. The triangulation $\tau_{F'}$ consists of simplices with vertex set of the form $\theta \cup \beta$ where θ are the vertices of a simplex of τ_F and $\beta \subset K_{F'}$. To see that this process takes care of all the faces lost and created we make the following comments. First of all the faces lost are F and its subfaces. This means there will be no simplex with vertex set having λ_F as a subset. This is exactly what happens here. The new faces created are subfaces of the intersection of new facet (created by the blowup) with faces having as vertex one of the vertices of F. These faces intersected F prior to the blow up in some F' and so the new faces formed correspond to the simplices with vertex set that are subset of the union $\theta \cup \beta$ discussed above.

12.3. **E-polynomial.** The following has been taken from the paper of Batyrev and Dais [5]. Let X be an algebraic variety over $\mathbb{C}$ which is not necessarily compact or smooth. Denote by $h^{p,q}(H^k_c(X))$ the dimension of the (p,q) Hodge component of the k-th cohomology with compact supports. We define

$$e^{p,q}(X) = \Sigma_{k\geq 0}(-1)^k h^{p,q}(H^k_c(X)). \tag{12.8}$$

The polynomial

$$E(X; u, v) := \Sigma_{p,q} e^{p,q}(X) u^p v^q \tag{12.9}$$

is called E-polynomial of X.

Remark 12.2. *If the Hodge structure is pure, for example in the case of smooth projective toric varieties, then the coefficients $e^{p,q}(X)$ of the E-polynomial of X are related to the usual Hodge numbers by $e^{p,q}(X) = (-1)^{p+q}h^{p,q}(X)$*

We state the following theorem without proof.

Theorem 12.3. *(Proposition 3.4 in [5]) Let X be a disjoint union of locally closed subvarieties X_j, $j \in J$, where $J \subset \mathbb{N}$. Then*

$$E(X; u, v) = \Sigma_{j\in J} E(X_j; u, v) \tag{12.10}$$

12.4. **Ehrhart power series.** Let Δ be a lattice polyhedron and $k\Delta := \{kx \mid x \in \Delta\}$. Let $l(k\Delta)$ be the number of lattice points of $k\Delta$. Then the Ehrhart power series

$$P_{\Delta}(t) = \Sigma_{(k\geq 0)} l(k\Delta)t^k \tag{12.11}$$

Definition 12.6. *Let Δ_F be a (d-1) dimensional lattice polyhedron defining a d-dimensional toric singularity . It is well-known (see, for instance,* [5], *Theorem 5.4) that $P_{\Delta_F}(t)$ can be written in the form,*

$$P_{\Delta_F}(t) = \frac{\psi_0(\Delta_F) + \psi_1(\Delta_F)t + \ldots + \psi_{d-1}(\Delta_F)t^{d-1}}{(1-t)^d} \tag{12.12}$$

where $\psi_0(\Delta_F), \ldots, \psi_{d-1}(\Delta_F)$ are non-negative integers defined in (12.5).

12.5. **More on quasi-SL orbifolds .** Let $\mathbf{X}$ be a compact quasi-SL $2n$-dimensional quasitoric orbifold. Let $\text{Sing}(X)$ be the set of singular points of X. Consider the set $I = \{i \in \mathbb{N} \mid i \leq \text{number of faces in the polytope of } X\}$. We can index the set of faces by the set I. Call the inverse image of the interior of the face F_i as X_i. It can be easily seen that this gives a stratification of the orbifold where each stratum X_i is diffeomorphic to a complex torus.

It is easily seen that

$$W(\Delta_{F_i}, uv) = \Sigma_{X_j \geq X_i} \widetilde{W}(\Delta_{F_j}, uv) \tag{12.13}$$

where $X_j \geq X_i$ if $\overline{X_j} \supset X_i$. The above result is true because the coefficient of each term in the left hand side can be broken in to ones with different heights (see definition (12.2), equations (12.6), (12.3) and(12.7). The ones with height equal to the codimension of X_i contribute to $\widetilde{W}(\Delta_{F_i}, uv)$). These come from $G^{\circ}_{F_i}$. Use the decomposition $G_{F_i} = \bigsqcup_{F_j \supseteq F_i} G^{\circ}_{F_j}$ to observe that terms with lesser heights correspond to higher X_j.

12.6. **Poincaré Polynomial.** Recall that

$$H^*_{CR}(\mathbf{X}, \mathbb{R}) = \bigoplus_{F \leq P} \bigoplus_{g \in G^{\circ}_F} H^{*-2\iota(g)}(S(F,g), \mathbb{R}) \tag{12.14}$$

where $S(F,g)$ is homeomorphic to the inverse image of the face F.

Definition 12.7. *The Poincaré polynomial of a cohomology of X is a polynomial $P(X)(t)$ where the coefficient of t^d is the rank of the degree d cohomology group. We denote by $PP(X)(v)$ the Poincaré polynomial of the ordinary singular cohomology and $PP_{CR}(X)(v)$ as the Poincaré polynomial of the Chen-Ruan cohomology of* $\mathbf{X}$.

Now if X is a projective toric orbifold, it has pure Hodge structure. Since the Zariski closure of the X_i are the suborbifolds corresponding to the faces, from (12.14), (12.6) and (12.3), we have

$$PP_{CR}(X)(v) = \Sigma_{i \in I} PP(\overline{X_i})(v)\widetilde{W}(\Delta_{F_i}, v^2). \tag{12.15}$$

12.7. **Correspondence in quasitoric orbifolds.** Take a quasi-SL quasitoric orbifold $\mathbf{X}$. A slight perturbation makes the polytope P associated with the orbifold into a rational ploytope (see section 5.1.3.in [6]), and with suitable dilations make it into an integral polytope P' which is combinatorially equivalent to P. From the normal fan of P' we get a projective toric orbifold X' whose polytope is P'. (The orbifold structure of X' is determined by its analytic structure and we may conveniently refrain from using bold-face notation.) Putting $u = v$ in Theorem (12.3) we have

$$E(X'; v, v) = \Sigma_{i \in I} E(X'_i; v, v) \tag{12.16}$$

In the left hand side the coefficient of v^k is the sum of $e^{p,q}(X')$ where $p + q = k$. Since X' is Kahler the Hodge structure is pure and from remark (12.2) it follows $e^{p,q}(X') = (-1)^{p+q} h^{p,q}(X')$. Since toric orbifolds (see section 4 of [23]) have zero odd cohomology only the coefficient of v^{2k} terms are nonzero. By Baily's Hodge decomposition (see [3]), the Hodge numbers $h^{p,q}$ for $p + q = 2k$ add up to the $2k$-th Betti number of singular cohomology group. So the left hand side is the Poincaré polynomial of the ordinary cohomology, giving

$$PP(X')(v) = \Sigma_{i \in I} E(X'_i; v, v). \tag{12.17}$$

It is known from section 4 of [23] that the Betti numbers depend on the combinatorial equivalence class of the polytope P'. As P' is combinatorially equivalent to P, the left hand side equals the Poincaré polynomial of the quasitoric orbifold $\mathbf{X}$. The right hand side is a sum of E-polynomials of a number of tori. Since the number of tori of each dimension is the same by combinatorial equivalence of the polytopes, we have,

$$PP(X)(v) = \Sigma_{i \in I} E(X_i, v, v) \tag{12.18}$$

where $\bigsqcup X_i$ is the stratification by tori of the quasitoric orbifold $\mathbf{X}$. Now from 12.14 we get,

$$PP_{CR}(X)(v) = \Sigma_{i \in I} PP(\overline{X_i})(v) \widetilde{W}(\Delta_{F_i}, v^2) \tag{12.19}$$

Using 12.18 we have

$$PP_{CR}(X)(v) = \Sigma_{i \in I} \Sigma_{X_j \leq X_i} E(X_j, v, v) \widetilde{W}(\Delta_{F_i}, v^2) \tag{12.20}$$

Interchanging the order of summation, and using 12.13 we have

$$PP_{CR}(X)(v) = \Sigma_{j \in I} E(X_j, v, v) W(\Delta_{F_j}, v^2) \tag{12.21}$$

Theorem 12.4. *Suppose $\mathbf{X}$ is a quasi-SL quasitoric orbifold, and $\hat{\mathbf{X}}$ a crepant blowup. Then*

$$PP_{CR}(X)(v) = PP_{CR}(\hat{X})(v)) \tag{12.22}$$

Proof. Let $\rho : \hat{X} \to X$ be a crepant blowdown. We set $\hat{X}_i := \rho^{-1}(X_i)$. Then $\hat{X}_i$ has a natural stratification by products $X_i \times ((\mathbb{C}^*)^{codim(\theta)})$ induced by the triangulation,

$$\Delta_{F_i} = \cup_{\theta \in \tau_i} \theta \tag{12.23}$$

where τ_i consists of all simplices which intersect the interior of Δ_{F_i}, and $codim(\theta)$ denotes the codimension of θ in Δ_{F_i}.

Note that the E-polyinomial of a k-dimensional complex torus is $(v^2-1)^k$.

From (12.12) we have

$$W(\Delta_{F_i}; v^2) = P_{\Delta_{F_i}}(v^2)(1-v^2)^d \tag{12.24}$$

where d is the dimension of the face F_i. Consider the triangulation (12.23) of Δ_{F_i}. By counting lattice points using (12.12) and applying the inclusion exclusion principle we have

$$P_{\Delta_{F_i}}(v^2) = \sum_{\theta\in\tau_i}(-1)^{codim(\theta)}P_\theta(v^2) = \sum_{\theta\in\tau_i}(-1)^{codim(\theta)}W(\theta, v^2)(1-v^2)^{-dim(\theta)} \tag{12.25}$$

Multiplying both sides by $(1-v^2)^d$, we obtain

$$W(\Delta_{F_i}; v^2) = \Sigma_{\theta\in\tau_i}(v^2-1)^{codim(\theta)}W(\theta, v^2) \tag{12.26}$$

Since we are dealing with simplices θ which intersect the interior of Δ_{F_i} each stratum of $\widehat{X}$ is counted once. This is because each stratum corresponds to the interior of a face and for each face we have a simplex and it will lie in the interior of exactly one of the original (pre-triangulation) polyhedrons. Thus the equation (12.21) applied to $\widehat{X}$ gives

$$PP_{CR}(\hat{X})(v) = \Sigma_{i\in I}E(X_i; v, v)\Sigma_{\theta\in\tau_i}(v^2-1)^{codim(\theta)}W(\theta; v^2) \tag{12.27}$$

Now using (12.26)

$$PP_{CR}(\hat{X})(v)) = \Sigma_{i\in I}E(X_i; v, v)W(\Delta_{F_i}; v^2) = PP_{CR}(X)(v) \tag{12.28}$$

□

13. **Example**

We will consider the weighted projective space $\mathbf{X} = \mathbb{P}(1,3,3,3,1)$ which is a toric variety. The generators of the one dimensional cones of the fan of X are $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$ and $e_5 = (-1,-3,-3,-3)$. $\mathbf{X}$ may be realized as a quasitoric orbifold with the 4-dimensional simplex as the polytope and the e_is as characteristic vectors. However $\mathbb{P}(1,3,3,3,1)$ is not an SL orbifold and this choice of characteristic vectors coming from the fan does not make it an omnioriented quasi-SL quasitoric orbifold. So we choose a different omniorientation.

To be precise, by the correspondence established in [17], we can consider $\mathbf{X}$ as a symplectic toric orbifold with a simple rational moment polytope P whose facets have inward normal vectors $e_1, \ldots, e_5$. The moment polytope may be identified with the orbit space of the torus action. The denominations of the polytope are related to the choice of the symplectic form and is not important for us. Denote the facet of P with normal vector e_i by F_i. We assign the characteristic vectors as follows

$$\lambda_i = \begin{cases} e_i & \text{if } 1 \le i \le 4 \\ -e_5 & \text{if } i = 5. \end{cases} \tag{13.1}$$

The singular locus of $\mathbf{X}$ is the subset $X(F)$ where $F = F_1 \cap F_5$. The group G_F is isomorphic to $\mathbb{Z}_3$ and

$$G_F^\circ = \{g = \frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_5, g^2 = \frac{1}{3}\lambda_1 + \frac{2}{3}\lambda_5\} = \{(1,1,1,1),(1,2,2,2)\}. \tag{13.2}$$

Thus there are only two twisted sectors $S(F,g)$ and $S(F,g^2)$, each of age one. Since F is a triangle, the 4-dimensional quasitoric orbifold $\mathbf{X}(F)$ has $h^0 = h^2 = h^4 = 1$. Therefore each twisted sector contributes one to $h^k_{CR}(\mathbf{X})$ for $k = 2,4,6$.

We consider a crepant blowup $\mathbf{Y}$ of $\mathbf{X}$ along $X(F)$ with $\lambda_0 = (1,1,1,1)$. The singular locus of $\mathbf{Y}$ equals $Y(H)$ where $H = F_0 \cap F_5$. $G_H \cong \mathbb{Z}_2$ and $G_H^\circ = \{h = \frac{1}{2}\lambda_0 + \frac{1}{2}\lambda_5\} = \{(1,2,2,2)\}$. The age one twisted sector $S(H,h)$ contributes one to $h^k_{CR}(\mathbf{Y})$ for $k = 2,4,6$. But $h^2_{CR}(\mathbf{Y})$ also has an additional contribution from the new facet. Therefore $h^2_{CR}(\mathbf{Y}) = h^2_{CR}(\mathbf{X})$. Then by Poincaré duality, h^6_{CR} are also equal. Finally by conservation of Euler characteristic we get equality of h^4_{CR}.

It is also possible to directly ascertain the change in the ordinary Betti numbers due to blowup. The new facet F_0 is diffeomorphic to $F \times [0,1]$. So the new polytope has three extra vertices. We can arrange them to have indices $1,2,3$ and keep indices of other vertices unchanged, see [23] for definition of index. This means that ordinary homology, and therefore cohomology, of Y is richer than that of X by a generator in degrees $2,4,6$.

If we perform a further blowup of $\mathbf{Y}$ along F_0 with $(1,2,2,2)$ the new characteristic vector, we obtain a quasitoric manifold Z. It is easy to observe that Betti numbers of Chen-Ruan cohomologies of $\mathbf{Y}$ and Z are equal. If we switched the choice of characteristic vectors for the two blowups, McKay correspondence for Betti numbers would still hold.

Finally consider other choices of omniorientation that could make $\mathbf{X}$ quasi-SL. Switching the sign(s) of λ_2, λ_3 or λ_4 does not affect quasi-SLness or the calculations of Betti numbers. Another option is to take $\lambda_1 = -e_1$ and $\lambda_5 = e_5$. The calculations for this choice are analogous to the ones above.

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