$\mathcal{O}$-asymptotic classes of finite structures, pseudofinite dimension and forking

Darío Alejandro García
Matemático, Ph.D.(c)
Code: 200718197

Universidad de los Andes
Facultad de Ciencias
Departamento de Matemáticas
Bogotá, D.C.
12 de Diciembre, 2014

$\mathcal{O}$-asymptotic classes of finite structures, pseudofinite dimension and forking

DARÍO ALEJANDRO GARCÍA
MATEMÁTICO, PH.D.(C)
CODE: 200718197

DISERTACIÓN PRESENTADA PARA OPTAR AL TÍTULO DE
DOCTOR EN CIENCIAS - MATEMÁTICAS

ADVISOR
ALF ONSHUUS NIÑO, PH.D.
PROFESOR TITULAR - UNIVERSIDAD DE LOS ANDES

COADVISOR
THOMAS W. SCANLON, PH.D.
PROFESSOR - UNIVERSITY OF CALIFORNIA-BERKELEY

RESEARCH LINE
TEORÍA DE MODELOS

UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS
BOGOTÁ, D.C.
12 DE DICIEMBRE, 2014

Title in English

$\mathcal{O}$-asymptotic classes of finite structures, pseudofinite dimension and forking

Abstract: My research aims to study the of ultraproducts of finite structures and the study of forking, pseudofinite dimensions and other model-theoretic properties, specifically in pseudofinite structures and classes of finite linearly ordered structures. The main results obtained during my Ph.D can be separated in two main topics: *Pseudofinite dimensions and forking*, and *$\mathcal{O}$-asymptotic classes of finite structures.* Studying classes of finite structures (e.g 1-dimensional asymptotic classes) one can ask whether the notions of pseudofinite dimensions of Hrushovski and Wagner provide information about independence relations and other model-theoretic properties in their ultraproducts. In this setting, I proved that an instance of dividing in an ultraproduct of finite structures can be realized as a decrease in the pseudofinite dimension; thus implying, as a corollary, a generalization of a well-known result in 1-dimensional asymptotic classes; namely, that every infinite ultraproduct of models in such a class is supersimple of U-rank 1.

In the study of classes of finite linearly ordered structures, I stated the definition of *$\mathcal{O}$-asymptotic classes* as a way to meld ideas from 1-dimensional asymptotic classes and $\mathcal{O}$-minimality. The main examples of these classes are the class of finite linear orders and the class of cyclic groups $\mathbb{Z}/(2N+1)\mathbb{Z}$ with the natural order inherited by the order in the integers when we take the representatives $-N < -(N-1) < \cdots < -1 < 0 < 1 < \cdots < N-1 < N$. Results obtained include: a cell-decomposition result for $\mathcal{O}$-asymptotic classes melding ideas from the combinatorial cell decomposition for 1-dimensional asymptotic classes, and the cell decomposition theorem in $\mathcal{O}$-minimal structures; and a classification of the ultraproducts of $\mathcal{O}$-asymptotic classes: if every ultraproduct of a class $\mathcal{C}$ is o-minimal, then $\mathcal{C}$ is an $\mathcal{O}$-asymptotic class; every infinite ultraproduct of structures in an $\mathcal{O}$-asymptotic class is superrosy of U-thorn-rank 1 and NTP_2 of inp-rank 1.

I also present a preliminary collection of results towards isolate conditions under which dense $\mathcal{O}$-minimal structures can be obtained as quotients of ultraproducts of $\mathcal{O}$-asymptotic classes.

Keywords: Finite structures, ultraproducts, forking, counting, measure, pseudofinite dimension, linear orders, asymptotic classes, $\mathcal{O}$-asymptotic classes, $\mathcal{O}$-minimality, rosy, NTP_2.

Título en español

Clases $\mathcal{O}$-asintóticas de estructuras finitas, dimensión pseudofinita y bifurcación

Resumen: El objetivo de mi investigación es el estudio de ultraproductos de estructuras finitas y el estudio de bifurcación, dimensión pseudofinita y otras propiedades modelo-teóricas, específicamente en estructuras pseudofinitas y classes de estructuras finitas linealmente ordenadas.

Estudiando clases de estructuras finitas (por ejemplo clases 1-dimensionales asintóticas) una pregunta natural es saber bajo qué condiciones las nociones de dimensión pseudofinita definidas por Hrushovski y Wagner brindan información sobre relaciones de independencia y otras propiedades modelo-teóricas acerca de sus ultraproductos. En este sentido, presento aquí una demostración de que una instancia de división en un ultraproducto de estructuras finitas puede ser detectada por un descenso de la dimensión pseudofinita, obteniendo como corolario una generalización de un resultado conocido para clases 1-dimensionales asintóticas: Todo ultraproducto infinito de una clase 1-dimensional asintótica es supersimple de U-rango 1.

En el estudio de clases de estructuras finitas linealmente ordenadas, se establece la noción de *clases $\mathcal{O}$-asintóticas* como una forma de mezclar ideas provenientes de clases 1-dimensionales asintóticas y estructuras $\mathcal{O}$-minimales. Los principales ejemplos de estas clases son la clase de órdenes lineales finitos y la clase de grupos cíclicos $\mathbb{Z}/(2N+1)\mathbb{Z}$ con el orden heredado de los enteros al tomar representantes $-N < -(N-1) < \cdots < -1 < 0 < 1 < \cdots < N-1 < N$. Entre los resultados obtenidos se incluyen: un resultado de descomposición de celdas para clases $\mathcal{O}$-asintóticas - que mezcla ideas de la descomposición de celdas combinatorial para clases 1-dimensionales y el teorema de descomposición de celdas en estructuras $\mathcal{O}$-minimales- y una clasificación de los ultraproductos de clases $\mathcal{O}$-asintóticas: si cada ultraproducto de una clase $\mathcal{C}$ es $\mathcal{O}$-minimal, entonces $\mathcal{C}$ es una clase $\mathcal{O}$-asintótica; además, todo ultraproducto infinito de una clase $\mathcal{O}$-asintótica es superrosy de U-thorn-rango 1, y NTP_2 de inp-rango 1.

También se presentan algunos resultados preliminares que buscan encontrar condiciones bajo las cuales estructuras $\mathcal{O}$-minimales densas pueden obtenerse como cocientes de ultraproductos de clases $\mathcal{O}$-asintóticas.

Palabras clave: Estructuras finitas, ultraproductos, bifurcación, conteo, medida, dimensión pseudofinita, órdenes lineales, clases asintóticas, clases $\mathcal{O}$-asintóticas, $\mathcal{O}$-minimalidad, rosy, NTP_2.

Acceptation Note

Jury
Sergei Starchenko

Jury
Amador Martín-Pizarro

Jury
Alexander Berenstein

Advisor
Alf Onshuus Niño

Coadvisor
Thomas W. Scanlon

Bogotá, D.C., 12 de Diciembre de 2014

To my family, for the unconditional support, patience and love through all these years.

Agradecimientos

Muchas personas me han ayudado directa o indirectamente durante mi doctorado y durante la elaboración de este trabajo.

En primer lugar, quisiera expresar mi más profundo agradecimiento a mis asesores Alf Onshuus y Tom Scanlon por su apoyo continuo durante mis estudios de Doctorado y mi investigación, por su paciencia, motivación, generosidad y su inmenso conocimiento. Ellos me guiaron durante todo el tiempo de investigación y escritura de este trabajo, y no puedo imaginar haber tenido mejores asesores y mentores.

También quisiera agradecer al grupo de Lógica de Bogotá, especialmente a los profesores Alexander Berenstein y Andrés Villaveces, por abrirme las puertas al mundo de la Teoría de Modelos y por todas las discusiones informales pero fructíferas que tuve con ellos durante mi época de estudiante de posgrado.

Mis sinceros agradecimientos también al grupo al grupo de Lógica en la Universidad de California - Berkeley, especialmente a Alex Kruckman, Will Johnson, Michael Wan y James Freitag - por su amabilidad y por la hospitalidad brindada durante mi pasantía. Así mismo, quisiera agradecer a los profesores y estudiantes que participaron del semestre *Model Theory, Number Theory and Geometry* en el MSRI por todas las discusiones que me ayudaron a crecer como profesional y como persona. Entre ellos quisiera mencionar a los profesores Charles Steinhorn, Dugald

Macpherson, Ward Henson, Carol Wood, Sergei Starchenko, Maryanthe Malliaris, Cameron Hill, Vincent Guingona, y a los estudiantes Santiago Camacho, Caroline Terry, Gabriel Conant, Silvain Rideau y Samaria Montenegro.

Gracias también a los profesores Adolfo Quiroz and Tristram Bogart en la Universidad de los Andes, por ayudarme a lidiar con mi ignorancia en algunos temas de Combinatoria y Teoría de la Medida.

Le agradezco también a los administrativos del departamento de Matemáticas en la Universidad de los Andes por toda la ayuda y paciencia que me han brindado durante estos años, especialmente a Lina M. García, Magneli Castillo, Luisa Vargas y Víctor López. También quisiera agradecer la labor de los administrativos en la Universidad de California - Berkeley, especialmente de Debby Haaxman y Kathy Santos, cuya labor fue de vital importancia durante mi pasantía.

A todos mis amigos (matemáticos y no matemáticos) en Colombia y en Berkeley (mis compañeros en "GreenHouse"), que gracias a su alegría me ayudaron a sobrepasar esos momentos de frustración tan propios de la investigación en matemáticas. En particular, quisiera agradecer a mis amigos de "GreenHouse" for hacerme sentir como en casa durante mi estadía en el exterior.

Por último y más importante, quisiera agradecer a mi familia (mis padres, mis hermanos, sobrinas y sobrinos) por su paciencia y amor incondicional, y por servirme de motivación durante todos estos años.

Acknowledgments

Various people have helped me directly or indirectly during my Ph.D and during the elaboration of this dissertation.

Foremost, I am deeply grateful to my advisors Alf Onshuus and Tom Scanlon for their continuous support during my Ph.D studies and my research, for their patience, encouragement, generosity and immense knowledge. They have guided me during all the research and writing of this dissertation, and I could not imagine having better advisors and mentors.

I also would like to thank the Logic Group of Bogotá, especially professors Alexander Berenstein and Andrés Villaveces for introducing me to Model Theory and for all the informal but fruitful discussions during this time as a graduate student.

My sincere thanks also goes to the Logic Group at University of California - Berkeley, especially to Alex Kruckman, Will Johnson, Michael Wan and James Freitag - for their kindness and hospitality during my internships. Additionally, I would like to thank all professors and students that participated in the semester *Model Theory, Number Theory and Geometry* at the MSRI for all the talks and conversations that allowed me to grow as a mathematician and as a person. Among them, I would like to mention professors Charles Steinhorn, Dugald Macpherson, Ward Henson, Carol Wood, Sergei Starchenko, Maryanthe Malliaris, Cameron Hill, Vincent Guingona, as well as the students Santiago Camacho, Caroline Terry, Gabriel Conant, Silvain Rideau and Samaria Montenegro.

I also thank to professors Adolfo Quiroz and Tristram Bogart at Universidad de los Andes, for helping me to deal with my ignorance in some topics in combinatorics and measure theory.

I would like to thank the staff at the Mathematics Department at Universidad de los Andes for all the help and patience that have provided to me during these years, especially to Lina M. García, Magneli Castillo, Luisa Vargas and Víctor López. Also, I would like to acknowledge the labor of the staff at University of California - Berkeley, especially Debby Haaxman and Kathy Santos, whose labor was vital during my internships at this university.

To all my friends (mathematicians and non-mathematicians) in Colombia and Berkeley whose joy and enthusiasm helped me to pass through those frustrating moments characteristic of the research in mathematics. Thanks also to my "Green-House" friends at Berkeley for making me feel like home during my life abroad.

And last but more important, I would like to thank my family (my parents, siblings, nices and nephews) for all their patience and unconditional love, and also for being my motivation during all these years.

Contents

Introduction

Model Theory and Pseudofinite model theory

Model theory is a branch of mathematical logic whose subject is the study the interaction between mathematical structures such as groups, fields, graphs and the formal languages used to describe them. The main concept here is the concept of definable sets, which can be understood as the solutions of first order formulas in a structure in a similar way as algebraic varieties correspond to the solutions of systems of polynomials in algebraic geometry.

As with most branches of mathematics, the growth and development of model theory have been fueled by its applications, especially to other areas in mathematics. Early examples of these applications are Tarki's elimination of quantifiers for real closed fields and the Ax-Kochen/Ershov model theory of henselian fields. More recently - and due to the development of geometric model theory [43], the model theory of groups, and $\mathcal{O}$-minimality - connections with algebraic and diophantine geometry have been found. As examples of such connections we can mention Hrushovski's proof of Mordell-Lang conjecture for function fields [5] or the proof of the André-Oort conjecture for $\mathbb{C}^n$ by Pila [41] which uses a wonderful result on estimates of rational points in definable sets that is obtained using tools from $\mathcal{O}$-minimality [42]. Most of the applications of model theory come in two stages: first by identifying abstract (often combinatorial) properties of first-order theories that make them more tractable or "tame", and second when we realize that theories of mathematically meaningful structures satisfy those properties. Examples of these properties are stability, its generalization simplicity, NIP (which stands for Not Independence Property, and another generalization of stability but orthogonal to simplicity), and more recently rosiness and NTP_2 (standing for Not Tree Property of the second kind). Structures with stable theories include algebraically closed fields (such as the complex field), modules, algebraic groups and free groups. The theory of pseudofinite fields (infinite fields satisfying the theory of the class of finite fields) are simple but not stable while $\mathcal{O}$-minimal structures (such as the real exponential field) and algebraically closed valued fields are unstable structures with NIP.

The leading idea behind the most recent applications from model theory to other areas has been the slogan proposed by Hrushovski: "model theory is the geography of tame mathematics" (see [25, page 38]), where model-theorists use *informally* the terms "tame" or "wild" to distinguish between having desirable or undesirable model-theoretic behavior.

The paradigmatic example of a "wild" structure is the semiring of the natural numbers, which was proved by Gödel to not having any complete computable axiomatization. However, it has been shown that even in extremely "wild" subjects such as number theory, the solution of problems frequently uses illuminating expeditions into tame territory.

Another recurrent idea in model theory is the existence of abstract notions of independence (non-forking, non-thorn-forking, stable domination, etc.) which generalize linear and algebraic independence but also come together with notions of dimension and measure for definable sets. These ideas were first studied in stable theories, but variations of these techniques have been applied in the wider contexts of simple, NIP, rosy or even NTP_2 theories.

In contrast, Finite Model Theory - the specialization of model theory to the study *finite* structures - has very different methods, and usually refers to a field of mathematics which has more to do with computer science than to classical mathematical structures. Using connections to automata theory, finite model theory has developed through a broad range of applications to problems in graph theory, complexity theory, database theory and artificial intelligence (see [24], [36]).

However, in the last five years or so, some model-theorists and non model-theorists have applied modern methods from infinite model theory to classes of finite structures via their limits, that most of the time are obtained using the ultraproduct construction, but also direct and inverse limits.

Structures that satisfy the same first-order properties of some ultraproduct of finite structures are called *pseudofinite*. Examples of pseudofinite theories are the pseudofinite fields, the Random Graph (which turns out to be equivalent to the ultraproduct of the Paley graphs), vector spaces, classical geometries over finite fields, among others. As examples of structures which are not pseudofinite we could mention the complex field, or the dense linear order $(\mathbb{Q}, <)$. In fact, any pseudofinite linear order must be discrete.

The fundamental theorem of ultraproducts is due to Jerzy Łoś, and provides a transference principle between the finite structures and their limits. Roughly speaking, Łoś' Theorem states that a formula is true in the ultraproduct M of the structures $\langle M_n : n \in \mathbb{N} \rangle$ if and only if it is true for "almost every" M_n.

When applied to ultraproducts of finite structures, Łoś' theorem presents an interesting duality between the finite structures and the infinite structures. We start with a family of finite structures and produce infinite first-order structure with the same properties. This kind of finite/infinite connection can sometimes be used to prove qualitative properties of large finite structures, and in the other direction, quantitative properties in the finite structures often induced desirable qualitative properties in their ultraproducts.

The idea is that the counting measure on a class of finite structures can be lifted using Łoś' theorem to give notions of dimension and measure on their ultraproduct. This allows ideas from geometric model theory to be used in the context of pseudofinite theories, and potentially we can prove results in finite combinatorics (of graphs, groups, fields, etc) by studying the corresponding properties in the ultraproducts.

This approach was used by Hrushovski in [29], but was better explored in his striking papers [27] and [28], where he applies ideas from geometric model theory to additive combinatorics, locally compact groups and linear approximate subgroups. In additive combinatorics, he proves a result closely related to the Freiman's problem for arbitrary groups (see [51, Theorem 5.4.4]).

Goldbring and Towsner developed in [23] the *Approximate Measure Logic*, a logical framework that serves as a formalization of connections between finitary combinatorics and diagonalization arguments in measure theory or ergodic theory that have appeared in various places throughout the literature (see [3]). Using AML-structures, Goldbring and Towsner gave proofs of the Furstenberg's correspondence principle, Szemerédi's Regularity Lemma, the triangle removal lemma, and Szemerédi's Theorem: every subset of the integers with positive density contains arbitrarily long arithmetic progressions (see [52]).

Szemerédi's Regularity Lemma seems to be a recurrent theme, and it has captured the attention of model-theorists especially in the last years when they realized that the addition (sometimes implicit) of model-theoretic assumptions gives rise to an improved version of this result. For instance, Tao used ultraproduct methods (pseudofinite fields) to get a strong version of the Szemerédi's Regularity Lemma for arbitrarily large finite fields, the so-called *Algebraic Regularity Lemma* (see [50, Lemma 5]). Using this result, he was able to obtain a classification for polynomials in finite fields of arbitrary large size and characteristic: a polynomial P over a finite field of large characteristic $\mathbf{F}$ is either a *moderate asymmetric expander* in the sense that $|P(A,B)| \gg |F|$ whenever $A, B \subset \mathbf{F}$ are large enough, or else P has the form $P(x,y) = Q(F(x)+G(y))$ or $P(x,y) = Q(F(x)G(y))$ for some polynomials Q, F, G. On the other hand, Malliaris and Shelah presented in [37] a strong version of the Szemerédi's Regularity Lemma for *stable graphs* (finite graphs such that the formula xRy does not have the k-order property for some k), on which they eliminated the *irregular* pairs and extract much larger indiscernibles (homogeneous subgraphs) than predicted by Ramsey theorem.

In the *finite to infinite* direction, Macpherson and Steinhorn define in [38] the concept of *1-dimensional asymptotic classes* of finite structures, in which they isolate quantitative conditions on the sizes of definable sets in a class of finite structures $\mathcal{C}$ which allows to obtain good model-theoretic properties of their infinite ultraproducts. Namely, they proved that the ultraproducts of such classes are supersimple of U-rank 1, and provide a criterion under which the ultraproducts of 1-dimensional asymptotic classes would be stable, thus superstable of U-rank 1. This definition was later generalized by Elwes [18] to a notion of N-dimensional asymptotic classes, of which the ultraproducts were supersimple of U-rank at most N.

There is a research program suggested during the semester program *Model Theory, Arithmetic Geometry and Number theory* (Spring 2014) at the Mathematical Sciences Research Institute in Berkeley, which in general terms can be stated as *develop geometric model theory for pseudofinite structures*. There is a variety of possible questions about what is the relationship between the different concepts in model theory (stability, NIP, simplicity, geometries coming from independence relations, etc) once the assumption of pseudofiniteness is added, and how these classical model-theoretic properties on the ultraproducts of a class of finite structures reflect on quantitative properties for the definable sets along the class.

The underlying philosophy here is the belief that a geography of tame fragments and tame classes of finite structures may yield some insight into finite model theory and more applications to finite (extremal) combinatorics and/or computer science.

Thesis work

Dimension theory is one of the most important concepts in model theory because it can be used to give a combinatorial description of the definable sets of first order structures, allowing also the use of inductive arguments to prove properties about definable sets. Even more, it is possible to get structural properties of the models of a first order theory T by assuming some bound on the different ranks associated to T.

One of the recurrent themes around the notions of rank is their relationship with forking-independence. It is often desired that any instance of forking (on types or formulas) can be detected by a decrease of some dimension in the same way that any instance of linear dependence is witnessed by a decrease in the linear dimension, or any algebraic dependence can be detected by analyzing the transcendence degree.

In [29], Hrushovski and Wagner defined the notion of *quasidimension* on a structure M as a way to generalize the concept of dimension allowing values in an ordered group instead of allowing only integer values. The main example is what I call "logarithmic pseudofinite dimension" which is defined on ultraproducts of finite structures by taking the logarithm of the cardinality of nonstandard finite sets and factor them out by the convex hull of the reals.

The purpose of Chapter 2 is to provide a connection between forking in a pseudofinite structure and the logarithmic pseudofinite dimension: *any instance of division in a pseudofinite structure is witnessed by a decrease of the pseudodimension.* This connection is used to get some known results in asymptotic classes of finite structures mentioned before in this introduction.

The definition of 1-dimensional asymptotic classes of finite structures imposes a restriction on the sizes of definable sets in *one* variable that provides notions of measure and dimension. Using this notion, Macpherson and Steinhorn obtained results about the control of the size of definable sets in many variables (combinatorial cell decomposition) as well as results concerning the behavior of the infinite ultraproducts of structures in such classes. For instance, they show that if every ultraproduct of the class $\mathcal{C}$ is strongly minimal, then $\mathcal{C}$ is a 1-dimensional asymptotic class, and also that every infinite ultraproduct of a 1-dimensional class is supersimple of U-rank 1.

The paradigmatic example of a class of finite structures which is not a 1-dimensional class is the class of all finite totally ordered sets. However, the only definable sets in the structures of this class (and in their ultraproducts) are finite unions of intervals and points implying that the structures involved are *$\mathcal{O}$-minimal.*

Macpherson and Steinhorn defined in [39] the *robust-classes of finite structures*, which is an attempt to bring an appropriate version of $\mathcal{O}$-minimality to classes of finite structures using direct limits instead of ultraproducts. Macpherson and Steinhorn (and Macpherson's student Marshall) developed some general model theory of robust chains and exhibited $(\mathbb{Q}, +, <)$ as the direct limit of an $\mathcal{O}$-minimal robust chain. They thereby showed that one can define a finitary version of $\mathcal{O}$-minimality that is not restricted to discrete orders, and can have algebraic content.

I propose a different approach through the notion of *$\mathcal{O}$-asymptotic classes of finite structures* which arises as an adaptation of the definition of 1-dimensional asymptotic classes in the context of linearly ordered finite structures, melding ideas of asymptotic classes and $\mathcal{O}$-minimality.

In Chapter 3 I present the definition of $\mathcal{O}$-asymptotic classes of finite structures. The main example is the class of totally ordered structures and the class of cyclic groups $\mathbb{Z}/(2N+1)\mathbb{Z}$ with the ordering given by $-\overline{N} < \cdots < -\overline{1} < \overline{0} < \overline{1} < \ldots < \overline{N}$. Such orders on cyclic groups are used to give proofs in additive combinatorics (see [50, Theorem 1.3] for instance).

I also prove a cell-decomposition result for $\mathcal{O}$-asymptotic classes (Theorem 3.1.7) combining ideas from $\mathcal{O}$-minimal cell decomposition and the combinatorial cell-decomposition of 1-dimensional asymptotic classes. Finally, I start the study of the model theory of infinite ultraproducts of structures in $\mathcal{O}$-asymptotic classes, obtaining the following results:

Proposition 3.3.1 *If every ultraproduct of a class of finite totally ordered structures is $\mathcal{O}$-minimal, then the class is $\mathcal{O}$-asymptotic.*

Theorem 3.3.2 *Every ultraproduct of an $\mathcal{O}$-asymptotic class is superrosy of $U^{þ}$-rank 1.*

Theorem 3.3.3 *Every ultraproduct of an $\mathcal{O}$-asymptotic class is NTP_2.*

An important restriction here comes from the fact that ultraproducts of finite linearly ordered structures are discrete, so groups such as $(\mathbb{Q}, +, <)$ or $(S^1, \cdot)$ cannot be obtained as ultraproducts of $\mathcal{O}$-asymptotic classes. However, once we pass to a (non-definable) quotient of the ultraproduct, dense structures may appear.

In Chapter 4 I present constructions that provide these dense quotients, as well as a preliminary collection of results whose purpose is to isolate conditions under which an $\mathcal{O}$-minimal structure (or a structure interpretable in an $\mathcal{O}$-minimal theory) can be obtained in the quotient of an ultraproduct of $\mathcal{O}$-asymptotic classes. In Chapter 5 I show more examples of classes of finite linearly ordered structures, together with some preliminary steps towards a proof that they are $\mathcal{O}$-asymptotic classes.

CHAPTER 1

Preliminaries

1.1 Notational conventions

In this section we set up some notation and conventions which will be used throughout this thesis

The Greek letters $\phi, \varphi, \psi, \theta$ will denote formulas in some ambient theory T, while α, β, μ, ν are used to denote real (sometimes hyperreal) numbers. In Chapter 2, the letters μ and δ will denote a measure operator and a dimension operator, respectively.

As usual, lower-case Roman letters will be used as follows: a, b, c, d, e are elements of a model; f, g, h are functions; i, j, k, l, m, n, s, t are integers or indices, p, q, r are types, u, v, w, x, y, z are variables ranging over elements. The remaining letters (that is the letter "o") are not used as mathematical symbols for clarity. The only exception to this convention will be during the proof of Theorem 3.1.7, where an unexpected amount of different indices is used.

The symbol $\mathcal{U}$ will always denote a non-principal ultrafilter (see Definition 1.3.1) over the index set of a class of finite structures, and we write $\prod_{\mathcal{U}} M_i$ for the ultraproduct of the elements $\langle M_i : i \in I \rangle$ (I infinite) with respect to the ultrafilter $\mathcal{U}$.

Capital letter C will always denote a real constant, and the calligraphic letter $\mathcal{C}$ will be used to denote a class of finite structures.

1.2 Basic definitions from model theory

Here we recall the basic setup of model theory, especially directed to an understanding of ultraproducts of finite structures and Classification Theory.
We assume basic knowledge in first-order logic. Throughout this thesis, $\mathcal{L}$ will denote a countable first-order language.

Definition 1.2.1. *Let $\mathcal{L}$ be a first order language, M an $\mathcal{L}$-structure, A a subset of M and $\bar{b}$ a tuple of elements from M.*

1. *A set of formulas Γ in variables $\overline{x}$ is* finitely satisfiable *if every finite subset $\Gamma_0 \subseteq \Gamma$ is satisfiable, which means there is a tuple $\bar{b} \in M$ such that $M \models \varphi(\bar{b})$ for every formula $\varphi(\bar{b}) \in \Gamma_0$.*

2. *A set of formulas Γ in variables $\overline{x}$ is* k-inconsistent *if there is no consistent subset of formulas from Γ of size k.*

3. *We say that a formula $\varphi(x)$ (possibly with parameters) is* algebraic *if there is some $N < \omega$ such that $M \models \exists x_1, \ldots, x_N \, \forall y(\varphi(y) \rightarrow (y = x_1 \vee \cdots \vee y = x_N))$.*

4. *We define the* algebraic closure of *A to be the set of all elements in M that realize an algebraic formula with parameters from A. This set will be denoted $acl(A)$.*

5. *The* definable closure of *A is the set of all elements b in M for which there is an algebraic formula $\varphi(x)$ with parameters from A such that $M \models \varphi(b) \wedge \forall z(\varphi(z) \rightarrow z = b)$. The definable closure will be denote by $dcl(A)$.*

Definition 1.2.2. *Let M be an $\mathcal{L}$-structure and $A \subseteq M$.*

1. *The set $S_n(A)$ consists of all maximal n-types, that is, maximal sets of formulas in n-variables which are finitely satisfiable in M. We denote as $S(A)$ to the union of all sets $S_n(A)$, $n < \omega$.*

2. *The* type of $\overline{b}$ over A in M *is the collection of all formulas $\varphi(\overline{x}; \overline{a})$ with parameters from A such that $M \models \varphi(\overline{b}; \overline{a})$.*

3. *We say that $\overline{b}$ realizes the type $p \in S_{|\overline{b}|}(A)$ if $p = tp(\overline{b}/A)$.*

We say that a structure M has *quantifier elimination* if for every formula $\phi(\overline{x})$ there is a quantifier free formula $\psi(\overline{x})$ such that $M \models \forall\overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$. Quantifier elimination is a very useful method in model-theory to understand the definable sets of an $\mathcal{L}$-structure M. In order to achieve quantifier elimination, sometimes an enriched language $\mathcal{L}' \supseteq \mathcal{L}$ is needed.
I would like to recall the following basic result: an $\mathcal{L}$-structure M has quantifier elimination if and only if every primitive existential formula $\phi(x_1, \ldots, x_n)$ (this is a formula of the form $\phi = \exists y(\psi(y; x_1, \ldots, x_n))$ with ψ a conjunction of a quantifier-free formulas) is equivalent to a quantifier-free formula $\theta(x_1, \ldots, x_n)$.

1.3 Pseudofinite structures, measures and dimensions

1.3.1 The Ultraproduct Construction

Now we present the ultraproduct construction, which will be a recurrent topic throughout this thesis. The following definitions, and more explanations and results can be found for example in [31].

Definition 1.3.1. *Let I be a non-empty set. A* filter on I *is a collection $\mathcal{F}$ of subsets of I satisfying the following conditions:*

1. *The collection $\mathcal{F}$ is closed under supersets, that is, if $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$ then $B \in \mathcal{F}$.*

2. *The collection $\mathcal{F}$ is closed under* finite *intersections: if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.*

3. *$I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.*

An ultrafilter on I is a filter $\mathcal{U}$ on I which is maximal with respect to $\subseteq$. Equivalently, a filter $\mathcal{U}$ is an ultrafilter if it satisfies the following extra condition

4. *For every subset X of I, either $X \in \mathcal{U}$ or $I - X \in \mathcal{U}$.*

Theorem 1.3.2 (Tarski, 1930)**.** *Every filter on a set I can be extended to an ultrafilter on I.*

For an infinite set I, a remarkable example of a filter on I is the *Frechét filter*, that can be defined as:

$$\mathcal{F}_{\text{Frechét}} = \{X \subseteq I : I - X \text{ is finite}\}.$$

An ultrafilter $\mathcal{U}$ on I extending the Frechét filter on I will be called *non-principal*. It is easy to show that if an ultrafilter $\mathcal{U}$ does not contain the Frechét filter, then it has the form $\mathcal{U} = \{X \subseteq I : i \in X\}$ for some element $i \in I$. Ultrafilters of this kind are called *principal*.

We now proceed to define the ultraproduct construction. First, let $\mathcal{U}$ be an ultrafilter on a set I, and for every $i \in I$ let M_i be an $\mathcal{L}$-structure.

Definition 1.3.3. *Let f, g be elements in the cartesian product $\prod\limits_{i \in I} M_i$. We say that f and g are $\mathcal{U}$-*equivalent *if the set $\{i \in I : f(i) = g(i)\}$ belongs to $\mathcal{U}$. We denote as $\prod\limits_{\mathcal{U}} M_i$ to the set of $\mathcal{U}$-equivalence classes of elements from $\prod\limits_{i \in I} M_i$.*

Note that as an abuse of notation we avoid the use of the set I in the notation $\prod\limits_{\mathcal{U}} M_i$, since it is implicitly mentioned in $\mathcal{U}$. Also, every element $a \in M$ can be described as $a = [a^i]_{\mathcal{U}}$, denoting in this way the fact that a is the $\mathcal{U}$-equivalence class of the function $f \in \prod_{i \in I} M_i$ given by $f(i) = a_i$.

Definition 1.3.4. *We define the* ultraproduct of the structures $\langle M_i : i \in I \rangle$ with respect to $\mathcal{U}$ *as the following* $\mathcal{L}$*-structure:*

- *Universe:* $M = \prod_{\mathcal{U}} M_i$.
- *If* c *is a constant symbol in* $\mathcal{L}$*, then* $c^M := [c^{M_i}]_{\mathcal{U}}$.
- *If* f *is an* n*-ary function symbol in* $\mathcal{L}$*, then*

$$f^M([a_1^i]_{\mathcal{U}}, \dots, [a_n^i]_{\mathcal{U}}) := [f^{M_i}(a_1^i, \dots, a_n^i)]_{\mathcal{U}}.$$

- *If* R *is an* n*-ary relation symbol in* $\mathcal{L}$*, then we define* $M \models R([a_1^i]_{\mathcal{U}}, \dots, [a_n^i]_{\mathcal{U}})$ *if and only if* $\{i \in I : M_i \models R(a_1^i, \dots, a_n^i)\} \in \mathcal{U}$.

Definition 1.3.5 (Asymptotic properties along $\mathcal{U}$)**.** *Let* $\mathcal{U}$ *be an ultrafilter on* ω *and* $\langle M_n : n < \omega \rangle$ *a collection of structures.*

1. *We say that a property* P *holds* for $\mathcal{U}$-almost every M_n *when the set* $\{n < \omega : M_n \text{ satisfies } P\}$ *belongs to* $\mathcal{U}$.
2. *We write* $|M_n| \to_{\mathcal{U}} \infty$ *as an abbreviation for the following: for every* $N < \omega$*, the set* $\{i \in I : |M_i| \geq N\}$ *belongs to* $\mathcal{U}$.
3. *Given a sequence* $\langle r_n : n < \omega \rangle$ *of real numbers, we write* $\lim_{n \to \mathcal{U}} r_n = s$ *to mean the following: For every* $\epsilon > 0$*, the set* $\{n < \omega : r_n \in (s - \epsilon, s + \epsilon)\}$ *belongs to* $\mathcal{U}$.

Ultraproducts of finite structures have turned out to be very important as a way to study the model theory of classes of finite structures using the powerful known methods and results coming from infinite model theory.

The fundamental theorem of ultraproducts is due to Jerzy Łoś, and provides a powerful transfer principle between the factor structures and their ultraproducts.

Theorem 1.3.6 (Łoś). *Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct with respect to an ultrafilter $\mathcal{U}$. Then for every first order $\mathcal{L}$-formula $\varphi(\overline{x})$ and every tuple $\overline{c} = ([c_1^i]_{i\in\mathcal{U}}, \ldots, [c_n^i]_{i\in\mathcal{U}})$ of elements in M, we have that*

$$M \models \varphi(\overline{c}) \text{ if and only if } \{i \in I : M_i \models \varphi(c_1^i, \ldots, c_n^i)\} \in \mathcal{U}.$$

Throughout this thesis, we will be especially interested in ultraproducts of finite structures with respect to an ultrafilter on a countable index set, that we usually will identify with ω.

Structures that are elementarily equivalent to ultraproducts of finite structures are called *pseudofinite*. Alternatively, we call a structure *pseudofinite* if every first-order sentence true in such structure has a realization in a finite structure. These two definitions coincide, as we will see with the next proposition.

Proposition 1.3.7. *Suppose $\mathcal{L}$ is a first-order language and M is an $\mathcal{L}$-structure. Let $\Gamma_{\mathcal{L}}$ be the first-order theory of all finite $\mathcal{L}$-structures (i.e., those sentences which are true in all finite $\mathcal{L}$-structures). The following are equivalent*

1. $M \models \Gamma_{\mathcal{L}}$.

2. *Every first-order $\mathcal{L}$-sentence which is true in M is true in a finite model.*

3. *There is a set $\{M_i : i \in I\}$ of finite $\mathcal{L}$-structures and an ultrafilter $\mathcal{U}$ on I such that*

$$M \equiv \prod_{\mathcal{U}} M_i$$

Proof. (1) $\Rightarrow$ (2): Assume ϕ is a sentence which is true in M. If there is no finite $\mathcal{L}$-structure satisfying ϕ, then for every finite $\mathcal{L}$-structure M_i we have that $M_i \models \neg\phi$. By the definition of $\Gamma_{\mathcal{L}}$, this implies that $\neg\phi \in \Gamma_{\mathcal{L}}$, contradicting that $M \models \Gamma_{\mathcal{L}}$.

(2) $\Rightarrow$ (3) : Consider the set T of $\mathcal{L}$-sentences that are true in M, and let I be the collection of finite subsets of T. For every $i = \{\phi_1, \ldots, \phi_n\} \in I$, let M_i be a finite $\mathcal{L}$-structure such that $M_i \models \phi_1 \wedge \cdots \wedge \phi_n$.

Let $\mathcal{F}_0$ be the collection of the sets $X_i = \{j \in I : M_j \models \phi \text{ for all } \phi \in i\}$. We will show that $\mathcal{F}_0$ satisfies the finite intersection property: if $X_i, X_j \in \mathcal{F}_0$, then the set

$$X_i \cap X_j = \{k \in I : M_k \models \phi \text{ for all } \phi \text{ in } i\} \cap \{k \in I : M_k \models \phi \text{ for all } \phi \text{ in } j\}$$

contains $X_{i \cap j}$. So $\mathcal{F}_0$ can be extended first to a filter and then to an ultrafilter (a maximal filter) $\mathcal{U}$ on I.

Now we show that $M \equiv \prod_{\mathcal{U}} M_i$. If $M \models \phi$, then the set $\{i \in I : M_i \models \phi\} = X_{\{\phi\}} \in \mathcal{U}$ and by Łoś' theorem we conclude that $\prod_{\mathcal{U}} M_i \models \phi$.

(3) $\Rightarrow$ (1): For every $\mathcal{L}$-sentence $\phi \in \Gamma_{\mathcal{L}}$, we have that $\{i \in I : M_i \models \phi\} = I \in \mathcal{U}$, and by Łoś' theorem we conclude that $M \models \phi$. $\square$

There is a deep result of model theory which can be used to get pseudofinite models: Cherlin, Harrington and Lachlan proved in [9] that all models of totally categorical theories (κ-categorial for each κ) are pseudofinite.

For instance, the theory of the Random Graph is pseudofinite, as opposed to the theory of dense linear orders which is not. In fact, any pseudofinite linearly ordered structure is discrete, as it will be shown in Theorem 1.3.9.

1.3.2 Countable saturation

In general, ultraproducts satisfy a useful property known as $\aleph_1$-*saturation* (also referred as *countable saturation*). Namely, every type over a countable set of parameters has a realization.

Proposition 1.3.8 (Theorem 6.1.1 in [6]). *($\aleph_1$-saturation) Let $M = \prod_{\mathcal{U}} M_i$ be an ultraproduct with respect to a non-principal ultrafilter $\mathcal{U}$ on ω, and $A \subseteq M$ be countable. Assume $p(\overline{x})$ is a (partial) type over A which is finitely satisfiable in M. Then $p(\overline{x})$ is realized in M.*

Proof. Suppose that $p(\overline{x}) = \{\varphi_m(\overline{x}) : m < \omega\}$ is an enumeration of the formulas in $p(\overline{x})$. Since $p(\overline{x})$ is finitely satisfiable, for every $k < \omega$ we have that the intersection $\varphi_1(M) \cap \cdots \cap \varphi_k(M)$ is non-empty. By Łoś' Theorem, this means that the set

$$S'_k := \{i \in \omega : M_i \models \exists \overline{x}\, (\varphi_1(\overline{x}) \wedge \cdots \wedge \varphi_k(\overline{x}))\} \in \mathcal{U}.$$

Let $S_k = S'_k \cap [k, \infty)$. Note that the sets S_k are $\mathcal{U}$-large, $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_k \supseteq \cdots$ and $\bigcap_{k=1}^{\infty} S_k = \emptyset$.

For any $i \in S_1$, let k_i denote the largest natural number k such that $i \in S_k$, and take $\overline{a}_i$ an element in $\varphi_1(M_i) \cap \cdots \cap \varphi_{k_i}(M_i)$. By construction, for each $m < \omega$ we have $\{i \in \omega : \overline{a}_i \in \varphi_m(M_i)\} \supseteq \{i \in \omega : m \leq k_i\} \supseteq S_1 \cap S_m = S_m \in \mathcal{U}$. Thus, the element $\overline{a} = [a_i]_{i \in \mathcal{U}} \in M$ realizes $p(\overline{x})$. □

As an example to illustrate the power provided by the $\aleph_1$-saturation of ultraproducts of finite structures we can characterize the order type of an ultraproduct of finite linearly ordered structures.

Theorem 1.3.9. *Let $\mathcal{C} = \{(M_i, <) : i \in \omega\}$ be a class of finite linearly ordered structures such that $|M_i| \to_{\mathcal{U}} \infty$, and take $M = \prod_{\mathcal{U}} M_i$ to be an infinite ultraproduct of the elements in the class. Then the order-type of $(M, <)$ is $\omega \oplus (\kappa \times \mathbb{Z}) \oplus \omega^*$ (with lexicographical order) where ω^* is the inverse-order in the natural numbers, and κ is a dense linear order without end points.*

Proof. By Łoś' Theorem, M has first and last element, and any other element beside those two must have a successor and a predecessor. Consider the (non first-order definable) equivalence relation given by aEb if and only if there is some $k \in \mathbb{Z}$ such that $S^k(a) = b$ (where S denotes the successor function).

There is a canonical linear order in M/E given by $[a]_E \leq [b]_E$ if and only if $a \leq b$. Let κ be the collection of classes $[a]_E$ such that a is not E-equivalent with neither the maximum nor the minimum of M, with the order inherited by M/E. It is enough to show that κ is dense. Let $[a]_E < [b]_E$ be two elements in κ, and consider the type $p(x) := \{S^n a < x < S^{-n} b : n < \omega\}$. Given $\Gamma_0 \subseteq^{fin} p(x)$, take $k = \max\{n : S^n a < x < S^{-n} b \in \Gamma_0\}$. Then, $S^{k+1}(a) \models \Gamma_0$.

Thus, $p(x)$ is finitely satisfiable, and by $\aleph_1$-saturation, there is an element $c \in M$ realizing p. It follows that $a < c < b$ and c is not equivalent with neither a nor b. So, $[a]_E < [c]_E < [b]_E$. □

The following result appears in [19], and the proof I present here is an adaptation of the proof given there.

Proposition 1.3.10. *If $M = \prod_{\mathcal{U}} M_i$ is an ultraproduct of finite structures with $I = \omega$ and $|M_i| \to_{\mathcal{U}} \infty$. Then $|M| = 2^{\aleph_0}$*

Proof. Note that

$$\left|\prod_{\mathcal{U}} M_i\right| \leq \left|\prod_{i\in\omega} M_i\right| \leq |\mathbb{N}^{\mathbb{N}}| = |\mathbb{R}|,$$

so it suffices to show that $|\mathbb{R}| \leq |\prod_{\mathcal{U}} M_i|$.

We will need the following combinatorial lemma:

Lemma 1.3.11. *There exists a family $\mathcal{F}$ of functions $f : \mathbb{N} \to \mathbb{N}$ such that:*

1. $|\mathcal{F}| = 2^{\aleph_0}$,
2. *For any $f \in \mathcal{F}$ and any $n \in \mathbb{N}$, $f(n) < 2^n$,*
3. *If $f \neq g$ are in $\mathcal{F}$, then $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite.*

Proof. Given a set $A \subseteq \mathbb{N}$ let $f_A : \mathbb{N} \to \mathbb{N}$ be given by

$$f_A(n) = \sum_{k<n} \chi_A(k) 2^k,$$

where χ_A is the characteristic function of A. Then the family $\mathcal{F} = \{f_A : A \subseteq \mathbb{N}\}$ satisfies the three conditions:

1. If $A \neq B$ are subsets of $\mathbb{N}$ and $n = \min(A \triangle B)$ (say $n \in A - B$ without loss of generality), then $f_A(n+1) = f_B(n+1) + 2^n \neq f_B(n+1)$. This shows that $f_A \neq f_B$ whenever $A, B \subseteq \mathbb{N}$ with $A \neq B$, and so $|\mathcal{F}| = 2^{\aleph_0}$.

2. For any $f_A \in \mathcal{F}$ and $n \in \mathbb{N}$,

$$f_A(n) = \sum_{k<n} \chi_A(k) 2^k \leq \sum_{k<n} 2^k = 2^n - 1 < 2^n.$$

3. If $f_A \neq f_B$ are elements in $\mathcal{F}$, $\{n \in \mathbb{N} : f_A(n) = f_B(n)\} = \{n \in \mathbb{N} : n < \min(A \triangle B)\}$ is finite.

□

Proof of Proposition 1.3.10:
Let $A_n = \{k \in \mathbb{N} : 2^n \leq |M_k| < 2^{n+1}\}$, so the sets A_n are not in $\mathcal{U}$ and partition $\mathbb{N}$. For each $k \in A_n$ let $\{a_{k,j} : j < 2^n\}$ be a list of 2^n distinct elements of M_k.

Let $\mathcal{F}$ be as in the Lemma 1.3.11. For $f \in \mathcal{F}$, let $h_f : \mathbb{N} \to \bigcup_k M_k$ be given by $h_f(k) = a_{k,f(n)}$, where n is such that $k \in A_n$.

Note that if $f \neq g$ are in $\mathcal{F}$, then

$$\{k \in \mathbb{N} : h_f(k) = h_g(k)\} = \bigcup \{A_n : n \in \mathbb{N}, f(n) = g(n)\}$$

is a finite union of sets not in $\mathcal{U}$. Hence, $[h_f]_{\mathcal{U}} \neq [h_g]_{\mathcal{U}}$, and we are done. □

When the index set is arbitrary (i.e., not necessarily ω), computing the exact size of ultraproducts is a very difficult problem even for ultraproducts of finite structures. The main references in this subject are the paper of Keisler, [30] and the work of Shelah, [49]. In the later paper, Shelah shows that if an ultraproduct of finite sets is infinite of size κ, then $\kappa^{\aleph_0} = \kappa$.

1.3.3 Measures and dimension in pseudofinite structures

Definition 1.3.12. *Let M be a non-empty finite set. We define the* normalized counting measure *on the subsets of M by*

$$m(X) = \frac{|X|}{|M|}.$$

The definition above also admits local version: if D is a non-empty subset of M, we may define the *counting measure localized on D* as

$$m_D(X) = \frac{|X \cap D|}{|D|}.$$

Assume now that $M = \prod_{\mathcal{U}} M_i$ is an ultraproduct of finite structures. It is possible to obtain a finitely-additive probability measure on the definable subsets of M from the normalized counting measure on each M_i. In fact, there are several ways to do so (all equivalent). The main feature of this kind of constructions is the analysis of the measure in M using the asymptotics of the measure in the finite structures M_i.

I present here an explicit construction that cover all the possible cases we are interested in. We can find this construction as presented here in [21].

Definition 1.3.13. *Let $\mathcal{L}$ be a first over language. $\mathcal{L}^+$ is the language of two-sorted structures with sorts D and OF containing:*

- *Formulas in $\mathcal{L}$ for the sort D.*
- *Formulas in the language $\{+, \cdot, 0, 1, <\}$ for the sort OF.*
- *For every $\mathcal{L}$-formula $\phi(\overline{x}, \overline{y})$, a function symbol $f_\phi : D^{|\overline{y}|} \longrightarrow OF$.*

The leading idea here is that the functions f_ϕ can be seen as functions on definable sets ($f_\phi(\overline{b})$ will correspond to $f(\phi(\overline{x}, \overline{b}))$ that take values in an ordered field. With this idea, the functions f_ϕ could correspond to cardinalities, measures (of several kinds), dimensions, etcetera. Even more, it might be possible to combine different kind of operators (e.g. measure and dimensions).

To construct the normalized counting measure on M, we associate to every finite $\mathcal{L}$-structure M_i an $\mathcal{L}^+$-structure K_i by putting:

- $D(K_i) = M_i$, with the same interpretations for formulas in $\mathcal{L}$.
- $OF(K_i) = (\mathbb{R}, +, \cdot, 0, 1, <)$.
- Given a formula $\phi(\overline{x};\overline{y})$ with $|\overline{x}| = r$, we make $f_\phi(\overline{b}) = m_{M_i^r}\left(\phi(M_i^r, \overline{b})\right) = \dfrac{|\phi(M_i^r;\overline{b})|}{|M_i|^r}$.

Consider now the ultraproduct $K = \prod_{\mathcal{U}} K_i$. Given a definable subset $X := \phi(M;\overline{b})$ of M, we define the real-valued function

$$m(\phi(M;\overline{b})) := \inf\{r \in \mathbb{Q} : K \models f_\phi(\overline{b}) \leq r\}.$$

It is rather routine to show that this is a finitely-additive real valued measure on the definable sets of M. Moreover, by Caratheodory's theorem, this measure extends uniquely to the σ-algebra generated by the definable sets of M. Caratheodory's theorem is the following statement, a proof of which can be found for example in [1, Section 1.3.10].

Theorem 1.3.14 (Caratheodory's extension theorem)**.** *Let $\mathcal{A} \subseteq \wp(X)$ be an algebra of subsets of X. Every pre-measure μ on $\mathcal{A}$ can be extended to a measure on $\sigma(\mathcal{A})$; i.e, the exists a measure ν on $\sigma(\mathcal{A})$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$.*

Moreover, if μ is σ-finite, then ν is uniquely determined by μ.

The most useful feature of this construction is the possibility to estimate the value of the measures of definable sets in the ultraproduct M once we know some estimates for the measures of the definable sets in the finite structures M_i. This transfer principle is a natural consequence of Łoś' theorem applied in the ultraproduct $K = \prod_{\mathcal{U}} K_i$.

Proposition 1.3.15 (cf. [27], Proposition 1.2)**.** *Let $\phi(\overline{x}, \overline{y}$ be an $\mathcal{L}$-formula.*

(i) If $K \models f_\phi(\bar{b}) \leq r$ then $m(\phi(\overline{x}, \bar{b})) \leq r$.

(ii) If $m(\phi(\overline{x}, \bar{b})) < r$ then $K \models f_\phi(\bar{b}) \leq r$.

(iii) If $K \models f_\phi(\bar{b}) \geq r$ then $m(\phi(\overline{x}, \bar{b})) \geq r$

Proof. (i) is clear by definition. For (ii), note that if

$$m(\phi(\overline{x}, \bar{b})) = \inf\{q \in \mathbb{Q} : K \models f_\phi(\bar{b}) \leq q\} < r,$$

there is a rational number $q < r$ such that $K \models f_\phi(\bar{b}) \leq q$, which clearly implies $K \models f_\phi(\bar{b}) \leq r$ since $f_\phi(\bar{b}), q, r$ are elements in the ordered field $OF(K)$.

For (iii), assume that $K \models f_\phi(\bar{b}) \geq r$. Then for every $r' < r$ we have that $K \models \neg(f_\phi(\bar{b}) \leq r')$. In particular, for every rational $r' < r$ we have that $r' \notin \{q \in \mathbb{Q} : K \models f_\phi(\bar{b}) \leq q\}$, that is, for every $r' < r$ we have $r' \leq m(\phi(\overline{x}, \bar{b}))$, which implies $m(\phi(\overline{x}, \bar{b})) \geq r$. □

For rational values, there can be some "infinitesimal disagreement" between $m(\phi(x, b))$ and $f_\phi(b)$ in K. For example, if $r = \dfrac{p}{q} \in \mathbb{Q} \cap (0, 1)$ we can define for $n < \omega$ the structure

$$(M_n, P(M_n)) := (\{1, \ldots, qn\}, \{1, \ldots, pn + 1\}).$$

We have that $f_P^{M_n} = \dfrac{pn+1}{qn} = \dfrac{p}{q} + \dfrac{1}{n}$ which implies $K_n \models \neg f_P \leq r$, and by Łoś' theorem, $K \models \neg f_P \leq r$.

On the other hand, $m(P(M)) = \inf\{r' \in \mathbb{Q} : K \models f_P \leq r'\} \leq \inf\{r + \frac{2}{n} : n < \omega\} = r$. It is important to mention that this infinitesimal disagreement can only happen when $m(\phi(\overline{x}, \bar{b}))$ takes a rational value, that is, if $m(\phi(\overline{x}, \bar{b})) = \alpha \in \mathbb{Q}$, then for a real number r we have $r \leq \alpha$ if and only if $r \leq f_\phi(\bar{b})$.

Goldbring and Towsner developed in [23] a conservative extension of the first order logic (so called *Approximate Measure Logic*) where they take this infinitesimal variation into consideration.

1.4 Uniform quantifier elimination

Consider the status of quantifier elimination for the theory of a finite structure M. If interpreted naively, quantifier elimination is trivial for a finite structure, since every subset of M^n is finite and thus quantifier-free definable (we can express it as a disjunction of formulas $x = a$). However, we can recover an interesting question in the following way: when we work in a class of finite structures $\mathcal{C} = \langle M_i : i < \omega \rangle$ where the cardinality of M_i grows as i tends to infinity, we may ask whether an $\mathcal{L}$-formula $\phi(\overline{x})$ is *uniformly* equivalent to a quantifier-free formula $\psi(\overline{x})$, that is, $M_i \models \forall x\, (\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$ for all $i < \omega$.

Another way to address this situation is to work on the ultraproducts $M = \prod_{\mathcal{U}} M_i$ and find elimination sets for the family of definable sets in the structures M_i. For instance, we obtain by Łoś' theorem that if every ultraproduct M has quantifier elimination, then for any formula $\phi(x)$ there is a finite set $\Phi(x)$ of quantifier free formulas so that for any $M_i \in \mathcal{C}$ there is some $\psi(x) \in \Phi$ with $M_i \models \forall x(\phi(x) \leftrightarrow \psi(x))$.

Now we present some examples of this phenomenon.

Finite linear orders: Let $\mathcal{C}_{ord} = \{M_i : i < \omega\}$ be a class of finite linear orders in the language $\mathcal{L} = \{<\}$. We can expand it to $\mathcal{L}^* = \{c, C, <, S, S^{-1}\}$ where c, C are constant symbols (which will represent the minimum and the maximum of the structure) and S, S^{-1} are unary functions (which will represent the successor and the predecessor functions, respectively, with the convention $S(C) = C$, $S^{-1}(c) = c$).

We claim the class $\mathcal{C}_{ord}$ has uniform quantifier elimination in the language $\mathcal{L}^*$:

In this language, the atomic formulas have the form $S^n(x) = S^m(x)$ or $S^n(x) < S^m(x)$. Consider now a primitive formula, which up to logical equivalence will have the form

$$\exists x \left(\bigwedge_{i<l} t_i < S^{p_i} x \wedge \bigwedge_{j<m} S^{q_j} x < u_j \wedge \bigwedge_{k<n} S^{r_k} x = v_k \right)$$

where t_i, q_j, v_k are all terms which does not include x as a variable. Note that we can uniformize all the indices p_i, q_j, r_k just by taking $N = \max\{p_i, q_j, r_k\}$, because we know that $t_i < S^{p_i} x$ is equivalent to $S^{N-p_i} t_i < S^N x$. So we can assume the formula has the form:

$$\exists x \left(\bigwedge_{i<l} t_i < S^N x \wedge \bigwedge_{j<m} S^N x < u_j \wedge \bigwedge_{k<n} S^N x = v_k \right).$$

If $n \neq 0$, then the formula is equivalent to

$$(v_0 \leq S^{-N}(C)) \wedge \bigwedge_{k<n} v_k = v_0 \wedge \bigwedge_{i<l} t_i < v_0 \wedge \bigwedge_{j<m} v_0 < u_j.$$

Assume now that $n = 0$. We will use induction on l, m:

- $l = 0$: The resulting formula $\exists x \left(\bigwedge_{j<m} S^N x < u_j \right)$ is equivalent to $\bigwedge_{j<m} S^N(c) < u_j$. Similarly when $m = 0$.

- $l = m = 1$: In this case the formula has the form $\exists x \, (t_0 < S^n x \wedge S^n x < u_0)$ which is equivalent to

$$S(t_0) < u_0 \wedge S^{n-1} c \leq t_0.$$

- $k > 1$: Note that the formula

$$\exists x \left(\bigwedge_{i<l} t_i < S^N x \wedge \bigwedge_{j<m} S^N x < u_j \right)$$

is equivalent to the formula

$$\left(t_0 < t_1 \wedge \exists x \left(\bigwedge_{1 \leq i < l} t_i < S^N x \wedge \bigwedge_{j<m} S^N x < u_j \right)\right)$$
$$\vee \left(\neg(t_0 < t_1) \wedge \exists x \left(t_0 < x \wedge \bigwedge_{2 \leq i < l} t_i < S^N x \wedge \bigwedge_{j<m} S^N x < u_j \right)\right)$$

in which each of the formulas with quantifiers have less than l conjunctions, and by induction hypothesis are equivalent to quantifier free formulas. Note that the same argument can be used in the case when $m > 1$. ✓

Abelian groups: In the characterization of the theory of abelian groups given by Szmielew, one of the main tools is the following:

Theorem 1.4.1 (Szmielew's definability theorem.)**.** (See [26, Appendix A2])
Let A be an abelian group and R an n-ary relation on A, definable without parameters in the language of groups. Then R is a boolean combination of relations of the form $\phi(A^n)$ where $\phi(\overline{x})$ is either $t(\overline{x}) = 0$ or $p^m | t(\overline{x})$ for some term t, prime p and positive integer m.

Therefore, if $\mathcal{C}$ is a class of abelian finite groups in the language $\mathcal{L} = \{0, +, -\}$, we can expand the language with predicates $R_{p,m}(y)$ (meaning $p^m | y$) to obtain uniform quantifier elimination along the class $\mathcal{C}$.

In the next section we present a uniform quantifier elimination result that will be used later in this thesis.

1.4.1 Uniform quantifier elimination for $\mathcal{C}_P$.

Let M_n be the finite structure given by $M_n = ([1, n \cdot 2^n], <, P)$ where $<$ is the usual order on $\mathbb{N}$ and P is a unary predicate interpreted as follows:

$$P(M_n) = \bigcup_{k=0}^{n-1} \left[k \cdot 2^n + 1, k \cdot 2^n + \frac{2^n}{2^k} \right].$$

Let $\mathcal{C}_P$ be the class of structures $\{M_n = ([1, n \cdot 2^n], <, P) : n < \omega\}$. In this section we will show that the class $\mathcal{C}_P$ has uniform quantifier elimination (in an expansion of the original language). This example will provide an example of a class of finite structures which is not $\mathcal{O}$-asymptotic, but whose ultraproducts are quasi-$\mathcal{O}$-minimal.

Consider the extension $\mathcal{L}^* = \{<, P, c, C, S, L, R\}$ of $\mathcal{L}$ where c, C are constant symbols for the least and greatest element respectively, S is an unary function interpreted as the successor function, and L, R are 0-definable unary functions interpreted in the following way: for $M \in \mathcal{C}_P$, $M \models y = L(x)$ if and only if

$$\begin{aligned} M \models &(y < x)\,(P(x) \wedge \neg P(y) \wedge \forall z (y < z < x \rightarrow P(z)) \\ &\vee (\neg P(x) \wedge P(y) \wedge \forall z (y < z < x \rightarrow \neg P(z)))\,. \end{aligned}$$

Similarly for $M \models y = R(x)$. Basically, $L(x)$ is the first change of sign for P where you go from x to the left. Similarly with the function $R(x)$.

Consider a primitive formula ϕ in the original language $\mathcal{L}$, say

$$\begin{aligned} \exists x \Bigg(&\bigwedge_{i<l} t_i < S^N x \wedge \bigwedge_{j<m} S^N x < u_j \wedge \bigwedge_{k<n} S^N x = v_k \\ &\wedge \bigwedge_{i,j,k} (P(t_i)^{p_i} \wedge P(u_j)^{p_j} \wedge P(v_k)^{p_k}) \wedge (P(x))^{p_x} \Bigg) \end{aligned}$$

where $p_i, p_j, p_k, p_x \in \{0, 1\}$ in the notation $\varphi^0 := \varphi, \varphi^1 := \neg\varphi$.

We have to show that this formula is equivalent to a formula which only depends on the terms t_i, u_j, v_k. As in the previous section, if $n > 0$ the formula ϕ is equivalent to $t_i = t_i$ or $t_i \neq t_i$, depending on whether the corresponding term $x = S^{-N} v_k$ is a witness of the existential formula or not.

Also, the first two conjunctions will place x into a union of intervals, so we may assume that ϕ has the form

$$\begin{aligned}\phi :&= \exists x \left(\left(\bigvee_{i<l} a_i < x < b_i \vee \bigvee_{j<n} x = c_j \right) \wedge \bigwedge_{i<l} (P(a_i)^{p_i^a} \wedge P(b_i)^{p_j^b} \wedge \bigwedge P(c_j)^{p_j^c}) \wedge (P(x))^{p_x} \right) \\ &\equiv \exists x \left(\left(\bigvee_{i<l} a_i < x < b_i \vee \bigvee_{j<n} x = c_j \right) \wedge (P(x))^{p_x} \right) \wedge \bigwedge_{i<l} (P(a_i)^{p_i^a} \wedge P(b_i)^{p_j^b} \wedge \bigwedge_{j<n} P(c_j)^{p_j^c}).\end{aligned}$$

So it is enough to find the equivalent formula corresponding to:

$$\begin{aligned}\phi :&= \exists x \left(\left(\bigvee_{i<l} a_i < x < b_i \vee \bigvee_{j<n} x = c_j \right) \wedge P(x)^{p_x} \right) \\ &\equiv \exists x \left(\bigvee_{i<l} (a_i < x < b_i \wedge P(x)^{p_x}) \vee \bigvee_{j<n} (x = c_j \wedge P(x)^{p_x}) \right) \\ &\equiv \bigvee_{i<l} \exists x (a_i < x < b_i \wedge P(x)^{p_x}) \vee \bigvee_{j<n} \exists x (x = c_j \wedge P(x)^{p_x}).\end{aligned}$$

Again, every formula $\exists x(x = c_j \wedge P(x)^p_x)$ is equivalent to either $c_j = c_j$ or $c_j \neq c_j$. On the other hand the formula $\exists x(a < x < b \wedge P(x))$ is equivalent to either $a \neq a$ (if $S(a) \geq b$) or

$$\widehat{\phi} := (\neg P(a) \wedge R(a) < b) \vee (P(a) \wedge R(a) \neq S(a)) \vee (P(a) \wedge R(a) = S(a) \wedge R(S(a)) < b).$$

1.5 Some Fundamentals on Classification Theory

Starting with ideas coming from his proof of Morley's conjecture for uncountable theories, Saharon Shelah developed in the seventies the field of *Classification Theory* which has been a major influence in the development of Model Theory, especially in the development of what is called *geometric model theory*. It can be described very naively as an attempt of classificate the first-order theories with respect to local combinatorial properties on formulas that encode global model-theoretic information of their structures (e.g., number of types), or with respect to the existence of independence relations in their models.

Two important concepts in Classification Theory are the notion of *forking* and the independence relation derived from it, and the classes of stable theories, simple theories and theories with NIP that serve as a preliminary classification of the most known examples of first-order theories.

Here we recall some of the main definitions that will be crucial for the understanding of this work. These notions will be used throughout this thesis when applied to the ultraproducts of finite structures.

1.5.1 Forking and Simple theories

Definition 1.5.1. *Let $\mathcal{L}$ be a first-order language, $\phi(\overline{x};\overline{y})$ an $\mathcal{L}$-formula and M an $\mathcal{L}$-structure.*

1. *We say that $\phi(\overline{x};\overline{y})$ has the* order property *relative to the theory of M if for every $n<\omega$ there are tuples $\langle \overline{a}_i, \overline{b}_i : i \leq n\rangle$ in M such that $M \models \phi(\overline{a_i};\overline{b_j})$ if and only if $i<j$.*

2. *We say that $\phi(\overline{x};\overline{y})$ has the* tree property *relative to the theory of M if there is some integer $k \geq 2$ such that the following hold: for every $n<\omega$ there are tuples $\langle \overline{a}_\sigma : \sigma \in \{0,\dots,n-1\}^{\leq n}\rangle$ such that:*

    - *For every $s \in \{0,\dots,n-1\}^{<n}$, the set $\{\phi(\overline{x};\overline{a}_{s^\frown i}) : i \leq n\}$ is k-inconsistent.*
    - *For every $f \in \{0,\dots,n-1\}^{n}$, the set $\{\phi(\overline{x};\overline{a}_{f\restriction j}) : j \leq n\}$ is consistent.*

It is customary in Classification Theory to define a desirable property as the negation of the undesirable properties. The following definition exemplifies this fact perfectly.

Definition 1.5.2. *Let M be an $\mathcal{L}$-structure.*

1. *We say that M is* stable *if no formula has the order property in M.*

2. *We say that M is* simple *if no formula witnesses the tree property in M.*

Similarly, we say that a complete theory T is stable (resp. simple) if every model of T is stable (resp. simple).

It is important to mention that when we work in models which are saturated enough, these properties (order property, tree property) have equivalent definitions in terms of the existence of infinite sequences in a combinatorial arrangement. We will not use the latter because we are dealing with classes of finite structures and their ultraproducts.

Also, it is important to mention that as soon as a partial order with infinite (or by compactness arbitrary long) chains is definable in a theory T, the theory is not simple.

This can be shown in the following way: assume $\phi(x, y)$ defines a partial order $\prec$ with infinite chains. Then there exists elements $b_1 \prec \ldots \prec b_{n^n}$. For $s = (k_1, \ldots, k_m) \in \{0, \ldots, n-1\}^{\leq n}$, consider the tuple $\overline{a_s} = a_s^- a_s^+$ defined as

$$a_s^- = b_{k_1 \cdot n^{n-1} + k_2 \cdot n^{n-2} + \ldots + k_m \cdot n^{n-1}} \qquad a_s^+ = b_{(k_1+1) \cdot n^{n-1} + k_2 \cdot n^{n-2} + \ldots + k_m \cdot n^{n-1}}$$

These tuples witness the tree-property (with $k = 2$) for the formula $\phi(x; y_1 y_2) := y_1 \prec x \prec y_2$.

Now, we proceed to define forking and the independence relation that comes from it.

Definition 1.5.3 (Forking)**.** *Let M be a saturated model.*

1. *A formula $\phi(x, b)$ is said to* divide over A *if there is an infinite sequence $\langle b_i : i < \omega \rangle$ such that:*

   - *For every $i < \omega$, $b_i \models tp(b/A)$.*

- *The set $\{\phi(x, b_i); i < \omega\}$ is k-inconsistent for some $k < \omega$.*

2. *A formula* forks over A *if it implies a finite disjunction of formulas (possibly over additional parameters) which divide over A.*

3. *A type* forks (divides) *over A if it implies a formula which forks (divides) over A.*

4. *We write $a \mathop{\smile\!\!\!\!\!\!|}_A B$ (read as a is independent of B over A) to denote that $tp(a/B)$ does not fork over A.*

For example, in any theory the formula $x = b$ will divide over A whenever $b \notin \operatorname{acl}(A)$. Moreover, this notion of independence generalizes the notion of algebraic independence in the complex field, as well as the concept of linear independence for vector spaces.

Forking is fundamental to the understanding of simple theories, as it was shown by Byunghan Kim and Anand Pillay in [32]. Roughly speaking, simple theories are exactly those theories for which non-forking has a nice independence-like behavior (for instance, $\mathop{\smile\!\!\!\!\!\!|}$ is symmetric, as in page 29), and this is the main reason why a lot of the research in model theory during the nineties was oriented to study simple theories.

Shelah introduced several ranks which among other things generalize Krull's dimension from commutative algebra as well as transcendence degree (in algebraically closed fields) or linear dimension in vector spaces. The idea behind these notions was trying to import some geometric intuition into the study of the definable sets of a given structure, or the study of definable sets among the class of structures of a given theory.

Definition 1.5.4. 1. *We say that $q \in S(B)$ is a* forking extension *of $p \in S(A)$ (with $A \subseteq B$) if q is an extension of p and the type q forks over A. Otherwise, we call it a non-forking extension of p.*

2. *We define the* U*-rank of types to be the foundation rank of forking. Namely,* $U(p(x)) \geq 0$ *if and only if* $p(x)$ *is consistent,* $U(p(x)) \geq \alpha + 1$ *if and only if there is a forking extension* $q(x)$ *of* $p(x)$ *such that* $U(q(x)) \geq \alpha$ *and for a limit ordinal* λ*,* $U(p(x)) \geq \lambda$ *if and only if* $U(p(x)) \geq \alpha$ *for every* $\alpha < \lambda$*.*

Definition 1.5.5. *A structure* M *is said to be* supersimple of U-rank n *if there is an* 1*-type* $p(x)$ *such that* $U(p(x)) = n$*, but there is no* 1*-type* $q(x)$ *with* $U(q(x)) \geq n+1$*.*

In order to have an specific example consider the set $A = \mathbb{Q}$ viewed as subset of $\mathbb{C}$ and the type $p(x) = tp(\pi/A)$ in the theory of algebraically closed fields. This type is determined simply by saying that π is not the solution of any non-zero polynomial with rational coefficients.
Consider now $B = \mathbb{Q}[\pi]$ and $q(x) = tp(\sqrt{1-\pi}/B)$. Note that $q(x)$ is a forking extension of $p(x)$ since we may take the formula $\phi(x, \pi) := x^2 + \pi = 1 \in q(x)$. If we consider an indiscernible sequence $\langle b_i : i < \omega \rangle$ with $b_0 = \pi$ over $A = \mathbb{Q}$ (that in this case will only correspond to a sequence of algebraically independent transcendentals) we will have that $\{x^2 + b_i = 1 : i < \omega\}$ is 2-inconsistent.
Therefore, we have that $U(q(x)) + 1 \leq U(p(x))$ (p has a forking extension). On the other hand, we know that trans.deg$(\mathbb{Q}(\pi)/\mathbb{Q}) = 1$, and trans.deg$(\mathbb{Q}(\sqrt{1-\pi})/\mathbb{Q}(\pi)) = 0$. In this sense, forking is witnessed by a drop of the dimension (either U-rank or the dimension given by the transcendence degree).

This idea that forking is witnessed by some sort of drop of dimension will be the motivating idea in Chapter 2, in which we will study the relationship between forking and the drop of what we will call *pseudofinite dimension* in ultraproducts of finite structures.

The following example is both basic and classical, and will be used later in this work.

Example 1.5.6. Let $\mathcal{L} = \{E\}$ be the language consisting in a single binary relation E. Let T be the theory saying that:

1. E is an equivalence relation.

2. Each class of E is infinite.

3. There are infinitely many distinct equivalent classes.

Let M be a model of T and consider an element $b \in M$. We will show that $U(tp(b/\emptyset)) \geq 2$, and for this it is enough to show that there is a dividing chain of length 2.

Choose a sequence $\langle a_i : i < \omega \rangle$ of elements in M such that $M \models \neg(a_i E a_j)$ for $i \neq j$, and let $b \neq a_1$ so that bEa_1. If $A = \{a_i : i < \omega\}$ and $B = A \cup \{b\}$. Then we have the following:

- *$p(x) = tp(b/A)$ divides over $\emptyset$:* Note that the formula $xEa_1 \in p(x)$, and the set of formulas $\{xEa_i : i < \omega\}$ is 2-inconsistent because the elements a_i are in different equivalence classes.

- *$q(x) = tp(b/B)$ is a dividing extension of $p(x)$:* The formula $x = b \in tp(b/A \cup \{b\})$ divides over A, as it is witnessed by a sequence of distinct elements in the same class of a_1.

So, since we have the dividing chain $tp(b/\emptyset) \subset p(x) \subset q(x)$, we conclude that $U(tp(b/\emptyset)) \geq 2$. It is not hard to show that, indeed, $U(tp(b/\emptyset)) = U(M) = 2$.

1.5.2 $\mathcal{O}$-minimal theories

Definition 1.5.7.

1. *Let $(M, <, \cdots)$ be an $\mathcal{L}$-structure for a language $\mathcal{L}$ extending the language or order $\mathcal{L}_{ord} = \{<\}$. The structure M is said to be $\mathcal{O}$-minimal if every definable*

subset of M is a finite union of intervals and points. This is, for every formula $\phi(x;\overline{y})$ (with $|x| = 1, |\overline{y}| = m$) and every $\overline{a} \in M^m$ the definable set $\phi(M;\overline{a})$ is a finite union of intervals and points.

2. *A theory is $\mathcal{O}$-minimal if every model of T is $\mathcal{O}$-minimal.*

It was proved by Knight, Pillay and Steinhorn in [33, Theorem 0.2] that for a complete theory T, if T is the theory of some linearly ordered $\mathcal{O}$-minimal structure then every model of T is $\mathcal{O}$-minimal.

The study of $\mathcal{O}$-minimal theories was initiated in [44]. A very good exposition about the most important results and examples can be found in [15]. Among the examples of $\mathcal{O}$-minimal structures we can find $(\mathbb{Q}, <)$, the real ordered field $(\mathbb{R}, +, \cdot, 0, 1, <)$, and several expansions of it such as the real-exponencial field $\mathbb{R}_{exp} = (\mathbb{R}, +, \cdot, 0, 1, \exp)$ or the real field expanded with all restricted subanalytic functions, $\mathbb{R}_{an}$.

As with the ordered field of the real numbers, in every $\mathcal{O}$-minimal structure there is a notion of dimension which is related to the geometric notion of dimension in which the dimension of a definable set is the maximal of the dimensions of boxes contained inside it. This definition of dimension may be extended to type-definable sets $p(x_1, \ldots, x_n)$ by taking $\dim(p) = \min\{\dim(\varphi(M^n)) : \varphi \in p\}$.

In some of the most classical model-theoretic contexts a notion of dimension plays a central role. Usually we also have that, once the definition imposes a condition only on definable sets in one variable, it is possible to obtain dimension-theoretic consequences for definable sets in several variables.

In the case of $\mathcal{O}$-minimal theories, the result providing information about higher-dimensional definable sets is the *Cell Decomposition Theorem* (see [15], Theorem 2.11). For this, we first have to give the following definition

Definition 1.5.8. *For every $n \in \mathbb{N}$, we define the k-*cells *in M^n by induction on n as follows:*

1. *A 0-cell in M is a point, a 1-cell in M is an open interval.*
2. *Assume $C \subseteq M^n$ is a k-cell.*
   - *If $f : C \longrightarrow M$ is a definable continuous function, then the graph of f is a k-cell in M^{n+1}.*
   - *If $f, g : C \longrightarrow M$ are definable continuous functions with $f(x) < g(x)$ for every $x \in C$ (f, g may be the constant functions $-\infty$, $+\infty$), then the set*

     $$\{(\overline{x}, y) \in M^n \times M : \overline{x} \in C, f(\overline{x}) < y < g(\overline{x})\}$$

     is a $(k+1)$-cell in M^{n+1}.

Definition 1.5.9. *A* cell-decomposition *of M^n is a collection of cells $\{Z_1, \ldots, Z_m\}$ such that every Z_i is a cell in M^n, $Z_i \cap Z_j = \emptyset$ for $i \neq j$, and $M^n = \bigcup_{i \leq m} Z_i$.*

Theorem 1.5.10 (Cell Decomposition Theorem)**.** *For every definable set $A = \varphi(M^n; \overline{a})$ there is a cell-decomposition of M^n such that A can be expressed as a finite union of cells in the decomposition.*

In every $\mathcal{O}$-minimal theory, algebraic closure coincides with definable closure, and we have a well-behaved notion of independence given by $\overline{a} = (a_1, \ldots, a_n)$ is independent from $\overline{b}$ over A if and only if for every $i < n$, if $a_{i+1} \in dcl(a_1 \ldots a_i A\overline{b})$ then $a_{i+1} \in dcl(a_1, \ldots, a_i A)$. Equivalently, we have that $\overline{a}$ is dependent of $\overline{b}$ over A if $\dim(tp(\overline{a}/A\overline{b})) < \dim(tp(\overline{a}/A))$.

So, as in the complex field, $\mathcal{O}$-minimal theories have a good notion of independence and any "dependence" may be witnessed with a drop of dimension. However, forking fails to provide a good notion of independence in this setting. (Note that any $\mathcal{O}$-minimal theory is not simple because of the presence of a linear order).

A weaker variant of $\mathcal{O}$-minimality is given by the concept of *quasi-$\mathcal{O}$-minimality*, and among the main examples we can find the structures $\mathcal{M} = (\mathbb{R}, <, Q)$ where Q is a predicate for the rational numbers, or the ordered group of integers $(\mathbb{Z}, +, <)$.

Definition 1.5.11. *A structure M is said to be* quasi-$\mathcal{O}$-minimal *if, for any $N \equiv M$, its definable subsets are exactly the Boolean combinations of $\emptyset$-definable subsets and intervals with endpoints in N.*

This definition was given by Belegradek, Peterzil and Wagner in [2], and they also proved the following syntactical characterization for quasi-$\mathcal{O}$-minimal theories that will be used in Chapter 3.

Theorem 1.5.12 (Belegradek, Peterzil, Wagner)**.** *A structure M is quasi-$\mathcal{O}$-minimal if and only if for every formula $\theta(x;\overline{y})$ there is a formula $\chi(x,\overline{y},\overline{z})$ of the form*

$$\bigvee_i (\phi_i(\overline{x}) \wedge \psi_i(\overline{y}) \wedge \rho_i(x;\overline{z})),$$

where each $\rho_i(x;\overline{z})$ is a conjunction of formulas of one of the forms $x = z, x < z$ and $z < x$, such that

$$M \models \forall\overline{y}\,\exists\overline{z}\,\forall x(\theta(x,\overline{y}) \leftrightarrow \chi(x,\overline{y},\overline{z})).$$

1.5.3 Theories with NIP and theories with NTP_2.

During the last two decades, theories with NIP have received the attention of model-theorists, who aim to extend notions and ideas coming from stability theory to this broader context. Generically stable types appear as a generalization of the properties of types in stable theories (such as definability, stationarity) to unstable contexts. These include in particular stable types, defined by Lascar and Poizat, as well as the *stably dominated types* that are at the heart of the model theory of algebraically closed valued fields. The use of Keisler measures and the concept of generically stable types/measures) have shown to be very important in the study of theories with NIP.

The class of theories with NTP_2 contains both simple theories and theories with NIP. Even if it was defined in [47, 48], its systematic study was initiated in [10].

Now we give the definitions of these classes of first-order theories.

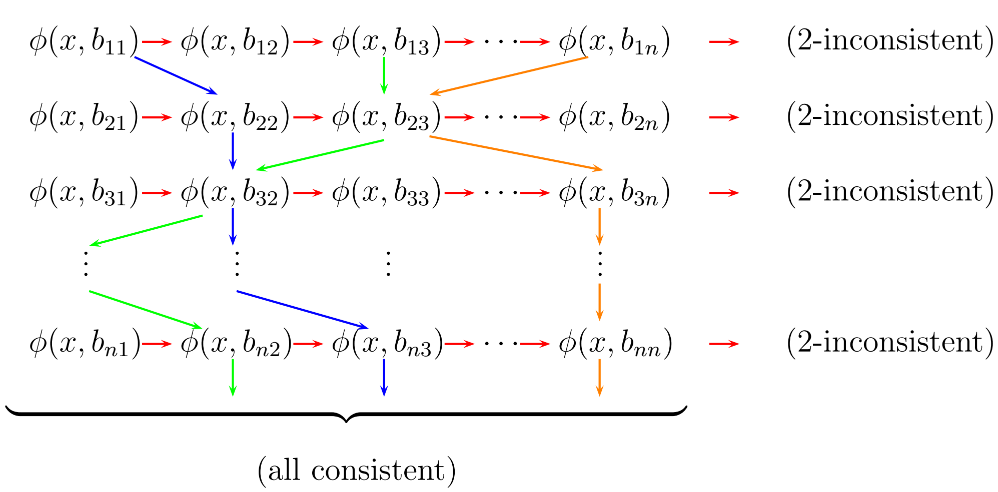
Definition 1.5.13.

1. *We say that* $\phi(\overline{x};\overline{y})$ *has the* independence property *relative to the theory of* M *if for every* $n < \omega$ *there are tuples* $\langle \overline{a}_i, \overline{b}_J : i \leq n, J \subseteq n \rangle$ *in* M *such that* $M \models \phi(\overline{a}_i; \overline{b}_J)$ *if and only if* $i \in J$.

2. *Let* M *be a saturated model. A formula* $\phi(x,y)$ *has* TP_2 *(standing for* tree property of the second kind*) relative to* M *if for every* $n < \omega$ *there is an array* $\langle b_{\alpha,i} : \alpha, i \geq n \rangle$ *of elements in* M *such that:*

   - *For every* $\alpha < \omega$*, the set* $\{\phi(x, b_{\alpha,i}) : i \leq n\}$ *is 2-inconsistent.*
   - *For any function* $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$*, the set* $\{\phi(x, b_{\alpha,f(\alpha)}) : \alpha \leq n\}$ *is consistent.*

A way to understand TP_2 is to think about an "arbitrary large rectangular array" of formulas for which every row is 2-inconsistent, but every "descending path" is consistent.



Definition 1.5.14.

1. *We say that M has* NIP *(resp. NTP$_2$) if no $\mathcal{L}$-formula witnesses the independence property (resp. TP$_2$) in M.*

2. *We say that a theory T has NIP (resp. NTP$_2$) if every model of T has NIP (resp. NTP$_2$).*

The class of theories with NIP include all stable theories, $\mathcal{O}$-minimal theories, the theory of Algebraically Closed Valued Fields, Presburger Arithmetic, and many others.

There are several recent results on classical theories that have NTP$_2$ but are not simple theories nor theories with NIP: they include ultraproducts of p-adics (see [12]) and the theory of the non-standard Frobenius automorphism acting on an algebraically closed valued field of equicharacteristic 0 (see [13]).

The following results about theories with NTP$_2$ will be used in Chapter 3.

Fact 1.5.15. *If T has TP$_2$, there is some formula $\phi(x, \overline{y})$ with $|x| = 1$ that has TP$_2$.*

Fact 1.5.16 (Chernikov, Lemma 2.2 in [10])**.** *Assume M is a saturated model of T and there is an array $\langle \overline{b}_{\alpha,i} : \alpha, i < \omega \rangle$ witnessing TP$_2$ (2-inconsistent rows, consistent descending paths) for the formula $\phi(x; \overline{y})$ in M. Then the rows may be assumed to be mutually indiscernible sequences.*

1.5.4 þ-Forking and Rosy Theories

The notion of þ-forking (which should be read as *thorn-forking*) was introduced in Alf Onshuus' Ph.D. Thesis and provides a generalization of forking to contexts in which non-forking failed to provide a nice independence relation (such as $\mathcal{O}$-minimal theories).

Definition 1.5.17 (þ-forking)**.** *Let M be a saturated model.*

1. *A formula* $\phi(\overline{x},\overline{b})$ strongly divides *over* A *if* $\overline{b}$ *is not algebraic over* A *and there is* $k<\omega$ *such that the set* $\{\phi(\overline{x},\overline{b}'):\overline{b}'\models tp(\overline{b}/A)\}$ *is* k*-inconsistent.*

2. *We say that* $\phi(\overline{x},\overline{b})$ þ-divides *over* A *if there is a finite tuple* $\overline{c}$ *such that* $\phi(\overline{x},\overline{b})$ *strongly divides over* $A\overline{c}$*.*

3. *We say that* $\phi(\overline{x},\overline{b})$ þ-forks *over* A *it there are formulas* $\psi_1(\overline{x},\overline{c}_1),\ldots,\psi_n(\overline{x},\overline{c}_n)$ *such that each* $\psi_i(\overline{x},\overline{c}_i)$ *þ-divides over* A *and* $\phi(\overline{x},\overline{b})\vdash\bigvee_{i=1}^{n}\psi_i(\overline{x},\overline{c}_i)$*.*

4. *A type* þ-forks (þ-divides) over A *if it implies a formula which þ-forks (þ-divides) over* A*.*

5. *We write* $\overline{a}\mathop{\smile\!\!\!\!\!\!|}^{þ}_{A}B$ *(read as* $\overline{a}$ is thorn-independent of B over A*) to denote that* $tp(\overline{a}/B)$ *does not þ-fork over* A*.*

Simple theories are those theories on which forking-independence has nice properties and can be characterized as theories where the forking-independence is symmetric (meaning that $\overline{a}\mathop{\smile\!\!\!\!\!\!|}_{A}\overline{b}$ if and only if $\overline{b}\mathop{\smile\!\!\!\!\!\!|}_{A}\overline{a}$). There is a class of theories called *rosy* for which the þ-forking has desirable properties, and can be also characterized as those theories where the þ-independence is a symmetric independence relation.

Definition 1.5.18. 1. *We say that* $q\in S(B)$ *is a* þ-forking extension *of* $p\in S(A)$ *(with* $A\subseteq B$*) if* q *is an extension of* p *and the type* q *þ-forks over* A*. Otherwise, we called it a non-þ-forking extension of* p*.*

2. *We define the* $U^{þ}$*-rank (read as U-thorn-rank) to be the foundation rank for þ-forking. So,* $U^{þ}(p(x))\geq 0$ *if and only if* $p(x)$ *is consistent,* $U^{þ}(p(x))\geq\alpha+1$ *if and only if there is a þ-forking extension* $q(x)$ *of* $p(x)$ *such that* $U(q(x))\geq\alpha$ *and for a limit ordinal* λ*,* $U^{þ}(p(x))\geq\lambda$ *if and only if* $U^{þ}(p(x))\geq\alpha$ *for every* $\alpha<\lambda$*.*

Definition 1.5.19. *A structure* M *is said to be* superrosy of U^{þ}-rank n *if there is an* 1*-type* $p(x)$ *such that* $U^{þ}(p(x))=n$*, but there is no* 1*-type* $q(x)$ *with* $U^{þ}(q(x))\geq n+1$*.*

The basic properties of þ-forking and superrosy theories were studied in [40]. It was shown there that in $\mathcal{O}$-minimal theories the notion of þ-independence coincides with the usual notion of independence, and the $\mathrm{U}^{þ}$-rank corresponds to the $\mathcal{O}$-minimal dimension on definable sets.

The class of rosy theories is the class of theories for which þ-forking has a nice behavior. Also, according to Definition 1.5.19 we might say that super-rosy of $\mathrm{U}^{þ}$-rank n if n is the length of the maximal þ-dividing chain for 1-types in M.

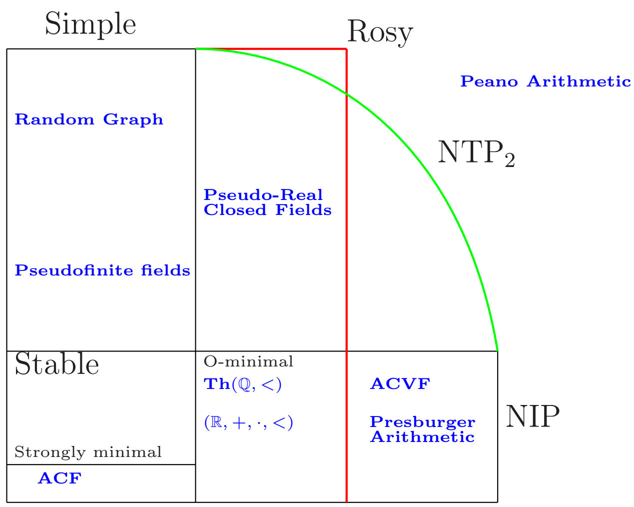
It is known that in the presence of a definable order, forking is very different from þ-forking. For example, in the theory $\mathrm{Th}(\mathbb{Q},<)$ we have that the formula $\varphi(x) := a < x < b$ divides over the empty set (despite the fact that the $\dim(\varphi(M)) = 1$), but it does not þ-fork over the empty set.

Onshuus proved that the notions of forking and þ-forking coincide for types in stable theories and in all simple theories under the stable-forking conjecture. Later, Ealy proved in [16] that forking is equivalent to þ-forking for types in simple theories which eliminates hyperimaginaries.

Onshuus, Usvyatsov and the author proved in [22] that these two notions coincide for generically stable types in arbitrary theories, which allows us to combine the power of forking-independence with the geometric intuition behind the þ-independence to the study of, for example, stably dominated types in algebraic closed valued fields.

1.5.5 Final remarks and the map of the universe

In order to give some motivation for some results that will be mentioned through this work, it might be helpful to show a part of what some model-theorists call "the map of the universe", which is nothing more than a representation of the universe of the first-order theories, with some regions corresponding to the dividing lines given by the different properties from the Classification Theory. A very incomplete map (which nevertheless contain all the properties that will be used through this thesis) is shown in the following figure:



It is a well known theorem of Shelah that a theory T is stable if and only if T is simple and has NIP (which implies the main three black rectangles on the picture). Stable theories contain all strongly minimal theories (such as the theory of Algebraically Closed Fields), but also contains many other interesting examples.

The class of rosy theories includes both $\mathcal{O}$-minimal theories (such as $Th(\mathbb{Q},<)$ or $(\mathbb{R},+,\cdot,<)$) and simple theories (such as the theory of pseudofinite fields or the theory of the Random Graph), while the class of theories with NTP$_2$ includes both theories with NIP and simple theories.

As an example of a rosy theory which has TP$_2$ we can mention the theory of the so-called *generic triangle-free random graph* (not in the picture). On the other hand, the theory of Algebraically Closed Valued Fields has NIP (thus NTP$_2$) but it is not rosy.

Beyond the NTP$_2$ and the rosy theories we can find for instance the theory of Peano Arithmetic or the theory of ZFC.

This picture shows that the two possible extensions of both $\mathcal{O}$-minimal theories and simple theories are the rosy theories and the theories with NTP$_2$. This fact is the motivation of the intended classification for the ultraproducts of $\mathcal{O}$-asymptotic classes that will appear in Chapter 3.

1.6 1-dimensional asymptotic classes

The asymptotic classes of finite structures appear as an attempt to isolate conditions on finite structures inspired by the notions of dimension, independence and complexity measures in model theory.

The starting point here is the celebrated theorem of Chatzidakis, van den Dries and Macintyre that appears in [7], and is a generalization (using methods of Ax) of the Lang-Weil estimates for number of points of varieties in finite fields that appeared in [35].

Theorem 1.6.1 (Chatzidakis, van den Dries, Macintyre). *Let $\varphi(\overline{x}, \overline{y})$ be a formula in the language $\mathcal{L}_{rings} = \{+, \times, -, 0, 1\}$ with $|\overline{x}| = n, |\overline{y}| = m$.*

Then, there is a positive constant C and a finite set D of pairs (d, μ) with $d \in \{0, 1, \ldots, n\}$ and $\mu \in \mathbb{Q}^{\geq 0}$ such that for each finite field $\mathbb{F}_q$ and $\overline{a} \in \mathbb{F}_q^m$, there is a pair $(d, \mu) \in D$ such that

$$||\varphi(\mathbb{F}_q^n; \overline{a}) - \mu\, q^d| \leq C q^{d-1/2}. \qquad (*)$$

Furthermore, for each $(d, \mu) \in D$ there is a formula $\varphi_{(d,\mu)}(\overline{x})$ which defines in each finite field $\mathbb{F}_q$ the set

$$\left\{\overline{a} \in \mathbb{F}_q^m : ||\varphi(\mathbb{F}_q^n; \overline{a}) - \mu\, q^d| \leq C q^{d-1/2}\right\}.$$

Macpherson and Steinhorn used this result to define what they called *1-dimensional asymptotic classes*, concept which is later generalized to *N-dimensional asymptotic classes* by Elwes in his Ph.D. dissertation. (also in [18])

Definition 1.6.2. *Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures where $\mathcal{L}$ is a finite language (in the sense of finitely many constant, relation and function symbols).*

We say that $\mathcal{C}$ is a 1-dimensional asymptotic class *if the following hold for every $m \in \mathbb{N}$ and every formula $\varphi(x, \overline{y})$, where $\overline{y} = (y_1, \cdots, y_m)$:*

1. *There is a positive constant C and a finite set $E \subseteq \mathbb{R}^{>0}$ such that for every $M \in \mathcal{C}$ and $\overline{a} \in M^m$, either $|\varphi(M, \overline{a})| \leq C$, or for some $\mu \in E$,*

$$||\varphi(M, \overline{a})| - \mu|M|| \leq C|M|^{1/2}.$$

2. *For every $\mu \in E$, there is an $\mathcal{L}$-formula $\varphi_\mu(\overline{y})$ such that for all $M \in \mathcal{C}$, $\varphi_\mu(M^m)$ is precisely the set of $\overline{a} \in M^m$ with*

$$||\varphi(M, \overline{a})| - \mu|M|| \leq C|M|^{1/2}.$$

Roughly speaking, the 1-dimensional asymptotic classes are classes of finite structures with a strong uniformity condition on the cardinality of its definable sets in one variable that reflects a similar behavior to that of the class of finite fields.

Note that the set of measures does not include $\mu = 0$. In fact, we might say that measure zero sets are those which are uniformly finite, with size bounded by the constant C.

As a very trivial example, we could consider the class of finite structures with just equality. The definable sets in this case will be either finite or cofinite (with uniform finiteness bounds for each formula) and therefore the only possible measures will be either 0 (meaning they are finite but uniformly bounded) or 1 in the case of cofinite (again, with a uniform bound on the number of points in the complement)

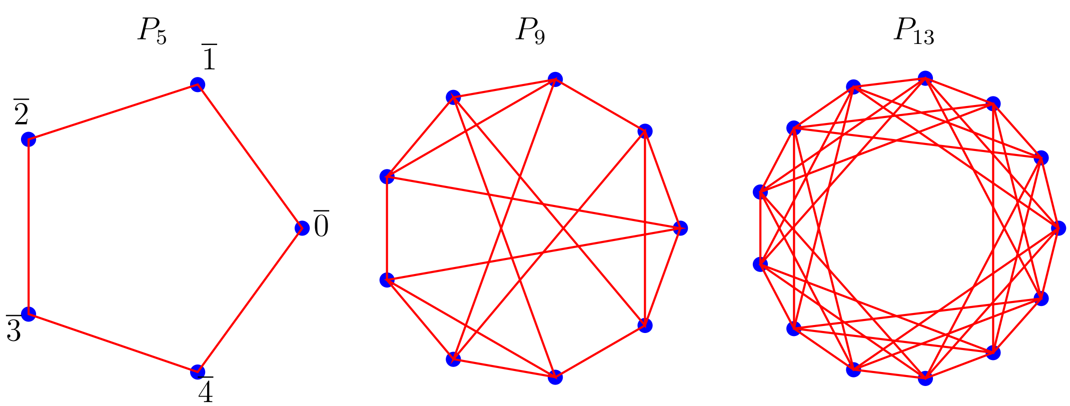
1.6.1 Examples

In what follows we present the main known examples of 1-dimensional asymptotic classes.

In order to prove that a class of finite structures is a 1-dimensional asymptotic class usually we must pass through two stages: first, we must describe the definable sets in one-variable along the structure -this usually uses a uniform quantifier elimination result -, and then we can compute the sizes to determine the possible measures.

Finite Fields: The seminal example of these classes is the class of finite fields, for which these conditions appear as the remarkable theorem of Chatzidakis, van den Dries and Macintyre mentioned earlier.

Paley graphs: For each prime $q \equiv 1 \pmod 4$, we can define the graph $P_q = (V, R)$ where $V = \mathbb{F}_q$ (the field with q elements) and the relation R is defined by xRy if and only if $x - y$ is a square in $\mathbb{F}_q$.



It was shown in [38] that the class $\mathcal{C} = \{P_q : q \equiv 1\ (mod\ 4),\ q \text{ prime}\}$ is a 1-dimensional asymptotic class. The main ingredient is the following result from Bollobás and Thomason (see Theorem 10 in Ch. XIII.2 of [4]).

Theorem 1.6.3. *Let U and W be disjoint sets of vertices of P_q with $|U \cup W| = m$. Let $v(U,W)$ be the number of vertices not in $U \cup W$ joined to each vertex of U and no vertex in W. Then,*

$$|v(U,W) - 2^{-m}q| \leq \frac{1}{2}(m - 2 + 2^{-m+1})q^{1/2} + \frac{m}{2}.$$

From this theorem it is possible to conclude that every infinite ultraproduct of Paley graphs is elementarily equivalent to the random graph (just because it satisfies the required axioms), from which we can deduce that every ultraproduct has quantifier elimination and thus there is uniform quantifier elimination on the class. Macpherson and Steinhorn use this fact and the quantitative nature of the Bollobás-Thomason result to obtain the appropriate measures for boolean combinations of formulas of the form $v(U,W)$ which are the possible definable sets in one-variable.

Cyclic groups: Let $\mathcal{C}_{cyc}$ be the class of cyclic groups, namely $\mathcal{C}_{cyc} = \{\mathbb{Z}/n\mathbb{Z} : n < \omega\}$. Macpherson and Steinhorn also proved in [38] that this class is a 1-dimensional asymptotic class. The quantifier elimination result is given by the Szmielew's theorem mentioned in Section 1.4. From there they made a careful computation to know what are the possible measures for the given definable sets, as well as the appropriate definition formulas.

Difference fields: In [46], Ryten proved that certain class of difference fields give more examples of 1-dimensional asymptotic classes. Recall that a *difference field* is a pair (F, σ) where F is a field and σ is an automorphism of F.

Now, fix a prime p and integers $m \geq 1, n > 1$ with $(m,n) = 1$. Let $\mathcal{C}_{m,n,p}$ be the collection of difference fields of the form $(\mathbb{F}_{p^{kn+m}}, \mathrm{Frob}^k)$ where Frob^k is the Frobenius automorphism on $\mathbb{F}_{p^{kn+m}}$, given by $\mathrm{Frob}^k(x) = x^p$.

Theorem 1.6.4 (Ryten)**.** *The class $\mathcal{C}_{m,n,p}$ is a 1-dimensional asymptotic class.*

Non-example: Finite linear orders. The class of finite linear orders is not a 1-dimensional asymptotic class, since the formula $x < a$ can pick out an initial segment of M of arbitrary size as a varies. Otherwise, if $\mathcal{C}_{ord}$ is a 1-dimensional asymptotic class, we can take $\nu > 0$ to be the minimal positive possible measure for the formula $\varphi(x; a) := x < a$. For arbitrarily large n, consider the linear order given by $([0, n] \cap \mathbb{N}, <)$ and $a = [\![\frac{\nu}{2}]\!] \cdot n$. Then for any $\mu \in E_\varphi$ and fixed constant $C > 0$ we would have

$$||\varphi(M; a) - \mu|M|| \geq \mu \cdot n - \frac{\nu}{2}n \geq \frac{\nu}{2}n \geq 2Cn^{1/2} > C|M|^{1/2},$$

obtaining a contradiction.

Despite the fact that the class $\mathcal{C}_{ord}$ is not a 1-dimensional asymptotic class, its infinite ultraproducts are well understood and have very nice model-theoretic properties (they are discrete $\mathcal{O}$-minimal structures). This observation lead us to think that they might deserve a different treatment, namely, a different quantitative condition on its definable sets from which we can get some good properties on their ultraproducts.

Trying to find such quantitative condition (extending also the conditions given for 1-dimensional asymptotic classes) was the problem that motivated my work on $\mathcal{O}$-asymptotic classes, that I expose in Chapter 3.

1.6.2 Ultraproducts of 1-dimensional asymptotic classes

In [38], Macpherson and Steinhorn also obtained a classification of the infinite ultraproducts of structures in 1-dimensional asymptotic classes. Namely, they showed the following:

Proposition 1.6.5 (Lemma 2.5 in [38]). *Let $\mathcal{C}$ be a class of finite structures and suppose that every infinite ultraproduct of members of $\mathcal{C}$ is strongly minimal. Then $\mathcal{C}$ is a 1-dimensional asymptotic class.*

Theorem 1.6.6. *[Lemma 4.1 in [38]] Every infinite ultraproduct of members of a 1-dimensional asymptotic class is supersimple of U-rank 1.*

In particular, we will provide a new proof of Theorem 1.6.6 as an application of the main result of Chapter 2, and we will also obtain analogues of these results for $\mathcal{O}$-asymptotic classes of finite structures, in Chapter 3.

1.6.3 Combinatorial cell-decomposition

Using the definition of 1-dimensional asymptotic classes, which is a condition on the definable sets in only one variable, Macpherson and Steinhorn could also obtain results about the control of the size of definable sets in many variables as well as results concerning the behavior of the infinite ultraproducts of structures in such classes.

The following result might be viewed as a combinatorial cell decomposition for asymptotic classes in the sense that conditions on the definable sets in one variable lead to logical finiteness properties for n-variable definable sets.

Theorem 1.6.7 (Theorem 2.1 in [38]). *Suppose $\mathcal{C}$ is a 1-dimensional asymptotic class of finite $\mathcal{L}$-structures. Then for every formula $\varphi(\overline{x}, \overline{y})$ (where $\overline{x} = (x_1, \ldots, x_n)$, $\overline{y} = (y_1, \ldots, y_m)$) the following hold:*

(1) There is a positive constant C and a finite set of pairs $D = \{(d_i, \mu_i) : i \leq k\}$ with $d_i \in \{0, 1, \ldots, n\}$ and $\mu_i \in \mathbb{R}^{>0}$ such that, for every $M \in \mathcal{C}$ and $\overline{a} \in M^m$ there is a pair $(d, \mu) \in D$ such that

$$||\varphi(M^n; \overline{a})| - \mu |M|^d| \leq C|M|^{d-1/2}.$$

(2) For every $(d,\mu) \in D$ there is an $\mathcal{L}$-formula $\varphi_{d,\mu}(\overline{y})$ such that for every $M \in \mathcal{C}$,

$$\varphi_{d,\mu}(M^m) = \left\{ \overline{a} \in M^m : ||\varphi(M^n;\overline{a})| - \mu|M|^d| \leq C|M|^{d-1/2} \right\}.$$

CHAPTER 2

Pseudofinite dimensions and forking

In this chapter we present a connection between forking in a pseudofinite structure and the logarithmic pseudofinite dimension : any instance of dividing in a pseudofinite structure is witnessed by a decrease of the dimension. This relationship can be viewed as a generalization of the notion of rank in stable or simple theories, in the sense that forking can be witnessed by a drop in the dimension.

This connection will be used to get some known results in asymptotic classes of finite structures, as defined in Section 1.6.

Section 2.4 contains a calculation of the possible pseudofinite dimensions for the ultraproducts of 1-dimensional asymptotic classes, which can be easily generalized to the context of N-dimensional asymptotic classes. As a corollary, we obtain new proofs of the following known results from [38],[18]: Every infinite ultraproduct of the members of a 1-dimensional asymptotic class (resp. N-dimensional asymptotic class) is supersimple of U-rank 1. (resp. U-rank less than or equal to N).

We start the proof recalling some results in enumerative combinatorics and measure theory (Section 2.2) and use them to prove the main result in Section 2.3.

2.1 The logarithmic pseudofinite dimension

In this section we present the definition of quasidimension as presented in [29] and give the construction of the *logarithmic pseudofinite dimension*, also showing that it defines a quasidimension on the ultraproducts of finite structures.

Definition 2.1.1. *Let M be any structure. A* quasi-dimension *on M is a map δ from the class of definable sets into an ordered abelian group G, together with a formal element $-\infty$, satisfying:*

1. $\delta(\emptyset) = -\infty$, *and* $\delta(X) > -\infty$ *implies* $\delta(X) \geq 0$.

2. $\delta(X \cup Y) = \max\{\delta(X), \delta(Y)\}$.

3. *Let $g \in G \cup \{-\infty\}$, X a definable subset of M^k and π is the projection to some of the coordinates. If $\delta(\pi^{-1}(\overline{x})) \leq g$ for all $\overline{x} \in \pi(X)$, then $\delta(X) \leq \delta(\pi(X)) + g$.*

We will focus on the logarithmic pseudofinite dimension, which is a quasidimension defined on ultraproducts of finite structures. Consider the following construction:

Assume M is an infinite ultraproduct of finite structures $\langle M_i : i \in I\rangle$, with $|M_i| \to \infty$ with respect to the ultrafilter $\mathcal{U}$. For a definable set $X = \phi(M; \overline{a})$ with $\overline{a} = [\overline{a}_i]_{\mathcal{U}}$, there is a map

$$\begin{aligned} \log_i : \mathrm{Def}(M_i) &\longrightarrow \mathbb{R} \cup \{-\infty\} \\ \phi(M_i; \overline{a}_i) &\longmapsto \log(|\phi(M_i; \overline{a}_i)|) \end{aligned}$$

where log is the usual natural logarithm and $|\phi(M_i; \overline{a}_i)|$ represents the size of the definable set $\phi(M_i; \overline{a}_i)$. If $\phi(M_i; \overline{a}_i) = \emptyset$, then $\log_i(\phi(M_i; \overline{a}_i)) = -\infty$.

It is possible to take the ultraproduct of such functions and obtain a map

$$\log = \prod_{\mathcal{U}} \log_i : \mathrm{Def}(M) \longrightarrow {}^*\mathbb{R} \cup \{-\infty\}$$

$$X \longmapsto \log(|X(M)|) := [\log(|\phi(M_i; \overline{a}_i)|)]_{i \in \mathcal{U}}$$

where ${}^*\mathbb{R}$ is a non-standard real closed field. Let $\mathrm{Conv}(\mathbb{Z})$ be the convex hull of $\mathbb{Z}$ in ${}^*\mathbb{R}$ (a convex subgroup of ${}^*\mathbb{R}$) and $\pi : {}^*\mathbb{R} \cup \{-\infty\} \to ({}^*\mathbb{R}/\mathrm{Conv}(\mathbb{Z})) \cup \{-\infty\}$ the natural quotient map (with $\pi(-\infty) = -\infty$).

For a definable subset X of M, define

$$\delta(X) = \pi(\log(|X|)).$$

This is a way to measure "largeness" of the definable sets in M. For instance, note that $\delta(X) = 0$ if and only if $\log(|X|) \in \mathcal{C}$, which implies by compactness that $|\phi(M_i; \overline{a}_i)|$ is uniformly bounded by a fixed constant C on a $\mathcal{U}$-large set of indices.

Proposition 2.1.2. *The function δ defined for definable sets as $\delta(X) = \pi(\log(|X|))$ is a quasi-dimension on $M = \prod_{\mathcal{U}} M_i$.*

Proof. (1) Clearly $\delta(\emptyset) = -\infty$, and for any definable $X = \phi(M; \overline{a})$ with $\overline{a} = [\overline{a}_i]_{\mathcal{U}}$ we have that, if X is non-empty in the ultraproduct, then

$$\{i \in I : |\phi(M_i; \overline{a}_i)| \geq 1\} = \{i \in I : \log_i(|\phi(M_i; \overline{a}_i)|) \geq 0\} \in \mathcal{U}$$

which implies $\delta(X) \geq 0$.

(2) Let $X = \phi(M; \overline{a}), Y = \psi(M; \overline{b})$ be definable subsets of M, and assume without loss of generality that $\delta(X) \geq \delta(Y)$. By the construction of δ, this implies in particular that there is a fixed integer C (that may be assumed greater than 1) such that $|\phi(M_i; \overline{a}_i)| + C \geq |\psi(M_i; \overline{b}_i)|$ for $\mathcal{U}$-almost all $i \in I$. In those indices,

we have

$$|\phi(M_i;\overline{a}_i)| \leq |\phi(M_i;\overline{a}_i) \cup \psi(M_i;\overline{b}_i)| \leq 2|\phi(M_i;\overline{a}_i)| + C \leq 2C|\phi(M_i;\overline{a}_i)|$$
$$\log(|\phi(M_i;\overline{a}_i)|) \leq \log(|\phi(M_i;\overline{a}_i) \cup \psi(M_i;\overline{b}_i)|) \leq \log(2C) + \log(|\phi(M_i;\overline{a}_i)|)$$
$$0 \leq \log(|\phi(M_i;\overline{a}_i) \cup \psi(M_i;\overline{b}_i)|) - \log(|\phi(M_i;\overline{a}_i)|) \leq \log(2C).$$

So, $\delta(X \cup Y) = \pi(\log(|X \cup Y|)) = \pi(\log(|X|)) = \delta(X)$.

(3) Let X be a definable subsets of M^k and p be a projection to some of the coordinates and $\alpha \in {}^*\mathbb{R}/\mathcal{C}$. Also assume $\delta(p^{-1}(\overline{x})) \leq \alpha$ for all $\overline{x} \in p(X)$.

Since $X = \prod_{\mathcal{U}} \phi(M_i;\overline{a}_i)$ we have $\mathcal{U}$-almost everywhere that:

$$|\phi(M_i;\overline{a}_i)| \leq |p(\phi(M_i;\overline{a}_i))| \cdot \max\{|p^{-1}(\overline{x})| : \overline{x} \in p(\phi(M_i;\overline{a}_i))\}$$
$$\log(|\phi(M_i;\overline{a}_i)|) \leq \log(|p(\phi(M_i;\overline{a}_i))|) + \max\{\log(|p^{-1}(\overline{x})|) : \overline{x} \in p(\phi(M_i;\overline{a}_i))\}$$

and since the natural projection $\pi : {}^*\mathbb{R} \longrightarrow {}^*\mathbb{R}/\mathcal{C}$ is an order-preserving group homomorphism, we conclude that

$$\delta(X) \leq \delta(p(X)) + \sup\{\delta(p^{-1}(\overline{x})) : \overline{x} \in p(X)\} \leq \delta(p(X)) + \alpha.$$

□

2.2 Some lemmas from combinatorics and measure theory

The purpose of these sections is to show the relationship between the pseudofinite dimension defined in Section 2.1 and the forking relation inside the structure M. This relationship can be viewed as a generalization of the notion of rank in stable or simple theories, in the sense that forking can be witnessed by a drop in the dimension.

To prove that any instance of dividing is witnessed by a drop of the logarithmic pseudofinite dimension, we will use some more or less known results in enumerative combinatorics and measure theory. We include the proofs here for completeness.

We start with the following lemma:

Lemma 2.2.1. *If* $m \geq 2i+1$ *are integers, then* $\binom{m}{i} - \binom{m}{i+1} \leq 0$.

Proof. A heuristic argument can be given by considering the Pascal triangle and noticing that the coefficients in the $m-th$ row are $\binom{m}{i}$, and these coefficients are decreasing after the value in the middle. We can also do the following straightforward calculation:

$$\begin{aligned} \binom{m}{i} - \binom{m}{i+1} &= \frac{m!}{i!(m-i)!} - \frac{m!}{(i+1)!(m-i-1)!} \\ &= \frac{m!}{i!(m-i-1)!}\left(\frac{1}{m-i} - \frac{1}{i+1}\right) \\ &= \frac{m!}{i!(m-i-1)!}\left(\frac{2i+1-m}{(m-i)(i+1)}\right) \leq 0. \end{aligned}$$

□

Now we present a lemma in measure theory. Assume we have a measure space $(X, \mathcal{B}, \mu)$. Given measurable sets $A_1, \ldots, A_n$, we can define for $(k \geq 1)$ the S_k to be the sum of the measures of all k-intersections of $A_1, \ldots, A_n$, namely,

$$S_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mu(A_{i_1} \cap \cdots \cap A_{i_k}).$$

We know from the inclusion-exclusion principle that the measure of $\bigcup_{i=1}^{n} A_i$ is an alternating sum of S_k's. What we will prove now is that starting from a positive term of this sum, the result is positive.

Proposition 2.2.2 (Truncated inclusion-exclusion principle)**.** *Let X be a measure space and $A_1, \ldots, A_n$ be measurable sets, and let $S_1, \ldots, S_n$ as defined above. Then* *for every* $k \leq n/2$,

$$\sum_{i=2k+1}^{n} (-1)^{i-1} S_i \geq 0.$$

Proof. By the inclusion-exclusion principle we know that

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{m=1}^{n} (-1)^{m-1} S_m.$$

For every non-empty $W \subseteq \{1, \ldots, n\}$, define $E_W = \bigcap_{i \in W} A_i \cap \bigcap_{i \notin W} A_i^c$. The non-empty intersections of this kind are exactly the atoms of the algebra of sets generated by $A_1, \ldots, A_n$ that are contained in $\bigcup_{i=1}^{n} A_i$. Thus, they are disjoint and satisfy the following easy identities:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{W \subseteq \{1,\ldots,n\}} \mu(E_W), \qquad \mu(A_{i_1} \cap \cdots \cap A_{i_m}) = \sum_{i_1,\ldots,i_m \in W} \mu(E_W).$$

So the inclusion-exclusion principle states that

$$\begin{aligned}
\sum_{W \subseteq \{1,\ldots,n\}} \mu(E_W) &:= \sum_{m=1}^{n} (-1)^{m-1} S_m \\
&= \sum_{m=1}^{n} (-1)^{m-1} \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \mu(A_{i_1} \cap \cdots \cap A_{i_m}) \right) \\
&= \sum_{m=1}^{n} (-1)^{m-1} \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \left(\sum_{i_1,\ldots,i_m \in W} \mu(E_W) \right) \right) \\
&= \sum_{m=1}^{n} (-1)^{m-1} \left(\sum_{W \subseteq \{1,\ldots,n\}} \alpha_W^m \mu(E_W) \right)
\end{aligned}$$

where the coefficient α_W^m is the number of times that the summand $\mu(E_W)$ appears in S_m. Thus, we know that for every $W \subseteq \{1, \ldots, n\}$, $\sum_{m=1}^{n} (-1)^{m-1} \cdot \alpha_W^m = 1$.

We have three cases:

- If $|W| < 2k+1$, all the possible summands $\mu(E_W)$ are already in the sum $\sum_{i=1}^{2k}(-1)^{i-1}S_i$.

- If $|W| = 2k+1$, the coefficient of $\mu(E_W)$ in $\sum_{i=1}^{2k}(-1)^{i-1}S_i$ is 0, because there is exactly one more term $\mu(E_W)$ appearing in S_{2k+1} (the measure of the intersection $\bigcap_{i\in W} A_i$).

- If $|W| > 2k+1$: Note that in every S_i appear $\binom{|W|}{i}$ summands of the form $\mu(E_W)$ for a fixed $W \subseteq \{1,\dots,n\}$. So, the coefficient of $\mu(E_W)$ in $\sum_{i=1}^{2k}(-1)^{i-1}S_i$ is:

$$\beta_W = \sum_{m=1}^{2k}(-1)^{m-1}\binom{|W|}{i} = \sum_{j=1}^{k}\left[\binom{|W|}{2j-1} - \binom{|W|}{2j}\right] \leq 0$$

  by the previous lemma (note that $j \leq k$ implies $2j-1 \leq 2k-1 < |W|$).

Therefore, we obtain

$$\begin{aligned}
\sum_{W\subseteq\{1,\dots,n\}} \mu(E_W) &= \sum_{m=1}^{n}(-1)^{m-1}S_m \\
&= \sum_{m=1}^{2k}(-1)^{m-1}S_m + \sum_{m=2k+1}^{n}(-1)^{m-1}S_m \\
&= \sum_{|W|<2k+1} \mu(E_W) + 0 + \left(\sum_{|W|=2k+1} \beta_W \cdot \mu(E_W)\right) + \sum_{m=2k+1}^{n}(-1)^{m-1}S_m.
\end{aligned}$$

So,

$$\sum_{|W|>2k+1} \mu(E_W) = \sum_{|W|=2k+1} \beta_W \cdot \mu(E_W) + \sum_{m=2k+1}^{n}(-1)^{m-1}S_m,$$

and we conclude that

$$\sum_{m=2k+1}^{n}(-1)^{m-1}S_m = \sum_{|W|>2k+1} \mu(E_W) - \sum_{|W|=2k+1} \beta_W \mu(E_W)$$

which must be non-negative since all the coefficients β_W are less than or equal to 0. $\square$

The following measure-theoretic proposition will play a key role in the proof that dividing implies a decrease of pseudofinite dimension.

Proposition 2.2.3. *Let X be a probability space and fix $0 < \epsilon \leq \frac{1}{2}$. Let $\langle A_i : i < \omega \rangle$ be a sequence of measurable subsets of X such that $\mu(A_i) \geq \epsilon$ for every i.*

Then, for every $k < \omega$ there are $i_1 < i_2 < \ldots < i_k$ such that

$$\mu\left(\bigcap_{j=1}^{k} A_{i_j}\right) \geq \epsilon^{3^{k-1}}.$$

Proof. The proof will be by induction on k.

- $k = 1$: By hypothesis we have $\mu(A_i) \geq \epsilon = \epsilon^{3^{1-1}}$. ✓

- $k = 2$: Assume not, then $\mu(A_i \cap A_j) < \epsilon^{3^{2-1}} = \epsilon^3$ for every $i \neq j$. By the truncated inclusion-exclusion principle we know that for every $N \in \mathbb{N}$,

$$\begin{aligned} 1 \geq \mu\left(\bigcup_{i=1}^{N} A_i\right) &\geq \sum_{i=1}^{N} \mu(A_i) - \sum_{1 \leq i < j \leq N} \mu(A_i \cap A_j) \quad \text{(by Proposition 2.2.2)} \\ &\geq N\epsilon - \frac{N(N-1)}{2}\epsilon^3. \qquad (\dagger) \end{aligned}$$

Define the quadratic function given by

$$f(x) = x \cdot \epsilon - \frac{x(x-1)}{2}\epsilon^3 = -\frac{x^2}{2}\epsilon^3 + x\left(\epsilon + \frac{\epsilon^3}{2}\right).$$

This function achieves its maximum value at $x_0 = \dfrac{1}{\epsilon^2} + \dfrac{1}{2} > 0$, and by taking any integer $N \in [x_0 - 1, x_0]$ we have that

$$\begin{aligned}
f(N) \geq f(x_0 - 1) &= \left(\frac{1}{\epsilon^2} - \frac{1}{2}\right)\epsilon - \frac{\left(\frac{1}{\epsilon^2} - \frac{1}{2}\right)\left(\frac{1}{\epsilon^2} - \frac{3}{2}\right)}{2} \cdot \epsilon^3 \\
&= \frac{1}{\epsilon} - \frac{\epsilon}{2} - \frac{\frac{1}{\epsilon^4} - \frac{2}{\epsilon^2} + \frac{3}{4}}{2} \cdot \epsilon^3 \\
&= \frac{1}{\epsilon} - \frac{\epsilon}{2} - \frac{1}{2\epsilon} + \epsilon - \frac{3}{8}\epsilon^3 \\
&= \frac{1}{2\epsilon} + \frac{\epsilon}{2} - \frac{3}{8}\epsilon^3 \\
&\geq 1 + \epsilon\left(\frac{1}{2} - \frac{3}{8}\epsilon^2\right) \qquad [\text{because } \epsilon \leq \tfrac{1}{2}] \\
&> 1.
\end{aligned}$$

contradicting the inequality (†). ✓

Now, assume the induction hypothesis, which is that there is a tuple $(i_1, \ldots, i_k)$ satisfying

$$i_1 < \ldots < i_k \quad \text{and} \quad \mu\left(\bigcap_{j=1}^{k} A_{i_j}\right) \geq \epsilon^{3^{k-1}}. \qquad (*)$$

Claim: *There are infinitely many such tuples.*

Proof of the Claim: Assume not, and take ℓ to be the maximum of all indices appearing in the tuples $(i_1, \ldots, i_k)$ which satisfies (*). The sequence $\langle A_j : j \geq \ell + 1\rangle$ would contradict the induction hypothesis. ✓

Now, let $\langle \alpha_j : j < \omega\rangle$ be an enumeration of all tuples satisfying (*) and define $B_j = \bigcap_{i \in \alpha_j} A_i$. By construction, $\mu(B_j) \geq \epsilon^{3^{k-1}}$.

By the $k = 2$ case, there are indices $j_1 \neq j_2$ such that

$$\mu(B_{j_1} \cap B_{j_2}) \geq (\epsilon^{3^{k-1}})^3 = \epsilon^{3^{k-1} \cdot 3} = \epsilon^{3^k}$$

where j_1, j_2 are indices corresponding to two different tuples $\alpha_{j_1} \neq \alpha_{j_2}$. In particular, there are (at least) $k+1$ indices $i_1 < i_2 < \cdots < i_k < i_{k+1}$ such that

$$\mu\left(\bigcap_{j=1}^{k+1} A_{i_j}\right) \geq \mu(B_{j_1} \cap B_{j_2}) \geq \epsilon^{3^k} = \epsilon^{3^{(k+1)-1}}.$$

□

2.3 Dividing and drop of the pseudofinite dimension

With the results of the previous subsection, we are now able to give a proof of the main result of this Chapter. The setting, as in the definition of logarithmic pseudofinite dimension, is the following: $\langle M_i : i \in I \rangle$ is a family of finite structures, M is an infinite ultraproduct of the family and δ denotes the logarithmic pseudofinite dimension defined on definable subsets of M.

Theorem 2.3.1. *Let $X = \psi(x, \overline{a})$ be a definable subset of M and $\phi(x, \overline{b})$ a formula implying $\psi(x, \overline{a})$. If $\phi(x, \overline{b})$ divides over $\overline{a}$, then there exists $\overline{b}' \models tp(\overline{b}/\overline{a})$ such that $\delta(\phi(x, \overline{b}')) < \delta(X)$.*

Proof. Towards a contradiction, assume that for every $b' \models tp(\overline{b}/\overline{a})$ we have $\delta(\phi(x, b')) = \delta(X)$. Then for each $\overline{b}'$ there is $n_{b'} \in \mathbb{N}$ such that

$$\log(|X|) - \log(|\phi(x, b')|) < n_{b'}.$$

Thus,

$$\log\left(\frac{|X|}{|\phi(x, b')|}\right) < n_{b'}$$
$$\frac{|X|}{|\phi(x, b')|} < e^{n_{b'}}$$
$$e^{n_{b'}} |\phi(x, b')| \geq |X|.$$

In particular, there is $C_{b'} \in \mathbb{N}$ such that $C_{b'} \cdot |\phi(x,b)| \geq |X|$.

Claim: *There is a uniform bound C such that $C \cdot |\phi(x,b')| \geq |X|$ for every $b' \models tp(b/\overline{a})$.*

Proof of the Claim: If not, for every $n < \omega$ there is $\overline{b}_n \equiv_{\overline{a}} \overline{b}$ such that $\log(|X|) - \log(|\phi(x,\overline{b}_n)|) > n$. Consider as in Section 1.3.3 the multi-sorted structures given by

$$\mathcal{M}_i = \langle M_i, \mathbb{R}, \log_\varphi \rangle_{\varphi \in \mathcal{L}}$$

where $\log_\varphi$ is a function between different sorts interpreted as

$$\begin{aligned} \log_\varphi : M_i &\longrightarrow \mathbb{R} \\ \overline{b} &\longmapsto \log(|\varphi(x,\overline{b})|). \end{aligned}$$

Now, take the ultraproduct $M = \prod_{\mathcal{U}} M_i$ and consider the type

$$p(y) = tp(\overline{b}/\overline{a}) \cup \{\log_\psi(\overline{a}) - \log_\phi(\overline{b}) > n : n \in \mathbb{N}\}.$$

This type is finitely satisfiable in M, and by $\aleph_1$-saturation of the ultraproduct, there is $\overline{b}' \models p(y)$ which means $\overline{b}' \models tp(\overline{b}/\overline{a})$ and $\delta(\phi(x,\overline{b}')) < \delta(X)$, a contradiction. ✓

Let $C \in \mathbb{N}$ be such that $C \cdot |\phi(x,\overline{b}')| \geq |X|$ for every $\overline{b}' \equiv_{\overline{a}} \overline{b}$. Since $\phi(x,\overline{b})$ divides over $\overline{a}$, there is an indiscernible sequence $\langle \overline{b}_j : j < \omega \rangle$ (which can be assumed to be in M by $\aleph_1$-saturation) such that:

- $\overline{b}_i \models tp(\overline{b}/A)$.
- $\{\phi(x,\overline{b}_j) : j < \omega\}$ is k-inconsistent for some $k < \omega$.

Assume $\bar{b}_j = [\bar{b}_j^i]_{i\in\mathcal{U}}$. By the claim, $C \cdot |\phi(x, \bar{b}_j)| \geq |X|$ which implies that $\dfrac{|\phi(x, \bar{b}_j^i)|}{|X|} \geq \dfrac{1}{C}$ for $\mathcal{U}$-almost all i.

Consider the normalized counting measure localized on $\phi(M_i; \bar{a}_i)$ in each finite structure M_i, and the Loeb measure induced on M by these measures. We have that $\mu(X) = 1$ and $\langle \phi(M, \bar{b}_j) : j < \omega \rangle$ is a sequence of measurable sets with $\mu(\phi(M, \bar{b}_j)) \geq \frac{1}{C}$ for every $j < \omega$. By the Proposition 2.2.3 there are $j_1 < \ldots < j_k < \omega$ such that

$$\mu\left(\bigcap_{l=1}^{k} \phi(M, \bar{b}_{j_l})\right) \geq \frac{1}{C^{3^{k-1}}} > 0.$$

In particular, $\bigcap_{l=1}^{k} \phi(M, b_{j_l})$ is non-empty, contradicting k-inconsistency. □

The theorem above allows us to conclude that the number of possible different values for pseudofinite dimensions of definable sets is a bound for the length of dividing chains, providing also a bound for the U-rank in types. We will explore this idea in the Section 2.4.

We might think about two possible generalizations of Theorem 2.3.1: either changing dividing by forking or showing that the original formula (instead of replacing the parameters by a conjugate) has lower pseudofinite dimension. The following two examples show limitations for these attempts:

Example 2.3.2. Consider the class of finite structures $M_n = ([0, 2^n], E_n)$ where E_n is an equivalence relation with classes

$$[0, 2^{n-1} - 1], [2^{n-1}, 2^{n-1} + 2^{n-2} - 1], [2^{n-1} + 2^{n-2}, 2^{n-1} + 2^{n-2} + 2^{n-3} - 1],$$
$$\ldots, [2^{n-1} + 2^{n-2} + \cdots + 2^2, 2^n].$$

Let $M = \prod_{\mathcal{U}} M_n$ and $b = [\bar{0}]_{\mathcal{U}}$. In the ultraproduct M the relation symbol E is interpreted as an equivalence relation with infinitely many infinite classes, and as it was shown in Example 1.5.6 we have that the formula xEb divides over the empty set. Theorem 2.3.1 shows that there is a conjugate of b over $\emptyset$ witnessing a drop of pseudofinite dimension, but this drop is not witnessed by the formula xEb because

$$\log|M_n| - \log|xE_n 0| = \log(2^n) - \log(2^{n-1}) = \log 2 < 1$$

which implies that $\delta(M) = \delta(xEb)$.

Example 2.3.3. This example is an adaptation of the classical example of the circular order that shows that the formula $x = x$ may fork over the $\emptyset$. Consider the structure $M_n = (\mathbb{Z}/(3n)\mathbb{Z}, R)$ where R is a ternary relation interpreted in M_n as follows: $M_n \models R(w; \overline{a}, \overline{c})$ if and only if there are integers a', w', c' congruent with $a, w, c \pmod{3n}$ respectively, such that $a' < w' < c'$ and $|c' - a'| \geq n$.

Take $M = \prod_{\mathcal{U}} M_n$, and the elements $a := [a_n = 0]_{\mathcal{U}}, b := [b_n = n]_{\mathcal{U}} \in M$.

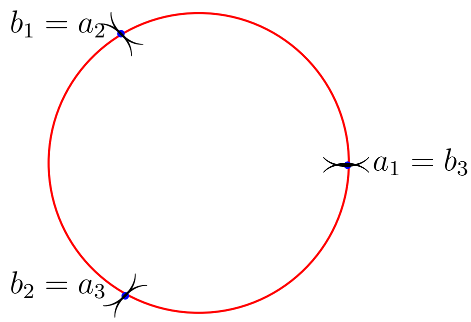
Claim: *The formula $R(x; a, b)$ divides over $\emptyset$.*

Proof of the Claim: On each M_n consider the sequence given by $\left\langle (a^n_k, b^n_k) = (n + k \cdot [\![\log n]\!], n + (k+1) \cdot [\![\log n]\!]) : k \leq \frac{n}{\log n} \right\rangle$, and consider in the ultraproduct the sequence given by $\langle (a_k, b_k) = ([a^n_k]_{n\in\mathcal{U}}, [b^n_k]_{n\in\mathcal{U}}) : k < \omega \rangle$. This is a sequence in $tp(a, b/\emptyset)$ which is indiscernible over the empty set, and by construction we have that the set of formulas $\{R(x; a_k, b_k) : i < \omega\}$ is 2-inconsistent. ✓

Consider the elements in the ultraproduct M given by $a_1 := [a^n_1 = 0]_{\mathcal{U}}$, $a_2 := [a^n_2 = n]_{\mathcal{U}} = b_1$ and $a_3 := [a^n_3 = 2n]_{\mathcal{U}} = b_2$ and $b_3 = a_1$. Note that the formula $x = x$ forks over $\emptyset$, because it implies the disjunction

$$\bigvee_{i=1}^{3} R(a_i, x, b_i) \vee \bigvee_{i=1}^{3} x = a_i$$

of formulas that divide over $\emptyset$.



However, the set of realizations of the formula of $x = x$ is M and it does not witness any drop of pseudofinite dimension ($\delta(M)$ is the maximal value of the pseudofinite dimension among subsets of M).

Even if Theorem 2.3.1 only applies for dividing formulas, there are natural settings where dividing is equivalent to forking. For example forking and dividing over arbitrary sets are equivalent in simple theories, and they are also equivalent over models in theories with NTP_2 [11].

2.4 Pseudofinite dimension and asymptotic classes

In general, the logarithmic pseudofinite dimension can take infinitely many values on the definable sets of $M = \prod_{\mathcal{U}} M_i$. For instance, this is the case for the class $\mathcal{C}_{ord}$ of finite linear orders, as it will be shown in Section 2.5.

The main feature of this example is that an infinite linear order can be defined on the ultraproducts, implying they are unstable non-simple and thus they have arbitrarily long dividing chains.

On the other hand, there are classes of finite structures with a better behavior of their ultraproducts. That is the case of the 1-dimensional asymptotic classes (and more generally of the N-dimensional asymptotic classes) whose definition appear in [38] and [18]. These classes are known to have supersimple ultraproducts of finite rank, which implies a finite bound on the length of dividing chains in their ultraproducts.

The purpose of this section is to show that the supersimplicity of these classes can be detected by the logarithmic pseudofinite dimension. For instance, we will show that for 1-dimensional classes (which ultraproducts are supersimple of U-rank 1) the only possible values for the pseudofinite dimension are $-\infty, 0$ and $\alpha = \delta(M)$.

Proposition 2.4.1. *Let $\mathcal{C}$ be a class of finite structures. If $\mathcal{C}$ satisfies the condition (1) in Definition 1.6.2, namely for every $m \in \mathbb{N}$ and every formula $\varphi(x, \overline{y})$ (where $\overline{y} = (y_1, \cdots, y_m)$) there is a positive constant C and a finite set $E \subseteq \mathbb{R}^{>0}$ such that for every $M \in \mathcal{C}$ and $\overline{a} \in M^m$, either $|\varphi(M, \overline{a})| \leq C$ or for some $\mu \in E$,*

$$||\varphi(M, \overline{a})| - \mu|M|| \leq C|M|^{1/2}.$$

Then for every infinite ultraproduct M of elements in $\mathcal{C}$ the only possible values for $\delta(X)$ while X varies among the non-empty definable subsets of M^1 are 0 and $\delta(M)$.

Proof. Let $\varphi(x, \overline{a})$ be a formula with parameters from the ultraproduct and take $\mu_1, \ldots, \mu_k > 0$ the possible measures in E_φ. Assume $M = \prod_{\mathcal{U}} M_i$ with $i \in I$ and $\overline{a} = [\overline{a_i}]_{i \in I}$.

For every $M_i \in \mathcal{C}$ one of the following hold:

- $|\varphi(M_i, \overline{a_i})| \leq C$
- $||\varphi(M_i, \overline{a_i})| - \mu_j|M_i|| \leq C|M_i|^{1/2}$ for some $j = 1, 2, \ldots, k$.

Consider the sets

$$\begin{aligned} A_0 &= \{i \in I : |\varphi(M_i, \overline{a_i})| \leq C\} \\ A_1 &= \{i \in I : ||\varphi(M_i, \overline{a_i})| - \mu_1|M_i|| \leq C|M_i|^{1/2}\} \\ &\vdots \\ A_k &= \{i \in I : ||\varphi(M_i, \overline{a_i})| - \mu_k|M_i|| \leq C|M_i|^{1/2}\}. \end{aligned}$$

Since $A_0 \cup A_1 \cup \cdots \cup A_k = I$, one of these sets belongs to $\mathcal{U}$ because $\mathcal{U}$ is an ultrafilter on I. We have to consider two cases:

- If $A_0 \in \mathcal{U}$, then $|\varphi(M_i, \overline{a_i})| \leq C$ (a.e. in $\mathcal{U}$) for some fixed $C > 0$, which implies $|\varphi(M, \overline{a})| \leq C$ and therefore

$$\delta(\varphi(M, \overline{a})) = \pi(\log(|\varphi(x, \overline{a})|)) \leq \pi(\log(C)) = 0.$$

- If $A_j \in \mathcal{U}$ for some $j = 1, 2, \ldots, k$ then we obtain

$$\mu_j|M_i| - C|M_i|^{1/2} \leq |\varphi(M_i, \overline{a_i})| \leq \mu_j|M_i| + C|M_i|^{1/2}.$$

Let $\mu_* = \min\{\mu_1, \ldots, \mu_k\}$ and $\mu^* = \max\{\mu_1, \ldots, \mu_k\}$. For every definable set $X = \prod_{\mathcal{U}} X_i$, either $\delta(X) = 0$ (because the corresponding set A_0 belong to $\mathcal{U}$) or

$$\mu_*|M_i| - C|M_i|^{1/2} \leq |X_i| \leq \mu^*|M_i| + C|M_i|^{1/2}.$$

So

$$|M_i|^{1/2}\left(\mu_*|M_i|^{1/2} - C\right) \leq |X_i| \leq |M_i|^{1/2}\left(\mu^*|M_i|^{1/2} + C\right)$$

$$|M_i|^{1/2}\left(\mu_*|M_i|^{1/2} - \frac{\mu_*}{2}|M_i|^{1/2}\right) \leq |X_i| \leq |M_i|^{1/2}\left(\mu^*|M_i|^{1/2} + \frac{\mu^*}{2}|M_i|^{1/2}\right)$$

(asymptotically, because $|M_i| \to \infty$)

$$|M_i|^{1/2}\left(\frac{\mu_*}{2}|M_i|^{1/2}\right) \leq |X_i| \leq |M_i|^{1/2}\left(\frac{3\mu^*}{2}|M_i|^{1/2}\right)$$

and taking logarithms we obtain

$$\begin{aligned}
&\frac{1}{2}\log(|M_i|) + \log\left(\frac{\mu_*}{2}\right) + \frac{1}{2}\log(|M_i|) \\
&\qquad \le \log(|X_i|) \le \frac{1}{2}\log(|M_i|) + \log\left(\frac{3\mu^*}{2}\right) + \frac{1}{2}\log(|M_i|), \\
&\log(|M_i|) + \log\left(\frac{\mu_*}{2}\right) \\
&\qquad \le \log(|X_i|) \le \log(|M_i|) + \log\left(\frac{3\mu^*}{2}\right), \\
&\pi(\log(|M_i|)) = \pi\left(\log(|M_i|) + \log\left(\frac{\mu_*}{2}\right)\right) \\
&\qquad \le \pi(\log(|X_i|)) \le \pi\left(\log(|M_i|) + \log\left(\frac{3\mu^*}{2}\right)\right) = \pi(\log(|M_i|)), \\
&\pi(\log(|M_i|)) \le \pi(\log(|X_i|)) \le \pi(\log(|M_i|)), \\
&\delta(M) \le \delta(X) \le \delta(M)
\end{aligned}$$

as we desired. □

Now, we present a new proof of the following known result that appears in [38, Lemma 4.1] (Theorem 1.6.6 in this thesis).

Corollary 2.4.2 (cf. [38], Lemma 4.1)**.** *Let $\mathcal{C}$ be a class of finite structures satisfying the condition (1) of Definition 1.6.2, and let M be an infinite ultraproduct of members of $\mathcal{C}$. Then $Th(M)$ is supersimple of U-rank 1.*

Proof. Assume $U(M) \ge 2$. Then there is an increasing chain of types $p_0 \subset p_1 \subset p_2$ such that p_{i+1} is a dividing extension of p_i for $i = 0, 1$. In particular, there are formulas $\phi_1(x, a_1) \in p_1$ which divides over A_0 and $\phi_2(x, a_2) \in p_2$ which divides over A_1. By Proposition 2.3.1 there are tuples $a_1' \equiv_{A_0} a_1$ and $a_2' \equiv_{A_1} a_2$ such that

$$\delta(\phi(x, a_2')) < \delta(\phi(x, a_1')) < \delta(M).$$

This contradicts Proposition 2.4.1 which states there are only two possible values for δ on non-empty definable subsets of M. □

2.5 Pseudofinite dimensions in classes of ordered structures.

In Section 2.4 we analyzed the behavior of the 1-dimensional asymptotic classes with respect to the logarithmic pseudofinite dimension, proving that the possible values for $\delta(X)$ in this case are $-\infty$ (for the empty set), 0 (if $\phi(M_i; \overline{a}_i)$ is uniformly finite through the class) or $\delta(M)$.
In the Chapter 3 we will work in the setting of classes of linearly ordered structures. Now we calculate the possible values for the pseudofinite dimension in this class.

Example 2.5.1. Consider the class of ordered structures $\mathcal{C}_{ord} = \langle M_n = [0, n] : n < \omega\rangle$. If α, β are real numbers with $0 \leq \alpha < \beta \leq 1$ we may define

$$X(M_n) = [0, n^\alpha], \qquad Y(M_n) = [0, n^\beta].$$

We will show that $\delta(X) < \delta(Y)$. Assume otherwise, then $\delta(X) = \delta(Y)$ which implies

$$\log |X| + N \geq \log |Y|$$

for some (fixed) $N \in \mathbb{N}$, i.e.,

$$\alpha \log n + N \geq \beta \log n$$
$$N \geq (\beta - \alpha) \log n$$

a contradiction because $\beta - \alpha > 0$ and n tends to ∞.

After Theorem 2.3.1, this example seems completely natural once we have in mind that the ultraproducts of elements in the class $\mathcal{C}_{ord}$ are infinite linear orders, and they are known to be an example where there are arbitrarily long forking chains. Another notion of pseudofinite dimension defined in [29] is the normalized pseudofinite dimension $\widehat{\delta}$, defined for a definable subset $X = \phi(M;\overline{a})$ as follows:

$$\widehat{\delta}(X) := \lim_{i\to\mathcal{U}} \frac{\log(|\phi(M_i;\overline{a}_i)|)}{\log(|M_i|)}$$

Unfortunately, Theorem 2.3.1 cannot be extended to this notion of pseudofinite dimension, but still the phenomenon of having infinitely many possible values for the class of ordered structures persists.

Example 2.5.2. Consider the class $\mathcal{C}_{ord}$ of finite linear orders and the structures $M_n = [0, n]$ which are elements of this class. For every $\gamma \in [0, 1]$, consider the definable set (with parameters) given by

$$X(M_n) = [0, [\![n^\gamma]\!]].$$

Using the normalized pseudodimension, we obtain that

$$\widehat{\delta}(X) = \lim_{\mathcal{U}} \frac{\log |X(M_n)|}{\log |M_n|} = \lim_{\mathcal{U}} \frac{\log n^\gamma}{\log n} = \gamma.$$

which implies there are too many possible values for the normalized pseudodimensions of definable sets in the class $\mathcal{C}_{ord}$.

2.6 A final remark

Independently from work of Macpherson and Steinhorn to isolate conditions on classes of finite structures under which their ultraproducts would be stable, supersimple, have a good independence relation (maybe related with forking), etc., I proved Theorem 2.3.1 which is a close connection between the independence relation of non-forking and the logarithmic pseudofinite dimension.

My results as presented here are very general in the sense that I was only using the pseudofiniteness of the structures, but as a consequence of such generality they are weak and cannot give more information about the ultraproducts unless more assumptions are added. The publication of my results in arXiv [20] and the fact that Macpherson and Steinhorn realized that we were working on similar problems was the beginning of a fruitful interaction in which, combining the study of the (at that moment unpublished) properties of Attainability (A), Strong Attainability (SA), Dimension Comparison (DC) with the soft analysis (measure-theoretic arguments) of my approach, we succeed in describing conditions under which the ultraproducts of finite structures will be (super) simple. Namely, (A) implies the structure is simple, (SA) implies that the structure is supersimple, and (SA) together with (DC) implies that the forking-independence is the same as the independence given by the pseudofinite dimension. These results, with applications to asymptotic classes, pseudofinite vector spaces classes of vector spaces and homocylic groups, together with some connections with the Independence Theorem [53, Section 2.5] and the description of possible applications to pseudofinite groups and generalizations of Tao's Algebraic Regularity Lemma will appear in [21].

Those results are part of a collaboration and will not be included in this thesis.

CHAPTER 3

$\mathcal{O}$-asymptotic classes of finite structures

It was shown in Chapter 1, the class of finite linearly ordered sets is not a 1-dimensional asymptotic class because the formula $x < a$ can pick out an arbitrary proper initial segment of a structures as a varies. A posteriori, we can give a structural reason for this: the ultraproducts of 1-dimensional asymptotic classes are supersimple, thus no order with infinite chains can be defined inside them.

On the other hand, the only definable sets in the structures of this class (and in their ultraproducts) are finite unions of intervals and points implying that the structures involved are *$\mathcal{O}$-minimal*. $\mathcal{O}$-minimality and its variants are properties that give a good structure theory for infinite ordered structures. Our aim here is to isolate conditions on classes of finite structures to get nice asymptotic properties, melding ideas of asymptotic classes and $\mathcal{O}$-minimality. With this idea in mind, we propose an adaptation of the definition of 1-dimensional asymptotic classes in the context of totally ordered structures.

In this chapter we will develop the theory of *$\mathcal{O}$-asymptotic classes of finite structures*, which is an adaptation of the definition of 1-dimensional asymptotic classes in the context of totally ordered structures. We also provide a collection of examples and place the ultraproducts of $\mathcal{O}$-asymptotic classes within the context of the Classification Theory (see Section 1.5, and the map in Section 1.5.5).

3.1 $\mathcal{O}$-asymptotic classes and cell decomposition results

Definition 3.1.1. *Let $\mathcal{C}$ be a class of finite linearly ordered structures in a language $\mathcal{L}$ containing $<$. We say $\mathcal{C}$ is a* **weak-$\mathcal{O}$**-asymptotic class *if for every $m \in \mathbb{N}$ and formula $\varphi(x; y_1, \ldots, y_m)$ there are constants $C > 0$ and $k \geq 1$ and a finite set $E = \{\overline{\mu}_i = (\mu_i^1, \ldots, \mu_i^k) : i \leq \ell\} \subseteq ([0,1])^k$ such that:*

1. *For every $M \in \mathcal{C}$ and $\overline{a} \in M^m$ there are elements*

$$c_0 = \min M < c_1 < \ldots < c_k = \max M$$

and a tuple $\overline{\mu} \in E$ such that:

(*) For every $i = 1, 2, \ldots, k$, either

$$\begin{cases} \mu_i = 0 \quad \textit{and} \quad |\varphi(M, \overline{a}) \cap (c_{i-1}, c_i)| \leq C \\ \qquad\qquad \textit{or} \\ \mu_i > 0 \quad \textit{and} \quad ||\varphi(M, \overline{a}) \cap (c_{i-1}, c_i)| - \mu_i |(c_{i-1}, c_i)|| \leq C |(c_{i-1}, c_i)|^{1/2} \end{cases}$$

2. *For every $\overline{\mu} \in E$ there is a formula $\varphi_{\overline{\mu}}(\overline{y}; z_1, \ldots, z_k)$ such that*

$$M \models \varphi_{\overline{\mu}}(\overline{a}; c_1, \ldots, c_k) \quad \textit{implies that (*) holds.}$$

Definition 3.1.2. *Let $\mathcal{C}$ be a class of finite linearly ordered structures in a language $\mathcal{L}$ containing $<$. We say $\mathcal{C}$ is an $\mathcal{O}$-*asymptotic class *if for every $m \in \mathbb{N}$ and formula $\varphi(x; y_1, \ldots, y_m)$ there are constants $C > 0$ and $k \geq 1$ and a finite set $E \subseteq ([0,1])^k$ such that:*

1. *For every $M \in \mathcal{C}$ and $\overline{a} \in M^m$ there are elements*

$$c_0 = \min M < c_1 < \ldots < c_k = \max M$$

and a tuple $\overline{\mu} \in E$ such that:

(*) For every $i = 1, 2, \ldots, k$, either

$$\begin{cases} \mu_i = 0 \quad \textit{and} \quad |\varphi(M, \overline{a}) \cap (c_{i-1}, c_i)| \leq C \\ \qquad\qquad \textit{or} \\ \mu_i > 0 \quad \textit{and} \quad \textit{for every } (u, v) \subseteq (c_{i-1}, c_i), \\ \qquad ||\varphi(M, \overline{a}) \cap (u, v)| - \mu_i|(u, v)|| \leq C|(u, v)|^{1/2} \end{cases}$$

2. *For every $\overline{\mu} \in E$ there is a formula $\varphi_{\overline{\mu}}(\overline{y}; z_1, \ldots, z_k)$ such that*

$$M \models \varphi_{\overline{\mu}}(\overline{a}; c_1, \ldots, c_k) \quad \textit{implies that (*) holds.}$$

Remark 3.1.3. *Roughly speaking, a class of finite ordered structures is weakly $\mathcal{O}$-asymptotic if every formula in one variable admits a decomposition into a uniform number of intervals such that on each interval it behaves like in 1-dimensional classes. $\mathcal{O}$-asymptoticity requires also this decomposition to be* uniform *in the sense that the definable set is uniformly distributed along the intervals (c_{i-1}, c_i). We believe that these two notions are equivalent, but we have not been able to prove this equivalence. The main difficulty here is the non-additive nature of the "error term".*

Notation: Assume we are working in an $\mathcal{O}$-asymptotic class.

- We call the formula $\varphi_{\overline{\mu}}$ the $\overline{\mu}$-definition of φ.

- We say that $\varphi(x; \overline{a})$ *admits a decomposition with proportion* $\overline{\mu}$ if there are $c_1, \ldots, c_k$ such that condition (1) of Definition 3.1.2 holds for $\varphi(x; \overline{a})$ and $c_1 < \cdots < c_k$, using the tuple of measures $\overline{\mu}$.

- When we say *by uniformity of* μ_i *in* (c,d) or *by the uniformity of distribution of* $\varphi(x;\overline{a})$ *in* (c,d) we mean that, for every $(u,v)\subseteq(c,d)$,

$$||\varphi(M,\overline{a})\cap(u,v)|-\mu_i|(u,v)||\leq C|(u,v)|^{1/2}.$$

  We will implicitly use either μ_i or $\varphi(x;\overline{a})$, but in any case it will be clear from the context.

Now we present a definition of $\mathcal{C}$-*cells*, that is analogous to the concept of cells in $\mathcal{O}$-minimal theories.

Definition 3.1.4. *Let* $\mathcal{C}$ *be an* $\mathcal{O}$*-asymptotic class and let* $(M,<,...)$ *be a structure in* $\mathcal{C}$.

1. *We define* $\mathcal{C}$-cells *inductively:*
   - *A* $(1)-\mathcal{C}$*-cell is a non-empty set of the form* (a,b).
   - *If* X *is a* $(i_1,\dots,i_m)-\mathcal{C}$*-cell and* $\varphi_{\overline{\mu}}(\overline{y};\overline{z})$ *with* $|\overline{y}|=m$ *is the* $\overline{\mu}$*-definition for some formula* $\varphi(x;\overline{y})$*, then for every* $\overline{d}$ *with* $|\overline{d}|=|\overline{z}|$ *we have that if the set* $X\cap\varphi_{\overline{\mu}}(\overline{y};\overline{d})$ *is non-empty, it is also a* $(i_1,\dots,i_m)-\mathcal{C}$*-cell.*
   - *Let* X *be a* $(i_1,i_2,\dots,i_m)-\mathcal{C}$*-cell.*
     - *If* $f:M^n\longrightarrow M$ *is a definable function, then* $\Gamma(f,X)=\{(\overline{x},f(\overline{x})):\overline{x}\in X\}$ *is a* $(i_1,\dots,i_m,0)-\mathcal{C}$*-cell.*
     - *If* $f,g:M^n\longrightarrow M$ *is are definable functions with* $f|_X<g|_X$*, then the set* $(f,g)_X=\{(\overline{x},y):f(\overline{x}))<y<g(\overline{x}):\overline{x}\in X\}$ *is a* $(i_1,\dots,i_m,1)-\mathcal{C}$*-cell.*
2. *A* $\mathcal{C}$-cell decomposition for M^n *is a partition* $\{Z_1,\dots,Z_k\}$ *of* M^n *such that every* Z_i *is a* $\mathcal{C}$*-cell.*
3. *If* Z *is a* $\mathcal{C}$*-cell in* M^k*, we can define* $dim(Z)$ *as*

$$dim(Z)=\min\left\{\sum_{j=1}^{k}i_j:\textit{ the cell } Z \textit{ can be expressed as a } (i_1,\dots,i_k)-\mathcal{C}\textit{-cell}\right\}$$

Definition 3.1.5. *We say that $Z \subseteq M^k$ is a* 1-cell *if there is some $j \leq k$ such that the projection of Z under the j-th coordinate is a bijection between Z and a (1)-$\mathcal{C}$-cell in M^1.*

Remark 3.1.6.

- *Note that by definition, for a $(i_1, \ldots, i_k) - \mathcal{C} -$cell it is always true that $i_1 = 1$. Also, it is possible that a (1)-cell is a point, and this will happen when $S^2(a) = b$ (with S the successor function).*

- *Moreover, a $(1,0) - \mathcal{C} -$ cell might also be described as a $(1,1) - \mathcal{C}$-cell. For instance, $\{x \in (a,b) : f(x) = c\} = \{x \in (a,b) : S^{-1}c < f(x) < Sc\}$.*

- *The concept of 1-cell (without parenthesis) should not be confused with (1)-cells. In general, 1-cells can be subsets of M^k for any k, while (1)-cells are always intervals in M^1, possibly intersected with formulas of the form $\varphi_{\overline{\mu}}(y; \overline{d})$.*

The main differences between the $\mathcal{C}$-cells and the cells in $\mathcal{O}$-minimal structures arise from the lack of continuity, the fact that we are dealing with discrete orders that are approximable by finite structures, and the intersection with $\overline{\mu}$-definitions. All these reasons make it difficult to apply the usual definition of cells in dense or discrete $\mathcal{O}$-minimal theories.

Note that the cells in this context are allowed to be the intersection of $\mathcal{O}$-minimal cells (the ones produced by intervals and definable functions) with some other definable sets (the ones given the definition for some formula φ and measure $\overline{\mu} \in E_{\varphi}$). Thus, these cells are similar to the cells in Presburger Arithmetic used by Cluckers in [14] to get the cell-decomposition theorem for $(\mathbb{Z}, +, <)$.

Given a formula $\varphi(x, \overline{y})$ and $\overline{a} \in M$, once the measure $\overline{\mu} \in E$ for $\varphi(x, \overline{a})$ is known, the condition (2) of being an $\mathcal{O}$-asymptotic class provides a canonical way to choose the decomposition into intervals for $\varphi(M; \overline{a})$:

Consider the definable functions $m_1^\varphi, \ldots, m_k^\varphi$ with parameters $\overline{a}$, defined as follows:

- $M \models m_1^\varphi(\overline{a}) = w$ if and only if

$$\begin{aligned} M \models & \exists w_2, \ldots, w_k \left(\varphi_{\overline{\mu}}(\overline{a}, w, w_2, \ldots, w_k)\right) \\ & \wedge \forall v \left(v < w \rightarrow \neg \exists w_2, \ldots, w_k \left(\varphi_{\overline{\mu}}(\overline{a}, w, w_2, \ldots, w_k)\right)\right) \end{aligned}$$

- $M \models m_2^\varphi(\overline{a}) = w$ if and only if

$$\begin{aligned} M \models & \exists w_3, \ldots, w_k \left(\varphi_{\overline{\mu}}(\overline{a}, m_1^\varphi(\overline{a}), w, \ldots, w_k)\right) \\ & \wedge \forall v \left(v < w \rightarrow \neg \exists w_3, \ldots, w_k \left(\varphi_{\overline{\mu}}(\overline{a}, m_1^\varphi(\overline{a}), w, w_3, \ldots, w_k)\right)\right) \end{aligned}$$

$$\vdots \qquad \vdots \qquad \vdots$$

- $M \models m_k^\varphi(\overline{a}) = w$ if and only if

$$\begin{aligned} M \models & \ \varphi_{\overline{\mu}}(\overline{a}, m_1^\varphi(\overline{a}), \ldots, m_{k-1}^\varphi(\overline{a}), w) \\ & \wedge \forall v (m_{k-1}^\varphi(\overline{a}) < v < w \rightarrow \neg \left(\varphi_{\overline{\mu}}(\overline{a}, m_1^\varphi(\overline{a}), m_2^\varphi(\overline{a}), \ldots, m_{k-1}^\varphi(\overline{a}), v)\right). \end{aligned}$$

Basically, $m_1^\varphi(\overline{a}) = w$ if and only if w is the first element that can be used in a "good decomposition" for $\varphi(x, \overline{a})$, and we may define the functions $m_i^\varphi(\overline{a})$ as the $i-th$ element in a "minimal" decomposition for $\varphi(x, \overline{a})$ $(i = 2, \ldots, k)$.
Note that these functions $m_i^\varphi(\overline{y})$ have higher-dimensional generalizations for n-variable formulas, obtained after using the induction hypothesis and the lexicographical order. We will denote these functions as $m_i^{\varphi,n}(\overline{y})$. The index n could be implicit if the formula $\varphi(\overline{x}; \overline{y})$ satisfies $|x| = n$, but we will omit sometimes the formula and for clarity will always put the index n when necessary.

These functions will play a key role in the proof of the following result, which can be seen as an analogue of Theorem 2.1 of [38], combined with ideas from cell decomposition in $\mathcal{O}$-minimal theories.

Theorem 3.1.7. *Suppose $\mathcal{C}$ is an $\mathcal{O}$-asymptotic class of finite $\mathcal{L}$-structures. Then for every $\mathcal{L}$-formula $\varphi(\overline{x},\overline{y})$ (where $|x| = n, |y| = m$) the following hold:*

1. *There is a positive constant $C > 0$, $k = k(\varphi, n) \in \mathbb{N}$ and a finite set D of tuples $\overline{\alpha}$ with $\overline{\alpha} \in [0,1]^k$ such that for every $M \in \mathcal{C}$ and $\overline{a} \in M^m$, there is a $\mathcal{C}$-cell decomposition $\{Z_1, \dots, Z_k\}$ of M^n into k cells such that:*

   (*): For every $i \leq k$, either

$$\begin{cases} \alpha_i = 0 \text{ and } |\varphi(M^n;\overline{a}) \cap Z_i| \leq C \\ \qquad\qquad or \\ \alpha_i \neq 0 \text{ and } ||\varphi(M^n;\overline{a}) \cap Z_i| - \alpha_i|Z_i|| \leq C|L_i|^{dim(Z_i)-1/2} \end{cases}$$

   where L_i is a 1-cell of maximal size contained in Z_i.

2. *For every $\overline{\alpha} \in D$ there are $\mathcal{L}$-formulas $\varphi_{\overline{\alpha}}(\overline{y};\overline{z})$ and $Z_1(\overline{x};\overline{z}), \dots, Z_k(\overline{x};\overline{z})$ such that $M \models \varphi_{\overline{\alpha}}(\overline{a};\overline{d})$ implies*

   - *$\{Z_1(\overline{x};\overline{d}), \dots, Z_k(\overline{x};\overline{d})\}$ is a cell decomposition of M^n.*
   - *(*) holds for this decomposition.*

Proof. We will prove this result by induction on n.

- <u>$n = 1$ ***case:***</u>

  Let $\varphi(x;\overline{y})$ be an $\mathcal{L}$-formula. By the definition of $\mathcal{O}$-asymptotic classes, there are constants $C > 0$, $k \geq 1$ and $E \subseteq^{fin} [0,1]^k$ such that:

  (i) For all $M \in \mathcal{C}$ and $\overline{a} \in M^m$ there are $c_0 = \min M < c_1 < \dots < c_k = \max M$ and $\overline{\mu} \in E$ such that for every $i = 1, 2, \dots, k$ either $\mu_i > 0$ and

$$||\varphi(M,\overline{a}) \cap (c_{i-1}, c_i)| - \mu_i(c_{i-1}, c_i)|| \leq C|(c_{i-1}, c_i)|^{1/2}, \quad (*)$$

  or $\mu_i = 0$ and $|\varphi(M,\overline{a}) \cap (c_{i-1}, c_i)| \leq C$. Here we can take $Z_{2i} = c_i$ and $Z_{2i+1} = (c_{i-1}, c_i)$ for $i = 0, \dots, k$ that clearly is a $\mathcal{C}$-cell-decomposition for M^1. Also, we can take D to be the finite set of measures $\overline{\alpha} \in [0,1]^{2k}$ given by $\alpha_{2i} = 0, \alpha_{2i+1} = \mu_i$, for $\overline{\mu} \in E$.

(ii) For every $\overline{\mu} \in E$ there is an $\mathcal{L}$-formula $\psi_{\overline{\mu}}(\overline{y}; z_1, \ldots, z_{k-1})$ such that $M \models \psi_{\overline{\mu}}(\overline{a}; c_1, \ldots, c_{k-1})$ implies (∗) holds. We can take now the formula

$$\varphi_{\overline{\mu}}(\overline{a}; c_1, \ldots, c_{k-1}) = \psi_{\overline{\mu}}(\overline{a}; c_1, \ldots, c_{k-1}) \wedge (\min M < c_1 < \ldots < c_{k-1} < \max M)$$

to ensure that $\{Z_i : i \leq 2k+1\}$ is a cell-decomposition of M^1.

- **Inductive case:**

Let $\varphi(z, \overline{x}; \overline{y})$ be an $\mathcal{L}$-formula with $(z, \overline{x})$ as object variables and $\overline{y}$ as parameter variables, $|\overline{x}| = n, |\overline{y}| = m$. The measures for this formula will be denoted by α, so when using the case $n = 1$ or the inductive hypothesis, we will be using different letters.

Let us fix $\overline{a}$. Considering $(\overline{x}; \overline{a})$ as parameter variables, we have by the $n = 1$ case that for every $(\overline{b}, \overline{a}) \in M^n \times M^m$, there are $c_0 = \min M < c_1 < \ldots < c_k = \max M$ in M and an element $\overline{\mu}_{\overline{b}} = (\mu^1, \ldots, \mu^k) \in E$ such that:

(*) For every $i = 1, 2, \ldots, k$, either

$$\begin{cases} \mu_i = 0 \;\text{ and }\; |\varphi(M, \overline{b}; \overline{a}) \cap (c_{i-1}, c_i)| \leq C \\ \qquad\qquad \text{or} \\ \mu_i > 0 \;\text{ and }\; ||\varphi(M, \overline{b}; \overline{a}) \cap (c_{i-1}, c_i)| - \mu_i |(c_{i-1}, c_i)|| \leq C|(c_{i-1}, c_i)|^{1/2}. \end{cases}$$

Furthermore, there are is a formula $\varphi_{\overline{\mu}_{\overline{b}}}(\overline{x}, \overline{y}, w_1, \ldots, w_k)$ such that $M \models \varphi_{\overline{\mu}_{\overline{b}}}(\overline{b}, \overline{a}, c_1, \ldots, c_k)$ if and only if (∗) holds.

Note that the formulas above depend only on the associated measure $\overline{\mu} \in E$. So, we can enumerate E by $E = \{\overline{\mu}_1, \ldots, \overline{\mu}_t\}$ and $\varphi_1, \ldots, \varphi_t$ the corresponding formulas. ($\varphi_i := \varphi_{\overline{\mu}_i}$). Consider the formulas

$$\phi_i(\overline{x}; \overline{y}) := \exists z_1, \ldots, z_k (\varphi_i(\overline{x}, \overline{y}; z_1, \ldots, z_k)).$$

Now, given $\overline{a} \in M^m$, the formulas $\phi_1(\overline{x};\overline{a}), \dots, \phi_t(\overline{x};\overline{a})$ form a partition of M^n. Also, by induction hypothesis, for each formula $\phi_i(\overline{x};\overline{y})$ there is a finite set $D_i = \{\overline{\nu}^i_1, \dots, \overline{\nu}^i_u\} \subseteq [0,1]^l$ and formulas $Z_{i,1}(\overline{x},\overline{y};\overline{w}), \dots, Z_{i,\ell}(\overline{x},\overline{y};\overline{w})$ such that:

For every $\overline{a} \in M^m$ there are $\overline{\nu} \in D_i$ and a tuple $\overline{d}$ satisfying:

- $\{Z_{i,1}(\overline{x};\overline{a},\overline{d}), \dots, Z_{i,l}(\overline{x};\overline{a},\overline{d})\}$ is a cell decomposition of M^n.
- $(*)_{i,n}$: For every $j \leq l$, either

$$\begin{cases} \nu_j = 0 \ \text{ and } \ |\varphi_i(M^n;\overline{a}) \cap Z_{i,j}(M^n;\overline{d})| \leq C \\ \qquad\qquad \text{or} \\ \nu_j \neq 0 \ \text{ and } \ ||\varphi_i(M^n;\overline{a}) \cap Z_{i,j}(M^n;\overline{d})| - \nu_j |Z_{i,j}(M^n;\overline{d})| \leq C|L_{i,j}(M^n;\overline{d})|^{1/2}. \end{cases}$$

where $L_{i,j}$ is the 1-cell of maximal size contained in $Z_{i,j}$.

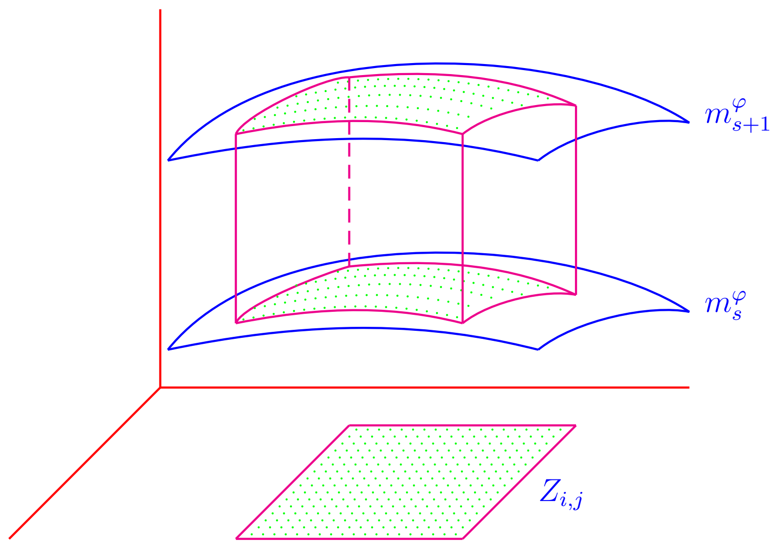
Furthermore, for every $\overline{\nu} \in D_i$ there is a formula $\chi^i_{\overline{\nu}}(\overline{y};\overline{z})$ such that $M \models \chi^i_{\overline{\nu}}(\overline{a};\overline{d})$ if and only if the conditions above hold. As we did before, we can enumerate these formulas and let $\chi^i_h(\overline{y};\overline{z})$ be the formula corresponding the the measure $\overline{\nu}_h \in D_i$ $(h \leq u)$.

Using the functions $m^{\phi,n}_i(\overline{y})$, we may assume that the formulas $Z_{i,j}(\overline{x};\overline{a},\overline{d})$ providing the cell-decomposition of M^n for ϕ_i can be written as $Z_{i,j}(\overline{x};\overline{a},m^n(\overline{a}))$, where we are omitting the indices i and ϕ since they are fixed through this proof.

Consider now the following sets of formulas:

$$\begin{aligned} &\mathfrak{X}(z,\overline{x};\overline{y}) \\ &:= \{Z_{i,j}(\overline{x};m^n(\overline{y})) \wedge \varphi_i(\overline{x};\overline{y},m^{\varphi_i}_1(\overline{x},\overline{y}),\dots,m^{\varphi_i}_k(\overline{x},\overline{y})) \wedge \left(m^{\varphi_i}_s(\overline{x},\overline{y}) < z < m^{\varphi_i}_{s+1}(\overline{x},\overline{y})\right)\} \\ &\cup \{Z_{i,j}(\overline{x};m^n(\overline{y})) \wedge \varphi_i(\overline{x};\overline{y},m^{\varphi_i}_1(\overline{x},\overline{y}),\dots,m^{\varphi_i}_k(\overline{x},\overline{y})) \wedge (m^{\varphi_i}_s(\overline{x},\overline{y}) = z)\} \end{aligned}$$

for $i \leq t, j \leq u, 0 \leq s \leq k$.



It is clear that for every $\overline{a} \in M^m$, the collection $\mathfrak{X}(z, \overline{x}; \overline{a})$ forms a $\mathcal{C}$-cell decomposition of M^{n+1}, because the sets defined by

$$Z_{i,j}(\overline{x}; m^n(\overline{a})) \wedge \varphi_i(\overline{x}, \overline{y}, m_1(\overline{x}), \ldots, m_k(\overline{x}, \overline{y}))$$

are $\mathcal{C}$-cells, $m^{\varphi_i}_s(\overline{x})$ are definable functions, and originally the formulas $Z_{i,j}$ and φ_i formed a partition of the space M^n.

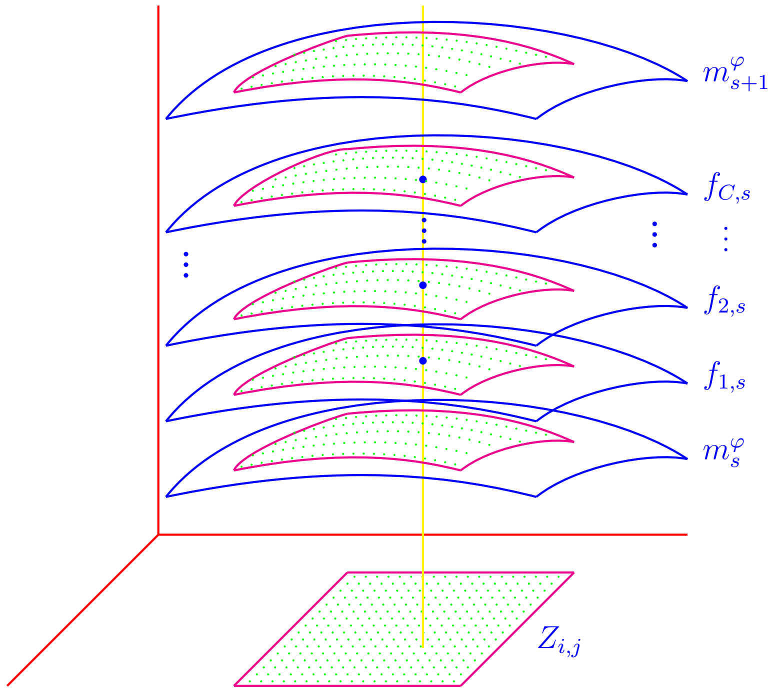
Consider now the following refinement of $\mathfrak{X}(z, \overline{x}; \overline{a})$:

(a) If $\mu^s_i = 0$, then for every $\overline{b} \models Z_{i,j}(\overline{x}; m^n(\overline{a})) \cap \phi_i(\overline{x}; \overline{a})$, the vertical fibers between $m_s(\overline{b}, \overline{a})$ and $m_s(\overline{b}, \overline{a})$ are uniformly finite. So, we have

$$|\{z : (m_s(\overline{b}, \overline{a}) < z < m_s(\overline{b}, \overline{a})) \wedge \varphi(z, \overline{b}; \overline{a})\}| \leq C$$

and we can take the following definable functions:

$$\begin{aligned}
f_{1,s}(\overline{x}) &= \min\{z : \ m_s(\overline{x}) < z < m_{s+1}(\overline{x}) \wedge \varphi(z, \overline{x}; \overline{a})\} \\
f_{2,s}(\overline{x}) &= \min\{z : \ f_{1,s}(\overline{x}) < z < m_{s+1}(\overline{x}) \wedge \varphi(z, \overline{x}; \overline{a})\} \\
\vdots & \qquad \vdots \\
f_{C,s}(\overline{x}) &= \min\{z : \ f_{C-1,s}(\overline{x}) < z < m_{s+1}(\overline{x}) \wedge \varphi(z, \overline{x}; \overline{a})\} \\
&= \max\{z : \ z < m_{s+1}(\overline{x}) \wedge \varphi(z, \overline{x}; \overline{a})\}.
\end{aligned}$$



Thus, we have a refinement of the original $\mathcal{C}$-cell given by the collection of $\mathcal{C}$-cells

$$\{Z_{i,j}(\overline{x}; m^n(\overline{a})) \wedge \varphi_i(\overline{x}; \overline{a}, m_1(\overline{x}, \overline{y}), \ldots, m_k(\overline{x}, \overline{y}))$$
$$\wedge\, (f_{r,s}(\overline{x}) < z < f_{r+1,s}(\overline{x})) : 0 \leq r \leq C+1\}$$
$$\cup\, \{Z_{i,j}(\overline{x}; m^n(\overline{a})) \wedge \varphi_i(\overline{x}; \overline{a}, m_1(\overline{x}, \overline{y}), \ldots, m_k(\overline{x}, \overline{y})) \wedge (f_{r,s}(\overline{x}) = z) : 0 \leq r \leq C+1\},$$

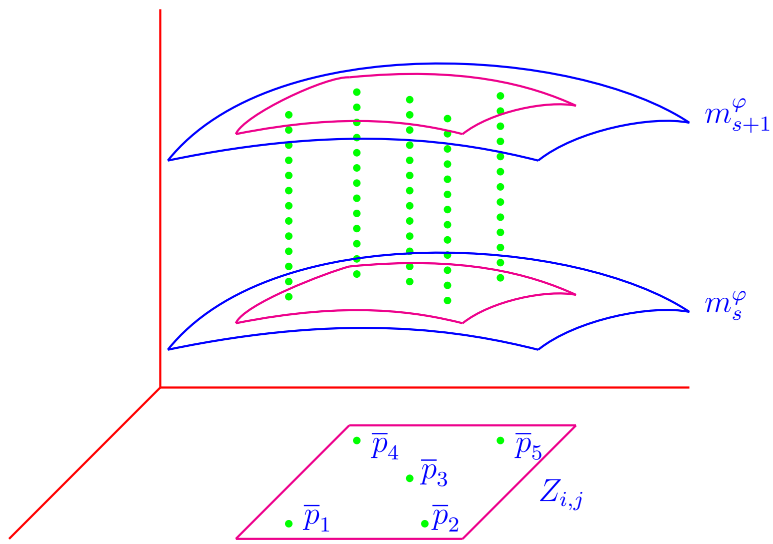
with $f_{0,s} = m_s, f_{C+1,s} = m_{s+1}$.

(b) If $\nu^i_{h,j} = 0$ and $\mu^s_i \neq 0$, then the $\mathcal{C}$-cell given by

$$\{\overline{x} \in M^n : Z_{i,j}(\overline{x}; \overline{a}, m^n(\overline{a})) \wedge \varphi_i(\overline{x}; \overline{a}, m_1(\overline{x}, \overline{a}), \ldots, m_s(\overline{x}, \overline{a}))\}$$

has at most C_n points, where C_n is a constant that might be different to the constant C of the case $n = 1$. Let $\overline{p}_1, \ldots, \overline{p}_{C_n}$ be an enumeration of such points. We then consider the refinement given by the collections

$$\{\{(z, \overline{p}_r) : m_s(\overline{p}_r) < z < m_{s+1}(\overline{p}_r)\} : r \leq C_n\} \cup \{(\overline{p}_r, z) : m_s(\overline{p}_r) : r \leq C_n\}.$$



Let $\widehat{\mathfrak{X}}(z,\overline{x};\overline{a})$ be the refinement of $\mathfrak{X}(z,\overline{x};\overline{a})$ after the processes (a) and (b), leaving any cell outside this cases unmodified. This still provides a $\mathcal{C}$-cell decomposition of M^{n+1}. For every $Z \in \widehat{\mathfrak{X}}(z,\overline{x},\overline{y})$ let L_Z be a 1-$\mathcal{C}$-cell contained in Z, with maximal size. We have the following cases:

- If Z is obtained after process (a) we consider two cases:
  - The $\mathcal{C}$-cell Z has the form

$$\{(z,\overline{x}) : Z_{i,j}(\overline{x};m^n(\overline{a}))\wedge\varphi_i(\overline{x};\overline{a},m_1(\overline{x},\overline{a}),\ldots,m_k(\overline{x};\overline{a}))\wedge f_{r,s}(\overline{x}) < z < f_{r+1,s}(\overline{x})\}$$

    and the intended measure will be $\alpha_Z = 0$. Clearly,

$$||\varphi(M^{n+1};\overline{a})\cap Z| - \alpha_Z|Z|| = 0 \leq C|L_Z|^{\dim Z - 1/2}. \qquad \checkmark$$

  - The $\mathcal{C}$-cell Z has the form

$$\{(z,\overline{x}) : Z_{i,j}(\overline{x};m^n(\overline{a}))\wedge\varphi_i(\overline{x};\overline{a},m_1(\overline{x},\overline{a}),\ldots,m_s(\overline{x};\overline{a}))\wedge f_{r,s}(\overline{x}) = z\}$$

    and the intended measure is $\alpha_Z = \nu^i_{h,j}$. Note that

$$\dim Z_{i,j} = \dim(Z_{i,j}\cap\varphi_i) = \dim(f_{r,s}(Z_{i,j}\cap\varphi_i)) = \dim Z.$$

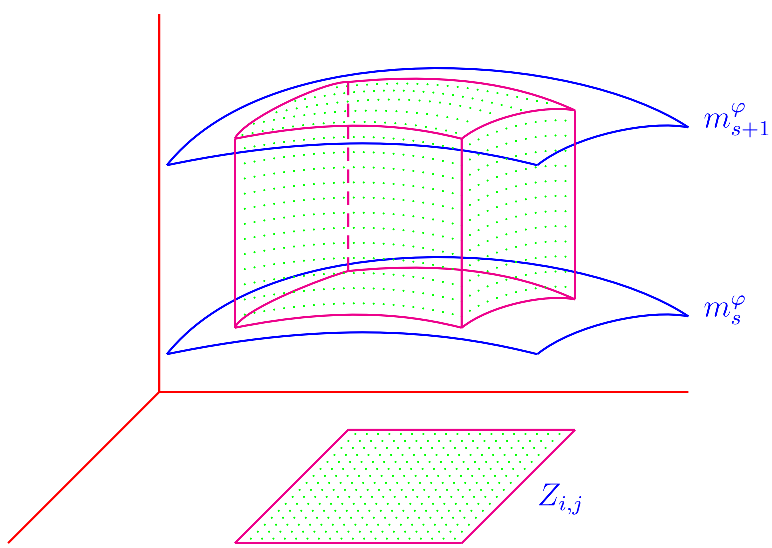
Let $L_{i,j}$ be a 1-$\mathcal{C}$-cell contained in $Z_{i,j}$ of maximal size. Then we obtain

$$\begin{aligned}||\varphi(M^{n+1};\overline{a}\cap Z| - \alpha_Z|Z|| = ||Z_{i,j}\cap\phi_i| - \nu^i_{h,j}|Z_{i,j}|| &\leq C_n\cdot|L_{i,j}|^{\dim Z_{i,j}-1/2}\\ &= C_n\cdot|\{(\overline{x};f_{r,s}(\overline{x}):\overline{x}\in L_{i,j}\}|^{\dim Z_{i,j}-1/2}\\ &\leq C_n\cdot|L|^{\dim Z-1/2}. \qquad \checkmark\end{aligned}$$

– If Z is obtained after process (b) we consider two cases again:

* The $\mathcal{C}$-cell Z has the form $Z=\{(\overline{p}_r,z):m_s(\overline{p}_r)<z<m_{s+1}(\overline{p}_r)\}$ and is itself a 1-$\mathcal{C}$-cell. Take $\alpha_Z=\mu^s_i>0$ to obtain

$$\begin{aligned}&||\varphi(M^{n+1};\overline{a})\cap Z| - \alpha_Z|Z||\\ &=||\varphi(M,\overline{p}_r;\overline{a})\cap(m_s(\overline{p}_r),m_{s+1}(\overline{p}_r))| - \mu^s_i|(m_s(\overline{p}_r),m_{s+1}(\overline{p}_r))||\\ &\leq C\cdot|(m_s(\overline{p}_r),m_{s+1}(\overline{p}_r))|^{1/2}\\ &= C\cdot|\{(z,\overline{p}_r):m_s(\overline{p}_r)<z<m_{s+1}(\overline{p}_r)\}|^{1/2}\\ &= C\cdot|L_Z|^{1/2}=C\cdot|L_Z|^{\dim Z-1/2}. \qquad \checkmark\end{aligned}$$



* The $\mathcal{C}$-cell Z has the form $Z=\{(m_s(\overline{p}_r),\overline{p}_r)\}$. In this case we can take the measure $\alpha_Z=0$. The desired inequality clearly holds if we consider the fact that the size of the maximal 1-$\mathcal{C}$-cell contained in a point is 1.

(c) If Z did not pass through neither process (a) nor (b), then $\mu_i^s, \nu_{i,h}^j > 0$. Again we have two cases:

* The $\mathcal{C}$-cell Z has the form

$$\{(z, \overline{x}) : Z_{i,j}(\overline{x}; m^n(\overline{a})) \wedge \varphi_i(\overline{x}; \overline{a}, m_1(\overline{x}, \overline{a}), \dots, m_k(\overline{x}; \overline{a})) \wedge z = m_s(\overline{a})\}.$$

  This case is analogue to the second case in (a), replacing the function $f_{r,s}$ by m_s. The intended measure would be $\alpha_Z = v_{i,h}^j$.

* The $\mathcal{C}$-cell Z has the form

$$\begin{aligned} Z = \{(z, \overline{x}) : &Z_{i,j}(\overline{x}; m^n(\overline{a})) \wedge \varphi_i(\overline{x}; \overline{a}, m_1(\overline{x}, \overline{a}), \dots, m_k(\overline{x}; \overline{a})) \\ &\wedge m_s(\overline{x}, \overline{a})) < z < m_{s+1}(\overline{x}, \overline{a})\}. \end{aligned}$$

  The intended measure in this case is $\alpha_Z = \mu_i^s$. Let L be the interval $(m_s(\overline{b}, \overline{a}), m_{s+1}(\overline{b}, \overline{a}))$ with $\overline{b} \in Z_{i,j} \cap \phi_i$ of maximal size. Note that $\dim Z = \dim Z_{i,j} + 1$ and we obtain

$$\begin{aligned}
&|\varphi(M^{n+1}; \overline{a}) \cap Z| \\
&= \sum_{\overline{b} \in Z_{i,j} \cap \phi_i} \left|\varphi(M, \overline{b}; \overline{a}) \cap (m_s(\overline{b}; \overline{a}), m_{s+1}(\overline{b}, \overline{a}))\right| \\
&\leq \sum_{\overline{b} \in Z_{i,j} \cap \phi_i} \left(\mu_i^s \cdot \left|(m_s(\overline{b}; \overline{a}), m_{s+1}(\overline{b}, \overline{a}))\right| + C \left|(m_s(\overline{b}; \overline{a}), m_{s+1}(\overline{b}, \overline{a}))\right|^{1/2}\right) \\
&\leq \mu_s^i \cdot \left(\sum_{\overline{b} \in Z_{i,j} \cap \phi_i} \left|(m_s(\overline{b}; \overline{a}), m_{s+1}(\overline{b}, \overline{a}))\right|\right) + C \cdot |L|^{1/2} |Z_{i,j} \cap \phi_i| \\
&\leq \mu_i^s \cdot |Z| + C_1 |L|^{1/2} \left(\nu_{i,h}^j \cdot |Z_{i,j}| + C_n |L_{i,j}|^{\dim Z_{i,j} - 1/2}\right) \\
&\leq \mu_i^s \cdot |Z| + C_1 \cdot \nu_{i,h}^j \cdot |L|^{1/2} |L_{i,j}|^{\dim Z_{i,j}} + C_1 C_n \cdot |L|^{1/2} |L_{i,j}|^{\dim Z_{i,j} - 1/2} \\
&\qquad\qquad\qquad (\text{because } |Z_{i,j}| \leq |L_{i,j}|^{\dim Z_{i,j}}) \\
&\leq \mu_s^i \cdot |Z| + 2C_1 \cdot C_n |L_Z|^{\dim Z_{i,j} + 1/2} \\
&\leq \mu_s^i \cdot |Z| + C_{n+1} |L_Z|^{(\dim Z_{i,j} + 1) - 1/2} = \mu_s^i \cdot |Z| + C_{n+1} |L_Z|^{\dim Z - 1/2}.
\end{aligned}$$

This completes the proof of the condition (1) for the inductive case.

In any case, the formulas $\chi^i_{\overline{\nu}}(\overline{a}, m^n(\overline{a}))$ prove that the cells in $\widehat{\mathfrak{X}}(\overline{a}, m^n(\overline{a}))$ are definable. So, the definability condition holds.

$\square$

3.2 Examples of $\mathcal{O}$-asymptotic classes

In this section we will present various examples of $\mathcal{O}$-asymptotic classes

Example 3.2.1. The class $\mathcal{C}_{ord}$ of finite linear orders is an $\mathcal{O}$-asymptotic class.

It was shown in Section 1.4 that this class admits uniform quantifier elimination. Moreover, every formula $\varphi(x, \overline{a})$ in one variable is a disjunction of formulas of the form

$$\bigwedge_i t_i(\overline{a}) < x \wedge \bigwedge_j x < u_j(\overline{a}) \ \wedge \bigwedge_l x = v_l(\overline{a})$$

(where t_i, u_j, v_k are terms in the expanded language $\mathcal{L}^* = \{<, c, C, S, S^{-1}\}$ that depend on the tuple $\overline{a}$) and it defines a finite union of intervals and points with a bound in the number of intervals. If $\varphi(x, \overline{a})$ defines at most k intervals, we may take $\langle c_i : i \leq k\rangle$ to be the end points of the intervals, and as measures all the possible vectors $\overline{\mu} \in \{0, 1\}^k$.

3.2.1 Cyclic groups with an ordering

Definition 3.2.2. *Given a natural number N, we consider the finite linearly ordered structure $\mathcal{Z}_N = (\mathbb{Z}/(2N+1)\mathbb{Z}, +, <)$ with the usual additive structure and the linear order defined as:*

$$-\widetilde{N} < -\widetilde{N-1} < \cdots < \widetilde{0} < \cdots \widetilde{N-1} < \widetilde{N}.$$

We will show that the class $\mathcal{C} = \{\mathcal{Z}_N : N < \omega\}$ is an $\mathcal{O}$-asymptotic class. First, we need a result that describes uniformly the definable sets for this class.

Lemma 3.2.3. *Let $\mathcal{C}_{ocyc} = \{\mathcal{Z}_N : N < \omega\}$. Then for every N and $\overline{a} \in \mathcal{Z}_N$ the formula $\varphi(x; \overline{a})$ is equivalent in $\mathcal{Z}_N$ (with a uniform number of disjunctions) to a boolean combination of formulas of the form:*

$$x = b, \quad x < b, \quad x > b, \text{ and } P_m(x + b)$$

where $P_m(y) := \exists t\,((0 < t < \cdots < mt = y) \vee (0 > t > \cdots > mt = y))$ for $m \geq 2$.

Proof. First, define the function

$$\begin{aligned} f : &\mathcal{Z}_N \longrightarrow \mathbb{Z} \\ &\widetilde{x} \longmapsto y \end{aligned}$$

where y is the unique integer such that $-N \leq y \leq N$ and $y \equiv x \mod (2N+1)$, and the following formulas as an intended interpretation of every $\mathcal{Z}_N$ into $(\mathbb{Z}, +, <, 0)$:

$$\begin{aligned} Z(x; N) &:= -N \leq x \leq N \\ S(x, y, z) &:= Z(x) \wedge Z(y) \wedge Z(z) \wedge (x + y = z \vee x + y - N = z \vee x + y + N = z) \\ O(x, y) &:= Z(x) \wedge Z(y) \wedge x < y. \end{aligned}$$

Given a formula $\varphi(x; \overline{y})$ and using these replacements and continuing inductively on the connectives and quantifiers (in the natural way) we obtain a formula $\widehat{\varphi}(x; \overline{y}, w)$ such that $\mathcal{Z}_N \models \varphi(b; \overline{a})$ if and only if $\mathbb{Z} \models \widehat{\varphi}(f(b); f(\overline{a}), N)$. By Presburger's Theorem, the formula $\widehat{\varphi}(x; \overline{y}, w)$ is equivalent in $\mathbb{Z}$ to a boolean combination of formulas of the form

$$nx = t(\overline{y}, w), \quad nx < t(\overline{y}, w), \quad t(\overline{y}, w) < nx, \quad D_m(nx + t(\overline{y}, w))$$

where $t(\overline{y}, w)$ is a term which has the form $\sum_{i=1}^{l} \alpha_i \cdot y_i + \beta w$ with $\alpha_i, \beta \in \mathbb{Z}$, and the formula $D_m(z)$ means "z is divisible by m".

Since n is fixed, if we replace $\overline{y}, w$ by $f(\overline{a}), N$, the first three formulas produce definable sets which are points or intervals in $\mathbb{Z} \cap [-N, N]$, and they will produce boolean combination of points and intervals in $\mathcal{Z}$ where the maximum number of points and intervals depends only on n.

Now, let $\phi(x) := D_m(nx + t(f(\overline{a}), N))$ and assume that $\mathbb{Z} \models \phi(x)$. We can write $t(f(\overline{a}), N) = mq + r$ for $0 \leq r < m$. So, there is some $z \in \mathbb{Z}$ such that $mz = nx + r$. We have three cases:

- $\underline{g.c.d.(n, m) = 1}$: there is a combination such that $\gamma_1 n + \gamma_2 m = 1$ and by writing $\gamma_1 r = mk + r_2$ (with $0 \leq r_2 < m$) we obtain:

$$\begin{aligned} \mathbb{Z} \models D_m(nx + t(f(\overline{a}), N)) &\Leftrightarrow mz = nx + r \text{ for some } z \in \mathbb{Z} \\ &\Leftrightarrow x \equiv -\gamma_1 r \ (mod\, m) \\ &\Leftrightarrow x + mk + r_2 \equiv 0 \ (mod\, m) \\ &\Leftrightarrow x + r_2 \equiv 0 \ (mod\, m) \end{aligned}$$

  So, the definable set $f^{-1}(\phi(\mathbb{Z})) \subseteq \mathcal{Z}_N$ is the set of realizations of the formula $P_m(x + f^{-1}(r_2))$, adding at most one point at each end.

- $\underline{g.c.d.(n, m) = h > 1 \text{ and } h|r}$: In this case we can divide both sides of the equation $mz = nx + r$ by h obtaining

$$\begin{aligned} \mathbb{Z} \models D_m(nx + t(f(\overline{a}), N)) &\Leftrightarrow mz = nx + r \text{ for some } z \in \mathbb{Z} \\ &\Leftrightarrow \left(\frac{m}{h}\right) z = \left(\frac{n}{h}\right) x + \left(\frac{r}{h}\right) \text{ for some } z \in \mathbb{Z} \\ &\Leftrightarrow m_1 z = n_1 x + r_1 \text{ for some } z \in \mathbb{Z} \end{aligned}$$

  where $m_1 = \dfrac{m}{h}, n_1 = \dfrac{m}{h}$ and $r_1 = \dfrac{r}{h}$. Since $m.c.d.(m_1, n_1) = 1$, we can continue as in the case above, proving that $f^{-1}(\phi(\mathbb{Z}))$ is the set of realizations of the formula $P_{m_1}(x+f^{-1}(r_2))$ (where r_2 is the chosen as above from m_1, n_1, r_1), adding at most one point at each end.

- g.c.d.$(n, m) = h > 1$ and $h \not| r$: In this case there is no solution to the equation $nx + r \equiv 0 \ (mod\, m)$, and $f^{-1}(\phi(\mathbb{Z})) = \emptyset$.

Therefore, the formula $\varphi(x; \overline{a})$ is equivalent in $\mathcal{Z}_N$ (for large enough N) to a combination of formulas of the form

$$x = b, \quad x < b, \quad x > b, \quad P_m(x+b),$$

with a bound in the number of intervals and points given by the number of disjunctions in the formula equivalent to $\widehat{\varphi}(x, \overline{y}, w)$ provided by Presburger's Theorem. $\square$

Proposition 3.2.4. *The class $\mathcal{C}_{ocyc}$ is an $\mathcal{O}$-asymptotic class.*

Proof. Let $\varphi(x; \overline{y})$ be a formula in the language $\mathcal{L} = \{+, <\}$. By the Lemma 3.2.3, $\varphi(x; \overline{y})$ is equivalent (uniformly in $\mathcal{C}_{ocyc}$) to a boolean combination of formulas of the form $x = z, x < z$ or $P_m(x + z)$ where z is a term depending on $\overline{y}$. This boolean combination can be assumed to be of the form

$$\bigvee_{0 \le i \le k} \left(a_i < x < b_i \wedge \bigwedge_j (P_{m_{i,j}}(x + c_{i,j}))^{\eta_{i,j}} \right)$$

where $\eta_{i,j}$ is either 0 or 1 (with the notation $\phi^0 = \phi, \phi^1 = \neg\phi$) and $b_i \le a_{i+1}$ for $i \le k-1$.

The decomposition is then given by $c_{2i} = a_i, c_{2i+1} = b_i$ and the possible measures on each interval are the elements $\left\{0, \frac{1}{m}, \frac{2}{m} \ldots, \frac{m-1}{m}, 1\right\}$ where m is the product of all the integers $m_{i,j}$.

$\square$

3.3 Ultraproducts in $\mathcal{O}$-asymptotic classes

In this section we present a series of results which will allow us to place the ultraproducts of structures in $\mathcal{O}$-asymptotic classes in the classification theory map, i.e., we will study model theoretic properties such as $\mathcal{O}$-minimality, NIP, NTP_2, rosiness for the non-trivial ultraproducts of $\mathcal{O}$-asymptotic classes.

Proposition 3.3.1. *Let $\mathcal{C}$ be a class of finite ordered structures and suppose that every infinite ultraproduct is $\mathcal{O}$-minimal. Then $\mathcal{C}$ is an $\mathcal{O}$-asymptotic class*

Proof. Let $\mathcal{C} = \{M_i : i \in \omega\}$ and assume $\prod_{\mathcal{U}} M_i$ is $\mathcal{O}$-minimal for all $\mathcal{U}$. We start with an easy claim:

- *Claim:* For each formula $\varphi(x; \overline{y})$ there is a bound k in the alternation number of $\varphi(x, \overline{a})$, along all tuples $\overline{a}$ in structures of the class $\mathcal{C}$.

  Proof of the claim: Assume not. Then for every $k < \omega$ there is M_{i_k} and $\overline{a}_k \in M_{i_k}$ such that

$$M_{i_k} \models \exists x_0, x_1, \ldots, x_{2k+1} \left(\bigwedge_{i=0}^{2k} x_i < x_{i+1} \bigwedge_{i=0}^{k} \varphi(x_{2i}, \overline{a}_k) \wedge \bigwedge_{i=0}^{k} \neg\varphi(x_{2i+1}, \overline{a}_k) \right).$$

  If $\mathcal{U}$ is a non-principal ultrafilter on ω containing the set $A = \{i_k : k < \omega\}$, then for $M = \prod_{\mathcal{U}} M_i$ and the tuple $\overline{a} = [\overline{a}_k]$ we obtain that

$$M \models \exists x_0, x_1, \ldots, x_{2k+1} \left(\bigwedge_{i=0}^{k} \varphi(x_{2i}, \overline{a}_k) \wedge \bigwedge_{i=0}^{k} \neg\varphi(x_{2i+1}, \overline{a}_k) \right)$$

  for every $k < \omega$, contradicting the $\mathcal{O}$-minimality of M, since $\varphi(M, \overline{a})$ cannot be a finite union of intervals and points. ✓

Let $\varphi(x; \overline{y})$ be a formula. Since there is a bound on the alternation number of $\varphi(x; \overline{y})$, there are also bounds k' and l on the number of intervals and points (respectively) that instances of the formula $\varphi(x : \overline{y})$ can define among the structures in $\mathcal{C}$. Define

$$\Phi(x, \overline{y}, \overline{z}) := \varphi(x; \overline{y}) \leftrightarrow \left(\bigvee_{i=0}^{k'} z_i < x < z_{i+1} \vee \bigvee_{i=k'+1}^{l} x = z_i \right).$$

Then, for every infinite ultraproduct M of structures in $\mathcal{C}$,

$$M \models \forall \overline{y}\, \exists z_0, \ldots, z'_k, z_{k'+1}, \ldots, z_{k'+l} \forall x \left(\Phi(x; \overline{y}, \overline{z}) \right).$$

In particular, there is $N_\varphi \in \mathbb{N}$ such that M_i satisfies the same sentence for every $i \geq N_\varphi$ (if not, we can construct an infinite ultraproduct in which the sentence does not hold).

So, we can take $C = N_\varphi$, $E = \{\overline{\mu} \subseteq [0,1]^{2k'} : \mu_i \in \{0,1\}\}$ and for every $\overline{a} \in M$, we can take $\overline{c} = (c_0, \ldots, c_{2k'+1})$ the corresponding tuple witnessing $M \models \forall x\, (\Phi(x; \overline{a}, \overline{c}))$. This shows that the class $\mathcal{C}$ satisfies the condition (1) of the definition.

For the condition (2) (the definability clause), it is enough to take the formulas

$$\psi_{\overline{\mu}}(\overline{y}; \overline{z}) := \forall x \left(\Phi(\overline{y}, \overline{z}) \right).$$

□

The previous result is not true if we replace the condition of $\mathcal{O}$-minimality by quasi-$\mathcal{O}$-minimality. To show this, consider the class $\mathcal{C}_P$ defined in Section 1.4.1. Every ultraproduct of elements in $\mathcal{C}_P$ is quasi-$\mathcal{O}$-minimal as it follows from the quantifier elimination for $\mathcal{C}_P$ done in Section 1.4.1 and Theorem 1.5.12.

Now we show that the class $\mathcal{C}_P = \{M_n : n < \omega\}$ is not an $\mathcal{O}$-asymptotic class.

Suppose $\mathcal{C}_P$ is $\mathcal{O}$-asymptotic, and let $\overline{c}^n = (c_0^n, \ldots, c_k^n)$ be the tuple in M_n witnessing the condition (1) for the formula $P(x)$. Let $\mu = \min\{\mu_i > 0 : \overline{\mu} \in E\}$ and take n is large enough so that $n > k \cdot \left(\dfrac{1 + C}{\mu} \right)^2$.

Since there are only k intervals, one of those intervals has size $|(c_i, c_{i+1})| = L \geq \frac{n}{k}2^n$. Then,

$$\frac{1}{\frac{n}{k}} + \frac{C}{\left(\frac{n}{k}\right)^{1/2}} < (1+C)\frac{1}{\left(\frac{n}{k}\right)^{1/2}} < (1+C)\frac{\mu}{1+C} = \mu$$

and we have

$$\begin{aligned}\frac{|P(x) \cap (c_i, c_{i+1})|}{|(c_i, c_{i+1})|} &< \frac{\sum_{k=1}^{L/2^n} 2^n \cdot \frac{1}{2^k}}{L} \leq \frac{2^n \sum_{k=1}^{L/2^n} \frac{1}{2^k}}{2^n \frac{n}{k}} \\ &\leq \frac{1}{\frac{n}{k}} < \mu - \frac{C}{\left(\frac{n}{k}\right)^{1/2}} < \mu - \frac{C}{|(c_i, c_{i+1})|^{1/2}}\end{aligned}$$

So,

$$|P(x) \cap (c_i, c_{i+1})| < \mu|(c_i, c_{i+1})| - C|(c_i, c_{i+1})|^{1/2}$$

a contradiction.

Since the leading idea in the definition of $\mathcal{O}$-asymptotic classes is to meld properties from one-dimensional classes (which ultraproducts are known to be simple and unstable in general) and $\mathcal{O}$-minimal theories (which are known to be unstable theories with NIP), we are not expecting the ultraproducts of $\mathcal{O}$-asymptotic classes to be neither simple nor with NIP. The two natural contexts that extends both simple and $\mathcal{O}$-minimal theories are rosy theories and theories with NTP_2. We will show that both of these are properties of the ultraproducts of $\mathcal{O}$-asymptotic classes.

Theorem 3.3.2. *Let $\mathcal{C}$ be an $\mathcal{O}$-asymptotic class. Then for every infinite ultraproduct M of elements of $\mathcal{C}$, $Th(M)$ is superrosy of $U^{þ}$-rank* 1.

Proof. Let $M = \prod_{i \in \mathcal{U}} M_i$ be an infinite ultraproduct of structures in an $\mathcal{O}$-asymptotic class $\mathcal{C}$. To prove that M is super rosy of $\text{U}^{þ}$-rank 1, it is enough to show that the only formulas $\phi(x, b)$ which þ-fork over the empty set are the algebraic formulas.

Assume otherwise. Then there is a tuple of parameters e and a formula $\theta(y, e) \in tp(b/c)$ such that the set

$$\{\phi(x, b') : b' \models \theta(y, e)\}$$

is k-inconsistent for some $k < \omega$.

Put $b = [b_i]_{i \in \mathcal{U}}$. Since $\mathcal{C}$ is $\mathcal{O}$-asymptotic, there is a finite set $E \subseteq [0, 1]^{l+1}$ and a constant C (both associated with the formula $\phi(x, y)$) such that for every $i < \omega$ there is a tuple $\overline{\mu} \in E$ and $c_1^{b_i} < c_2^{b_i} < \cdots < c_l^{b_i}$ satisfying:

1. Either $\mu_j = 0$ and $|\phi(M_i, b_i)| \leq C$, or $\mu_j > 0$ and

$$||\phi(x, b_i)| - \mu_j|(c_j^{b_i}, c_{j+1}^{b_i})|| \leq C|(c_j^{b_i}, c_{j+1}^{b_i})|^{1/2}.$$

2. There is a formula $\phi_{\overline{\mu}}(y, z_1, \ldots, z_l)$ such that

$$M_i \models \phi_{\overline{\mu}}(b_i, c_1, \ldots, c_l) \quad \text{implies} \quad (1) \text{ holds for } \overline{\mu} \text{ and } b_i, c_1, \ldots, c_l.$$

Furthermore, there is a unique tuple $\overline{\mu} \in E$ which works for b_i in an $\mathcal{U}$-large set of indices.

Take $\theta(y, e)$ to imply the formula

$$\exists^{>C \cdot l} x(\phi(x, y)) \wedge \exists z_1, \ldots, z_l(\phi_{\overline{\mu}}(y, z_1, \ldots, z_l)).$$

The first part of the conjunction implies that for $b' \models \theta(y, e)$, $\phi(x, b')$ is not algebraic. The second part ensures that the measure $\overline{\mu}$ works for $\phi(x, b'_i)$ and suitable $c_1^{b'_i}, \ldots, c_l^{b'_i}$ for an $\mathcal{U}$-large set $I_{b'}$. Since $\phi(x, b)$ is non-algebraic, $\overline{\mu} \neq \overrightarrow{0}$. Let μ_j be the first non-zero coordinate of the tuple $\overline{\mu}$.

Claim: *There is an infinite interval (α, β) contained in infinitely many of the intervals $(c_j^{b'}, c_{j+1}^{b'})$ with $b' \models \theta(y, e)$.*

Proof of the Claim: Consider the formula

$$L(w) := \exists y, z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_l(\theta(y, e) \wedge \phi_{\overline{\mu}}(y; z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_l)).$$

This formula defines all the left end points of the intervals $(c_j^{b'}, c_{j+1}^{b'})$ for $b' \models \theta(y, e)$. Furthermore, we may assume that different elements of $L(M)$ correspond to different elements in $\theta(M, e)$, using the function $f_\phi^j(y)$ corresponding to the j-th position of the minimal tuple that provides the decomposition for $\phi(x, b')$ through the structures M_i.

Consider the formula

$$\chi(y, w) := \exists z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_l(\phi_{\overline{\mu}}(y; z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_l)).$$

If $L(M)$ is finite, there is an element $c \in M$ such that $M \models \chi(b', c)$ for infinitely many $b' \models \theta(y, e)$. Thus, the type $p(w')$ given by

$$\{\exists z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_l(\phi_{\overline{\mu}}(b'; z_1, \ldots, z_{j-1}, c, z_{j+1}, \ldots, z_l) \wedge w' < z_{j+1}) : b' \models \chi(y, c)\}$$
$$\cup \{S^n(c) < w' : n < \omega\}$$

is finitely satisfiable. By compactness and $\aleph_1$-saturation of M, there is some $\beta \models p(w')$, and so (α, β) with $\alpha = c$ is the desired interval.

Assume now that $L(M)$ is infinite. We can conclude (using the same argument that we used before in the proof) that for an $\mathcal{U}$-large set of indices i, there is a decomposition $d_1^i, \ldots, d_{k_L}^i$ in M_i such that for some $h \leq k_L$ and $\nu_h > 0$ we have

$$||L(M) \cap (d_h^i, d_{h+1}^i)| - \nu_h|(d_h^i, d_{h+1}^i)|| \leq C|(d_h^i, d_{h+1}^i)|^{1/2}.$$

Note that for every element in $L(M) \cap (d_h, d_{h+1})$, there is an element r_c such that (c, r_c) is infinite and

$$M \models \exists z_1, \ldots, z_{j-1}, z_{j+2}, \ldots, z_l(\phi_{\overline{\mu}}(b'; z_1, \ldots, z_{j-1}, c, r_c, z_{j+2}, \ldots, z_l)).$$

Take $\widehat{c}_0 = \min(L(M) \cap (d_h, \infty))$ and set $\widehat{d}_0 = \min\{r_{\widehat{c}_0}, d_{h+1}\}$. Note that the interval $(\widehat{c}_0, \widehat{d}_0)$ is infinite.

Now assume $\widehat{c}_m, \widehat{d}_m$ have been already constructed satisfying $(\widehat{c}_m, \widehat{d}_m)$ is infinite.

Since $(\widehat{c}_m, \widehat{d}_m)$ is infinite, from the uniform distribution of the set $L(w)$ in the intervals (d^i_h, d^i_{h+1}) and the fact that $\nu_h > 0$, we can conclude that $L(M) \cap (\widehat{c}_m, \widehat{d}_m)$ is infinite. Let $\widehat{c}_{m+1}$ be the minimum element of this set, and $\widehat{d}_{m+1} = \min\{\widehat{d}_m, r_{\widehat{c}_{m+1}}\}$.

$(\widehat{c}_{m+1}, \widehat{d}_{m+1})$ *is infinite:* By construction, $(\widehat{c}_{m+1}, r_{\widehat{c}_{m+1}})$ is infinite. On the other hand, since $L(M) \cap (\widehat{c}_m, \widehat{d}_m)$ is infinite then $(\widehat{c}_{m+1}, \widehat{d}_m) \supseteq \Big(L(M) \cap (\widehat{c}_m, \widehat{d}_m)\Big) - \{\widehat{c}_{m+1}\}$ is infinite.

Since $\widehat{c}_m \in L(M)$, there is $b'_m \models \theta(y, e)$ such that $c^{b'_m}_j = \widehat{c}_m$. By construction,$(\widehat{c}_m, \widehat{d}_m) \subseteq (\widehat{c}_m, r_{\widehat{c}_m}) = (c^{b'_m}_j, c^{b'_m}_{j+1})$.

Consider now the type given by:

$$q(u, v) := \{\widehat{c}_m < u < S^m(u) < v < \widehat{d}_m : m < \omega\}.$$

Clearly, $q(u, v)$ is finitely satisfiable and since M is $\aleph_1$-saturated, there is a pair $(\alpha, \beta) \models q(u, v)$ in M. Therefore, (α, β) is infinite and $(\alpha, \beta) \subseteq (\widehat{c}_m, \widehat{d}_m) \subseteq (c^{b'_m}_j, c^{b'_m}_{j+1})$ for every $m < \omega$. This completes the proof of the claim. ✓

Now, consider the counting measure on M localized in (α, β), i.e., the measure given by

$$m(X) = \lim_{i \to \mathcal{U}} \frac{|X \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|}.$$

Let $\{b^t = [b^t_i] : t < \omega\}$ be a set of infinitely many elements in the ultraproduct such that $(\alpha, \beta) \subseteq (c_j^{b^t}, c_{j+1}^{b^t})$ and $b^t \models \theta(y, e)$, provided by the claim above. For every $t < \omega$, we have that

$$||\phi(x, b_i^t) \cap (\alpha_i, \beta_i)| - \mu_j |(\alpha_i, \beta_i)|| \leq C|(\alpha_i, \beta_i)|^{1/2} \quad \text{for } i \text{ in an } \mathcal{U}\text{-large set.}$$

In particular,

$$\begin{aligned} |\phi(x, b_t) \cap (\alpha_i, \beta_i)| &\geq \mu_j |(\alpha_i, \beta_i)| - C|(\alpha_i, \beta_i)|^{1/2} \\ \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} &\geq \mu_j - C|(\alpha_i, \beta_i)|^{-1/2}. \end{aligned}$$

Since (α, β) is infinite, we can take i large enough so that $\dfrac{C}{|(\alpha_i, \beta_i)|^{1/2}} \leq \dfrac{\mu_j}{2}$, obtaining

$$\begin{aligned} \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} &\geq \mu_j - \frac{\mu_j}{2} \\ \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} &\geq \frac{\mu_j}{2} \\ m(\phi(x, b_t)) = \lim_{i \to \mathcal{U}} \frac{|\phi(x, b_t) \cap (\alpha_i, \beta_i)|}{|(\alpha_i, \beta_i)|} &\geq \frac{\mu_j}{2}. \end{aligned}$$

Therefore, the sets $\langle \phi(x, b_t) : t < \omega \rangle$ are events in a probability space with $m(\phi(x, b_t)) \geq \epsilon = \dfrac{\mu_j}{2}$, then by Proposition 2.2.3, there are $b_{t_1}, \ldots, b_{t_k}$ with $t_1 < \cdots < t_k$ such that

$$m\left(\bigcap_{i=1}^{k} \phi(x, b_{t_k})\right) \geq \epsilon^{3^{k-1}}$$

in particular, $\{\phi(x, b_t) : t < \omega\}$ is not k-inconsistent, and neither is the set $\{\phi(x, b') : b' \models \theta(y, e)\}$. Contradiction. ∎

Theorem 3.3.3. *Every infinite ultraproduct of members of an $\mathcal{O}$-asymptotic class has NTP_2, with $inp - rank(M) = 1$.*

Proof. Let M be an infinite ultraproduct of members in an $\mathcal{O}$-asymptotic class, and suppose that the formula $\phi(x, \overline{y})$ with $|x| = 1$ witness TP_2 in M. If $inp{-}rank(M) \geq$

2, by Fact 1.5.16 there are mutually indiscernible sequences $\langle \overline{a}_i : i < \omega \rangle, \langle \overline{b}_i : i < \omega \rangle$ such that:

- The sets $\{\phi(x; \overline{a}_i) : i < \omega\}$ and $\{\phi(x; \overline{b}_i) : i < \omega\}$ are both 2-inconsistent.
- For every $i, j < \omega$, $\phi(M; \overline{a}_i) \cap \phi(M; \overline{b}_j) \neq \emptyset$.

For every formula $\phi(x; \overline{a}_i)$, there is a tuple of measures $\overline{\mu} \in E_\phi$ such that $\phi(M; \overline{a}_i)$ admits a decomposition with proportions $\overline{\mu}$. By the pigeonhole principle, there are tuples $\overline{\mu}, \overline{\nu} \in E$ such that $\phi(M; \overline{a}_i)$ admits a decomposition with proportion $\overline{\mu}$ for infinitely many $i < \omega$, and $\phi(M; \overline{b}_j)$ admits a decomposition with proportion $\overline{\nu}$ for infinitely many $j < \omega$. Without loss of generality, we can restrict the indiscernibles sequences to such indices.

Note that for every $i < \omega$ the decomposition of $\phi(M; \overline{a}_i)$ is given by elements $c_0^i < \ldots < c_k^i$, that can be assumed (as in Section 3.1) to be the images of the definable functions $m_t^\phi := f_t(\overline{a}_i) = c_t^i$ for $t \leq k$. Likewise, the decomposition of $\phi(M; \overline{b}_j)$ is given by the element $g_t(\overline{b}_i)$ for $t \leq k$.

Claim 1: Assume $\mu_t = 0$ for some $t < k$. Then for every $j < \omega$ the intersection $\phi(M, \overline{b}_j) \cap \phi(M; a_i) \cap (f_t(\overline{a}_i), f_{t+1}(\overline{a}_i))$ is empty.

Proof of the Claim 1: Otherwise, by mutual indiscernibility we have $\overline{b}_j \models \exists x(\phi(x; \overline{y}) \wedge \phi(x; \overline{a}_i) \wedge f_t(\overline{a}_i) < x < f_{t+1}(\overline{a}_i))$ for every $j < \omega$. However, since $\mu_t = 0$, $|\phi(M; \overline{a}_i) \cap (f_t(\overline{a}_i), f_{t+1}(\overline{a}_i))| \leq C$ and this contradicts 2-inconsistency of $\{\phi(x; \overline{b}_j) : j < \omega\}$.

Note that we similarly get the following: if $\nu_t = 0$ for some $t < k$, then for every $i < \omega$ we have $\phi(M; \overline{a}_i) \cap \phi(M; \overline{b}_j) \cap (g_t(\overline{b}_j), g_{t+1}(\overline{b}_j)) = \emptyset$. ✓

Consider t, l minimal such that $\phi(M; \overline{a}_i) \cap \phi(M; \overline{b}_j) \cap (f_t(\overline{a}_i), f_{t+1}(\overline{a}_i)) \neq \emptyset$ and $\phi(M; \overline{a}_i) \cap \phi(M; \overline{b}_j) \cap (g_l(\overline{b}_j), g_{l+1}(\overline{b}_j)) \neq \emptyset$. By the claim above, $\mu_t = \mu$ and $\nu_l = \nu$

are both positive.

Claim 2: For $i_1 < i_2 < \omega$, $(f_t(\overline{a}_{i_1}), f_{t+1}(\overline{a}_{i_1})) \cap (f_t(\overline{a}_{i_2}), f_{t+1}(\overline{a}_{i_2})) = \emptyset$.

Proof of Claim 2: Assume otherwise, and put $R = \frac{1}{\mu}$. By indiscernibility we have that for every $i < j < \omega$ the intersection $(f_t(\overline{a}_i), f_{t+1}(\overline{a}_i)) \cap (f_t(\overline{a}_j), f_{t+1}(\overline{a}_j))$ is non-empty. Thus, since the intersection of finitely many intervals is equal to the intersection of two intervals, there are $1 \leq i < j \leq R+1$ so that

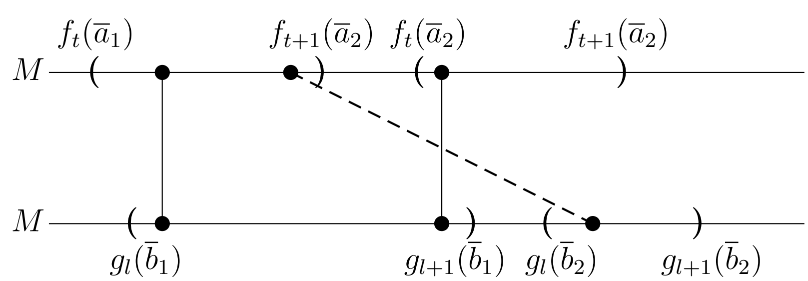
$$\bigcap_{i=1}^{R+1} (f_t(\overline{a}_i), f_{t+1}(\overline{a}_i)) = (f_t(\overline{a}_i), f_{t+1}(\overline{a}_i)) \cap (f_t(\overline{a}_j), f_{t+1}(\overline{a}_j)) := (u, v) \neq \emptyset.$$

By the uniform distribution of $\phi(M; a_i)$ along (u, v) we have that, if m is normalized counting measure with respect to (u, v), $m(\phi(M; \overline{a}_i)) = \mu > 0$, and by the 2-inconsistency of the sets $\phi(M; \overline{a}_i)$,

$$m\left(\bigcup_{i=1}^{R+1} \phi(M; \overline{a}_i)\right) = \sum_{i=1}^{R+1} m(\phi(M; \overline{a}_i)) = (R+1) \cdot \mu > 1,$$

which is a contradiction. ✓

Note that a similar conclusion can be obtained for the intervals $(g_l(\overline{b}_i), g_{l+1}(\overline{b}_i))$. So, without loss of generality, we may assume that $f_{t+1}(\overline{a}_1) < f_t(\overline{a}_2)$ and $g_{l+1}(\overline{b}_1) < g_l(\overline{b}_2)$, but this contradicts the fact that $(g_l(\overline{b}_1), g_{l+1}(\overline{b}_1)) \cap (f_t(\overline{a}_2), f_t(\overline{a}_2)) \neq \emptyset$ as we can see in the following diagram:



□

3.4 Recovering $\mathcal{O}$-minimality from $\mathcal{O}$-asymptotic classes

Let $\{M_i : i \in I\}$ be a collection of finite linearly ordered structures which contains structures of arbitrarily large cardinality, and let $M = \prod_{\mathcal{U}} M_i$ be its ultraproduct with respect to a non-principal ultrafilter $\mathcal{U}$ on I. It was shown in Section 3.3 that most of these ultraproducts are far from being $\mathcal{O}$-minimal.

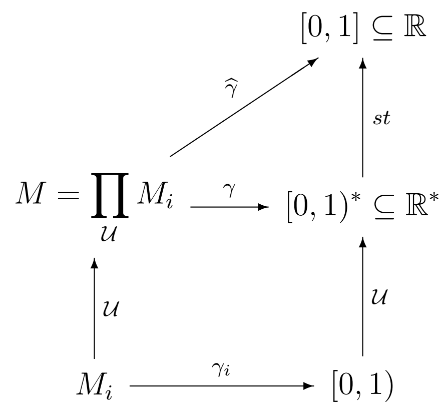
In this section we consider the problem to obtain dense $\mathcal{O}$-minimal structures from the ultraproducts of $\mathcal{O}$-asymptotic classes. Every infinite ultraproduct of structures in an $\mathcal{O}$-asymptotic class is discrete, with order-type $\omega \oplus (\kappa \times \mathbb{Z}) \oplus \omega^*$, where κ is a dense linear order without end points. However, there are two (non-definable) ways to obtain dense linear orders (with end points) as quotients of these ultraproducts. We will present both constructions, each with results that suggest there are some $\mathcal{O}$-minimal properties in this quotients.

The first of the two constructions has as image the real unit interval $[0, 1]$.

Construction 3.4.1. *If $|M_i| = k_i + 1 \in \mathbb{N}$, we may assume the universe of M_i to be $[0, k_i] = \{0, 1, 2, \ldots, k_i\}$. We can define the map γ_i to be:*

$$\begin{aligned} \gamma_i : \ M_i = [0, k_i] &\longrightarrow [0, 1) \subseteq \mathbb{R} \\ j &\longmapsto \frac{j}{|M_i|} = \frac{j}{k_i + 1}. \end{aligned}$$

These functions induce a map $\gamma = \prod_{\mathcal{U}} \gamma_i : M \longrightarrow [0, 1)^*$ (where the interval $[0, 1)^*$ is considered in the non-standard real field $\mathbb{R}^*$). By taking the standard map we *define* $\widehat{\gamma}(c) = st(\gamma(c))$ *for* $c \in M$*, obtaining the following diagram:*



Remark 3.4.2. *Note we can obtain the map* $\widehat{\gamma}$ *also with the equation*

$$\widehat{\gamma}(a) = st\left(\lim_{i \to \mathcal{U}} \frac{|\{x \in M_i : x \leq a\}|}{|M_i|}\right) = st(m([\min M, a]))$$

where m *is the normalized counting measure on* M*.*

The following is a straightforward result:

Proposition 3.4.3. *The map* $\widehat{\gamma}$ *is surjective.*

Proof. We start this proof by showing the following:
Claim: The set $\widehat{\gamma}(M)$ is dense in $[0,1)$.
Proof of the Claim: Let $\alpha \in [0,1)$ and $\epsilon > 0$ such that $(\alpha-\epsilon, \alpha+\epsilon) \subseteq [0,1)$. Since $\mathcal{U}$ is a non-principal ultrafilter on a countable index set (that we may identify with ω), there is an increasing sequence $\{k_n : n < \omega\}$ of natural numbers and a subsequence of structures $\{M_{i_n} : n < \omega\}$ such that $|M_{i_n}| = k_n + 1$ and $I = \{i_n : n < \omega\} \in \mathcal{U}$. Take $m_0 = \min\left\{k_n : \dfrac{1}{k_n+1} < \dfrac{\epsilon}{2}\right\}$. It is clear that $\{i_n : k_n \geq m_0\} \in \mathcal{U}$.
For every $n < \omega$, take $m_n = \min\left\{m \in \mathbb{N} : \dfrac{m}{k_n+1} > \alpha - \dfrac{\epsilon}{2}\right\}$. By the choice of k_n, m_n and ϵ we have $m_n \leq k_n$. Also, by the minimality of m_n, we have that $\dfrac{m_n}{k_n+1} < \alpha + \dfrac{\epsilon}{2}$, since otherwise we would have

$$\frac{m_n - 1}{k_n+1} = \frac{m_n}{k_n+1} - \frac{1}{k_n+1} \geq \left(\alpha + \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} = \alpha > \alpha - \frac{\epsilon}{2},$$

contradicting the minimality of m_n. Therefore, we obtain

$$\alpha - \frac{\epsilon}{2} < \gamma_{i_n}(S^{m_n}(0)) := \frac{m_n}{k_n+1} < \frac{m_n+1}{k_n+1} < \alpha + \epsilon.$$

Consider the element $u = (u_i)$ in M defined by:

$$u_i = \begin{cases} S^{m_n}(0), & \text{if } i = i_n \in I \\ 0, & \text{otherwise.} \end{cases}$$

By Łoś' Theorem we have $\alpha - \frac{\epsilon}{2} < \gamma(u) < \alpha + \frac{\epsilon}{2}$, and $\widehat{\gamma} \in (\alpha - \epsilon, \alpha + \epsilon)$. ✓

Now we prove $\widehat{\gamma}$ is surjective. Take $\alpha \in [0, 1)$. Since $\widehat{\gamma}(M)$ is dense in $[0, 1)$, there is an increasing sequence $\{\widehat{\gamma}(u_n) : n < \omega\}$ and a decreasing sequence $\{\widehat{\gamma}(v_n) : n < \omega\}$ such that

$$\lim_{n\to\infty} \widehat{\gamma}(u_n) = \alpha = \lim_{n\to\infty} \widehat{\gamma}(v_n).$$

Consider the type $p(x) = \{u_n < x : n < \omega\} \cup \{x_n < v_n : n < \omega\}$, which has to be realized in M because M is $\aleph_1$-saturated. Let c be a realization of $p(x)$. Since $\widehat{\gamma}$ is a order-preserving map, we obtain

$$\alpha = \lim_{n\to\infty} \widehat{\gamma}(u_n) \leq \widehat{\gamma}(c) \leq \lim_{n\to\infty} \widehat{\gamma}(v_n) = \alpha.$$

□

Proposition 3.4.4. *Let $\mathcal{C}$ be an $\mathcal{O}$-asymptotic class, and M an infinite ultraproduct of elements of $\mathcal{C}$. For every definable $X \subseteq M$, $\widehat{\gamma}(X)$ is a finite union of intervals and points.*

Proof. Assume $X = \varphi(M; \overline{a})$ for some $\mathcal{L}$-formula $\varphi(x, \overline{y})$ and some tuple $\overline{a} = (\overline{a}_i) \in M$. Let $C_\varphi > 0$ and $E_\varphi \subseteq [0, 1]^k$ be the constant and the set of tuples of real numbers associated to the formula φ. There is a fixed tuple $\overline{\mu} \in E_\varphi$ such that, for an $\mathcal{U}$-large set of indices i the following holds: There are elements $c^i_1 < \cdots < c^i_k \in M_i$ such that for every $j \leq k$ either $\mu_j = 0$ and $|\varphi(M_i, \overline{a}_i) \cap (c^i_j, c^i_{j+1})| \leq C$, or, $\mu_j \neq 0$ and $||\varphi(M, \overline{a}_i) \cap (c^i_j, c^i_{j+1})| - \mu_j |(c^i_j, c^i_{j+1})|| \leq C|(c^i_j, c^i_{j+1})|^{1/2}$.

Define $c_j = (c^i_j) \in M$ for $j \leq k$. It is enough to show that for every $j = 1, \ldots, k$ the set $\widehat{\gamma}(\varphi(M;\overline{a}) \cap (c_j, c_{j+1}))$ is a finite union of intervals and points. We have three cases:

- $\widehat{\gamma}(c_j) = \widehat{\gamma}(c_{j+1}) = \alpha$: In this case, $\widehat{\gamma}(\varphi(M;\overline{a}) \cap (c_j, c_{j+1})) = \{\alpha\}$.

- $\widehat{\gamma}(c_j) < \widehat{\gamma}(c_{j+1})$ *and* $\mu_j = 0$*:* In this case, $\varphi(M;\overline{a}) \cap (c_j, c_{j+1})$ consists of at most C points. So does $\widehat{\gamma}(\varphi(M;\overline{a}) \cap (c_j, c_{j+1}))$.

- $\widehat{\gamma}(c_j) < \widehat{\gamma}(c_{j+1})$ *and* $\mu_j > 0$*:* In this case, we will show that

  $$\widehat{\gamma}(\varphi(M;\overline{a}) \cap (c_j, c_{j+1})) = (\widehat{\gamma}(c_j), \widehat{\gamma}(c_{j+1})).$$

  Let $\alpha \in (\widehat{\gamma}(c_j), \widehat{\gamma}(c_{j+1}))$, and consider sequences $\langle q_n, r_n : n < \omega\rangle$ in $(\widehat{\gamma}(c_j), \widehat{\gamma}(c_{j+1}))$ converging to α, such that

  $$\widehat{\gamma}(c_j) < q_1 < \cdots < q_n < \cdots < \alpha < \cdots r_n < \cdots < r_1 < \widehat{\gamma}(c_{j+1}).$$

  By Proposition 3.4.3, there are elements $\langle (u_n, v_n) : n < \omega\rangle$ in M such that $\widehat{\gamma}(u_n) = q_n$, and $\widehat{\gamma}(v_n) = r_n$ for $n < \omega$. Since $\widehat{\gamma}$ cannot distinguish between two elements in M whose distance is finite, the intervals (u_n, v_n) are infinite in M. Consider the partial type $p(x) := \{u_n < x : n < \omega\} \cup \{x < v_n : n < \omega\} \cup \{\varphi(x, \overline{a})\}$. By the uniform distribution of $\varphi(x, \overline{a})$ (with measure μ_j) we can conclude that $X \cap (u_n, v_n)$ is non-empty, thus $p(x)$ is finitely satisfiable. Since M is $\aleph_1$-saturated, there is $w \models p(x)$ in M, and we have that

  $$\alpha = \lim_{n\to\infty} q_n = \lim_{n\to\infty} \widehat{\gamma}(u_n) \leq \widehat{\gamma}(u) \leq \lim_{n\to\infty} \widehat{\gamma}(v_n) = \lim_{n\to\infty} r_n = \alpha.$$

  This completes the proof.

□

Note that the images of definable subsets of M^1 under the map $\widehat{\gamma}$ are finite union of points and *closed* intervals.

It is possible to extend the map $\widehat{\gamma}$ to higher dimensional definable sets in a natural way.

Definition 3.4.5.

(i) For a definable set $X \subseteq M^n$, we define the set $\widehat{\gamma}(X)$ to be

$$\widehat{\gamma}(X) := \{(\widehat{\gamma}(a_1), \ldots, \widehat{\gamma}(a_n)) : (a_1, \ldots, a_n) \in X\}.$$

(ii) In particular, for a definable function $f : M \longrightarrow M$ we define $\widehat{\gamma}(f)$ to be the set

$$\widehat{\gamma}(f) := \{(\widehat{\gamma}(a), \widehat{\gamma}(f(a)) : a \in M\}.$$

The images of definable subsets of M^n induce a structure on $I = [0, 1]$ by taking the boolean algebra generated by the images of definable subsets of M^n. A question that arises is the following:

Question 3.4.6. *Under which conditions is the structures induced by M on I $\mathcal{O}$-minimal?*

Conjecture 3.4.7. *Assume $f : M \longrightarrow M$ is a definable function and $M = \prod_{\mathcal{U}} M_i$ is a ultraproduct of elements in an $\mathcal{O}$-asymptotic class. Then, there are finitely many continuous functions $f_1, \ldots, f_k$ with $f_i : [0, 1] \longrightarrow [0, 1]$ such that $\gamma(f) \subseteq \bigcup_{i=1}^{k} graph(f_i)$.*

Proposition 3.4.8. *Let $f : M \longrightarrow M$ be a definable function. Then for every $\alpha \in [0, 1]$, the set $X_\alpha := \{\beta \in\ [0, 1] : (\alpha, \beta) \in \widehat{\gamma}(f)\}$ is finite.*

Proof. Assume X_α is infinite for some $\alpha \in [0, 1]$. Then there is a sequence of different elements $\{\beta_i : i < \omega\rangle\} \subseteq X_\alpha$. Without loss of generality, we may assume that $\langle \beta_i : i < \omega \rangle$ is an increasing sequence.
Take a and $\langle a_i, b_i : i < \omega \rangle$ be elements of M such that:

- $f(a_i) = b_i$

- $\gamma(b_i) = \beta_i$

- $a \neq a_i$ and $\gamma(a) = \gamma(a_i) = \alpha$.

Since $\langle \beta_i : i < \omega \rangle$ is an increasing sequence in the real interval $[0, 1]$, we can take $\beta = \gamma(b)$ to be the limit of the sequence $\langle \beta_i : i < \omega \rangle$.
Consider the formula

$$\varphi(y; u, v, w, z) := u < y < v \wedge \exists x(w < x < z \wedge f(x) = y).$$

By $\mathcal{O}$-asymptoticity, we can take $\nu > 0$ minimal between all possible positive numbers appearing in the tuples of E_φ.
Let δ_1, δ_2 be elements in $[0, 1]$ such that $\delta_1 < \alpha < \delta_2$ and $\frac{\nu}{2}|(\beta_1, \beta)| > |(\delta_1, \delta_2)|$, and take d_1, d_2 in M such that $\gamma(d_1) = \delta_1$ and $\gamma(d_2) = \delta_2$. Consider the formula $\varphi(y; b_1, b, d_1, d_2)$. Since we have that $\varphi(M; b_1, b, d_1, d_2)$ is infinite (it contains all the elements b_i for $i \geq 2$), it holds for $\mathcal{U}$-almost all i that

$$\begin{aligned} |\varphi(M; b_1^i, b^i, d_1^i, d_2^i)| &\geq \nu \cdot |(b_1^i, b^i)| - C|(b_1^i, b^i)|^{1/2} \\ &\geq \nu \cdot |(b_1^i, b^i)| - \frac{\nu}{2}|(b_1^i, b^i)| \\ &\qquad\qquad (\text{because } |(b_1^i, b^i)| \geq (2C/\nu)^2 \text{ for } \mathcal{U}\text{-almost all } i) \\ &\geq \frac{\nu}{2}|(b_1^i, b^i)|. \end{aligned}$$

Also, since f is a function, $|\varphi(M; b_1^i, b^i, d_1^i, d_2^i)| \leq |(d_1^i, d_2^i)|$ for $\mathcal{U}$-almost all i.
Thus,

$$\frac{|(d_1^i, d_2^i)|}{|M_i|} \geq \frac{|\varphi(M_i; b_1^i, b^i, d_1^i, d_2)|}{|M_i|} \geq \frac{\nu}{2}\frac{|(b_1^i, b^i)|}{|M_i|},$$

and taking the limit across the ultraproduct we obtain $(\delta_2 - \delta_1) \geq \frac{\nu}{2}(\beta_2 - \beta_1)$, contradicting the choice of δ_1, δ_2. □

Now we present the second construction, which has as image an $\aleph_1$-saturated dense linear order with endpoints.

Construction 3.4.9. *Consider the ($\bigvee$-definable) equivalence relation E given by xEy if and only if there is some $n < \omega$ such that $x \in [S^{-n}(y), S^n(y)]$, that is, if* *and only if the distance between x and y is finite. We consider the projection map $\pi_E : M \longrightarrow M/E$. Note that M/E is a dense linear order with endpoints, that we can describe as $\{-\infty\} \cup \kappa \cup \{+\infty\}$.*

Proposition 3.4.10. *Let X be a definable subset of M^1. Then $\pi_E(X)$ is a finite union of intervals and points.*

Proof. X admits a decomposition into l intervals, witnessed by $c_0 < \cdots < c_l$. Let $\overline{\mu} \in [0,1]^l$ be the tuple of measures which is recurrent for X through the ultraproduct. If $\mu_i = 0$, then $X \cap (c_{i-1}, c_i)$ is finite (with at most C points) and $\pi_E(X \cap [c_{i-1}, c_i])$ has at most C points. If $\mu_i > 0$, then by uniformity there is n large enough such that if $(a,b) \subseteq (c_{i-1}, c_i)$ and $|(a,b)| \geq n$ then $X \cap (a,b) \neq \emptyset$. So, if $\pi_E(c_{i-1}) < \alpha = \pi_E(a) < \pi_E(c_i)$, take $b \in X \cap (a, S^n(a))$ to obtain $\pi_E(a) = \alpha = \pi_E(b) \in \pi_E(X)$. Therefore, $\pi_E(X \cap [c_{i-1}, c_i]) = [\pi_E(c_{i-1}), \pi_E(c_i)]$ and by taking the union we conclude that

$$\pi_E(X) = \bigcup_{i=1}^{k} \pi_E\left(X \cap [c_{i-1}, c_i]\right)$$

is a finite union of intervals and points. □

Again, the images of definable subsets of M^1 under the map π_E are finite union of points and *closed* intervals.

As before, we can generalize the map π_E to higher dimensional definable sets in a natural way.

Definition 3.4.11.

(i) For a definable set $X \subseteq M^n$, we define the set $\pi_E(X)$ to be

$$\pi_E(X) := \{(\pi_E(a_1), \ldots, \pi_E(a_n)) : (a_1, \ldots, a_n) \in X\}.$$

(ii) In particular, for a definable function $f : M \longrightarrow M$ we define $\pi_E(f)$ to be the set

$$\pi_E(f) := \{(\pi_E(a), \pi_E(f(a)) : a \in M\}$$

It is possible to induce a structure on $I = M/E$ by taking the boolean algebra generated by the images of definable subsets of M^n under π_E. So we can ask the following:

Question 3.4.12. *Under which conditions the structures induced by M on I is $\mathcal{O}$-minimal?*

Definition 3.4.13. *A definable function $f : M \longrightarrow M$ is said to be* pseudocontinuous *if there is a natural number $N = N_f$ such that for every $a \in M$,*

$$f(S(a)) \in [S^{-N}(f(a)), S^N(f(a))]$$

As before, we define $\pi_E(f)$ to be the set given by

$$(\overline{x}, \overline{y}) \in \pi_E(f) \;\Leftrightarrow \text{there are } x \text{ and } y \text{ such that } \pi_E(x) = \overline{x}, \pi_E(y) = \overline{y} \text{ and } f(x) = y$$

Proposition 3.4.14. *If f is a pseudocontinuous function, then $\pi_E(f)$ is a function on M/E.*

Proof. Assume $(\alpha, \beta_1), (\alpha, \beta_2) \in \pi_E(f)$, and take $a < a' \in M$ such that $\pi_E(a) = \alpha = \pi_E(a')$, $\beta_1 = \pi_E(f(a))$ and $\beta_2 = \pi_E(f(a'))$. Let β be an element in M/E with $\beta_1 < \beta < \beta_2$. Since $\pi_E(a) = \pi_E(a')$, we have that there is $m < \omega$ such that $a' = S^m(a)$. If $N = N_f$ is the natural number provided by the pseudocontinuity of f, we have that $f(a') \in [S^{-(N \cdot m)}(f(a)), S^{(N \cdot m)}(f(a))]$, concluding that $f(a)Ef(a')$, that is, $\beta_1 = \beta_2$. □

Proposition 3.4.15. *Let $f : M \longrightarrow M$ be a pseudocontinuous function. Then $\pi_E(f) : M/E \longrightarrow M/E$ is continuous (in the sense of the order topology).*

Proof. Put $g := \pi_E(f)$ and let α be an arbitrary element of M/E. Consider an interval (β_1, β_2) containing $g(\alpha)$. Let a, b_1, b_2 be elements in M such that $\pi_E(a) =$

$\alpha, \pi_E(b_1) = \beta_1$ and $\pi_E(b_2) = \beta_2$ and consider the formulas

$$\varphi_1^-(x) := x < a \wedge f(x) \geq b_2, \qquad \varphi_2^-(x) := x < a \wedge f(x) \leq b_1,$$
$$\varphi_1^+(x) := x > a \wedge f(x) \geq b_2, \qquad \varphi_2^+(x) := x > a \wedge f(x) \leq b_1.$$

If all these sets are empty, then g is continuous in α because π_E is an increasing function.

Let $a_1 = \max \varphi_1^-(M) \cup \varphi_2^-(M)$ and $a_2 = \min \varphi_1^+(M) \cup \varphi_2^+(M)$. By maximality of a_1 and minimality of a_2 we have that $a_1 < x < a_2$ implies $f(x) \in (b_1, b_2)$. If we prove that $\pi_E(a_1) < \alpha < \pi_E(a_2)$ we are done. Assume for example that $\pi_E(a_1) = \alpha$, then there is a natural number k such that $a = S^k(a_1)$. Since f is pseudocontinuous, there is a natural number N such that

$$f(a) \in [S^{-(N\cdot k)}(f(a_1)), S^{(N\cdot k)}(f(a_1))].$$

If β_1', β_2' are elements in M/E such that $\beta_1 < \beta_1' < g(\alpha) < \beta_2' < \beta_2$, we would have $g(\alpha) = \pi_E(f(a)) = \pi_E(f(a_1))$, and thus $f(a_1) \in [b_1, b_2]$, a contradiction. □

Conjecture 3.4.16. *Assume $f : M \longrightarrow M$ is a definable function and $M = \prod_{\mathcal{U}} M_i$ is a ultraproduct of elements in an $\mathcal{O}$-asymptotic class. Then, there are pseudocontinuous functions $f_1, \ldots, f_k$ with $f_i : M \longrightarrow M$ such that $graph(f) \subseteq \bigcup_{i=1}^{k} graph(f_i)$.*

CHAPTER 4

Further results

4.1 More examples

Let α be an irrational real number, with $0 < \alpha < 1$, and $f : \mathbb{N} \to \mathbb{N}$ be the function given by $f(x) = [\![\alpha \cdot x]\!]$, where $[\![*]\!]$ denotes the greatest integer less than or equal to $*$.

Consider the class $\mathcal{C}_\alpha$ given by structures $M_n = \langle\ [0, n], <_n, f_n\rangle$ where $<_n$ is the usual order on $[0, n]$ and $f_n(x)$ is the restriction of f to $[0, n]$.

The purpose of this section is to provide results towards a proof that the class $\mathcal{C}_\alpha$ is an $\mathcal{O}$-asymptotic class. First, we start with the following proposition.

Lemma 4.1.1. *Let M be a structure in the class $\mathcal{C}_\alpha$.*

1. *For every $s < \omega$, the image of the function f^s is $[0, f^s(\max M)]$.*
2. *Take $N = [\![\frac{1}{\alpha}]\!]$. Then for every $a \in M$, either $|f^{-1}(a)| = 0$, $|f^{-1}(a)| = N$ or $|f^{-1}(a)| = N + 1$.*

Proof. 1. By induction on s:

- Case $s = 1$: Let $k = \min(M \backslash f(M))$. We wil show that $k = f(\max M)+1$. Since $f(0) = 0$ we know that $k > 0$. By minimality, we have that $k - 1 = [\![\alpha m]\!]$ for some $m \in M$. Assume such m to be maximal, then

$$\begin{aligned} k - 1 = [\![\alpha m]\!] < k < k + 1 &\leq [\![\alpha(m+1)]\!] \\ \alpha m < k < k + 1 &\leq \alpha m + \alpha \\ 0 < k - \alpha m < (k - \alpha m) + 1 &\leq \alpha, \end{aligned}$$

a contradiction because $\alpha < 1$. Thus, the image of f is an interval, whose maximum clearly is $f(\max M)$.

- Case $s + 1$: Assume that the function f^s has image $[0, f^s(\max M)]$, and let k be minimal in the set $M - f^{s+1}(M)$. If $k < f^{s+1}(\max M)$, then we have for some maximal $m < \max M$ that

$$\begin{aligned} k - 1 = f^{s+1}(m) < k < k + 1 \leq f^{s+1}(m+1) \\ k - 1 = [\![\alpha \cdot f^s(m)]\!] < k < k + 1 \leq [\![\alpha \cdot f^s(m+1)]\!]. \end{aligned}$$

By the case $s = 1$, there is some p with $f^s(m) \leq p \leq f^s(m+1)$ such that $k = [\![\alpha \cdot p]\!]$. By the induction hypothesis, $p = f^s(m)$ or $f^s(m+1)$ (also, because f^s is a non-decreasing function). This is a contradiction.

2. By (1), if a is not in $[0, f(\max M)]$, then $f^{-1}(a) = \emptyset$. Assume now that $a \leq f(\max M)$ and suppose that $|f^{-1}(a)| \geq N + 2$. We have that for some k,

$$\begin{aligned} a = [\![\alpha \cdot k]\!] = [\![\alpha(k + N + 1)]\!] &< a + 1 \\ a \leq \alpha k < \alpha k + \alpha(N+1) &< a + 1 \end{aligned}$$

a contradiction, because $\alpha(N + 1) > 1$.

Suppose now that $|f^{-1}(a)| \leq N-1$. Then for some k we have

$$\alpha k < a \leq \alpha(k+1) < \alpha(k+N-1) < a+1 < \alpha(k+N)$$
$$0 < a - \alpha k < \alpha < \alpha(N-1) < (a-\alpha k)+1 < \alpha N$$

a contradiction, because $\alpha N < 1$.

So, if $a \in f(M_n)$, either $|f^{-1}(a)| = N$ of $|f^{-1}(a)| = N+1$.

□

Proposition 4.1.2. *The class $\mathcal{C}_\alpha = M_n : n < \omega$ admits uniform quantifier elimination in the language $\mathcal{L}' = \{<, f, S, S^{-1}, \min M, \max M\}$.*

Proof. Note that the atomic formulas in the original language have the form $f^m x = f^n y$ or $f^m x < f^n y$ for some $m, n < \omega$.

Consider a primitive existential formula of the form

$$\phi := \exists x \left(\bigwedge_i f^{m_i} x = f^{n_i} a_i \wedge \bigwedge_j f^{h_j} x \neq f^{l_j} b_j \wedge \bigwedge_k f^{p_k} x < f^{q_k} c_k \wedge \bigwedge_t f^{r_t} x > f^{s_t} d_t \right)$$

We now give some simplifications of the original formula. First, we can replace the terms which do not contain x by new parameters (e.g replace $f^{n_i} a_i$ simply by a_j). We also can assume that there are no conjuncts of the form $f^h x \neq b$ because we can replace them by $(f^h x < b \vee f^h x > b)$ and since we are dealing with existential formulas, after distributing, we obtain a conjunction of formulas only of the forms $f^m x = a$, $f^p x < c$ an $f^r x > d$.
Since the order is discrete, we have that $f^p x < c$ if and only if $\min M \leq f^p x \leq S^{-1}(c)$ and $f^r x > d$ if and only if $S(d) \leq f^r x \leq \max M$.

With these changes, the existential formula has the form

$$\phi = \exists x \left(\bigwedge_{i \leq k} f^{m_i} x = a_i \wedge \bigwedge_{j \leq \ell} c_j \leq f^{n_j} x \leq d_j \right).$$

Assume, without loss of generality that $m_1 = \min\{m_1, \ldots, m_k\}$ and $n_1 = \min\{n_1, \ldots, n_\ell\}$.

By Lemma 4.1.1 we know that for every s, the range of the function f^s is $[0, f^s(\max M)]$. If $m_1 \leq n_1$, then for every $i \leq k$, $j \leq \ell$ we have that $f^{m_i}x = a_i$ if and only if $f^{m_i - m_1}(a_1) = a_i$ and $c_j \leq f^{n_j}x \leq d_j$ if and only if $c_j \leq f^{n_j - m_1}(a_1) \leq d_j$. In this case the formula ϕ is equivalent to

$$\left(0 \leq a_1 \leq f^{m_1}(\max M) \wedge \bigwedge_{i \leq k} f^{m_i - m_1}(a_1) = a_i \wedge \bigwedge_{j \leq \ell} c_j \leq f^{n_j - m_1}(a_1) \leq d_j\right).$$

Assume now that $n_1 < m_1$, the formula ϕ is equivalent to

$$\left(\bigwedge_{j \leq \ell}(f^{n_j - n_1}(c_1) \leq d_j) \wedge \bigwedge_{i \leq k}(f^{m_1 - m_i}(a_1) = a_i) \wedge \bigwedge_{\{j : n_j \geq m\}} (c_j \leq f^{n_j - m_1}(a_1) \leq d_j)\right).$$

This completes the proof of uniform quantifier elimination in the class $\mathcal{C}_\alpha$. □

Now, suppose that $\varphi(x; \overline{a})$ is a formula in the language $\mathcal{L}' = \{\min, \max, f, S, S^{-1}, <\}$. By quantifier elimination, we may assume that $\varphi(x; \overline{a})$ has the form

$$\bigvee_i \bigwedge_j \psi_{ij}(x; \overline{a})$$

where the formulas $\psi_{ij}(x; \overline{a})$ are either atomic formulas or negation of atomic formulas.

By adding conjunctions, we may assume the disjunctives to have empty intersections. So, in order to show that $\mathcal{C}_\alpha$ is an $\mathcal{O}$-asymptotic class it is enough to check on the conjunctions of atomic formulas and their negations.

Thus, the formulas we are interested in have the form

$$\phi(x;\overline{a}) := \bigwedge_{i<m_1} t_1^i(x) = t_2^i(x) \ \wedge \ \bigwedge_{j<m_2} t_1^j(x) \neq t_2^j(x) \ \wedge \ \bigwedge_{k<m_3} t^k(x) = u^k(\overline{a})$$
$$\wedge \ \bigwedge_{h<m_4} t^h(x) \neq u^h(\overline{a}) \ \wedge \ \bigwedge_{l<m_5} t_1^l(x) < t_2^l(x) \ \wedge \ \bigwedge_{n<m_6} t^n(x) < u^n(\overline{a})$$

where all the t's and u's are terms in the language $\mathcal{L}'$.

The following propositions will help us to reduce the cases:

Proposition 4.1.3. *If $m_3 \neq 0$, then the formula $\phi(x;\overline{a})$ has a uniform decomposition into intervals, with measure $\mu = 0$.*

Proof. Suppose $m_1 \neq 0$, then $\phi(x;\overline{a})$ implies $t(x) = u(\overline{a})$ for some terms $t = t_k, u = u_k$ from $\mathcal{L}'$. Since there are only unary functions in $\mathcal{L}'$, we may assume that

$$t(x) := f^{k_1} S^{m_1} \cdots f^{k_r} S^{m_r}(x)$$

with $k_i \in \mathbb{N}, m_i \in \mathbb{Z}$

Let us call $b = u(\overline{a})$. Then $t(x) = b$ if and only if $x \in (S^{-m_r}(f^{-k_r} \cdots (S^{-m_1}(f^{-k_1}(b)))))$, but by Lemma 4.1.1(3), there is a bound $C = C_{k_1,m_1,\ldots,k_r,m_r}$ (depending only on $k_1, m_1, \ldots, k_r, m_r$) such that for any $b \in M$,

$$\left|S^{-m_r}(f^{-k_r} \cdots (S^{-m_1}(f^{-k_1}(b))))\right| \leq C$$

Thus, the formula $\phi(x;\overline{a})$ would have uniform distribution in every $M \in \mathcal{C}_\alpha$, with measure $\mu = 0$ and $C = C_{k_1,m_1,\ldots,k_r,m_r}$. □

So, assume that $m_3 = 0$. Also, note that by the linear ordering the formulas with terms of the form $t_1 \neq t_2, t_1 \neq u$ might be changed by $t_1 < t_2 \vee t_2 < t_1, t_1 < u \vee u < t_1$ respectively. Thus, by augmenting the number of conjunctives (and using the symbols max and min if necessary) we can assume without loss of generality that $\phi(x;\overline{a})$ has the form:

$$\phi(x;\overline{a}) := \bigwedge_{i<m} t_1^i(x) = t_2^i(x) \ \wedge \ \bigwedge_{j<n} t_1^j(x) < t_2^j(x) \ \wedge \ \bigwedge_{k<r} c_k < t^k(x) < d_k$$

with c_k, d_k being terms in the language $\mathcal{L}'(\overline{a})$.

Now we start calculating the measure of the formulas in these conjunctions.

Proposition 4.1.4. *Each of the conjunctions $a < t(x) < b$ has a uniform distribution with measure $\overline{\mu} = (0, 1, 0)$.*

Proof. We can assume that the term t has the form $t(x) := f^{m_1}S^{n_1} \cdots f^{m_k}S^{n_k}(x)$ with $m_i \in \mathbb{N}, n_j \in \mathbb{Z}$.

Therefore, $a < t(x) < b$ if and only if $x \in (S^{-n_k}f^{-m_k} \cdots S^{-n_1}f^{-m_1})(a, b)$, but since f^k is a non-decreasing function for every k, and S is an increasing bijection from $M - \{\max M\}$ to $M - \{\min M\}$, this set has the form $[c_1, c_2]$ for some $c_1 \leq c_2$. Thus, we have a decomposition for the formula $a < t(x) < b$.

□

Lemma 4.1.5. *If $m > 0$, the formula $f^m S^n x \geq x$ has a uniform distribution with measure zero and $C = \dfrac{\alpha^m \cdot n}{1 - \alpha^m}$.*

Proof. Note that

$$\begin{aligned} f^m(S^n x) \geq x &\Rightarrow \alpha^m(x + n) \geq f^m(S^n x) \geq x \\ &\Rightarrow x(1 - \alpha^m) \geq \alpha^m \cdot n \\ &\Rightarrow x \leq \frac{\alpha^m \cdot n}{1 - \alpha^m}. \end{aligned}$$

So, the conclusion follows. □

Lemma 4.1.6. *For every $L \in \mathbb{Z}$, and every $\mathcal{L}'$-term $t(x)$ which mention at least once the function f, the formula $t(x) \geq S^L(x)$ has uniform distribution with measure zero and a constant C depending on the term $t(x)$*

Proof. Let us write $t(x) := f^{m_1}S^{n_1}\cdots f^{m_k}S^{n_k}(x)$. Then

$$\begin{aligned} t(x) &:= f^{m_1}S^{n_1}\cdots f^{m_k}S^{n_k}(x) \geq S^L x \\ &\Rightarrow \alpha^{m_1}\left(\alpha^{m_2}\left(\cdots\left(\alpha^{m_k}(x+n_k)+\cdots+n_2\right)+n_1\right) \geq S^L(x)\right. \\ &\Rightarrow \left(\prod_{i\leq k}\alpha^{m_i}\right)x+\alpha^{m_1}n_1+\alpha^{m_1}\alpha^{m_2}n_2+\cdots\alpha^{m_1}\alpha^{m_2}\cdots\alpha^{m_k}n_k \geq x+L \\ &\Rightarrow x \geq \frac{\left(\sum_{j\leq k}\left(\prod_{i\leq j}\alpha^{m_i}\right)n_j\right)-L}{1-\prod_{i\leq k}\alpha^{m_i}}, \end{aligned}$$

and the formula $t(x) \geq S^L(x)$ has uniform distribution with measure zero and constant

$$C = \frac{\left(\sum_{j\leq k}\left(\prod_{i\leq j}\alpha^{m_i}\right)n_j\right)-L}{1-\prod_{i\leq k}\alpha^{m_i}}$$

□

Remark 4.1.7. *Note that the Lemma above implies in particular that the formula $t(x) = S^L(x)$ has also uniform distribution, with measure zero.*

Now we start calculating the measure of some formulas that can be defined. First, let start with the formula $\varphi(x) := f(x) < f(Sx)$.

Claim 1: We have $\lim_{n\to\infty}\frac{|\varphi(M_n)|}{|M_n|} = \alpha$.

Proof of Claim 1: Note that every jump in the function corresponds to a realization of $\varphi(x)$, and can be uniquely associated to a value in the image of the function f. This correspondence is almost surjective, except for the value 0 in the image, so

$$\lim_{n\to\infty}\frac{|\varphi(M_n)|}{|M_n|} = \lim_{n\to\infty}\frac{[\![\alpha\cdot n]\!]}{n+1} = \alpha.$$

Moreover,

$$||\varphi(M_n)|-\alpha|M_n|| \leq ||([\![\alpha\cdot n]\!])|-\alpha\cdot(n+1)| \leq ||([\![\alpha\cdot n]\!])-\alpha\cdot n|+|\alpha| \leq 2. \qquad \checkmark$$

The uniformity in the distribution follows by taking $C = N[\![\frac{1}{\alpha}]\!]$. ✓

Consider now the formula $\phi(x) := f(x) = f(S^M x)$ for $M \leq [\![\frac{1}{\alpha}]\!]$. To find the limit asymptotic counting measure of $\phi(M_n)$ as n tends to infinity, we will use the following more or less known result in number theory, that can be found for example in [34, Chapter 3]:

Fact 4.1.8 (Weyl's Equidistribution Theorem)**.** *Let α be an irrational real number, and consider the sequence $z_n = \alpha \cdot n - [\![\alpha \cdot n]\!]$ (the fractional part of $\alpha \cdot n$). Then, for every $0 \leq a < b \leq 1$,*

$$\lim_{n\to\infty} \frac{|\{k \leq n : a \leq z_k \leq b\}|}{n} = b - a.$$

Claim 2: For every M, $f(x) = f(x + M)$ if and only if the fractional part of $\alpha \cdot x$ is less than $1 - \alpha M$.

Proof of Claim 2: Assume that $\alpha x - [\![\alpha x]\!] < 1 - \alpha M$. Then we have

$$[\![\alpha x]\!] < [\![\alpha x]\!] + \alpha M < \alpha(x + M) < [\![\alpha x]\!] + 1.$$

This implies that $f(x) = [\![\alpha x]\!] = [\![\alpha(x + M)]\!] = f(x + M)$. For the other direction, assume that $\alpha x - [\![\alpha x]\!] \geq 1 - \alpha M$. Then $\alpha x + \alpha M \geq [\![\alpha x]\!] + 1$ and we obtain $f(x + M) = [\![\alpha(x + M)]\!] \geq f(x) + 1 > f(x)$. ✓

Using Claim 2 and Weyl's Equidistribution Theorem, we have that

$$\lim_{n\to\infty} \frac{\phi(M_n)}{|M_n|} = \lim_{n\to\infty} \frac{\{k \leq n : 0 < z_k < 1 - \alpha M\}}{n} = 1 - \alpha M.$$

Bibliography

[1] R. Ash. *Probability and Measure Theory.* Academic Press. Second Edition. 1999

[2] O. Belegradek, Y. Peterzil, F. Wagner. *Quasi-O-minimal structures.* The Journal of Symbolic Logic, Vol. 65, No. 3 (Sep., 2000), pp. 1115-1132.

[3] V. Bergelson, A. Blass, M. Di Nasso, R. Jin. *Ultrafilters across Mathematics.* Contemporary Mathematics, 530. American Mathematical Society. 2010.

[4] B. Bollobás. *Random Graphs.* Academic Press. New York, 1985.

[5] E. Bouscaren. *Model Theory and Algebraic Geometry: An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture.* Lecture notes in mathematics, 1696. Springer. 1999.

[6] C.C. Chang, H.J. Keisler. *Model Theory.* Studies in Logic and the Foundations of Mathematics. Volume 73. Springer. 1992.

[7] Z. Chatzidakis, L. van den Dries, A.J. Macintyre. *Definable sets over finite fields.* J. Reine Angew. Math. 427 (1992), 107–135.

[8] G. Cherlin. E. Hrushovski. *Finite strucutres with few types.* Princeton University Press. Princeton and Oxford. 2003

[9] G. Cherlin, L. Harrington, and A. H. Lachlan. $\aleph_0$ *-categorical,* $\aleph_0$*-stable structures.* Annals of Pure and Applied Logic, 28(2):103–135. 1985

[10] A. Chernikov. *Theories without the tree property of the second kind.* Annals of Pure and Applied Logic. Volume 165, Issue 2, February 2014, Pages 695–723

[11] A. Chernikov, I. Kaplan. *Forking and dividing in* NTP_2 *theories.* J. Symbolic Logic. Volume 77, Issue 1 (2012), 1-20.

[12] A. Chernikov, I. Kaplan, P. Simon. *Groups and fields with* NTP_2. Submitted. arXiv:1212.6213 [math.LO].

[13] A. Chernikov, M. Hils. *Valued difference fields and* NTP_2. Israel Journal of Mathematics, 1-29. Jun 2014).

[14] R. Cluckers. *Presburger sets and p-minimal fields*. Journal of Symbolic Logic Volume 68, Issue 1 (2003), 153-162.

[15] L. van den Dries. *Tame topology and o-minimal structures*. London Mathematical Society. Lecture Note Series 248. Cambridge University Press. 1998

[16] C. Ealy. *Thorn Forking in Simple Theories and a Manin-Mumford Theorem for T-Modules*. Ph.D. Thesis. University of California - Berkeley. 2004

[17] C. Ealy, A. Onshuus. *Characterizing Rosy theories*. The Journal of Symbolic Logic. Volume 72, Number 3. Sept. 2007.

[18] R. Elwes. *Asymptotic classes of finite structures.* Journal of Symbolic Logic. Volume 72, Issue 2 (2007), 418-438.

[19] T. Frayne, A. Morel, D. Scott. *Reduced direct products*. Fundamenta Mathematica, 51 (1962), 195–228

[20] D. García. *A note on pseudofinite dimension and forking*. Available in `http://arxiv.org/pdf/1402.5212v1.pdf`.

[21] D. García, D. Macpherson, C. Steinhorn. *Pseudofinite structures and simplicity.* Preprint available in `http://arxiv.org/pdf/1409.8635.pdf`.

[22] D. García, A. Onshuus, A. Usvyatsov. *Generic stability, forking and thorn-forking*. Transactions of the American Mathematical Society.Volume 365, Number 1, January 2013, Pages 1–22.

[23] I. Goldbring. H. Towsner. *An approximate logic for measures*. Israel Journal of Mathematics (to appear). Preprint. `arXiv:1106.2854v1`. June 2011.

[24] E. Grädel, P.G. Kolaitis, L. Libkin, M. Marx, J. Spencer, M.Y.Vardi, Y. Venema, S. Weinstein. *Finite Model Theory and its Applications.* Texts in Theoretical Computer Science. Springer (2007).

[25] D. Haskell, A. Pillay, C. Steinhorn. *Model Theory, Algebra and Geometry.* Mathematical Sciences Research Institute Publications, 39. 2000.

[26] W. Hodges. *Model Theory.* Encyclopedia of mathematics and its applications. Cambridge University Press. 1994

[27] E. Hrushovski. *Stable group theory and approximate subgroups*. Journal of the American Mathematical Society. Volume 25. Number 1. January 2012. Pages 189-243.

[28] E. Hrushovski. *On Pseudo-Finite Dimensions.* Notre Dame Journal of Formal Logic. Volume 54 (2013), no. 3-4, 463–495.

[29] E. Hrushovski, F. Wagner. *Counting and dimensions.* Model Theory with applications to Algebra and Analysis. 2008

[30] H.J. Keisler. *Ultraproducts of finite sets.* The Journal of Symbolic Logic, 32 (1967), 47–57.

[31] H.J. Keisler. *The Ultraproduct Construction*

[32] B. Kim. *Simple First Order Theories.* Ph.D Thesis. University of Notre Dame. 1996.

[33] J. Knight, A. Pillay, C. Steinhorn. *Definable Sets in Ordered Structures, II.* Transactions of the American Mathematical Society, Vol. 295, No. 2 (Jun., 1986), pp. 593-605

[34] T.W. Körner. *Fourier Analysis.* Cambridge University Press. 1988

[35] S. Lang, A. Weyl. *Number of points of varieties of finite fields.* American Journal of Mathematics. Vol. 76, No. 4 (Oct., 1954), pp. 819-827

[36] L. Libkin. *Elements of Finite Model Theory.* Springer. 2012.

[37] M. Malliaris, S. Shelah. *Regularity lemmas for stable graphs.* Trans. Amer. Math. Soc. 366 (2014), 1551-1585.

[38] D. Macpherson, C. Steinhorn. *One-dimensional asymptotic classes of finite structures.* Transactions of the American Mathematical Society. Volume 360, pages 411-448. 2007.

[39] D. Macpherson, C. Steinhorn. *Definability in classes of finite structures.* In Finite and Algorithmic Model Theory. pp 140-176. London Mathematicatical Society, No. 379. Cambridge University Press. 2011.

[40] A. Onshuus. *Properties and consequences of thorn-independence.* J. Symbolic Logic **71** (2006), 1-21.

[41] J. Pila. *O-minimality and the André-Oort conjecture for* $\mathbb{C}^n$. Annals of Mathematics, 173. (2001). 1779-1840.

[42] J. Pila and A. Wilkie. *The rational points of a definable set.* Duke Math. J., 133 (2006), 591–616.

[43] A. Pillay. *Geometric stability theory.* Oxford Logic Guides, 32. Oxford University Press. 1996.

[44] A. Pillay, C. Steinhorn. *Definable sets in ordered structures.* Bulletin of the American Mathematical Society. Volume 11, Number 1, July 1984.

[45] B. Poizat. *A course in model theory: an Introduction to Contemporary Mathematical Logic.* Universitext. Springer, New York. 2000.

[46] J. Ryten. *Model Theory of Finit Difference Fields and Simple Groups.* Ph.D. Thesis. University of Leeds. 2007.

[47] S. Shelah. *Classification theory and the number of non-isomorphic models*. Studies in Logic and the Foundations of Mathematics **92**. North Holland, Amsterdam. 1990

[48] S. Shelah. *Simple unstable theories*. Annals of Mathematical Logic Volume 19, Issue 3, December 1980, Pages 177–203.

[49] S. Shelah. *On the cardinality of ultraproduct of finite sets*. The Journal of Symbolic Logic, 35 (1) (Mar., 1970), 83–84.

[50] T. Tao. *Expanding polynomials over finite fields of large characteristic, and a regularity lemma for definable sets*. Available at `http://arxiv.org/pdf/1211.2894v4.pdf`.

[51] T. Tao. V.H. Vu *Additive Combinatorics*. Cambridge studies in advances mathematics [105]. Cambridge University Press. 2006.

[52] H. Towsner. *A model theoretic proof of Szemerédi's Theorem*. Available in arxiv: `http://arxiv.org/pdf/1002.4456v3.pdf`

[53] F. Wagner. *Simple theories*. Mathematic and its Applications. Kluwer Academic Publishers. 2000