Canonical Group Quantization and Rotation Generators for Indistinguishable Particles

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Abstract

Recently, a condition for the validity of the Spin-Statistics connection in non relativistic quantum mechanics has been obtained [Kuc04]. This condition involves certain relations between the angular momentum operators of a single particle and of a system of two indistinguishable particles. Assuming its validity, this characterization of the Spin-Statistics connection in terms of angular momentum operators could provide a clue to a physically motivated condition leading to a proof of the Spin-Statistics theorem in non relativistic quantum mechanics. In this work (as a first step in that direction) we study Isham's canonical group approach to the quantization of configuration spaces that are homogeneous spaces [Ish84]. The basic idea is to find unitary representations of certain Lie groups acting canonically on the phase spaces of these configuration spaces. As a first example, we study the quantization of a two-sphere. We consider the representations of the canonical group for this space and obtain explicit formulas for the representations of its infinitesimal generators. The so-obtained angular momentum operators are a quantum version of Poincaré's angular momentum vector and can be interpreted as the angular momentum operators of an electron coupled to the (external) field of a magnetic monopole. In a second example, we construct the angular momentum operators associated to the two dimensional real projective space and discuss their meaning in the context of quantum indistinguishability of two spin-zero particles. In particular we show that these angular momentum operators (arising naturally from the canonical group quantization of the projective space) coincide with those considered in [Re06] and are thus related to the transported spin basis constructed by Berry and Robbins [BR97].

Antes de nós nos mesmos arvoredos
Passava o vento, quando havia vento,
E as folhas não mexiam
De outro modo do que hoje.
Ricardo Reis

Pater ipse colendi
haud facilem esse uiam uoluit, primusque per artem
mouit agros curis acuens mortalia corda
nec torpere graui passus sua regna ueterno.
Vergilius, Georgica I 120

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Chapter 1

Introduction and Statement of results

The understanding of the relation between the spin and the statistics of identical particles in the context of non-relativistic quantum mechanics seems to be unsatisfactory. There are at least two reasons for this. First, even though the spin-statistics theorem has found rigorous proofs in the context of relativistic quantum field theory, this has not been the case in non relativistic quantum mechanics. The original proof of Pauli [Pau40] is based on ideas that go beyond the non-relativistic quantum mechanics domain and is valid for free (quantum) fields. The spin-statistics relation arises in quantum field theory as a consequence of the Poincaré invariance of the theory, but it is perfectly possible to construct, say, a Galilei-invariant quantum field theory of spinless fermions. Although it seems natural to study the spin-statistics relation in relativistic quantum theory, there are many physical phenomena that depend crucially on it and still can be described using only non relativistic quantum theory. This is the case, for example, with the Fractional Quantum Hall effect. Hence, as Feynman once said [FLS65], the spin-statistics theorem *"appears to be one of the few places in physics where there is a rule which can be stated in very simple words, but for which no one has found a simple and easy explanation... This probably means that we do not have a complete understanding of the fundamental principles involved"*.

Additionally, there is an intriguing relation between quantum indistinguishability, topology and statistics. Since all proofs of the spin-statistics theorem are of a more analytical nature, it would be very interesting to find a connection between these two approaches: one which is more topological, the other more analytical [Tsc89].

The original proof of Pauli of the spin-statistics theorem dates back to the decade of 1940. He proved, on the one hand, that quantization according to the exclusion principle for particles with integer spin is not possible. On the other hand, he demonstrated that it is possible to quantize the theory for half-integral spins according to Bose-statistics, but the energy of the system would be negative. The conclusion of Pauli is that the connection between spin and statistics is one of the most important consequences of the special theory of relativity. After the arrival of what is called "axiomatic quantum field theory", new mathematically rigorous proofs were developed; for details, see [SW00, DS98] and references therein. There are also recent approaches to the spin-statistics relation within the framework of algebraic quantum field theory [Ku95, Haa01].

In the non-relativistic context, it has not been possible to prove a spin-statistics theorem, although many attempts have been made, using "configuration space techniques". The original ideas on which most of these works are based, were put forward in the works of Laidlaw and de Witt [LD71] and of Leinaas and Myrheim [LM77]. In these works the fact that, as a consequence of Gibb's paradox, even at a classical level the configuration space of a system of identical particles must take their indistinguishability into account was exploited to show that new features of a topological nature arise in any quantization scheme whose starting point is the classical configuration space. Their main result was that, in 3-dimensional space, identical particles of spin zero must be either Bosons or Fermions. In 2-dimensional space, their works show that identical particles follow anyonic statistics. Very briefly, the idea is the following: The configuration space for a system of identical particles is not the Cartesian product $\Re^{3N}$, but the space obtained by identifying points in $\Re^{3N}$ representing the same physical configuration, where the same physical configuration means the impossibility to distinguish identical particles. Therefore, the configuration space can written as

$$Q_N = \tilde{Q}_N / S_N, \tag{1.1}$$

where

$$\tilde{Q}_N = \left\{ (r_1, ..., r_N) \in \Re^{3N} | r_i \neq r_j \,\text{for all pairs}\, (i,j) \right\}, \tag{1.2}$$

and where S_N is the permutation group. The non-coincidence condition $r_i \neq r_j$ is included in the definition in order to make $\tilde{Q}_N$ a manifold and to avoid the coincidence of two different particles. A discussion of the physical status of this condition can be found in [BS92]. Finally, the quotient space is the set of equivalence classes of points in $\tilde{Q}_N$ under permutations.

The fact that this space is multiply-connected implies that the standard definition of the probability amplitude in terms of a sum over paths is not consistent (paths belonging to different homotopy classes produce different relative phases in the integrand of the path integral upon a gauge transformation of the classical Lagrangian, thus making the amplitude an ill-defined quantity). The solution to this problem is provided by a re-definition of the amplitude in terms of partial amplitudes. To each homotopy class α, a partial amplitude K^α is assigned, where the sum is performed only over paths belonging to the given class α. The total amplitude is then defined to be a weighted sum of the form

$$K = \sum_{\alpha \in \pi_1(\mathcal{Q})} \chi(\alpha) K^\alpha. \tag{1.3}$$

Laidlaw and DeWitt showed that the factors $\chi(\alpha)$ must be unitary one-dimensional representations (characters) of the fundamental group of the configuration space. Thus there are, in 3 dimensions, only two possibilities:

$$\chi^1(\alpha) = 1 \text{ for all } \alpha, \tag{1.4}$$
$$\chi^2(\alpha) = \pm 1 \text{ depending on wether } \alpha \text{ is an even or odd permutation,} \tag{1.5}$$

corresponding to Bose and Fermi statistics, respectively.

The proposal of Leinaas and Myrheim was based, in mathematical terms, in considering the wave function to be a cross-section of some vector bundle on Q_N. The Fermi-Bose alternative (for spin zero particles) arises in this case as a consequence of the topological properties of the vector bundles involved. In the case of two identical particles, it can be proven that

$$Q_2 \approx \Re^3 \times \Re^+ \times RP^2,$$

where RP^2 is the two dimensional projective space. From a physical point of view, this topological equivalence can be understood if we describe the dynamics of the system of two indistinguishable particles using center of mass and relative coordinates.

As mentioned above, in the last years a significative effort has been made, in order to gain a better understanding of the relation between spin and statistics in a non-relativistic context (see, for example, [BR97, Kuc04, KM05]). Nevertheless, many of the attempts to derive the spin-statistics relation in non relativistic quantum mechanics are based on assumptions that are not very clear from a physical point of view. Others (like [Pe03]) lacking a sound mathematical basis, lead easily to confusions that do not contribute to a real advance in our understanding of the subject.

In the work of Berry and Robbins [BR97], the quantum mechanics of two identical particles of spin S is reformulated using a *transported spin basis*. In the case of two particles, this spin basis can be explicitly constructed using the Schwinger representation of spins and it leads to the correct relation between spin and statistics. But, because of the fact that other constructions leading to the wrong relation can be envisaged, their work cannot be considered a proof. The extension of the construction to more than two particles has been considered in [HR04]. However, the approach itself is based on assumptions that are inconsistent from a mathematical point of view [PPRS04].

Three years ago, Anastopoulos [Ana04] studied the spin-statistics relation from a different point of view. His proposal is based on the idea of working on the *classical phase space* rather than on the configuration space. He suggests some additional conditions from which it seems to be possible to derive the spin-statistics relation. There are many other approaches to the spin-statistics problem in non relativistic quantum mechanics (for a review of many of these proposals, see [DS98]) and consequently an on-going discussion about the possibility of obtaining a better understanding of the problem.

Among these recent works, the one by Kuckert [Kuc04] is particularly appealing. Working on configuration spaces like the one defined above, he states and proves the following theorem:

Theorem 1.1. *In three spatial dimensions, the spin-statistics connection holds if and only if there exists a unitary operator* $U : \mathcal{H}^{\uparrow} \to \mathcal{H}^{\uparrow\uparrow}_{\mathrm{rel}}$ *such that*

$$\jmath_z \mid_{\mathcal{H}_+} = 2U(J_z)U^* \mid_{\mathcal{H}_+} .$$

Here, $\mathcal{H}^{\uparrow}$ is the restriction of the one-particle Hilbert space to states of maximum angular momentum states (along the z axis). Analogously, $\mathcal{H}^{\uparrow\uparrow}_{\mathrm{rel}}$ is the restriction of the two-particle Hilbert space to states of maximum angular momentum and $\jmath_z$ is the third component of the relative angular momentum operator that acts on the two particle Hilbert space. J_z is the total angular momentum operator acting on the one particle Hilbert space. Finally, $\mathcal{H}_+$ is the subspace of $\mathcal{H}^{\uparrow\uparrow}_{\mathrm{rel}}$ of states of positive parity with respect the parity operation $(x, y, z) \to (x, y, -z)$.

One of the main motivations of the present thesis, is the study of the above stated theorem. First of all, the theorem was stated and proven using local coordinates. Therefore, it is necessary to explore the possibility of proving it within a global, differential-geometric formulation. Now, if the theorem would turn out to hold in this context, an interesting possibility would arise: The existence or non existence of the operator U could be explored, thus leading to a definite statement about the spin-statistics connection in non relativistic quantum mechanics.

A first step in this direction is the construction of the corresponding Hilbert spaces and angular momentum operators. In this thesis we will consider only the case of spinless particles, where a quantization approach can be applied without having to consider a classical model for spin. We will construct the possible Hilbert space representations of the angular momentum operators for two indistinguishable particles, obtaining explicit formulas that could be used to study the existence of the operator U. In order to construct these angular momentum operators, we adopt the canonical group quantization approach of C. Isham [Ish84]. The reason for choosing this method is that it provides a physically motivated point of view from which the relationship between geometric structures, classical and quantum symmetries and quantization becomes particularly clear.

It should be kept in mind that such constructions, which employ a good amount of geometry and topology, are not something obvious. For this reason, we first explore the well-known example of the magnetic monopole. The angular momentum operators for this system are well-known but, to the best of our knowledge, they have not been derived in this way.

Statement of results

Having discussed the motivations for and the theoretical framework of the present work, we will summarize the main results of the present thesis.

- Using the techniques explained in Isham's work, we explicitly construct the Hilbert space representations of the possible quantizations of the two-sphere. In particular, we obtain the following (local) expression for the part of the representation of the canonical group that pertains to the rotation group ($SU(2)$):

$$U(g)\,|\psi(x)\rangle = \left(\frac{\beta z + \bar{\alpha}}{|\beta z + \bar{\alpha}|}\right)^n |\psi(g^{-1}\cdot x)\rangle\,. \tag{1.6}$$

  Here, ψ is a local section of a line bundle over the two-sphere, z is a local coordinate for the sphere and g is an element of $SU(2)$ that, written in matrix form, takes the following form:

$$g = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}, \qquad \text{with } |\alpha|^2 + |\beta|^2 = 1.$$

- We obtain a local formula for the infinitesimal generators of the operators $U(g)$. These operators are indexed by an integer number n that arises during one of the steps of the quantization procedure and that is closely related to Dirac's quantization condition for the electron charge. These infinitesimal generators are a quantum version of Poincaré's angular momentum vector:

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} - n\hat{\mathbf{K}}$$

  and can be interpreted as the angular momentum operators of an electron attached to the field of a magnetic monopole.

- As explained before, the configuration space of two indistinguishable particles is the two dimensional real projective space. This space can be described as a homogeneous space, that is, as the quotient of two Lie groups. There are two possible (scalar) quantizations of this configuration space, which are given by spaces of sections on associated line bundles of the form ($n = \pm 1$)

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & SU(2)\times_{\mathcal{U}_n}\mathbb{C} \\ & & \big\downarrow{\scriptstyle \pi_F} \\ & & RP^2. \end{array}$$

  For the quantization that is realized on the non trivial line bundle, we obtain the corresponding angular momentum operators and show that they can be represented as the infinitesimal generators of an $SU(2)$ action given by the following expression:

$$\mathcal{D}(g) = \begin{pmatrix} 1/2(\alpha^2+\bar{\alpha}^2-\beta^2-\bar{\beta}^2) & -i(\alpha^2-\bar{\alpha}^2-\beta^2+\bar{\beta}^2) & \alpha\beta+\bar{\alpha}\bar{\beta} \\ -i(\alpha^2-\bar{\alpha}^2+\beta^2-\bar{\beta}^2) & 1/2(\alpha^2+\bar{\alpha}^2+\beta^2+\bar{\beta}^2) & -i(\alpha\bar{\beta}-\bar{\alpha}\beta) \\ -(\alpha\bar{\beta}+\bar{\alpha}\beta) & -i(\alpha\beta-\bar{\alpha}\bar{\beta}) & (|\alpha|^2-|\beta|^2) \end{pmatrix}.$$

Before finishing this introductory chapter, we give a brief description of the contents of the present work:

- In chapter 2 we review the basic facts about Isham's proposal, called the canonical group quantization. The main purpose of this chapter is to present the theoretical background, which is the basis of the work done in the following chapters. We show how to find the canonical group in systems whose configuration space is a quotient of Lie groups, and study its representations. At the end of this chapter, we consider configuration spaces whose canonical groups are a semidirect product between a Lie group and a real vector space.

- Chapter 3 is devoted to the quantization of the two sphere. As a configuration space, S^2 is a homogeneous space, and its canonical group is $\mathcal{G} = R^3 \circledS SU(2)$. The method explained in chapter 2 is used here to find the representations of the unitary group $SU(2)$.

- In chapter 4 we study the configuration space of two indistinguishable spinless particles, that is, the two dimensional projective space. As in the case of the two-sphere, we construct the angular momentum operators associated to this physical system.

- We finish this document with three appendices where some mathematical tools needed for the understanding of the work are explained.

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Chapter 2

The general scheme of quantization

It is well known that in a quantum theory we must have to work with at least four basic rules. These rules are the following:

1) In a quantum theory, the probabilistic information is coded into a unit vector in a complex Hilbert space $\mathcal{H}$. For pure states, this vector represents the state of the system and describe completely its physical properties: mean values of observables or possible results of measurements. Because there is a set of superselection rules, not all vectors in $\mathcal{H}$ may represent physically realizable systems. For example, it is not possible to produce a state which is a superposition of states of different electrical charges or different baryon numbers [SW00].

2) Observables of the system are represented in mathematical terms by self-adjoint operators acting on $\mathcal{H}$. If we have that $F(O)$ is a function of the observable O and if $\hat{O}$ is the operator associated to O then $F(O)$ is represented by $F(\hat{O})$.

3) The average $\langle\hat{O}\rangle$ of the possible results of experimental measurements performed on the observable O in the state $|\psi\rangle$ is given by the expression

$$\langle\hat{O}\rangle = \frac{\langle\psi|\hat{O}|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{2.1}$$

The average value $\langle\hat{O}\rangle$ is unchanged if $|\psi\rangle$ is multiplied by a nonvanishing but arbitrary complex number. Thus, a pure state corresponds to a ray in a Hilbert space. A ray is a subset of the form

$$|\psi\rangle_{\mathbb{C}} := \{|\phi\rangle \in \mathcal{H}|\, |\phi\rangle = \lambda\, |\psi\rangle\,, \lambda \in \mathbb{C}\}\,.$$

4) In a closed system the state vectors at different times are unitarily related by Schrödinger's equation.

$$|\psi(t_1)\rangle = U(t_1 - t_0)|\psi(t_0)\rangle. \tag{2.2}$$

The unitary operator $U(t)$ is written in terms of a self-adjoint operator $\hat{H}$ known as the Hamiltonian operator of the system. Thus,

$$U(t) = e^{-it\hat{H}}.$$

The dynamical evolution $U(t)|\psi\rangle$ is a flow of vectors in the Hilbert space.

In this sense, the dynamical evolution of a quantum system is analogous to the classical flow of the Hamiltonian vector field ξ_f in a phase space M. This suggests that quantizing a given classical system involve some type of relation between symplectic transformations of the phase space M and unitary operators on $\mathcal{H}$. Moreover, the global topological structure of the classical configuration space should play a non-trivial role in the quantization of the system; for example, as was explained before, the topology of the phase space determines the type of statistic that must be chosen.

In this chapter, we will explain the group theoretical approach to quantization proposed in the article by Chris Isham, from the Les Houches session of 1983 [Ish84]. In the first part, we will review Isham´s implementation of Dirac´s quantization conditions. In the second, we will explain how to quantize classical systems whose configuration spaces are homogeneous spaces.

2.1 Dirac's quantization conditions

Roughly speaking, quantization of a classical system described by means of a symplectic manifold (M, ω) involves the replacement of classical observables f by operators $\hat{f}$ in such a way that the Poisson bracket of two observables is mapped to the commutator of the corresponding observables [Woo97]. More precisely, quantization can be described as a map $C^\infty(M, \Re) \to \mathrm{OP}(\mathcal{H}_Q)$ such that

(1) The map $f \mapsto \hat{f}$ is linear.

(2) If f is constant, then $\hat{f}$ must correspond to the multiplication operator.

(3) If $\{f_1, f_2\} = f_3$ then $[\hat{f}_1, \hat{f}_2] = -i\hbar \hat{f}_3$.

These conditions imply that the set of quantum operators $\mathrm{OP}(\mathcal{H})$ form a Hilbert space representation of the set of classical observables $C^\infty(M, \Re)$ and that the Poisson brackets (denoted by $\{,\}$) are the classical analogue of the quantum commutators (denoted by $[,]$). In other words, we are looking for a *representation* of the classical algebra of observables on the symplectic space M on the quantum algebra of observables defined on the Hilbert space $\mathcal{H}$. Thus, any program of quantization must solve the problem of finding conditions for the existence of the following injective map

$$\begin{aligned} \mathcal{O} : C^\infty(M, \Re) &\to \mathrm{OP}(\mathcal{H}_Q) \\ f &\mapsto \mathcal{O}_f := \hat{f}, \end{aligned}$$

which satisfies Dirac's conditions. As has been proven by Van Hove, a quantization of all classical observables in this sense does not exist [BW97]. As explained at length in [Ish84], a possible quantization algorithm can be envisaged on an arbitrary, finite dimensional, symplectic manifold M. It consists (mainly) in the following steps:

(1) Find a finite dimensional Lie group $\mathcal{C}$ which is related (in a way to be explained later) to a group ($\mathcal{G}$) of symplectic transformations of the symplectic manifold M and whose Lie algebra $\mathcal{L}(\mathcal{C})$ is

   (a) isomorphic to a sub-algebra of the Poisson bracket algebra $(C^{\infty}(M, \Re), \{,\})$ and

   (b) big enough to generate (in a way to be specified) a sufficiently large set of classical observables.

(2) Study the irreducible, unitary representations of the canonical quantization group $\mathcal{C}$. Thus, the classical objects are symplectomorphisms of the phase space which should be represented, at the infinitesimal level, by self-adjoint operators on quantum Hilbert spaces. In this way, we implement the idea of replacing Poisson brackets by operators commutators.

This general program suggests that the quantization of observables is made possible by the assumption that $\mathcal{L}(\mathcal{C})$ generates, in some sense, the set of classical observables. Actually, we do not expect to be able to quantize the full algebra $C^{\infty}(M, \Re)$, but we demand that the family of classical observables generated by $\mathcal{L}(\mathcal{C})$ turns out to be as big as possible.

2.2 Step 1: Identification of the Canonical Group $\mathcal{G}$

The first step of Isham's scheme consists in trying to associate the Lie algebra of the Lie group $\mathcal{G}$ of symplectic transformations with functions on M. More precisely, one looks for a Lie algebra homomorphism

$$\begin{aligned} P : \mathcal{L}(\mathcal{G}) &\rightarrow C^{\infty}(M, \Re) \\ A &\mapsto P^{A}. \end{aligned} \tag{2.3}$$

In this way, we are able to define a quantization map by using a representation U of the group and assigning to each smooth function, in the image of P, the self-adjoint generator obtained from U. The idea is to relate $\mathcal{L}(\mathcal{G})$ to some subalgebra of the Poisson bracket algebra of observables and this can be done by using the well known homomorphism between the set of smooth functions over M and the set of Hamiltonian vector fields. The map

$$\begin{aligned} j : C^{\infty}(M, \Re) &\rightarrow \mathrm{Ham\,VF}(M) \\ f &\rightarrow -\xi_f, \end{aligned} \tag{2.4}$$

where $\iota_{\xi_f}\omega = \mathrm{d}f$, is a Lie algebra homomorphism with kernel $\Re$, the set of constant functions on phase space. Each ξ_f generates a local one-parameter group of symplectic transformations and, if all these vector fields are complete, they produce a global group ($\mathcal{G}$) of symplectic transformations of the phase space M. Since classical observables must be associated with an element of the Lie algebra, one wants to construct an inverse function that maps $C^\infty(M, \Re)$ onto $\mathcal{L}(\mathcal{G})$. That is, we have already constructed the sequence

$$C^\infty(M, \Re) \xrightarrow{\ j\ } \mathrm{Ham\,VF}(M) \longrightarrow \mathcal{L}(\mathcal{G});$$

then, we are looking for an element $A \in \mathcal{L}(\mathcal{G})$ that induces a Hamiltonian vector field such that $j \circ P = \gamma$, as it will be explained. In this way, given the map j it will be possible to construct $\mathcal{C}$ and a homomorphism $j^{'} : \mathcal{C} \to \mathcal{G}$ whose kernel will be the commutative group $\Re$.

Given a Lie group acting on a phase space by symplectic transformations, one can construct a map $\gamma : \mathcal{L}(\mathcal{G}) \to \mathrm{Ham\,VF}(M)$ as follows. Let $\Re \to \mathcal{G}$ be a map sending $t \in \Re$ to $e^{-tA} \in \mathcal{G}$ where $A \in \mathcal{L}(\mathcal{G})$. This generates a one-parameter subgroup of symplectic transformations on M, given by

$$\phi_t^A(x) := e^{-tA} \cdot x \qquad x \in M,$$

and the tangent vectors to the flow lines form a vector field γ^A, defined through its action on smooth functions:

$$\gamma_x^A(f) = \frac{d}{dt} f(\phi_t^A(x))|_{t=0}. \tag{2.5}$$

From this construction, it can be shown that the map $\gamma : A \in \mathcal{L}(\mathcal{G}) \to \gamma^A \in \mathrm{Ham\,VF}(M)$ is a Lie algebra homomorphism. Since we are assuming that ϕ_t^A is a symplectic transformation, it preserves the symplectic form ω, that is, $\phi_t^{A*}\omega = \omega$. Using the relation $\phi_t^{A*}(L_{\gamma^A}\omega) = \frac{d}{dt}\phi_t^{A*}\omega = 0$, we conclude that the Lie derivative $L_{\gamma^A}\omega$ vanishes. This is a necessary but not sufficient condition in order that γ^A be a Hamiltonian vector field. In fact, the vector field γ^A is Hamiltonian if it satisfies the relation $\iota_{\gamma^A}\omega = df$ for some function f. But, at the same time, $\iota_{\gamma^A}\omega$ is a closed one form. To see this, note that

$$L_{\gamma^A}\omega = \iota_{\gamma^A} \circ \mathrm{d}\omega + \mathrm{d} \circ \iota_{\gamma^A}\omega = \mathrm{d} \circ \iota_{\gamma^A}\omega = 0.$$

Then, a sufficient condition in order that the image of the map γ is entirely contained in the set of Hamiltonian vector fields, is that every closed form on M is also exact. In other words, the phase spaces that can be quantized, using this scheme, are such that the first cohomology group of M vanishes [Re06].

The situation can be summarized by the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Re & \longrightarrow & C^\infty(M, \Re) & \xrightarrow{\ j\ } & \mathrm{HamVF}(M) & \longrightarrow & 0. \\ & & & & & \nwarrow^{P} & \uparrow \gamma & & \\ & & & & & & \mathcal{L}(\mathcal{G}) & & \end{array}$$

The first row represents a short exact sequence, since the kernel of the map j is the set of constant functions on phase space, that is, $\Re$. In addition, if the first cohomology group of the phase space vanishes, the map γ is a Lie algebra isomorphism from $\mathcal{L}(\mathcal{G})$ into the set of Hamiltonian vector fields. To achieve the correspondence between classical observables and the Lie algebra $\mathcal{L}(\mathcal{G})$, P must be linear and also a Lie algebra homomorphism. In other words, P must satisfy

$$\left\{P^A, P^B\right\} = P^{[A,B]}. \tag{2.6}$$

Since every exact sequence of vector spaces splits, there is no difficulty in finding a linear map P such that the diagram commutes [Ish84], that is, in obtaining for $A \in \mathcal{L}(\mathcal{G})$

$$\gamma^A = -\xi_{P^A}. \tag{2.7}$$

This is a natural requirement, since the map j assigns a Hamiltonian vector field to every function on the phase space. We are looking for a way to map the Lie algebra of $C^\infty(M, \Re)$ homomorphically by j onto $\mathcal{L}(\mathcal{G})$, and it is natural to expect that, if $\gamma(A) = j(f)$, then $P^A = f$. Nevertheless, P does not in general respect the relation (2.6). In fact, since $\ker j = \Re$, each P^A is defined only up to an arbitrary constant, and these constants must be chosen carefully.

From the requirement that γ is a Lie algebra homomorphism, we obtain:

$$\xi_{\{P^A, P^B\}} = \xi_{P^{[A,B]}}. \tag{2.8}$$

However, if two Hamiltonian vector fields coincide, that is, if $\xi_f = \xi_g$, the only thing we can conclude is that $f = g + \text{constant}$, and, in particular, equation (2.8) implies only that

$$z(A, B) = \left\{P^A, P^B\right\} - P^{[A,B]}$$

is a constant. The conclusion is: although we have some freedom in the choice of P it is restricted by the condition that P must be linear. In general, any two maps P' and P related by

$$P'^A := P^A + d(A),$$

where d belongs to the dual of $\mathcal{L}(\mathcal{G})$, satisfy equation (2.7). Thus, the problem to be solved is as follows. Can we find a $d \in \mathcal{L}(\mathcal{G})^*$ such that the new function P' will produce functions $z(A, B)$ that do vanish for all A and B, that is,

$$\left\{P'^A, P'^B\right\} = P'^{[A,B]}? \tag{2.9}$$

In the context of geometric quantization this is a well known problem and is called the moment map problem [Woo97]. Its standard solution is as follows.

First of all, there are some important properties of the 2-cocycle $z(A, B)$. For every $A, B, C \in \mathcal{L}(\mathcal{G})$ we have:

(a) $z(A, B) = -z(B, A)$ and

(b) $z([A,B],C)+z([B,C],A)+z([C,A],B)=0$.

The first condition says that a cocycle is a skew-symmetric bilinear form $z:\mathcal{L}(\mathcal{G})\times\mathcal{L}(\mathcal{G})\to\Re$, and the second one, is a version of the Jacobi identity.

The easiest way of seeing how the cocycle works is to compute the Poisson brackets $\left\{P'^A,P'^B\right\}$. In this case, we obtain

$$\left\{P'^A,P'^B\right\}-P'^{[A,B]}=\left\{P^A,P^B\right\}-P^{[A,B]}-d([A,B])=z(A,B)-d([A,B]).$$

Hence the closure (2.9) can be realized if and only if the cocycle has the property:

$$z(A,B)=d([A,B])$$

for all $A,B\in\mathcal{L}(\mathcal{G})$ and for some $d\in\mathcal{L}(\mathcal{G})^*$; in this case, z is called a two-coboundary. We conclude at least two things. Every two-coboundary is also a two-cocycle. Saying that two two-cocycles z and z' are equivalent if they differ by a two-coboundary, that is, $z-z'=d$, we define a unique cohomology class in $H^2(\mathcal{L}(\mathcal{G}),\Re)$. Thus, we see that the group $\mathcal{G}$ and its action on M by P define a class in the second cohomology group of its Lie algebra. And if the cocycle can be made to vanish, we obtain the desired Lie algebra isomorphism $A\to P^A$. In this way, we can quantize by fixing a representation of $\mathcal{G}$, since we can assign a self-adjoint generator to every observable $P^A\in C^\infty(M,\Re)$. In this case, it is possible to use the group $\mathcal{G}$ as the canonical quantization group $\mathcal{C}$.

2.3 Step 2: Representations of $\mathcal{G}$

The next step of our program of quantization is to study the representations of the canonical group. In general, this is not an obvious step, since the canonical group has non-trivial properties: it could be not compact, for example. However, we do not need to consider the general case but only the class of groups that arise in physical systems. Actually, in this work we only need to concern ourselves with homogeneous spaces of the type $\mathcal{Q}=G/H$, where G and H are Lie groups. As it will be shown in the next section, the canonical group of this type of configuration spaces is $\mathcal{G}=W\circledS G$, where W is a real vector space, and it is sufficient to study just Mackey's theory of the representations of semidirect products.

As will be explained later, when $\mathcal{G}=W\circledS G$, the space $\mathcal{Q}$ is one of the orbits generated by the action of G in W^*. In this space, the group G acts transitively and effectively, and, therefore, its irreducible representations can provide the basic self-adjoint observables. The simplest representation of a Lie group G acting on a manifold $\mathcal{Q}$ is defined on the set of complex-valued functions on $\mathcal{Q}$ by the well-known expression:

$$(U(g)\psi)(x):=\psi(g^{-1}\cdot x). \tag{2.10}$$

We are interested in unitary representations, but unitarity does not follows from the above definition. In fact, notice that

$$
\begin{aligned}
\langle U(g)\psi, U(g)\phi\rangle &= \int_{\mathcal{Q}} \psi^*(g^{-1}\cdot x)\phi(g^{-1}\cdot x)d\mu(x) \\
&= \int_{\mathcal{Q}} \psi^*(x)\phi(x)d\mu(g^{-1}\cdot x). \qquad (2.11)
\end{aligned}
$$

Hence, the representation will be unitary if and only if the measure μ is G-invariant, that is, $d\mu(g^{-1}\cdot x) = d\mu(x)$. There are cases for which such invariant measures exist; for example, Lebesgue measure in $\Re^3$ is invariant under the action of $SO(3)$.

State vectors in quantum theory are complex-valued functions on the classical configuration space or, in general, on a Mackey orbit $\mathcal{O}$ in W. But if the space $\mathcal{Q}$ has a suitable topological structure, the states can be generalized to include cross-sections of vector bundles E over $\mathcal{Q}$. The set of cross sections of a vector bundle has a natural vector space structure, and we define as the Hilbert space (the completion of) the set of square-integrable smooth sections of the bundle: $L^2(\Gamma(E); d\mu)$ [Ish84]. A natural analogue on cross-sections of the equation (2.10) could be

$$(U(g)\Psi)(x) := \Psi(g^{-1}\cdot x),$$

where $\Psi \in L^2(\Gamma(E))$. But this is not a well defined expression, since the left and the right-hand sides correspond to elements in different fibers; $\mathbb{C}_x$ and $\mathbb{C}_{g^{-1}\cdot x}$ respectively. How can we solve this problem? The answer is that, given an action on $\mathcal{Q}$, we need a lift $l_g^{\uparrow}$ of G on the vector bundle in such a way that for each $x \in \mathcal{Q}$, $l_g^{\uparrow}$ maps linearly the fiber $\mathbb{C}_x$ onto $\mathbb{C}_{g\cdot x}$. Then, the correct expression for the representation of the group is

$$(U(g)\Psi)(x) := l_g^{\uparrow}\Psi(g^{-1}\cdot x),$$

which is now well defined, since both sides correspond to elements of the same fiber.

$$l_g^{\uparrow} : \mathbb{C}_x \to \mathbb{C}_{g\cdot x}$$

The problem we are considering is to look for a left G-action on the total space of the associated bundle $\mathbb{C} \to E \to \mathcal{Q}$, called a G-lift of the G-action l_g on $\mathcal{Q}$, such that the following diagram commutes

$$
\begin{array}{ccc}
E & \xrightarrow{l_g^{\uparrow}} & E \\
\pi\downarrow & & \pi\downarrow \\
\mathcal{Q} & \xrightarrow{l_g} & \mathcal{Q}
\end{array}
\qquad\qquad \pi \circ l_g^{\uparrow} = l_g \circ \pi,
$$

where $l_g(x) = g\cdot x$. To be a representation, U must be a group homomorphism, that is, $U(g_1)U(g_2) = U(g_1 g_2)$. We obtain:

$$
\begin{aligned}
(U(g_1)U(g_2)\Psi)(x) &:= l_{g_1}^{\uparrow}(U(g_2)\Psi)(g_1^{-1}\cdot x) = (l_{g_1}^{\uparrow}\circ l_{g_2}^{\uparrow})\Psi(g_2^{-1}g_1^{-1}\cdot x) \\
&\equiv l_{g_1 g_2}^{\uparrow}\Psi((g_1 g_2)^{-1}\cdot x),
\end{aligned}
$$

and, therefore, $l^{\uparrow}_{g_1} \circ l^{\uparrow}_{g_2} = l^{\uparrow}_{g_1 g_2}$.

With this, we have, in principle, completed the scheme of quantization of homogeneous spaces. In the next section, we will develop a general approach to physical systems whose configuration spaces are homogeneous spaces.

2.4 Homogeneous spaces

Lie groups and their algebras appear in theoretical physics as symmetry groups of dynamical systems. By Noether's theorem, these symmetries are associated with conservation laws. For example, if a physical system is invariant under rotations then its angular momentum is conserved [SW86]. In our work, we will consider classical systems whose configuration space is a Lie group quotient space. This quotient a has principal bundle structure, and has well known properties. For example, the right action of a subgroup H of a Lie group G defines a topological space. In this chapter, we will study the fibre bundle structure of such quotient spaces, and show how to find the representations of their canonical groups.

2.4.1 Principal bundles

Definition 2.1. [Ish03] A bundle $(G, \pi, \mathcal{Q}, H)$, is called a **principal bundle** if G is a right H-space and if $(G, \pi, \mathcal{Q}, H)$ is isomorphic to the bundle $(G, \rho, G/H, H)$, where G/H is the orbit space of the H-action on G and ρ is the usual projection map:

$$\begin{array}{ccc} G & \xrightarrow{id} & G \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\rho}\big\downarrow \\ \mathcal{Q} & \xrightarrow{\simeq} & G/H \end{array}$$

By construction, H is called the structure group of the bundle.

The statement that a homogeneous space (for our purposes just a quotient of the form G/H) has principal fibre bundle structure is something that has to be proven. Let H be a closed Lie subgroup of a Lie group G. In order to show that

$$\begin{array}{c} G \\ {\scriptstyle\pi}\big\downarrow \\ \mathcal{Q} = G/H \end{array}$$

is a principal bundle with total space G, fibre H and base space $\mathcal{Q} = G/H$, we define: first, the right action of H on G by $g \mapsto gh$, $g \in G$ and $h \in H$; and second, the projection $\pi : G \longrightarrow \mathcal{Q} = G/H$ by the map $\pi : g \mapsto [g] = \{gh \,|\, h \in H\}$. The goal is to show that there exists an open cover $\{U_i\}$ and a biyective map between $\pi^{-1}(U_i)$ and $U_i \times H$ for each i. To do this, we define a map

$$\pi^{-1}(U_i) \xrightarrow{f_i} H$$

on each local chart U_i such that

$$f_i(s_i([g])) = e, \qquad \forall g \in \pi^{-1}(U_i),$$

where $s_i : U_i \longrightarrow \pi^{-1}(U_i)$ is any local section. In general, the existence of these local sections is not something obvious. For an existence proof see [Ste51]. The functions f_i must preserve the right action of H, i.e. the last relation transforms under the action of H in this way:

$$f_i(s_i([g])h) = eh = h, \qquad \forall g \in \pi^{-1}(U_i).$$

H acts transitively on each fibre. If $g \in [g]$ for $[g] \in U_i$, then there exists at least one $a \in H$ such that $s_i([g])a = g$. Expressing a in terms of g, we obtain $a = g(s_i([g]))^{-1}$ and then

$$f_i(g) = f_i(s_i([g])a) = g(s_i([g]))^{-1}.$$

In this way we can define a local trivialization $\phi_i : U_i \times H \longrightarrow \pi^{-1}(U_i)$ by

$$\phi_i^{-1}(g) = ([g], f_i(g)), \tag{2.12}$$

and its inverse is obviously

$$\phi_i([g], a) = s_i([g])a. \tag{2.13}$$

2.4.2 The canonical group $(C^\infty(\mathcal{Q}, \Re)/\Re) \circledS \operatorname{Diff} \mathcal{Q}$

The general problem of the quantization of a system which configuration space is of the form $\mathcal{Q} = G/H$ has been considered by Isham [Ish84]. The idea is to consider two natural classes of transformations on the cotangent bundle $M = T^*\mathcal{Q}$. The first class is the one induced by diffeomorphisms of $\mathcal{Q}$. If $\phi \in \operatorname{Diff} \mathcal{Q}$, using the differential of ϕ (push-forward), we obtain a transformation ϕ^* (pull-back) on the cotangent bundle defined by:

$$\langle \phi^* l, v \rangle_q = \langle l, \phi_* v \rangle_{\phi(q)},$$

where $l \in T^*_{\phi(q)}\mathcal{Q}$ and $v \in T_q\mathcal{Q}$. Although the map $l \mapsto \phi^* l$ is a symplectic transformation on $T^*\mathcal{Q}$, this action is not transitive, and, for this reason, this transformation must be complemented by a second group of transformations: translations along the fibers. Given $h \in C^\infty(\mathcal{Q}, \Re)$, we can use its exterior derivative to map $l \in T^*_q\mathcal{Q}$ to $l - (\mathrm{d}h)_q \in T^*_q\mathcal{Q}$. Notice that since h only acts through its differential, two functions $h \in C^\infty(\mathcal{Q}, \Re)$ and $h' \in C^\infty(\mathcal{Q}, \Re)$ satisfying $h - h' =$ constant, produce the same translation along a fibre [Re06]. In this way, we are considering the quotient group $C^\infty(\mathcal{Q}, \Re)/\Re$.

We thus have the following group actions on $T^*\mathcal{Q}$:

$$\begin{array}{rcll} \operatorname{Diff} \mathcal{Q} \times T^*\mathcal{Q} & \rightarrow & T^*\mathcal{Q} & \\ (\phi, l) & \mapsto & \phi^{-1*} l & l \in T^*_q \mathcal{Q} \end{array} \tag{2.14}$$

and

$$
\begin{array}{rcl}
C^\infty(\mathcal{Q},\Re)/\Re \times T^*\mathcal{Q} & \to & T^*\mathcal{Q} \\
([h], l) & \mapsto & l - (\mathrm{d}h)_q \qquad l \in T^*_q\mathcal{Q}.
\end{array} \tag{2.15}
$$

The vector fields induced by both actions are hamiltonian. To see this, let γ^x be the vector field on $T^*\mathcal{Q}$ associated with the subgroup of diffeomorphisms generated by the vector field x on $\mathcal{Q}$. The action of $\mathrm{Diff}\,\mathcal{Q}$ on $T^*\mathcal{Q}$ preserves the Liouville form θ, and hence $L_{\gamma^x}\theta = 0$; then, by Cartan's formula,

$$
i_{\gamma^x}\mathrm{d}\theta + \mathrm{d}(i_{\gamma^x}\theta) = \omega(\gamma^x, \cdot) + \mathrm{d}\,\langle\theta, \gamma^x\rangle = 0,
$$

and thus $\iota_{\gamma^x}\omega = -\mathrm{d}\,\langle\theta,\gamma^x\rangle$. It follows that γ^x is a Hamiltonian vector field with the generating function $\langle\theta,\gamma^x\rangle$, that is, $\gamma^x = \xi_{\langle\theta,\gamma^x\rangle} \in \mathrm{HamVF}(M)$. Now let $[h] \in C^\infty(\mathcal{Q},\Re)/\Re$; then, in local symplectic coordinates the second action is given by:

$$
(q^i, p_j) \mapsto \Big(q^i, p_j - \frac{\partial h(q)}{\partial q^j}\Big),
$$

and generates a flow in the phase space whose vector field is given by

$$
\gamma^h_{(q^i,p_j)}(f) = \frac{d}{dt} f\Big(q^i, p_j + t\frac{\partial h(q)}{\partial q^j}\Big)\Big|_{t=0} = \frac{\partial h(q)}{\partial q^j}\frac{\partial}{\partial p_j} f.
$$

Then $\iota_{\gamma^h}\omega = -\mathrm{d}h$, that is, the vector field generated by this action is also Hamiltonian.

Both actions, therefore, are transitive and symplectic, and it is natural to try to express them as an action of a single group. In fact, the translations along the fibers can be combined with $\mathrm{Diff}\,\mathcal{Q}$ to give a left action on $T^*\mathcal{Q}$ defined by:

$$
\tau_{(\alpha,\phi)}(l) = \phi^{-1*}(l) - (\mathrm{d}\alpha)_{\phi(q)},
$$

where $l \in T^*_q\mathcal{Q}$, $\alpha \in C^\infty(\mathcal{Q},\Re)/\Re$ and $\phi \in \mathrm{Diff}\,\mathcal{Q}$. The group law is determined by the requirement $\tau_{(\alpha_2,\phi_2)} \circ \tau_{(\alpha_1,\phi_1)} = \tau_{(\alpha_2,\phi_2)\cdot(\alpha_1,\phi_1)}$. Then,

$$
\begin{array}{rcl}
\tau_{(\alpha_2,\phi_2)} \circ \tau_{(\alpha_1,\phi_1)}(l) & = & \tau_{(\alpha_2,\phi_2)}(\phi_1^{-1*}(l) - (\mathrm{d}\alpha_1)_{\phi_1(q)}) \\
& = & \phi_2^{-1*}(\phi_1^{-1*}(l) - (\mathrm{d}\alpha_1)_{\phi_1(q)}) - (\mathrm{d}\alpha_2)_{(\phi_2\circ\phi_1)(q)} \\
& = & ((\phi_2\circ\phi_1)^{-1})^*(l) - \mathrm{d}(\phi_2^{-1*}\alpha_1 + \alpha_2)_{(\phi_2\circ\phi_1)(q)} \\
& = & \tau_{(\alpha_1\circ\phi_2^{-1*}+\alpha_2,\phi_2\circ\phi_1)}(l).
\end{array} \tag{2.16}
$$

The structure obtained is, then, that of a semi-direct product of the form

$$
(C^\infty(\mathcal{Q},\Re)/\Re) \circledS \mathrm{Diff}\,\mathcal{Q}.
$$

Now we are in position to construct the canonical group of homogeneous spaces of the type $\mathcal{Q} = G/H$. The idea is, in general lines, to find a suitable finite dimensional vector space $W \subset C^{\infty}(\mathcal{Q}, \Re)/\Re$ and a finite dimensional Lie group G of transformations of $\mathcal{Q}$, in such a way that $W \circledS G \subset (C^{\infty}(\mathcal{Q}, \Re)/\Re) \circledS \mathrm{Diff}\,\mathcal{Q}$. Since this subgroup is still transitive and symplectic, will serve as a basis for a quantum theory of homogeneous spaces.

Let V be a finite-dimensional real vector space and let $g \mapsto R(g)$ be a representation of G on V. On the cotangent bundle $T^*V \simeq V \times V^*$, the dual map $R^*(g)$ of G on V^* is defined by: $\langle R^*(g)w, v\rangle = \langle w, R(g)v\rangle$, where $w \in V^*$ and $v \in V$. Note that V^* may be made to act on T^*V by translations along the fibers, that is, given $w^{'} \in V^*$ we can map $(u, w) \in V \times V^*$ to $(u, w - w^{'}) \in V \times V^*$, inducing a combined action of $V^* \circledS G$ in $T^*V \simeq V \times V^*$ defined by:

$$\tau^{'}_{w',g}(u, w) = (R(g)u, R^*(g^{-1})w - w^{'}).$$

If we regard a dual vector $w \in V^*$ as a function

$$\begin{aligned} h^w : V &\longrightarrow \Re \\ v &\longmapsto \langle w, v\rangle, \end{aligned}$$

we can consider $V^* \circledS G$ as a subgroup of $(C^{\infty}(V, \Re)/\Re) \circledS \mathrm{Diff}V$. The action of this group then satisfies all requirements of Isham's quantization program, with the exception of transitivity. Nevertheless, the vector space V decomposes into G-orbits and if $\mathcal{O}_v$ denotes the orbit generated by the vector $v \in V$, then G acts transitively on $\mathcal{O}_n$ which stabilizer is, say, H_v. By construction, $\mathcal{O}_v$ is isomorphic to G/H_v. Let $\mathcal{Q}_v = G/H_v$, then our Lie group G acts on $\mathcal{Q}_v$ in a natural way:

$$g^{'} \cdot [g] = \left[g^{'} g\right]$$

and, therefore, G is itself a subgroup of $\mathrm{Diff}\,\mathcal{Q}_v$.

The action of $V^* \circledS G$ on $T^*(G/H_n)$ can be defined by

$$\tau_{w',g}(u, w) := (R(g)u, R^*(g^{-1})w - w^{'}), \tag{2.17}$$

where $(u, w) \in T^*(G/H_n)$ and where the action of $w^{'}$ is to be understood as the action of (the equivalence class) of a function on $\mathcal{Q}_v$, namely the restriction of $h^{w^{'}}$ to the orbit $\mathcal{O}_v$.

In this way we can say that the canonical group of a configuration space of the form G/H is $V^* \circledS G$, where V is a vector space.

Chapter 3

The Magnetic Monopole

In this chapter we will study the techniques explained above, beginning with a very concrete example and its known solution. Specifically, we will consider the problem of a point electric charge moving in the magnetic field generated by a fixed magnetic monopole. Although Maxwell's equations deny the possibility of a point magnetic charge, there are at least three reasons for studying it:

(1) As Dirac [Dir31] pointed out, the existence of magnetic charges imply the quantization of the electric charge.

(2) A spin one-half fermion can be realized as a scalar particle that is constrained to move in the magnetic field generated by a magnetic monopole. There seems to be applications of this basic fact to the study of entanglement [Bas06].

(3) Of special relevance to this work, the angular momentum theory in the magnetic monopole case presents a rich structure, highlighting the geometric and topological nature of the problem [BL81]. Our expectation is that the analogous situation in the case of indistinguishable particles may lead to an advance in the understanding of the spin-statistics connection.

Dirac's procedure and his reasoning have a deep topological and geometrical basis, and the problem is best described using fibre bundle techniques. It is our aim to show how the general scheme of quantization works in this case.

In the first part of the chapter, we will start with the magnetic monopole problem as described classically. In the second one, we will demonstrate that the quantization of the electric and monopolar charge arises as a consequence of the topological and geometric structure behind the Dirac monopole. And in the last one, we will use fibre bundle techniques to reproduce the same results.

3.1 The classical picture

Let us consider a monopole of strength g placed at $\mathbf{r} = 0$. In the framework of nonrelativistic classical physics, it produces a radial magnetic vector field:

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3}.$$

The vector potential $\mathbf{A}$ must satisfy $\mathbf{B} = \nabla \times \mathbf{A}$. This equation can be solved for $\mathbf{A}$, but only locally. In fact, in order to avoid singularities, we have to work with more than one vector potential. We define $\mathbf{A}^S$ in the northern hemisphere and $\mathbf{A}^N$ in the southern hemisphere of the sphere surrounding the monopole. For $(\xi_3 \neq -r)$, we have:

$$\mathbf{A}^S = \frac{g}{r(r+\xi_3)}(-\xi_1, \xi_2, 0), \tag{3.1}$$

and for $(\xi_3 \neq r)$:

$$\mathbf{A}^N = \frac{g}{r(r-\xi_3)}(\xi_2, -\xi_1, 0). \tag{3.2}$$

This satisfactorily eliminates the singularity from each region. Since we are working on a sphere, we said that the configuration space of the monopole is the two sphere S^2.

To formulate the problem of a point electric charge attached to this magnetic field in terms of hamiltonian mechanics requires first the definition of the lagrangian L:

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + e\dot{\mathbf{x}} \cdot \mathbf{A},$$

where m and e are the mass and the charge of the electric particle. In this formulation, the momentum $\mathbf{p}$ conjugated to $\mathbf{x}$ is defined by $\mathbf{p} = \frac{\partial L}{\partial \mathbf{x}}$. The hamiltonian is:

$$H = \mathbf{p} \cdot \dot{\mathbf{x}} - L = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2. \tag{3.3}$$

One is tempted to define intuitively $\mathbf{J}_{\rm orb} = m\mathbf{x} \times \dot{\mathbf{x}}$ as the orbital angular momentum of the system. Nevertheless, there is something wrong with this expression, since this is not a constant of motion. To see this, note that

$$\frac{d}{dt}\mathbf{J} = m\mathbf{x} \times \ddot{\mathbf{x}} = eg\,\mathbf{x} \times \frac{\dot{\mathbf{x}} \times \mathbf{x}}{r^3} \neq 0,$$

where we have used the Lorentz force law: $m\ddot{\mathbf{x}} = e\dot{\mathbf{x}} \times \mathbf{B}$. The problem is that the direction of the Lorentz force is radial; then, the angular momentum must be a conserved quantity.

Another source of angular momentum is the electromagnetic field itself. Then, in order that the angular momentum is a constant of motion, we can compute the angular momentum of the total electromagnetic field produced by both charges. As

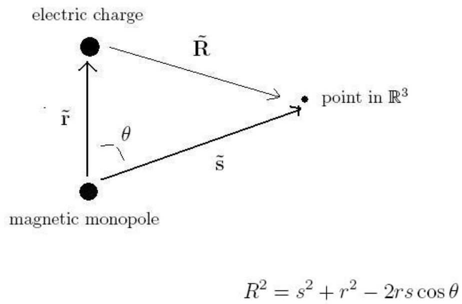


Figure 3.1: Electromagnetic fields produced by an electric charge and a magnetic monopole.

it is sketched in the figure 3.1, we have for the angular momentum of the fields:

$$\begin{aligned}
\mathbf{J}_{\text{field}} &= \int_{\text{all space}} d\mathbf{s}\,[\mathbf{s}\times[\mathbf{E}\times\mathbf{B}]] = eg\int_{\text{all space}} d\mathbf{s}\left[\mathbf{s}\times\left[\frac{\mathbf{R}}{R^3}\times\frac{\mathbf{s}}{s^3}\right]\right] \\
&= -eg\int_{\text{all space}} d\mathbf{s}\frac{[\mathbf{s}\times[\mathbf{r}\times\mathbf{s}]]}{R^3 s^3} \\
&= eg\int_{\text{all space}} d\mathbf{s}\frac{[s^2\mathbf{r} - rs\cos\theta\mathbf{s}]]}{R^3 s^3} = -\frac{eg}{c}\frac{\mathbf{r}}{r} \equiv -(eg/c)\hat{e}_r \qquad (3.4)
\end{aligned}$$

The consequence is that, instead of the expression $m\mathbf{x}\times\dot{\mathbf{x}}$, the angular momentum integrals are:

$$\mathbf{J} = \mathbf{J}_{\text{orb}} + \mathbf{J}_{\text{field}} = m\mathbf{x}\times\dot{\mathbf{x}} - (eg/c)\hat{e}_r. \qquad (3.5)$$

It is not a difficult task to verify that $\mathbf{J}$ is an integral of motion. In fact, $d\mathbf{J}/dt = 0$.

In Hamiltonian terms, we find that the canonical linear momentum is:

$$\mathbf{p}^\alpha = m\dot{\mathbf{x}} + (e/c)\mathbf{A}^\alpha.$$

Introducing this result into the equation (3.5), we obtain:

$$\mathbf{J}^\alpha = \mathbf{x}\times(\mathbf{p} - (e/c)\mathbf{A}^\alpha) - (eg/c)\hat{e}_r = \mathbf{L} - (eg/c)\mathbf{K}^\alpha, \qquad (3.6)$$

where

$$\begin{aligned}
\mathbf{L} &= \mathbf{x}\times\mathbf{p}, \\
\mathbf{K}^S &= (1/g)(\mathbf{x}\times\mathbf{A}^S) + \hat{e}_r = \frac{\mathbf{x} + r\hat{e}_3}{r + \xi_3} \\
\mathbf{K}^N &= (1/g)(\mathbf{x}\times\mathbf{A}^N) + \hat{e}_r = \frac{\mathbf{x} - r\hat{e}_3}{r - \xi_3}.
\end{aligned}$$

In other words, the three integrals of the motion for the northern hemisphere are:

$$\begin{aligned} J_1^S &= L_1 - \mu\frac{\xi_1}{r+\xi_3}, \\ J_2^S &= L_2 - \mu\frac{\xi_2}{r+\xi_3}, \\ J_3^S &= L_3 - \mu, \end{aligned} \tag{3.7}$$

and for the southern hemisphere:

$$\begin{aligned} J_1^N &= L_1 - \mu\frac{\xi_1}{r-\xi_3}, \\ J_2^N &= L_2 - \mu\frac{\xi_2}{r-\xi_3}, \\ J_3^N &= L_3 + \mu, \end{aligned} \tag{3.8}$$

where $\mu = (eg/c)$.

3.2 The quantum mechanical picture

In the overlap region (intersection between the two hemispheres), the two vector potentials do not match up. But, since the mangetic field is defined in all the space, we must have:

$$\nabla \times \mathbf{A}^S = \nabla \times \mathbf{A}^N.$$

That is, the two vector potentials must differ at most by a gauge transformation:

$$\mathbf{A}^S - \mathbf{A}^N = 2g\nabla\phi,$$

where we have parameterized the sphere with the usual angles (θ, ϕ). Wave functions ψ^N and ψ^S are defined piecewise in each local chart, and in the overlap region they are related by a phase factor [Nak03]:

$$\psi^S = e^{-2i\mu\phi/\hbar}\psi^N. \tag{3.9}$$

In order that the wave function be single-valued, the phase factor integrated around a closed loop in the overlap region must satisfy that $e^{-2i\mu\phi/\hbar} = e^{-2i\mu\phi/\hbar}e^{-2i\mu(2\pi)/\hbar}$. Hence, the expression (3.9) forces us to take

$$\frac{2\mu}{\hbar} = n \qquad n \in \mathbb{Z}.$$

This is Dirac's quantization condition of the electric and magnetic charges.

3.3 Quantization of $\mathcal{Q} = S^2$

In this section, we explore a simple but fundamental example: the sphere S^2. This topological space is the simplest possible even-dimensional compact spin manifold, and it is the configuration space in which the magnetic monopole problem is defined. The sphere S^2 can be described as the base space of a principal bundle with S^1 as structure group. In fact, the $U(1)$ action on $SU(2)$ gives rise to the bundle

$$\begin{array}{ccc} U(1) & \longrightarrow & SU(2) \\ & & \downarrow \\ & & SU(2)/U(1) \simeq S^2, \end{array}$$

which is isomorphic to the Hopf bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2. \end{array}$$

Although there exists a set of important isomorphisms: $U(1) \cong S^1$, $SU(2) \cong S^3$ and $\mathbb{C}P^1 \cong S^2$, for practical purposes, we will distinguish between them. Taking this into account, we can describe the two-sphere in two different ways: as the homogeneous space $SU(2)/U(1)$ and as the complex projective space $\mathbb{C}P^1$. Before quantizing, we will describe carefully the topology of the space involved. And, after that, we will implement the group theoretical approach to quantization as was described in the last chapter.

3.3.1 S^2 as a homogeneous space

As is well known, $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 | |z_0|^2 + |z_1|^2 = 1\}$. As a set, S^3 is basically the same as $SU(2)$. The bijection is clear if we describe $SU(2)$ in the following way:

$$SU(2) = \left\{ \begin{pmatrix} z_0 & \bar{z}_1 \\ -z_1 & \bar{z}_0 \end{pmatrix} \mid |z_0|^2 + |z_1|^2 = 1 \right\}.$$

Then, we are able to write each element of $SU(2)$ as the pair (z_o, z_1), and now we have a well defined product of two elements in the three sphere.

With this, we can consider the following subgroup of $SU(2)$:

$$U(1) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \mid \lambda\bar{\lambda} = 1 \right\}.$$

The right action $(SU(2) \times U(1)) \to SU(2)$ is

$$\begin{pmatrix} z_0 & \bar{z}_1 \\ -z_1 & \bar{z}_0 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} z_0\lambda & \bar{z}_1\bar{\lambda} \\ -z_1\lambda & \bar{z}_0\bar{\lambda} \end{pmatrix}.$$

In S^3 this action takes the form $(z_0, z_1) \cdot \lambda = (z_0\lambda, z_1\lambda)$. With this, we now know that the elements of the quotient are orbits of the form:

$$[z_0 : z_1] = \left\{(z_0\lambda, z_1\lambda) \,|\, (z_0, z_1) \in SU(2) \text{ and } \lambda \in S^1\right\}. \tag{3.10}$$

To exhibit the fibre bundle structure of the sphere, we define, first, two natural local charts:

$$\begin{aligned} U_N &= S^2 - \{N\}, && \text{The sphere minus the North Pole,} \\ U_S &= S^2 - \{S\}, && \text{The sphere minus the South Pole.} \end{aligned}$$

The projection of the principal bundle is the Hopf map, and is constructed as follows. The unit two-sphere embedded in $\Re^3$ is expressed as

$$S^2 = \left\{(\xi_1, \xi_2, \xi_3) \in \Re^3 \,|\, (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = 1\right\}.$$

Define $z_0 = x_1 + ix_2$ and $z_1 = x_3 + ix_4$. The Hopf map $\pi : SU(2) \longrightarrow S^2$ is defined by the following relations[1]

$$\xi_1 = 2(x_1x_3 + x_2x_4), \tag{3.11}$$
$$\xi_2 = 2(x_2x_3 - x_1x_4), \tag{3.12}$$
$$\xi_3 = (x_1)^2 + (x_2)^2 - (x_3)^2 - (x_4)^2. \tag{3.13}$$

Let (X, Y) be the usual stereographic projection coordinates of a point located in the southern hemisphere U_N of S^2 from the North Pole, that is, from the point $\xi_3 = 1$. Taking a complex plane whose coordinate is given by $Z = X + iY$, we find that in local coordinates

$$\begin{aligned} X + iY &= \frac{\xi_1}{1-\xi_3} + \frac{i\xi_2}{1-\xi_3} = \frac{\xi_1 + i\xi_2}{1-\xi_3} \\ &= \frac{2(x_1x_3 + x_2x_4 + ix_2x_3 - ix_1x_4)}{1 - (x_1)^2 - (x_2)^2 + (x_3)^2 + (x_4)^2} = \frac{x_1 + ix_2}{x_3 + ix_4} = \frac{z_0}{z_1} \equiv z. \end{aligned}$$

In the same way, let (U, V) be the usual stereographic projection coordinates of a point located in the northern hemisphere U_S of S^2 from the South Pole, that is from the point $\xi_3 = -1$. We find that in local coordinates

$$\begin{aligned} U + iV &= \frac{\xi_1}{1+\xi_3} - \frac{i\xi_2}{1+\xi_3} = \frac{\xi_1 - i\xi_2}{1+\xi_3} \\ &= \frac{2(x_1x_3 + x_2x_4 - ix_2x_3 + ix_1x_4)}{1 + (x_1)^2 + (x_2)^2 - (x_3)^2 - (x_4)^2} = \frac{x_3 + ix_4}{x_1 + ix_2} = \frac{z_1}{z_0} \equiv \zeta. \end{aligned}$$

[1] Note that we can verify that

$$\sum_{i=1}^{3} (\xi_i)^2 = \Big(\sum_{i=1}^{4} (x_i)^2\Big)^2 = 1.$$

On the equator $U_N \cap U_S$, $z = 1/\zeta$. We can consider this parametrization of the sphere from a slightly different point of view. In fact, we can use polar coordinates (θ, ϕ), and, in this way, stereographic coordinates become:

$$z = \frac{\xi_1}{1-\xi_3} + \frac{i\xi_2}{1-\xi_3} = \frac{e^{i\phi}\sin\theta}{1-\cos\theta},$$

and

$$\zeta = \frac{\xi_1}{1+\xi_3} - \frac{i\xi_2}{1+\xi_3} = \frac{e^{-i\phi}\sin\theta}{1+\cos\theta},$$

This coordinates are more useful to study the monopole problem. In fact, polar coordinates are the natural ones when writing down the expressions for angular momentum operators and potential vectors.

In the Hopf bundle

$$\begin{array}{ccc} U(1) & \longrightarrow & SU(2) \\ & & \Big\downarrow \pi \\ & & S^2 \end{array}$$

the projection $\pi : SU(2) \to S^2$ is defined on U_N by

$$(z_0, z_1) \mapsto [z_0 : z_1] \equiv \frac{z_0}{z_1} = z,$$

and in U_S by

$$(z_0, z_1) \mapsto [z_0 : z_1] \equiv \frac{z_1}{z_0} = \zeta.$$

Using the construction for principal bundles explained in the last chapter, we must define local trivializations. To do this, we need to define the map:

$$\pi^{-1}(U_i) \xrightarrow{f_i} U(1)$$

on each local chart U_i. Let $s_i : U_i \subset S^2 \longrightarrow \pi^{-1}(U_i) \subset S^3$ a local section, defined on U_N by

$$[z_0 : z_1] \mapsto (z_0, z_1) \cdot \frac{\overline{z}_1}{|z_1|}$$

and defined on U_S by

$$[z_0 : z_1] \mapsto (z_0, z_1) \cdot \frac{\overline{z}_0}{|z_0|}.$$

Define for $g \in SU(2)$

$$f_N(g) = g \cdot s_N([g])^{-1} = \begin{pmatrix} z_0 & \bar{z}_1 \\ -z_1 & \bar{z}_0 \end{pmatrix} \begin{pmatrix} \frac{z_1}{|z_1|} & 0 \\ 0 & \frac{\bar{z}_1}{|z_1|} \end{pmatrix} \begin{pmatrix} \bar{z}_0 & -\bar{z}_1 \\ z_1 & z_0 \end{pmatrix} = \begin{pmatrix} \frac{z_1}{|z_1|} & 0 \\ 0 & \frac{\bar{z}_1}{|z_1|} \end{pmatrix},$$

and

$$f_S(g) = g \cdot s_S([g])^{-1} = \begin{pmatrix} z_0 & \bar{z}_1 \\ -z_1 & \bar{z}_0 \end{pmatrix} \begin{pmatrix} \frac{z_0}{|z_0|} & 0 \\ 0 & \frac{\bar{z}_0}{|z_0|} \end{pmatrix} \begin{pmatrix} \bar{z}_0 & -\bar{z}_1 \\ z_1 & z_0 \end{pmatrix} = \begin{pmatrix} \frac{z_0}{|z_0|} & 0 \\ 0 & \frac{\bar{z}_0}{|z_0|} \end{pmatrix}.$$

Then, to finish the definition of the principal bundle structure we define local trivializations $\phi_i : U_i \times H \longrightarrow \pi^{-1}(U_i)$ by

$$\phi_N^{-1}(z_0, z_1) = ([z_0 : z_1], f_N(z_0, z_1))) = (z, \frac{z_1}{|z_1|}) \tag{3.14}$$

and

$$\phi_S^{-1}(z_0, z_1) = ([z_0 : z_1], f_S(z_0, z_1))) = (\zeta, \frac{z_0}{|z_0|}). \tag{3.15}$$

3.3.2 Construction of the angular momentum operators

To proceed to quantize the two-sphere, we first need to define the complex line bundle associated to the Hopf principal bundle. That is, we need to study the bundle

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & SU(2) \times_{\mathcal{U}_n} \mathbb{C} \\ & & \downarrow \pi_F \\ & & S^2 \end{array}$$

The representation $\mathcal{U}_n : U(1) \to GL(\mathbb{C}, 1)$ is taken to be $\mathcal{U}_n(e^{i\phi}) = e^{in\phi}$.

As we already know, the projection $\pi_F : S^3 \times_{\mathcal{U}} \mathbb{C} \to S^2$ is defined by

$$[(z^0, z^1), v] \mapsto \pi(z^0, z^1).$$

The representation of $U(1)$ is indexed by one natural number. A local trivialization of the associated bundle must contain this number n; then, we can define

$$\begin{array}{rcl} \varphi_N : \pi_F^{-1}(U_N) & \rightarrow & U_N \times \mathbb{C} \\ \left[\begin{pmatrix} z_0 & \bar{z}_1 \\ -z_1 & \bar{z}_0 \end{pmatrix}, v\right] & \mapsto & \left([z_0 : z_1], \left(\frac{z_1}{|z_1|}\right)^n v\right) \end{array} \tag{3.16}$$

This trivialization is independent of the chosen representantive. In fact,

$$((z_0, z_1), v) \sim \left((z_0, z_1) \cdot e^{i\phi}, e^{-in\phi} v\right)$$

but

$$\varphi_1 \left[\left(z_0 e^{i\phi}, z_1 e^{i\phi}\right), e^{-in\phi} v\right] = \left([z_0 \lambda : z_1 \lambda], \left(\frac{e^{i\phi} z_1}{|e^{i\phi} z_1|}\right)^n e^{-in\phi} v\right) = \left([z_0 : z_1], \left(\frac{z_1}{|z_1|}\right)^n v\right).$$

In the same way, we define

$$\begin{array}{rcl} \varphi_N^{-1} : U_N \times \mathbb{C} & \rightarrow & \pi_F^{-1}(U_N) \\ ([z_0 : z_1], w) & \mapsto & \left[(z_0, z_1), \left(\frac{z_1}{|z_1|}\right)^{-n} w\right], \end{array} \tag{3.17}$$

$$
\begin{aligned}
\varphi_S : \pi_F^{-1}(U_S) &\;\to\; U_S \times \mathbb{C} \\
((z_0, z_1), v] &\;\mapsto\; \left([z_0 : z_1], \left(\frac{z_0}{|z_0|} \right)^n v \right)
\end{aligned}
\tag{3.18}
$$

and

$$
\begin{aligned}
\varphi_S^{-1} : U_S \times \mathbb{C} &\;\to\; \pi_F^{-1}(U_S) \\
([z_0 : z_1], w) &\;\mapsto\; \left[(z_0, z_1), \left(\frac{z_0}{|z_0|} \right)^{-n} w \right].
\end{aligned}
\tag{3.19}
$$

In this way, we have for the transition functions of the bundle:

$$
(\varphi_S \circ \varphi_N^{-1})([z_0 : z_1], w) = \left([z_0 : z_1], \left(\frac{z}{|z|} \right)^n w \right), \tag{3.20}
$$

then, $g_{SN}([z_0 : z_1]) = (z/|z|)^n$.

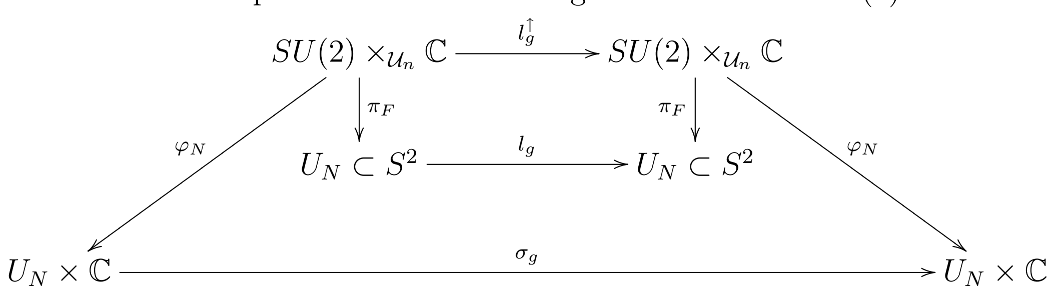
As a configuration space, S^2 is a homogeneous space, and its canonical group is $\mathcal{G} = R^3 \circledS SU(2)$, the semidirect product between a three dimensional vector space and the unitary group. The multiplication law of the group is given by the relation:

$$
(w_2, R_2) \cdot (w_1, R_1) = (w_2 + R_2 w_1, R_2 R_1).
$$

To construct angular momentum operators, we have to study the representations of $SU(2)$. To do this, we need a lifting of the form:

$$
\begin{array}{ccc}
SU(2) \times_{\mathcal{U}_n} \mathbb{C} & \xrightarrow{\;l_g^{\uparrow}\;} & SU(2) \times_{\mathcal{U}_n} \mathbb{C} \\
{\scriptstyle \pi_F} \big\downarrow & & {\scriptstyle \pi_F} \big\downarrow \\
S^2 & \xrightarrow{\;l_g\;} & S^2
\end{array},
$$

where $l_g[g'] := [gg']$ and $l_g^{\uparrow}([g', v]) = [gg', v]$. The diagram below might help to keep track of the different maps involved in the lifting of the action of $SU(2)$.



Since the diagram must commute, we conclude that $\sigma_g = \varphi_N \circ l_g^{\uparrow} \circ \varphi_N^{-1}$. The group $SU(2)$ acts on the sphere by Möbius transformations:

$$
g \cdot z = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix} \cdot z := \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}}.
$$

These are the rotations on the sphere, since $SU(2)$ is the double covering of $SO(3)$. The group $SU(2)$ acts on cross sections as follows:

$$\begin{aligned}
\sigma_g([z_0 : z_1], w) &= (\varphi_N \circ l_g^{\uparrow} \circ \varphi_N^{-1})([z_0 : z_1], w) \\
&= (\varphi_N \circ l_g^{\uparrow}) \left(\left[(z_0, z_1), (z_1/|z_1|)^{-n}\, w\right]\right) \\
&= \varphi_N \left(\left[(\alpha, \beta) \cdot (z_0, z_1), (z_1/|z_1|)^{-n}\, w\right]\right) \\
&= \left([z_0' : z_1'], \left(\frac{\beta z + \alpha}{|\beta z + \alpha|}\right)^n w\right), \qquad (3.21)
\end{aligned}$$

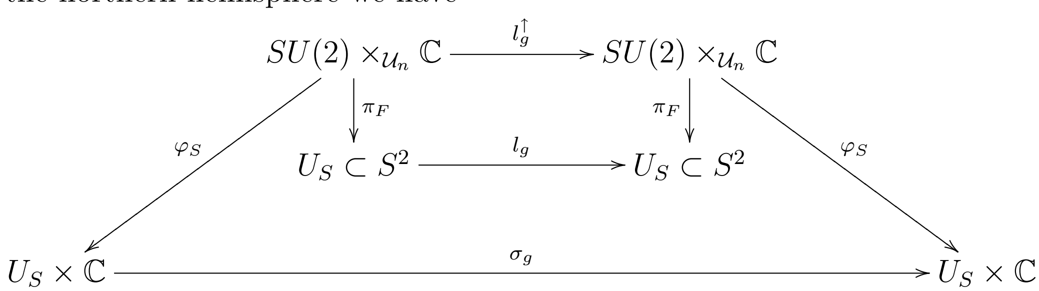
where $z_0' = \alpha z_0 - \bar{\beta} z_1$ and $z_1' = \beta z_0 + \bar{\alpha} z_1$. In this way, a section over this associated bundle $\Psi(x) = (x, |\psi(x)\rangle)$, $x \in \mathcal{Q}$, transforms under an action of the group $SU(2)$ in the following way:

$$\begin{aligned}
(U(g)\Psi)(x) &= \sigma_g(\Psi(g^{-1} \cdot x)) = \sigma_g(g^{-1} \cdot x, \left|\psi(g^{-1} \cdot x)\right\rangle) \\
&= \left(x, \left(\frac{\beta z + \bar{\alpha}}{|\beta z + \bar{\alpha}|}\right)^n \left|\psi(g^{-1} \cdot x)\right\rangle\right). \qquad (3.22)
\end{aligned}$$

This result provides a geometrical point of view to transform spinors. In fact, $SU(2)$ acts on quantum states defined over the southern hemisphere of the sphere in a non-trivial way:

$$|\psi(x)\rangle \mapsto \left(\frac{\beta z + \bar{\alpha}}{|\beta z + \bar{\alpha}|}\right)^n \left|\psi(g^{-1} \cdot x)\right\rangle .$$

In the northern hemisphere we have



$$\begin{aligned}
\sigma_g([z_0 : z_1], w) &= (\varphi_S \circ l_g^{\uparrow} \circ \varphi_S^{-1})([z_0 : z_1], w) \\
&= (\varphi_S \circ l_g^{\uparrow}) \left(\left[(z_0, z_1), (z_0/|z_0|)^{-n}\, w\right]\right) \\
&= \varphi_S \left(\left[(\alpha, \beta) \cdot (z_0, z_1), (z_0/|z_0|)^{-n}\, w\right]\right) \\
&= \left([z_0' : z_1'], \left(\frac{\alpha - \zeta\bar{\beta}}{|\alpha - \zeta\bar{\beta}|}\right)^n w\right), \qquad (3.23)
\end{aligned}$$

and the cross sections transform as

$$\begin{aligned}
(U(g)\Psi)(x) &= \sigma_g(\Psi(g^{-1} \cdot x)) = \sigma_g(g^{-1} \cdot x, \left|\psi(g^{-1} \cdot x)\right\rangle) \\
&= \left(x, \left(\frac{\alpha - \zeta\bar{\beta}}{|\alpha - \zeta\bar{\beta}|}\right)^n \left|\psi(g^{-1} \cdot x)\right\rangle\right). \qquad (3.24)
\end{aligned}$$

Again, $SU(2)$ acts on quantum states defined over the northern hemisphere of the sphere in a non-trivial way:

$$|\psi(x)\rangle \mapsto \left(\frac{\alpha-\zeta\bar{\beta}}{|\alpha-\zeta\bar{\beta}|}\right)^n \left|\psi(g^{-1}\cdot x)\right\rangle.$$

To see other derivations of rules of transformations of spinors see [GBVF01].

3.3.3 Infinitesimal generators

In the southern hemisphere

Let

$$\omega = \left(\frac{\beta z+\bar{\alpha}}{|\beta z+\bar{\alpha}|}\right)^n \tag{3.25}$$

To find the infinitesimal generators of the representation

$$|\psi(x)\rangle \mapsto \omega\left|\psi(g^{-1}\cdot x)\right\rangle.$$

we have to make an infinitesimal variation around the identity. We introduce a parametrization $\omega(t)$ and $g(t)$ in such a way that $\omega(0)=1$ and $g(0)=1$. Then, the infinitesimal generators satisfy:

$$\begin{aligned} -i(\tilde{\omega}+\tilde{L})\,|\psi(x)\rangle &= \frac{d}{dt}\bigg|_{t=0}\left(\omega(t)\left|\psi(g(t)^{-1}\cdot x)\right\rangle\right) \\ &= \left(\frac{d}{dt}\bigg|_{t=0}\omega(t)\right)|\psi(x)\rangle + \frac{d}{dt}\bigg|_{t=0}\left|\psi(g(t)^{-1}\cdot x)\right\rangle \end{aligned} \tag{3.26}$$

Angular momentum in z axis

In this case $\alpha = e^{it}$ and $\beta = 0$. Then $\omega = e^{-nit}$ Its infinitesimal generator is

$$-i\tilde{\omega}_z = \frac{d\omega}{dt}\bigg|_{t=0} = -in.$$

We can study the part $\psi(g^{-1}\cdot x)$ in the traditional way, that is, let $g\in SO(3)$. For example, let g the generator of rotation around the z axis be:

$$g = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

then, the action of this matrix in a wave function is

$$\psi(g^{-1}\cdot x) = \psi(\xi_1\cos t - \xi_2\sin t, \xi_1\sin t + \xi_2\cos t, \xi_3).$$

Its infinitesimal generator is given by:

$$\frac{d}{dt}\bigg|_{t=0}\psi(\xi_1\cos t - \xi_2\sin t, \xi_1\sin t + \xi_2\cos t, \xi_3) = \left(-\xi_2\frac{\partial}{\partial\xi_1} + \xi_1\frac{\partial}{\partial\xi_2}\right)\psi(x).$$

In this way

$$\hat{J}_3^N = \hat{L}_3^N + n.$$

Angular momentum in y axis

In this case both parameters α and β are reals numbers. Then, $\alpha = \cos t$ and $\beta = \sin t$. Thus,

$$\begin{aligned}\tilde{\omega}_y &= \left.\frac{d\omega}{dt}\right|_{t=0} = \left.\frac{d}{dt}\right|_{t=0}\left(\frac{\cos t + \sin t z}{\cos t + \sin t\bar{z}}\right)^{n/2} \\ &= in\,(z-\bar{z}) = in\frac{\xi_2}{1-\xi_3}. \end{aligned} \tag{3.27}$$

In this way

$$\hat{J}_2^N = \hat{L}_2^N - n\frac{\xi_2}{1-\xi_3}.$$

To recover the expression of J_1^S we compute the commutator between the two above generators. In this way, we have recovered the expression for the Poincaré vector written in equation (3.8). In general we have that

$$\hat{\mathbf{J}}^N = \hat{\mathbf{L}}^N - n\hat{\mathbf{K}}^N.$$

In the northern hemisphere

The situation for the northern hemisphere is quite similar. Let

$$\omega = \left(\frac{\alpha - \zeta\bar{\beta}}{|\alpha - \zeta\bar{\beta}|}\right)^n \tag{3.28}$$

To find the infinitesimal generators of the representation

$$|\psi(x)\rangle \mapsto \omega\left|\psi(g^{-1}\cdot x)\right\rangle.$$

we have to make, again, an infinitesimal variation around the identity. Then, the infinitesimal generators satisfy:

$$\begin{aligned}-i(\tilde{\omega}+\tilde{L})\,|\psi(x)\rangle &= \left.\frac{d}{dt}\right|_{t=0}\left(\omega(t)\left|\psi(g(t)^{-1}\cdot x)\right\rangle\right) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0}\omega(t)\right)|\psi(x)\rangle + \left.\frac{d}{dt}\right|_{t=0}\left|\psi(g(t)^{-1}\cdot x)\right\rangle \end{aligned} \tag{3.29}$$

Angular momentum in z axis

In this case $\alpha = e^{it}$ and $\beta = 0$. Then $\omega = e^{nit}$ Its infinitesimal generator is

$$-i\tilde{\omega}_z = \left.\frac{d\omega}{dt}\right|_{t=0} = in.$$

In this way, its infinitesimal generator is

$$\hat{J}_3^S = \hat{L}_3^S - n.$$

Angular momentum in y axis

In this case both parameters α and β are reals numbers. Then, $\alpha = \cos t$ and $\beta = \sin t$. Thus,

$$\begin{aligned} \tilde{\omega}_y &= \left.\frac{d\omega}{dt}\right|_{t=0} = \left.\frac{d}{dt}\right|_{t=0} \left(\frac{\cos t - \zeta \sin t}{\cos t - \bar{\zeta} \sin t} \right)^{n/2} \\ &= in\left(\bar{\zeta} - \zeta\right) = in\frac{\xi_2}{1+\xi_3}. \end{aligned} \tag{3.30}$$

In this way

$$\hat{J}_2^S = \hat{L}_2^S - n\frac{\xi_2}{1+\xi_3}.$$

To recover the expression of J_1^S we compute the commutator between the two above generators. In this way, we have recovered the expression for the Poincaré vector written in equation (3.7):

$$\hat{\mathbf{J}}^S = \hat{\mathbf{L}}^S - n\hat{\mathbf{K}}^S.$$

Remark 3.1 (Quantization of the electric charge)**.** Notice that the quantum mechanical representation of Poincaré's angular momentum vector that we have obtained forces us to take

$$\mu = n.$$

This is, in fact, an expression of the quantization of the electric charge.

Chapter 4

Indistinguishable particles

As an application of the methods discussed in chapter 2 we will work out the case of the projective space. As it was explained in the discussion of chapter 1 this is the space to be considered when studying the problem of two indistinguishable particles. This is an interesting case, since complex line bundles on it are naturally associated to $SU(2)$-bundle structures [Kel06].

4.1 Projective space RP^2

Since the projective space can be seen as a principal bundle with discrete fibres:

$$\begin{array}{ccc} Z_2 & \longrightarrow & S^2 \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & RP^2 \end{array}$$

any quantization of it will give rise to wave functions taking values in Z_2, which we will think of as an expression of the spin-statistics relation. In fact, if we define the associated bundle

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & SU(2) \times_{\mathcal{U}_\pm} \mathbb{C} \\ & & \Big\downarrow{\scriptstyle \pi_F} \\ & & RP^2 \end{array}$$

where $\mathcal{U}_\pm$ denotes the two possible representations of Z_2, from eq. (C.4) we have the following isomorphism

$$\Gamma(SU(2) \times_{\mathcal{U}_\pm} \mathbb{C}) \approx \{\psi \in C^\infty(SU(2), \mathbb{C}) | \psi(pk) = \mathcal{U}_\pm(k^{-1})\psi(p) \quad \text{for all} \;\; k \in S_2\}. \tag{4.1}$$

This is an example of the spin-statistics relation for two indistiguishable particles:

$$\psi(z_0, z_1) = (\pm 1)\psi(z_1, z_0).$$

In these lines, we will study the topological structure of the projective space. We already know that $S^2 = \{(\xi_1, \xi_2, \xi_3) \in \Re^3 | \sum(\xi_i)^2 = 1\}$ and $RP^2 = S^2/\sim$ where

$(\xi_1, \xi_2, \xi_3) \sim (\xi_1', \xi_2', \xi_3')$ if and only if $\xi_i = \gamma \xi_i'$, with $\gamma \in \{1, -1\}$. Let us introduce the following coordinate neighborhoods:

$$U_\alpha = \{[\vec{\xi}] = [\xi_1, \xi_2, \xi_3] \in RP^2 | \xi_\alpha \neq 0\} \tag{4.2}$$

Local trivializations are functions that satisfy $\phi_\alpha^{-1} : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times Z_2$ defined by

$$[\xi_1, \xi_2, \xi_3] \mapsto ([\vec{\xi}], \mathrm{sgn}(\xi_\alpha))$$

and satisfy $\phi_\alpha : U_\alpha \times Z_2 \longrightarrow \pi^{-1}(U_\alpha)$ defined by

$$([\vec{\xi}], \lambda) \mapsto \lambda \, \mathrm{sgn}\,(\xi_\beta)\vec{\xi}.$$

In this way, we have

$$\phi_\beta^{-1} \circ \phi_\alpha : ([\vec{\xi}], \lambda) \mapsto ([\vec{\xi}], \lambda \, \mathrm{sgn}(\xi_\alpha) \, \mathrm{sgn}\,(\xi_\beta)).$$

and the transition functions are

$$g_{\alpha\beta}(\vec{\xi}) = \mathrm{sgn}(\xi_\alpha)\mathrm{sgn}(\xi_\beta).$$

The projective space can also be described as a homogeneous space $SU(2)/H$ (for details, see appendix B). Studying the projective space in this way lead us to use the techniques employed in the case of S^2

4.2 Quantization of RP^2

Let us define the bundle L_-, a sub-bundle of the trivial bundle $RP^2 \times \mathbb{C}^3$. The construction is as follows [Re06]. The total space of the bundle is defined by:

$$E(L_-) = \left\{ ([x]\,, \lambda \,|\phi(x)\rangle) \in RP^2 \times \mathbb{C}^3 \,|\, \lambda \in \mathbb{C} \,\mathrm{and}\, x \in [x] \right\},$$

where

$$|\phi(x)\rangle = (x_1, x_2, x_3).$$

This definition carries some ambiguity, since the choice of λ is not unique. The function $|\phi(x)\rangle$ is well-defined only over the sphere, not over the elements of the projective plane. Nevertheless, the kets $|\phi(x)\rangle$ satisfy an antisymmetric relation, that is, $|\phi(-x)\rangle = -|\phi(x)\rangle$. The value of λ is determined by the choice of the representative, and, then, an element $y \in E(L_-)$ is described in exactly two ways:

$$y = ([x]\,, \lambda \,|\phi(x)\rangle) = ([x]\,, (-\lambda) \,|\phi(-x)\rangle)\,.$$

Since $E(L_-)$ is embedded in the total space of a trivial bundle, the projection map is obviously defined as:

$$\pi\left([x]\,, \lambda \,|\phi(x)\rangle\right) := [x]\,,$$

and local trivializations can be defined on the local charts defined in eq. (4.2):

$$\begin{array}{rcl}\varphi_\alpha : \pi^{-1}(U_\alpha) & \to & U_\alpha \times \mathbb{C} \\ ([x]\,, \lambda\,|\phi(x)\rangle) & \mapsto & ([x]\,, \mathrm{sign}(x_\alpha)\lambda)\,, \end{array} \tag{4.3}$$

and

$$\begin{array}{rcl}\varphi_\beta^{-1} : U_\beta \times \mathbb{C} & \to & \pi^{-1}(U_\alpha) \\ ([x]\,, w) & \mapsto & ([x]\,, \mathrm{sign}(x_\beta) w\,|\phi(x)\rangle)\,. \end{array} \tag{4.4}$$

Note that both maps are well defined because, by definition, $[x] \in U_\alpha$ if and only if $x_\alpha \neq 0$.

The associated complex line bundle to the principal bundle $(SU(2), SU(2)/H, H)$ is the following

$$\begin{array}{ccc}\mathbb{C} & \longrightarrow & SU(2) \times_{\mathcal{U}_n} \mathbb{C} \\ & & \Big\downarrow{\scriptstyle \pi_F} \\ & & RP^2 \end{array}$$

where the total space is defined by

$$SU(2) \times_{\mathcal{U}_n} \mathbb{C} = \{[g, v]\,|\, g \in SU(2) \,\mathrm{and}\, v \in \mathbb{C}\}\,.$$

The equivalence class is the one defined by the action of the group

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \begin{pmatrix} 0 & \bar{\lambda} \\ -\lambda & 0 \end{pmatrix} \mid\ |\lambda|^2 = 1 \right\}.$$

We denote the orbits of this action as $[[z_0 : z_1]]$. There are two one-dimensional representations of this group. The first one is the trivial representation, that is, $h \mapsto 1$ for all $h \in H$. To define the second representation, we define de following map:

$$\begin{array}{rcl}\kappa : \ H & \longrightarrow & \mathbb{C} \\ \\ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} & \longmapsto & 1 \\ \begin{pmatrix} 0 & \bar{\lambda} \\ -\lambda & 0 \end{pmatrix} & \longmapsto & -1. \end{array}$$

It is easy to show that this map defines a representation.

We define local trivializations in terms of the Hopf map $x : SU(2) \to S^2$ by:

$$\begin{array}{rcl}\vartheta_\alpha : \pi^{-1}(U_\alpha) & \to & U_\alpha \times \mathbb{C} \\ [(z_0, z_1), v] & \mapsto & ([[z_0 : z_1]], \mathrm{sgn}\,(x_\alpha(z_0, z_1))\, v) \end{array} \tag{4.5}$$

$$\begin{array}{rcl}\vartheta_\beta^{-1} : U_\alpha \times \mathbb{C} & \to & \pi^{-1}(U_\beta) \\ ([[z_0, z_1]], w] & \mapsto & ((z_0, z_1), \mathrm{sgn}\,(x_\beta(z_0, z_1))\, w)\,. \end{array} \tag{4.6}$$

We are looking for an isomorphism

$$\begin{array}{rcl} \Phi_\alpha : \pi^{-1}(U_\alpha) \subset SU(2) \times_\kappa \mathbb{C} & \to & \pi^{-1}(U_\alpha) \subset L_- \\ [g, v] & \mapsto & \left([x(g)]\,, v\,|\psi(x(g))\rangle\right). \end{array}$$

This definition is independent of the chosen representative. In fact, if we take any element of the equivalence class we have

$$[gh, \kappa(h)v] \mapsto \left([x(g)]\,, \kappa(h)v\,|\psi(x(gh))\rangle\right) = \left([x(g)]\,, v\,|\psi(x(g))\rangle\right).$$

The inverse function is:

$$\begin{array}{rcl} \Phi_\alpha^{-1} : \pi^{-1}(U_\alpha) \subset L_- & \to & \pi^{-1}(U_\alpha) \subset SU(2) \times_\kappa \mathbb{C} \\ \left([x]\,, \lambda\,|\psi(x)\rangle\right) & \mapsto & [(z_0, z_1), \lambda]\,, \end{array} \tag{4.7}$$

where the element in $SU(2)$ is chosen in such a way that $\pi(z_0, z_1) = x$. For antisymmetric wave functions we have the following bundle isomorphism:

$$\begin{array}{ccc} L_- & \xrightarrow{\tau_g} & L_- \\ \downarrow{\scriptstyle \Phi^{-1}} & & \downarrow{\scriptstyle \Phi^{-1}} \\ SU(2) \times_{\mathcal{U}_n} \mathbb{C} & \xrightarrow{l_g^{\uparrow}} & SU(2) \times_{\mathcal{U}_n} \mathbb{C} \\ \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi_F} \\ U_\alpha \subset RP^2 & \xrightarrow{l_g} & U_\alpha \subset RP^2 \end{array}$$

Since the diagram commutes, $\tau_g = \Phi \circ l_g^{\uparrow} \circ \Phi^{-1}$. Then,

$$\begin{aligned} \tau_g\left([x]\,, \lambda\,|\phi(x)\rangle\right) &= \left(\Phi \circ l_g^{\uparrow} \circ \Phi^{-1}\right)\left([x]\,, \lambda\,|\phi(x)\rangle\right) \\ &= \left(\Phi \circ l_g^{\uparrow}\right)\left[(z_0, z_1)\,, \lambda\right] \\ &= \Phi\left[(\alpha, \beta) \cdot (z_0, z_1)\,, \lambda\right] \\ &= \left(\left[x(z_0', z_1')\right], \lambda\left|\phi(x(z_0', z_1'))\right\rangle\right), \end{aligned} \tag{4.8}$$

where, as in the magnetic monopole case, $z_0' = \alpha z_0 - \bar{\beta} z_1$ and $z_1' = \beta z_0 + \bar{\alpha} z_1$. We can write the result obtained above in the following way:

$$\tau_g\left([x]\,, \lambda\,|\phi(x)\rangle\right) = \left(\left[x(z_0', z_1')\right], \lambda\mathcal{D}(g)\,|\phi(x)\rangle\right), \tag{4.9}$$

where

$$\mathcal{D}(g) = \begin{pmatrix} 1/2(\alpha^2 + \bar{\alpha}^2 - \beta^2 - \bar{\beta}^2) & -i(\alpha^2 - \bar{\alpha}^2 - \beta^2 + \bar{\beta}^2) & \alpha\beta + \bar{\alpha}\bar{\beta} \\ -i(\alpha^2 - \bar{\alpha}^2 + \beta^2 - \bar{\beta}^2) & 1/2(\alpha^2 + \bar{\alpha}^2 + \beta^2 + \bar{\beta}^2) & -i(\alpha\bar{\beta} - \bar{\alpha}\beta) \\ -(\alpha\bar{\beta} + \bar{\alpha}\beta) & -i(\alpha\beta - \bar{\alpha}\bar{\beta}) & (|\alpha|^2 - |\beta|^2) \end{pmatrix}.$$

4.3 Discussion

In this section we will briefly discuss the result obtained in the previous one. Let us define a vector-valued function on the sphere, as follows:

$$|\psi\rangle = \sqrt{\frac{4\pi}{3}} \begin{pmatrix} Y_{1,-1} \\ -Y_{1,0} \\ Y_{1,1} \end{pmatrix}.$$

This function transforms in an equivariant way with respect to the action of $SU(2)$ on the sphere, i.e. it satisfies [Pas01]:

$$|\psi(g \cdot x)\rangle = \mathcal{D}^{(1)}(g)|\psi(x)\rangle,$$

where $\mathcal{D}^{(1)}$ stands for the three dimensional irreducible, unitary representation of $SU(2)$. It is possible to show that the projector $|\psi\rangle\langle\psi|$ is isomorphic to the module of sections on the line bundle L_- [Pas01]. Moreover, it can be shown that the transported spin basis of Berry and Robbins, when expressed in the total angular momentum basis, decomposes in a sum of terms each one of which gives place to a line bundle over the projective space. When the corresponding bundle is L_-, the associated projector (which is obtained directly from the operator $U(r)$ defining the transported basis [BR97]) exactly coincides with $|\psi\rangle\langle\psi|$ [Re06]. The $SU(2)$ equivariance property of $|\psi\rangle$ immediately implies that the line bundle corresponding to the projector $|\psi\rangle\langle\psi|$ has the structure of an equivariant $SU(2)$-bundle. This structure can be used to define the transformation properties of a section, with respect to the action of $SU(2)$. In the present work, we have derived these same transformation properties using a different representation of the bundle L_-. The fact that the angular momentum operators for a quantum theory based on RP^2 are precisely the infinitesimal generators of the unitary operator induced by the lift of the $SU(2)$ action is something that follows naturally from the quantization approach we have applied. It is therefore satisfying to confirm that these operators coincide with the ones that are intrinsically present in the definition of the projector $|\psi\rangle\langle\psi|$. At this point, all we can affirm is that, both from the physical and mathematical points of view, these *are* the correct angular momentum operators. This is the starting point for a detailed study of the theorem stated by Kuckert [Kuc04]. Our hope is that some kind of obstruction involving the angular momentum operators (arising from the topology of the configuration space) makes it impossible to have spin zero fermions (in the context of Kuckert's work, this would amount to have a definite statement about the existence or non existence of the unitary operator U relating the angular momentum operators of a two particle system to those of a one particle system). The answer to this interesting question is not clear at this point, and will be left for future work.

Chapter 5

Final discussion and Perspectives

In this work, our efforts have been directed towards the understanding of a general scheme of quantization and its relevance for the understanding of the spin-statistics relation. We have chosen Isham's method, basically, because its geometrical approach to the problem is a way to understand the relation between symmetry groups on the classical phase space and the algebra of operators acting on Hilbert space. Its emphasis on the search for the "correct" canonical commutation relations for a given configuration space makes the method particularly appealing for our purposes. Even though the initial motivation of Isham was to study problems related with the quantization of gravity, the method has provided a unifying point of view for many of the well-known topological effects in physics. We have used his techniques to find angular momentum operators for the quantization the two-sphere and the real projective space. In the case of the two-sphere, we obtained a quantum version of Poincaré's angular momentum vector. For simplicity, it is natural to consider the problem of indistinguishability first at the level of two particle system. In the case of the projective space, the obtained operators coincide with those considered in [Re06] and are therefore equivalent to the transported spin operators postulated by Berry and Robbins some time ago [BR97]. There are, therefore, no fundamentally new results in our approximation. But, as Feynman once said, *"there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage"* [Fey48].

Having obtained the angular momentum operators from a quantization scheme, one should have, in principle, full control of the underlying Hilbert space representation and its relation to the topology of the configuration space. As mentioned in the first chapter, this could open interesting possibilities regarding the spin-statistics connection, if a link with the work of Kuckert [Kuc04] could be established.

The question of how to quantize a classical system where an angle ϕ is involved in its description has been controversial since the beginning of quantum mechanics. Basically, the problem is that this parameter is a multivalued function defined on the corresponding phase space. In the case of a configuration space of the form $\mathcal{Q} = S^1$, the phase space has the topological global structure of a cylinder $S^1 \times \Re$, and the Poisson algebra of the three functions $\sin\phi, \cos\phi$ and p_ϕ obey the Lie algebra commutation relations of the euclidian group $SO(2) \circledS \Re^2$. The irreducible unitary representations of this group allow for fractional angular momentum of the form: $l = \hbar(n + \delta)$, $n \in \mathbb{Z}$ and $\delta \in [0, 1)$. Very recently, proposals to look for orbital angular momentum operator in connection to quantum optics of laser modes in external magnetic fields have appeared, using techniques quite similar to those that we have used in this work [Kas05]. The relevance of these possibility should not be overlooked, not only because of its theoretical attractiveness, but also because there seems to be strong experimental evidence for it [OMV+05].

This suggests that applications to the field of quantum optics of techniques like the ones studied in these thesis could be a promising research field.

Appendix A

Spinor map

The special unitary group $SU(2)$ and the rotation group $SO(3)$ have remarkable relationships between them. $SO(3)$ is the group of 3×3 matrices R satisfying $R^T = R^{-1}$ and $\det(R) = 1$. Its elements are linear transformations whose action rotate a vector in $\Re^3$. Every rotation can be parameterized by an axis of rotation $\hat{n} = (x, y, z)$, with $|x|^2 + |y|^2 + |z|^2 = 1$, and an angle of rotation about this axis, ψ. Then, we define every matrix in the rotation group $R(\psi, \hat{n})$ by

$$R(\psi, \hat{n}) = e^{\psi N} = Id + (\sin\psi)N + (1 - \cos\psi)N^2, \tag{A.1}$$

where Id is the identity matrix and

$$N = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix},$$

is a skew-symmetric matrix lying in the Lie algebra $so(3)$, with the following properties:

$$N^2 = \begin{pmatrix} -(y^2 + z^2) & xy & xz \\ xy & -(x^2 + z^2) & yz \\ xz & yz & -(x^2 + y^2) \end{pmatrix}$$

and $N^3 = -N$, $N^4 = -N^2$, $N^5 = N$, ... The matrix $R(\psi, \hat{n})$ arises from geometrical considerations. It is, in fact, the rotation through an angle ψ about an axis along $\hat{n}$.

Lemma A.1. [Nab97] For any $\psi \in \Re$ and any $\hat{n} = (x, y, z)$, with $|x|^2+|y|^2+|z|^2 = 1$, $R(\psi, \hat{n})$ is in $SO(3)$. Conversely, given an $R \in SO(3)$ there exists a unique real number $\psi \in [0, 2\pi]$ and a unique unit vector $\hat{n} \in \Re^3$ such that $R = R(\psi, \hat{n})$.

The set of all 2×2 unitary matrices with determinant equals to one forms a group, since it is closed under multiplication. The special unitary group $SU(2)$ is defined by:

$$SU(2) = \left\{ U \in GL(2, \mathbb{C}) | UU^\dagger = UU^\dagger = 1, \, \det U = 1 \right\}.$$

Typical members of this group are matrices of the form

$$\mathbf{u} = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}, \quad \text{with } |\alpha|^2 + |\beta|^2 = 1.$$

Since α and β are complex numbers, the condition imposed to the determinant leaves three independet parameters. In fact, we can write all 2×2 unitary matrices in terms of an axis of rotation and an angle around this axis, in the same way that the members of the rotation group. Let $\mathbf{r}$ be a vector in $\Re^3$. This vector can be written as a unitary matrix using the three Pauli spin matrices, that is,

$$\mathbf{h} = \mathbf{r} \cdot \sigma = r_x \sigma_x + r_y \sigma_y + r_z \sigma_z = \begin{pmatrix} r_z & r_x - i r_y \\ r_x + i r_y & -r_z \end{pmatrix}.$$

Let us transform $\mathbf{h}$ into a new matrix $\mathbf{h}'$ by a unitary transformation using a matrix from the group $SU(2)$. Then

$$\mathbf{h}' = \mathbf{u}\mathbf{h}\mathbf{u}^{-\mathbf{1}} \equiv \mathbf{r}' \cdot \sigma.$$

In this way, associated with each matrix $\mathbf{u} \in SU(2)$ there is a 3×3 matrix $\mathbf{R}(\mathbf{u})$ which transforms the vector $\mathbf{r}$ into the vector $\mathbf{r}'$.

A.1 The map $SU(2) \to SO(3)$

Lemma A.2. [Nab97] Let $\mathbf{u} = \begin{pmatrix} a+ib & c+id \\ -c-id & a-ib \end{pmatrix}$ be a unitary matrix in $SU(2)$, then the matrix $R_{\mathbf{u}} : \Re^3 \to \Re^3$ relative to the three Pauli matrices is

$$\begin{pmatrix} a^2 - b^2 - c^2 + d^2 & 2ab + 2cd & -2ac + 2bd \\ -2ab + 2cd & a^2 - b^2 + c^2 - d^2 & 2ad + 2bc \\ 2ac - 2bd & 2bc - 2ad & a^2 + b^2 - c^2 - d^2 \end{pmatrix}.$$

With this construction we are able to define the **spinor map**

$$\begin{aligned} \text{Spin} : SU(2) &\to SO(3) \\ u(\psi, \hat{n}) &\to R(\psi, \hat{n}), \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} u(\psi, \hat{n}) &= \cos(\psi/2) Id - i \sin(\psi/2)(x\sigma_x + y\sigma_y + z\sigma_z) \\ &= \begin{pmatrix} \cos(\psi/2) - i\sin(\psi/2) z & -y\sin(\psi/2) - ix\sin(\psi/2) \\ y\sin(\psi/2) - ix\sin(\psi/2) & \cos(\psi/2) + i\sin(\psi/2) z \end{pmatrix}. \end{aligned} \tag{A.3}$$

The spinor map is a two-one map, carrying $\pm u$ onto the same element of $SO(3)$. If we identify $\mathbb{Z}_2$ with the subgroup of $SU(2)$ generated by the kernel of the spin map: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, then we have an isomorphism between the quotient group $SU(2)/\mathbb{Z}_2$ an the rotation group $SO(3)$.

A.2 Special matrices in $SU(2)$

A rotation around the z axis is represented by a matrix of the form

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since this is a subgroup of $SO(3)$, by means of the spinor map, we can associated each matrix of this form with a matrix in $SU(2)$. In fact, by the equation (A.1) we can find an angle ψ and a unit vector $\hat{n}$ such that

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = Id + (\sin\psi)N + (1-\cos\psi)N^2.$$

It will show in the appendix B that the corresponding matrix in $SU(2)$ is:

$$\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \tag{A.4}$$

A rotation around the y axis is represented by a matrix of the form

$$\begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}.$$

Since this is, again, a subgroup of $SO(3)$, by means of the spinor map, we can associated each matrix of this form with a matrix in $SU(2)$. In fact, by the equation (A.1) we can find an angle ψ and a unit vector $\hat{n}$ such that

$$\begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} = Id + (\sin\psi)N + (1-\cos\psi)N^2.$$

The equation for the element R_{22} is $1 = 1-(1-\cos\psi)(x^2+z^2)$. Then, $x^2+z^2=0$. Since this is the sum of two positive numbers, the only solution is $x=z=0$. This implies that $y^2=1$, and we choose, $z=-1$. Finally, the equation for the element R_{11} is $\cos\phi = 1-(1-\cos\psi)$; then $\psi=\phi$. The conclusion is

$$\begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} = R(\phi,(0,1,0)).$$

The corresponding matrix in $SU(2)$ is, then:

$$\begin{pmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{pmatrix}. \tag{A.5}$$

In this case, α and β are both real numbers.

Appendix B

RP^2 as a homogeneous space

The projective plane RP^2 can be visualized as the quotient

$$\frac{SO(3)}{O(2)},$$

where $O(2)$ is the group of orthogonal transformations. This group can be written as the union of two disjoint sets: the matrices with determinant 1 and the matrices with determinant -1. Then, in order to show that the projective space is a homogeneous space, we write the orthogonal group $O(2)$ as a subgroup of $SO(3)$:

$$\left\{ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -\cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} \mid 0 \leq \phi \leq 2\pi \right\}.$$

Since this is a subgroup of $SO(3)$, by means of the spinor map, we can associated each matrix in $O(2)$ with a matrix in $SU(2)$. In fact, by the equation (A.1) we can find an angle ψ and a unit vector $\hat{n}$ such that

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = Id + (\sin\psi)N + (1-\cos\psi)N^2.$$

The equation for the element R_{33} is $1 = 1-(1-\cos\psi)(x^2+y^2)$. Then, $x^2+y^2 = 0$. Since this is the sum of two positive numbers, the only solution is $x = y = 0$. This implies that $z^2 = 1$, and we choose, for reasons to be clear in the next step, $z = -1$. Finally, the equation for the element R_{11} is $\cos\phi = 1-(1-\cos\psi)$; then $\psi = \phi$. The conclusion is

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = R(\phi, (-1, 0, 0)).$$

The corresponding matrix in $SU(2)$ is, then:

$$\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \tag{B.1}$$

In the same way, we also can find an angle ψ and a unit vector $\hat{n}$ such that

$$\begin{pmatrix} -\cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} = Id + (\sin\psi)N + (1-\cos\psi)N^2.$$

This matrix to the square is equal to the identity. We can use this fact to find the value of the angle:

$$Id = \left[Id + (\sin\psi)N + (1-\cos\psi)N^2\right]^2 = Id + \sin\psi(\sin\psi + 1 - 2\cos\psi)N^2.$$

The equality is satisfied if $\sin\psi = 0$, then $\psi = \pi$. The equation for the element R_{33} is $-1 = 1 - 2(x^2+y^2)$, and the conclusion is that $x^2+y^2 = 1$; it follows that $z = 0$. Finally, the equations for the elements R_{11} and R_{22} are:

$$-\cos\phi = 1 - 2y^2 \tag{B.2}$$

$$\sin\phi = 1 - 2x^2, \tag{B.3}$$

then we have that $x = \sin(\phi/2)$ and $y = \cos(\phi/2)$. The conclusion is

$$\begin{pmatrix} -\cos\phi & \sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} = R(\pi, (\cos(\phi/2), \cos(\phi/2), 0)).$$

The corresponding matrix in $SU(2)$ is, then:

$$\begin{pmatrix} 0 & ie^{i\phi/2} \\ -ie^{-i\phi/2} & 0 \end{pmatrix}. \tag{B.4}$$

By the spinor map, we have written the projective space as a homogeneous space of the form

$$\frac{SU(2)}{H}, \tag{B.5}$$

where

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \begin{pmatrix} 0 & \bar{\lambda} \\ -\lambda & 0 \end{pmatrix} \mid |\lambda|^2 = 1 \right\},$$

is a subgroup of $SU(2)$. The orbits of this actions are generated by the action of H in the following way:

$$(\alpha,\beta)\cdot\lambda = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \alpha\lambda & \bar{\beta}\bar{\lambda} \\ -\beta\lambda & \bar{\alpha}\bar{\lambda} \end{pmatrix} = (\alpha\lambda, \beta\lambda)$$

and

$$(\alpha,\beta)\cdot\lambda = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 0 & \bar{\lambda} \\ -\lambda & 0 \end{pmatrix} = \begin{pmatrix} -\bar{\beta}\lambda & \alpha\bar{\lambda} \\ -\bar{\alpha}\lambda & -\beta\bar{\lambda} \end{pmatrix} = (-\bar{\beta}\lambda, \bar{\alpha}\lambda).$$

Appendix C

Associated Bundles

The basic idea in the theory of associated bundles is that given a particular principal bundle $(G, \pi, \mathcal{M}, H)$ with structure group H, we can form a fibre bundle with fibre F for each space F on which H acts as a group of transformations.

Definition C.1. [Ish03] Let X be a right H-space and let Y be a left H-space. Then the H-product of X and Y is the space of orbits of the H-action on the Cartesian product $X \times Y$. Thus, we define an equivalence relation on $X \times Y$ in which $(x, y) \sim (u, v)$ if and only if there exists $h \in H$ such that $u = xh$ and $v = h^{-1}y$.

Definition C.2. Let $\xi = (G, \pi, \mathcal{M}, H)$ be a principal H-bundle and let F be a left H-space. Define $G \times_H F = G \times F / \sim$, where $(g, v) \cdot h = (gh, h^{-1}v)$ and define a map $\pi_F : G \times_H F \to \mathcal{M}$ by $\pi_F([g, v]) = \pi(g)$. Then $\xi[F] = (G \times_H F, \pi_F, \mathcal{M})$ is a fibre bundle over $\mathcal{M}$, with fibre F. This is said to be associated to the principal bundle.

In particular, if $\mathcal{U} : H \longrightarrow Gl(\mathbb{C}^n)$ is a unitary representation of H on $\mathbb{C}^n$ we can construct the associated vector bundle

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & G \times_{\mathcal{U}} \mathbb{C}^n \\ & & \Big\downarrow \pi_F \\ & & \mathcal{M} \end{array}$$

where $(p, v) \equiv (ph, \mathcal{U}(h^{-1})v)$. The fibre of this bundle is the vector space $\mathbb{C}^n$.

Cross-sections

The cross-sections $\Gamma(P \times_{\mathcal{U}} F)$ are of considerable importance and are related to some special type of functions. Let $\psi \in C^\infty(P, F)$ with the property

$$\psi(pk) = \mathcal{U}(k^{-1})\psi(p) \qquad \text{for all} \quad k \in K. \tag{C.1}$$

Then we can define a corresponding cross-section S_ψ by the following relation

$$S_\psi(x) = [p_x, \psi(p_x)], \qquad \text{for any} \quad p_x \in \pi^{-1}(x) = K_x. \tag{C.2}$$

This definition is independent of the choice of the point p_x in the fibre K_x. In fact, let $p_x = pk$ where also $p \in \pi^{-1}(x)$

$$S_\psi(x) = [p_x, \psi(p_x)] = [pk, \psi(pk)] = [p, \psi(p)].$$

Conversely, given a cross section $S \in \Gamma(P \times_{\mathcal{U}} F)$ define $\psi_S \in C^\infty(P, F)$ by the relation

$$S(\pi(p)) = [p, \psi_S(p)]. \tag{C.3}$$

In this case $[p, \psi_S(p)] = S(\pi(p)) = S(\pi(pk)) = [pk, \psi_S(pk)] = [p, \mathcal{U}(k)\psi_S(pk)]$ and hence $\psi_S(p) = \mathcal{U}(k)\psi_S(pk)$. Thus we have arrived to the following isomorphism:

$$\Gamma(P \times_{\mathcal{U}} F) \approx \{\psi \in C^\infty(P, F) | \psi(pk) = \mathcal{U}(k^{-1})\psi(p) \quad \text{for all} \quad k \in K\}. \tag{C.4}$$

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